$$I_n(x) = \int \sin^n x dx = \int \sin^{n-1} \sin x dx = [u := \sin^{n-1} x, dv := \sin x dx, v = -\cos x] = -\cos x \sin^{n-1} x +$$

$$+(n-1) \int \cos^2 x \sin^{n-2} x dx = -\cos x \sin^{n-1} x + (n-1) \left(\int \sin^{n-2} x dx - \int \sin^n x dx \right) =$$

$$= -\cos x \sin^{n-1} x + (n-1) \left(I_{n-2}(x) - I_n(x) \right)$$

$$I_n = -\cos x \sin^{n-1} x + (n-1) \left(I_{n-2}(x) - I_n(x) \right)$$

$$I_n = \frac{-\cos x \sin^{n-1} x + (n-1) I_{n-2}(x)}{n}$$

$$I_0 = x$$

$$I_1 = \cos x$$

$$I_2 = \frac{-\cos x \sin x + I_0}{2} = \frac{-\cos x \sin x + x}{2}$$

$$\left(\frac{-\cos x \sin x + x}{2} \right)' = \frac{1}{2}((\sin 2x)' + 1) = \frac{1}{2}(-\cos 2x \cdot 2 + 1) = -\cos 2x + 1 = \sin^2 x$$

$$\int \frac{dx}{(x^2+1)^n} = [u = \frac{1}{(x^2+1)^n}, dv = dx, du = -n \frac{1}{(x^2+1)^{n+1}} 2x dx] =$$

$$= \frac{1}{(x^2+1)^n} + \int xn \frac{1}{(x^2+1)^{n+1}} 2x dx = \frac{1}{(x^2+1)^n} + \int n \frac{x^2+1-1}{(x^2+1)^{n+1}} 2dx =$$

$$= \frac{1}{(x^2+1)^n} + 2n \left(I_n - I_{n+1} \right)$$

$$I_{n+1} = \frac{1}{2n} \left(\frac{x}{(x^2+1)^n} + (2n-1) I_n \right)$$

$$I_0 = x \quad I_1 = \operatorname{arctg} x \quad I_2 = \frac{1}{2} \left(\frac{x}{x^2+1} + \operatorname{arctg} \right)$$

Проверка

$$\frac{1}{2} \left(\frac{x}{x^2+1} + \operatorname{arctg} \right)' = \frac{1}{2} \left(\frac{x^2+1-2x^2}{(x^2+1)^2} + \frac{1}{x^2+1} \right) = \frac{1}{(x^2+1)^2}$$

1 Интегралы простейших дробей

1.
$$\int \frac{1}{x-a}dx = \ln|x-a| + C$$

2. $n \neq 1$
$$\frac{dx}{(x-a)^n} = \frac{1}{(1-n)(x-a)^n} + C$$

3.
$$\frac{dx}{x^2+1} = \operatorname{arctg} x$$

4.
$$\frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln|x^2+1|$$

5.
$$\int \frac{dx}{(x^2+1)^n}$$

6.
$$\int \frac{xdx}{(x^2+1)^n} = \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^n} = \frac{1}{2}\frac{1}{(1-n)(x^2+1)^{n-1}}$$

$$\int \frac{Ex+F}{x^2+bx+c}dx; b^2-4c<0$$

$$[\text{замена}] = \int \frac{At+B}{t^2+1}$$

$$\left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

$$\left(\frac{Ex+F}{\left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}}\right) = \frac{1}{c-\frac{b^2}{4}}\int \frac{Ex+F}{\frac{(x+\frac{b}{2})^2}{c-\frac{b^2}{4}}+1}dx = \left[t = \frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4}}}\right]$$

$$\int \frac{\tilde{E}t+\tilde{F}}{t^2+1}dt$$

$$\int \frac{3x+4}{x^2+2x+2}dx = \int \frac{3x+4}{(x+1)^2+1}dx = [x+1=t, dx=dt] = \int \frac{3(t-1)+4}{t^2+1}dt =$$

$$= \frac{3}{2}\ln|t^2+1| + \operatorname{arctg} t + C$$

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{t}{a}$$

2 Интегрирование рациональных дробей

$\frac{P_n(x)}{Q_m(x)}$ — рациональная дробь.

$$Q_m := \prod_{k=1}^{N}(x-a_k)^{r_k} \cdot \prod_{k=1}^{M}(x^2+b_kx+c_k)^{s_k}$$

$$\frac{P_n(x)}{Q_m(x)} = \text{ целая часть, если } n \geq m + \sum_{k=1}^{N}\sum_{j=1}^{r_k}\frac{A_{kj}}{(x-a_k)^j} + \sum_{k=1}^{M}\sum_{j=1}^{s_k}\frac{B_{kj}x+C_{kj}}{(x^2+b_kx+c_k)^j}$$

$$\frac{x^3+1}{x^3-5x^2+6x}$$

Выделим целую часть: $\frac{x^3+1}{x^3-5x^2+6x} = 1+\frac{5x^2-6x+1}{x^3-5x^2+6x} = 1+\frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-3}$

Это метод неопределенных коэффициентов.

$$= \frac{A(x^2-5x+6)+B(x^2-3x)+C(x^2-2x)}{x(x-2)(x-3)}$$

$$x^2 : A+B+C = 5$$

$$x : -5A-3B-2C = 6$$

$$1 : 6A = 1$$

$$A = \frac{1}{6}$$

$$B+C = \frac{29}{6}$$

$$-\frac{5}{6}-\frac{58}{6}-B = 6$$

$$-\frac{63}{6}+\frac{36}{6} = B$$

$$B = \frac{-27}{6}$$

$$C = \frac{56}{6}$$

$$\int\frac{x^3+1}{x^3-5x^2+6x}dx = x+\frac{1}{6}\ln|x|-\frac{27}{6}\ln|x-2|+\frac{56}{6}\ln|x-3|+C$$

$$\int\frac{dx}{x^3+1} = \int\frac{dx}{(x+1)(x^2-x+1)}$$

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$x^2=0:A+B$$

$$x=0:-A+B+C$$

$$1=1:A+C$$

$$A=-B\quad C=\frac{2}{3}\quad A=\frac{1}{3}\quad B=-\frac{1}{3}$$

$$\int\frac{dx}{x^3+1}=\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{-x+2}{x^2-x+1}dx$$

1 способ

$$\int\frac{-x+2}{x^2-x+1}dx=\int\frac{-x+2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx=[t:=x-\frac{1}{2}]=$$

$$\int\frac{-t+\frac{3}{2}}{t^2+\frac{3}{4}}dx=-\frac{1}{2}\ln|t^2+\frac{3}{4}|+\frac{3}{2}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2t}{sqrt3}$$

2 способ

$$\int\frac{-x+2}{x^2-x+1}dx=\int\frac{-\frac{1}{2}(2x-1)+\frac{3}{2}}{x^2-x+1}dx=-\frac{1}{2}\ln|x^2-x+1|+\frac{3}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=-\frac{1}{2}\ln|x^2-x+1|+\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$\int\frac{dx}{x^4+1}=\int\frac{dx}{(x^2+1)^2-2x^2}=\int\frac{dx}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}$$

$$\frac{1}{x^4+1}=\frac{Ax+B}{x^2-\sqrt{2}x+1}+\frac{Cx+D}{x^2+\sqrt{2}x+1}=\frac{(Ax+B)(x^2+\sqrt{2}x+1)+(Cx+D)(x^2-\sqrt{2}x+1)}{x^4+1}$$

$$x^3:0=A+C$$

$$x^2:0=B+D+\sqrt{2}A-\sqrt{2}C$$

$$x:0=A+C+\sqrt{2}B-\sqrt{2}D$$

$$1:1=B+D$$

$$B=D\quad B=0.5=D\quad A=C=\frac{1}{2\sqrt{2}}$$

$$\int\frac{\frac{1}{2\sqrt{2}}x+0.5}{x^2-\sqrt{2}x+1}dx=\int\frac{\frac{1}{4\sqrt{2}}(2x-\sqrt{2})+\frac{3}{4}}{x^2-\sqrt{2}x+1}dx=$$

$$= \frac{1}{4\sqrt{3}} \ln\Big(x^2 + \sqrt{2}x + 1\Big) + \int \frac{\frac{3}{4}}{x^2 - \sqrt{2}x + 1} dx = \int \frac{d(x - \frac{\sqrt{2}}{2})}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} = \sqrt{2} \operatorname{arctg}(\sqrt{(}x - \frac{\sqrt{2}}{2}))$$

$$\int \frac{6x^5 + 6x + 1}{x^6 + 3x^2 + x + 8} dx = \ln|x^6 + 3x^2 + x + 8| + C$$

3 Универсальная тригонометрическая подстановка

$$\int \frac{dx}{2\sin x - \cos x + 5}$$

$$\operatorname{tg} \frac{x}{2} =: t$$

$$x = 2 \operatorname{arctg} t$$

$$dx = \frac{2}{1 + t^2} dt$$

$$\frac{1}{\cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = 1 + t^2$$

$$\sin x = 2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2} = \frac{2t}{1 + t^2}$$