The Collatz Conjecture

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This is my take on the **Collatz Conjecture**, hopefully I dont sound mad by the end of this article. Either way, I hope you have a good read. Happy Mathin' :))

1 Introduction

This is one of the most simple yet famous and unsolved puzzles in mathematics. Mathematicians have spent over decades on this problem, but have not been able to find a solution. "Mathematics is not ripe enough to solve such a problem yet"- Paul Erdos

1.1 Problem Statement

- This conjecture states a simple rule, take any positive integer **n** and apply it to the following function:

$$f(n) = \begin{cases} n/2 & \text{if n is even} \\ 3n+1 & \text{if n is odd} \end{cases}$$

Now, do this for several iterations after picking an initial particular number.

2 The Conjecture

Now, what this Conjecture says is that no matter what positive integer we begin with and apply the operation, the value will always end up to 1 and will eventually end up in a $4- > 2- > 1$ loop.

This can be seen by taking a couple of examples:

- Starting with 5 we get: $5- > 16- > 8- > 4- > 2- > 1$ and then the $4- > 2- > 1$ loop
- Starting with 6 we get: $6- > 3- > 10- > 5- > 16- > 8- > 4- > 2- > 1$

This is also seen for other positive integers.
But here is the question, is this true for **all** positive integers?

3 My Take on this Conjecture

Mathematicians have been able to prove this conjecture upto the number $2^(68)$, but what about numbers more than that?.Well,when I heard about this and read a bit more about it,even I was amazed as to how just two simple rules on any number can have such an outcome and honestly I felt quite strange.But, when i spent more time thinking about this, I think I was able to come to a couple of observations/conclusions. These might not even be close when it comes to proving the conjecture, but something is better than nothing :))

3.1 Convergence of $f(n)$

With the help of a bit of backtracking, I noticed that to reach 1 in the end, we first need to reach 2 since we cant reach 1 with the help of $3n + 1$,so for that we need to reach 4 or 8 or 16.. and so on. If you notice the numbers above, they all are powers of 2, which shows that to get to the 1, at some point in the iterations, you need to arrive to a number which is a power of 2, which will finally lead to 1. This can also be seen for the case of 3, once we reach 16, repeated division leads to 1. In the case of 21, it immediately reaches 1 since after a single iteration it reaches the number 32. Now if this conjecture is true upto a large range of positive integers, it would be fair to say that for each n, **at some point the function $f(n)$ will always converge to a number which will be of the form 2^k**, which will end up to 1.

Hence, if we begin with any n:
$f(n)- > 2^k$ for some positive integer $k >= 2$

3.2 How many iterations?

This was a small interesting thing that I came across,but I felt it made some sort of sense. It is that **on an average, the number of iterations required to reach 1 for an odd number is more than the number of iterations required for an even number**. Well this could make sense if you think about it intuitively.While performing the given operation, **odd** numbers are multiplied by a factor of '3', hence scaling the number to somewhat higher than the original number, hence increasing the number to be feeded in for the second iteration.Wheareas for an **even** number, the operation scales it by a factor of '1/2', hence reducing the value of the output.From the perspective of a **second** intuition, using the argument in **3.1**, an even number will always be closer to an integer power of 2 in comparison to an odd number, since the prime factorisation of an even number already includes some power of 2, it would not be wrong to think that an even number would tentatively take lesser time(iterations) to reach 1.Another way of think of what ive told is that if you pick any odd natural number at random, and pick either the even number before or after, there is a high chance that the number of iterations for the odd number would greater than that for the even number.I also tried doing this visually,here is what I did:

Let $g(n)$ = **number of iterations of $f(n)$ starting with the n required to reach 1 for the first time after atleast one iteration**.

Now,I plotted the graph of $g(n)$ and n for n lying between 1 and 10.

Desmos for Plot of $g(n)$ and n.

This graph clearly shows that most odd numbers on an average require more number of iterations compared to even numbers.It can also be seen that except for 5, the other odd numbers need more number of iterations than both their neighbouring even numbers.

3.3 Nested Iterations and Links

Lets just say I am able to write this section only because of my foolishness.I came across this only because of trying to brute force myself into claim this conjecture wrong.Anyways, all is well that ends well I think.

What I was able to notice was that while performing the Iterations for a particular number, we indirectly get another readymade iteration for a different number, and that particular iteration is **nested** inside the iteration for the initial number.This can be seen with the following iteration:- $3- > 10- > 5- > 16- > 8- > 4- > 2- > 1$. Clearly these are the iterations for the number 3.But if we ignore the first iteration for sometime, we get all the iterations for the number 10.Similarly if we ignore the first 2 iterations, we get the iterations for the number 5 and so on.Hence showing **Nested Iterations**.

Another way to generalize it is:
If n is odd, in the iteration for n we always obtain the iteration for the number $3n + 1$
. If n is even, in the iteration for n we always obtain the iteration for $n/2$

Relating it to **3.2**, we observe:

$g(n) = g(3n + 1) + 1$ for n odd
$g(n) = g(n/2) + 1$ for n even

These Nested Iterations when plotted on paper, form branches of several links between number and all these branches finally terminate at $4- > 2- > 1$.

4 The End

Well,fortunately or Unfortunately,I'll let you decide,but this article comes to an end here.This is all I could tell you about my personal conclusions and observations on the **Collatz Conjecture**.I both wish and do not wish at the same time that you start spending time on this problem. Hope this article was worth a read.

Signing off:))
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