Minimum Vertex Cover Problem

1. Write fractional relaxation for an arbitrary graph G = (V, E)

- Given a graph G = (V, E) and the vertex weights = Weights(V)
  Minimum vertex cover problem for graph G as an Integer Linear Programming by using a variable x_v for each vertex v, taking 0 or 1 values.
- $x_v = 0$ means $v \notin$ Cover C, $x_v = 1$ means $v \in$ Cover C
- The weight of the solution, which we want to minimize, is $\Sigma_{v \in V}\, x_v * W(v)$ such that $x_u + x_v >= 1$ for each edge (u, v)
- This gives the Integer Linear Programming as
  Minimize $\Sigma_{v \in V}\, x_v * W(v)$
  Subject to $x_u + x_v >= 1$ for all $(u, v) \in$ Edges E
  $X_v <= 1$ for all $v \in$ Vertices V
  $X_v \in N$ for all $v \in$ Vertices V
  $X_v \in \{0, 1\}$
- Relaxing Integer Linear Programming to a linear program
  Minimize $\Sigma_{v \in V}\, x_v * W(v)$
  Subject to $x_u + x_v >= 1$ for all $(u, v) \in$ Edges E
  $X_v <= 1$ for all $v \in$ Vertices V
  $X_v >= 0$ for all $v \in$ Vertices V
- Suppose x* is an optimal solution to the linear program, rounding each value to the closest integer
  i.e., $x'_v = 1$ if $x^*_v >= ½$
  $x'_v = 0$ if $x^*_v < ½$
- The set of valid vertex cover C, because of each edge (u, v) has the conditions
  $x^*_u + x^*_v >= 1$
  so at least one of x^*_u or x^*_v must be at least ½ and at least one of u or v belongs to Cover C
- The cost of cover is at most twice the optimum

2. Dual linear program for an arbitrary graph G = (V, E)

- For the dual, the number of variables is equal to the number of vertices in the minimum vertex cover
- Using weak duality, we convert minimum vertex cover to maximum matching in a bipartite graph
- As we know, the LP for minimum vertex cover is

Minimize $\Sigma v \in V$ xv*W(v)

Subject to $x_u + x_v >= 1$ for all (u, v) ∈ Edges E

$X_v >= 0$ for all v ∈ Vertices V

- Its dual has one variable y(u,v) for edge (u, v) and it is

Maximize Σ (u, v) ∈V y(u,v)

Subject to Σ u: (u, v) ∈V y(u,v) <= Weights(v) for all v ∈ Vertices V

y(u,v) >= 0 for all (u, v) ∈ Edges E

- Assigning a non-negative charge to each edge such that the total charge Overall the edges are as large as possible.
- The sum of charges is a lower bound to the weight of the minimum vertex cover in the weighted graph, G = (V, E) with weights W

3. ILP corresponding to the dual LP (Maximum matching)

- Using weak duality, we convert minimum vertex cover to maximum matching in a bipartite graph
- The input is a graph with each edge having a positive weight W_{uv}
- A maximum weighted bipartite matching is defined as a perfect matching where the sum of values of the edges in the matching have a maximum value.
- The size of such a matching is n^2.
- graph is not complete bipartite then missing edges are inserted with the value zero
- The Variable x_{uv} is defined as,

  $x_{uv} = 0$ if the edge (u, v) belongs to the matching

  $x_{uv} = 1$ if the edge (u, v) does not belong to the matching
- ILP for this graph is Maximum $\Sigma W_{uv}\, x_{uv}$

For all u, $\Sigma x_{uw} = 1$

For all v, $\Sigma x_{pv} = 1$

For all u, v $x_{uv} >= 0$ is integral

4. Write in words the graph problem corresponding the dual ILP
   - A matching in a Bipartite Graph is a set of the edges chosen in such a way that no two edges share an endpoint.
   - A maximum matching is a matching of maximum size (maximum number of edges).
   - In a maximum matching, if any edge is added to it, it is no longer matching.
   - There can be more than one maximum matchings for a given Bipartite Graph.
   - Given a bipartite graph G = (A ∪ B, E), find an S ⊆ A × B that is a matching and is as large as possible.
   - A matching *M* is maximal if and only if there does not exist an augmenting path with respect to *M*.
   - Algorithm

     bipartiteMatch (u, visited, assign)

     Input: Starting node, visited list to keep track, assign the list to assign node with another node.

     Output: Returns true when a matching for vertex u is possible.

```
Begin
   for all vertex v, which are adjacent with u, do
        if v is not visited, then
               mark v as visited
              if v is not assigned, or bipartiteMatch(assign[v], visited, assign) is true,
then
                  assign[v] := u
                   return true
        done

           return false
```

End

maxMatch (graph)

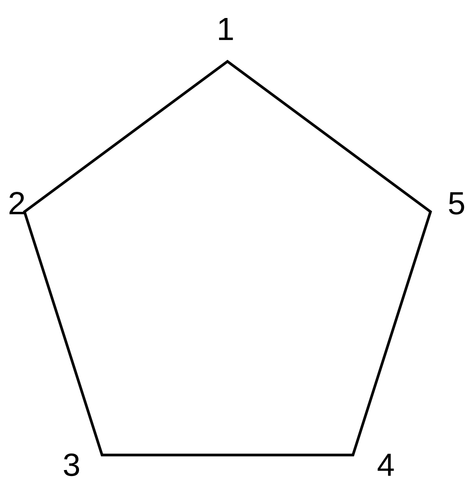
Input: The given graph.

Output: The maximum number of the match.

```
Begin
  initially no vertex is assigned
  count := 0
  for all applicant u in M, do
    make all node as unvisited
    if bipartiteMatch(u, visited, assign), then
      increase count by 1
  done
End
```

5. Given a graph C5 with edges 1-2, 2-3, 3-4, 4-5, 5-1
   - write ILP for MVCP and the dual ILP
   - give the optimal solution for the MVCP and for the dual ILP
   - give the optimal fractional for the MVCP and for the dual ILP

Solution:



a. **Minimum Vertex Cover Problem**

Objective: Minimize $\Sigma_{a \in V}$ weight (V) * x_a

Assuming weights of all the vertices are equal to 1

$x_1 + x_2 + x_3 + x_4 + x_5$

Subject to $x_1 + x_2 >= 1$, $x_2 + x_3 >= 1$, $x_3 + x_4 >= 1$, $x_4 + x_5 >= 1$, $x_5 + x_1 >= 1$

We must add 0/1 constraints for the variables to state that the values come from 0 or 1. So, $x_i \in \{0, 1\}$, $\forall$ i $\in$ V is the additional constraint

Solving the vertex cover problem, we get the minimum vertex cover

set as {1, 2, 4}

thus, we have the integer constraints as Such that $x_1 = x_2 = x_4 = 1$, $x_3 = x_5 = 0$ and the cover cost is **3**

Maximum Matching Problem

1) Objective: Maximize Σ (u, v) $\in$V y(u,v)

$y_1 + y_2 + y_3 + y_4 + y_5 = y^T B$

Subject to Σ u: (u, v) $\in$V y(u,v) <= Weights(v) for all v $\in$ Vertices V

y(u,v) >= 0 for all (u, v) $\in$ Edges E

y_1 $x_1 + x_2 <= 1$

y_2 $x_2 + x_3 <= 1$

y_3 $x_3 + x_4 <= 1$

y_4 $x_4 + x_5 <= 1$

y_5 $x_1 + x_5 <= 1$

$x_1 * (y_1 + y_5) + x_2 * (y_1 + y_2) + x_3 * (y_2 + y_3) + x_4 * (y_3 + y_4) + x_5 * (y_4 + y_5)$ is Maximized

$y_1 + y_5 >= 1$

$y_1 + y_2 >= 1$

$y_2 + y_3 >= 1$

$y_3 + y_4 >= 1$

$y_4 + y_5 >= 1$

$y_2 = y_3 = 1$ and $y_1 = y_4 = y_5 = 0$ The matching set is ([1,2], [3,4]) Assuming all edge weights are equal to 1 we get the solution as 2

b. Optimal solution for Vertex Cover

Considering the greedy approach (starting with the vertex having minimal weight), we get $x_1 = x_2 = x_4 = 1$ and $x_3 = x_5 = 0$ so, the sub-optimal cost is 3 whereas optimal $x_1 = x_2 = x_4 = 1$ and $x_3 = x_5 = 0$ cost is also 3 **Greedy Approach value <= 2 * optimal value** 3 <= 2 * 3, 3 <= 6

Optimal Solution for Maximum matching

Considering the greedy approach here too, $y_2 = y_3 = 1$ and $y_1 = y_4 = y_5 = 0$ The matching set is ([1,2], [3,4]) Assuming all edge weights are 1 we get 2

Greedy approach <= 2 * optimal value

2 <= 2*2

2 <= 4

c. Fractional optimal Solution for Minimum Vertex Cover

y_1 : $x_1 + x_2 >= 1$, y_1 ; $x_2 + x_3 >= 1$, $x_3 + x_4 >= 1$, $x_4 + x_5 >= 1$, $x_5 + x_1 >= 1$ are the integral constraints for graph

adding all the constraints we get,

$x_1 + x_2 + x_3 + x_4 + x_5 >= 5/2$

$x_1 + x_2 + x_3 + x_4 + x_5 >= 2.5$

⇨ $x_1 = 0.5$, $x_2 = 0.5$, $x_3 = 0.5$, $x_4 = 0.5$, $x_5 = 0.5$

Fractional Optimal Solution for Maximum matching

$y_1 + y_5 >= 1$

$y_1 + y_2 >= 1$

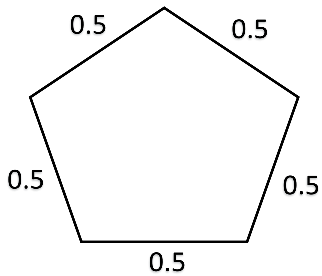
$y_2 + y_3 >= 1$

$y_3 + y_4 >= 1$

$y_4 + y_5 >= 1$ are the integral constraints

$2y_1 + 2y_2 + 2y_4 + 2y_3 + 2y_5 >= 5$

$y_1 + y_2 + y_4 + y_3 + y_5 >= 5/2$



$y_1 = 0.5, y_2 = 0.5, y_3 = 0.5, y_4 = 0.5, y_5 = 0.5$

Here, the fractional solutions for minimum vertex cover and its dual (maximum matching) are same.