Question: prove that any coin graph can be colored into 4 colors.

Solution: A coin graph is a graph with vertices related to non-overlapping quarters (coins) and edges related to touching pairs of coins.

Proof – 1:

- Consider the following set of propositional variables $\{X_{n,i} : 1 \leq i \leq 4 \wedge n \in N\}$. We are deducing $X_{n,i}$ as the n^{th} coin has color i. Let Σ be the following set of sentences:
  a. $X_{n,1} \vee X_{n,2} \vee X_{n,3} \vee X_{n,4}$ $\forall$ $n \in N$, - shows that every coin gets a color,
  b. $\neg(X_{n,i} \wedge X_{n,j})$, $\forall$ $1 \leq i < j \leq 4$ and $n \in N$, - shows that each coin gets at most one color
  c. $\neg(X_{n,i} \wedge X_{m,i})$, $\forall$ $1 \leq i \leq 4$ and all pair of adjacent coins X_n and X_m - shows that no two adjacent coins get the same color.
- ΣΣ is finitely satisfiable by hypothesis, so by efficiency, is satisfiable. Any truth valuation satisfying gives the decided coloring.

Proof – 2:

- According to Four-color theorem, given any separation of a plane into neighboring regions, producing a figure called a coin-graph, not more than four colors are to be required to color the portions of the graph such that no 2 neighboring regions have the same color. Adjacent means two regions share a common margin curve segment, not merely a corner where three or more regions meet.

- If the four-color assumption were false, there would be at least one-coin arrangement with the least possible number of regions that requires five colors. The proof showed that such a nominal counterexample could not exist, using two technical concepts:

  1. A mandatory set is a set of configurations such that every coin graph that fulfills some mandatory conditions for being a smallest non-4-colorable triangulation (such as having minimum degree 5) need to have at the most one configuration from this set.
  2. A reducible configuration is an arrangement of coins that cannot occur in a minimal counterexample. If a coin placement contains a reducible configuration, then the graph can be reduced to a smaller graph. If the smaller graph can be colored with four colors, then the original graph can also. This signifies that if the

original graph cannot be colored with four colors, the smaller graph can't either, or so the original graph is not nominal.

- Suppose v, e, and f are the number of vertices, edges, and regions (faces). Since each region is triangular and two regions share each edge, we have that 2edges (e) = 3faces(f). This together with Euler's formula, v − e + f = 2, can be used to show that 6v − 2e = 12.

- If v_n is the number of vertices of degree *n* and *D* is the maximum degree of any vertex,

$$6v - 2e = 6\sum_{i=1}^{D} v_i - \sum_{i=1}^{D} iv_i = \sum_{i=1}^{D}(6-i)v_i = 12.$$

  But since 12 > 0 and 6 − *i* ≤ 0 for all *i* ≥ 6, this determines that there is at least one vertex of degree 5 or less.

- If there is a graph requiring five colors, then there is a minimum such diagram, where taking out any vertex makes it four-colorable. Let this graph be G. G cannot have a vertex of degree 3 or less because if d(v) ≤ 3, we can remove vertex v from G, the smaller graph can be four-colorable, then add back v and extend the four-coloring to it by choosing a color different from its neighbors.

  Thus, any coin graph can be four-colored.