Arithmatic and Geometric Progression

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1 ARITHMATIC PROGRESSION

In mathematics, an arithmetic progression (AP) or arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant. For instance, the sequence 5, 7, 9, 11, 13, 15 . . . is an arithmetic progression with common difference of 2.
If the initial term of an arithmetic progression is a_1 and the common difference of successive members is d, then the nth term of the sequence (a_n) is given by:

$$a_n = a_1 + (n-1)d,$$

and in general

$$a_n = a_m + (n-m)d.$$

1.1 SUM OF AP

The sum of the members of a finite arithmetic progression is called an arithmetic series. For example, consider the sum:

$$2 + 5 + 8 + 11 + 14$$

This sum can be found quickly by taking the number n of terms being added (here 5), multiplying by the sum of the first and last number in the progression (here 2 + 14 = 16), and dividing by 2:

$$\frac{n(a_1 + a_n)}{2}$$

1.2 DERIVATION

To derive the above formula, begin by expressing the arithmetic series in two different ways [1]:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$

$$S_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \ldots + (a_n - 2d) + (a_n - d) + a_n.$$

Adding both sides of the two equations, all terms involving d cancel:

$$2S_n = n(a_1 + a_n).$$

Dividing both sides by 2 produces a common form of the equation:

$$S_n = \frac{n}{2}(a_1 + a_n).$$

An alternate form results from re-inserting the substitution $a_n = a_1 + (n-1)d$:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d].$$

Furthermore the mean value of the series can be calculated via: S_n/n:

$$\overline{n} = \frac{a_1 + a_n}{2}.$$

2 GEOMETRIC PROGRESSION

In mathematics, a geometric progression, also known as a geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54, ... is a geometric progression with common ratio 3. Similarly 10, 5, 2.5, 1.25, ... is a geometric sequence with common ratio $\frac{1}{2}$.

Examples of a geometric sequence are powers r^k of a fixed number r, such as 2^k and 3^k. The general form of a geometric sequence is

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

where $r \neq 0$ is the common ratio and a is a scale factor, equal to the sequence's start value.

2.1 SUM OF GP

A geometric series is the sum of the numbers in a geometric progression. For example:

$$2 + 10 + 50 + 250 = 2 + 2 \times 5 + 2 \times 5^2 + 2 \times 5^3,$$

Letting a be the first term (here 2), m be the number of terms (here 4), and r be the constant that each term is multiplied by to get the next term (here 5), the sum is given by:

$$\frac{a(1 - r^m)}{1 - r}$$

In the example above, this gives:

$$2 + 10 + 50 + 250 = \frac{2(1 - 5^4)}{1 - 5} = \frac{-1248}{-4} = 312.$$

2.2 DERIVATION OF SUM OF GP

To derive this formula, first write a general geometric series as [2]:

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}.$$

We can find a simpler formula for this sum by multiplying both sides of the above equation by $1 - r$, and we'll see that

$$\begin{aligned}
(1 - r)\sum_{k=1}^{n} ar^{k-1} &= (1 - r)(a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}) \\
&= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\
&- ar - ar^2 - ar^3 - \ldots - ar^{n-1} - ar^n \\
&= a - ar^n
\end{aligned}$$

since all the other terms cancel. If $r \neq 1$, we can rearrange the above to get the convenient formula for a geometric series that computes the sum of n terms:

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}.$$

References

[1] Wikipedia. Arithmetic progressions - wikipedia the free encyclopedia, 2015.

[2] Wikipedia. Geometric progressions - wikipedia the free encyclopedia, 2015.