If there are any symbols or terminology you do not recognize then please let us know.

(1) Give the truth table for the following propositions

Expression	**Meaning**
$a \wedge b$	a and b
$a \vee b$	a or b
$a \oplus b$	a xor b
$\neg a$ (or $\bar{a}$)	not a
$a \implies b$	a implies b, or: if a then b
$a \iff b$	a and b are equivalent, or: "a if and only if b"

It is usual to apply these "bit-wise" to the bits of integers, e.g. $\texttt{0011} \oplus \texttt{0101} = \texttt{0110}$.

Solution

$\neg a$	a	b	$a \wedge b$	$a \vee b$	$a \oplus b$	$a \implies b$	$a \iff b$
1	0	0	0	0	0	1	1
1	0	1	0	1	1	1	0
0	1	0	0	1	1	0	0
0	1	1	1	1	0	1	1

(2) Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of **natural numbers**, and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of **integers**.

Consider the following set definitions

- $A = \{a \in \{1, 2, 3, 4\} \mid (a < 2) \vee (a > 3)\}$
- $B = \{a \in \mathbb{N} \mid a < 9\}$
- $C = \{a \in \mathbb{N} \mid a > 2 \wedge a < 7\}$
- $D = \{i \in \mathbb{Z} \mid i^2 \leq 9\}$

a) Give an explicit enumeration for each set, i.e. write down the elements in the form $\{x_1, x_2, \ldots\}$.

b) What is the cardinality of each set?

c) Which of these sets are subsets of at least one other set?

Solution

a)
- $A = \{1, 4\}$
- $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $C = \{3, 4, 5, 6\}$
- $D = \{-3, -2, -1, 0, 1, 2, 3\}$

b)
- $\#A = 2$ ($\#A$ is also denoted by $|A|$)
- $\#B = 8$
- $\#C = 4$
- $\#D = 7$

c) $A \subset B$ and $C \subset B$.

(3) If the set A is $\{1, 3, 4\}$ and the set B is $\{3, 5\}$, write down:

Expression	**Meaning**
$A \cup B$	union of A and B
$A \cap B$	intersection of A and B
$A - B$	A minus B
$A \times B$	Cartesian product of A and B: set of all possible pairs (a, b) where $a \in A$ and $b \in B$
2^B (or $\mathcal{P}(B)$)	power set of B: set of all subsets of B

Solution

- $A \cup B = \{1, 3, 4, 5\}$
- $A \cap B = \{3\}$
- $A - B = \{1, 4\}$
- $A \times B = \{(1, 3), (1, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$
- $2^B = \{\emptyset, \{3\}, \{5\}, \{3, 5\}\}$

(4) Draw the (undirected) graph $G = (V, E)$, where

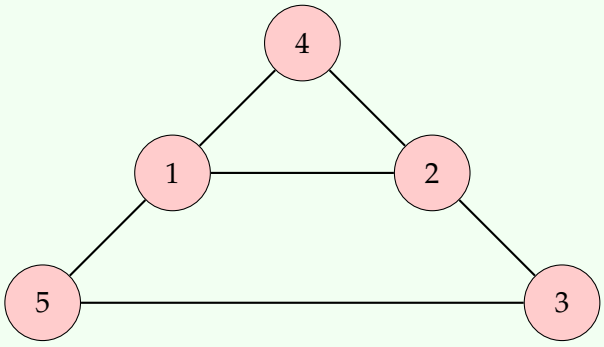
$$\begin{aligned} V &= \{1, 2, 3, 4, 5\} \\ E &= \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 5), (1, 5)\} \end{aligned}$$

a) Is the graph connected?

b) What about the graph $G' = (V', E')$, where $V' = \{1, 2, 3, 4\}$ and $E' = \{(1, 3), (2, 4)\}$?

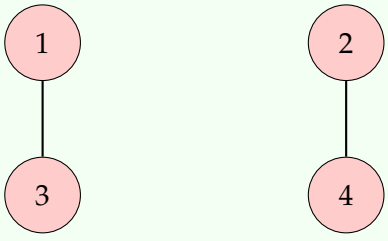
Solution

G:



G is connected.

G' (pronounced "G prime"):



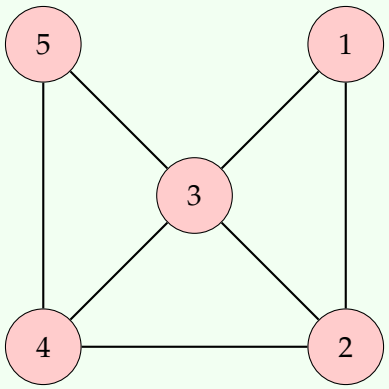
G' is not connected.

(5) Draw the graph $G = (V, E)$, where $V = \{1, \ldots, 5\}$ and

$$E = \{(a, b) \mid a, b \in V \wedge (a < b < a + 3)\}.$$

Solution

a	b
1	2, 3
2	3, 4
3	4, 5
4	5
5	



(6) Express the following expressions using O-notation

- $x + 5$
- 2016
- $543x + x^3 + 13$
- $2784x + 132 \times 1074$
- $x^{578} + 4685 + 2^x$
- $x^2 + x(\log x)^2 + 35$
- $x + x \log^2 x + 35$
- $2016^x + x^x + x!$
- $x^{86754} + x!$

Solution

$$\begin{aligned}
x + 5 &= O(x) \\
2016 &= O(1) \\
543x + x^3 + 13 &= O(x^3) \\
2784x + 132 \times 1074 &= O(x) \\
x^{578} + 4685 + 2^x &= O(2^x) \\
x^2 + x(\log x)^2 + 35 &= O(x^2) \\
x + x \log^2 x + 35 &= O(x \log^2 x) \\
2016^x + x^x + x! &= O(x^x) \\
x^{86754} + x! &= O(x!)
\end{aligned}$$