Blockhouse Work Trial Task - Question 2:
A Mathematical Algorithm for Optimal Trade Scheduling

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I tried this from whatever I understand, using my own research and best efforts. With more support and professional guidance, or in a team, I believe the results could be even more robust, and more advanced microstructure dynamics or market idiosyncrasies could be explored.

1 Mathematical Model and Solution

Our objective is to efficiently schedule a large metaorder of size S over discrete time intervals (buckets), such that we minimize expected temporary transaction costs (market impact), fully recognizing dynamic market liquidity.

For every interval i, we estimate the instantaneous impact function from data as:

$$g_{t_i}(x) = \alpha_i x^\gamma$$

where x is the trade size for interval i, α_i is a positive, time-varying impact parameter empirically estimated from the trade/book data (see Q1), and γ is the concavity exponent (set as 0.5 based on theoretical and empirical evidence, as discussed in Q1).

The cumulative temporary impact for the N-interval execution schedule $\{x_i\}$ is:

$$\text{Impact} = \sum_{i=1}^{N} \alpha_i x_i^\gamma$$

subject to:

$$\sum_{i=1}^{N} x_i = S \qquad \text{and} \qquad x_i \geq 0 \ \forall i$$

Why a Power-Law Model?

The power-law model $g_{t_i}(x) = \alpha_i x^\gamma$ is both mathematically and empirically justified:

- Real order book depth and price movement data show sublinear scaling: the marginal cost of executing additional size diminishes as the size increases, which is naturally captured by $0 < \gamma < 1$.
- The model enables direct adaptation to liquidity: α_i fluctuates based on book conditions and competition in each bucket, favoring trading more when and where liquidity is better.
- In practical execution, this model allows a strictly convex (for $0 < \gamma < 1$) optimization problem with a unique, closed form solution, making it robust and computationally efficient.

A strictly linear impact ($g_{t_i}(x) = \beta_i x$) is too simplistic: it implies the same marginal cost in all buckets, disregarding book depth variation, as shown empirically in Q1 and economic studies (Almgren, Bouchaud). Piecewise, kernel, or nonparametric models are not used because they would be extremely noisy or statistically unstable given the short rolling data windows, and lead to poorly interpretable allocations.

Optimization Solution

We use the Lagrangian method to solve for the minimum impact schedule. Setting up the multiplier augmented function:

$$\mathcal{L} = \sum_{i=1}^{N} \alpha_i x_i^{\gamma} - \lambda \left(\sum_{i=1}^{N} x_i - S \right)$$

Setting first derivatives to zero, for each i:

$$\gamma \alpha_i x_i^{\gamma-1} = \lambda \Rightarrow x_i = \left(\frac{\lambda}{\gamma \alpha_i} \right)^{1/(\gamma-1)}$$

Summing over i to enforce $\sum x_i = S$ yields the scaling constant. Therefore,

$$x_i^* = S \times \frac{w_i}{\sum_j w_j} \quad \text{with} \quad w_i = (\gamma \alpha_i)^{1/(\gamma-1)}$$

This solution is implemented in code for all estimated intervals.

2 Results and Practical Interpretation

Using the alpha series $\{\alpha_i\}$ derived from SOUN, FROG, and CRWV over rolling time windows (from Q1), we input these to the algorithm and scheduled $S = 50,000$ shares. The optimal allocation heavily favored intervals with low α_i (high liquidity), and minimized trading during temporary spikes in impact (illiquidity).

The constructed **aggregate execution schedule** (see Figure 1 in Cell 1 of `Q2.ipynb`) clearly shows that more size is allocated to intervals with lower execution cost. The **cumulative execution schedule** (see Figure 2 in Cell 1 of `Q2.ipynb`) advances smoothly and reliably to the total target S, never exceeding it, and adapts to real time changes in alpha.

This model does not impose a uniform, VWAP-style schedule, nor spread orders too thinly. It is "liquidity-seeking": allocations vary with market opportunity, which is crucial for cost minimization and risk management in real practice and institutional trading systems.

3 Why Not Other Models?

- **Linear Impact:** As discussed, leads to schedules that do not adapt to liquidity, missing both low-cost opportunities and failing to avoid high-cost periods.

- **Static/Uniform (VWAP):** Ignores information about book variation, often raising realized cost during market shocks, as observed in both the alpha time series and schedule results.

- **Nonparametric/Ad hoc:** Require vast data per bucket and, in practice, are too unstable for fast-changing, real-world microstructure applications.

References

- Bouchaud, J.-P., Farmer, J.D., & Lillo, F. (2009). How Markets Slowly Digest Changes in Supply and Demand. `https://arxiv.org/abs/0809.4554`
- Almgren, R. & Chriss, T. (2000). Optimal Execution of Portfolio Transactions.
- Code and figures: see `Q2.ipynb` or attached repository.