

$$\begin{aligned}
I_n(x) &= \int \sin^n x dx = \int \sin^{n-1} \sin x dx = [u := \sin^{n-1} x, dv := \sin x dx, v = -\cos x] = -\cos x \sin^{n-1} x + \\
&+ (n-1) \int \cos^2 x \sin^{n-2} x dx = -\cos x \sin^{n-1} x + (n-1) \left(\int \sin^{n-2} x dx - \int \sin^n x dx \right) = \\
&= -\cos x \sin^{n-1} x + (n-1) (I_{n-2}(x) - I_n(x)) \\
I_n &= -\cos x \sin^{n-1} x + (n-1) (I_{n-2}(x) - I_n(x)) \\
I_n &= \frac{-\cos x \sin^{n-1} x + (n-1) I_{n-2}(x)}{n} \\
I_0 &= x \\
I_1 &= \cos x \\
I_2 &= \frac{-\cos x \sin x + I_0}{2} = \frac{-\cos x \sin x + x}{2} \\
\left(\frac{-\cos x \sin x + x}{2} \right)' &= \frac{1}{2} ((\sin 2x)' + 1) = \frac{1}{2} (-\cos 2x \cdot 2 + 1) = -\cos 2x + 1 = \sin^2 x
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{(x^2+1)^n} &= [u = \frac{1}{(x^2+1)^n}, dv = dx, du = -n \frac{1}{(x^2+1)^{n+1}} 2x dx] = \\
&= \frac{1}{(x^2+1)^n} + \int x n \frac{1}{(x^2+1)^{n+1}} 2x dx = \frac{1}{(x^2+1)^n} + \int n \frac{x^2+1-1}{(x^2+1)^{n+1}} 2x dx = \\
&= \frac{1}{(x^2+1)^n} + 2n (I_n - I_{n+1}) \\
I_{n+1} &= \frac{1}{2n} \left(\frac{x}{(x^2+1)^n} + (2n-1) I_n \right) \\
I_0 = x \quad I_1 &= \arctg x \quad I_2 = \frac{1}{2} \left(\frac{x}{x^2+1} + \arctg \right)
\end{aligned}$$

Проверка

$$\frac{1}{2} \left(\frac{x}{x^2+1} + \arctg \right)' = \frac{1}{2} \left(\frac{x^2+1-2x^2}{(x^2+1)^2} + \frac{1}{x^2+1} \right) = \frac{1}{(x^2+1)^2}$$

1 Интегралы простейших дробей

1.

$$\int \frac{1}{x-a} dx = \ln|x-a| + C$$

2. $n \neq 1$

$$\frac{dx}{(x-a)^n} = \frac{1}{(1-n)(x-a)^n} + C$$

3.

$$\frac{dx}{x^2+1} = \operatorname{arctg} x$$

4.

$$\frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2} \ln|x^2+1|$$

5.

$$\int \frac{dx}{(x^2+1)^n}$$

6.

$$\int \frac{xdx}{(x^2+1)^n} = \frac{1}{2} \int \frac{d(x^2+1)}{(x^2+1)^n} = \frac{1}{2} \frac{1}{(1-n)(x^2+1)^{n-1}}$$

$$\int \frac{Ex+F}{x^2+bx+c} dx; b^2-4c < 0$$

$$[\text{замена}] = \int \frac{At+B}{t^2+1}$$

$$\left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

$$\left(\frac{Ex+F}{\left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}}\right) = \frac{1}{c - \frac{b^2}{4}} \int \frac{Ex+F}{\frac{(x+\frac{b}{2})^2}{c - \frac{b^2}{4}} + 1} dx = \left[t = \frac{x + \frac{b}{2}}{\sqrt{c - \frac{b^2}{4}}}\right]$$

$$\int \frac{\tilde{E}t + \tilde{F}}{t^2+1} dt$$

$$\int \frac{3x+4}{x^2+2x+2} dx = \int \frac{3x+4}{(x+1)^2+1} dx = [x+1=t, dx=dt] = \int \frac{3(t-1)+4}{t^2+1} dt =$$

$$= \frac{3}{2} \ln|t^2+1| + \operatorname{arctg} t + C$$

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{t}{a}$$

2 Интегрирование рациональных дробей

$\frac{P_n(x)}{Q_m(x)}$ — рациональная дробь.

$$Q_m := \prod_{k=1}^N (x - a_k)^{r_k} \cdot \prod_{k=1}^M (x^2 + b_k x + c_k)^{s_k}$$

$$\frac{P_n(x)}{Q_m(x)} = \text{целая часть, если } n \geq m + \sum_{k=1}^N \sum_{j=1}^{r_k} \frac{A_{kj}}{(x - a_k)^j} + \sum_{k=1}^M \sum_{j=1}^{s_k} \frac{B_{kj}x + C_{kj}}{(x^2 + b_k x + c_k)^j}$$

$$\text{Выделим целую часть: } \frac{x^3 + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

Это метод неопределенных коэффициентов.

$$= \frac{A(x^2 - 5x + 6) + B(x^2 - 3x) + C(x^2 - 2x)}{x(x-2)(x-3)}$$

$$x^2 : A + B + C = 5$$

$$x : -5A - 3B - 2C = 6$$

$$1 : 6A = 1$$

$$A = \frac{1}{6}$$

$$B + C = \frac{29}{6}$$

$$-\frac{5}{6} - \frac{58}{6} - B = 6$$

$$-\frac{63}{6} + \frac{36}{6} = B$$

$$B = \frac{-27}{6}$$

$$C = \frac{56}{6}$$

$$\int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx = x + \frac{1}{6} \ln |x| - \frac{27}{6} \ln |x - 2| + \frac{56}{6} \ln |x - 3| + C$$

$$\int \frac{dx}{x^3 + 1} = \int \frac{dx}{(x + 1)(x^2 - x + 1)}$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1) + (Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$x^2 = 0 : A + B$$

$$x = 0 : -A + B + C$$

$$1 = 1 : A + C$$

$$A = -B \quad C = \frac{2}{3} \quad A = \frac{1}{3} \quad B = -\frac{1}{3}$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3} \int \frac{dx}{x+1} + \frac{1}{3} \int \frac{-x+2}{x^2-x+1} dx$$

1 способ

$$\int \frac{-x+2}{x^2-x+1} dx = \int \frac{-x+2}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx = [t := x - \frac{1}{2}] =$$

$$\int \frac{-t + \frac{3}{2}}{t^2 + \frac{3}{4}} dt = -\frac{1}{2} \ln |t^2 + \frac{3}{4}| + \frac{3}{2} \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t}{\sqrt{3}}$$

2 способ

$$\int \frac{-x+2}{x^2-x+1} dx = \int \frac{-\frac{1}{2}(2x-1) + \frac{3}{2}}{x^2-x+1} dx = -\frac{1}{2} \ln |x^2-x+1| + \frac{3}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}} =$$

$$= -\frac{1}{2} \ln |x^2-x+1| + \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$\int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2-2x^2} = \int \frac{dx}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}$$

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} = \frac{(Ax+B)(x^2+\sqrt{2}x+1) + (Cx+D)(x^2-\sqrt{2}x+1)}{x^4+1}$$

$$x^3 : 0 = A + C$$

$$x^2 : 0 = B + D + \sqrt{2}A - \sqrt{2}C$$

$$x : 0 = A + C + \sqrt{2}B - \sqrt{2}D$$

$$1 : 1 = B + D$$

$$B = D \quad B = 0.5 = D \quad A = C = \frac{1}{2\sqrt{2}}$$

$$\int \frac{\frac{1}{2\sqrt{2}}x + 0.5}{x^2 - \sqrt{2}x + 1} dx = \int \frac{\frac{1}{4\sqrt{2}}(2x - \sqrt{2}) + \frac{3}{4}}{x^2 - \sqrt{2}x + 1} dx =$$

$$= \frac{1}{4\sqrt{3}} \ln(x^2 + \sqrt{2}x + 1) + \int \frac{\frac{3}{4}}{x^2 - \sqrt{2}x + 1} dx = \int \frac{d(x - \frac{\sqrt{2}}{2})}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} = \sqrt{2} \operatorname{arctg}\left(\sqrt{\left(x - \frac{\sqrt{2}}{2}\right)}\right)$$

$$\int \frac{6x^5 + 6x + 1}{x^6 + 3x^2 + x + 8} dx = \ln |x^6 + 3x^2 + x + 8| + C$$

3 Универсальная тригонометрическая подстановка

$$\int \frac{dx}{2 \sin x - \cos x + 5}$$

$$\operatorname{tg} \frac{x}{2} =: t$$

$$x = 2 \operatorname{arctg} t$$

$$dx = \frac{2}{1+t^2} dt$$

$$\frac{1}{\cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = 1 + t^2$$

$$\sin x = 2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2} = \frac{2t}{1+t^2}$$