classmate
Date
Page

# Discrete Structures

## Sets SETS

**Sets :-** A set is a collection of well defined and different objects. By the word, well defined we mean that we are given a rule with the help of which we can say whether perticular object belongs to the set or not. The word different implies that repeatition of objects is not allowed.
It is denoted by capital letters.
Eg :- A, B, C -- etc. Eg :- A = {a, b, c}.

**Elements of sets :-**
Each object of the set is called element of set.
Eg :- A set of days of a week.
A set of even integers.

These elements of sets are denoted by small letters.
Eg :- a, b, c -- etc.

**Methods of designating a set :-**

(i) **Tabular, Roster or Enumeration form Method.**

When we represent a set by listing all it's elements within (curlie brackets) { }, so seperated by (comas), is called tabular, roster or enumeration form.
Eg :- A = {a, e, i, o, u}.

(ii) **Selector, Setbuilder or Rule Method.**

When we represent a set by defining all its elements

by its defining property, it is called selector, set builder or rule method.
Eg :- $\{x : x$ is set of vowels$\}$

Membership of a set :-
If an object $x$ is a member of the set A we write $x$ belongs to A $(x \in A)$. Similarly if $x$ is not a member of A we write $x$ does not belong to A $(x \notin A)$ to show that $x$ is not member of A.

Types of Sets :-
Finite set :- A set is said to be finite if it has finite number of elements.
Eg :- $A = \{2, 4, 6, 8\}$

Infinite set :- A set is said to be infinite if it has infinite number of elements.
Eg :- $B = \{x : x$ is an odd integers$\}$.

Singleton set :- A set is containing only one element is called singleton set or unit set.
Eg :- $A = \{x : x$ is a prime minister of India$\}$

Empty, Null, void set :- A set which contains no element is called null set. It is denoted by $\phi$

Eg :- $A = \{x : x$ is a positive integer satisfying $x^2 = \frac{1}{4}\}$

$A = \{\}$.
Both are equal to $\phi$

Scanned by CamScanner

Ques:- Determine which of the following sets are equal to:-
$\phi$, $\{0\}$, $\{\phi\}$.

Ans:- Each set is different from other. The set $\{0\}$ contains one element i.e. the number 0.
The set $\phi$ contains no element. It is empty set.
The $\{\phi\}$ also contains one element ie, null set.

**Subset** :- Let A and B two set. If all the elements of set A are present in set B than A is called subset of B.

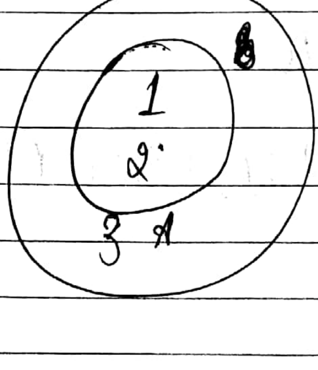
**Superset** :- and B is called superset of A (above).

Note:- If A is not a subset of B then it is written as $A \not\subset B$.

Eg:- $A = \{1, 2\}$   $D = \{4, 6\}$
$B = \{1, 2, 3\}$
$D \not\subset B$ → not subset.
$A \subset B$ — subset

**Proper and Improper Subsets** :-
Let A and B be two sets. If A is a subset of B and B is not equal to A then A is called proper subset of B.

If A is subset of B and both are equal, A is called improper subset of B.

Eg:-



ie. There is at least 1 element in B which is not present in A.

Eg:- Proper subset
$A = \{1, 2\}$, $B = \{1, 2, 3\}$
$A \subset B$ and $A \neq B$

Improper subset
$A = \{1, 2, 3\}$, $B = \{1, 3, 2\}$
$A \subset B$ and $A = B$

Equality of Sets:-
Two set A and B are said to be equal if both have same elements. In other words if two sets A and B are equal when every element of A is an element of B and every element of B is an element of A.
i.e. if $A \subset B$ & $B \subset A$
then $A = B$

Eg:- $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$B = \{x : x$ is a natural number & $1 \leq x \leq 10\}$
then $A = B$

Power Set:- The power set of a finite set is the set of all subsets of the given set.
Power set of A is denoted by P(A) if A has n elements then P(A) has $2^n$ elements.

Eg:- $A = \{1, 2, 3\}$
$P(A) = 2^n = 2^3 = 8$ elements.

$P(A) = \{\phi, 1, 2, 3, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$

classmate
Date ______
Page ______

Universal Set :- The members of all sets under investigation usually belong to some large set called universal set.
Eg :- In human population studies, the universal set consists of all the people in the world.

Compareable and non comparable sets :-
Two sets are said to be comparable if one of the two sets is subset of the other. Otherwise non-comparable.

Eg :- $A = \{2, 3, 5\}$ , $B = \{2, 3, 5, 6\}$ , $C = \{1, 5\}$

Here $A \subset B$ $\therefore$ A & B are comparable.
On the other hand $A \not\subset C$ and $C \not\subset B$, $C \not\subset A$ and $B \not\subset C$. So, these are non-comparable.

Order / cardinality :- Number of different elements in a finite or infinite set is known as its cardinality.
Denoted by $O(A)$ $\longrightarrow$ Order of A.

Let $A = \{2, 3, 5\}$.
$O(A) = 3$.

Equivalent Sets :- Two finite sets A and B are said to be equivalent sets if the total number of elements in A is equal to total number of elements in B.

Eg :- $A = \{1, 2, 3, 4, 6\}$ and $B = \{2, 1, 7, 9, 12\}$
Therefore $O(A) = O(B) = 5$, implies A and B are equivalent sets

## Complement of sets :-
Let A be the subset of universal set U then the complement of A is the set of all those elements of U which do not belong to A

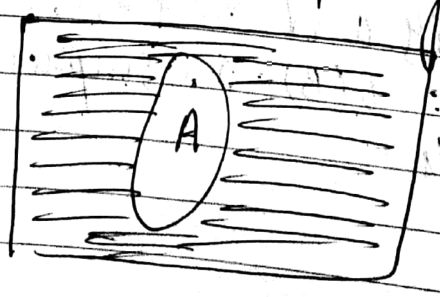
Eg :- $U = \{2, 4, 6, 8, 10\}$
$A = \{4, 8\}$
$A^c = \{2, 6, 10\}$.

Ques :- Prove that $(A^c)^c = A$

Ans :- Let U be universal set having elements
$U = \{a, b, c, d\}$
and A be subset of U having elements
$A = \{b, d\}$
consider L.H.S
$A^c = \{a, c\}$
$(A^c)^c = \{b, d\} = A$
Hence proved.

## Venn diagram :-
The relation between two sets can be explained by certain diagram called venn diagram. In a venn diagram universal set U is represented by rectangle and any subset of U is represented by a circle within a rectangle U.

Eg :- $A^c$ is the shaded region in the diagram



Combination of Sets :-

(i) Union of two sets :- If A and B are two sets then their union is the set consisting of all the elements of A together with all the elements of B.

eg :- $A = \{1, 2, 3\}$ , $B = \{3, 4, 5\}$
$A \cup B = \{1, 2, 3, 4, 5\}$

(ii) Intersection of two sets :- The intersection of two sets A and B is the set of elements common to A and B.

eg :- $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$
$A \cap B = \{3\}$

(iii) Disjoint Sets :- If A and B are two sets such that $A \cap B = \phi$, Then A and B are disjoint sets.

eg :- $A = \{1, 2\}$, $B = \{3, 4\}$, $A \cap B = \phi$.

(iv) Difference of two sets :- The difference of two sets A and B is the set of those elements of A which do not belong to B.

Eg :- $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $A - B = \{1, 2\}$.

$\boxed{\text{Note :- } A - B \neq B - A}$

Ques) Prove that $(A^c)^c = A$.

Let $S = (A^c)^c = \{x : x \notin \bar{A}\}$
$= \{x : x \in A\}$
$= A.$

Hence proved.

## Symmetric difference of two sets :-

If $A$ and $B$ are any two sets then $(A-B) \cup (B-A)$ is called symmetric difference of $A$ and $B$. It is denoted by $A \Delta B$

Eg:- $A = \{1,2,3\}$, $B = \{3,4\}$

$A - B = \{1,2\}$, $B - A = \{4\}$

$A \Delta B = (A-B) \cup (B-A) = \{1,2,4\}$.

## Some fundamental laws of Algebra of sets :-

(i) ~~Impo~~ Idempotent law :- If $A$ is any set then

(i) $A \cup A = A$

(ii) $A \cap A = A$.

Proof :- (i) $A \cup A = \{x : x \in A \cup A\} = \{x : x \in A \text{ or } x \in A\}$

$= \{x : x \in A\} = A = R.H.S$.

(ii) $A \cap A = \{x : x \in A \cap A\} = \{x : x \in A \text{ and } x \in A\}$

$= \{x : x \in A\} = A = R.H.S$

(ii) Identity law :- If $A$ is any set then

(i) $A \cup \phi = A$ and

(ii) $A \cap \phi = A$

Proof :-

(i) $L.H.S = A \cup \phi = \{x : x \in A \cup \phi\} = \{x : x \in A \text{ or } x \in \phi\}$

$= \{x : x \in A\}$ [$\because x \notin \phi$ as $\phi$ is null set]

$= A = R.H.S$

(ii) $L.H.S$ $A \cap \phi = \{x : x \in A \cap \phi\} = \{x : x \in A \text{ and } x \in \phi\}$

$= \{x : x \in A\}$ [$\phi$ is null set]

$= A = R.H.S$

Classmate

Date ______
Page ______

(iii) Commutative law :- If A and B any two sets then
(i) $A \cup B = B \cup A$, (ii) $A \cap B = B \cap A$.

Proof :- (i)
L.H.S $\{x : x \in A \cup B\} = \{x : x \in A$ or $x \in B\}$.
$= \{x : x \in B$ or $x \in A\} = x \in (B \cup A)\}$
$= B \cup A =$ R.H.S.

(ii) L.H.S $= \{x : x \in A \cap B\} = \{x : x \in A$ and $x \in B\}$
$= \{x : x \in B$ and $x \in A\} = \{x \in (B \cap A)\}$
$B \cap A =$ R.H.S.

(iv) Associative law :- If A, B, C are any three sets then
(i) $A \cup (B \cup C) = (A \cup B) \cup C$, (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

~~Proof :-~~
~~L.H.S $\{x : x \in$~~

(V) Distributive law :- If A, B, C are any three sets then
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Scanned by CamScanner

classmate

Date

Page

(vi) Involuntary law :- If A be any set then $(A^C)^C = A$

Proof :- L.H.S

$(A^C)^C = \{x : x \notin A^C\}$

$= \{x : x \in A\}$

$= A = R.H.S.$

It is also called double compliment

1

(vii) Component law :- (i) $A \cup A^C = U$

(ii) $A \cap A^C = \phi$

(viii) Absorption law :- (i) $A \cup (A \cap B) = A$

(ii) $A \cap (A \cup B) = A$

Imp. State and Proofe :- De-Morgan's Law.

De-Morgan's law states that let A and B be two subsets of U then

(i) $(A \cup B)^C = A^C \cap B^C$

(ii) $(A \cap B)^C = A^C \cup B^C$.

Proof :- (i) $(A \cup B)^C = A^C \cap B^C$

L.H.S $= (A \cup B)^C = \{x : x \in (A \cup B)^C\}$

$= \{x : x \notin (A \cup B)\}$

$= \{x : x \notin A \text{ and } x \notin B\} = \{x : x \in A^C \text{ and } x \in B^C\}$

$= \{x : x \in (A^C \cap B^C)\} = A^C \cap B^C = R.H.S$

(ii) $(A \cap B)^C = A^C \cup B^C$

L.H.S $= (A \cap B)^C = \{x : x \in (A \cap B)^C\}$

$= \{x : x \notin (A \cap B)\} = \{x : x \notin A \text{ or } x \notin B\}$

$= \{x : x \in A^C \text{ or } x \in B^C\} = \{x : x \in (A^C \cup B^C)\}$

$= A^C \cup B^C = L.H.S.$

classmate
Date
Page

De Morgan's law for difference of sets:-
If A, B, C are any three sets then
(i) $A-(B\cup C) = (A-B)\cap(A-C)$
(ii) $A-(B\cap C) = (A-B)\cup(A-C)$
Proof:-
(ii) $A-(B\cup C) = (A-B)\cup(A-C)$
L.H.S. $A-(B\cap C)$
$= \{x : x \in A-(B\cap C)\}$
$= \{x : x \in A \text{ and } x \notin (B\cap C)\}$
$= \{x : x \in A \text{ and } x \in (B\cap C)^c\}$
$= \{x : x \in A \text{ and } x \in B^c) \cup (x \in A \text{ and } x \in C^c)\}$
$= \{x : x \in (A\cap B^c) \cup x \in (A\cap C^c)\}$
$= (A\cap B^c) \cup (A\cap C^c)$
$= (A-B)\cup(A-C)$ $(A-B = A\cap B^c)$.

(i) $A-(B\cup C) = (A-B)\cap(A-C)$
L.H.S $A-(B\cup C)$
$= \{x : x \in A-(B\cup C)\} = \{x : x \in A \text{ or } x \notin (B\cup C)\}$
$= \{x : x \in A \text{ or } x \notin (B\cup C)\} = \{x : x \in A \text{ or } \cancel{x \in} x \notin B \text{ and } x \notin C\}$
$= \{x : x \in A \text{ or } x \notin (B\cup C)^c\}$
$= \{x : x \in A \text{ or } x \in B^c \text{ and } x \in C^c\}$
$= \{x : x \in A \text{ or } x \in B^c \cap C^c\}$
$= \{x : x \in A \text{ or } B^c\} \cap \{x : x \in A \text{ or } C^c\} = \{x : x \in (A\cup B^c)\cap(A\cup C^c)\}$
$= (A-B)\cap(A-C)$

classmate
Date
Page

Laws and Results:-

1.) If $A \subset B$ and $B \subset C$ then $A \subset C$.

2.) $(A \cup B) = (A / B) \cup (B / A) \cup (A \cap B)$

minus

3.) If $A \subset B$ then $P(A) \subset P(B)$

4.) $(A \cup B) \cup (A \cup B^c) = A$

5.) If $(A \cup B) \cup (A \cap B^c) = A \cap (B \cup B^c)$
then $A \cap U = A$ = R.H.S

6.) If $A \subseteq B$ and $B \cap C = \phi$ then $A \cap C = \phi$

7.) $A' - B' = B - A$

8.) $(A - B) - C = A - (B \cup C)$

9.) $(A - B) - C = (A - C) - B$

10.) $P(A \cap B) = P(A) \cap P(B)$

(3.) L.H.S $P(A \cap B)$

Let $x \in P(A \cap B)$

$\Rightarrow x \subset A \cap B \Leftrightarrow x \subseteq A$ and $x \subseteq B$

$\Leftrightarrow x \in P(A)$ and $x \in P(B)$

$\Leftrightarrow x \in \{P(A) \cap P(B)\}$

$P(A \cap B) = P(A) \cap P(B)$

Important Questions

Ques: If $A \subset B$ and $B \subset C$ then $A \subset C$

Given:- $A \subset B$, $B \subset A$

To prove:- $A \subset C$

Proof: $\{x : x \in A\}$ - (1)

$\{x : x \in B\}$ $(\because A \subset B)$ gives

$x \in C$ - (2) $(\because B \subset C)$

From (1) & (2)

$\Rightarrow A \subset C$.

Ques:- If $A \subset B$ ~~then~~ and $B \cap C = \phi$ then, $A \cap C = \phi$.

Given $A \subset B$

$B \cap C = \phi$

Proove:- $A \cap C = \phi$ i.e. if $x \in A$ then $x \notin C$.

Proof:- Consider $x \in A$ -(1)

$x \in B$ $[A \subset B]$

$x \notin C$ $[A \cap C]$ -(2)

From (1) and (2)

$x \in A$, $x \notin C$

then $A \cap C = \phi$

Ques:- If $A^c - B^c = B - A$

Proof:- consider L.H.S $= A^c - B^c$

$= (A^c \cap (B^c))^c = [A - B = A \cap B^c]$

$= B \cap A^c =$

$= B - A^c = R.H.S$.

Ques:- Prove that $P(A \cap B) = P(A) \cap P(B)$

Ans:- Let $x \in P(A \cap B)$ -(1)

$\Rightarrow x \subseteq A \cap B$

$\Rightarrow x \subset A$ and $x \subset B$

$\Rightarrow x \in P(A)$ and $x \in P(B)$

$\Rightarrow x \in P(A) \cap P(B)$ -(2)

From (1) and (2)

We get $P(A \cap B) \subseteq P(A) \cap P(B)$ -(3)

Similarly $P(A) \cap P(B) \subset P(A \cap B)$ -(4)

$P(A \cap B) = P(A) \cap P(B)$.

classmate
Date
Page

Ques:- Two finite sets have m and n elements. The total number of subsets of first set is 56 more than the total no. of second set. Find values of m & n.

Ans:- Let A be the first set with m elements and B be the second set with n elements. Then as given in ques. $P(A) = P(B) + 56$.

$2^m = 2^n + 56$

One possible answer is
$m = 6$ and $n = 3$.

Ques:- If $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{1, 2, 3, 6\}$
Find $\bigcup A_i$ and $\bigcap A_i$, $i = 1, 2, 3$

Ans:- $\bigcup A_i = (A_1 \cup A_2 \cup A_3) = \{1, 2, 3, 6\}$
$\bigcap A_i = (A_1 \cap A_2 \cap A_3) = \{2\}$

Ques:- Show that for any two sets A and B,
$A - (A \cap B) = A - B$

Ans:- $A - (A \cap B) = \{x : x \in A - (A \cap B)\}$
$= \{x : x \in A \ \& \ x \notin A \cap B\}$
$= \{x : x \in A \ \& \ (x \notin A \text{ or } x \notin B)\}$
$= \{x : (x \in A \text{ and } x \notin A) \text{ or } (x \notin B \text{ and } x \in A)\}$
$= \{x : x \in \phi \text{ or } (x \in A \ \& \ x \notin B)\}$
$= \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in A - B\}$
$A - B$

Ques:- Proove that $(A - B) \cap B = \phi$

Ans:- Let L.H.S $= (A - B) \cap B$
$(A \cap B^c) \cap B$
$A \cap (B^c \cap B)$
$A \cap \phi$
$= \phi =$ R.H.S

Scanned by CamScanner

classmate
Date
Page

Ques:- If A and B are disjoint sets then Prove A-B and B-A are disjoined.

Ans:- We have to prove $(A-B) \cap (B-A) = \phi$

Let L.H.S $(A-B) \cap (B-A)$

$(A-B^c) \cap (B-A^c)$

$[A \cap (B^c \cap B) \cap A^c]$

$\Rightarrow [A \cap \phi \cap A^c] = (\phi)$

$\phi = R.H.S.$

Ques:- Show that $A \subset B$ iff $B^c \subset A^c$

Ans:- consider $A \subset B$

To prove:- $B^c \subset A^c$

Proof:- Let $x \in B^c$ - (1)

$\Rightarrow x \notin B$

$\Rightarrow x \notin A \Rightarrow x \in A^c$ - (2)

From (1) & (2).

$B^c \subset A^c$

Vice versa of above Ques.

To prove:- $A \subset B$.

Proof:- Let $x \in A$ (3)

$\Rightarrow x \notin A^c$

$\Rightarrow x \notin B^c. \Rightarrow x \in B$ - (4)

From (3) and (4)

$A \subset B$.

Ques:- $A \cup B = A \cap B$

iff $A = B$

Ans:- consider $A \cup B = A \cap B$

To prove:- $A = B$ ie. $A \subset B < B < A$.

Proof:- ~~To~~ Firstly, we will prove that $A < B$

Let, $x \in A$.

Scanned by CamScanner

$\Rightarrow x \in A \cup B$

$\Rightarrow x \in A \cap B$ (given)

$\Rightarrow x \in A$ and $x \in B$ - (2)

$\therefore A < B$ - (3)

Similarly, we can prove that $B < A$ - (4)

$\therefore A = B$.

Now, consider $A = B$.

To prove :- $A \cup B = A \cap B$

$L.H.S = A \cup B = A \cup A$ (given)

$= A$ - (5)

$R.H.S = A \cap B = A \cap A$ (given)

$= A$ - (6)

From (5) & (6)

$A \cup B = A \cap B$.

Ques:- If A, B and C are three sets then show that :

$(A-B)-C = A-(B \cup C)$

Ans:- L.H.S

$A-B-C = A-(B \cup C)$

Consider L.H.S

$(A-B)-C$

$\Rightarrow (A-B) \cap C^c$

$\Rightarrow (A \cap B^c) \cap C^c$

$\Rightarrow A \cap [B \cup C]^c \Rightarrow [B^c \cap C^c] = (B \cup C)^c$

$\Rightarrow A-[B \cup C]$

$= R.H.S$

Partition of sets :- A partition of non-empty set A is a collection $P = \{A_1, A_2, A_3 \ldots\}$ of subsets of A if and only if P

(i) $A = A_1 \cup A_2 \cup A_3 \ldots$

(ii) $A_i \cap A_j = \phi$ for $i \neq j$

Eg :- Let $A = \{a, b, c\}$ be any set then $P_1 = \{\{a\}, \{b, c\}\}$

$P_2 = \{a, b, c\}$

$P_3 = \{\{b\}, \{a, c\}\}$, $P_4 = \{\{c\}, \{a, b\}\}$ are partitions of set A.

Minset or fundamental product of sets :-

Let $\{B_1, B_2, \ldots B_n\}$ be the set of subsets of given set A. Let a set of the form $D_1 \cap D_2 \cap \ldots D_n$ where each $D_i$ may be either ~~between~~ $B_i$ compliment.

For all $i = 1, 2 \ldots 10$ is called fundamental product of those sets or minset minterm generated by $D_1, D_2 \ldots D_n$.

Minterm set generated by two sets $A_1$ and $A_2$ are

$D_1 = \{A_1 \cap A_2$

$D_2 = A_1 \cap A_2'$ $\qquad A' \rightarrow A$'s compliment

$D_3 = A_1' \cap A_2$

$D_4 = A_1' \cap A_2'$

# No. of Minterms generated by N sets $= 2^n$

# Dual of minset is called maxset

# Dual is obtained by changing intersection into union

Scanned by CamScanner

classmate
Date
Page

# Discrete Structures

## RELATIONS

Cross partition :- If $A = \{A_1, A_2 + \ldots A_m\}$ and $B = \{B_1, B_2 \ldots B_n\}$ be partition of set X then the set $P = A_I \cap B_I$ ~~wh~~ from $\{1, 2, 3 \ldots n\}$ & I varies from $\{I, 2 \ldots n\}$.

Ordered pair :- By an ordered pair (a, b) of elements we need a pair a belongs to A and b belongs to B i.e. $a \in A$ and $b \in B$.
The ordered pair AB & BA are different unless a = B also ~~ab~~ (a, b) = (C, d) iff a = c & b = d
also if ordered pair $(a, b) \in A \times B$
i.e. $a \in A$, $b \in B$.

Cartesian product / cross product :- The sets of all ordered pair (a, b) such that $a \in A$, $b \in B$ is called cartesian product of two sets. A and B is called cartesian product of A and B and is denoted by $A \times B$ and is defined as $A \times B = \{(a, b)$. $a \in A$ and $b \in B$ if $A \times B \times C = \{(a, b)$ & $a \in A$, $b \in B$, $c \in C\}$

Ques :- Prove that :- ~~A X (BUC~~ $A \subset B$ & $C \subset D \Rightarrow$
$A \times C \subset (B \times D)$

Ans :- Let (x, y) be any member of $A \times C$
$\therefore (x, y) \in (A \times C) \Rightarrow x \in A$ & $y \in C$
$\Rightarrow x \in B$ & $y \in D \Rightarrow A = y \in C$
$[\because A \subset B$ & $C \subset D]$

Scanned by CamScanner

Relations :- A relation R from a set A to set B is defined as subset of A×B. If R is a relation from a set A to set B and if ab ∈ R for a ∈ A & b ∈ B. Then we say that a is related to b and we write it as aRb. If ab ∉ R, then we say that a is not related to b and we write it as a $\not R$ b.

$A = \{1, 2\}$ , $B = \{3, 4\}$
$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$
$R < A \times B$.

Domain and Range of Relation :- If R is an relation from a set A to set B then the set of first component of the elements of R is called domain of R and set of second components of the element of R is called range of R. Thus domain of R equals to element A where (a, b) belongs to R.

$\{a : (a, b) \in R\}$

Range of $R = \{b : (a, b) \in R\}$

If R is a relation from a set A to set A then R is called relation on A. Thus a relation on set A is defined as any subset of A×A.

eg :- Let $A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let $R = \{(1,2), (2,2), (3,2), (3,3)\}$

$R \subseteq A \times A$ ∴ R is a relation on set A.

Date
Page

Types of Relations :-

1. Inverse relation :- If $R \subseteq A \to B$ is a relation then inverse relation $R^{-1} : B \to A$ i.e. if $(a,b) \in R$ then $(b,a) \in R^{-1}$.

2. Identity relation :- If a relation is defined on a set itself then it is identity relation i.e.
$R : A \to A$
$R = \{(a,a) : a \in A\}$, if $A = \{1,2\}$
$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
Identity relation has the elements $\{(x,y) : x \in A$ and $y \in A$ & $x = y\}$
If A has two elements then identity relation has two elements

3. Void or empty relation :- The relation which has no elements is a void relation or empty relation

4. Universal relation :- Cross product of two sets is called universal relation.

Properties of Relations :-

1. Reflexive Relation :- A ~~reflexive~~ relation R on set A is called reflexive relation for $(x,x) \in R \; \forall x \in A$. That is if $x R x$ for every $x \in R$
Eg :- If $A = \{1,2\}$
$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
Let $R = \{(1,1), (2,2)\}$
The R is a reflexive relation on A

classmate
Date
Page

2. Symmetric relation :- A relation R on a set A is called symmetric relation if $aRb \Rightarrow bRa$ where $a, b \in A$ i.e. if $(a,b) \in R \Rightarrow (b,a) \in R$ where $a, b \in A$
Eg :- Let $A = \{1,2\}$, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

Let $R = \{(1,1), (1,2), (2,1)\}$ is symmetric
Inverse of every element must be present in set along with element itself.

3. Transitive relation :- A relation R on a set A is called a transitive relation if $aRb, bRc \Rightarrow aRc$ $\forall a, b, c \in A$
i.e. if $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$ where $a, b, c \in A$
$R = \{(1,2), (1,3), (3,2)\}$

4. Irreflexive relation :- A Relation R on a set A is called irreflexive relation iff $a \not R a$ $\forall a \in A$
(same elements not allowed)
Eg :- Let $A = \{1,2\}$, $A \times A = \{(1,1), (2,1), (2,2), (1,2)\}$
$R = \{(1,2), (2,1)\}$

5. Asymmetric relation :- A relation R on a set A is called asymmetric relation if $aRb$ and $b \not R a$.
Eg :- Let $A = \{1,2\}$, $A \times A = \{(1,1), (2,1), (2,2), (1,2)\}$
$R = \{(1,2), (1,1), (2,2)\}$

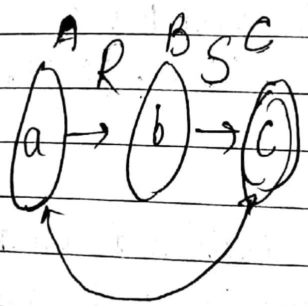
6. Anti symmetric relation :- A relation R on a set A is called anti ~~relation~~ symmetric if $aRb$ and $bRa \Rightarrow a=b$
$(a,b) \in R$, $(b,a) \in R$
Eg :- Let $A = \{1,2\}$, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
$R = \{(1,1), (2,2)\}$

Scanned by CamScanner

7 Equivalence relation :- A relation R on set A is called equivalence relation if R is reflexive, symmetric and transitive.

8. Partial Order relation :- A relation R on set A is said to be partial order relation if it satisfies following three conditions :-
→ Reflexive
→ Anti Symmetric
→ Transitive

Composition of Relation :-
Let A, B, C be three sets and R be a relation from A to B and let S be a relation from B to C i.e. R is a subset of A×B and S is a subset of B×C then R and S give rise to a relation from A to C denoted by SOR defined by a(SOR)c if for some b ∈ B we have aRb & bRc i.e. SOR = ordered pair AC,
SOR = {(a,c) • ∃ b∈B for which (a,b)∈R & (b,c)∈S}
The relation ROS is called composition of R and S.




Ternary relation :- A ~~ternary~~ relation is a set of ordered tuples. In particular, if S is a set the subset of $S^3 = S \times S \times S$ is called ternary relation.

classmate
Date
Page

n-ary relation :- n-ary relation is a set of tuples in perticular if $S$ is a set, then a subset of $S^n = S \times S \times S \dots$ n is called n-ary relation.

Equivalence Class :-
Let $R$ be an equivalence relation on a non-empty set $X$. Let $a \in X$, then the equivalence class of $a$ is denoted by $[a]$ is defined as follows :-
$[a] = \{b \in X : (a, b) \in R\}$

Let $A$ be a set of non-zero integers and equivalence $(\approx)$ be the relation defined on $A \times A$ as $(a, b) \approx (c, d)$ iff $ad = bc$ and $a + d = b + c$

(1.) $\approx$ is reflexive
Since $ab = ba \ \forall a, b \in A$
$\Rightarrow (a, b) \approx (a, b) \ \forall (a, b) \in A \times A$
$\therefore \approx$ is reflexive

(2.) $\approx$ is symmetric
Let $(a, b) \approx (c, d)$ where $(a, b), (c, d) \in A \times A$
$\Rightarrow ad = bc$
$\Rightarrow da = cb$
$\Rightarrow (c, d) \approx (a, b)$. This is Symmetric

(3.) $\approx$ is transitive
Let $(a, b) \approx (c, d)$ & $(c, d) \approx (e, f)$;
$(a, b), (c, d), (e, f) \in A \times A$
$\Rightarrow ad = bc$ & $cf = de$
$\Rightarrow adcf = bcde$ (L.H.S × L.H.S & R.H.S × R.H.S)
$\Rightarrow af = be$
$\Rightarrow (a, b) \approx (e, f)$
$\therefore \approx$ is transitive

Ques:- Each of the following defines a relation on the set N of positive integers

R : $x > y$   S : $x+y=10$   T : $x+4y=10$ for all $x, y \in N$

Determine which of the relations are

(i) reflexive (ii) symmetric (iii) transitive, (iv) Antisymmetric

Ans :- (i) Reflexive

for $x > y$

Put $x = 6$, $y = 4$ in S   where $x > y$

S : $6 + 4 = 10$

Hence L.H.S = R.H.S

Relation is reflexive

where

Put $x = 6$, $y = 1$ in T   $x > y$.

T : $6 + 4(1) = 10$

Hence L.H.S = R.H.S

Relation is reflexive

(ii) Symmetric

Put $x = 6$, $y = 4$ in S

$S = 10$

But for $y = 6$ and $x = 4 \in S$ because $x > y \in R$

So this is not symmetric

Similarly in second equation also x can't be less then y $x \not< y$, So the relation is not symmetric

(iii) Transitive

If $(7,3) \in S$, $(3,6) \in S$ then $(7,6) \in S$

which is not possible so relation can't be ~~symmetric~~ transitive

Similarly T can't be transitive.

Date
Page

## Congruence Modulo relation :-

consider the set of integers. X is congruent to y modulo m ie. $x \equiv y \pmod m$ if and only if $x-y$ is divided by m ie. $m/x-y$.

1. $\equiv$ Relation is reflexive

Since $m/x-x \ \forall x \in Z$

$\Rightarrow x \equiv x \pmod m$

$\therefore$ Reflexive relation

2. $\equiv$ Relation is Symmetric

Let $m/x-y$

$\Rightarrow x-y = mc$ where c is any Integer

$\Rightarrow y-x = mc \Rightarrow m(-c)$

$\Rightarrow m/y-x$

$\Rightarrow y \equiv x \pmod m$

3. $\equiv$ relation is ~~transitive~~

Let $x \equiv y \pmod m$ & $y = z \pmod m$

$\Rightarrow m/x-y$ & $m/y-z$

$\Rightarrow x-y = mc$ & $y-z = md$ where c and d are integers

Now $x-z = y+mc - y+md = m(c+d)$

$\Rightarrow m/x-z = c+d$.

$= z \pmod m = x$.

Compatible relation :- A relation R on set A is said to be compatible relation if it is reflexive and symmetric.

Ques:- Let $A = \{a, b, c\}$ and let
$R = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$
where $R$ is clearly an equivalence relation. Find equivalence classes of the elements of $A$.

Ans:- The equivalence classes of the elements of $A$ are:
$[a] = \{a, b\}$
$[b] = \{b, a\} = [a]$
$[c] = \{c\}$
The rank is 2.

Rank:- The rank of $R$ is the number of distinct equivalence class of ~~A~~ with respect to $R$.

Ques:- Let $A = \{1, 2, 3, 4\}$ and let
$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\}$

Ans:- $R$ to be reflexive
Since $(1,1), (2,2), (3,3), (4,4) \in R$
So it is reflexive

Symmetric.
Since $(1,2)$ $(2,1) \in R$
Similarly $(2,3), (3,2)$, $(1,3), (3,1) \in R$
So it is symmetric

Transitive
Since $(1,2)$ & $(2,1) \in R$ which implies $(1,1) \in R$.
$(1,3), (2,3) \in R$ so $(2,1) \in R$ so it is transitive.

Hence $R$ is an equivalence ~~that~~ relation

[1] or R(1) = {1, 2, 3}
[2] or R(2) = {1, 2, 3} = [1] or R(1)
[3] or R(3) = {3, 1, 2} = [1] or R[1]
[4] or R(4) = {4}
The rank is 2.

Ques:- A = {1, 2, 3, 4, 5}, B = {a, b, c, f}, C = {a, 5}
find (AUC) - (AUC) x B

Ans:- {1, 2, 3, 4, a, 5} = AUC
{1, 2, 3, 4, a, 5} x {a, b, c, f} = (AUC) x B

{(1, a), (1, b), (1, c), (1, f), (2, a), (2, b), (2, c), (2, f), (3, a), (3, b), (3, c), (3, f), (4, a), (4, b), (4, c), (4, f), (a, a), (a, b), (a, c), (a, f), (5, a), (5, b), (5, c), (5, f)}

(AUC) - (AUC) x B = {1, 2, 3, 4, a, 5} = AUC

Article :- If R and S are two equivalence relation on set A then R∩S is also an equivalence relation on A.

Proof :- Given R is a relation on A it means R is subset of A x A ($R \subset A \times A$)
Similarly ($S \subset A \times A$) S is subset of A x A
∴ Therefore R∩S is a relation

(i) Reflexive :-
Let $x \in A$ and since R, S are reflexive relations.
Therefore if $x \in A$ implies $(x, x) \in R$ & $(x, x) \in S$
$\Rightarrow (x, x) \in R \cap S$
∴ R∩S is reflexive

Scanned by CamScanner

(ii) Symmetric

Let $x, y \in A$ . S.T:- $(x, y) \in R \cap S$

$\therefore (x, y) \in R$ & $(x, y) \in S$

As R & S are symm., Rel on A

$\therefore (y, x) \in R$ & $(y, x) \in S$

$\Rightarrow (y, x) \in R \cap S$

$\Rightarrow R \cap S$ is symm. on A

(iii) Transitive

Let $x, y, z \in A$

Show that :- $(x, y) \in R \cap S$,

$(y, z) \in R \cap S$

To prove :- $(x, z) \in R \cap S$

Now :- $(x, y) \in R$ & $(x, y) \in S$

Also $(y, z) \in R$ & $(y, z) \in S$

R & S are transitive relation on A

$(x, y) \in R$ & $(x, z) \in S$

$\therefore (x, z) \in R \cap S \Rightarrow R \cap S$ is transitive

Ques:- Proove that a relation R on set A is symmetric if and only if $R = R^{-1}$

Ans:- Proof:-

If R is symmetric

$R = R^{-1}$ and thus $(R^{-1})^{-1} = R = R^{-1}$ which means $R^{-1}$ is also symmetric

$(a, b) \in R^{-1}$ only when $R \notin (b, a)$

$(b, a) \in R^{-1}$ only when $R \notin (a, b)$

Hence it is symmetric

classmate

Date ____

Page ____

Ques:- # Let A, B and C be three sets $R: A \to B$, & $S: B \to C$ then prove that $(SOR)^{-1} = R^{-1} O S^{-1}$.

Proove:- Let $a \in A$, $b \in B$, $c \in C$

Since $R:- A \to B$

$\therefore R \subset A \times B$

also

$S: B \to C$

$\therefore S \subset B \times C$

$\therefore A \times B = \{(a, b)\} = R \Rightarrow B \times C = \{(b, c)\} = S$

Since $(a, b) \in R$ & $(b, c) \in S$

$\Rightarrow (a, c) \in SoR$

$\Rightarrow (c, a) \in (SOR)^{-1}$ - (1)

Since $(a, b) \in R$ & $(b, c) \in S$

$\therefore (b, a) \in R^{-1}$ & $(c, b) \in S^{-1}$

$(c, b) \in S^{-1}$ & $(b, a) \in R^{-1}$

$\therefore (c, a) \in R^{-1} o S^{-1}$ -(2)

From 1 & 2

$(SOR)^{-1} \subset R^{-1} O S^{-1}$ -(3)

Similarly we can Proove that

$(R O S^{-1}) \subset (SOR)^{-1}$ -(4)

From (3) & (4)

$(SOR)^{-1} = R^{-1} O S^{-1}$.

Discreate Structures

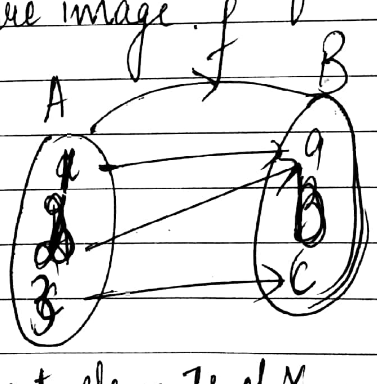
# FUNCTIONS

**Function:**
Let A and B be non-empty sets. A function f from A to B denoted by ($f: A \to B$) defined as a rule which associates each element of set A to a unique element of set A. Function is also called a mapping or transformation

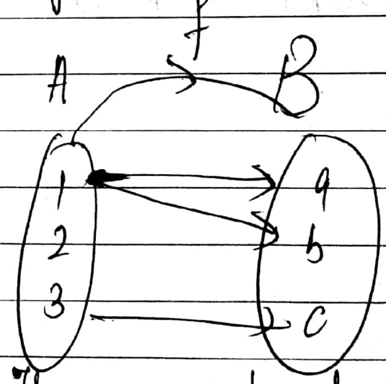
**Arguement:-** The element A is called the arguement of the function f.

**Value:-** $f(a)$ is value of function A and is called image of A

**Note:** 1. Each element of A has 1 and only 1 image.
2. Pre-Image of an element of B may or may not exists.
3. An element of a set B may have more than 1 pre image.




Different elements of X may be associated with same element of Y.




There may be element of Y which is not associated with any element of X

**Domain :-** Domain of a function is a set A on which function is defined $f(A \to B)$ is called domain.

**Range :-** Range of f means set of images of $f(x)$ set B in which element image of set A is called range.

**Types of functions :-**

(i) **One-One function :-** It is also called injective function. A function is called one-one function if each element of A has different image i.e. $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$.

(ii) **Onto function :-** It is also called bijective function. A function from A to B is called Onto if function ~~is one-one and onto~~ an element from set B has at least 1 pre image.

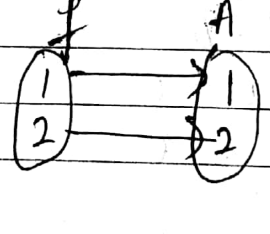
(iii) **Inverse of a function :-** If a fnf : $A \to B$ is bijective then it exists is defined as $f^{-1}: B \to A$ and is written as $f^{-1}(y) = x$ iff $f(x) = y$

(iv) **Many-one function :-** A function $f: A \to B$ is called many to one if there exists two or more! More element in set A having same images in B.

(v) **Equal function :-** Two functions f and g from set A to set B are equal if $f(a) = g(a) \ \forall a \in A$

(vi) **Identity function :-** A function $f: A \to A$ is called identity function if each element of set A has image on itself. i.e. $f(a) = a \ \forall a \in A$.

For eg:- A




$f(1) = 1, \ f(2) = 2$

(vii) Into function :- A function $f: A \to B$ is called into function if range of $f$ is not equal to c domain set of A.

(viii) constant function :-
let $f: A \to B$, then it is said to be a constant function if every element of mapped on to the same element of B.

constant Mapping :-
If the range of $f$ has only one element then $f$ is called constant mapping.

Composition of function :-
let $f: A \to B$ and $g: B \to C$ be two functions then composition $gof$ is defined if $R_f < D_g$ and $fog$ if $R_g < D_f$

Eg:- let $x = \{p, q, r, s\}$
$y_1 = \{x, y, z, t\}$
$y_2 = \{x, y, z, t, s\}$
$z = \{a, b\}$
if $: x \to y$, $g: y_2 \to z$

Ques: A function $f$ from $x$ to $y_1$, $g: y_2 \to z$ is defined as
$f(p) = f(q) = z$
$f(r) = x$, $f(s) = y$, $g(x) = g(y) = a$, $g(z) = g(t) = g(s) = b$
find $fog$ & $gof$

$gof = g(f(p)) = g(z) = b$
$g(f(q)) = g(z) = b$
$g(f(r)) = g(x) = a$
$g(f(s)) = g(y) = a$

classmate

Date ______

Page ______

## Consider a function $f: N \to N$ where $N$ is the set of Natural Numbers defined as $f(n) = n^2 + n + 1$. Prove that function is one-one but not onto.

Proof: Since $f: N \to N$ is a function, when $N$ is a set of Natural No. ~~defined as~~ Also $f(n) = n^2+n+1$. Firstly prove that function is one-one ~~but not onto~~.

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + x_1 + \cancel{1} = x_2^2 + x_2 + \cancel{1}$

$\Rightarrow x_1^2 + x_1 - x_2^2 - x_2 = 0$

$\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0$

$\Rightarrow (x_1 + x_2)(x_1 - x_2) + (x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow (x_1 + x_2) + 1 \neq 0$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

The function one one when $x_1 = x_2$

To prove that function is not onto.

$\forall y \in N$ (R.H.S), $\exists x \in N$ (L.H.S)

Such that $f(x) = y$

$\Rightarrow x^2 + x + 1 = y \Rightarrow x^2 + x + (1-y) = 0$

$x = \frac{-1 \pm \sqrt{-3-4y}}{2}$ function is not onto

# Descrete Structures

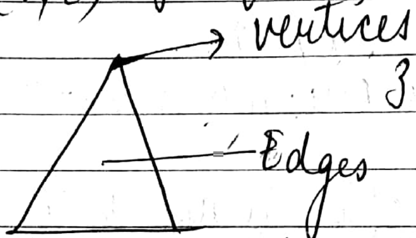
# GRAPHS

Graph :- A graph G consists of two things:

(i) A set $V = V(G)$ whose elements are called vertices, points or nodes

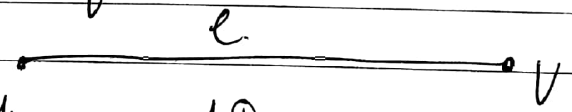
(ii) A set of $E = E(G)$ of unordered pairs distinct vertices called edges of G. We denoted such a graph by $G(V, E)$

eg :-

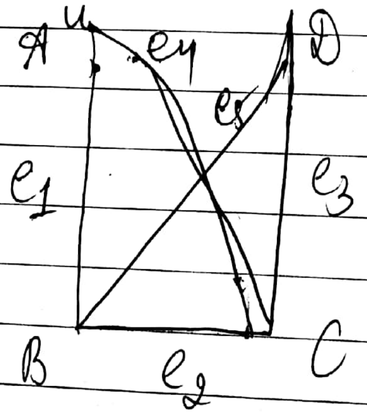



3 vertices & 3 edges

Vertices u & V are said to be adjacent if there is an edge $e = \{u, v\}$. In such a case u & v are called end points of e and e is said to connect u and v. Edge is said to be incident on each of its end points u and v.



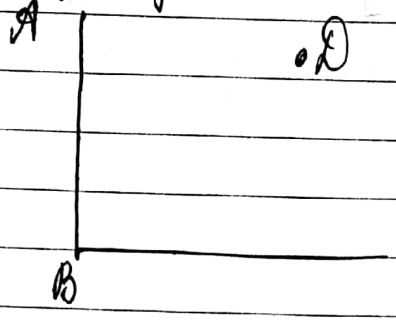

eg :-




V consists of vertices A, B, C, D. E consists of edges $e_1, e_2, e_3, e_4, e_5$

Loop :- An edge that is incident from and into itself is called loop.

Isolated Vertex :- A vertex of graph G which is not the end vertices of any edge in G is called isolated vertex.

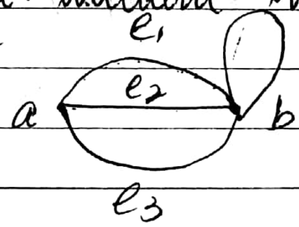



D is isolated Vertex.

Parallel edges :- If two or more edges of graph G have same end vertices then these edges are called parallel edges.

Degree of vertex :- Degree of vertex v in graph G written as deg v = number of edges in G which contain V i.e. which are incident on V.

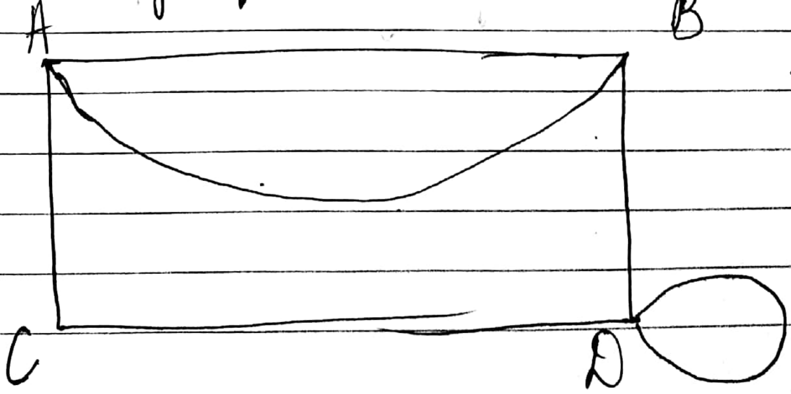
Eg :-




deg(a) = 3
deg(b) = 5

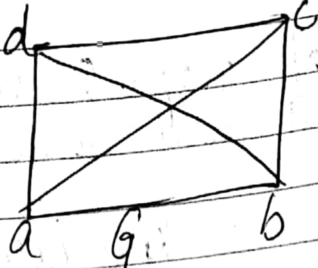
Degree of loop :- A loop contributed 2 to the degree of vertex since that vertex serves as both end points of the loop. It is always equal to 2.

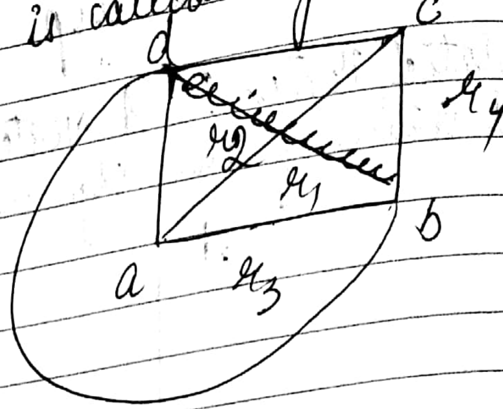
Eg :-



Region :- A given map will divide the planar into connected area is called region.

Eg :-

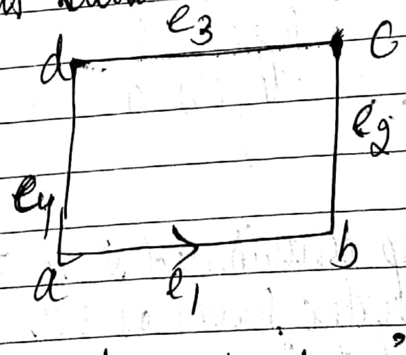




deg($r_1$) 3
deg($r_2$) 3
deg($r_3$) 3
deg($r_4$) = 3/12

Map :- A particular planner representation of a finite planar multigraph is called map.

Adjacent Vertex :- Vertex U and V are called if there exists an edge between u and v in such a case u and v are called end points of edge and the edge is said to be Incident on (u, v)



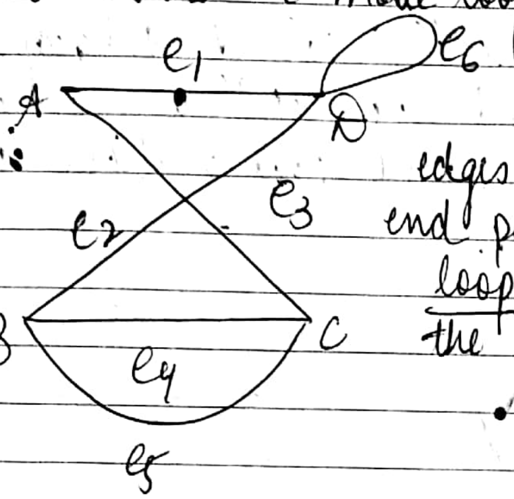
Length :- Number of edges in a path.

## Types of graphs :-

(i) Simple graph :- A graph which has neither loop nor parallel edges is called simple graph.

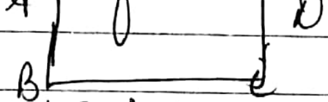
(ii) Multigraph :- A multigraph G consists of vertices and edges & A multigraph G also consists of set V of vertices and set E of edges except that E may contain multiple edges i.e. edges connecting the same end points and E may contain 1 or more loop.

Eg :-




Edges $e_4$ and $e_5$ are multiple edges. Since they connect same end points. Edge $e_6$ is called loop. Since its end points are the same vertex.

(iii) Finite graph :- A multigraph is said to be finite if it has finite number of vertices and finite number of edges. Eg :-



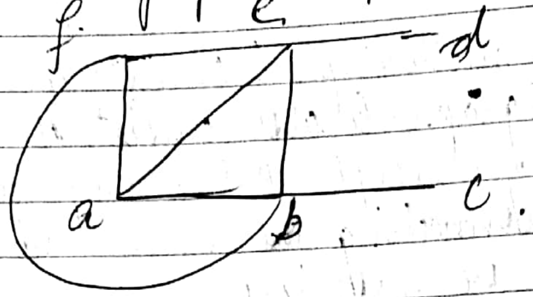

(iv) Trivial graph :- The trivial graph is the graph with one vertex and no edge.

Eg :- $\cdot e$

(v) Empty graph / Null Graph :- The empty graph is the graph which have no vertex and no edges. Also called null graph.

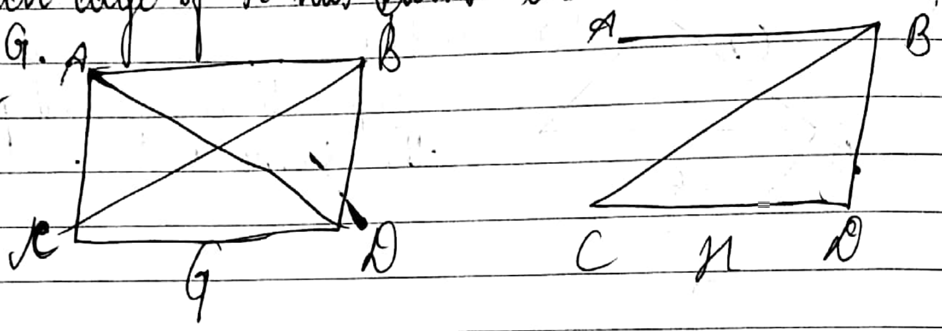
(vi) Plannar graph :- A graph or a multigraph which can be drawn in the plane such that its edges do not cross each other is called plannar graph. Tree graphs are plannar always

Eg :-



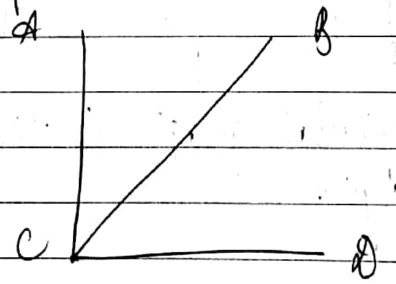

(vii) Sub graph :- A graph H is called subgraph of G if all the vertices and edges of H are in G and each edge of H has same end vertices in H as in G.

Eg :-



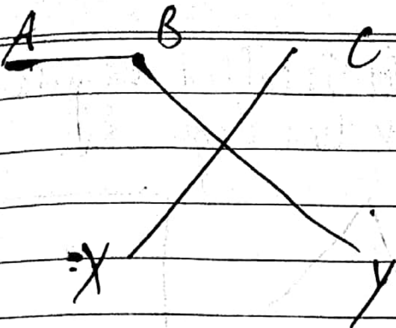

(viii) Connected Graph :- A graph G is connected if there is a path between any two of its vertices

Eg :-




(ix) Disconnected Graph :- The graph G is disconnected if there is no path between any two of its vertices.

(ix) Eg:-

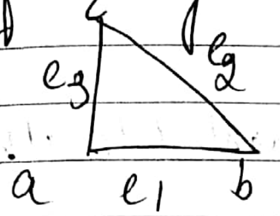



Since, ABY are connected & X&C are connected but there is no path from ABY to either C or X.

A graph which is not connected is also called not connected graph.

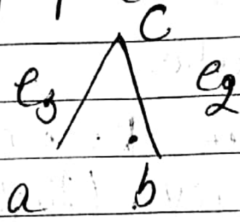
(x) Sub graph (G-E):- The graph obtained by deleting the edge E from edge set is called sub graph (G-E)

Eg:-




$V = \{a, b, c\}$

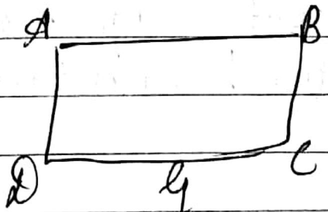
$E = \{e_1, e_2, e_3\}$



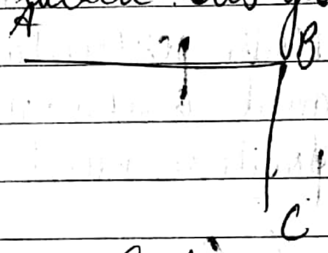

$G - e_1$

(xi) Sub graph (G-V):- The graph obtained by deleting the vertex V from edge set and deleting all edges in E(G) which are incident of V is called sub graph (G-V)

Eg:-

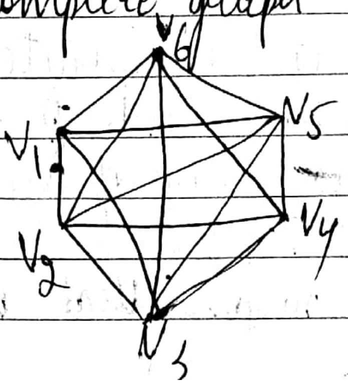



G



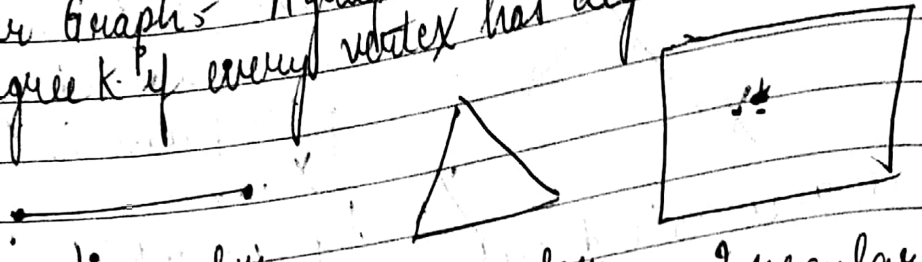

G-A

(xii) Complete graph:- A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus, a complete graph G must be connected. the complete graph with n vertices is denoted by $K_n$




$K_6$

(XIII) Regular Graph :- A graph G is called regular graph of degree K if every vertex has degree K.



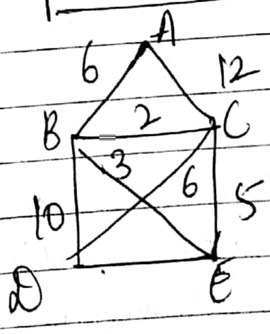
K regular    2 regular    2 regular

2 regular    2 regular

where K = no. of vertices connected to other vertex. (Highest)

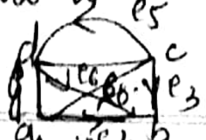
(XIV) Labelled / Weighted graph :- A graph G is called labelled graph if its edges or vertices are assigned data of one kind or another.
In other particular, if each ~~vertex~~ edge e of G is assigned a non-negative number $l(e)$ is called positive number.

Eg :-



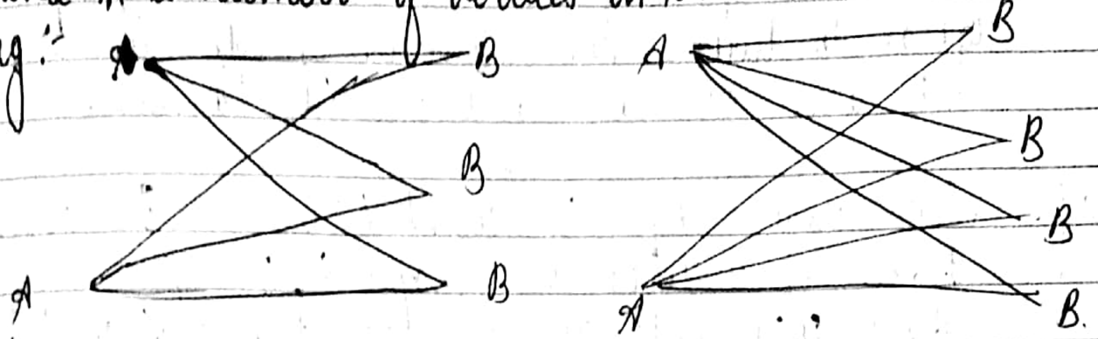

(XV) Bipartite Graph :- A graph G is called bipartite graph if its vertex V can be partitioned into two subsets M and N. Such that each edge of connects a vertex of M to a vertex of N by a complete bipartite graph. We mean that each

(XX) Traversable Multiple :- Graph drawn without any breaks in curve and without repeating edges.




vertex of M is connected to each vertex of N. This graph is denoted by $K_{m,n}$ where m is number of vertices in M and n is number of vertices in N.

Eg :-




A graph in which each vertex of m is connected to each vertex of n is called complete bipartite graph.
Denoted by $K_{m,n}$.
In other words no edge joining two vertex in same vertex

(XVi) Discrete graph :- A graph $D_n$ is called discrete graph if it has n number of vertices

Eg :-

$D_5$

Here $D_5$ is discrete graph with 5 vertices

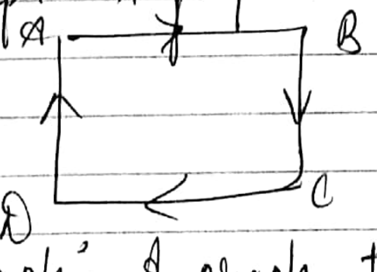
(xvii) Linear graph :- A graph $L_n$ with n vertices $\{V_i, V_{i+1}\}$ for $1 \leq i < n$ edges is called linear graph.

Eg :-

$L_6$

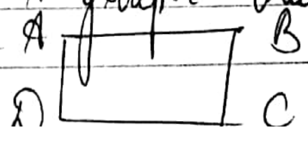
(xviii) Directed graph :- A graph which have directions

Eg :-




(XiX) Undirected graph :- A graph that doesnot have directions

Eg :-



classmate
Date
Page

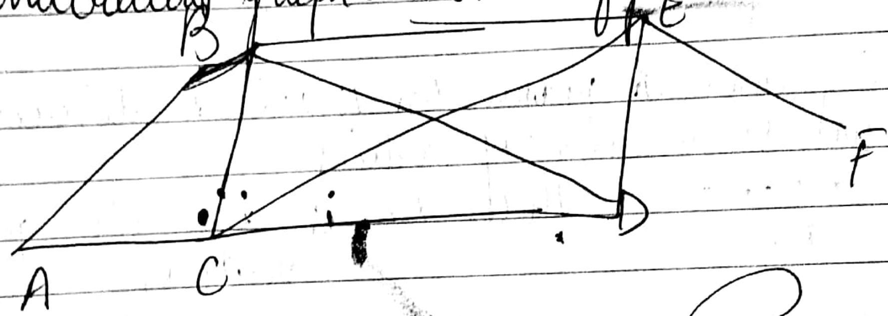
First theorems on graph theory :-

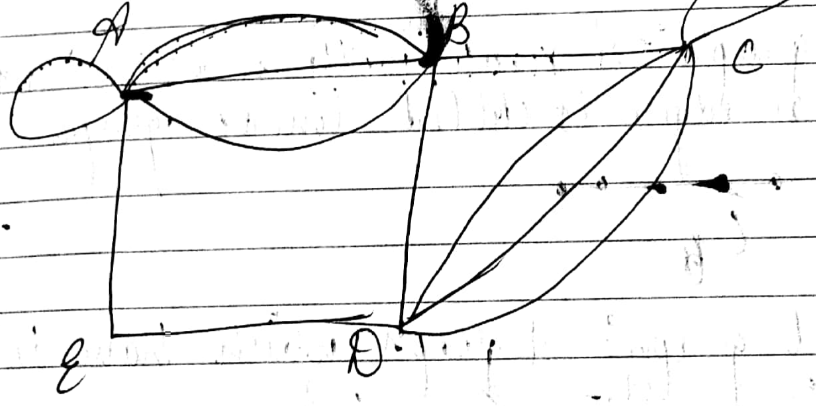
State :- The sum of degree of vertices of a graph is equal to twice the number of edges in G. This is also called hand shaking lemma.

Proof :- Since each edge is counted twice in counting degree of vertices of graph G. Therefore all the edges contribute to sum of degree of vertices.

Therefore $\boxed{\sum deg\ v_i = 2e}$

Eg :- Find number of vertices, the number of edges and degree of each vertex in the foll$^n$ undirected graph. Also verify above theorem.



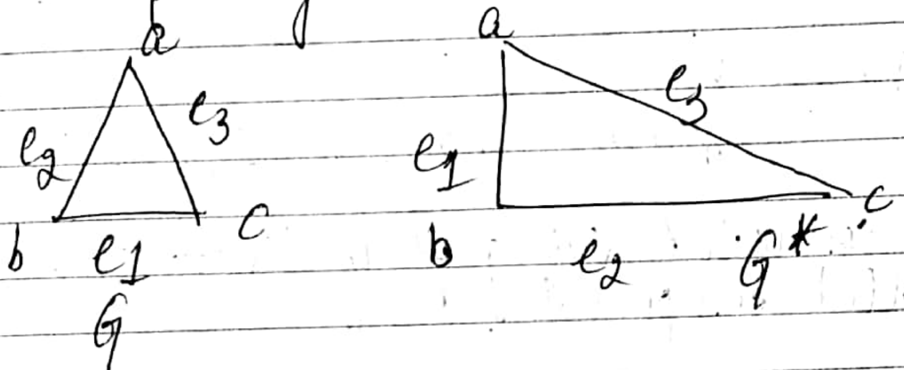
→ **Isomorphism of Graph :-** Suppose $G(V, E)$ & $G^*(V^*, E^*)$ are graphs, then these two graphs are said to be isomorphic if they have.

(i) Same number of vertices

(ii) Same no. of edges

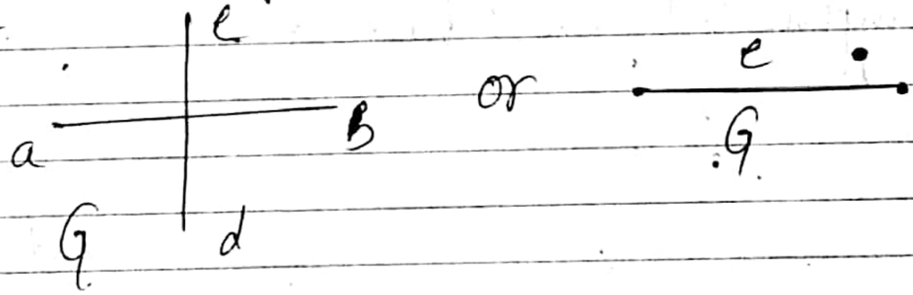
(iii) Corresponding vertices with same degree.

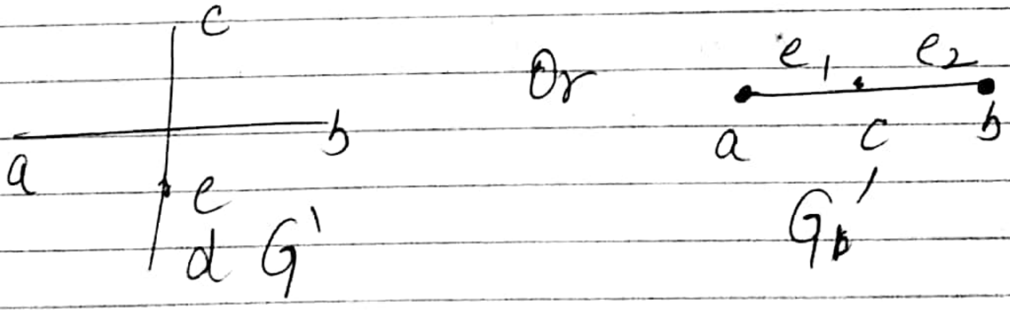
Eg :-




→ **Homomorphic Graph :-** Given any graph G we can obtain a new graph by dividing an edge of G with additional vertices.

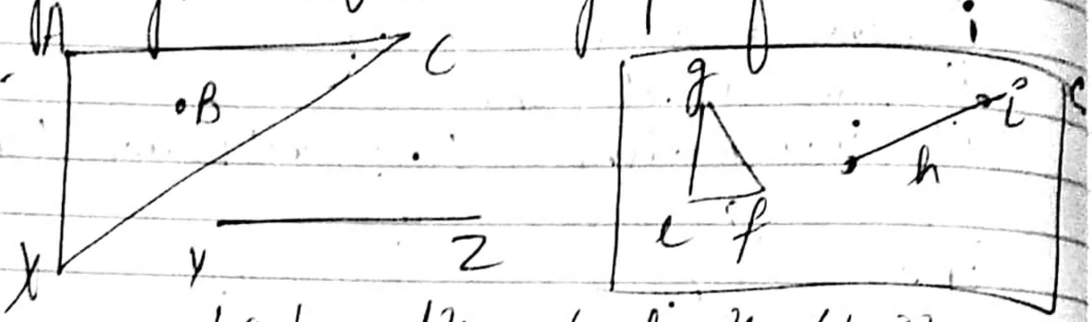
Eg :-



**Connected Components :-** A connected components of G is a sub graph of G which is not contained in any larger connected sub graph of G.

Eg :-

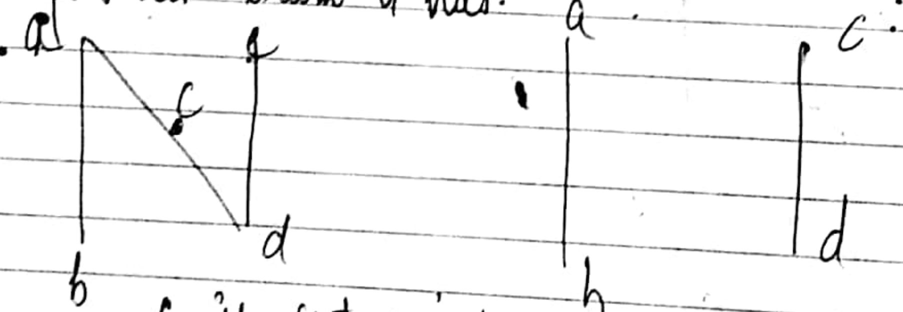


$\{a, b, c, d\}$, $\{e, f, g\}$, $\{h, i\}$

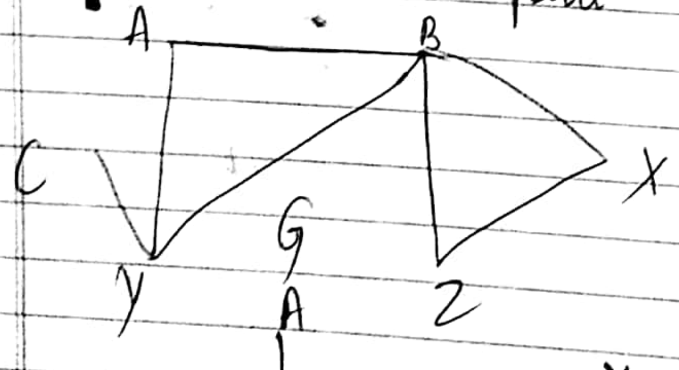
Each connected sub graph of disconnected graph is called <u>component</u>.

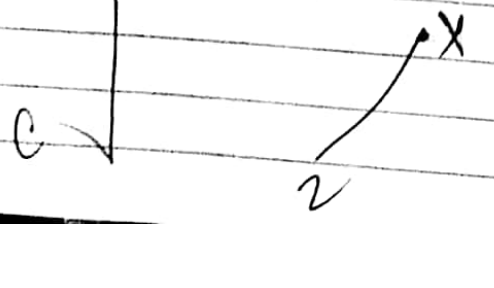
**Cut point :-** A vertex V is called a cut point for G if G-V is disconnected. V is a cut point for any Graph if G-V has more connected components than G has.

Eg :-



c is cut point

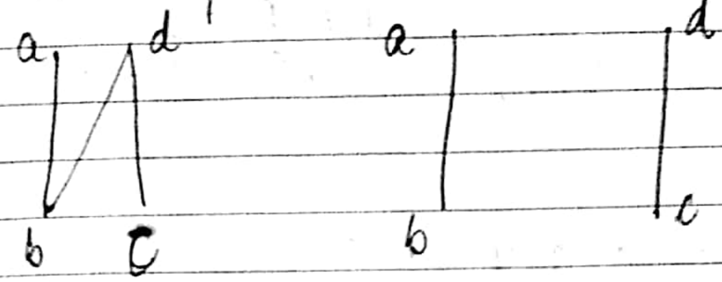




G-B.

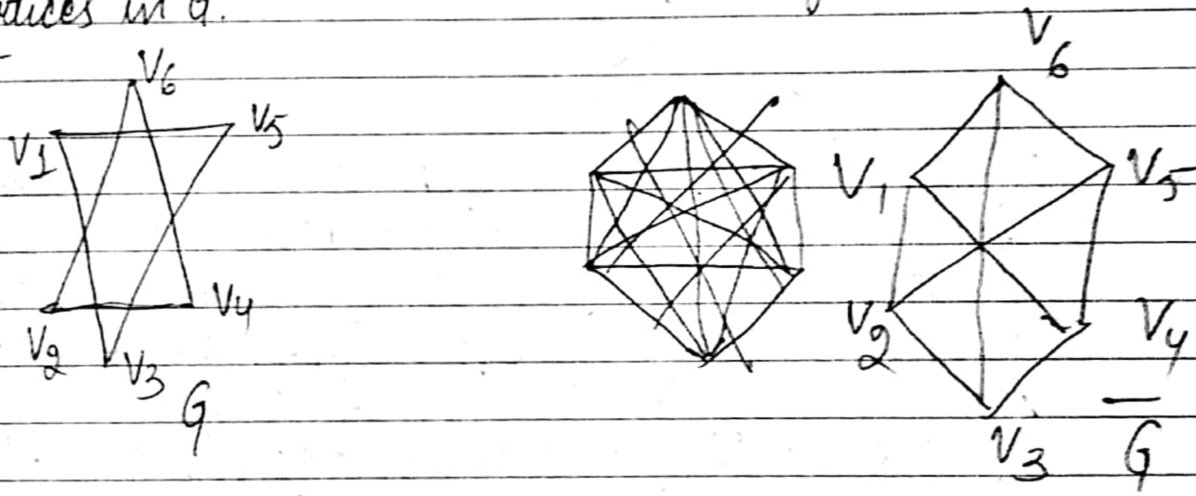
Bridges :- An edge E is a bridge for G if G-E is disconnected. In general e is a bridge for any graph G if G-e has more connected components than G has.

Eg:-




complement of a graph :- The complement of a graph G is denoted by $\overline{G}$ and is defined as simple graph with a vertex Z same as vertex Z of G together with edge set satisfying the property that there is an edge between two vertices in $\overline{G}$ when there is no edge between these vertices in G.

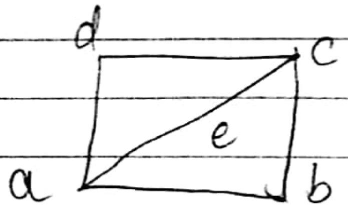
Eg:-




Distance :- Consider a connected graph G. The distance between vertices u and v is G written as length of shortest path between u and v. ie d(u,v)

Pendent Vertex :- A vertex whose degree is 1 is called pendent Vertex

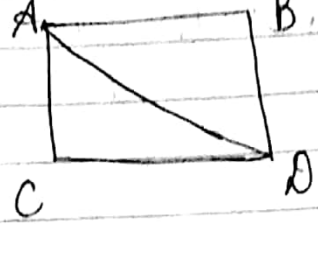
Eg:-




e is pendent Vertex

Diameter :- The diameter of G written as diam(G) is maximum distance between any two points in G.

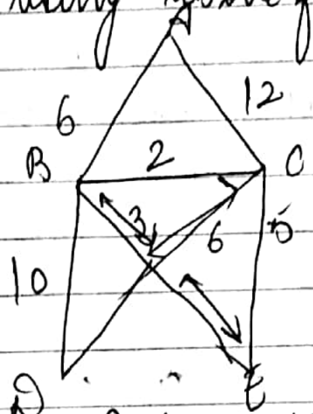
Eg :-




d(A, D) = 1
diam(G) = 2

Ques :- Let G be a labelled graph with length assigned to its edges. Explain the maximum path from A to D problem using above figure as example.

~~Ans~~ :- ~~Let A~~




Ans :- Let A and D be vertices in G. The minimum path problem refers to finding a path of minimum length between A and D where length of path is sum of length of its edges. There are 6 simple paths from A to D. (A, B, D) = 16,
(A, C, B, D) = 24, (A, B, E, C, D) = 20, (A, C, D) = 18,
(A, B, C, D) = 14, (A, C, E, B, D) = 30.
Therefore (A, B, C, D) = 14.

Ques :- Show that every connected graph G may be viewed as a waited or labelled graph. What is the minimum path in such a graph.

Ans :- Here we can assume that every edge in G has length 1, then minimum path alpha ($\alpha$) between vertices P and Q is a path of minimum length in the original sense. i.e. a path with a minimum number of edges.

Theorem :- Proove that in a graph, Number of vertices of odd degree is even.

Proof :- Let $v_1, v_2 - - - v_n$ be n vertices and $e_1, e_2 - - - e_n$ be edges in graph G. Then by first theorem on graph theorey.

$$\sum_{i=1}^{n} deg(v_i) = 2e \quad - (1)$$

Now divide the sum on L.H.S of first in two parts.

(i) One part contain the sum of degree of vertices which have even degree

(ii) Second part contain the sum of degree of vertices which have odd degree.

Then eq (1) can be written as :-

$$\sum_{even} deg(v_i) + \sum_{odd} deg(v_i) = 2e \quad - (2)$$

Since, the R.H.S of (2) is an even no. also $\sum_{even} deg(v_i)$ is even. This implies $\sum_{odd} (v_i)$ is also even.

Because the sum of odd numbers can only be even if $v$'s are even in number i.e. The sum of degree of vertices having odd degree is even. Therefore number of vertices having odd ~~of~~ degree must be even.

Scanned by CamScanner

classmate
Date
Page

**Theorem:-** Proove that maximum degree of any vertex in a simple graph having $n$ vertices is $n-1$.

**Proof:-** Since in a simple graph, There are no parallel edge and no loop, Therefore a vertex can be connected to remaining $(n-1)$ vertices at most by $(n-1)$ edges. Hence the maximum degree of any vertex in a simple graph having $n$ vertices is $n-1$

**Theorem:-** Show that maximum no. of edges in a simple graph with $n$ vertices is $\frac{n(n-1)}{2}$

**Proof:-** Let G be a simple graph with $n$ vertices.
By handshaking lemma, we know that

$$\sum_{i=1}^{n} deg(v_i) = 2e$$

where $e$ is the number of edges in the graph G

$$d(v_1) + d(v_2) + \text{---} + d(v_n) = 2e$$

We know that maximum degree of each vertex in the graph G can be at the most $(n-1)$.
Therefore, above eq$^n$ reduces to

$$\underbrace{(n-1) + (n-1) + \text{———} + (n-1)}_{n \text{ times}} = 2e$$

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2}$$

Hence proved

Ques:- Find no. of edges in the graph $K_8$, $K_{12}$ and $K_{15}$.

Ans:- $K_8 = e = \frac{n(n-1)}{2}$

$$e = \frac{\cancel{8}^{4} \times 7}{\cancel{2}} = 28.$$

$$K_{12}, e = \frac{\cancel{12}^{6} \times 11}{\cancel{2}} = 66$$

$$K_{15}, e = \frac{15 \times \cancel{14}^{7}}{\cancel{2}} = 105$$

Ques:- A graph G has 21 edges, 3 vertices of degree 4 and all other vertices are of degree 3. Find number of vertices in G.

Ans:- Let n be number of vertices in G, Then by first theorem

$$\sum_{i=1}^{n} (\deg v_i) = 2e$$

where e is number of edges.

Let $v_1, v_2, v_3$ be vertices of degree 4 and $v_4, v_5 + \text{--} v_n$ be the remaining vertices of degree 3.

$$\sum_{i=3}^{n} \deg(v_i) + \sum_{i=4}^{n} d(v_i) = 2(21)$$

$$3 \times 4 + (n-3)(3) = 42$$

$$12 + 3n - 9 = 42$$

$$3n + 3 = 42$$

$$3n = 39$$

$$\boxed{n = 13}$$

Scanned by CamScanner

Ques:- Find K, if a K regular graph with 8 vertices has 12 edges & also draw K regular graph.

Ans:- We know that a graph will be K regular if degree of all the vertices in G are same and equal to K.

Number of vertices $v = 8$

Number of edges $e = 12$

Therefore by first theorem

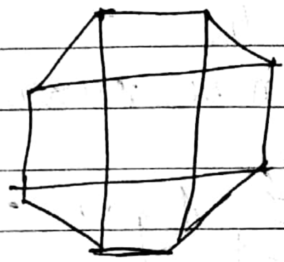
$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$v = 8$, $\deg = K$, $e = 12$

$$\sum_{i=1}^{8} K = 2(12) \Rightarrow 8K = 2(12)$$

$$8K = 24$$

$$\boxed{K = 3}$$



Ques:- Do there exists ($\exists$) a graph with 6 vertices with degree 3, 2, 4, 1, 3, 2 resp.

Ans:- $n = 6$.

Let e be the number of edges in the graph.

Therefore by first theorem.

$$\sum_{i=1}^{n} \deg(v_i) = 2$$

$$\sum_{i=1}^{n} 3+2+4+1+3+2 = 2e$$

$$\Rightarrow 2e = 15$$

$e = \frac{15}{2}$ Edges can't be in fraction so it is not possible

Ques:- Is it possible to draw a simple graph with 4 vertices and 7 edges.

Ans:- In the simple graph with n vertices, maximum no. of edges will be $\frac{n(n-1)}{2}$. Hence a simple graph with 4 vertices will have atmost $\frac{4 \times 3}{2} = 6$ edges.

Therefore a simple graph with 4 vertices cannot have 7 edges. Hence such a graph does not exists.

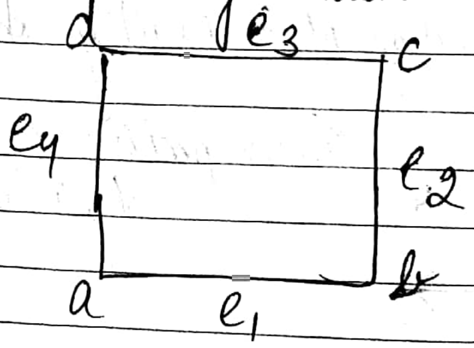
## Graph colouring:- Consider a graph G, a vertex colouring or simple colouring is an assignment to the vertices of G such that adjacent vertices have different colour. We say that G is n colourable if there exits colouring of G which uses n colours.

Chromatic Number:- The minimum number of colours needed to paint a graph is called chromatic number of G denoted by $\chi(G)$.

Walk:- A walk in a graph G is a finite sequence of vertices and edges. Vertex $V_0$ is called initial vertex and starting vertex and vertex $V_k$ is called terminal vertex and final vertex. $V_0$ and $V_k$ need not be distinct. The no. of edges in a walk is called the length of the walk.

Vertices and edges can be represented in a walk.

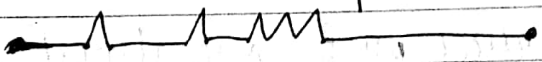
Eg:-



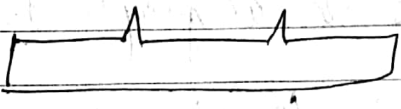

$W = a, e_1, b, e_2, c, e_3, d, e_4.$

Types of Walk :-

Open Walk :- If a walk begins and end with different vertices is called open walk.

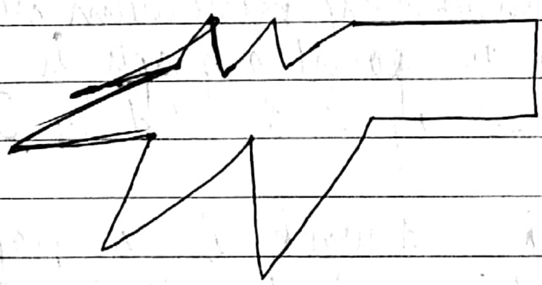


Closed Walk :- If the initial and terminal vertices of a walk are same, it is called closed walk.



Trivial Walk :- A walk containing no. edge and has length 0 is called trivial walk.

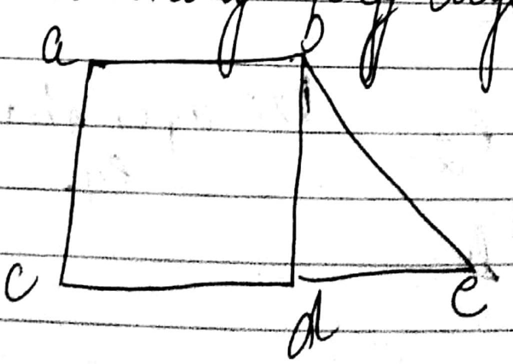
Path :- An open-walk in which no vertex appear more than one is called path



Circuit or closed path or cycle :- A circuit is a closed walk in which no vertex except initial and terminal vertex appears more than 1.

Euler path :- Euler path in G is a simple path containing every edge of G exactly once.

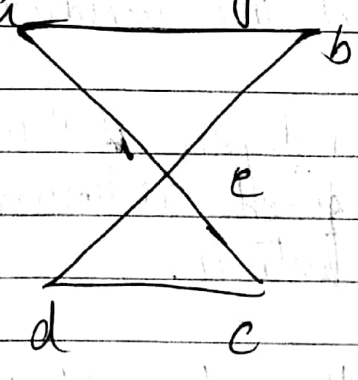
Eg :-




Euler path : a, c, d, e, b, d, a, b.

Euler Circuit :- Euler circuit in a graph G is a simple circuit containing every edge of G exactly once

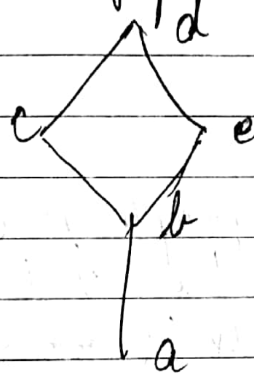
Eg :-




Euler circuit :- a, b, c, d, e, b, a.

Hamiltonian path :- A simple path in a graph G that passes through every vertex exactly once is called hamiltonian path.

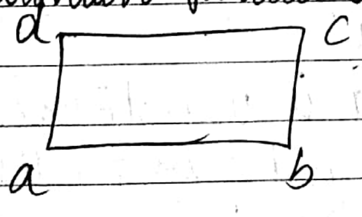
Eg :-




a, b, c, d, e

Hamiltonian Circuit :- A simple circuit in a graph G that passes through every vertex exactly once is called hamiltonian circuit.

Eg :-




HC :- a, b, c, d, a

Source :- A vertex v in G is called source if it has positive out degree and 0 in degree.

Sink :- A vertex v is called sink if it has positive in degree and 0 out degree

Out degree (Outdeg) :- The out degree of a vertex V of G written as Outdeg(V). It is defined as the number of edges which are incident out of u.

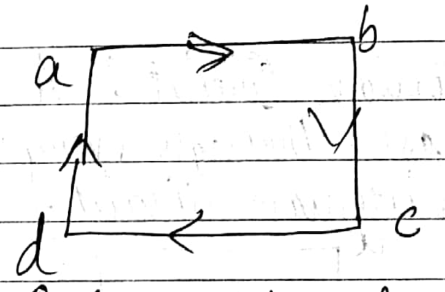
Indegree (Indeg) :- The in degree of V written as Indeg(V). It is defined as number of edges which are incident into u.

Connectivity :- There are three types of connectivities. Connectivity extends the concept of adjacency and is essentially a form of concatenated adjacency

Types :-

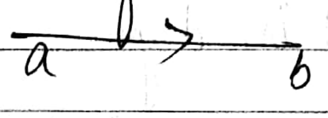
(i) Strongly connected :- A directed graph is strongly connected if there is a path from vertex A to B and from B to A ie, each is reachable from the other.

Eg :-



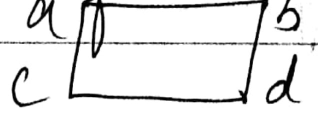
(ii) Unilaterally connected :- G is unilaterally connected if any pair of vertices A and B in G, there is a path from A to B and B to A ie, one of them is reachable from the other.

Eg :-



(iii) Weakly connected :- A directed graph is weakly connected if its undirected graph is connected

Eg :-



Imp Euler formula :-

State :- Let M be a connected map with V vertices, E edges and R regions then $V - E + R = 2$.

Proof :- We prove the theorem by applying Induction method on vertices and edges.

Case 1 :- Subcase 1 : When there is only 1 vertex fig (i)

fig (i)

Here $V = 1$, $E = 0$, $R = 1$

$\therefore V - E + R = 1 - 0 + 1 = 2 = R.H.S$

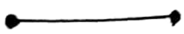
a

fig (ii)

fig (ii) Here $V = 1$, $E = 1$, $R = 2$.

$\therefore V - E + R = 1 - 1 + 2 = 2 = R.H.S$

Subcase 2 :- When there is only 1 edge

fig (iii)  $V = 2$, $E = 1$, $R = 1$

Result is true for V vertices & E edges is increased by 1 ie. $V = V + 1$

Result is true in every case

Case 2 :- Assume result is true for $n = k$ ie result is true for V vertices and E edges ie $V - E + R = 2$

Case 3 :- Now assume that result is true for V vertices & E edges is increased by 1 ie. $V = V + 1$

Now construct edge from new vertices to anyone of existing vertex.

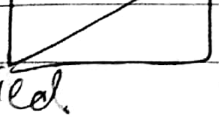
Subcase 1 :- In such a manner that the graph will remain planar

$V' = V + 1$

$E' = (E + 1)$, $R' = R$

$\therefore V' - E' + R' = V + \cancel{1} - E - \cancel{1} + R = V - E + R = 2$ (As in case 2)

Subcase 2 :- Join two existing vertices in such a manner that graph will still remains planar



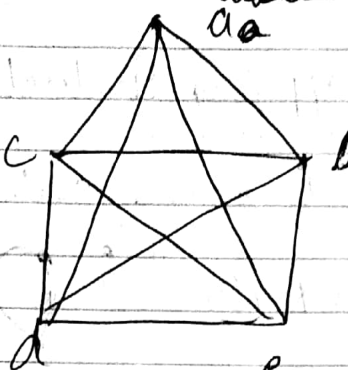
When edge is increased by 1

$E' = E + 1$, $R' = R + 1$, $V' = V$

Hence Verified. $V - E + R = 2$ (case 2)

Ques: Proove that graph $K_5$ is not planar or $K_5$ is 5 colourable

Ans :-




∴ No. of Vertices in $K_5$ ⇒ $V = 5$

Edges(E) = 10

If $K_5$ is a planar graph then

$E \leq 3V - 6$

$10 \leq 3(5) - 6$

$10 \leq 15 - 6$

$10 \leq 9$

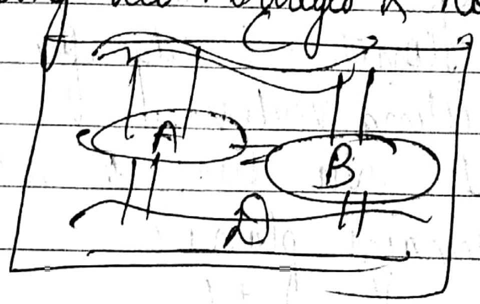
Which is not possible, therefore it is not planar

Results :-

(i) $2E \geq 3R$

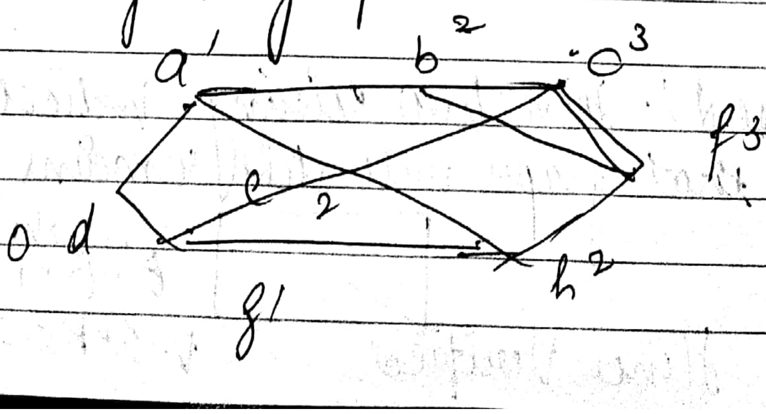
(ii) $E \leq 3V - 6$

Konigsberg Bridge Problem :- Begining anywhere and ending anywhere can a person walk through the town crossing all 7 bridges & not crossing any bridge twice




Euler proved that such a walk is impossible.

BST :- Algo for finding shortest path b/w the two vertices in unweighted graph

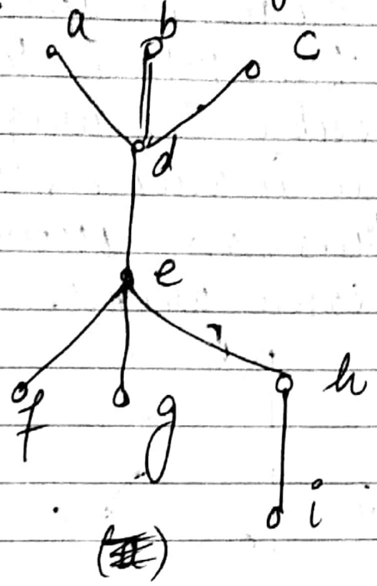


# Discreate Structures

## TREES

**Tree :-** A connected graph that contains no simple circuit.

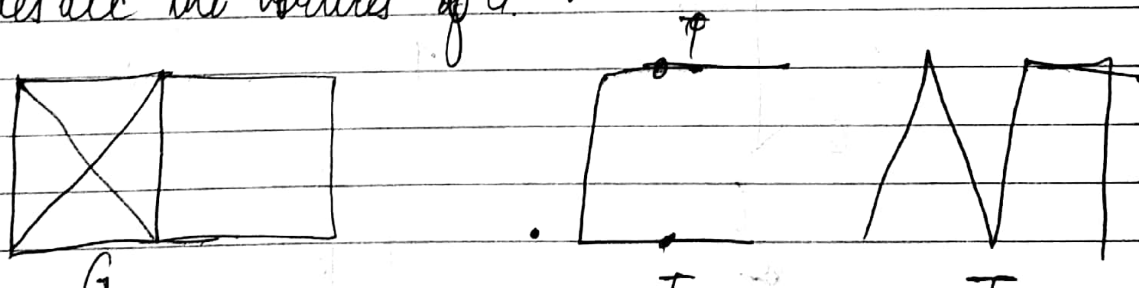
Eg :-




**Forest :-** collection of disjoint trees.

**Spanning tree :-** A sub graph T of a connected graph G is called spanning tree of G if T is a tree and includes all the vertices of G.

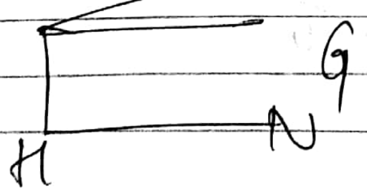
Eg :-



G $\qquad$ $T_1$ $\qquad$ $T_2$

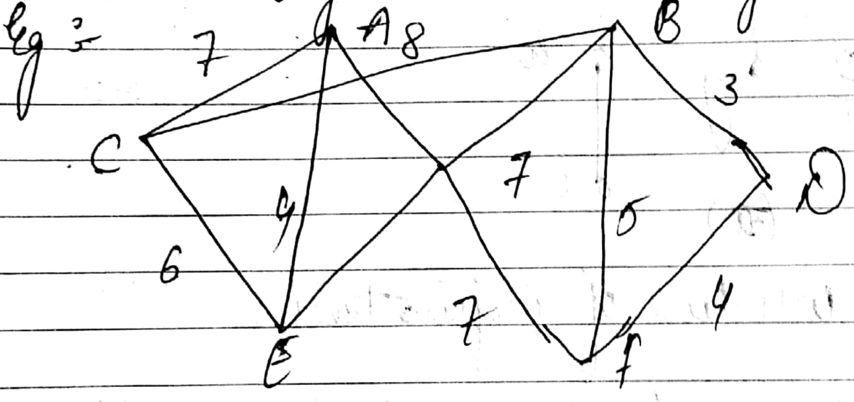
**Tree Graph :-** The tree T is called a tree if it is connected and has no cycle.

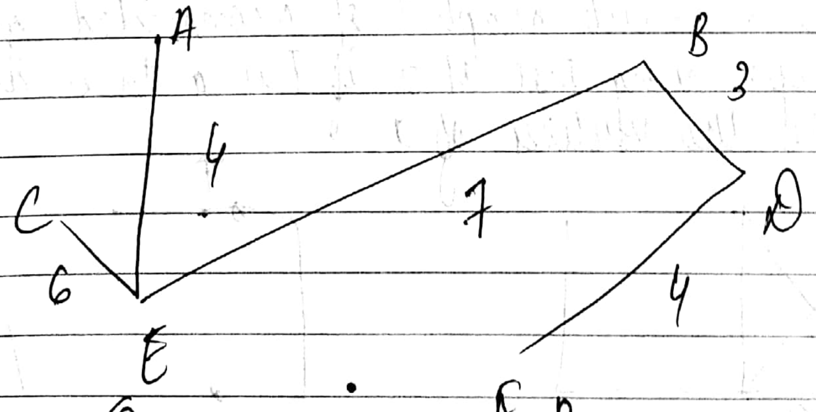
Eg :-

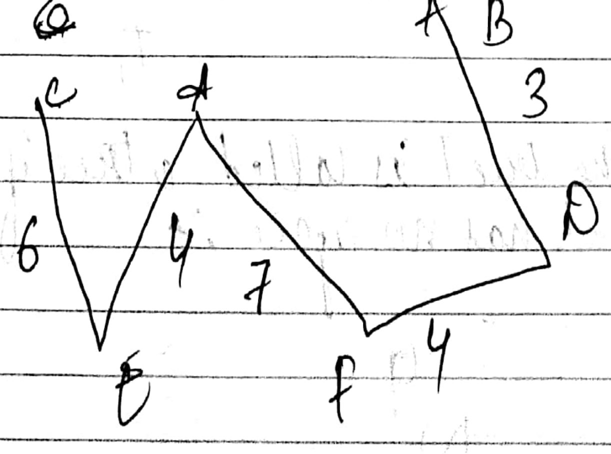


Minimal spanning tree :- Suppose G is connected weighted graph i.e., each graph of G is assigned a non-negative number called weight of the edge then any spanning T of G is assigned a total weight obtained by adding the weight of the edges in G. A minimal spanning tree of G is a spanning tree whose total weight is as small as possible. The weight of the minimal spanning tree is unique, different minimal spanning trees can occur when two or more edges have same weight.

Eg :-



Discrete

# Inclusion Exclusion Principle

Let P & Q be any 2 disjoint sets. Then,
$|P \cup Q| = |P| + |Q| - |P \cap Q|$

Basic counting Principle :- If we have to perform n operations and their are $n_1$ different methods to perform 1st operation, $n_2$ different ways to perform 2nd operation, $n_3$ different ways to perform 3rd operation and so on. Then, total no. of ways to perform n operation will be $n_1 \times n_2 \times \ldots \times n_n$. This is known as Basic counting principle.

There are two ~~ways~~ counting principles :-

Sum Rule                    Product Rule.

i) Sum Rule :- If there are two jobs such that they can be performed independently in $m$ and $n$ ways. Then no. of ways in which either of two jobs can be performed is $m+n$ ways. (Independent)

(ii) Product Rule :- If there are two jobs such that one of them can be done in $m$ ways and when it has been done second job can be done in $n$ ways. Then, 2 jobs can be done in $m \times n$ ways.

Ques :- 1. Survey conducted 500 people
Tennis Player = 300
Football Player = 250
Tennis & Football player = 100.

Ans :- $|T| = 300$, $|F| = 250$, $|T \cap F| = 100$

~~Basic~~ count Principle

$$|T \cup F| = |T| + |F| - |T \cap F|$$
$$= 300 + 250 - 100$$
$$= 450$$

Ques 2:- Total students = 130
Mathematics = 60
Physics = 51
Maths & Physics = 30
Chemistry = 54
Chemistry & Mathematics = 26
Chemistry & Physics = 21
PCM = 12

$|M| = 60,\ |P| = 51,\ |P \cap M| = 30,\ |C| = 54,$
$|C \cap M| = 26,\ |C \cap P| = 21;\ |P \cap C \cap M| = 12$

Ans:- (i) Students studying bio = $|\overline{P \cup C \cup M}|$

$|P \cup C \cup M| = |M| + |P| + |C| - |P \cap M| - |P \cap C| - |C \cap M| + |P \cap C \cap M|$
$= 60 + 51 + 54 - 30 - 21 - 26 + 12$
$= 60 + 105 - 77 + 12$
$= 177 - 77 = 100.$

$|\overline{P \cup C \cup M}| = 130 - 100 = 30$

(ii) Not studying chem, students studying maths but not physics

$|M| = |M| - |M \cap P| - |M \cap C| + |P \cap C \cap M|$
$= 60 - 30 - 26 + 12$
$= 72 - 56 = 16.$

(iii) Students not studying PCM = 30 (Bio)

classmate

Date ______

Page ______

Ques 3:- A class has
10 Boys, 8 girls. Cal. the
Teacher wants to select either boy or a girl. ~~total~~
ways. (max)

Ans:- ~~~~ ~~~~ $m+n = 18$.

Ques 4:- If teacher wants to select a boy and a girl

Ans:- $m \times n = 8 \times 10 = 80$ ways

Ques 5:- 3 students for classical, 5 scholarships
5 = Maths, 4 = P.Ed.
One of these can be awarded (Cal. Ways)

Ans:- Sum rule = $3+5+4 = 12$ ways.

Ques 6:- Question paper contains 4 Questions that have 2 possible answers and 3 with 5 possible answers. Way cal.

Ans:- $2^4 \times 5^3 = (\quad)$ ways.

# RECURRENCE RELATIONS.

A recurrence relation is a functional relational b/w independent variable x & dependent variable f(x) and difference of variance order of f(x), is called recurrence relation. Eg:- $a_n = a_{n-1} + a_{n-2}$

**Order** :- Highest subscript - lowest subscript
$= n-(n-2) = 2$

Degree :- Highest power of highest subscript term

Note :- 1) If degree = 1, relation is called linear recurrence relation otherwise non-linear

2) If we have 0 on R.H.S then the relation is called homogeneous recurrence relation otherwise Non-homogenous.

3) If characteristics roots are real & Distinct say $\alpha$ & $\beta$ then sol$^n$ is

$$a_n = a_1(\alpha)^n + a_2(\beta)^n$$

→ For 3 roots

$$a_n = a_1(\alpha)^n + a_2(\beta)^n + a_3(\gamma)^n$$

4.) If roots are equal then sol$^n$ is

$$a_n = (a_1 + a_2 n)(\alpha)^n$$

→ if 2 roots are $\alpha_1, \alpha_1, \beta$ then

$$a_n = (a_1 + a_2 n)(\alpha_1)^n + a_3(\beta)^n$$

→ if 3 roots are equal

~~$a_n = a_1(\alpha+i\beta)^n + a_2(\alpha-i\beta)^n$~~

$a_n = (a_1 + a_2 n + a_3 n^3)(\alpha)^n$

5.) if roots in form $\alpha \pm i\beta$ then

$a_n = a_1(\alpha+i\beta)^n + a_2(\alpha-i\beta)^n$

Sums

1. Solve $S_n - 5S_{n-1} + 6S_{n-2} = 0$

A.E = $x^n - 5x^{n-1} + 6x^{n-2} = 0$

characteristic eq$^n$ = $\frac{x^n}{x^{n-2}} - \frac{5x^{n-1}}{x^{n-2}} + \frac{6x^{n-2}}{x^{n-2}} = 0$

$= x^2 - 5x + 6 = 0$

$= x^2 - 3x - 2x + 6 = 0$

$= x(x-3) - 2(x-3) = 0 \Rightarrow (x-2)(x-3) = 0$

$= x = 2, 3$

Sol$^n$ = $a_1 2^n + a_2 3^n$

2. $S_n - 9S_{n-1} + 18S_{n-2} = 0$, if $S_0 = 1$, $S_1 = 4$

A.E = $x^n - 9x^{n-1} + 18x^{n-2} = 0$

characteristic eq$^n$:- $\frac{x^n}{x^{n-2}} - \frac{9x^{n-1}}{x^{n-2}} + \frac{18x^{n-2}}{x^{n-2}} = 0$

$= x^2 - 9x + 18 = 0$

$= x^2 - 6x - 3x + 18 = 0$

$\Rightarrow x(x-6) - 3(x-6) = 0$

$\Rightarrow x = 3, 6$

$(S_n)$ Sol$^n$ = $a_1 3^n + a_2 6^n$

If $S_0 = 1$

$S_0 = a_1 3^0 + a_2 6^0$

$1 = a_1(1) + a_2(1)$

$\Rightarrow a_1 + a_2 = 1 \quad - (1)$

If $S_1 = 4$

$S_1 = a_1 3^1 + a_2 6^1$

$4 = 3a_1 + 6a_2 \quad - (2)$

Solving (1) & (2)

$3a_1 + 3a_2 = 3$

$\ominus 3a_1 \ominus 6a_2 = \ominus 4$

$- \quad - \quad -$

$-3a_2 = -1$

$a_2 = \frac{1}{3}$

$a_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$S_n = \frac{2}{3}(3)^n + \left(\frac{1}{3}\right)(6)^n$

(4) $S_n - 8S_{n-1} + 16 S_{n-2} = 0$

$A.E = x^n - 8x^{n-1} + 16x^{n-2} = 0$

$$C.E = \frac{x^n}{x^{n-2}} - \frac{8x^{n-1}}{x^{n-2}} + \frac{16x^{n-2}}{x^{n-2}} = 0$$

$$\Rightarrow x^2 - 8x + 16 = 0$$
$$\Rightarrow x^2 - 4x - 4x + 16 = 0$$
$$\Rightarrow x(x-4) - 4(x-4) = 0$$
$$\Rightarrow (x-4)(x-4) = 0$$
$$x = 4, 4$$

$a_n = (a_1 + n a_2) 4^n$ ($\because$ Repeated roots)

Sol$^n$. of non-homogeneous finite order linear Recurrence Relation

Let $S_n + a_1 S_{n-1} + a_2 S_{n-2} + \cdots a_r S_{n-r} = f(n)$ — (1)
be a non-homogeneous recurrence relation of order r. Its complete solution has 2 parts

→ Homogeneous sol$^n$. of corresponding homogeneous recurrence relation.

→ Particular sol$^n$. of given non-homogeneous recurrence relation which depends upon the function

Step 1:- Find homogeneous sol$^n$. of recurrence relation taking $f(n) = 0$ i.e. $S_n + a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_r S_{n-r} = 0$

Step 2:- Particular solⁿ of given non-homogeneous recurrence relation is obtained by using following rule:-

Rule 1: When $f(n)$ is constant take $S_n = p$ as particular solution where $p$ is a constant therefore, eqⁿ ① becomes

$$p + a_1 p + a_2 p + \cdots a_n p = f(n)$$

$$\Rightarrow p = \frac{f(n)}{a_1 + a_2 + a_3 - - + a_n}$$

but $1 + a_1 + a_2 + \cdots + a_n \neq 0$
If $1 + a_1 + a_2 + \cdots + a_n = 0$
then P.S cannot be $S_n = p$

In that situation $S_n = np$ will be particular solⁿ. to find $p$ put $S_n = np$ in ① eqⁿ. If it is again not possible to find $p$ take $S_n = n^2 p$.

① Solve $S_n + 5S_{n-1} = 9$, $S_0 = 8$.

Ans:- The given eqⁿ. is non-homogeneous
∴ Solⁿ will consist of 2 parts.
→ complementary fxn corresponding to homogeneous solⁿ (taking R.H.S = 0)
∴ The given eqⁿ becomes

$$S_n + 5S_{n-1} = 0$$

$$x^n + 5x^{n-1} = 0$$

$$\frac{x^n}{x^{n-1}} + \frac{5x^{n-1}}{x^{n-1}} = 0 \Rightarrow x + 5 = 0$$

$$\boxed{x = -5}$$

C.F = ∴ $S_n^h = a_1(-5)^n$ — ①

$S_0 = a_1(-5)^0 = a_1$

$\Rightarrow \boxed{a_1 = 6}$

$$\boxed{(S_n^{(h)}) = 6(-5)^n}$$

which is again case of failure.

Taking $S_n^{(p)} = n^2 p$

$S_{n-1} = (n-1)^2 p$

$S_{n-2} = (n-2)^2 p$

$\therefore S_n - 2S_{n-1} + S_{n-2} = 12$

$n^2 p - 2(n-1)^2 p + (n-2)^2 p = 12$

$p[n^2 - 2(n-1)^2 + (n-2)^2] = 12$

$p[n^2 - 2(n^2 - 2n + 1) + (n-2)^2] = 12$

$\cancel{p[n^2 - 2n^2 - 2 + 2n + n^2 - 2n + 4] = 12}$

$\cancel{p[-2n + 2] = 12}$

$\cancel{p[n}$

$p[\cancel{n^2} - \cancel{2n^2} + \cancel{4n} - 2 + \cancel{n^2} - \cancel{4n} + 4] = 12$

$p(2) = 12$

$\boxed{p = 6}$

$P.S = S_n^{(p)} = 6n^2$

General Sol$^n$ :- $(a_1 + a_2 n)(1)^n + 6n^2$

$(a_1 + a_2 n) + 6n^2$

$\boxed{a^n \to pI : pa^n}$ [even if $a^n$ is multiplied by any no]

Ques :- Solve :- $S_n - 4S_{n-1} + 3S_{n-2} = 3^n$ — (1)

Ans :- Non-homogeneous

$S_n - 4S_{n-1} + 3S_{n-2} = 0$

$x^n - 4x^{n-1} + 3x^{n-2} = 0$

$\frac{x^n}{x^{n-2}} - \frac{4x^{n-1}}{x^{n-2}} + \frac{3x^{n-2}}{x^{n-2}} = 0$

$x^2 - 4x + 3 = 0$

$x = 1, 3$

$S_n^{(n)} = a_1(1)^n + a_2(3)^n$ — (2)

Particular solⁿ :-

$PS = S_n = p3^n$

$S_{n-1} = p3^{n-1} = p3^n/3$

$S_{n-2} = p3^{n-2} = p3^n/(3^2)$

$\therefore\ p3^n - 4\left(\frac{p3^n}{3}\right) + \cancel{3}\left(\frac{p3^n}{3^{\cancel{2}}}\right) = 3^n$

$p3^n - 4p3^{n-1} + p3^{n-1} = 3^n$

$p3^n - 3^n = \cancel{p3^n(4-1)}\ 3p3^{n-1}$

$\cancel{3^n}(p-1) = p\cancel{3^n}$

$p-1 = p$

$p - p = 1$

$0 = 1$

case of failure.

$P.S = S_n = np3^n$

$S_{n-1} = (n-1)p3^{n-1}$

$S_{n-2} = (n-2)p3^{n-2}$

$np3^n - 4\left(\frac{p3^n}{3}\right)(n-1) + 3(n-2)p(3^{n-2}) = 3^n$

$np3^n - 4p3^{n-1}(n-1) + \cancel{3}(n-2)p\left(\frac{3^n}{3^{\cancel{2}}}\right) = 3^n$

$np3^n - 4p3^{n-1}(n-1) + (n-2)p(3^{n-1}) = 3^n$

$p3^{n-1}[3n - 4(n-1) + (n-2)] = 3^n$

$p3^{n-1}[\cancel{3n} - \cancel{4n} + 4 + \cancel{n} - 2] = 3^n$

$p3^{n-1}[2] = 3^n$

$2p = \frac{3^n}{3^{n-1}}$

$2p = 3$

$\boxed{p = \frac{3}{2}}$

Put this value in P.S of $= \frac{3^{n+1}}{2} \cdot n$

$P.S = S_n{}^P = \frac{3}{2}(3^n)(n) = \frac{3^{n+1}}{2} \cdot n$

$\therefore$ complete solution :- $a_1(1)^n + a_2(3)^n + \frac{3^{n+1}}{2} \cdot n$

Case III : Polymonial $Eq^n$,

$a_n + b \rightarrow P_n + Q$ , $an^2 + bn + c \rightarrow Pn^2 + Qn^2 + R$

(1.) $S_n - 7S_{n-1} + 10S_{n-2} = 6 + 8n$.

Homogenous $Sol^n$

$S_n - 7S_{n-1} + 10S_{n-2} = 0$

$x^n - 7x^{n-1} + 10x^{n-2} = 0$

$\frac{x^n}{x^{n-2}} - \frac{7x^{n-1}}{x^{n-2}} + \frac{10x^{n-2}}{x^{n-2}} = \frac{0}{x^{n-2}}$

$x^2 - 7x + 10 = 0$

$x = 5, 2$

$S_n{}^{(n)} = a_1(2)^n + a_2(5)^n$

Particular $sol^n$

$Pn + Q$

$S_n{}^{(P)} = Pn + Q$

$S_{n-1} = P(n-1) + Q$

$S_{n-2} = P(n-2) + Q$

$Pn - 7(P(n-1) + Q) + 10(P(n-2) + Q) = 6 + 8n$

$Pn - 7Pn + 7P - 7Q + 10Pn - 20P + 10Q = 6 + 8n$

$P[n - 7n + 7 + 10 - 20] + Q[-7 + 10] = 6 + 8n$

$P(-6n - 3) + Q(3) = 6 + 8n$

$P(n-1) + Q$

$Pn - 7(Pn - P + Q) + 10(Pn - 2P + Q) = 6 + 8n$

$Pn - 7Pn + 7P - 7Q + 10Pn - 20P + 10Q = 6 + 8n$

$4Pn - 13P + 3Q = 6 + 8n$

$P(4n-13)+4Q = 6+8n$

$4Pn - 13P + 4Q = 8n + 6$

$4P = 8$

$\therefore P = \frac{8}{4} = 2$

$\boxed{P = 2}$

$\Rightarrow -26 + 4Q = 6$

$4Q = 32$

$\boxed{Q = 8}$

$\boxed{P.I = 2n+8}$

General / complete sol$^n$ $= S_n^{(H)} + S_n^{(P)}$

$= a_1(2)^n + a_2(5)^n + (2n+8)$ Ans

Q6

# GENERATING FUNCTIONS

Let $S$ be any sequence with terms $S_0, S_1, \dots$ generating function of sequence as infinite series.

$$G(S,Z) = \sum_{n=0}^{\infty} S_n Z^n$$

$$= S_0 + S_1 \cdot Z + S_2 Z^2 + \dots$$

# Generating function of some standard sequence

1. $S_n = a, \; n \geq 0$

$$G(S,Z) = \sum_{n=0}^{\infty} S_n Z^n = \sum_{n=0}^{\infty} a \cdot Z^n = a \sum_{n=0}^{\infty} Z^n$$

$$= a(1 + Z + Z^2 + \dots)$$

$$= a\left[\frac{1}{1-z}\right] \quad \left(\because S_\infty = \frac{a}{1-x}\right)$$

2. $S_n = b^n, \; n \geq 0$ then $G(S,z) = \dfrac{1}{1-bz}$

3. $S_n = C \cdot b^n, \; G(S,z) = \dfrac{C}{1-bz}$

# Proove that Generating function of sum of of two sequences is equal to sum of their Generating function.

OR

If $S_n = a_n + b_n$ then $G(S,z) = G(a,z) + G(b,z)$

Solⁿ: Given $S_n = a_n + b_n$

$$G(S_z) = \sum_{n=0}^{\infty} S_n z^n = \sum_{n=0} (a_n + b_n) z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n$$

: Averag

classmate

Date ____
Page ____

$\therefore G(S,z) = G(0,z) + G(6,z)$

Ques:- Find sequence whose generating function is

$$\frac{6-29z}{1-11z+30z^2}$$

Sol$^n$:- $G(S,z) = \frac{6-29z}{1-11z+30z^2} = \frac{6-29z}{(1-6z)(1-5z)}$

Put $z = 1/6$, $z = 1/5$ — Partial addition

$$= \frac{6-\frac{29}{6}}{(1-6z)(1-\frac{5}{6})} + \frac{6-\frac{29}{5}}{(1-\frac{6}{5})(1-5z)}$$

$$= \frac{7}{1-6z} - \frac{1}{1-5z} = 7\cdot 6^n - 1\cdot 5^n$$

$\Sigma S_n \cdot z^n$

$\Sigma a_i z^n$

Ques:- Solve :- $S_n + 5S_{n-1} = 9$, $S(0) = 6$

Ans:- The solution will consist of two parts:- homogeneous and particular sol$^n$.

Homogeneous sol$^n$. :-

The homogeneous sol$^n$ of recurrence relation

$S_n + 5S_{n-1} = 0$

C.E $= x^n + 5x^{n-1} = 0$

$x + 5 = 0$

$\boxed{x = -5}$

$S_n = a_1(-5)^n$

$\frac{6-29z}{(1-6z)(1-5z)} = \frac{A}{(1-6z)} + \frac{B}{(1-5z)}$

Particular Sol$^n$ :- $f(n) = 9$

Put $S_n = p$ in given eq$^n$:

$\therefore p + 5p = 9$

$p = 9/6 = 3/2$

$\Rightarrow \boxed{S_n = 3/2}$

Scanned by CamScanner

Generating $sol^n$ $= S_n + S_n^{(p)}$
$$= a_1(-5)^n + \frac{3}{2} \quad -①$$

Now, $S(0) = 6$

Put $n=0$ in ①

$$6 = a_1(-5)^0 + \frac{3}{2}$$

$$6 = a_1 + \frac{3}{2} \Rightarrow a_1 = 6 - \frac{3}{2} = \frac{9}{2}$$

$$\therefore G.S = S_n = \frac{9}{2}(-5)^n + \frac{3}{2}$$

Ques:- Solve:- $S_n - 7S_{n-1} + 10S_{n-2} = 6 + 8n \quad -①$

$Sol^n$:- Homogeneous $sol^n$

$$S_n - 7S_{n-1} + 10S_{n-2} = 6 + 8n$$

$$CE:- x^n - 7x^{n-1} + 10x^{n-2} = 0$$

$$x^2 - 7x + 10 = 0$$

$$x^2 - 5x - 2x + 10 = 0$$

$$x = 5, 2$$

$$S_n = a_1(5)^n + a_2\, 2^n$$

P.S $\Rightarrow f(n) = 6 + 8n$

Put $S_n = p$ in given $eq^n$ for $n=0$

$$\therefore p - 7p + 10p = 6 + 8(0)$$

$$p - 7p + 10p = 6$$

$$4p = 6$$

$$p = \frac{3}{2} \qquad \boxed{S_n = \frac{3}{2}}$$

$$G.S = a_1(5)^n + \frac{3}{2} \quad -①$$

# Group Theory

Algebraic Structure :- If there exist a system such that it consists of non-empty set and one or more operations are performed on that set, then the system is called algebraic structure. It is generally denoted by $<A, *>$ where $A$ is a non-empty set and $*$ are operations on $A$ and algebraic system is also called algebraic structure because the operation on set $A$ is defined a structure on elements on $A$

Group :- Let us consider an algebraic system/structure $<G, *>$ where $*$ is binary operation on $G$, then the system $<G, *>$ is said to form a group if it satisfies following properties.

i) Closure or closed property :- $\forall a, b \in G, a * b \in G$

(ii) Associative law :- $\forall a, b, c \in G, (a*b)*c = a*(b*c)$

(iii) Existence of identity :- $\forall a \in G, \exists, e \in G$

s.t $a * e = a = e * a$

where $e$ is the identity element of G / it is always unique in case of addition, identity element is zero. in case of multiplication it is one

(iv) Existence of Inverse :- for any $a \in G$ $\exists b \in G$

s.t. $a * b = e = b * a$

In case of addition, inverse element is $-a$

In case of multiplication, inverse element is $1/a$

Date
Page

$\langle I, + \rangle$ is a set of integers where I is a set of integer that form infinite abillion group.

Abelian group:- In addition to above properties if there exists one more property ie commutative law ie

$\forall a, b \in G$ and $a * b = b * a$ hold then the group is said to be abelian group.

- closer
- commutative
- Associative
- Existance of Identity
- Existance of Inverse

Ques:- Let $\langle G, * \rangle$ be a group then prove that identity element is also unique

Ans:- Since $\langle G, * \rangle$ is a group to identity elements Since e is the identity element of G and f is any other element

$\therefore e * f = f = f * e$ - (1)

Also since f is identity element of G and e is any other element

$f * e = e = e * f$ = (2)

From (1) & (2) $e = f$

$\therefore$ Identity element is Unique.

Theorem:- Prove that inverse of an element of group G is always unique

Proof:- Since $\langle G, * \rangle$ is a group
Let e be the identity element of G
Let a be any element of G

If $b$ is inverse of $a$ then $a*b = e = b*a$ — ①
& $c$ is inverse of $a$ then $a*c = e = c*a$ — ②

Consider $b = b*e$

$= b*(a*c)$ [using ②]

$= (b*a)*c$

$= e*c$ [using ①]

$\therefore \boxed{b = c}$

$\therefore$ Inverse of $a$ is Unique

Theorem :- Prove that a non-empty subset $H$ of a group $G$ is a subgroup of $G$ iff $a \in H, b \in H \Rightarrow ab^{-1} \in H$

Subgroup :- If $G$ is a group and $H$ be any non-empty subset of $G$ then $H$ is a subgroup of $G$ if $H$ satisfies all the properties of $G$ under same operation of $G$.

Proof :- Since $G$ is a group & $H$ be its subset

Let $H$ be a subgroup of $G$

$\therefore H$ satisfies all the properties of $G$

Let $a \in H$ & $b \in H$

Since $b \in H$ and $H$ is a sub group

$\therefore b^{-1} \in H$ ($b^{-1}$ is inverse of $b$)

$\therefore a \in H$ & $b^{-1} \in H$

$\Rightarrow \boxed{ab^{-1} \in H}$ (by closure property)

Conversely assume that $a \in H, b \in H \Rightarrow (ab^{-1}) \in H$

To Prove :- $H$ is a subgroup of $G$

Assume that $b = a$

$\therefore$ By giving condition $a \in H, b = a \in H$

$aa^{-1} \in H$

$e \in H$.

Identity element exists

$\therefore e \in H$

also $a \in H$

By given condition $a^{-1}e \in H$

$\therefore a^{-1} \in H$

$\therefore$ Inverse of element exists

Since $b \in H$

$\therefore b^{-1} \in H$

Since $a \in H$, & $b^{-1} \in H$

Acc to given condition

$a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$

Closure property Exists

Since, $H$ is a subset of $G$

$\therefore$ All the elements of $H$ are in $G$.

Composition in $G$ is associated

$\therefore$ Composition in $H$ must be associative

$\therefore$ $H$ is a subgroup of $G$

Theorem 4:- Prove that union of two subgroups is again a subgroup iff one is contained in the other

To Prove: $H_1 \cup H_2$ is a subgroup of $G$

Since $H_1 \subset H_2$

$\therefore H_1 \cup H_2 = H_2$

& $H_2$ is a subgroup of $G$.

$\therefore (H_1 \cup H_2)$ is a subgroup of $G$.

Conversely Assume $(H_1 \cup H_2)$ is subgroup of $G$

To prove: $H_1 \subset H_2$ & $H_2 \subset H_1$

if possible suppose that

$H_1 \not\subset H_2$ & $H_2 \not\subset H_1$

Now $H_1 \not\subset H_2$

There exists an element $a \in H_1$ s.t. $a \notin H_2$

$a \in H_1 \Rightarrow a \in (H_1 \cup H_2)$ — (B)

Also $H_2 \not\subset H_1$

$\therefore \exists$ an element $b \in H_2$, st $b \notin H_1$

$b \in H_2 \Rightarrow b \in (H_1 \cup H_2)$ — (A)

$a \in (H_1 \cup H_2)$

$b \in (H_1 \cup H_2)$

$(H_1 \cup H_2)$ is a subgroup of G.

$\therefore$ By closure prop :-

$ab \in (H_1 \cup H_2)$ — (1)

$a \in H_1$ and $H_1$ is sub group

$\therefore a^{-1} \in H_1$ — (2)

$ab \in (H_1 \cup H_2) \Rightarrow ab \in H_1,\ ab \in H_2$

$a^{-1}(ab) \in H_1$) (By closure prop)

$\Rightarrow (a^{-1}a) b \in H_1$ (Associative prop)

$\Rightarrow eb \in H_1$

$\Rightarrow b \in H_1$

Which is a contradiction to eq (A)

→ Similarly, we can prove that $A \in H_2$ which will be a contradiction to (B)

$\therefore$ Our supposition is wrong

$\boxed{H_1 \subset H_2 \ \& \ H_2 \subset H_1}$

**Coset** :- Let $G$ be a group and $H$ be its subgroup. Let $a$ be any element of $G$ then the set $H(a)$ $Ha$ is called right coset of $H$ in $G$ generated by $a$ and is defined as $Ha = \{ha : h \in H\}$

Similarly $aH$ will be left coset & $= \{ah ; h \in H\}$

**Note** :- If $H$ is a subgroup of $G$ and $h \in H$ then
$$hH = Hh = H$$

$\Rightarrow$ Prove that :- Any two right cosets of $H$ in $G$ are either disjoint or identical

Proof :- Since $G$ is a group and $H$ be its subgroup
Let $Ha$, $Hb$ are two right cosets of $H$ in $G$ generated by $a, b$ resp.

$\therefore$ Either $Ha \cap Hb = \phi$ or
$Ha \cap Hb \neq \phi$

If $Ha \cap Hb = \phi$
$\therefore$ 2 right cosets are disjoint in this case
$\therefore$ Result is true in this case

If $Ha \cap Hb \neq \phi$
We will prove that $Ha = Hb$
Since $Ha \cap Hb \neq \phi$
$\therefore$ There exist an element $x \in (Ha \cap Hb)$
$\Rightarrow x \in Ha$ & $x \in Hb$
$\Rightarrow x = h_1 a$ and $x = h_2 b$ (where $h_1, h_2 \in H$)
$-①$ $-②$

From ① & ②
$h_1 a = h_2 b$

Pre operating on b.s by $h^{-1}$,
$h_1^{-1}(h_1 a) = h_1^{-1}(h_2 b)$ $\quad [h \in H \Rightarrow h^{-1} \in H]$

classmate
Date
Page

$(h_1^{-1} h_1) a = (h_1^{-1} h_2) b$ (Associative Prop)
$ea = hb$ (where $h = h_1^{-1} h_2 \in H$) → ②
$a = hb$

Again Pre operating on b.s by H
⇒ $H a = H(hb)$
$Ha = Hb$ [From note]
∴ H

Cyclic group :- Let G be a group in which the composition on G is multiplication then G is said to be cyclic if for some $a \in G$ every element of G can be expressed in the form of $a^n$ where n is any integer then G is called cyclic and a is called generator of G

For eg :- $G = \{1, -1, i, -i\}$
$= \{i^4, i^2, i^1, i^3\} = \{<i>\}$

Theorem :- Prove that every cyclic group is abelian
Proof :- Let G be cyclic group
∴ G is $= a^n$ where n is any integer
To prove :- G is abellian
i.e to prove if $(x, y) \in G$ then $xy = yx$
$x \in G \Rightarrow x = \{a^r : r$ is any int$\}$
$y \in G \Rightarrow y = \{a^s : s$ is any int$\}$
$xy = a^r a^s$
$= a^{r+s}$
$= a^{s+r}$
$= a^s \cdot a^r$
$\boxed{xy = yx}$ G is abelian Grp

classmate
Date
Page

# MonoiDs

Semi group :- If any set under the operation * satisfies these properties
Closure property, Associative ~~property~~ law then that set is said to form a semi group

Monoid :- If a set satisfies the following properties under the operation *
1. Closure prop.
2. Associative law
3. Existence law of identity then the set is said to form a monoid

Homomorphism :- Let G and G' be two groups then the function $f: G \to G'$ is said to form homomorphism if $f(ab) = f(a).f(b), \forall a, b \in G$
in case of addition
$f(a+b) = f(a) + f(b)$

Monomorphism :- Let G and G' be two groups then the function $f: G \to G'$ is said to form Monomorphism if f is homomorphism and One-One

Homomorphic Image :- Let ___ form
G' is called homomorphic image if f is homomorphic and onto

Theorem 2:- If $f: G \to G'$ is homomorphism ~~then prove~~ that $\ker f$ is a normal subgroup of $G$

Automorphism :- Let — — form ~~Kom.~~ Automorphism if $f$ is homomorphism, one-one and onto.

Theorem:- If a fxn $f: G \to G'$ is homomorphism then
1. $f(e) = e'$ [where $e$ is the identity element of $G$ and $e'$ is the identity element of $G'$]
2. $f(a^{-1}) = [f(a)]^{-1}$ $\forall a \in G$

Proof:- Since $G$ is a group and $f: G \to G'$ is a homomorphism

$\therefore f(e) = e'$

$\therefore \ker f \neq \phi$ $\qquad e \in \ker f$

Now, we will prove that $\ker f$ is a subgroup of $G$

i.e. to prove $\forall x, y \in \ker f$

i.e. to prove if $x, y \in \ker f$ then $f(xy^{-1}) = e'$

Since $x \in \ker f \Rightarrow f(x) = e'$

$y \in \ker f \Rightarrow f(y) = e'$

Consider $f(xy^{-1}) = f(x)\, f(y^{-1})$
$= f(x) \cdot [f(y)]^{-1}$
$= e' \cdot (e')^{-1}$
$= e' e$
$= e$

$\therefore xy^{-1} \in \ker f$

$\therefore \ker f$ is a subgroup of $G$

Now we will prove that $\ker f$ is a normal subgrp of $G$

ie to prove that $\forall g \in G$, $h \in \ker f$, $ghg^{-1} \in \ker f$

ie to prove $f(ghg^{-1}) = e'$

Proof 1:- Since $f: G \to G'$ is homomorphism

$\therefore f(ab) = f(a) \cdot f(b) \quad \forall (a,b) \in G$

(1) consider $f(e) = f(e \cdot e)$

$f(e) = f(e) \cdot f(e)$ — (1)

also $f(e) = f(e) \cdot e'$ — (2)

from (1) & (2)

$f(e)\cancel{f(e)} = \cancel{f(e)} \cdot e'$

$\boxed{f(e) = e'}$ Hence proved.

(2) consider $f(a) \cdot f(a^{-1}) = f(a \cdot a^{-1}) = f(e) = e'$ — (3)

Similarly $f(a^{-1}) \cdot f(a) = e'$ — (4)

from (3) & (4)

$f(a) f(a^{-1}) = e' = f(a^{-1}) f(a)$

Inverse of $f(a)$ is $f(a^{-1})$

$\Rightarrow [f(a)]^{-1} = f(a^{-1})$ Hence proved

Kernel of Homomorphism:- Let $f: G \to G'$ is homomorphism

then Kernel of $f(\ker f) = \{x \in G : f(x) = e'\}$

{eg $a \in \ker f$, $f(a) = e'$}

Theorem 2:- If $f: G \to G'$ is homomorphism then prove that $\ker f$ is a normal subgroup of $G$.

Now $f(h) = e'$

$$f(ghg^{-1}) = f(g) \cdot f(h) \cdot f(g^{-1})$$
$$= f(g) \cdot e' \left[f_1(g)\right]^{-1}$$
$$= f(g) \cdot [f(g)]^{-1}$$
$$= e'$$

$\therefore xy^{-1} \in \ker f$

$\therefore \ker f$ is a subgroup of $G$.

Now we will prove that $\ker f$ is a normal subgroup of $G$ ie. to prove $\forall g \in G, h \in \ker f$,

$ghg^{-1} \in \ker f$

ie to prove $f(ghg^{-1}) = e'$

Now $f(h) = e'$

$$f(ghg^{-1}) = f(g) \cdot f(h) \cdot f(g^{-1})$$
$$= f(g) \cdot e' \left[f(g)\right]^{-1}$$
$$= f(g) \cdot [f(g)]^{-1}$$
$$= e'$$

$\therefore ghg^{-1} \in \ker f$

$\therefore \ker f$ is a normal subgroup of $G$

Note:- $Z_m$ is integer modulo $m$ and is defined as

$$Z_m = \{0, 1, \ldots (m-1)\} +_m, \times_m \}$$

where $+_m$ or $a +_m b$ = remainder after $(a+b) \div m$

Similarly $\times_m$ or $a \times_m b$ = remainder after $(a * b)/m$

Theorem :- Let G be a reduced residue system modulo m and $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ group under multiplication modulo ~~15~~ 15

(1) Find the multiplication table of G [redacted]

(2) Find $2^{-1}$, $7^{-1}$, $11^{-1}$

(3) Find the order of G and the subgroup generated by (a) 2, (b) 7, (c) 11

(4) is G cyclic

1.

| $\times_{15}$ | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
| 2 | 2 | 4 | 8 | 14 | 1 | 7 | 11 | 13 |
| 4 | 4 | 8 | 1 | 13 | 2 | 14 | 7 | 11 |
| 7 | 7 | 14 | 13 | 4 | 11 | 2 | 1 | 8 |
| 8 | 8 | 1 | 2 | 11 | 4 | 13 | 14 | 7 |
| 11 | 11 | 7 | 14 | 2 | 13 | 1 | 3 | 4 |
| 13 | 13 | 11 | 7 | 1 | 14 | 3 | 4 | 2 |
| 14 | 14 | 13 | 11 | 8 | 7 | 4 | 2 | 1 |

2. $\rightarrow$ $2^{-1}$ is 8 , $7^{-1}$ is 13 , $11^{-1}$ is 11

(3) (a) $(2) \times_{15} (2) = 4$

$(2) \times_{15} (2) \times_{15} (2) = 4 \times_{15} 2 = 8$

$(2) \times_{15} (2) \times_{15} (2) \times_{15} (2) = 8 (\times_{15}) 2 = 1$

Order of 2 = 4

Subgroup generated by 2 is $\{2^0, 2^1, 2^2, 2^3\} =$

$\{1, 2, 4, 8\}$

4. The group $G$ is cyclic if there exist an element whose order is = Order of $G$.
Here Order of $G = 8$
Since there doesn't exist any element whose order is 8
$\therefore$ $G$ is not Cyclic

3 (b) $7 \times_{15} 7 = 4$
$7 \times_{15} 7 \times_{15} 7 = (4) \times_{15} (7) = 13$
$7 \times_{15} (7) \times_{15} (7) \times_{15} (7) = 13 \times_{15} 7 = 1$
Order of $7 = 4$.
Subgroup generated 7 is $\{7^0, 7^1, 7^2, 7^3\}$ = $\{1, 7, 49, 343\}$

(C) $11 \times_{15} 11 = 1$
Order of $11 = 2$
Subgroup generated by 11 is $\{11^0, 11^1\} = \{1, 121\}$

Quotient Group :- Let $G$ be a group and $H$ be a normal subgroup of $G$. Let $G/H$ denotes the left or right cosets of $H$ in $G$ then $G/H$ is a Quotient group under the coset multiplication
$(aH)(bH) = abH$

Note :-
If relation of congruence is an equivalence relation then it will partition $G$ into mutually disjoint classes such that every two members of the same class are congruent and the Union of all these disjoint classes will be equal to $G$ and order of $H$ is equal to order of

$H(a)$ i.e.

$$O(H) = O(aH) = O(Ha)$$

Theorem :- State and prove Lagrange theorem

Statement :-

The Order of Subgroup of a finite group is division of Order of the group i.e. if G is a group and H be its subgroup of G then order of H divides Order of G

Proof :- Since G is a finite group.

Let Order of $G = m$

Also, H is a subgroup of G

Let order of $H = n$

$$\left[ n/m = \frac{m}{n} \right]$$

To prove :- $n/m$

Define a congruence relation on G as

$$a \equiv b \pmod{H} \text{ iff } ab^{-1} \in H \quad - (A)$$

To prove :- relation is an equivalence relation

(i) To prove reflexive

Since $e \in H$

$\Rightarrow aa^{-1} \in H \ \forall a \in H$

Comparing with eq (A)

$a \equiv a \pmod{H}$

Thus, rel is reflexive.

(2) To prove symmetric

i.e. to prove if $a \equiv b \pmod{H}$ then $b \equiv a \pmod{H}$

Consider $a \equiv b \pmod{H}$

$ab^{-1} \in H$

$(ab^{-1})^{-1} \in H$

$\Rightarrow ba^{-1} \in H$

Scanned by CamScanner

classmate
Date
Page

Comparing with eq (A)
$b \equiv a \pmod H$
Thus, rel is symmetric

(3) To prove rel. is transitive
i.e. to prove if $a \equiv b \pmod H$, $b \equiv c \pmod H$
then $a \equiv c \pmod H$
Consider $a \equiv b \pmod H$ & $b \equiv c \pmod H$
Comparing with eq (A)
$(ab^{-1}) \in H$ & $(bc^{-1}) \in H$
$(ab^{-1})(bc^{-1}) \in H$ (Closure prop)
$ac^{-1} \in H$
Comparing with eq (A)
$a \equiv c \pmod H$
Thus, rel. is transitive
∴ Relation of congruence is an equivalence relation
G is partitioned into mutually disjoint equivalence classes st. every two members of the same class are congruent
$= \{x \in G \text{ s.t } x \equiv a \pmod H\}$
$= \{x \in G \cdot xa^{-1} \in H\} = \{x \in G \cdot x \in Ha\}$

∴ $G = Ha_1 \cup Ha_2 \cup Ha_3 \text{ —— } \cup Ha_r$

$\Rightarrow O(G) = O(Ha_1 \cup Ha_2 \cup \text{——} \cup Ha_r)$
$\Rightarrow O(G) = O(Ha_1) + O(Ha_2) + \text{——} + O(Ha_r)$
$\Rightarrow O(G) = O(H) + O(H) \text{ ——— } O(H)$ [Acc. to Note]
$\Rightarrow m = \underbrace{n + n + \text{——} n}_{r \text{ times}}$

$\Rightarrow m = rn$
$\Rightarrow n | m$
∴ $O(H)$ divides $O(G)$

# RINGS

**Rings :-** A non-empty set R together with the operation addition & multiplication is said to form a ring if the foll. prop's are satisfied :-

(1.) The set R undergo operation Addition forms an abelian group. $(R, +)$

(1.) Closure property $\forall a, b \in R$, $a+b \in R$
(2) Associative law $\forall a, b, c \in R$, $(a+b)+c = a+(b+c)$
(3) Existence of Identity $\forall a, b \in R$, $a+0 = a = 0+a$
(4) Existence of Inverse $\forall a \in R$, $\exists (-a) \in R$ s.t $a+(-a) = 0 = (-a)+a$
(5) Commutative law $\forall a, b \in R$, $a+b = b+a$.

**Rings with Unity :-** If there exists $1 \in R$ such that $1 \cdot a = a$ for every $a \in R$ then R is called Ring with Unity and 1 is called Unity of R

**Commutative or Abelian Ring :-** If $a \cdot b = b \cdot a$ $\forall a, b \in R$ then ring R is called abelian ~~grou~~ ring

**Zero divisors :-** Let R be a ring if there exists non-zero element $a, b \in R$ such that $a \cdot b = 0$ then these are called zero divisors

**Integral Domain :-** A commutative ring with Unity without zero divisors is called Integral domain ie if $a \times b = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

Date
Page

Division Ring :- A ring R with Unity is called division ring if for each non zero $a \in R$ such that $a \cdot b = 1 = b \cdot a$
This is called Multiplicative Inverse

Field - A commutative division ring is called field.

Subring :- Let R be a ring and S be any non-empty subset of R then S is called Subring if S itself is a ring under the same composition in R.

Left Ideal :- Consider a ring R and U be any non-empty subset of R then U is called left ideal of R if (1) U is subgroup of R under addition
ie if $\forall a, b \in U$ $a - b \in U$
(2) $r \times a \in U$ $\forall r \in R, a \in U$

Right Ideal :- Consider a ring R & U be any non-empty subset of R then U is called right ideal if (1) U is subgroup of R under addition
ie if $\forall a, b \in U$ $a \in U$
(2) $ar \in U$ $\forall a \in U, r \in R$

Ideal :- Consider a ring R & U be any non-empty subset of R then U is called Ideal if
(1) $\forall a, b \in U$, $a - b \in U$
(2) $r \times a \in U$ $\forall r \in R, a \in U$
(3) $ar \in U$, $\forall r \in R, a \in U$

Theorem: If $U$ is an ideal of $R$ where $R$ is a ring and $1 \in U$ then $U = R$

Proof: Given that :-

$R$ is a ring and $U$ is an ideal of $R$

To prove :- $U = R$ ie, to prove $U \subset R$, $R \subset U$

Since $U$ is an ideal of $R$

$\therefore U \subset R$ - (1)

Let $x \in R$ - (2)

$1 \in U$ (Given)

$\therefore x \cdot 1 \in U$ (ideal cond.)

$\Rightarrow x \in U$ - (3)

from (2) & (3)

$R \subset U$ - (4)

from (1) & (4)

$\boxed{U = R}$

Quotient Ring :- Let $R$ be a ring and $A$ is ideal of $R$ then the set denoted by $R/A$ or

$$\frac{R}{A} = \{A + r : r \in R\}$$

Together with two composition

(1) $(A + r_1) + (A + r_2) = A + (r_1 + r_2)$

(2) $(A + r_1) \times (A + r_2) = A + (r_1 \times r_2)$

$\forall\ r_1, r_2 \in R$

Ques:- If $J$ is an ideal of ring $R$ then show that
(1) if $R$ is commutative then $R/J$ is commutative
(2) if $R$ has unity $1$ then $R/J$ has unity $J+1$

Sol$^n$:- Since $R$ is a ring and $J$ is an ideal of $R$

$\therefore R/J = \{J+r : r \in R\}$

Together with 2 compositions

(1) To prove:- $R/J$ is commutative

Let $J+r_1, J+r_2 \in R/J$ $(r_1, r_2 \in R)$

To prove:- $(J+r_1)(J+r_2) = (J+r_2)\cdot(J+r_1)$

$$(J+r_1)\cdot(J+r_2) = J+(r_1 \cdot r_2)$$
$$= J+(r_2 \cdot r_1)$$
$$= (J+r_2)\cdot(J+r_1)$$

$\therefore R/J$ is commutative

(2) To Prove:- $(J+1)$ is Unity of $R/J$

i.e. To prove $(J+r)\cdot(J+1) = J+r$

Consider L.H.S

$$(J+r)\cdot(J+1) = J+(r\cdot 1)$$
$$J+r = \text{R.H.S}$$

→ Ring homomorphism:- Let $R_1$ and $R_2$ be two Rings and mapping $f: R_1 \to R_1$ is called Ring Homomorphism if

1. $f(a+b) = f(a)+f(b)$
2. $f(ab) = f(a)\cdot f(b)$ $\forall a,b \in R_1$

Monomorphism - If $f$ is one-one and homomorphism

Epimorphism - If $f$ is homomorphism & Onto

Isomorphism - If $f$ is homomorphism, One-One & Onto

Endomorphism - If $f$ is homomorphism and $R_1 = R_2$

Automorphism - If $f$ is Isomorphism and $R_1 \cong R_2$

Scanned by CamScanner

classmate
Date
Page

Kernel of Ring homomorphism :- Let $R_1$ and $R_2$ be two rings and the mapping $f: R_1 \to R_2$ is homomorphism then (Kernel f) (Ker f) $= \{x \in R_1 : f(x) = 0'\}$ where $0'$ is the zero element of $R_2$

Boolean Ring :- A ring R is said to be boolean ring if $x^2 = x \ \forall x \in R$

Ques :- If R is a Boolean ring then show that
(i) $2x = 0 \ \forall x \in R$
(ii) If $x + y = 0$ then $x = y$
(iii) R is commutative

Sol (i) Let $x \in R$

$x + x \in R$ ($\because$ R is a ring) (Closure prop)

$\Rightarrow (x+x)^2 = x + x$ ($\because$ R is a Boolean ring)

$= (x+x)(x+x) = (x+x)$

$= x^2 + x^2 + x^2 + x^2 = x + x$

$= x + x + x + x = x + x$

$= x + x = 0$

$= 2x = 0$

(ii) Given that $x + y = 0$
by (i) $x + x = 0$
$x + x = x + y$
$\Rightarrow x = y$ or $y = x$

(iii) Let $x, y \in R$
$x + y \in R$ (closure prop.)
$(x+y)^2 = x + y$ (boolean law)
$(x+y)(x+y) = (x+y)$
$x^2 + xy + yx + y^2 = x + y$

$$\cancel{x}+\overset{xy}{\cancel{yx}}+yx+\cancel{y}=x+y \quad [\because R \text{ is Boolean Ring}]$$

$$xy+\cancel{xyx}=0$$

$$xy=-yx \quad \text{(ii)}$$

Unit:- Let $(R,+,\cdot)$ $R$ be a ring with unity any element $a\in R$ is said to be unit or invertical if for each non-zero, $a\in R$ there exists $(\exists)$ $b\in R$ such that $a\cdot b = b\cdot a$

Note:- An element $m$ is called Zero divisor of $Z_n$ if G.C.D of $(m,n)$ is $no \neq 1$ or they are not relatively prime

Ques:- Find the zero divisors of $Z_{15}$

$Z_{15}=\{0,1,2,\ldots,14\}$

Ans

$15 \Rightarrow 3\times 5$ ] no common element GCD = 1

$2\to 2$ ]

$3\to 3$ ] G.CD = 0 $\longrightarrow$ Zero divisor

$4\to 2\times 2$

$5\to 5 \longrightarrow$

$6\to 2\times 3 \longrightarrow$

$7\to 7$

$8\to 2\times 2\times 2$

$9\to 3\times 3 \longrightarrow$

$10\to 2\times 5 \longrightarrow$

$11\to 11$

$12\to 2\times 2\times 3 \longrightarrow$

$13\to 13$

$14 \Rightarrow 7\times 2$

Hence Zero divisors are

3, 5, 6, 9, 10, 12

Ques:- Consider the Ring $Z_{10}$ of integers modulo 10

(i) find the units of $Z_{10}$

(ii) find $-3, -8, 3^{-1}$

(iii) let $f(x) = 2x^2+4x+4$ with roots of $f(x)$ over $Z_{10}$

(i) $10 \to 2 \times 5$

$Z_{10} = \{0, 1, 2 - - 9\}$

$1 \to 1$

$2 \to 2$ (GCD = 0)

$3 \to 3$ (GCD = 1) $\to$ Unit

$4 \to 2 \times 2$

$5 \to \cancel{2x}$

$6 \to 2 \times 3$

$7 \to 7$ $\longrightarrow$ Unit

$8 \to 2 \times 2 \times 2$

$9 \to 3 \times 3$ $\longrightarrow$ Unit

$\therefore$ Units are:- 1, 3, 7, 9

(ii) $-3 \to$ Inverse of 3 under addition

$3^{-1} \to$ Inverse of 3 under multiplication

Addition

| $+_{10}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 1 | | | | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | | | | | | | | | | |

$3^{-1} = 1$

$8^{-1} = 2$

Multiplication

| $\times_{10}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 1 | | | | | | | | | | |
| 2 | 6 | | | | | | | | | |
| 3 | 0 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | | |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | | | | | | | | | | |
| 9 | | | | | | | | | | |
| 10 | | | | | | | | | | |

$3^{-1} \Rightarrow 7$ $\qquad$ $a \times \boxed{7} = 1$

(iii) $f(x) = 2x^2 + 4x + 4$

$f(0) = 4$ $\qquad$ $f(0) \mid 10 = 4$

$f(1) = 10$ $\qquad$ $f(1) \mid 10 = 0 \rightarrow$ Root

Roots of $f(x)$ will be those elements from 0 to 9 which will give the result 0

| | Remainder |
|---|---|
| $f(2)$ | 20 |
| $f(3)$ | 34 |
| $f(4)$ | 52 |
| $f(5)$ | 74 |
| $f(6)$ | 100 |
| $f(7)$ | 130 |
| $f(8)$ | 164 |
| $f(9)$ | 202 |