3. We saw closure properties allowed new kernels to be created from existing kernels. Prove the statements below regarding these closure properties, or give counter-examples to disprove them. Assume $\mathbf{x}, \mathbf{z} \in \mathcal{X} = \mathbb{R}^d$.
   (a) If K_1 is a kernel on $\mathcal{X}$, then $K(\mathbf{x}, \mathbf{z}) = e^{K_1(\mathbf{x}, \mathbf{z})}$ is also a kernel.
   (b) $K(\mathbf{x}, \mathbf{z}) = e^{(\|\mathbf{x}\|^2 + \|\mathbf{z}\|^2)} \cdot (\frac{\mathbf{x}^T\mathbf{z}}{\|\mathbf{x}\|^2\|\mathbf{z}\|^2})$ is a kernel.
   (c) $K(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{d} min(|\mathbf{x}_i|, |\mathbf{z}_i|)$ is a kernel

[30]

Q3
(c) To Prove $K(x, z) = \sum_{i=1}^{d} min(|x_i|, |z_i|)$ is a Kernel.

Proof : Try to prove that kernel matrix is PSd.
$K_1(x_1, x_2) = min(|x_1|, |x_2|)$ $x_1, x_2 \in R$

Consider $x_1, x_2, \ldots \ldots x_m \in \mathcal{X}$

Without loss of generality, let
$0 \le x_1 \le x_2 \ldots \ldots \le x_m$

Consider a Matrix R

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & -1 \cdots & 0 \\ 0 & \cdots & & \cdots & 1 \; -1 \\ 0 & \cdots & & \cdots & 1 \end{bmatrix}$$

Consider Matrix P

$$P = \begin{bmatrix} |x_1| & & & 0 \\ & |x_2| - |x_1| & & \\ & & \ddots & \\ 0 & & & |x_m| - |x_{m-1}| \end{bmatrix}$$

$P = R^T K_1 R$

P is diagnol, and PSd.
R is full rank

$\Rightarrow K_1$ is PSd

$\Rightarrow K_1 = \min(|x_1|, |x_2|)$ is a valid kernel

Consider $K(x,z) = \sum_{i=1}^{d} \min(|x_i|,$

$$= \sum_{i=1}^{d} K_i(x,z)$$

where $K_i(x,z) = \min(x[i], z[i])$

$\because$ Summation of kernels is a kernel in itself

$\Rightarrow \quad K(x,z) = \sum_{i=1}^{d} \min(|x_i|, |z_i|)$ is a valid kernel.

$\underline{Q_3}$

(a) If K_1 is a kernel then $e^{K_1(x,z)}$ is a valid kernel.

Proof: Consider infinite series expansion [Euler Series] of e^x

$$K(x,z) = e^{K_1(x,z)} = 1 + K_1(x,z) + \frac{K_1(x,z)^2}{2!} + \frac{K_1(x,z)^3}{3!} + \cdots\cdots$$

Consider the Gram Matrix K. Claim is that K is p.s.d.

Let $u \geq 0$ be any vector

Consider $u^T K u = \sum_{i}^{m} \sum_{j}^{m} u_i u_j K_{ij}$

$$= \sum_{i}^{m} \sum_{j}^{m} u_i u_j K(x_i, x_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} u_i u_j \left(1 + K_1(x_i,x_j) + \frac{K_1(x_i,x_j)^2}{2!} + \cdots + \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} u_i u_j + \sum\sum u_i u_j K_1(x_i,x_j) + \sum u_i u_j \frac{K_1(x_i,x_j)^2}{2!} + \cdots\cdots$$

$\because$ Product of two kernels in a kernel itself.

$$= \underbrace{u^T 1 u}_{\geq 0} + \underbrace{u^T K_1 u}_{\geq 0} + \underbrace{\frac{u^T K_2 u}{2!}}_{\geq 0} + \underbrace{\frac{u^T K_3 u}{3!}}_{\geq 0} + \cdots\cdots +$$

where K_i is the kernel matrix corresponding to kernel function

≥ 0

$$K_i(x_i', x_j') = K_1(x_i', x_j')^i$$

$\Rightarrow$ K is a p.s.d.

$\Rightarrow$ $K(x, z) = e^{K_1(x,z)}$ (where K_1 is a kernel), is a valid kernel.

Hence Proved

(b) $K(\mathbf{x}, \mathbf{z}) = e^{(\|\mathbf{x}\|^2 + \|\mathbf{z}\|^2)} \cdot (\frac{\mathbf{x}^T\mathbf{z}}{\|\mathbf{x}\|^2\|\mathbf{z}\|^2})$ is a kernel.

Consider $e^{\|x\|^2 + \|z\|^2} = e^{\|x\|^2} \cdot e^{\|z\|^2}$

$= f(x) \cdot f(z)$ where $f(x) = e^{\|x\|^2}$

From closure properties of Kernels $e^{\|x\|^2 + \|z\|^2}$ is a valid Kernel.

$\frac{x^T z}{\|x\|^2 \|z\|^2}$: Consider $\phi(x) = \frac{x}{\|x\|^2}$

$$K'(x, z) = \langle \phi(x), \phi(x) \rangle$$

$$= \frac{x^T z}{\|x\|^2 \|z\|^2}$$

We could find a valid mapping $\phi(x)$ corresponding to

$K'(x, z) = \frac{x^T z}{\|x\|^2 \|z\|^2}$, $\therefore$ It is a valid Kernel.

From closure properties of Kernels Product of two kernels is a Kernel in it self

$\therefore$ $K(x, z) = e^{(\|x\|^2 + \|z\|^2)} \cdot \frac{x^T z}{\|x\|^2 \|z\|^2}$ is a valid Kernel