

CS 7545

Machine Learning Theory

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Preface

This is the organized course material of CS 7545 Machine Learning Theory from Georgia Institute of Technology. The material is based on the course in Fall 2019. The instructor of this course is professor Jacob Abernethy and the course website can be found via this link.

This course contains many theoretical aspects of machine learning, including decision making, online learning, boosting, regret minimization, etc. Students will get familiar with the frontiers of theoretical machine learning research after finishing this course. It is a totally theoretical course and no programming will be covered.

CS 7545 is a graduate level course, thus it has several prerequisites. Familiarity with Algorithm analysis, linear algebra, probability and statistics is a must to understand the contents of this course. Convex analysis background ~~is strongly recommended but not required~~ is also very important.

Before studying any algorithm and theory, we will first review some math aspects and point out some notations that we will use in this course material. Afterwards, we will talk about convex analysis and deviation bound. Online learning algorithms will be covered next. Finally, we will talk about statistical learning theory.

Thanks to professor Jacob Abernethy for his instruction.

Chapter 1

Math Review

In this chapter, we will briefly review several aspects of linear algebra, norm, etc. The following notations will be used throughout the course.

- M : Matrix (size $\mathbb{R}^{m \times n}$).
- M^T : Transpose Matrix of M .
- \mathbf{x} : Vector (size \mathbb{R}^n).
- \mathbf{x}_i : i -th element of vector \mathbf{x} (size \mathbb{R}).

1.1 Positive Semidefinite and Positive Definite Matrices

We will first define Positive semidefinite (PSD) and positive definite matrices.

Definition 1.1 (Positive Semidefinite Matrix). A matrix $M \in \mathbb{R}^{n \times n}$ is said to be positive semidefinite (PSD) if it satisfies the following two conditions.

1. M is a symmetric matrix. That is $M^T = M$.
2. For all $\mathbf{x} \in \mathbb{R}^n$, we have that $\mathbf{x}^T M \mathbf{x} \geq 0$. This condition is equivalent to all eigenvalues of M being non-negative.

We can denote matrix M being positive semidefinite (PSD) as $M \succeq 0$.

Definition 1.2 (Positive Definite Matrix). Similar to above definition, a matrix M is said to be positive definite (PD) if it satisfies the following two conditions:

1. M is a symmetric matrix. That is $M^T = M$.
2. For all $\mathbf{x} \in \mathbb{R}^n$, we have that $\mathbf{x}^T M \mathbf{x} > 0$. This condition is equivalent to all eigenvalues of M being positive.

We can denote matrix M being positive definite (PD) as $M \succ 0$.

Notice that the only difference between positive semidefinite matrices and positive definite matrices is whether the eigenvalues could be zero. In addition, all PD matrices are PSD, while PSD matrices may not be PD.

Notation 1.3. We denote $M \preceq 0$ if $-M \succeq 0$ and $M \prec 0$ if $-M \succ 0$.

1.2 Norm

Definition 1.4 (Norm). Norm is defined as a function $\|\cdot\| : \mathbb{R}^n \rightarrow [0, \infty]$ which satisfies the following conditions:

1. **Identity of Indiscernibles:** $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$.
2. **Absolute Homogeneity:** $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
3. **Triangle Inequality:** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Some famous norm functions can be seen following:

1. ℓ_1 norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |\mathbf{x}_i|$.
2. ℓ_2 norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n \mathbf{x}_i^2}$.
3. ℓ_∞ norm: $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |\mathbf{x}_i|$.
4. ℓ_p norm: $\|\mathbf{x}\|_p = \sqrt[p]{\sum_{i=1}^n |\mathbf{x}_i|^p}$.
5. M norm (suppose $M \in \mathbb{R}^{n \times n}$): $\|\mathbf{x}\|_M = \sqrt[2]{\mathbf{x}^T M \mathbf{x}}$.

We can see that ℓ_1 , ℓ_2 and ℓ_∞ are special case of ℓ_p norm.

We will next prove that ℓ_∞ norm satisfies the three condition in norm definition.

Proof. We will just use the norm definition to finish the proof.

1. Identity of Indiscernibles:

Since $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |\mathbf{x}_i|$, $\|\mathbf{x}\|_\infty = 0$ if and only if $\mathbf{x}_i = 0$ for all $i \in [1, n]$.

2. Absolute Homogeneity:

$$\|\alpha\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |\alpha\mathbf{x}_i| = |\alpha| \max_{1 \leq i \leq n} |\mathbf{x}_i| = |\alpha|\|\mathbf{x}\|_\infty.$$

3. Triangle Inequality:

$$\|\mathbf{x} + \mathbf{y}\|_\infty = \max_{1 \leq i \leq n} |\mathbf{x}_i + \mathbf{y}_i| \leq \max_{1 \leq i \leq n} (|\mathbf{x}_i| + |\mathbf{y}_i|) \leq \max_{1 \leq i \leq n} |\mathbf{x}_i| + \max_{1 \leq j \leq n} |\mathbf{y}_j| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

□

Definition 1.5 (Dual Norm). Given a norm $\|\cdot\|$, we can define its dual norm $\|\cdot\|_*$ as following:

$$\|\mathbf{y}\|_* = \sup_{\mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|=1} \langle \mathbf{x}, \mathbf{y} \rangle \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ is the inner product of two vectors.

Claim 1.6. *The ℓ_2 norm's dual norm is ℓ_2 norm itself.*

Proof. For any vector $\mathbf{y} \in \mathbb{R}^n$, we have that

$$\|\mathbf{y}\|_{2,*} = \sup_{\mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|_2=1} \langle \mathbf{x}, \mathbf{y} \rangle = \sup_{\mathbf{x} \in \mathbb{R}^n} \langle \frac{\mathbf{x}}{\|\mathbf{x}\|_2}, \mathbf{y} \rangle = \frac{1}{\|\mathbf{y}\|_2} \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{y}\|_2.$$

Notice that here the inner product is maximized when two vectors ($\frac{\mathbf{x}}{\|\mathbf{x}\|_2}$ and \mathbf{y}) take the same direction. \square

Claim 1.7. *The ℓ_p norm is dual to ℓ_q norm if and only if the following equation holds:*

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (1.2)$$

Proof. TBD \square

Claim 1.8. *For a PD Matrix M , the dual norm of M norm is M^{-1} norm, where M^{-1} is the inverse matrix of M .*

Proof. TBD \square

Theorem 1.9 (Hölders Inequality). *Suppose $\|\cdot\|$ and $\|\cdot\|_*$ are a pair of dual norms. Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we can have that*

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|_*. \quad (1.3)$$

Proof. By definition,

$$\|\mathbf{x}\| \|\mathbf{y}\|_* = \|\mathbf{x}\| \sup_{\mathbf{z} \in \mathbb{R}^n, \|\mathbf{z}\|_2=1} \langle \mathbf{z}, \mathbf{y} \rangle \geq \|\mathbf{x}\| \langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Here we use the fact that the supremum over \mathbf{z} must be larger or equal than taking any vector \mathbf{x} . \square

This theorem will be very useful in the upcoming chapters.

Chapter 2

Convex Analysis

2.1 Convex Set and Convex Function

In this section, we will review some concepts in convex analysis.

First, let us define some basic and useful notations.

Definition 2.1 (Gradient). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function and let $\mathbf{x} \in \mathbb{R}^n$. We can defined the gradient of f at \mathbf{x} as

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right). \quad (2.1)$$

Remark 2.2. Notice that here we define gradient as a row vector ($\mathbb{R}^{1 \times n}$). However, for convenience, we may abuse this notation and use it as a column vector ($\mathbb{R}^{n \times 1}$) in the following contents.

Definition 2.3 (Hessian). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice differentiable function and let $\mathbf{x} \in \mathbb{R}^n$. Then the Hessian of f at \mathbf{x} is defined as

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix} \quad (2.2)$$

Notice that if f is a twice differentiable function and $\mathbf{x} \in \mathbb{R}^n$, then $\nabla^2 f(\mathbf{x})$ is a symmetric matrix since for all $i, j \in [1, n]$, it has

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}).$$

Next, we will talk about convexity, its definition and several useful facts.

Definition 2.4 (Convex Set). Let $\mathcal{K} \subset \mathbb{R}^n$, set \mathcal{K} is called a convex set if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$ and $\forall t \in [0, 1]$, we have

$$t\mathbf{x} + (1-t)\mathbf{y} \in \mathcal{K}. \quad (2.3)$$

The meaning of this definition is that for any two arbitrary points in the set \mathcal{K} , the straight line between these two points are all contained in \mathcal{K} .

Definition 2.5 (Convex Function). Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{K} \rightarrow \mathbb{R}^n$ be a differentiable function. Then function f is convex if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$ and $\forall t \in [0, 1]$, we have that

$$f((1-t)\mathbf{x} + t\mathbf{y}) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y}). \quad (2.4)$$

Claim 2.6. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{K} \rightarrow \mathbb{R}^n$ be a differentiable function. Then function f is convex if and only if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$, we have that

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \langle f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle. \quad (2.5)$$

An intuition behind this claim is that f is convex if and only if the first order approximation of f at any point \mathbf{y} is not larger than the function itself.

Claim 2.7. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{K} \rightarrow \mathbb{R}$ be a twice differentiable function. Then f is convex if and only if $\forall \mathbf{x} \in \mathcal{K}$,

$$\nabla^2 f(\mathbf{x}) \succeq 0. \quad (2.6)$$

Remark 2.8. It can be seen that above three formulas (2.4), (2.5) and (2.6) are equivalent. However, in reality, if f is twice differentiable, using (2.6) is much simpler than using (2.4) or (2.5), because it only contains one variable in \mathcal{K} and we only need to determine if $\nabla^2 f(\mathbf{x})$ is PSD or not.

In facts, many functions satisfy this condition. Some examples are $f(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$, $f(\mathbf{x}) = 1$, etc.

Proposition 2.9.

1. If f is convex and g is convex, then $f + g$ is convex.
2. If f is convex, then $\forall \alpha \geq 0$, αf is convex.
3. If f is convex and g is convex, then $\max\{f, g\}$ is convex.
4. If $g(\mathbf{x}, \mathbf{y})$ is jointly convex in \mathbf{x}, \mathbf{y} , then $f(\mathbf{x}) = \inf_{\mathbf{y}} g(\mathbf{x}, \mathbf{y})$ is convex.

Definition 2.10 (Concave Function). Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex set and $f : \mathcal{K} \rightarrow \mathbb{R}$, then f is concave if $-f$ is convex.

We can see that one example of concave function is \log function.

Definition 2.11 (Lipschitz). Let $\|\cdot\|$ be a norm, $c \geq 0$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Then f is called c -Lipschitz with respect to $\|\cdot\|$ if $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq c\|\mathbf{x} - \mathbf{y}\|. \quad (2.7)$$

Intuitively, it means that the difference between function value is smaller than the difference between the norm of difference between two points times some constant c .

Claim 2.12. If f is a differentiable function, then f is c -Lipschitz with respect to $\|\cdot\|$ if and only if $\forall \mathbf{x} \in \mathbb{R}^n$,

$$\|f(\mathbf{x})\|_* \leq c \quad (2.8)$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$.

Proof. We first prove the 'if' part.

Let $\|\cdot\|$ be a norm on \mathbb{R}^n and $c \geq 0$, let $\|\nabla f(\mathbf{x})\|_* \leq c \ \forall \mathbf{x} \in \mathbb{R}^n$. Then, by the mean value theorem, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, there exists $t \in [0, 1]$ such that

$$f(\mathbf{x}) = f(\mathbf{y}) + \langle \nabla f((1-t)\mathbf{x} + t\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

Then, by Hölder's inequality,

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{y})| &= |\langle \nabla f((1-t)\mathbf{x} + t\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle| \\ &\leq \|\nabla f((1-t)\mathbf{x} + t\mathbf{y})\|_* \|\mathbf{x} - \mathbf{y}\| \\ &\leq c \|\mathbf{x} - \mathbf{y}\|. \end{aligned}$$

That is, f is c -Lipschitz with respect to $\|\cdot\|$.

We will next prove the 'only if' part.

Suppose that f is c -Lipschitz with respect to $\|\cdot\|$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then we have

$$\begin{aligned} \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{y}) - f(\mathbf{x})}{h} \\ &\leq \lim_{h \rightarrow 0} \frac{c \|\mathbf{x} + h\mathbf{y} - \mathbf{x}\|}{h} \\ &= c \lim_{h \rightarrow 0} \frac{h \|\mathbf{y}\|}{h} \\ &= c \|\mathbf{y}\|. \end{aligned}$$

Notice that $\langle \nabla f(\mathbf{x}), \mathbf{y} \rangle$ is the directional derivative of f at \mathbf{x} in the direction of \mathbf{y} . Then,

$$\|\nabla f(\mathbf{x})\|_* = \sup_{\|\mathbf{y}\| \leq 1} \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle \leq \sup_{\|\mathbf{y}\| \leq 1} c \|\mathbf{y}\| = c.$$

Therefore, $\forall \mathbf{x} \in \mathbb{R}^n$, $\|\nabla f(\mathbf{x})\|_* \leq c$. □

Theorem 2.13 (Jensen's Inequality). Let $\mathcal{K} \in \mathbb{R}^n$ be a convex set, X be a random variable on \mathcal{K} and $f : \mathcal{K} \rightarrow \mathbb{R}$ be a convex function, then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \tag{2.9}$$

where \mathbb{E} denotes the expectation of a random variable.

Theorem 2.14 (Young's Inequality). Let $p, q > 0$ and satisfy that $\frac{1}{p} + \frac{1}{q} = 1$. Then $\forall a, b > 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.10}$$

Proof. Since $-\log$ is a convex function, then $\forall a, b > 0$,

$$\begin{aligned} \log(ab) &= \log(a) + \log(b) \\ &= \frac{p}{p} \log(a) + \frac{q}{q} \log(b) \\ &= \frac{1}{p} \log(a^p) + \frac{1}{q} \log(b^q) \\ &\leq \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right). \end{aligned}$$

Then apply exp to both side and since exp function is monotonically increasing, we can have that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Definition 2.15 (Strongly Convex). Let \mathcal{K} be a convex set, $f : \mathcal{K} \rightarrow \mathbb{R}$ be a differentiable function and $\alpha > 0$. Then f is α -strongly convex with respect to $\|\cdot\|$ if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$,

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{\alpha}{2} \|\mathbf{x} - \mathbf{y}\|^2. \quad (2.11)$$

Notice that if f is strongly convex, then it must be a convex function. In addition, a strongly convex function grows at least quadratically.

Claim 2.16. Let $f : \text{dom}(f) \rightarrow \mathbb{R}$ be a twice differentiable function, then f is α -strongly convex if and only if $\forall \mathbf{x} \in \text{dom}(f)$,

$$\nabla^2 f(\mathbf{x}) - \alpha I \succeq 0.$$

Definition 2.17 (Smooth). Let $f : \text{dom}(f) \rightarrow \mathbb{R}$ be a differentiable function, let $\alpha > 0$, then f is α -smooth if $\forall \mathbf{x}, \mathbf{y} \in \text{dom}(f)$,

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{\alpha}{2} \|\mathbf{x} - \mathbf{y}\|^2.$$

Notice that in contrast to strongly convex, a smooth function grows **at most** quadratically.

Claim 2.18. We have a similar claim for smooth function to strongly convex function. That is, let $f : \text{dom}(f) \rightarrow \mathbb{R}$ be a twice differentiable function, then f is α -smooth if and only if $\forall \mathbf{x} \in \text{dom}(f)$, we have

$$\nabla^2 f(\mathbf{x}) - \alpha I \preceq 0.$$

Notice that although smooth seems to be contrary to strongly convex, a function f can be both α -strongly convex and β -smooth for some α and β . We will give an example next.

Example 2.18.1. Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix and let $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T M \mathbf{x}$. Denote λ_{\min} as the smallest eigenvalue of M and λ_{\max} as the largest eigenvalue of M . Then we can prove easily that M is λ_{\min} -strongly convex and λ_{\max} -smooth.

Proof. First, by taking derivative, it can be seen that $\nabla f(\mathbf{x}) = \mathbf{x}^T M$. Then, take the second derivative and we can get $\nabla^2 f(\mathbf{x}) = M$. By Claim 2.18 and Claim 2.16, we can show that f is λ_{\min} -strongly convex and λ_{\max} -smooth. □

2.2 Bregman Divergence

In this section, we will define Bregman Divergence and see its relation with convexity analysis.

Definition 2.19. Given a function $f : \mathcal{U} \rightarrow \mathbb{R}$, we can define its Bregman Divergence as

$$D_f(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle. \quad (2.12)$$

Example 2.19.1. Let $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|_2^2$, then

$$D_f(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2.$$

Notice that only in this situation is Bregman divergence a quadratic function.

Example 2.19.2. Let $\Delta^n = \{\mathbf{p} \in \mathbb{R}^n \mid \sum_{i=1}^n p_i = 1, p_i \geq 0\}$ be the simplex with dimension n . Let $f(\mathbf{p}) = \sum_{i=1}^n p_i \log p_i$ be the entropy function and suppose $f(0) = 0$. Then

$$D_f(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} = KL(\mathbf{p} \parallel \mathbf{q}),$$

where KL is Kullback-Leibler divergence.

We will next talk about some facts about Bregman divergence.

Fact 2.19.1.

1. If f is convex, then $D_f(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
2. f is μ -strongly convex if and only if $D_f(\mathbf{x}, \mathbf{y}) \geq \frac{\mu}{2} \|\mathbf{x} - \mathbf{y}\|^2$.
3. f is β -smooth if and only if $D_f(\mathbf{x}, \mathbf{y}) \leq \frac{\beta}{2} \|\mathbf{x} - \mathbf{y}\|^2$.
4. If $D_f(\mathbf{x}, \mathbf{y}) = D_f(\mathbf{y}, \mathbf{x})$, then f is quadratic.
5. Pinsker's Inequality: $KL(\mathbf{p}, \mathbf{q}) \geq \frac{1}{2} \|\mathbf{p} - \mathbf{q}\|_1^2$. It means that entropy function is 1-strongly convex in $\|\cdot\|_1$.

Definition 2.20 (Fenchel Dual Conjugate). Let f be convex, the Fenchel dual conjugate of f is defined as

$$f^*(\theta) = \sup_{\mathbf{x} \in \text{dom}(f)} [\langle \mathbf{x}, \theta \rangle - f(\mathbf{x})]. \quad (2.13)$$

Claim 2.21. $f^*(\theta)$ is convex.

Proof. It can be seen that $g_{\mathbf{x}}(\theta) = \langle \mathbf{x}, \theta \rangle - f(\mathbf{x})$ is a linear function. Linear function is convex. In addition, $f^*(\theta) = \sup_{\mathbf{x} \in \text{dom}(f)} g_{\mathbf{x}}(\theta)$ is the supreme of convex function, which means that f is also convex. □

Example 2.21.1. If $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|_2^2$, then $f^*(\theta) = \frac{1}{2} \|\theta\|_2^2$.

Example 2.21.2. If $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T M \mathbf{x}$ and M is a positive definite matrix,

$$f^*(\theta) = \sup_{\mathbf{x} \in \text{dom}(f)} \left[\langle \mathbf{x}, \theta \rangle - \frac{1}{2} \mathbf{x}^T M \mathbf{x} \right].$$

Let $g(\mathbf{x}) = \langle \mathbf{x}, \theta \rangle - \frac{1}{2} \mathbf{x}^T M \mathbf{x}$. Take the derivative over \mathbf{x} and we can get that

$$\begin{aligned} \nabla_{\mathbf{x}} g(\mathbf{x}) &= 0 \\ \Leftrightarrow \theta - M\mathbf{x} &= 0 \\ \Leftrightarrow \mathbf{x} &= M^{-1}\theta. \end{aligned}$$

Substitute the value of \mathbf{x} into $f^*(\theta)$ and we can get that

$$\begin{aligned} f^*(\theta) &= \theta^T M^{-1} \theta - \frac{1}{2} (M^{-1} \theta)^T M M^{-1} \theta \\ &= \frac{1}{2} \theta^T M^{-1} \theta. \end{aligned}$$

Example 2.21.3. Let $f(x) = \frac{1}{p} \|\mathbf{x}\|_p^p = \frac{1}{p} \sum_{i=1}^n x_i^p$, then

$$f^*(\theta) = \frac{1}{q} \|\theta\|_q^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

There is one famous inequality for Fenchel dual conjugate called Fenchel-Yang Inequality.

Theorem 2.22 (Fenchel-Yang Inequality). $\forall \mathbf{x} \in \text{dom}(f)$ and $\forall \theta \in \text{dom}(f^*)$, it holds that

$$f(\mathbf{x}) + f^*(\mathbf{x}) \geq \langle \mathbf{x}, \theta \rangle. \quad (2.14)$$

Proof. By the definition of Fenchel dual conjugate, $\forall \mathbf{x}$ and θ ,

$$\begin{aligned} f^*(\theta) + f(\mathbf{x}) &= \sup_{\mathbf{y}} [\langle \mathbf{y}, \theta \rangle - f(\mathbf{y})] + f(\mathbf{x}) \\ &\geq \langle \mathbf{x}, \theta \rangle - f(\mathbf{x}) + f(\mathbf{x}) \\ &= \langle \mathbf{x}, \theta \rangle. \end{aligned}$$

□

Corollary 2.23. By the inequality above, we can get directly that

$$\frac{1}{p} \|\mathbf{x}\|_p^p + \frac{1}{q} \|\theta\|_q^q \geq \langle \mathbf{x}, \theta \rangle.$$

I will next discuss some facts about Fenchel dual.

Fact 2.23.1.

1. If f is closed, then $(f^*)^* = f$.
2. $\nabla f(\nabla f^*(\theta)) = \theta$ and $\nabla f^*(\nabla f(\mathbf{x})) = \mathbf{x}$.
3. If f is differentiable, then $D_f(\mathbf{x}, \mathbf{y}) = D_{f^*}(\nabla f(\mathbf{y}), \nabla f(\mathbf{x}))$.
4. If f is μ -strongly convex with respect to $\|\cdot\|$, then f^* is $\frac{1}{\mu}$ -smooth with respect to $\|\cdot\|_*$, where $\|\cdot\|_*$ is dual to $\|\cdot\|$.

Chapter 3

Deviation Bound

In this chapter several famous deviation bounds are mentioned. These bounds will be heavily used in later chapters.

First of all, some simple definitions from measure theory will be discussed.

Definition 3.1 (Random Variable). A random variable X is a measurable function from a sigma algebra $\Omega \rightarrow \mathbb{R}$.

The definition of measurable function and sigma algebra is included in measure theory, which will not be discussed here.

Definition 3.2 (Cumulative Distribution Function). The CDF (cumulative distribution function) of a random variable X is defined as

$$F(t) := \Pr(X \leq t). \quad (3.1)$$

Definition 3.3 (Probability Density Function). If $F(t)$ is differentiable, then the PDF (probability density function) is defined as

$$f(t) = F'(t). \quad (3.2)$$

Definition 3.4 (Variance). The variance of a random variable X is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mu)]^2, \quad (3.3)$$

where $\mu = \mathbb{E}(X)$ is the expectation of X .

After the above definitions, we will next introduce some useful deviation bounds.

Theorem 3.5 (Markov's Inequality). Let X be a random variable and $X \geq 0$, then $\forall t \geq 0$, we have that

$$\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}. \quad (3.4)$$

Proof. The proof of above theorem is not very complex.

First of all, let $Z_t := \mathbf{1}(X \geq t) \cdot t$, where $\mathbf{1}(\cdot)$ is the indicator function.

Notice that $X \geq Z_t$ for all t . The reason is that if $X < t$, then $Z_t = 0 \leq X$. On the contrary, if $X \geq t$, then $Z_t = t \leq X$.

Use this, we can know that

$$\begin{aligned}\mathbb{E}(X) &\geq \mathbb{E}(Z_t) \\ &= \mathbb{E}[\mathbf{1}(X \geq t)] \cdot t \\ &= \Pr(X \geq t) \cdot t\end{aligned}$$

This directly shows that $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. □

Since $\Pr(X \geq t) = 1 - F(t)$, we can know from above theorem that $F(t) \geq 1 - \frac{\mathbb{E}(X)}{t}$.

Theorem 3.6 (Chebyshev's Inequality). *Let $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$, then $\Pr(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$.*

It turns out that this theorem can be easily proved with Markov Inequality.

Proof.

$$\begin{aligned}\Pr(|X - \mu| \geq t) &= \Pr(|X - \mu|^2 \geq t^2) \\ &\leq \frac{\mathbb{E}[(X - \mu)^2]}{t^2} \\ &= \frac{\sigma^2}{t^2}.\end{aligned}$$

□

Next, we will define gaussian distribution (also called normal distribution).

Theorem 3.7 (Gaussian Distribution). *Let's say $X \sim N(\mu, \sigma^2)$, it means that the pdf of X is*

$$P(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

This distribution is heavily used in many different areas.

Fact 3.7.1. *If $X \sim N(0, \sigma)$, and $t > 0$, then $\mathbb{E}[\exp(tX)] = \exp(\frac{t^2\sigma^2}{2})$.*

This can be proved just by taking integral of this pdf function.

Next, we will define subgaussian distribution, which will be used later in some inequalities.

Definition 3.8 (Subgaussian Distribution). *A random variable X with $\mathbb{E}(X) = 0$ is a subgaussian distribution with respect to variance proxy σ^2 if*

$$\mathbb{E}[\exp(tX)] \leq \exp\left(\frac{t^2\sigma^2}{2}\right). \tag{3.5}$$

We will then give an example of subgaussian distribution.

Example 3.8.1. *Let X be a bounded random variable, that is, $a \leq X \leq b$ and $\mathbb{E}(X) = 0$. Then X is subgaussian with respect to variance proxy $\frac{(b-a)^2}{4}$, which means*

$$\mathbb{E}[\exp(tX)] \leq \exp\left(\frac{t^2(b-a)^2}{8}\right).$$

This is also called Hoeffding's Lemma.

Claim 3.9. *If a random variable X is subgaussian with respect to variance proxy σ^2 , then*

$$P(|X| > t) \leq 2 \exp\left(-\frac{t^2}{2\sigma^2}\right). \quad (3.6)$$

Using subgaussian, we can introduce Hoeffding's inequality, which is extremely helpful in later material.

Theorem 3.10 (Hoeffding's Inequality). *Let X_1, \dots, X_n be independent random variables, with $\mathbb{E}(X_i) = \mu_i$ and $a_i \leq X_i \leq b_i$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ and $\mu = \frac{1}{n} \sum_{i=1}^n \mu_i$. It holds that*

$$\Pr(|\bar{X}_n - \mu| > t) \leq 2 \exp\left(-\frac{2n^2 t^2}{\sum_{i=1}^n (a_i - b_i)^2}\right). \quad (3.7)$$

We will next give the proof of this theorem.

Proof. Notice that $\Pr(|\bar{X}_n - \mu| > t) \leq \Pr(\bar{X}_n - \mu > t) + \Pr(\bar{X}_n - \mu < -t)$. Then, we just need to prove that $\Pr(\bar{X}_n - \mu > t) \leq \exp\left(-\frac{2n^2 t^2}{\sum_{i=1}^n (a_i - b_i)^2}\right)$.

$$\begin{aligned} \Pr(\bar{X}_n - \mu > t) &= \Pr(\exp(s(\bar{X}_n - \mu)) > \exp(st)) \\ (\text{Markov Inequality}) &\leq \frac{\mathbb{E}[\exp(s(\bar{X}_n - \mu))]}{\exp(st)} \\ &= \exp(-st) \mathbb{E}[\exp(s(\bar{X}_n - \mu))] \\ &= \exp(-st) \mathbb{E}[\Pi_{i=1}^n \exp(\frac{s}{n}(X_i - \mu_i))] \\ (\text{Independence}) &= \exp(-st) \Pi_{i=1}^n \mathbb{E}[\exp(\frac{s}{n}(X_i - \mu_i))] \\ (X_i - \mu_i \text{ Subgaussian}) &\leq \exp(-st) \Pi_{i=1}^n \exp(\frac{s^2}{8n^2}(b_i - a_i)^2) \\ &= \exp(\frac{s^2}{8n^2} \sum_{i=1}^n (a_i - b_i)^2 - st) \\ (s \text{ takes } \frac{4tn^2}{\sum_{i=1}^n (a_i - b_i)^2}) &= \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (a_i - b_i)^2}\right) \end{aligned}$$

Similarly, we can prove that $\Pr(\bar{X}_n - \mu < -t) \leq \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (a_i - b_i)^2}\right)$. Combine the above two conclusions and we can get that $\Pr(|\bar{X}_n - \mu| > t) \leq 2 \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (a_i - b_i)^2}\right)$. \square

Hoeffding's Inequality is very useful in proving other theorems. To see that, let's use it to prove the following claim.

Claim 3.11. *Let X_1, \dots, X_n be independent random variable and $0 \leq X_i \leq 1$, then with probability at least $1 - \delta$, it holds that*

$$\left| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] \right| \leq \sqrt{\frac{\log 2/\delta}{2n}}. \quad (3.8)$$

Proof. Let $\delta = 2 \exp(-\frac{2n^2 t^2}{\sum_{i=1}^n (a_i - b_i)^2}) = 2 \exp(-2nt^2)$ since $a_i = 0$ and $b_i = 1$. Then we can get that $t = \sqrt{\frac{\log 2/\delta}{2n}}$. By using Hoeffding's Inequality, we can get that

$$\Pr(|\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[\frac{1}{n} \sum_{i=1}^n X_i]| \geq \frac{\log 2/\delta}{2n}) \leq \delta.$$

Therefore, we can get that $\Pr(|\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[\frac{1}{n} \sum_{i=1}^n X_i]| \leq \frac{\log 2/\delta}{2n}) \geq 1 - \delta$. \square

Afterwards, let's will introduction the concept of Martingale.

Definition 3.12 (Martingale). A sequence of random variables Z_0, Z_1, \dots, Z_n is a Martingale sequence if the following two conditions are satisfied for any i

1. $\mathbb{E}[|Z_i|] < \infty$
2. $\mathbb{E}[Z_i | Z_1, \dots, Z_{i-1}] = Z_{i-1}$

Next let's see two examples of Martingale.

Example 3.12.1. Let X_1, \dots, X_n be bounded i.i.d (independent and identically distributed) random variables, and $\mathbb{E}[X_i] = 0$. Let $Z_i = \sum_{j=1}^i X_j$, and set $Z_0 = 0$, then Z_0, \dots, Z_n is a martingale.

Proof.

$$\begin{aligned} \mathbb{E}[Z_i | Z_1, \dots, Z_{i-1}] &= \mathbb{E}[Z_i | X_1, \dots, X_{i-1}] \\ &= \mathbb{E}[X_1 + \dots + X_i | X_1, \dots, X_{i-1}] \\ &= \mathbb{E}[X_1 + \dots, X_{i-1} | X_1, \dots, X_{i-1}] + \mathbb{E}[X_i | X_1, \dots, X_{i-1}] \\ &= X_1 + \dots + X_{i-1} + \mathbb{E}[X_i] \\ &= Z_{i-1} + 0 \\ &= Z_{i-1}. \end{aligned}$$

\square

Example 3.12.2. Let X_1, \dots, X_n be i.i.d, with $\mathbb{E}[X_i] = 0$ and $\text{Var}(X_i) = \sigma^2$. Let $S_i = \sum_{j=1}^i X_j$ and $Z_i = S_i^2 - i\sigma^2$. Then Z_i is a martingale.

Proof. Notice that

$$\begin{aligned} \mathbb{E}[X_n^2] &= \text{Var}(X_n) + \mathbb{E}[X_n]^2 \\ &= \text{Var}(X_n) \\ &= \sigma^2. \end{aligned}$$

In addition,

$$\begin{aligned} \mathbb{E}[S_{n-1} X_n | X_1, \dots, X_{n-1}] &= \mathbb{E}[X_n] \mathbb{E}[S_{n-1} | X_1, \dots, X_{n-1}] \\ &= 0. \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\mathbb{E}[Z_n | X_1, \dots, X_{n-1}] &= \mathbb{E}[S_n^2 - n\sigma^2 | X_1, \dots, X_{n-1}] \\
&= \mathbb{E}[(X_n + S_{n-1})^2 - n\sigma^2 | X_1, \dots, X_{n-1}] \\
&= \mathbb{E}[X_n^2 + S_{n-1}^2 + 2X_n S_{n-1} - n\sigma^2 | X_1, \dots, X_{n-1}] \\
&= \mathbb{E}[\sigma^2 + S_{n-1}^2 - n\sigma^2 | X_1, \dots, X_{n-1}] \\
&= S_{n-1}^2 - (n-1)\sigma^2 \\
&= Z_{n-1}.
\end{aligned}$$

□

Finally, we come to the last inequality of this chapter, which combines Hoeffding's Lemma and definition of Martingale.

Theorem 3.13 (Azuma's Inequality). *Let Z_0, Z_1, \dots, Z_n be a martingale, with $|Z_i - Z_{i-1}| \leq c_i$ for some c_1, \dots, c_n . Then we have that*

$$\Pr(Z_n - Z_0 > t) \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^n c_i^2}\right). \quad (3.9)$$

Proof.

$$\begin{aligned}
\Pr(Z_n - Z_0 > t) &= \Pr(\exp(s(Z_n - Z_0)) > \exp(st)) \\
&\leq \mathbb{E}[\exp(s(Z_n - Z_0))] \exp(-st) \\
&= \exp(-st) \mathbb{E}[\exp(s(Z_{n-1} - Z_0)) \exp(s(Z_n - Z_{n-1}))] \\
&= \exp(-st) \mathbb{E}[\mathbb{E}[\exp(s(Z_{n-1} - Z_0)) \exp(s(Z_n - Z_{n-1})) | Z_0, \dots, Z_{n-1}]] \\
&= \exp(-st) \mathbb{E}[\exp(s(Z_{n-1} - Z_0)) \mathbb{E}[\exp(s(Z_n - Z_{n-1})) | Z_0, \dots, Z_{n-1}]]
\end{aligned}$$

Since $|Z_n - Z_{n-1}| \leq c_n$, we can get that $Z_n - Z_{n-1} \in [-c_n, c_n]$. In addition, we know that $\mathbb{E}[Z_n - Z_{n-1} | Z_0, \dots, Z_{n-1}] = Z_{n-1} - Z_{n-1} = 0$. Therefore, by using Hoeffding's Lemma, we know that

$$\begin{aligned}
\Pr(Z_n - Z_0 > t) &\leq \exp(-st) \mathbb{E}[\exp(s(Z_{n-1} - Z_0)) \exp\left(\frac{s^2 c_n^2}{2}\right)] \\
&= \exp(-st + \frac{s^2 c_n^2}{2}) \mathbb{E}[\exp(s(Z_{n-1} - Z_0))] \\
&\leq \exp(-st + \frac{s^2 c_n^2}{2}) \exp\left(\frac{s^2 c_{n-1}^2}{2}\right) \mathbb{E}[\exp(s(Z_{n-2} - Z_0))] \\
&\dots \\
&\leq \exp(-st + \frac{s^2}{2} \sum_{i=1}^n c_i^2)
\end{aligned}$$

Let $s = \frac{t^2}{\sum_{i=1}^n c_i^2}$ and we can get that $\Pr(Z_n - Z_0 > t) \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^n c_i^2}\right)$. □

We can see that the above proof is a little bit similar to the proof of Hoeffding's Inequality in that they both take the exponential of both sides and then use Markov Inequality to transform probability into expectation. Afterwards, they perform some kind of relaxation for the expectation.

Chapter 4

Online Learning Algorithm

In this chapter we will talk about the basics of online learning algorithm step by step. We will first introduce the online learning setting and an algorithm called halving algorithm, which is simple but has strong assumption. Afterwards, we will relax the assumption and introduction other algorithms such like exponential weighted algorithm.

4.1 Online Learning and Halving Algorithm

First of all, let's introduce a simple setting of online learning. In this setting,

- Goal: predict rain/shine.
- Have a set of N weather experts.
- On each day t , expert i predicts $x_i^t \in \{0, 1\}$.
- Based on the predictions of experts, the algorithm predicts $\hat{y}^t \in \{0, 1\}$.
- Nature reveals $y^t \in \{0, 1\}$.
- The number of mistakes increases by one if $\hat{y}^t \neq y^t$.
- Assume there exists an perfect expert j , such that $x_j^t = y^t$ for all t .

We hope to develop an algorithm \mathcal{A} such that we can get an upper bound of total number of mistakes. In this setting, we can find a simple yet good algorithm called Halving algorithm such that the total number of mistakes is no larger than $\log N$.

Algorithm 1 Halving Algorithm

Let a set $C_1 = \{1, 2, \dots, N\}$.

For $t = 1, 2, \dots$

 Observe $x_i^t, \forall i \in C_t$.

$\hat{y}^t = \text{round}(\frac{1}{|C_t|} \sum_{i \in C_t} x_i^t)$

 Let $C_{t+1} = C_t$

 For all $i \in C_t$

 If $x_i^t \neq y_t$, remove i from C_{t+1} .

Basically, the algorithm will output the round of the average over all experts' output in the set. Afterwards, it will remove all the experts whose output is not the same as ground truth and continue. For this algorithm, we can have the following claim.

Claim 4.1. *For halving algorithm, it satisfies that the total number of mistakes is no larger than $\log_2 N$.*

Proof. The proof is quite simple. Initially we have $|C_1| = N$. Each time the number of mistakes increases, we can have that $|C_{t+1}|/|C_t| \leq 1/2$ since more than half of experts produces the different prediction than ground truth.

Therefore, if the number of mistakes reaches $\log_2 N$, we will have that $|C_t| = 1$, which means the only expert in the set is the perfect expert and no more mistake will be made. \square

Next, let's see an example of using halving algorithm to solve some problem.

Example 4.1.1 (Betting on sports).

Problem setting: let's say there are n sport teams. On each round t , two teams i_t, j_t play a match. Algorithm \mathcal{A} needs to predict if i_t beats j_t or vice versa. Then games happens, either i_t or j_t wins.

Assume there exists a permutation $\pi^* \in S_n$, i_t beats j_t if and only if $\pi^*(i_t) > \pi^*(j_t)$.

Now we need an algorithm that can minimize the number of mistakes to gain more revenue. We can design our algorithm by reducing to halving algorithm as following:

Algorithm 2 Betting Algorithm

Treat every $\pi \in S_n$ as an expert.
Let the prediction of expert i at round t be $x_i^t = \mathbf{1}[\pi(i_t) > \pi(j_t)]$.
Let the output of nature as $y_t = \mathbf{1}[\pi^*(i_t) > \pi^*(j_t)]$.
Run halving algorithm to the set of experts.

By halving algorithm, we know that the total number of mistakes is no larger than $\log_2 |S_n|$. In this case it is $\log_2 n! = O(n \log n)$.

Halving algorithm is very intuitive and the upper bound is also very satisfying. However, the problem is that it assumes that there is a perfect experts that will not make any mistakes. This assumption is too strong and in most cases it is not satisfied. To handle this issue, in next section we introduction exponential weights algorithm. It removes the perfect expert assumption thus can generalize to more situations.

4.2 Exponential Weights Algorithm

In this section, we change the assumption. We no longer assume that there exists a perfect expert, and we introduce two new notations.

- $M_T(i) = \sum_{t=1}^T \mathbf{1}[x_i^t \neq y^t]$
- $M_T(\mathcal{A}) = \sum_{t=1}^T \mathbf{1}[\hat{y}^t \neq y^t]$

In the other words, $M_T(i)$ is the number of mistakes expert i makes up to time T while $M_T(\mathcal{A})$ is the number of mistakes the algorithm makes.

Since we have remove the assumption that there is an expert who will never make mistake, we need a smoother algorithm than halving algorithm because in halving algorithm we are zero-tolerant to any mistake. Before talking about exponential weighted algorithm, let's first talk about weighted majority algorithm.

Algorithm 3 Weighted Majority Algorithm

Let $w_i^1 = 1$, for $i = 1, \dots, N$.

Let $\epsilon \in (0, 1)$ be a parameter we choose.

For $t = 1, 2, \dots$

Algorithm predicts $\hat{y}^t = \text{round}\left(\frac{\sum_{i=1}^N w_i^t x_i^t}{\sum_{i=1}^N w_i^t}\right)$

For $i = 1, \dots, N$
 $w_i^{t+1} = w_i^t (1 - \epsilon)^{\mathbf{1}_{[x_i^t \neq y^t]}}$

Intuitively, different from halving algorithm, in weighted majority algorithm, if an expert makes a mistake, we will reduce its weight rather than remove it from the set. The expert will still have contribution to the algorithm but the weight is smaller. One can also see that halving algorithm is a special case of weighted majority algorithm when $\epsilon = 1$.

For this algorithm, we have the following theorem:

Theorem 4.2. *For any ϵ and expert i , no matter what the sequence y^1, \dots, y^T is, we can have that*

$$M_T(WMA) \leq \frac{2 \log_e N}{\epsilon} + 2(1 + \epsilon)M_T(i). \quad (4.1)$$

To prove this theorem, we will need the following lemma first:

Lemma 4.3. *The following inequalities are valid.*

1. $\log(1 + x) \leq x$.
2. $1 + x \leq \exp(x)$.
3. $\exp(\alpha x) \leq 1 + (\exp(\alpha) - 1)x$, for $x \in [0, 1]$.
4. $-\log(1 + x) \leq -x + x^2$, for $x \in [-1, \frac{1}{2}]$.

There are many kinds of proof to this lemma online, thus we will omit the proof here. Next, we will go directly into the proof to this theorem.

Proof. Let $\Phi_t = \sum_{i=1}^N w_i^t$. Notice that $\Phi_1 = N$. Furthermore, since all weights are non-negative, we can have that $\Phi_{T+1} \geq w_i^{T+1} = (1 - \epsilon)^{M_T(i)}$.

We now take a look at the case when WMA makes a mistake at round t . In that case, we know that at least half of total weights are wrong. That is, at least $\frac{\sum_{i=1}^N w_i^t}{2}$ will shrink. By the updating

formula of weights in WMA, we can have that

$$\begin{aligned}
\Phi_{t+1} &\leq \frac{\sum_{i=1}^N w_i^t}{2} + (1 - \epsilon) \frac{\sum_{i=1}^N w_i^t}{2} \\
&= (1 - \frac{\epsilon}{2}) \sum_{i=1}^N w_i^t \\
&= (1 - \frac{\epsilon}{2}) \Phi_t.
\end{aligned}$$

Therefore, $\Phi_{T+1} \leq \Phi_0 (1 - \frac{\epsilon}{2})^{M_T(WMA)} = N (1 - \frac{\epsilon}{2})^{M_T(WMA)}$.

Now, combine above inequalities and we can have that

$$(1 - \epsilon)^{M_T(i)} \leq \Phi_{T+1} \leq N (1 - \frac{\epsilon}{2})^{M_T(WMA)}.$$

Take log on both left side and right side we can get that

$$M_T(i) \log(1 - \epsilon) \leq \log(N) + M_T(WMA) \log(1 - \frac{\epsilon}{2}).$$

Use inequality 1 and 4 in the lemma and we get

$$M_T(i)(\epsilon + \epsilon^2) \geq -\log(N) + \frac{\epsilon}{2} M_T(WMA).$$

This directly indicates that

$$M_T(WMA) \leq \frac{2 \log N}{\epsilon} + 2(1 + \epsilon) M_T(i).$$

□

Since i can take any value, we know that $M_T(WMA) \leq \frac{2 \log N}{\epsilon} + 2(1 + \epsilon) M_T(i^*)$ where i^* is the best expert that makes least number of mistakes

Following this theorem, by setting $\epsilon = \sqrt{\frac{\log N}{M_T(i^*)}}$, we reach that

$$\begin{aligned}
M_T(WMA) &\leq \frac{2 \log N}{\sqrt{\frac{\log N}{M_T(i^*)}}} + 2(1 + \sqrt{\frac{\log N}{M_T(i^*)}}) M_T(i^*) \\
&= 2\sqrt{M_T(i^*) \log N} + 2M_T(i^*) + 2\sqrt{\frac{\log N}{M_T(i^*)}} M_T(i^*) \\
&= 2M_T(i^*) + 4\sqrt{M_T(i^*) \log N}.
\end{aligned}$$

This concludes our discussion about weighted majority algorithm. We will now start discussing about exponential weighted algorithm. Before talking about the algorithm details, let's first introduce two new settings for this algorithm.

- Setting 1: Continuous Prediction
 - At time t , each expert i predicts $x_i^t \in [0, 1]$.

- Algorithm predicts $\hat{y}^t \in [0, 1]$.
- Nature outcomes $y^t \in \{0, 1\}$.
- We have a convex loss function $l : [0, 1] \times \{0, 1\} \rightarrow \mathbb{R}$.
- The loss of expert i at t is $l(x_i^t, y^t)$.
- The loss of algorithm at t is $l(\hat{y}^t, y^t)$.
- Define $L_t(i) = \sum_{s=1}^t l(x_i^s, y^s)$.
- Define $L_t(\text{Alg}) = \sum_{s=1}^t l(\hat{y}^s, y^s)$.

We can see that $L_t(i) = L_{t-1}(i) + l(x_i^t, y^t)$. We can give some examples of loss functions.

1. Absolute loss: $l(\hat{y}, y) = |\hat{y} - y|$.
 2. Square loss: $l(\hat{y}, y) = (\hat{y} - y)^2$.
 3. Log loss: $l(\hat{y}, y) = -y \log \hat{y} - (1 - y) \log(1 - \hat{y})$.
- Setting 2: Hedge/Action Setting
 - There are N actions.
 - Algorithm must (randomly) select an action i_t on day t .
 - This is equivalent that algorithm selects a distribution $\mathbf{p}^t \in \Delta_N$.
 - Nature chooses losses $l^t = [l_1^t, \dots, l_N^t] \in [0, 1]^N$, where l_i^t means the cost of choosing i .
 - Then, the expected loss to this algorithm is defined as $p^t l^t = \mathbb{E}[l_i^t]$.
 - Define $L_t(i) = \sum_{s=1}^t l_i^s$.
 - Define $L_t(\text{Alg}) = \sum_{s=1}^t p^s l^s$.

To continue, we define the regret of an algorithm, which is the cost function that we want to minimize.

Definition 4.4 (Regret). *The regret of an algorithm is defined as*

$$\text{Regret}_T(\text{Alg}) = L_T(\text{Alg}) - \min_{i \in [N]} L_T(i). \quad (4.2)$$

Now we will give the algorithm details of Exponential Weights Algorithm:

Algorithm 4 Exponential Weights Algorithm

Let $w_i^1 = 1$, for $i = 1, \dots, N$. Choose $\eta > 0$.

For $t = 1, \dots, T$

 For $i = 1, \dots, N$

$$w_i^t = \exp(-\eta L_t(i))$$

 Prediction Setting:

 Observe x_1^t, \dots, x_N^t

$$\text{Output } \hat{y}^t = \left(\sum_{i=1}^N w_i^t x_i^t \right) / \left(\sum_{i=1}^N w_i^t \right)$$

 Observed y^t .

 Hedge Setting:

$$\text{Output } \mathbf{p}^t = \frac{1}{\sum_{i=1}^N w_i^t} [w_1^t, \dots, w_N^t]$$

 Observed $l^t \in [0, 1]^N$.

For prediction setting, we can see that it is the same as WMA algorithm except the way it updates the weights. We have the following theorem for Exponential Weights Algorithm:

Theorem 4.5. *For any sequence of inputs and any choice of η , we have that*

$$L_T(EWA) \leq \frac{\eta L_T(i) + \log N}{1 - \exp(-\eta)} \quad (4.3)$$

Corollary 4.6. *For an excellent choice of η , we have that*

$$L_T(EWA) - L_T(i) \leq \log N + 2\sqrt{L_T(i^*) \log N}. \quad (4.4)$$

It means that the regret of algorithm is not larger than $\log N + 2\sqrt{L_T(i^) \log N}$.*

Compare it with the result of weighted majority algorithm, we can see that EWA has much better performance when $M_T(i^*) \gg \log N$. Next, before proving the theorem, let's first prove a lemma that will be used.

Lemma 4.7. *Let X be a random variable in $[0, 1]$, then $\log(\mathbb{E}[\exp(sX)]) \leq (\exp(s) - 1)\mathbb{E}[X]$.*

Proof. With the inequalities in Lemma 4.3, we can show that

$$\begin{aligned} \log(\mathbb{E}[\exp(sX)]) &\leq \log(\mathbb{E}[1 + (\exp(s) - 1)X]) \\ &\leq \log(1 + (\exp(s) - 1)\mathbb{E}[X]) \\ &\leq ((\exp(s) - 1)\mathbb{E}[X]) \end{aligned}$$

□

Next, let's prove theorem 4.5. We will prove the algorithm in Hedge setting, but the proof is almost the same for prediction setting.

Proof. First of all, let's define a random variable X_t as $X_t = l(x_i^t, y^t)$ with probability $\frac{w_i^t}{\sum_{j=1}^N w_j^t}$.

Then, like in the proof of WMA, we also define a function $\Phi_t = -\log(\sum_{i=1}^N w_i^t)$. We can get that

$$\begin{aligned}
\Phi_{t+1} - \Phi_t &= -\log\left(\frac{\sum_{i=1}^N w_i^{t+1}}{\sum_{i=1}^N w_i^t}\right) \\
&= -\log\left(\frac{\sum_{i=1}^N w_i^t \exp(-\eta l(x_i^t, y^t))}{\sum_{i=1}^N w_i^t}\right) \\
&= -\log\left(\sum_{i=1}^N \frac{w_i^t}{\sum_{j=1}^N w_j^t} \exp(-\eta l(x_i^t, y^t))\right) \\
&= -\log(\mathbb{E}[\exp(-\eta X_t)]) \\
&\geq (1 - \exp(-\eta))\mathbb{E}[X_t] \\
&= (1 - \exp(-\eta))\left(\sum_{i=1}^N \frac{w_i^t}{\sum_{j=1}^N w_j^t} l(x_i^t, y^t)\right) \\
&\stackrel{(Jensen's\ Inequality)}{\geq} (1 - \exp(-\eta))l\left(\sum_{i=1}^N \frac{w_i^t x_i^t}{\sum_{j=1}^N w_j^t}, y^t\right) \\
&= (1 - \exp(-\eta))l(\hat{y}^t, y^t).
\end{aligned}$$

In addition, we know that $\Phi_1 = -\log N$ and $\Phi_{T+1} = -\log\left(\sum_{i=1}^N \exp(-\eta L_T(i))\right) \leq \eta L_T(i)$ for any i . Therefore,

$$\begin{aligned}
\log N + \eta L_T(i) &\geq \Phi_T - \Phi_1 \\
&= \sum_{t=1}^T (\Phi_{t+1} - \Phi_t) \\
&= \sum_{t=1}^T (1 - \exp(-\eta))l(\hat{y}^t, y^t) \\
&= (1 - \exp(-\eta))L_T(EWA).
\end{aligned}$$

Therefore, we can have that for any i ,

$$L_T(EWA) \leq \frac{\log N + \eta L_T(i)}{1 - \exp(-\eta)}.$$

□

From corollary 4.6, we know that by choosing η carefully, we can have that $\text{Regret}_T(EWA) \leq \log N + 2\sqrt{L_T(i^*) \log N}$. If the loss function $l(x_i^s, y^s) \in [0, 1]$, it holds that

$$\begin{aligned}
\frac{L_T(EWA) - L_T(i)}{T} &\leq \frac{1}{T}(\log N + 2\sqrt{L_T(i^*) \log N}) \\
&= O\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

It shows that $\frac{\text{Regret}_T}{T}$ will go to zero as $T \rightarrow \infty$.

Hedge setting and prediction setting belong to a category called full information setting. It means besides the choice algorithm makes, it also knows the loss of all alternative choices. Next we will talk about Perception Algorithms.

4.3 Perception

We now introduce a new setting called linear prediction setting:

- At time t , observe $x^t \in \mathbb{R}^d$ with $\|x^t\|_1 \leq 1$.
- Define linear predictor as a function $h_w(\cdot)$ parameterized by $w \in \mathbb{R}^d$ and $h_w(\cdot) = \text{sign}(w \cdot x)$.
- Predict $\hat{y}^t \in \{-1, 1\}$ using some linear predictor.
- Outcome $y^t \in \{-1, 1\}$.

We assume that for some γ , $\exists w$ such that $\|w\|_2 \leq 1$ and $(w \cdot x^t)y^t > \gamma$ for any t . This is equivalent to $\exists w$, such that $\|w\|_2^2 \leq \frac{1}{\gamma^2}$ and $(w \cdot x^t)y^t > 1$ for any t .

Intuitively, this assumption means that there exists a perfect linear predictor with respect to margin γ . Based on this assumption, we can start talking about perception algorithm.

Algorithm 5 Perception

Let $w^1 = 0 \in \mathbb{R}^d$
For $t = 1, \dots, T$
 $\hat{y}^t = \text{sign}(w^t \cdot x^t)$.
 Observe y^t .
 If $y^t(w^t \cdot x^t) > 0$, then $w^{t+1} = w^t$.
 Else, $w^{t+1} = w^t + x^t y^t$.

This algorithm is the same as gradient descent algorithm with loss function $l(w; (x, y)) = \max\{0, -(w \cdot x)y\}$, and

$$w^{t+1} = w^t - \nabla l(w^t; (x^t, y^t)).$$

For Perception Algorithm, the following theorem holds that

Theorem 4.8. Let $M_T = \sum_{i=1}^T \mathbf{1}[y^i(w^i \cdot x^i) < 0]$. Assume $\exists w^*$ such that $\|w^*\| \leq \frac{1}{\gamma}$ and $(w^* \cdot x^t)y^t \geq 1$, for any t . Then

$$M_T \leq \frac{1}{\gamma^2}. \tag{4.5}$$

Proof. Suppose w^* satisfies the assumption in theorem, then define $\Phi_t = \|w^* - w^t\|^2$.

Notice that

$$\begin{aligned} \Phi_1 &= \|w^* - w^1\|^2 = \|w^*\|^2 \\ &\leq \frac{1}{\gamma^2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\gamma^2} &\geq \Phi_1 - \Phi_{T+1} \\
&= \sum_{t=1}^T (\Phi_t - \Phi_{t+1}) \\
&= \sum_{t=1}^T (\|w^* - w^t\|^2 - \|w^* - w^{t+1}\|^2) \\
&= \sum_{t:\text{mistake}(t)} (\|w^* - w^t\|^2 - \|w^* - w^t - x^t y^t\|^2) \\
&= \sum_{t:\text{mistake}(t)} 2(w^* - w^t)(x^t y^t) - (y^t)^2 \|x^t\|^2 \\
&= \sum_{t:\text{mistake}(t)} 2(w^* - w^t)(x^t y^t) - \|x^t\|^2 \\
&\geq \sum_{t:\text{mistake}(t)} 2(w^* - w^t)(x^t y^t) - 1 \\
(y^t(w^t \cdot x^t) < 0) &\geq \sum_{t:\text{mistake}(t)} 2(w^*)(x^t y^t) - 1 \\
(y^t(w^* \cdot x^t) \geq 1) &\geq \sum_{t:\text{mistake}(t)} 1 \\
&= M_T.
\end{aligned}$$

Therefore, we can get directly that $M_T \leq \frac{1}{\gamma^2}$.

□

Chapter 5

Game Theory

In this chapter, we will talk about the basics of game theory. First of all, let's give some definitions in game theory.

Definition 5.1 (Bimatrix Game). A two player bimatrix game is defined by matrix $M \in \mathbb{R}^{n \times m}$ and $N \in \mathbb{R}^{n \times m}$.

Each player (typically randomly) selects distribution $\mathbf{p} \in \Delta_n$, $\mathbf{q} \in \Delta_m$. If player 1 chooses $i \in [n]$ and player 2 chooses $j \in [m]$, then player 1 gets utility $U^1(i, j) = M_{ij}$ and player 2 gets utility $U^2(i, j) = N_{ij}$. Therefore, the utilities for their choices \mathbf{p} and \mathbf{q} is defined as an expectation

$$\begin{aligned} U^1(p, q) &= \mathbb{E}[M_{ij}] = \mathbf{p}^T M \mathbf{q} \\ U^2(p, q) &= \mathbb{E}[N_{ij}] = \mathbf{p}^T N \mathbf{q} \end{aligned} \tag{5.1}$$

In addition, if $M + N = 0 \in \mathbb{R}^{n \times m}$, then the game is called zero-sum game.

Next, we will give a definition of Nash's equilibrium.

Definition 5.2 (Nash's Equilibrium). Given $M, N \in \mathbb{R}^{n \times n}$, $\mathbf{p} \in \Delta_n$ and $\mathbf{q} \in \Delta_m$, (\mathbf{p}, \mathbf{q}) is called a Nash equilibrium if the following two inequalities hold:

$$\begin{aligned} \mathbf{p}^T M \mathbf{q} &\geq \tilde{\mathbf{p}}^T M \mathbf{q}, \quad \forall \tilde{\mathbf{p}} \in \Delta_n \\ \mathbf{p}^T N \mathbf{q} &\geq \mathbf{p}^T N \tilde{\mathbf{q}}, \quad \forall \tilde{\mathbf{q}} \in \Delta_m \end{aligned} \tag{5.2}$$

Intuitively, it means that (\mathbf{p}, \mathbf{q}) is a Nash equilibrium if and only if both players reach the best utility they can get at that point. That is, they cannot increase their utility by changing their strategy.

We will next give two famous theorems in game theory.

Theorem 5.3 (Nash's Theorem). For every bimatrix game M and N , there always exists a Nash's equilibrium.

Proof. The proof of this theorem relies on Brouwer's Fixed Pointer theorem.

In Brouwer's Fixed Point theorem, it states that if a function $f : \mathcal{K} \rightarrow \mathcal{K}$ is continuous, then $\exists x \in \mathcal{K}$ such that $f(x) = x$.

The proof of Nash's theorem is just to find correct \mathcal{K} and f , then the fixed pointer x is the equilibrium. \square

Theorem 5.4 (Maximum Theorem (Von Neumann)). *The following equation holds for any zero sum game defined by $M \in \mathbb{R}^{n \times m}$*

$$\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} = \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}. \quad (5.3)$$

Proof. To prove this equality, we will prove that either side is no larger the other side. That is,

$$\begin{aligned} \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} &\leq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} \\ \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} &\geq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} \end{aligned}$$

We will first prove \geq which is easier than \leq .

Let $\mathbf{p}_* = \arg \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q}$ and $\mathbf{q}_* = \arg \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}$. Then we can have that

$$\begin{aligned} \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} &= \max_{\mathbf{q} \in \Delta_m} \mathbf{p}_*^T M \mathbf{q} \geq \mathbf{p}_*^T M \mathbf{q}_* \\ \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} &= \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}_* \leq \mathbf{p}_*^T M \mathbf{q}_* \end{aligned}$$

Combine the above two equations we can have that

$$\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \geq \mathbf{p}_*^T M \mathbf{q}_* \geq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}.$$

Next we continue to prove that $\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \leq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}$.

To do so, we will utilize exponential weighted algorithm.

Imagine a repeated game:

For $t = 1, \dots, T$
 \mathbf{p}_t chosen by P_1 .
 \mathbf{q}_t chosen by P_2 .
 \mathbf{p}_t observes loss vector $l_t = M \mathbf{q}_t$.
 \mathbf{q}_t observes loss vector $h_t = -M^T \mathbf{p}_t$.
Players update to p_{t+1}, q_{t+1} via EWA for these loss vectors.

Then, we can have

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbf{p}_t^T M \mathbf{q}_t &= \frac{1}{T} \sum_{t=1}^T \mathbf{p}_t^T l_t \\ &= \min_{\mathbf{p} \in \Delta_n} \frac{1}{T} \sum_{t=1}^T \mathbf{p}^T l_t + \frac{\text{Regret}_T}{T} \\ &= \min_{\mathbf{p} \in \Delta_n} \frac{1}{T} \sum_{t=1}^T \mathbf{p}^T M \mathbf{q}_t + \epsilon_T^p \\ &= \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \bar{\mathbf{q}} + \epsilon_T^p \\ &\leq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} + \epsilon_T^p. \end{aligned}$$

Similarly,

$$\begin{aligned}
-\frac{1}{T} \sum_{t=1}^T \mathbf{p}_t^T M \mathbf{q}_t &= \frac{1}{T} \sum_{t=1}^T \mathbf{q}_t^T h_t \\
&= \min_{\mathbf{q} \in \Delta_m} \sum_{t=1}^T \frac{1}{T} \mathbf{q}^T h_t + \epsilon_T^q \\
&= - \max_{\mathbf{q} \in \Delta_m} \sum_{t=1}^T \frac{1}{T} \mathbf{p}^t M \mathbf{q} + \epsilon_T^q \\
&= - \max_{\mathbf{q} \in \Delta_m} \bar{\mathbf{p}}^T M \mathbf{q} + \epsilon_T^q \\
&\leq - \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} + \epsilon_T^q
\end{aligned}$$

Combine the above two inequalities we can get that

$$\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} - \epsilon_T^q \leq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} + \epsilon_T^p$$

This indicates that $\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \leq \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}$.

Therefore, we can conclude that $\min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} = \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}$. \square

Corollary 5.5. $\bar{\mathbf{p}}_T, \bar{\mathbf{q}}_T$ are ϵ approximate to Nash equilibrium, where $\epsilon = \epsilon_T^q + \epsilon_T^p$. Therefore, the above proof also gives an algorithm to calculate Nash equilibrium.

Corollary 5.6. Minimax theorem can also directly proves Nash's theorem by setting the Nash equilibrium $(\mathbf{p}_*, \mathbf{q}_*)$ as following:

$$\begin{aligned}
\mathbf{p}_* &= \arg \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \\
\mathbf{q}_* &= \arg \max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q}.
\end{aligned}$$

We can then prove that

$$\begin{aligned}
\mathbf{p}_*^T (-M) \mathbf{q}_* &\geq \tilde{\mathbf{p}}^T (-M) \mathbf{q}_*, \quad \forall \tilde{\mathbf{p}} \in \Delta_n \\
\mathbf{p}_*^T M \mathbf{q}_* &\geq \mathbf{p}_*^T M \tilde{\mathbf{q}}, \quad \forall \tilde{\mathbf{q}} \in \Delta_m.
\end{aligned}$$

That concludes our discussion about game theory.

Chapter 6

Boosting

In this chapter, we will talk about boosting, its definition and some theorems related to it. We will first give a definition of boosting.

Definition 6.1 (Boosting). *There are a set of data $\{(x_1, y_1), \dots, (x_n, y_n)\} \subset X \times \{-1, 1\}$ and a class of weak learners \mathcal{H} . Each $h \in \mathcal{H}$ maps $X \rightarrow \{-1, 1\}$. These learners are "weak" means that typically, you should not expect to find a "good" $h \in \mathcal{H}$ for any particular problem.*

The problem in boosting is that maybe we can combine many h s from \mathcal{H} to get an ensemble function $F(x) = \text{sign}(\sum_{i=1}^n q_i h_i(x))$ that has good performance on problem.

To continue, let's define two conditions in boosting: Weak Lemma Condition and Strong Lemma Condition.

Definition 6.2 (Weak Lemma Condition). *The weak lemma condition states that for any distribution $\mathbf{p} \in \Delta_n$, there exists some $h \in \mathcal{H}$ such that*

$$\Pr_{(x_i, y_i) \sim \mathbf{p}} [h(x_i) = y_i] = \sum_{i=1}^n p_i \mathbf{1}[h(x_i) = y_i] \geq \frac{1}{2} + \gamma. \quad (6.1)$$

In other words, it means that there exists a learner whose probability of being right is at least $\frac{1}{2} + \gamma$ for any distribution \mathbf{p} .

Definition 6.3 (Strong Lemma Condition). *The strong lemma condition states that $\exists \mathbf{q} \in \Delta(\mathcal{H})$ such that for every (x, y) in sample,*

$$\sum_{h \in \mathcal{H}} \mathbf{q}(h) \cdot h(x_i) y_i > 0. \quad (6.2)$$

That is, $F(x) = y$ for any data (x, y) in sample.

Intuitively, it means that there exists a way to combine all the weak learner, such that the ensemble function $F(x)$ can output the correct value for any data in the data set.

We will next use the theorems from game theory to prove a famous theorem in boosting:

Theorem 6.4. *Weak lemma condition is identical to strong lemma condition. That is, weak lemma condition is satisfied if and only if strong lemma condition is satisfied.*

Before going into the proof of this theorem, we will first try to transform WLC (Weak Lemma Condition) and SLC (Strong Lemma Condition) into a game theory format.

Let $M \in \mathbb{R}^{n \times m}$ be a matrix such that $M_{ij} = y_i h_j(x_i)$.

Then WLC means that $\forall \mathbf{p} \in \Delta_n, \exists i \in [m]$, such that $\mathbf{p}^T M \mathbf{e}_i \geq 2\gamma$. That is

$$\min_{\mathbf{p} \in \Delta_n} \max_{j \in [m]} \mathbf{p}^T M \mathbf{e}_j \geq 2\gamma.$$

Since $\mathbf{p}^T M$ is a vector, which means $\min_{\mathbf{p} \in \Delta_n} \max_{j \in [m]} \mathbf{p}^T M \mathbf{e}_j = \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \geq 2\gamma > 0$.

Similarly, SLC means that $\exists \mathbf{q} \in \Delta_m, \forall \mathbf{p} \in \Delta_n$, it has $\mathbf{p}^T M \mathbf{q} > 0$.

That is

$$\max_{\mathbf{q} \in \Delta_m} \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \mathbf{q} > 0.$$

By Minimax Theorem, we can know that these two statements are equivalent to each other.

Finally, in the last part of this chapter, we try to solve the problem that let M be any payoff matrix bounded in $[0, 1] \in \mathbb{R}^{n \times m}$, we want to find an approximate min-max pair $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$ satisfying the following condition:

$$\begin{aligned} \max_{\mathbf{q} \in \Delta_m} \tilde{\mathbf{p}}^T M \mathbf{q} &\leq V^* + \epsilon \\ \min_{\mathbf{p} \in \Delta_n} \mathbf{p}^T M \tilde{\mathbf{q}} &\leq V^* - \epsilon \end{aligned}$$

,

where $V^* = \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q}$.

The reason we want to do so is the following theorem:

Theorem 6.5. *For boosting game, if we find a $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$, which are ϵ -approximate Nash equilibrium, and $\epsilon < 2\gamma$, then $\forall i$, it holds that*

$$F_{\tilde{\mathbf{q}}}(x_i) = y_i. \quad (6.3)$$

Proof. By assumption, we know that $\forall i$,

$$\mathbf{e}_i^T M \tilde{\mathbf{q}} \geq V^* - \epsilon.$$

By weak lemma condition, we can get that $V^* = \min_{\mathbf{p} \in \Delta_n} \max_{\mathbf{q} \in \Delta_m} \mathbf{p}^T M \mathbf{q} \geq 2\gamma$.

Therefore,

$$\mathbf{e}_i^T M \tilde{\mathbf{q}} \geq 2\gamma - \epsilon > 0.$$

Equivalently, it means that $y_i \sum_{j=1}^m \tilde{\mathbf{q}}_j h_j(x_i) > 0$ for any (x_i, y_i) .

Thus, $\forall i, F_{\tilde{\mathbf{q}}}(x_i) = y_i$. □

After this, we try to solve this zero-sum game using exponential weights algorithm.

The following theorem holds for this algorithm:

```

For  $t = 1, \dots, T$ 
  For  $i = 1, \dots, n$ 
     $\mathbf{p}_t(i) = \exp(-\eta \sum_{s=1}^{t-1} e_i^T M \mathbf{q}_s) / Z.$ 
  For  $j = 1, \dots, m$ 
     $\mathbf{q}_t(j) = \exp(-\eta \sum_{s=1}^{t-1} \mathbf{p}_s^T M e_j) / Z'.$ 
Return  $\mathbf{p}, \mathbf{q} = (\frac{1}{T} \sum_{t=1}^T \mathbf{p}_t, \frac{1}{T} \sum_{t=1}^T \mathbf{q}_t).$ 

```

Theorem 6.6. $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$ are an ϵ -approx Nash Equilibrium, where $\epsilon = \frac{Reg_T^{\mathbf{p}} + Reg_T^{\mathbf{q}}}{T}.$

Corollary 6.7. Consider new version with $\mathbf{q}_t = \arg \max_{\mathbf{q} \in \Delta_m} \mathbf{p}_t^T M \mathbf{q}$, which indicates that $Reg_T^{\mathbf{q}} \leq 0$.
That means $\epsilon = \frac{Reg_T^{\mathbf{p}}}{T}.$

Now we use this algorithm in boosting:

Algorithm 6 In Boosting

```

For  $t = 1, \dots, T$ 
  For  $i = 1, \dots, n$ 
     $\mathbf{p}_t(i) = \exp(-\eta \sum_{s=1}^{t-1} y_i h_{js}(x_i)) / Z.$ 
   $\mathbf{q}_t(j) = \arg \max_{e_j} \mathbf{p}_t^T M e_j.$ 
   $= \arg \max_j \sum_{i=1}^n \mathbf{p}_t(i) h_j(x_i) = e_{\text{Best Weak Learner at } t}.$ 
Return  $\mathbf{p}, \mathbf{q} = (\frac{1}{T} \sum_{t=1}^T \mathbf{p}_t, \frac{1}{T} \sum_{t=1}^T \mathbf{q}_t).$ 

```

The last question in this chapter remained to be answered is how many rounds are needed to reach $\epsilon < 2\gamma$.

By theorem 4.5, we know that when it reaches $\epsilon < 2\gamma$, it has

$$\begin{aligned}
\epsilon &= \frac{Reg_T^{\mathbf{p}}}{T} \\
&\leq \frac{\log N + \sqrt{T \log N}}{T} \\
&< 2\gamma.
\end{aligned}$$

It means that it needs $T > \frac{\log N}{4\gamma^2}$ rounds.

That ends our discussion about boosting.

Chapter 7

Online Convex Optimization

In this chapter, we will talk about online convex optimization. We will focus on the definition of this problem, some algorithms to solve it and their complexities.

To begin with, let's talk about the framework used in online convex optimization.

7.1 Framework

- Given $\mathcal{K} \subset \mathbb{R}^d$, and \mathcal{K} is bounded, convex and closed.
- At each round t , learner selects $x_t \in \mathcal{K}$.
- Nature reveals a convex function $f_t : \mathcal{K} \rightarrow \mathbb{R}$.
- Learner wants to minimize $Regret_T = \sum_{t=1}^T f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^T f_t(x)$.

Let's see three examples about online convex optimization.

Example 7.0.1 (Prediction with respect to expert advice). *The prediction setting we talked about in online learning algorithm is an example of online convex optimization, where*

- $\mathcal{K} = \Delta_N$.
- $f_t(\tilde{w}) = l(\sum_{i=1}^N \tilde{w}_i x_i^t, y^t)$.

Example 7.0.2 (Online Portfolio Selection). *In this setting, we have*

- $\mathcal{K} = \Delta_N$.
- w^t is the distribution of n stocks fluctuations.
- r^t is the vector of price, where $r^t(i) = \frac{Price_t(stock_i)}{Price_{t-1}(stock_i)}$.
- $f_t(w^t) = -\log(w^t r^t)$.
- $\sum_{t=1}^T f_t(w^t) = -\log(\Pi_{t=1}^T (w^t r^t))$.

- $\min_{w \in \Delta_n} \sum_{t=1}^T f_t(w) = -\log(\max_{w \in \Delta_n} \Pi_{t=1}^T(wr^t)) = \text{negative log wealth of best portfolio.}$

In this example, basically the loss function is minimized when money is put into the stock whose value will grow higher at time t .

In addition, there exists an algorithm such that for any sequence r^1, \dots, r^t , we can have that

$$\sum_{t=1}^T f_t(w^t) - \min_{w \in \Delta_n} \sum_{t=1}^T f_t(w) \leq n \log T.$$

Example 7.0.3 (Online Regression). In this example, basically

- $\mathcal{K} = 2\text{-norm ball } \Theta$.
- $f_t(\theta) = (\theta x_t - y_t)^2, (x_t, y_t) \in (\mathbb{R}^n \times \mathbb{R})$.

Online convex optimization is not an easy problem, it combines ideas from

- Optimization
- Statistical Learning

We have the following claim for (OCO) online convex optimization:

Claim 7.1. *OCO is harder than vanilla optimization.*

Proof. The goal of vanilla is to find the solution to $\arg \min_{x \in \mathcal{K}} \Phi(x)$, where $\Phi(\cdot)$ is a convex function.

To prove this claim, let's take any OCO algorithm \mathcal{A} , and take the following procedures.

Initialize $x_1 \in \mathcal{K}$
For $t = 1, \dots, T$:
 $x_t = \mathcal{A}(f_1, \dots, f_{t-1})$
Let $f_t(\cdot) = \Phi(\cdot)$ [or $= \langle \nabla \Phi(x_t), \cdot \rangle$]
Output $\frac{1}{T} \sum_{t=1}^T x_t = \bar{x}_T$.

By this algorithm, we can show that $\Phi(\bar{x}_T) - \min_{x \in \mathcal{K}} \Phi(x) \leq \frac{\text{Regret}_T(\mathcal{A})}{T}$, since

$$\begin{aligned} \Phi(\bar{x}_T) &\leq \frac{1}{T} \sum_{t=1}^T \Phi(x_t) \\ &= \frac{1}{T} \sum_{t=1}^T f_t(x_t) \\ &\leq \frac{1}{T} \min_{x \in \mathcal{K}} \sum_{t=1}^T f_t(x_t) + \frac{\text{Regret}_T(\mathcal{A})}{T} \\ &\leq \frac{1}{T} \min_{x \in \mathcal{K}} \sum_{t=1}^T \Phi(x_t) + \frac{\text{Regret}_T(\mathcal{A})}{T} \\ &= \min_{x \in \mathcal{K}} \Phi(x) + \frac{\text{Regret}_T(\mathcal{A})}{T}. \end{aligned}$$

□

Therefore, if we can solve online convex optimization problem, then we can also solve vanilla optimization.

Next, we will talk about Online-to-Batch Conversion, which use online optimization to solve batch minimization. The setting is following:

- X - Data Space.
- Y - Label Space.
- D - Distribution over $X \times Y$.
- $\Theta \in \mathbb{R}^d$ - Bounded parameter space.
- $l : \Theta \times X \times Y \rightarrow \mathbb{R}$ - Loss function and convex in Θ .

We can pick any OCO algorithm \mathcal{A} and use it to generalize to batch minimization as following:

```

Initialize  $\theta_1 \in \Theta$ 
For  $t = 1, \dots, T$ :
     $\theta_t = \mathcal{A}(f_1, \dots, f_{t-1})$ 
    Let  $f_t(\cdot) = l(\cdot; (x_t, y_t))$ 
Output  $\bar{\theta}_T = \frac{1}{T} \sum_{t=1}^T \theta_t$ 

```

By this generalization, we can have that

Theorem 7.2. *Let risk $R(\theta) = \mathbb{E}_{(x,y) \in D}[l(\theta; (x, y))]$, then we can have*

$$\mathbb{E}[R(\bar{\theta}_T)] - \min_{\theta \in \Theta} R(\theta) \leq \mathbb{E}\left[\frac{\text{Regret}_T(\mathcal{A})}{T}\right]. \quad (7.1)$$

Afterwards, we will introduce several algorithms in online gradient descent. The first one is called online gradient descent.

7.2 Online Convex Optimization Algorithm

Algorithm 7 Online Gradient Descent

```

Initialize  $x_1 \in \mathcal{K}$ 
For  $t = 1, \dots, T$ :
     $x_{t+1} = \Pi_{\mathcal{K}}(x_t - \eta_t \nabla f_t(x_t))$ 

```

where $\Pi_{\mathcal{K}}(\cdot)$ is defined as the projection operator and $\Pi_{\mathcal{K}}(y) = \arg \min_{x \in \mathcal{K}} \|y - x\|_2$.

This is a very simple algorithm similar to projected gradient algorithm, except that at each step, we use f_t to calculate the gradient rather than a fixed f .

The following convergence theorem is associated with online gradient descent:

Theorem 7.3. Assume that f_t is G -Lipschitz ($\|\nabla f_t\| \leq G$), and $\forall x, y \in D$, $\|x - y\| \leq D$. Then, with $\eta_t = \frac{D}{G\sqrt{t}}$, we can get that

$$\text{Regret}_T(\text{OGD}) \leq \frac{3}{2}GD\sqrt{T}. \quad (7.2)$$

Before proving this theorem, we will first introduce a lemma about the projection operator.

Lemma 7.4 (Pythagorean Theorem for Bregman Divergences). For any $x \in \mathcal{K}$,

$$\|x - \Pi_{\mathcal{K}}(y)\| \leq \|x - y\|.$$

Then, let's start proving the convergence of OGD.

Proof. Let $x^* = \arg \min_{x \in \mathcal{K}} \sum_{t=1}^T f_t(x)$, and $\nabla_t = \nabla f_t(x_t)$.

By the convexity of $f_t(\cdot)$, we know that

$$f_t(x_t) - f_t(x^*) \leq \nabla_t^T(x_t - x^*).$$

By lemma 7.4 and the update function of OGD, we know that

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &= \|\Pi_{\mathcal{K}}(x_t - \eta_t \nabla f_t(x_t)) - x^*\|^2 \\ &\leq \|x_t - \eta_t \nabla f_t(x_t) - x^*\|^2 \\ &= \|x_t - x^*\|^2 + \|\eta_t \nabla f_t(x_t)\|^2 - 2(\eta_t \nabla f_t(x_t))^T(x_t - x^*). \end{aligned}$$

Therefore, we can get that

$$\begin{aligned} f_t(x_t) - f_t(x^*) &\leq \nabla_t^T(x_t - x^*) \\ &\leq \frac{1}{2\eta_t}(\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) + \frac{\eta_t}{2} \|\nabla f_t(x_t)\|^2 \\ &\leq \frac{1}{2\eta_t}(\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) + \frac{\eta_t G^2}{2} \end{aligned}$$

Define $\frac{1}{\eta_0} = 0$, and by summing all terms we can have

$$\begin{aligned} \text{Reg}_T &= \sum_{i=1}^T (f_t(x_t) - f_t(x^*)) \\ &\leq \sum_{i=1}^T \left(\frac{1}{2\eta_t} (\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) + \frac{\eta_t G^2}{2} \right) \\ &= \frac{1}{2} \sum_{i=1}^T \|x_t - x^*\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \sum_{i=1}^T \frac{\eta_t G^2}{2} \\ &= \frac{\|x_t - x^*\|^2}{2} \sum_{i=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{G^2}{2} \sum_{i=1}^T \eta_t \\ &= \frac{\|x_t - x^*\|^2}{2} \frac{1}{\eta_T} + \frac{G^2}{2} \sum_{i=1}^T \eta_t \end{aligned}$$

By letting $\eta_t = \frac{D}{G\sqrt{t}}$, we can have that

$$\begin{aligned}
Reg_T &\leq \frac{\|x_t - x^*\|^2}{2} \frac{1}{\frac{D}{G\sqrt{T}}} + \frac{G^2}{2} \sum_{i=1}^T \frac{D}{G\sqrt{t}} \\
&\leq \frac{DG\sqrt{T}}{2} + \frac{GD}{2} \sum_{i=1}^T \frac{1}{\sqrt{t}} \\
&\leq \frac{DG\sqrt{T}}{2} + GD\sqrt{T} \\
&= \frac{3}{2}GD\sqrt{T}.
\end{aligned}$$

□

Corollary 7.5. *Gradient Descent algorithm for minimization one convex function $f(\cdot)$ with respect to averaging converge at a rate of $O(\frac{GD}{\sqrt{T}})$.*

It's surprise to find that the convergence rate of OGD is the same as normal gradient descent algorithm when they output the averaging.

Next, we will talk about Stochastic Gradient Descent, its setting and convergence rate. We will see that we can use OGD to prove its convergence rate easily.

In stochastic gradient descent, basically we want to minimize $\arg \min_{x \in \mathcal{K}} f(x)$, where $f(x) = \mathbb{E}_\xi[h(x; \xi)]$.

The basic procedures in SGD is following:

- Sample ξ_t from data set.
- $x_{t+1} \leftarrow x_t - \eta_t \nabla h(x_t; \xi_t)$.
- Output $\bar{x}_t = \frac{1}{T} \sum_{i=1}^T x_t$.

We can have the following claim about SDG:

Claim 7.6. *Stochastic Gradient Descent converges at $O(\frac{DG}{\sqrt{T}})$.*

Proof. Notice that

$$\begin{aligned}
\mathbb{E}_{\xi_{1:t}}[f(\bar{x}_T)] &\leq \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T f(x_t) \right] \\
&= \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T \mathbb{E}[f(x_t) | \xi_1, \dots, \xi_{t-1}] \right] \\
&= \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T \mathbb{E}[\mathbb{E}_\xi[h(x_t; \xi_{1:t})] | \xi_1, \dots, \xi_{t-1}] \right] \\
&= \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T \mathbb{E}[\mathbb{E}_\xi[h(x_t; \xi_t)] | \xi_1, \dots, \xi_{t-1}] \right] \\
&= \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T \mathbb{E}[h(x_t; \xi_t) | \xi_1, \dots, \xi_{t-1}] \right]
\end{aligned}$$

Let $f_t(x) = h(x_t; \xi_t)$, then

$$\begin{aligned}
\mathbb{E}_{\xi_{1:t}}[f(\bar{x}_T)] &\leq \frac{1}{T} \mathbb{E}_{\xi_{1:t}} \left[\sum_{t=1}^T \mathbb{E}[f_t(x_t) | \xi_1, \dots, \xi_{t-1}] \right] \\
&= \mathbb{E}_{\xi_{1:t}} \left[\frac{1}{T} \sum_{t=1}^T f_t(x_t) \right] \\
&\leq \mathbb{E}_{\xi_{1:t}} \left[\frac{1}{T} \sum_{t=1}^T f_t(x^*) \right] + \frac{\text{Regret}_T(\text{OGD})}{T} \\
&= f(x^*) + \frac{3DG}{2\sqrt{T}}.
\end{aligned}$$

□

Therefore, the convergence rate $\mathbb{E}_{\xi_{1:t}}[f(\bar{x}_T)] - f(x^*) \leq \frac{3DG}{2\sqrt{T}}$.

We now conclude the three settings we talked about.

- D - some distribution on data.
- $f(x, \xi)$ - a loss function of x given data ξ .
- $F(x) - \mathbb{E}_{\xi \in D}[f(x; \xi)]$.

Imagine now we have a sample $\hat{D} = \{\xi_{1:n}\}$ and the empirical loss function is defined as $\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n f(x; \xi_i)$.

If the goal of learning is

$$\text{minimize } \hat{F}(x),$$

then its stochastic gradient descent.

If the goal of optimization is

$$\text{minimize } F(x),$$

then its online to batch minimization.

Their procedure is almost the same:

For $t = 1, \dots, T$:
 Sample $\xi_t \sim D$ (or \hat{D})
 $x_{t+1} = x_t - \eta_t \nabla f(x_t; \xi_t)$.

We have the following two claims for our algorithm.

Claim 7.7. Let $\bar{x}_T = \frac{1}{T} \sum_{t=1}^T x_t$, then we have

$$\mathbb{E}_{\xi_{1:n}}[F(\bar{x}_T) - F(x^*)] \leq O\left(\frac{GD}{\sqrt{T}}\right). \quad (7.3)$$

Corollary 7.8. *It is possible to get a high probability bound.*

Claim 7.9. *Let $\bar{x}_T = \frac{1}{T} \sum_{t=1}^T x_t$, then we have*

$$\mathbb{E}_{\xi_{1:n}}[\hat{F}(\bar{x}_T) - \hat{F}(x^*)] \leq O\left(\frac{GD}{\sqrt{T}}\right). \quad (7.4)$$

We have proven claim 7.9 before, therefore, we will only prove claim 7.7 here.

Proof. We can see that

$$\begin{aligned} \mathbb{E}_{\xi_{1:n}}[F(\bar{x}_T) - F(x^*)] &\leq \mathbb{E}_{\xi_{1:n}}\left[\frac{1}{T} \sum_{t=1}^T F(x_t) - F(x^*)\right] \\ &= \mathbb{E}_{\xi_{1:n}}\left[\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi \sim D}[f(x_t; \xi) - f(x^*; \xi)]\right] \\ &= \mathbb{E}_{\xi_{1:n}}\left[\frac{1}{T} \sum_{t=1}^T \mathbb{E}_{\xi_t}[f(x_t; \xi_t) - f(x^*; \xi_t)]\right] \\ &= \mathbb{E}_{\xi_{1:n}}\left[\frac{1}{T} \sum_{t=1}^T f(x_t; \xi_t) - f(x^*; \xi_t)\right] \\ &= \mathbb{E}\left[\frac{\text{Regret}_T}{T}\right] \\ &\leq \frac{3GD}{2\sqrt{T}}. \end{aligned}$$

□

Note that the proof is not algorithm specific. That is, any low-regret algorithm can help us prove the theorem. The proof can guarantee the convergence of order $\mathbb{E}[\frac{\text{Reg}_T}{T}]$ for any algorithm, and $\mathbb{E}[\frac{\text{Reg}_T}{T}]$ will be small for a low-regret algorithm.

Another thing we need to consider is that we have shown that standard gradient descent converges at $O(\frac{GD}{\sqrt{T}})$, which is the same as SGD. Why is SGD better then?

The answer is that in each iteration, standard gradient descent will take $O(n)$ time while SGD will only use constant time. The time complexity in each iteration is the reason why SGD overtakes standard gradient descent.

Next, we will talk about a new online convex optimization algorithm, called Mirror Descent. The setting is following:

- Given $\mathcal{K} \subset \mathbb{R}^d$ and a regularizer $R(\cdot)$.
- $\mathcal{K} \subset \text{int}(\text{dom}(R))$
- R is λ -strongly convex (w.r.t $\|\cdot\|$).
- Let $\nabla_t = \nabla f_t(x_t)$ at time t .

Then the algorithm is

Algorithm 8 Mirror Descent

$x_1 \in \mathcal{K}$ arbitrary.

For $t = 1, \dots, T$:

$$x_{t+1} = \arg \min_{x \in \mathcal{K}} \eta_t \langle \nabla_t, x \rangle + D_R(x, x_t)$$

where $D_R(x, x_t)$ is the Bregman Divergence

$$D_R(x, y) = R(x) - R(y) - \langle \nabla R(y), x - y \rangle.$$

Since R is λ -strongly convex, we know that $D_R(x, y) \geq \frac{\lambda}{2} \|x - y\|^2$.

Let's break down the update function of mirror descent. The term $\langle \nabla_t, x \rangle$ tends to move away from gradient. For example, if gradient $\nabla_t = [1, -1]$, then x would be $[-\infty, \infty]$. To avoid infinity, we introduce $D_R(x, x_t)$, which will increase if x moves too far away from x_t . Therefore, overall we can reach a balance of both parts.

Before proving its convergence, let's first introduce some lemmas.

Lemma 7.10. *For any $x, y, z \in \mathcal{K}$, it holds that*

$$D_R(z, x) + D_R(x, y) - D_R(z, y) = \langle \nabla R(y) - \nabla R(x), z - x \rangle. \quad (7.5)$$

Proof. By substituting $D_R(z, x)$ with the expression of Bregman Divergence, we can get the equality directly. \square

Lemma 7.11 (First-order Optimal Condition). *If $x^* = \arg \min_{x \in \mathcal{K}} \Phi(x)$, then $\forall u \in \mathcal{K}$, we have*

$$\langle \nabla \Phi(x^*), u - x^* \rangle \geq 0. \quad (7.6)$$

Let's denote $\Phi(x) = \eta_t \langle \nabla_t, x \rangle + D_R(x, x_t)$ and apply to mirror descent. In mirror descent, we have $\nabla \Phi(x) = \eta_t \nabla_t + \nabla R(x) - \nabla R(x_t)$. Then by first-order optimal condition, we have that $\forall u \in \mathcal{K}$

$$\langle \eta_t \nabla_t + \nabla R(x_{t+1}) - \nabla R(x_t), u - x_{t+1} \rangle \geq 0.$$

Lemma 7.12. *Furthermore, notice that $\forall v, \gamma$, we have*

$$\langle v, \gamma \rangle \leq \|v\| \cdot \|\gamma\|_* = \left(\frac{1}{\sqrt{\lambda}} \|v\| \right) (\sqrt{\lambda} \|\gamma\|_*) \leq \frac{\|v\|^2}{2\lambda} + \frac{\|\gamma\|_*^2 \lambda}{2}.$$

The last inequality is Holder's inequality.

Lemma 7.13. *The following inequality holds for any $u \in \mathcal{K}$,*

$$\eta_t (f_t(x_t) - f_t(u)) \leq \eta_t \langle \nabla_t, x_t - u \rangle \leq D_R(u, x_t) - D_R(u, x_{t+1}) + \frac{\eta_t^2}{2\lambda} \|\nabla_t\|_*^2.$$

Proof. The first inequality $\eta_t (f_t(x_t) - f_t(u)) \leq \eta_t \langle \nabla_t, x_t - u \rangle$ is indicated by the convexity of f . Now let's look at the second inequality.

$$\begin{aligned} \eta_t \langle \nabla_t, x_t - u \rangle &= \langle \nabla R(x_t) - \nabla R(x_{t+1}) - \eta_t \nabla_t, u - x_{t+1} \rangle \\ &\quad - \langle \nabla R(x_t) - \nabla R(x_{t+1}), u - x_{t+1} \rangle \\ &\quad + \langle \eta_t \nabla_t, x_t - x_{t+1} \rangle. \end{aligned}$$

By Lemma 7.11, we know that $\langle \nabla R(x_t) - \nabla R(x_{t+1}), u - x_{t+1} \rangle \leq 0$. Therefore, we have

$$\begin{aligned}
\eta_t \langle \nabla_t, x_t - u \rangle &\leq -\langle \nabla R(x_t) - \nabla R(x_{t+1}), u - x_{t+1} \rangle + \langle \eta_t \nabla_t, x_t - x_{t+1} \rangle \\
&\stackrel{(\text{Lemma 7.10})}{=} D_R(u, x_t) - D_R(u, x_{t+1}) - D_R(x_{t+1}, x_t) + \langle \eta_t \nabla_t, x_t - x_{t+1} \rangle \\
&\stackrel{(\text{Lemma 7.12})}{\leq} D_R(u, x_t) - D_R(u, x_{t+1}) - D_R(x_{t+1}, x_t) + \frac{\|\eta_t \nabla_t\|_*^2}{2\lambda} + \frac{\|x_t - x_{t+1}\|^2 \lambda}{2} \\
&\stackrel{(\lambda - \text{Convexity of } D)}{\leq} D_R(u, x_t) - D_R(u, x_{t+1}) + \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda}.
\end{aligned}$$

□

After all these lemma, we come to this theorem

Theorem 7.14. *For Mirror Descent, for any $u \in \mathcal{K}$, let η_1, \dots, η_t be decreasing, and let $d^2 = \max_{t=1, \dots, T} D_R(u, x_t)$, then we have*

$$\sum_{t=1}^T (f_t(x_t) - f_t(u)) \leq \frac{d^2}{\eta_T} + \frac{1}{2\lambda} \sum_{t=1}^T \eta_t \|\nabla_t\|_*^2. \quad (7.7)$$

Corollary 7.15. *Let $\eta_t = \frac{d\sqrt{\lambda}}{\sqrt{\sum_{s=1}^t \|\nabla_s\|_*^2}}$, then*

$$\text{Regret}_T(\text{MD}) \leq 2 \frac{d}{\sqrt{\lambda}} \sqrt{\sum_{t=1}^T \|\nabla_t\|_*^2}.$$

Proof. By Lemma 7.13, we have that

$$\begin{aligned}
\sum_{t=1}^T (f_t(x_t) - f_t(u)) &\leq \sum_{t=1}^T \frac{1}{\eta_t} (D_R(u, x_t) - D_R(u, x_{t+1}) + \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda}) \\
&= \frac{1}{\eta_1} D_R(u, x_1) - \frac{1}{\eta_T} D_R(u, x_{T+1}) + \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) D_R(u, x_{t+1}) + \sum_{t=1}^T \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda} \\
&\leq \frac{1}{\eta_1} d^2 + \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) D_R(u, x_{t+1}) + \sum_{t=1}^T \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda} \\
&\leq \frac{1}{\eta_1} d^2 + d^2 \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) + \sum_{t=1}^T \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda} \\
&\leq \frac{d^2}{\eta_1} + \frac{d^2}{\eta_T} - \frac{d^2}{\eta_1} + \sum_{t=1}^T \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda} \\
&= \frac{d^2}{\eta_T} + \sum_{t=1}^T \frac{\eta_t^2 \|\nabla_t\|_*^2}{2\lambda}
\end{aligned}$$

□

Online mirror descent can extend to many other online convex optimization algorithms, let's see two examples.

Example 7.15.1. Let \mathcal{K} be any convex set, and $R(x) = \frac{1}{2}\|x\|^2$. By the update function of OMD, we can get that

$$\begin{aligned} x_{t+1} &= \arg \min_{x \in \mathcal{K}} \eta_t \langle x, \nabla_t \rangle + D_R(x, x_t) \\ &= \Pi_{\mathcal{K}}(x_t - \eta \nabla_t) \end{aligned}$$

which is online gradient descent.

Example 7.15.2. Let $K = \Delta_n$ and $R(x) = \sum_{i=1}^n x_i \log x_i$. Then

$$x_{t+1} = \arg \min_{x \in \mathcal{K}} \eta_t \langle x, \nabla_t \rangle + \sum_{i=1}^n x_i \log \frac{x_i}{x_t(i)}.$$

By first order optimal condition, we can have that $(\eta_t \nabla_t)_i = \log \frac{x(i)}{x_t(i)}$, which means

$$x(i) = x_t(i) \exp(-\eta_t \nabla_t(i)) / Z.$$

It turns out that it is just exponential weights algorithm.

Why do we choose this function R ? Because entropy function is 1-strongly convex with respect to $\|\cdot\|_1$. Then $d^2 = \max_t D_R(u, x_t) \leq \log n$ and

$$\text{Regret}_T \leq \log n \sqrt{\sum_{t=1}^T \|\nabla_t\|_\infty^2}.$$

Finally, we will talk about the last algorithm in online convex optimization: Follow the Regularized Leader algorithm.

Algorithm 9 Follow the Regularized Leader

$x_1 \in \mathcal{K}$ arbitrary.

Regularizer $R(x)$

For $t = 1, \dots, T$:

$$(V1): x_{t+1} = \arg \min_{x \in \mathcal{K}} \eta \sum_{s=1}^t f_s(x) + R(x)$$

$$(V2): x_{t+1} = \arg \min_{x \in \mathcal{K}} \eta \sum_{s=1}^t \langle \nabla f_s(x_s), x \rangle + R(x)$$

Fact 7.15.1.

1. If $f_s(x)$ is linear in x , then FTL can have linear regret.
2. The second version of FTRL is just OMD as long as $x_t \in \text{int}(\mathcal{K})$.

Next we will prove the second fact.

Proof. If $x_t \in \text{int}(\mathcal{K})$, then for FTRL we have

$$\nabla R(x_{t+1}) = -\eta \sum_{s=1}^t \nabla f_s(x_s).$$

For OMD, we have

$$\begin{aligned}
\nabla R(x_{t+1}) &= \nabla R(x_t) - \eta \nabla_t f_t(x_t) \\
&= \nabla R(x_{t-1}) - \eta (\nabla_t f_t(x_t) + \nabla_t f_{t-1}(x_{t-1})) \\
&\dots \\
&= -\eta \sum_{s=1}^t \nabla f_s(x_s).
\end{aligned}$$

□

Therefore, OMD is just FTRL when $x \in \mathcal{K}$. This also shows that the convergence rate of FTRL is as good as that of OMD.

This concludes our discussion about online convex optimization. Next, we will discuss Multi-Armed Bandit, which tries to balance exploration and exploitation.

Chapter 8

Multi-Armed Bandit