**Week 5 - Assignment 7**

**1. Prove or disprove the statement “All birds can fly.”**

Let’s assume that the claim “All birds can fly” is true. We know that kiwis cannot fly. We know that kiwis are birds. Since there exists a bird “kiwi” that cannot fly, we can conclude that the given claim “All birds can fly” is false.

**2. Prove or disprove the claim (∀x, y ∈ R)[(x − y)2 > 0].**

We are going to prove that the given claim (∀x, y ∈ R)[(x − y)2 > 0] is false. Let’s assume that the given claim is true. This means that the equivalent statement below is also true.

¬(∃x, y ∈ R)[(x − y)2 ≤ 0]

That is, there is no value of x or y in the set of real numbers where (x−y)2 is less than or equal to zero.

However, when x = y:

(x − y) = 0

(x − y)

2 = 0

It follows that there is at least one value of x and y in the set of real numbers where (x − y)2 = 0. This contradicts with our initial assumption.

Hence, we can conclude that the claim is false.

**3. Prove that between any two unequal rationals there is a third rational.**

The given claim can be translated symbolically as follows:

(∀x, y ∈ Q)[(x < y) ⇒ (∃z ∈ Q)[x < z < y]]

Let’s assume that the given claim is false. Let’s assume that the two unequal rational numbers, x = p/q and y =r/s, do not have another rational number between them and x > y. The variables p, q, r and s are non-zero integers.

Clearly, z = x−y/2 satisfies y < z < x.

Given our statement, z cannot be a rational number since it exists between

x and y.

z = (p/q – r/s) / 2

z = (ps−rq / qs) / 2

z = ps – rq / 2qs

We know that the denominator of the above equation cannot be zero since both q and s are non-zero.

We also know that x − y > 0. So,

p/q – r/s > 0

So, ps – qr / rs > 0

Since rs > 0, it follows that ps − qr > 0.

By definition, z is a rational number since it can now be expressed as a ratio of two non-zero integers. This contradicts our assumption that there are no rational numbers between x and y. It follows that between any two unequal rational numbers, there exists another rational number.

**4. Explain why proving φ ⇒ ψ and ψ ⇒ φ establishes the truth of φ ⇔ ψ.**

Lets assume that the φ ⇒ ψ ∧ ψ ⇒ φ is true.

When φ is true, we know ψ is true from the first part of the above proposition φ ⇒ ψ (A).

When φ is false, from the second part of the equation, we know ψ has to be false. (The antecedent cannot be true if the consequent is false) (B).

Translating the above conclusions (A) and (B), we have:

(φ ⇒ ψ) ∧ (¬φ ⇒ ¬ψ)

That however is the defining property of a bi-conditional. This shows that the truth value of a bi-conditional can be established by knowledge of forward and reverse conditionals.

**5. Explain why proving φ ⇒ ψ and (¬φ) ⇒ (¬φ) establishes the truth of φ ⇔ ψ**

**6. Prove that if five investors split a payout of $2M, at least one investor receives at least $400, 000.**

Let v, w, x, y and z represent the shares of each of the investors.

Lets assume that everyone’s share is less that $400,000.

v < $400,000

w < $400,000

x < $400,000

y < $400,000

z < $400,000

Summing them,

v + w + x + y + z < $2M (A)

Since the investors are splitting the payout and there is no indication of any money remaining: v + w + x + y + z = $2M (B)

Clearly, A and B are in conflict. So our assumption that everyone’s share is less than $400,000 must be false. Hence, we can conclude that at least one investor gets at least $400, 000.

7. Prove that √3 is irrational.

First, a lemma to prove: “If p2 is divisible by 3, then p is also divisible by 3”.

Assume that p is the product of prime factors p1, p2, p3, .., pn.

Then,

p2 = p × p

p2 = (p1 × p2 × .. × pn) × (p1 × p2 × .. × pn)

But, we know that p2 is divisible by 3. This means one of (p1, p2, .., pn) has to be 3.

Which implies p is divisible by 3.

**8. Write down the converses of the following conditional statements:**

**a) If the Dollar falls, the Yuan will rise.**

If the Yuan rises, the dollar will fall.

**b) If x < y then -y < -x. (For x,y real numbers.)**

If -y < -x then x < y. (For x,y real numbers.)

**c) If two triangles are congruent they have the same area.**

If two triangles have the same area, they are congruent.

d) The quadratic equation ax2 + bx + c = 0 has a solution whenever b2 ≥ 4ac. (Where a,b,c,x denote real numbers and a != 0).

b2 ≥ 4ac whenever the quadratic equation ax2 + bx + c = 0 has a solution.

**e) Let ABCD be a quadrilateral. If the opposite sides of ABCD are pairwise equal, then the opposite angles are pairwise equal.**

Let ABCD be a quadrilateral. If the opposite angles are pairwise equal, then the opposite sides are pairwise equal.

**f) Let ABCD be a quadrilateral. If all four sides of ABCD are equal, then all four angles are equal.**

Let ABCD be a quadrilateral. If all four angles of ABCD are equal, then all four sides are equal.

**g) If n is not divisible by 3 then n2 + 5 is divisible by 3. (For n a natural number.)**

If n2 + 5 is divisible by 3, then n is not divisible by 3.

**9. Discounting the first example, which of the statements in the previous question are true, for which the converse is true, and which are equivalent?**

**Prove your answers.**

**b) If x < y then -y < -x. (For x,y real numbers.)**

If -y < -x then x < y. (For x,y real numbers.)

Original is true. Converse is true. They are equivalent.

Proof:

Claim:

(x < y) ⇔ (−y < −x)

Assume x < y. Now, subtract (x + y) from both sides.

We get:

x − (x + y) < y − (x + y)

−y < −x

Assume −y < −x. Now, add (x + y) to both sides.

We get:

−y + (x + y) < −x + (x + y)

x < y

Since we were able to prove the conditional and its converse, it is imperative that the relationship is equivalent.

**c) If two triangles are congruent they have the same area.**

If two triangles have the same area, they are congruent.

Original is true. Converse is false. They are not equivalent.

Proof

A) If two triangles are congruent they have the same area.

When two triangles are congruent, their sides have the same length

and they have the same angles. The area of a triangle is:

Lets say the triangles have sides a, b, c and angles A, B, C where the

angles correspond to the opposite sides.

Then, the area of the triangle is given by:

½ × a × b × sin(C)

Since by definition, the triangles have the same sides and angles, it is impossible for the above formula to give two different values.

B) If two triangles have the same area, they are congruent.

Lets say that the first triangle has sides a, b, c and angles A, B, C.

The second one has p, q, r and angles P, Q, R.

The given claim states that:

½ × a × b × sin(C) = ½ × p × q × sin(R)

Simplifying we get,

a × b × sin(C) = p × q × sin(R)

Given four numbers a, b, c and d, it is possible that:

(a × b = c × d) ∧ (a 6= c) ∧ (a 6= d) ∧ (b 6= c) ∧ (b 6= d)

For example, if we take two right triangles,

a = 4, b = 5, C = 90◦ and p = 10, q = 2, C = 90◦

They have the same area of 10. But, they are not congruent as their sides are not equal.

Since there exists at least one case where the area is the same but the sides are not, the statement “If two triangles have the same area, they are congruent” is false.

Therefore, the statements are not equivalent.

**d) The quadratic equation ax2 + bx + c = 0 has a solution whenever b2 ≥ 4ac. (Where a,b,c,x denote real numbers and a != 0)**

b2 ≥ 4ac whenever the quadratic equation ax2 + bx + c = 0 has a solution.

Original is true. Converse is true. They are equivalent.

Proof

The solution of a quadratic equation ax2 + bx + c is:

x = −b ± √b2 − 4ac / 2a

Given that x is a real number, the quadratic equation will not have a solution when the denominator is 0 or if the √ is taken on a negative number.

A) Assume b2 > 4ac. We also know that a > 0.

Let m = b2 − 4ac where m ∈ R+

Since m is positive, √m ∈ R.

Then,

x = −b± √m / 2a which has a real solution.

B) Assume x ∈ R. (The quadratic equation has a solution).

Let p = √b2 − 4ac.

Then,

x = −b±p / 2a

p = ±(2ax + b)

Since we know a, b and x are real numbers, p must be real too.

So, √b2 − 4ac is real.

In order for √q to be real, q has to be positive.

This means b2 − 4ac must be positive. Hence b2 > 4ac.

Since the original statement and its converse are true, the statements

are equivalent.

**e) Let ABCD be a quadrilateral. If the opposite sides of ABCD are pairwise equal, then the opposite angles are pairwise equal.**

Let ABCD be a quadrilateral. If the opposite angles are pairwise equal, then the opposite sides are pairwise equal.

Original is true. Converse is true. They are equivalent

Proof

Dividing the quadrilateral into two triangles: ABC and ACD, we form a line between AC.

A) Assume that the opposite sides are equal. This means:

AB = CD and BC = AD.

So the triangles ABC and ACD have three equal sides. (AB = CD, BC = AD and AC is common). This means they are congruent.

We know that congruent triangles have equal angles. This means: ∠ABC = ∠ADC

Similarly, we can prove that ∠DAB = ∠DCB by taking the triangles

ADB and BDC.

So, we can conclude that if the opposite sides are pairwise equal, then

the opposite angles are pairwise equal too.

B) Assume that the opposite angles are equal. This means: ∠ABC = ∠ADC and ∠DAB = ∠DCB

We know that sum of all the angles of a triangle is 180◦.

So, in triangle ABC, we have:∠CAB + ∠ABC + ∠BCA = 180◦

In triangle ACD, we have:

∠DAC + ∠ADC + ∠DCA = 180◦

Since we know ∠ABC = ∠ADC, we can conclude that: ∠CAB + ∠BCA = ∠DAC + ∠DCA(1)

We know that: ∠DAC + ∠CAB = ∠DAB and ∠DCA + ∠BCA = ∠DCB

Since ∠DAB = ∠DCB, we have: ∠DAC + ∠CAB = ∠DCA + ∠BCA

(or)

∠DCA = ∠DAC + ∠CAB − ∠BCA(2)

Substituting (2) in (1), we have: ∠CAB + ∠BCA = ∠DAC + ∠DAC + ∠CAB − ∠BCA

Simplifying, ∠BCA = ∠DAC.

Similarly, we can prove that ∠ACD = ∠CAB.

Now we can see that the triangles ABC and ADC are congruent since they have three equal angles. This means they also have three equal sides.

So, AB = DC and AD = BC. If a quadrilateral has opposite angles that are pairwise equal, then it must have pairwise opposite sides that are pairwise equal.

Since we have proved that the conditional and its converse are true, we can conclude that the statements are equivalent.

**f) Let ABCD be a quadrilateral. If all four sides of ABCD are equal, then all four angles are equal.**

Let ABCD be a quadrilateral. If all four angles of ABCD are equal, then all four sides are equal.

Original is true. Converse is true. They are equivalent

Proof

The proof will be similar to (e). We have establish that the inner triangles are congruent to prove that not only the opposite angles are equal, but the adjacent angles are also equal.

**g) If n is not divisible by 3 then n 2 + 5 is divisible by 3. (For n a natural number.)**

If n2 + 5 is divisible by 3, then n is not divisible by 3.

Original is true. Converse is true. They are equivalent

Proof

A) Assume n is not divisible by 3.

If n is not divisible by 3 then n2 is not divisible by 3. If a number is not divisible by 3, then the number before that or after that has to be divisbile by 3. When a number m is not divisible by 3, then it is

divisible by at least one of the following: (m − 1) (m + 1)

This means, the product of the above is definitely divisible by 3.

Given that n is not divisible by 3, we have: (n − 1) × (n + 1) is divisible by 3. This means:

n2 − 1 is divisible by 3.

Adding any multiple of 3 to a number divisible by 3 preserves its divisibility. Adding 6 to the above equation, we get: n2 − 1 + 6 = n2 + 5 is divisible by 3.

B) Assume n2 + 5 is divisible by 3.

Subtracting any multiple of 3 from a number does not affect its divisibility. Using this, lets subract 6 from the given equation: n2 + 5 − 6 is divisible by 3.

n2 − 1 is divisible by 3.

(n + 1)(n − 1) is divisible by 3.

For the above equation to be true, either n − 1 or n + 1 have to be divisible by 3. If a number is divisible by 3, then the number preceding it or succeding it cannot be divisible by 3. Since one of (n + 1) or (n − 1) is divisible by 3 for sure, it is not possible for n to be divisible by 3.

**10. Prove or disprove the statement “An integer n is divisible by 12 if and only if n3 is divisible by 12.”**

Assume n is divisible by 12. Then n can be expressed as p × 12 where p is any integer.

n = p × 12

Cubing on both sides we get:

n3 = (p × 12)3

n3 = p3 × 123

n3 = p3 × 12 × 12 × 12

Since 12 has to be a factor of n3 , we can conclude that if n is divisible by 12, then n3 is divisible by 12.

Assume that n3 is divisible by 12.

Let’s say n is made up prime factors (p1, p2, .., pn). Then, we can see that:

n3 = (p1 × p2 × .. × pn)3

n3 = p31 × p32 × ... × p3n

We know that n3 is divisible by 12. This means it must be divisible by factors of 12 - namely 2, 2 and 3.

So, there must exist pr, ps, pt in (p1, p2, .., pn) such that pr = 2, ps = 2and pt = 3.

Since pr, ps, pt are factors of n, n must also be divisible by 12.

**11. Let r, s be irrationals. For each of the following, say whether the given number is necessarily irrational, and prove your answer. (The last one is tricky to do by elementary means. I’ll give a solution in Lecture 8, but you should definitely try it first. Give it half an hour of focused thought.)**

1. r + 3

This has to be irrational.

Proof

Assume r + 3 is rational. Then there must exist two integers p and q that

are not equal to zero such that:

r + 3 = p/q

r = p/q − 3

r = p − 3q /q

This means r can be represented as a ratio of two integers. However, that contradicts with our premise that r is irrational. So our assumption that r + 3 is rational must be false.

2. 5r

This has to be irrational.

Proof

5r = p /q

4 = p/5q

**3. r + s**

This is not necessarily irrational.

Proof

Assume s = -r, in this case, r + s = 0 - which is not irrational.

Since there exists at least two numbers such that r + s is not irrational, we can conclude that r + s is not necessarily irrational.

**4. rs**

This is not necessarily irrational.

Proof

Assume

s = 1/r

In this case r × s becomes 1 - which is not irrational. Since there exists at least two numbers such that rs is not irrational, we can conclude that rs is not necessarily irrational.

**5. √r**

This has to be irrational.

Proof

Assume √r is rational. This means, for any two non-zero integers p and q:

√r = p/q

r = p2/q2

But this means r can be expressed as a ratio of two integers. This contradicts with our premise that r is irrational. So, it is not possible that √r is rational given that r is rational.

**6. rs**

**12. Let m and n be integers. Prove that:**

**a) If m and n are even, then m + n is even**.

Since m is even, we can represent it as 2r where r is an integer. Similarly we can represent n as 2s.

m + n = 2r + 2s

m + n = 2(r + s)

From the above equation, we can see that m + n is divisible by

2. This means m has to be even.

**b) If m and n are even, then mn is divisible by 4.**

Since m is even, we can represent it as 2r where r is an integer. Similarly we can represent n as 2s.

mn = 2r × 2s

mn = 4(rs)

From the above equation, we can see that mn is divisible by 2.

**c) If m and n are odd, then m + n is even.**

Since m is odd, we can represent m as 2r + 1. Similarly, we can represent n as 2s + 1.

m + n = (2r + 1) + (2s + 1)

m + n = 2r + 2s + 2

m + n = 2(r + s + 1)

Clearly, m + n is divisible by 2. Hence it has to be even.

**d) If one of m,n is even and the other is odd, then m + n is odd.**

Assume m is even. It can be represented as 2r for any integer r. Since one of them is even and other odd, lets assume n is odd and represent it as 2s + 1 where s is an integer.

Then,

m + n = 2r + 2s + 1

m + n = 2(r + s) + 1

Given that m + n results in an even number incremented by 1, we can conlude that m + n is odd.

**e) If one of m,n is even and the other is odd, then mn is even.**

Assume m is even. It can be represented as 2r for any integer r. Since one of them is even and other odd, lets assume n is odd and represent it as 2s + 1 where s is an integer.

Then,

mn = 2r × (2s + 1)

From the above equation its clear that mn has a factor in 2. So, mn must be even.