Some Elements of Learning Theory

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- A brief introduction to statistical learning

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- We do some (short) proofs

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- Mathematical model of learning and conditions characterizing what can be learned
- Guidelines to practitioners (e.g., choice of learning bias, control of overfitting)
- Principled and successful algorithms (SVM, Boosting)

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- Learning algorithm: given a loss function, maps finite training sets to predictors

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- Zero-one loss: $f^*(\boldsymbol{x}) = 2\mathbb{I}\{\eta(\boldsymbol{x}) \geq 1/2\} - 1$ and $\ell_{\mathcal{D}}(f^*) = \mathbb{E}\Big[\min\{\eta(\boldsymbol{X}), 1 - \eta(\boldsymbol{X})\}\Big]$
  where $\eta(\boldsymbol{x}) = \mathbb{P}(Y = 1 \mid \boldsymbol{X} = \boldsymbol{x})$

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  - Show that A can compress the training set (compression implies learning)

Success stories: Characterization of sample complexity

What is the training set size $m_{\mathcal{H}}$ necessary and sufficient to ensure

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- Majority vote over a set of consistent predictors achieves upper bound in the realizable case

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- New data is being generated all the time
- The train-test model of statistical learning is ill-suited for learning on data streams
- After observing a new data point, predictors should be incrementally adjusted at a constant cost

History bits

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- Volodya Vovk independently develops a related framework (Aggregating strategies, 1990)
- Similar ideas also independently emerged in game theory and information theory

The online learning protocol

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For $t = 1, 2, \dots$

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- Computation of h_{t+1} relies on local information
- No stochastic assumptions on the stream

Regret

Sequential risk

Given a convex loss ℓ and a stream $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \ldots$, the sequential risk of A is

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- A sequential counterpart to the variance error in statistical learning

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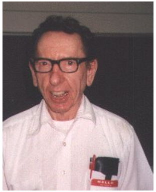
$$R_T = \sum_{t=1}^{T} \ell(y_t, h_t(\boldsymbol{x}_t)) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell(y_t, h(\boldsymbol{x}_t))$$

- A sequential counterpart to the variance error in statistical learning
- Can we ensure $\dfrac{R_T}{T} \to 0$ as $T \to \infty$ for all streams?

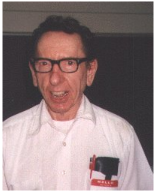
Online learning as a repeated game



Learning to play a game (1956)

- Theory of repeated games pioneered by James Hannan and David Blackwell

Online learning as a repeated game

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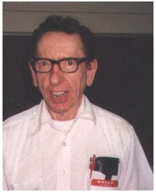

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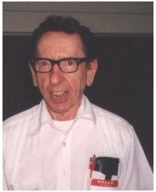

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- Let $\mathcal{H}$ be the d-dimensional simplex Δ_d

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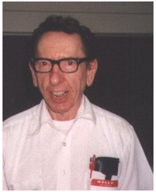

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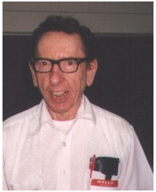


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- This is a linear loss with bounded coefficients $\ell_t(i) \in [0, 1]$

Prediction with expert advice

A sequential decision problem

- d actions
- Unknown deterministic assignment of losses to actions $\boldsymbol{\ell}_t = (\ell_t(1), \dots, \ell_t(d)) \in [0,1]^d$ for each time step t

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For $t = 1, 2, \dots$

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Regret

$$R_T = \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p}_t - \min_{\boldsymbol{p} \in \Delta_d} \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p}$$

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$$R_T = \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p}_t - \min_{\boldsymbol{p}\in\Delta_d} \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p} = \mathbb{E}\left[\sum_{t=1}^{T} \ell_t(I_t)\right] - \min_{i=1,\dots,d} \sum_{t=1}^{T} \ell_t(i)$$

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- $\ell_t(i) \to L_t(i) \in \{0,1\}$ independent random coin flip
- For any player strategy $\mathbb{E}\left[\sum_{t=1}^{T} L_t(I_t)\right] = \frac{T}{2}$
- Then the expected regret is

$$\mathbb{E}\left[\max_{i=1,\dots,d} \sum_{t=1}^{T} \left(\frac{1}{2} - L_t(i)\right)\right] = \boxed{(1 - o(1))\sqrt{\frac{T \ln d}{2}}}$$

for $d, T \to \infty$

Exponentially weighted forecaster (Hedge)

At time t pick action $I_t = i$ with probability proportional to

$$\exp\left(-\eta \sum_{s=1}^{t-1} \ell_s(i)\right)$$

the sum at the exponent is the total loss of action i up to the previous time step

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- If $\eta = \sqrt{\frac{\ln d}{8T}}$ then $\boxed{R_T \leq \sqrt{\frac{T \ln d}{2}}}$

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- We prove this later in a more general setting

The bandit problem: playing an unknown game

- d actions
- Unknown deterministic assignment of losses to actions $\boldsymbol{\ell}_t = (\ell_t(1), \dots, \ell_t(d)) \in [0,1]^d$ for each time step t

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?   .3   ?   ?   ?   ?   ?   ?   ?   ?

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2. Player gets feedback information: Only $\ell_t(I_t)$ is revealed

A growing range of applications

- Ad placement

A growing range of applications

- Ad placement
- Dynamic content/layout optimization

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- Ad placement
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- Real time bidding

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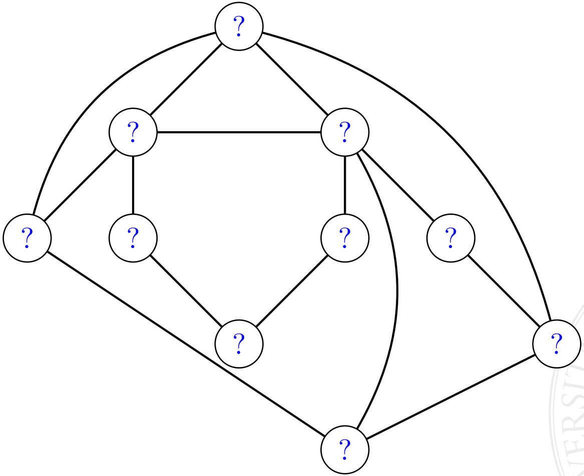
An observability graph over actions



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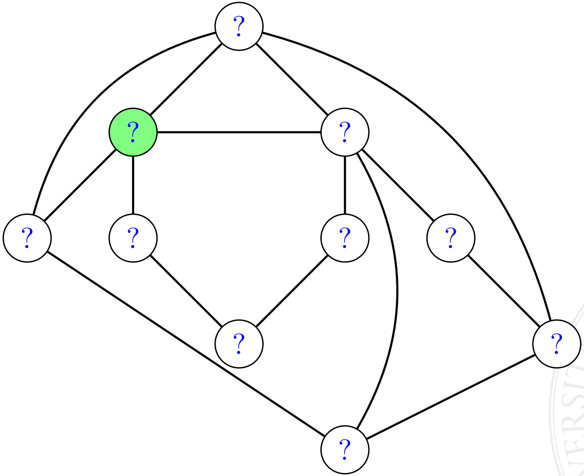

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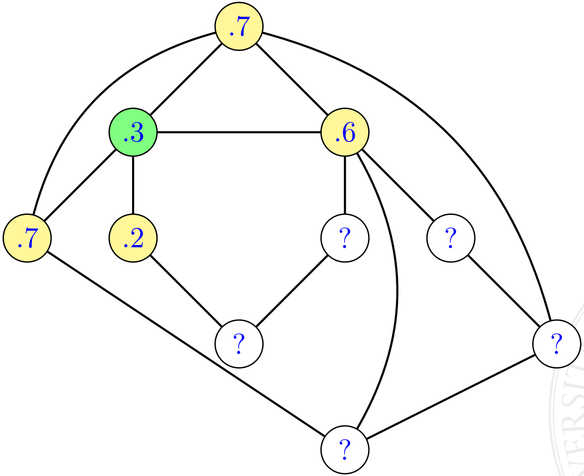
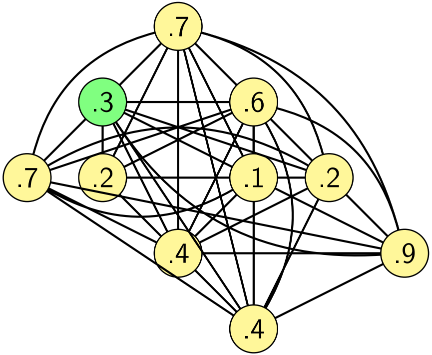


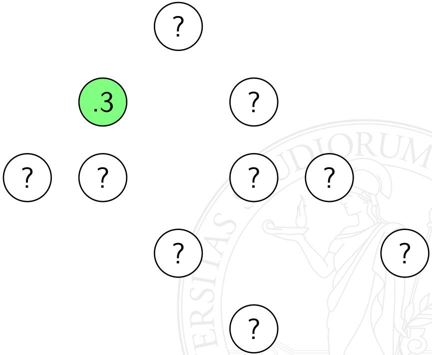
$\ell_t(i)$ is observed iff $I_t \in \{i\} \cup \mathcal{N}_G(i)$

Recovering expert and bandit settings

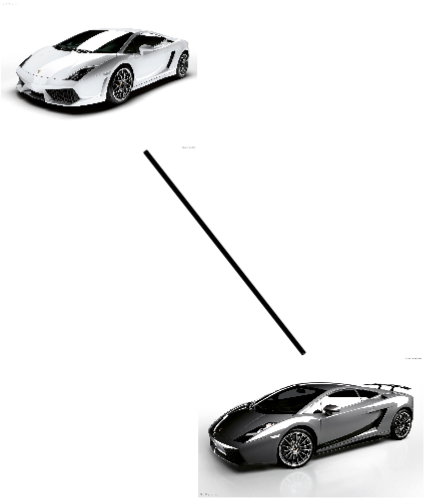
Experts: clique

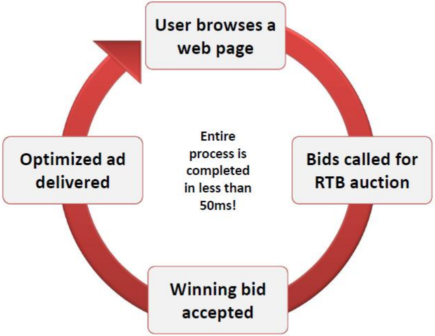


Bandits: edgeless graph



Relationships between actions





Hedge revisited on an observability graph G

Player's strategy must use loss estimates

- $$p_t(i) \quad \propto \quad \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) \qquad i = 1, \dots, d$$

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Importance sampling estimator

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$$\mathbb{E}_t\left[\widehat{\ell}_t(i)\right] = \frac{\ell_t(i)}{\mathbb{P}_t(\ell_t(i) \text{ observed})} \times \mathbb{P}_t(\ell_t(i) \text{ observed}) + 0 = \ell_t(i)$$

$$\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right] = \frac{\ell_t(i)^2}{\mathbb{P}_t(\ell_t(i) \text{ observed})^2} \times \mathbb{P}_t(\ell_t(i) \text{ observed}) + 0 = \frac{\ell_t(i)^2}{\mathbb{P}_t(\ell_t(i) \text{ observed})}$$

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Regret analysis

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{d} \frac{w_{t+1}(i)}{W_t} \qquad p_t(i) = \frac{1}{W_t} \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) = \frac{w_t(i)}{W_t} \quad \text{is a r.v.!}$$

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$$= \sum_{i=1}^{d} \frac{w_t(i)}{W_t} \exp(-\eta\, \widehat{\ell}_t(i)) \qquad \text{(because } w_{t+1}(i) = e^{-\eta \sum_{s=1}^{t-1} \hat{\ell}_s(i) - \eta \hat{\ell}_t(i)}\text{)}$$

Regret analysis

$$
\begin{aligned}
\frac{W_{t+1}}{W_t} &= \sum_{i=1}^{d} \frac{w_{t+1}(i)}{W_t} \\
&= \sum_{i=1}^{d} \frac{w_t(i)}{W_t} \exp\big(-\eta\,\widehat{\ell}_t(i)\big) \\
&= \sum_{i=1}^{d} p_t(i) \exp\big(-\eta\,\widehat{\ell}_t(i)\big)
\end{aligned}
$$

$p_t(i) = \dfrac{1}{W_t} \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) = \dfrac{w_t(i)}{W_t}$ is a r.v.!

(because $w_{t+1}(i) = e^{-\eta \sum_{s=1}^{t-1} \hat{\ell}_s(i) - \eta \hat{\ell}_t(i)}$)

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$$= \sum_{i=1}^{d} p_t(i) \exp\left(-\eta\, \widehat{\ell}_t(i)\right)$$

$$\leq \sum_{i=0}^{d} p_t(i) \left(1 - \eta\, \widehat{\ell}_t(i) + \frac{\left(\eta\, \widehat{\ell}_t(i)\right)^2}{2}\right) \qquad \left(\text{using } e^{-x} \leq 1 - x + x^2/2 \text{ for all } x \geq 0\right)$$

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$$\leq 1 - \eta \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) + \frac{\eta^2}{2} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

Regret analysis (cont.)

Taking logs, using $\ln(1+x) \leq x$, and summing over $t = 1, \dots, T$ yields

$$\ln \frac{W_{T+1}}{W_1} \leq -\eta \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) + \frac{\eta^2}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

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Moreover, for any fixed action k, we also have

$$\ln \frac{W_{T+1}}{W_1} \ge \ln \frac{w_{T+1}(k)}{W_1} = -\eta \sum_{t=1}^{T} \widehat{\ell}_t(k) - \ln d$$

Regret analysis (cont.)

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Putting together and dividing both sides by $\eta > 0$ gives

$$\sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) - \sum_{t=1}^{T} \widehat{\ell}_t(k) \le \frac{\ln d}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

Regret analysis (cont.)

Recall where we were:

$$\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\widehat{\ell}_t(i) - \sum_{t=1}^{T}\widehat{\ell}_t(k) \le \frac{\ln d}{\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\widehat{\ell}_t(i)^2$$

Regret analysis (cont.)

Recall where we were:

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Take expectation w.r.t. $I_1, \dots, I_T$

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Regret analysis (cont.)

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Loss estimates are unbiased:

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This is just the regret

$$R_T = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\ell_t(i) - \sum_{t=1}^{T}\ell_t(k)\right] \le \frac{\ln d}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right]\right]$$

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\end{aligned}$$

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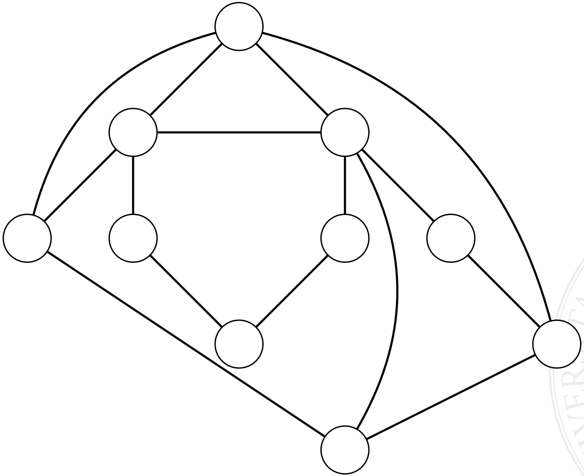
&\le \frac{\ln d}{\eta} + \frac{\eta}{2} T\,\alpha(G) && \text{(cool graph-theoretic fact)}
\end{aligned}
$$

$\alpha(G)$ is the independence number of G

Independence number $\alpha(G)$

The size of the largest independent set in G



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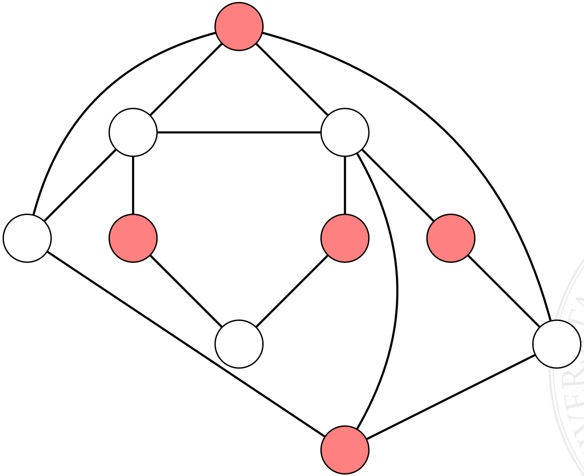

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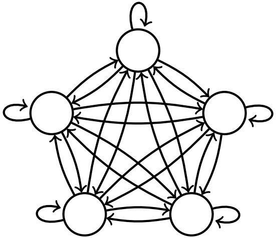
Special cases

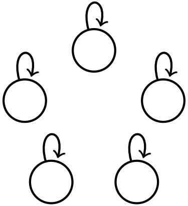
Experts (clique): $\alpha(G) = 1$ $R_T \leq \sqrt{T \ln d}$ Hedge algorithm

Bandits (edgeless graph): $\alpha(G) = d$ $R_T \leq \sqrt{T d \ln d}$ Exp3 algorithm

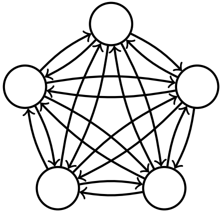
More general feedback models



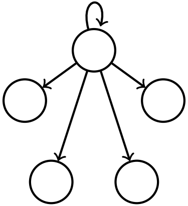
Experts



Bandits



Cops & Robbers



Revealing Action

A general gap theorem

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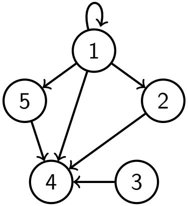
  3. Impossible games: $\Theta(T)$



Online convex optimization

Model space $\mathbb{V} \subseteq \mathbb{R}^d$ convex, closed, and nonempty

For $t = 1, 2, \dots$

1. The current $h_t \in \mathcal{H}$ is tested on the next data point $(\boldsymbol{x}_t, y_t)$ in the stream
2. A is charged with loss $\ell(y_t, h_t(\boldsymbol{x}_t))$
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Stochastic gradient descent

Minimization of training error

$$\min_{\boldsymbol{w} \in \mathbb{V}} \sum_{i=1}^{m} \ell(\boldsymbol{w}, (\boldsymbol{x}_i, y_i))$$

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- Run online algorithm on the sequence of loss functions $\ell_t = \ell_t(\cdot, (\boldsymbol{X}_t, Y_t))$

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&\geq LD\sqrt{\frac{T}{8}} && \text{(Khintchine inequality)}
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The geometry of $\mathbb{V}$ matters

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The Bregman divergence B_ψ measures a generalized squared distance between $\boldsymbol{w}, \boldsymbol{w}_t \in \mathbb{V}$

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A differentiable $\psi : \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex on $\mathbb{V}$ with respect to $\|\cdot\|$ if

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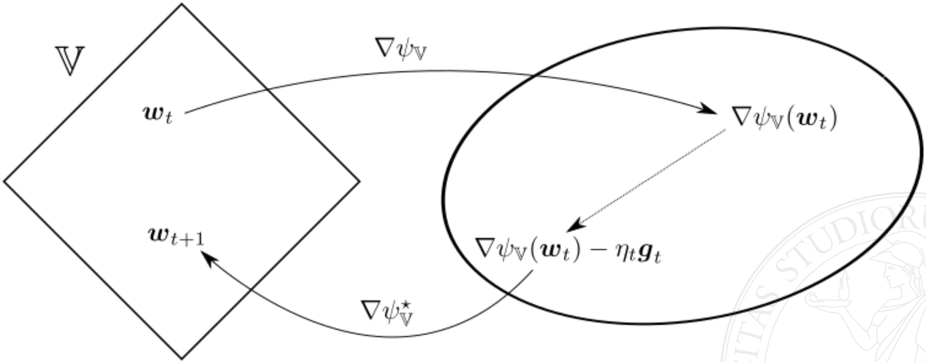
- $\nabla\psi^\star_{\mathbb{V}}$ is the functional inverse of $\nabla\psi_{\mathbb{V}}$

The mirror step



$$\boldsymbol{w}_{t+1} = \nabla\psi^\star_{\mathbb{V}}\Big(\nabla\psi_{\mathbb{V}}(\boldsymbol{w}_t) - \eta_t \underbrace{\nabla\ell_t(\boldsymbol{w}_t)}_{\boldsymbol{g}_t}\Big)$$

Regret analysis

Two basic inequalities

$\boldsymbol{g}_t = \nabla \ell_t(\boldsymbol{w}_t)$

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Regret analysis (cont.)

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- We proved $R_T(\boldsymbol{u}) \leq \frac{D^2}{\eta_T} + \frac{1}{2\mu}\sum_{t=1}^{T} \eta_t \left\|\boldsymbol{g}_t\right\|_\star^2$

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- OMD analysis for constant learning rate: $R_T(\boldsymbol{q}) \leq \dfrac{B_\psi(\boldsymbol{q}, \boldsymbol{p}_1)}{\eta} + \dfrac{\eta}{2} \sum_{t=1}^T \|\boldsymbol{g}_t\|_\star^2$

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- We can interpolate between OGD and EG using a p-norm as a mirror map:
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- Choosing $p = \frac{2\ln d}{2\ln d - 1}$ gives bound similar to EG without the tuning problem

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- By Jensen's inequality $$\underbrace{\sum_{i=1}^{d} \sqrt{\sum_{t=1}^{T} g_{t,i}^2}}_{\text{AdaGrad}} \leq \underbrace{\sqrt{d} \sqrt{\sum_{t=1}^{T} \|\boldsymbol{g}_t\|_2^2}}_{\text{OGD}}$$

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The confidence ellipsoid

Fix a sequence of contexts $C_1, \dots, C_t$ and choices $\boldsymbol{x}_s \in C_s$, $s = 1, \dots, t$

RLS estimate

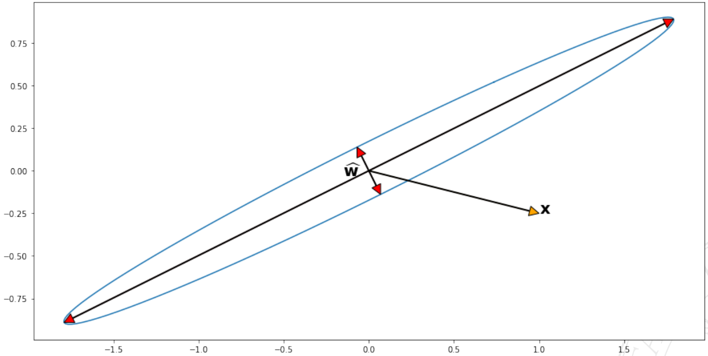
$$\widehat{\boldsymbol{w}}_t = V_t^{-1} \sum_{s=1}^{t} Y_s \boldsymbol{x}_s \qquad V_t = \lambda I_d + \underbrace{[\boldsymbol{x}_1, \dots, \boldsymbol{x}_t]}_{d \times t} [\boldsymbol{x}_1, \dots, \boldsymbol{x}_t]^\top$$

With high probability, $\boldsymbol{w} \in \mathcal{E}_t \equiv \left\{ \boldsymbol{u} \in \mathbb{R}^d \,:\, \|\boldsymbol{u} - \widehat{\boldsymbol{w}}\|_{V_t} \le \beta_t \right\}$

β_t of order $D + R\sqrt{1 + d \ln\left(1 + \frac{t}{d}\right)}$

Think of $\mathcal{E}_t$ as a d-dimensional confidence interval

The LinUCB/OFUL algorithm



Optimism in the face of uncertainty

$$\boldsymbol{x}_{t+1} = \operatorname*{argmax}_{\boldsymbol{x} \in C_{t+1}} \max_{\boldsymbol{u} \in \mathcal{E}_t} \boldsymbol{u}^\top \boldsymbol{x} = \operatorname*{argmax}_{\boldsymbol{x} \in C_t} \left(\widehat{\boldsymbol{w}}_t^\top \boldsymbol{x} + \beta_t \, \|\boldsymbol{x}\|_{V_t^{-1}} \right)$$

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