Add the missing arithmetic operators ($+, -, \times, /$) and parentheses to the following expression to make it true:

$$3 \quad 1 \quad 3 \quad 6 \quad = \quad 8$$

$$(3+1)/3\times 6 \quad = \quad (3+1)/(3/6) \quad = \quad 8$$

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- Not easy to find, but easy to verify. (NP certificates)
- Could do exhaustive search (Arithmetic trees), but may be easier to “guess then check.”
- Think outside the box: think about fractions, not only integers.

You have a 5 litre jug and a 3 litre jug, and an unlimited supply of water, but no measuring cups. How would you come up with exactly 4 litre of water?

Action	A (3L Jug)	B (5L Jug)

You have a 5 litre jug and a 3 litre jug, and an unlimited supply of water, but no measuring cups. How would you come up with exactly 4 litre of water?

Action	A (3L Jug)	B (5L Jug)
	0	0

You have a 5 litre jug and a 3 litre jug, and an unlimited supply of water, but no measuring cups. How would you come up with exactly 4 litre of water?

Action	A (3L Jug)	B (5L Jug)
	0	0
Fill A	3	0

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Action	A (3L Jug)	B (5L Jug)
	0	0
Fill A	3	0
Empty A into B	0	3

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	0	0
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Empty A into B	0	3
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Action	A (3L Jug)	B (5L Jug)
	0	0
Fill A	3	0
Empty A into B	0	3
Fill A	3	3
Fill B from A (2L)	1	5

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Empty B	1	0

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Empty A into B	0	3
Fill A	3	3
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Empty B	1	0
Empty A into B	0	1

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Fill A	3	1
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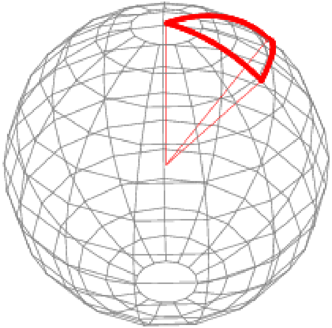
Clever Bear walked one mile due south. Then he changed direction and walked one mile due east. Then he turned again to the left and walked one mile due north. . . he was surprised to find himself exactly at the point he started from!
What is the colour of Clever Bear?

White (North pole), spherical triangle.



- Think outside the box.

There are 100 closed lockers in a hallway. A man begins by opening all one hundred lockers. Next, he closes every second locker. Then he goes to every third locker and closes it if it is open or opens it if it is closed. He continues like this until his 100th pass in the hallway, in which he only changes the state of locker number 100. How many lockers will be left open at the end?

10 lockers (Lockers $1^2, 2^2, 3^3, \ldots, 10^2$ because locker ℓ remains open iff it has an odd number of divisors, which only happens if ℓ is a perfect square).

```
lockers = [False for i in range(101)]

for run in range(1,101):
    for locker in range(run,101,run):
        lockers[ locker ] = not lockers[ locker ]

for i in range(101):
    if lockers[i]:
        print(i)
```

- Quick code helps find the pattern. Use maths to justify.

Little Alice has 10 pockets and £44 in £1 coins. She wants to put her coins in her pockets so distributed that each pocket contains a different number of pounds. Can she do so?

No.

Minimum is:

$$0 + 1 + 2 + \cdots + 9 = 10(0 + 9)/2 = 45 > 44.$$

- Sometimes we have to prove a solution does not exist.

We are next to a building of 100 floors. We have two identical eggs, and we are wondering what is the highest floor from which we can drop them without them breaking. How do we find out while minimizing the number of drops (in the worst case scenario). How about if we had 3 eggs? 4 eggs? etc. What is the optimal number of eggs to minimize the number of trials?

$10 + 9 = 19$.

Use "baby-step giant-step": use first egg at floors $10, 20, \ldots, 100$. Once it breaks at floor $10k$ we then know it will break at $10k - 9, \ldots, 10k - 1$. Use second egg 9 times at most

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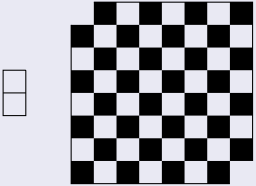
- Special cases, sub-problems, generalizations.

Below is an 8×8 chess board in which two diagonally opposite corners have been cut off.



You are given lots of dominoes, such that each domino can cover exactly two squares. Can you cover the entire board with dominoes? (No dominoes are allowed to be partly outside the board.) Can you prove your answer? (Show an example solution if this is possible, or show that it is impossible.)

No.

There are more black squares than white squares. (Each domino covers exactly one black and one white).

- Idea of invariant.