Deuteron smearing in CJ

Alberto Accardi, Ishara Fernando, Shujie Li

June 29th, 2020 (updated: March 4, 2024)

Abstract

This is a collection of notes documenting the smearing formalism implemented in the CJ `fitpack` package, with special attention to a comparison with the formalism utilized by AKP. We will also discuss in detail offshell, HT and TMC corrections and their interplay.

Contents

Chapter 1

On-shell F_2 smearing: CJ implementation

We discuss the implementation of F_2 smearing in the CJ code. For clarity, the more involved formulas for other structure functions will be discussed separately. This Chapter deals, in particular, with the case of an on-shell nucleon.

1.1 Integrand

The integrand of the smearing function:

$$f_{N/D} = f_{N/D}\big(y_D, p_T^2; \gamma\big) \ , \tag{1.1}$$

understood as a function of y_D and p_T^2, with γ a parameter. The latter is defined as

$$\gamma \overset{r.f.}{\equiv} \frac{q_z}{\nu} \overset{inv.}{=} \sqrt{1 + 4x_D^2 \frac{M_D^2}{Q^2}} = \sqrt{1 + 4x^2 \frac{m_N^2}{Q^2}} \ , \tag{1.2}$$

and the Lorentz invariant x and y momentum fractions are defined as

$$x_D \equiv Q^2 / P_D \cdot q \quad \text{and} \quad x \equiv \frac{M_D}{m_N} x_D \tag{1.3}$$

$$y_D \equiv \frac{p \cdot q}{P_D \cdot q} \quad \text{and} \quad y \equiv \frac{M_D}{m_N} y_D \tag{1.4}$$

at the Deuteron (left) and nucleon (right) levels. γ can then be interpreted as the Bjorken limit parameter: $\gamma \to 1$ as $Q^2 \to \infty$ with x_D (or x) fixed.

The following definitions and variables are common to the relativistic and non-relativistic cases. The names of the variables correspond to the names given in the code, with the exception of the spectator energy which is called here E_s, but is denoted by E_p in the code:

$$p_v^2 \equiv \vec{p}^{\,2} = p_T^2 + p_z^2 \tag{1.5}$$

$$p^2 \equiv p^\mu p_\mu = p_0^2 - p_v^2 \tag{1.6}$$

$$M_D \equiv 2m_N + \epsilon_D \tag{1.7}$$

$$\epsilon_D \equiv -2.224575 \text{ MeV} \tag{1.8}$$

$$m_N \equiv 938.91897 \text{ MeV} \tag{1.9}$$

The following quantities are also formally invariant in the two cases, but implicitly depend on the chosen (non)relativistic approximation for the spectator energy E_s:

$$p_0 \overset{r.f}{=} M_D - E_s \tag{1.10}$$

$$\epsilon \overset{r.f}{=} M_D - E_s - m_N \tag{1.11}$$

The first one is the active nucleon's energy, the second one is the active nucleon's "separation energy".

What are these?

We can now explicitly write the WBA and WBAREL formulas.

Non-relativistic case (WBA):

$$f(y_D, p_T^2; \gamma)^{WBA} = \underbrace{\theta(y_D)\,\theta(y_D^{\max} - y)}_{\text{kinematic limits}} \times \underbrace{\theta(p_v^{\text{cut}} - p_v)}_{\text{numerical cut}} \times \underbrace{\left[\frac{1}{4}\,\frac{\gamma^2 m_N + (1-y_D)M_D - E_s}{\gamma m_N^2}\right]}_{\text{Jacobian}} \times \underbrace{\left[1 + \frac{\gamma p_z}{m_N}\right]}_{\text{flux}} \times |\phi(\vec{p})|^2 \times \frac{1}{\gamma^2}\left[1 + \frac{(\gamma^2-1)}{\left(y_D\, M_D/m_N\right)^2}\left(1 + \frac{2\epsilon}{m_N} + \frac{p_v^2 - 3p_z^2}{2m_N^2}\right)\right] \tag{1.12}$$

where, in order of appearance,

$$y_D^{max} = \left(1 + \frac{\gamma^2}{2} + \frac{\epsilon_D}{m_N} - \frac{p_T^2}{2m_N^2}\right)\left(\frac{m_N}{M_D}\right) \tag{1.13}$$

$$E_s = m_N + \frac{\vec{p}_v^{\,2}}{2m_N} \tag{1.14}$$

$$p_z = m_N\left(\gamma - \sqrt{2\frac{M_D}{m_N}(y_D^{max} - y_D)}\right) \tag{1.15}$$

Relativistic case (WBAREL):

$$f(y_D, p_T^2, \gamma)^{WBAREL} = \underbrace{\theta(y_D)\,\theta(y_D^{\max} - y)}_{\text{kinematic limits}} \times \underbrace{\theta(p_v^{\text{cut}} - p_v)}_{\text{numerical cut}} \times \underbrace{\left[\frac{1}{4}\,\frac{\gamma M_D E_s}{(1-y_D)M_D + (\gamma^2-1)E_s}\right]}_{\text{Jacobian}} \times \underbrace{\left[1 + \frac{\gamma p_z}{m_N}\right]}_{\text{flux}} \times |\phi(\vec{p})|^2 \times \frac{1}{\gamma^2}\left[1 + \frac{(\gamma^2-1)}{\left(y_D\, M_D/m_N\right)^2}\left(\left(1 + \frac{\epsilon}{m_N}\right)^2 + \frac{p_v^2 - 3p_z^2}{2m_N^2}\right)\right] \tag{1.16}$$

where, in order of appearance,

$$y_D^{max} = \infty \qquad \left[y_D \leq 1 \text{ imposed when calculating } F_{2D}\right] \tag{1.17}$$

$$E_s = \sqrt{p_v^2 + m_N^2} \tag{1.18}$$

$$p_z = \begin{cases} \frac{1}{(\gamma^2-1)}\left[-M_D(1-y_D)\gamma + \sqrt{(1-y_D)^2 M_D^2 + (\gamma^2-1)(p_T^2 + m_N^2)}\right] & \text{for } \gamma > 1 \\ \frac{1}{2(1-y_0^D)M_D}\left[p_T^2 + m_N^2 - (1-y_0^D)^2 M_D^2\right] & \text{for } \gamma = 1 \end{cases} \tag{1.19}$$

Numerical cutoff

In all computations we set the following phase space cutoff:

$$p_v \leq p_v^{\rm cut} = 1000 \text{ MeV} \ . \tag{1.20}$$

(Note that $p_v^{\rm cut}$ is denoted as p_v^{max} in the code, but is only a numerical cutoff.)

Normalization conditions

The nucleon wave function ϕ is normalized such that $1 = \int dp_v p_v^2 |\phi|(p_v)|^2$. This normalization absorbs the angular integration in the definition of ϕ. Another common normalization, not used here, is $1 = \int d^3p|\psi|^2$, where $\psi = \phi/\sqrt{4\pi}$.

Baryon number conservation is imposed by multiplying the integrand by $N_{\rm baryon} = M/p_0$. This factor is already included in the "flux factor" defined below. (The code also allows for $N_{\rm baryon} = $ const., but this is not discussed in detail here.)

Notes:

- The non-relativistic Jacobian is obtained by retaining only the $O(\vec{p}^2/m_N^2)$ terms in the relativistic Jacobian. The same is true for the last lines in Eqs.(1.12) and (1.16). Note that ϵ/m_N is considered to be of the same order of $\vec{p}^2/m/N^2$.

- In fact, the last lines in Eqs.(1.12) and (1.16) are specific for the calculation of the F_{2D} structure function. Formulae for F_{1D}, F_{LD}, and F_{3D} will be discussed separately.

- The function $f(y_D, p_T^2)$ defined in the CJ code is related to the the $f(y, p_T^2)$ function discussed by Sergey K. in his notes [1] by

$$dy_D \, f^{\rm CJ}(y_D, p_T^2) = dy \, f^{\rm AKP}(y, p_T^2) \ , \tag{1.21}$$

  so that

$$f^{\rm CJ}(y_D) = \frac{M_D}{m_N} f^{\rm AKP}(y) \ . \tag{1.22}$$

- The non relativistic Jacobian coincides with AKP's. Indeed, one can explicitly verify that

$$\frac{\gamma m_N^2}{\gamma^2 m_N + (1-y_D)M_D - E_s} = \frac{m_N^2}{\sqrt{t^2 - p_T^2}} \ , \tag{1.23}$$

  where

$$t^2 = 2m_N M_D (y_D^{\rm max} - y_D) = 2m_N^2 \Big(\underbrace{1 + \frac{\gamma^2}{2} + \frac{\epsilon_D}{m_N}}_{=\bar{y}_{\rm max}} - y \Big) - p_T^2 \ . \tag{1.24}$$

  (For easier comparison to the AKP formula, in the last line we used $y = \frac{M_D}{m_N} y_D$.)

Coding notes:

- The $f(y_D, p_T^2)$ integrand is coded in the `DN` subroutine.

- **[Archaeology.]** In `fitpack` versions prior to June 15, 2020, the Jacobian was (wrongly) written

$$\text{Jacobian} = E_s \frac{m_N}{M_D} \frac{1}{2\gamma(1-y_0)} \tag{1.25}$$

  with the deuteron-level light-cone momentum fraction y_0 defined as

$$y_0 \equiv \frac{p^+}{P_D^+} = \frac{y_D}{1+\gamma} \left(1 + \sqrt{1 + (\gamma^2 - 1)\frac{(p^2 + p_T^2)}{M_D^2 y_D^2}} \right) \ . \tag{1.26}$$

  This was wrong for 2 reasons:

1. The $dy_D/dy = m/M_D$ change of variable Jacobian was forgotten. In practice, however, this mistake was compensated by the additional normalization numerically imposed on the $, p_T^2$-integrated smearing function $\mathcal{S}(y_D)$, see Section 1.2.
2. The formula was obtained by mixing and matching a calculation valid at any γ with one valid only at $\gamma = 1$.

1.2 Smearing function

The smearing function $\mathcal{S}(y_D;\gamma)$ is obtained by integration over d^2p_T:

$$\mathcal{S}(y_D;\gamma) = \mathcal{N}\int_0^{p_T^{\max}(y_D)} 2\,p_T\,dp_T\,f(y_D,p_T^2;\gamma)\ , \tag{1.27}$$

where

$$p_T^{\max}(y_D) = \begin{cases} \min\left[p_v^{\rm cut}, \sqrt{2}m_N\left(1+\dfrac{\gamma^2}{2}+\dfrac{\epsilon_D}{m_N}-\dfrac{M_D}{m_N}y_D\right)\right] & \text{for WBA} \\ p_T^{\rm cut} & \text{for WBAREL} \end{cases} \tag{1.28}$$

Normalization:

The normalization factor $\mathcal{N}$ is defined such that

$$1 = \int dy_D\,\mathcal{S}(y_D;\gamma=1)\ . \tag{1.29}$$

This condition is actually analytically guaranteed by the baryon number normalization condition that is included in the flux factor of the integrand, see Eqs. (1.12) and (1.16), so that $\mathcal{N}=1$. However, we also numerically impose Eq. (1.29):

- as a safety measure, and to compensate for the effects of $|\vec{p}|$ cutoffs in numerical representations of actual wave functions (that have long power-law tails).
- as a way to normalize the WJC-1 and WJC-1 wave functions, that contain a non-relativistic d-wave component;
- as a way to calculate $N_{\rm baryon} = \text{const}$ when that option is chosen.

Coding notes:

- The dp_T^2 integration (1.27) is performed by the `smearfn_1D` subroutine included in `smearfns.f`
- The normalization (1.29) is numerically imposed by the program `smeargrids.f`, that calculates and writes to files the (y,γ) grids for the requested smearing functions.
- [**Archaeology.**] The numerical calculation of $\mathcal{N}$ corrected, in `fitpack` version prior to 15 June 2020, for the mistaken 1/2 factor in the Jacobians of Eqs (1.12) and (1.16).

1.3 Deuteron F_2 structure function

The Deuteron F_2 structure function is then calculated as

$$F_{2D}(x,Q^2) = \int_{x_D}^{y_D^{\max}} dy_D \mathcal{S}(y_D;\gamma) F_{2N}\left(\frac{x_D}{y_D},Q^2\right)\ , \tag{1.30}$$

where, as a reminder, $x_D = (M_D/m_N)x$. The upper limit of integration is

$$y_D^{\max} = \begin{cases} \frac{m_N}{M_D}\left(1+\frac{\gamma^2}{2}+\frac{\epsilon_D}{m_N}\right) & \text{for WBA} \\ 1 & \text{for WBAREL} \end{cases} \tag{1.31}$$

Coding notes:

- The smearing integral (1.30) is implemented in the `smearF2` subroutine included in `nucl/deuteronSF15.f`
- The nonrelativistic $y_D^{\max}$ is not currently implemented.
- In the relativistic case with $\gamma > 1$, $y_D^{\max} = 1$ is utilized as a numerical approximation to ∞.

1.4 Kinematic constraints

Kinematic constraints arise from:

- 4-momentum and baryon number conservation;
- Defining "DIS" as the production of at least one pion.

In spectator approximation, 4 momentum conservation can be applied, respectively to the whole DIS on Deuteron diagram and to the top part only, *i.e.*, to the photon-nucleon interaction. On ethan can show that [2]

$$(m_N + m_\pi)^2 \leq W_N^2 \leq W_D^2 + m_N^2 - M_D^2 \ , \tag{1.32}$$

where $W_N^2 \equiv (p+q)^2$ and $W_D^2 \equiv (P_D + q)^2$). The l.h.s. inequality is the "pion threshold" condition discussed by Sergey Kulagin [1] and included in the AKP smearing calculations.

Rewriting this in terms of the invariant momentum fractions y, and neglecting ϵ_D compared to M_D for simplicity, one finds

$$x_D \left(1 + \frac{2m_N m_\pi + m_\pi^2}{Q^2}\right) \leq y_D + x_D \frac{m_N^2 - p^2}{Q^2} \leq 1 \ , \tag{1.33}$$

where the lower bound corresponds to the pion production threshold. As the nucleon is bound in the Deuteron, one also finds

$$p^2 \leq m_N^2 \ . \tag{1.34}$$

Eqns (1.33) and (1.34) provide limits for the integration over the (y, p^2) nucleon phase space, and are illustrated in Figure 1.1. For various values of x_D and Q^2. The lower bound in the (y, p_T^2) plane is illustrated in Figure 1 of Ref. [1].

Note:

- As one can see from Figure 1.1, the upper bound can be quite safely neglected, and the lower bound can be rewritten in terms of $x_N = x_D/y$:

$$x_N \leq \frac{1}{1 + \frac{2m_N m_\pi + m_\pi^2}{Q^2} + \frac{p^2 - m_N^2}{Q^2}} \equiv \widetilde{x}_\pi(p^2) \ , \tag{1.35}$$

  where the offshell pion threshold $\widetilde{x}_\pi$ reproduces the well-known result in the $p^2 \to m_N^2$ on-shell limit:

$$\widetilde{x}_\pi(p^2) \xrightarrow{p^2 \to m_N^2} \frac{1}{1 + \frac{2m_N m_\pi + m_\pi^2}{Q^2}} \equiv x_\pi \ . \tag{1.36}$$

- In the Bjorken limit, we obtain a simplified, p^2-independent limit for y_D

$$x_D \leq y_D \leq 1 \ . \tag{1.37}$$

Possible implementation in the 1D smearing formalism

The p^2- and x_D-dependent bounds (1.33) can be implemented in a straightforward way in a 3D smearing formalism. For a 1D smearing implementation one needs to resort to two simplifications:

- First, neglect the upper bound, and implement only the x_D-independent bound (1.36). (This is not absolutely necessary, but considering x_D-dependent bounds would add an x_D dependence to the smearing function $f = f(y, p_T^2; \gamma, x_D)$, and require a time-consuming 3D interpolation of the smearing function $\mathcal{S} = \mathcal{S}(y_D; \gamma; x_D)$.)
- Then, we could approximate $p^2 \approx m_N^2$ in Eq. (1.36) and implement

$$x_N \leq \frac{1}{1 + \frac{2m_N m_\pi + m_\pi^2}{Q^2}} \tag{1.38}$$

  directly in F_{2N}:

$$F_{2N}(x_N, Q^2) \rightsquigarrow F_{2N}(x_N, Q^2)\, \theta(x_\pi - x_N) \ . \tag{1.39}$$

Currently in CJ15: pion threshold is neglected

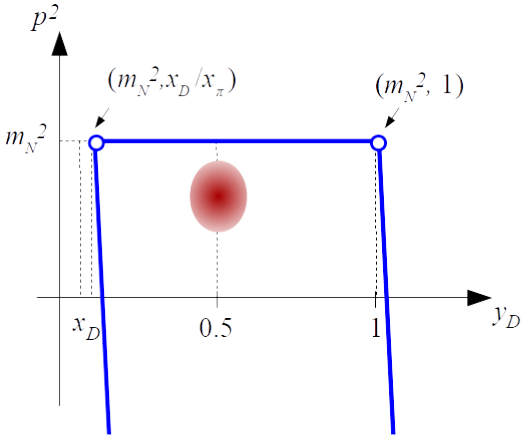
Currently, in the CJ code, the pion threshold is ignored and the bounds simplified to $x_D \leq y_D \leq 1$. The upper bound simplification is harmless also at finite Q^2 because $y_D \approx 1$ is far away from the region where the wave function has its dominant support. The lower bound simplification is only valid at sufficiently small x_D and sufficiently large Q^2 values.

The threshold $x_N \leq 1$ (as in a massless pion limit) is guaranteed for non mass corrected structure function by the fact that PDFs$\rightarrow 0$ as $x \rightarrow 1$, but is violated – as is well known – in most TMC prescriptions.

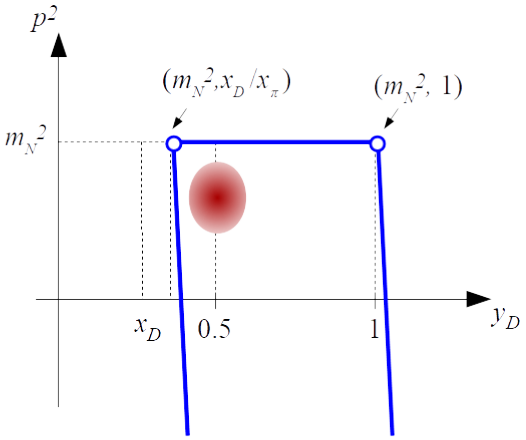
Since the aim in CJ is, thusfar, to perform global PDF fits away from the resonance region, *i.e.*, at $(p_D+q)^2 \gtrsim 3GeV^2$, we do not think the discussed approximate treatment of the thresholds is important.

Future CJ15: with approximate pion threshold

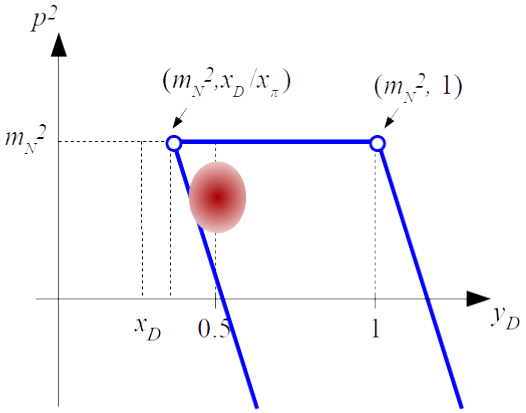
Nonetheless, we could think about the discussed approximate implementation of the pion threshold, and check the effect this has on teh PDF fits. It will also be important for future Quark-Hadron Duality studies, in order to more reliably extrapolate the DIS calculations to the resonance region.



(a) small x_D, large Q^2



(b) large x_D, large Q^2



(c) large x_D, small Q^2

Figure 1.1: Schematic representation of the (y, p^2) phase space boundaries for DIS on Deuteron at various values of x_D and Q^2. The red blob indicates the region where the TMD smearing function f has dominant support. **(a)** Small x_D, large Q^2 case. **(b)** Large x_D, large Q^2 case.**(c)** Large x_D, small Q^2 case; in this case, the pion threshold boundary (leftmost slanted line) cannot be neglected.

Chapter 2

Off-shell F_2 smearing

We discuss here the implementation of off-shell corrections for DIS on bound nucleons. Note that in CJ, off-shell effets are implemented at the PDF level. This has teh advantage to be compatible with off-shell corrections for other obseravbles, *e.g.*, Drell-Yan cross sections in $p + d$ scattering. We will discuss the connection to a DIS structure-function-level implementation.

2.1 Off-shell PDFs

Let's recall the factorized formula for an on-shell F_2 structure function:

$$F_{2N}(x_N, Q^2) = \int_{x_N}^1 \frac{dz}{z} \sum_i C_2^i\Big(\frac{x_N}{z}, \frac{Q^2}{\mu^2}\Big)\phi_i(z, \mu^2) \tag{2.1}$$

$$= \int_{x_N}^1 \frac{dz}{z} \sum_i C_2^i\Big(\frac{x_N}{z}\Big)\phi_i(z, Q^2) \ , \tag{2.2}$$

where i is a parton flavor index, and $C_2^i(x, Q^2/\mu^2)$ is the hard scattering coefficient for the F_2 structure function, which depends on the parton's light-cone fractional momentum x, and a factorization scale μ^2. The parton distribution $\phi_1(z, \mu^2)$ depends on μ^2 through DGLAP evolution. In DIS, one typically chooses typically chosen $\mu^2 = Q^2$, which is the choice adopted in the second line and throughout the rest of this report.

In CJ, we assume that the off-shell deformation of a nucleon's structure functions actually happens because of an off-shell deformation of its PDFs:

$$\phi_i(x, Q^2) \ \rightsquigarrow \ \widetilde{\phi}_i(x, Q^2, p^2) = \phi_i(x, Q^2)\Big(1 + \delta\phi_i(x, Q^2)\,\frac{p^2 - m_N^2}{m_N^2}\Big) + \mathcal{O}\Big(\frac{p^2 - m_N^2}{m_N^2}\Big)^2 \ . \tag{2.3}$$

Here and thereafter, we use a tilde above a function to denote it is an offhell function and explicitly depends on p^2, and satisfies

$$\widetilde{\phi}(x, Q^2, p^2) \xrightarrow{p^2 \to m_N^2} \phi(x, Q^2) \ . \tag{2.4}$$

The p^2 dependence is unknown, and we treat it in an *offshell expansion* in powers of the nucleon virtuality $v = (p^2 - m_N^2)/m_N^2$, with the PDF-level off-shell function $\delta\phi$ defined as

$$\delta\phi_i(x, Q^2) \equiv \frac{m_N^2}{\phi(x, Q^2)} \frac{d}{dp^2} \widetilde{\phi}(x, Q^2, p^2)\Big|_{p^2 = m_N^2} \tag{2.5}$$

$$= \frac{d}{d\log p^2} \log \widetilde{\phi}(x, Q^2, p^2)\Big|_{p^2 = m_N^2} \tag{2.6}$$

and treated as an unknown non-perturbative function. Note that the only dependence of $\widetilde{\phi}$ on p^2 comes from its functional dependence in the third argument, and therefore $d_{p^2}\widetilde{\phi} = \partial_{p^2}\widetilde{\phi}$. Care needs to be taken when we will discuss TMCs, since additional sources of p^2 dependence will be introduced.

In the CJ15 fit [3], we take 2 approximations for the PDF-level off0shell function $\delta\phi$:

- Since $\delta\phi_i$ are defined as a ratio of PDFs, the Q^2 scale evolution approximately cancels, and we can approxiamate

$$\delta\phi_i(x,Q^2) \approx \delta\phi_i(x) \ . \tag{2.7}$$

  With this choice, the offshell $\widetilde{\phi}_i$ PDFs obey the same DGLAP evolution equations as the on-shell ones.

- We furthermore assume flavor independence – for simplicity, and because we do not really have observables that would be allow flavor separation):

$$\delta\phi_i = \delta\phi \quad \forall i \ . \tag{2.8}$$

In the CJ12 fit [4] we actually used a model to a priori determine the flavor dependence, and only fitted the strength of the effect. In CJ15, instead, we fitted the shape of the flavor-independent off-shell function:

$$\delta\phi_{CJ15}(x) = N(x-x_0)(x-x_1)(1+x_0-x) \tag{2.9}$$

with x_1 fixed by the valence quark normalization condition [3]. In later fits [5], we found this parametrization to be overconstrained, and we considered a generic polynomial

$$\delta\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{2.10}$$

In practice, we found that good fits can only be obtained with up to third degree polynomials. By comparing fitted second and third order polynomials one can actually see that the data looses constraining power at $x \gtrsim 0.6$.

2.2 Off-shell nucleon structure function

The offshell $\widetilde{\phi}_i$ PDFs are then used in the factorized formula (2.2) to obtain an *off-hsell structure function*:

$$\widetilde{F}_{2N}(x_N,Q^2,p^2) \equiv \int_{x_N}^1 \frac{dz}{z} \sum_i C_2^i\Big(\frac{x_N}{z}\Big)\widetilde{\phi}_i(z,Q^2,p^2) \tag{2.11}$$

$$= F_{2N}(x_N,Q^2)\Big[1+\delta f(x_N,Q^2)\,\frac{p^2-m_N^2}{m_N^2}\Big] \ , \tag{2.12}$$

where

$$\delta f(x_N,Q^2) \equiv \frac{\int_{x_N}^1 \frac{dz}{z}\left[\sum_i C_2^i\big(\frac{x_N}{z}\big)\phi_i(z,Q^2)\right]\,\delta\phi(z)}{\int_{x_N}^1 \frac{dz}{z}\left[\sum_i C_2^i\big(\frac{x_N}{z}\big)\phi_i(z,Q^2)\right]} \ . \tag{2.13}$$

Note that:

- The Deuteron-level δf off-shell function is formally analogous to the parton-level $\delta\phi$ in the sense that

$$\delta f(x,Q^2) = \frac{d}{dp^2}\log\widetilde{F}(x,Q^2,p^2)\Big|_{p^2=m_N^2} \tag{2.14}$$

- The Q^2 dependence largely cancels between the numerator and the denominator, so that one can expect

$$\delta f(x_N,Q^2) \approx \delta f(x_N) \ ; \tag{2.15}$$

- At LO in α_s, the hard scattering coefficient is proportional to a delta-function, $C_2^i(z) = \delta(z-1)$, and we find:

$$\delta f(x_N)\big|_{LO} = \delta\phi(x_N) \ . \tag{2.16}$$

  We can therefore expect that the NLO deformations will not qualitatively alter this, so that

$$\delta f(x_N)\big|_{NLO} \approx \delta\phi(x_N) \ . \tag{2.17}$$

  Nonetheless the correspondence is not one to one.

- In the CJ15 paper [3] and in the recent AKP/CJ studies [5] we have in fact plotted what we here call $\delta\phi$ instead of δf.

2.3 AKP: 3D, structure function-level offshell smearing

In AKP [6, 7], the offshell expansion is performed directly at the structure function level, see Eq. (2.12), and δf is parametrized instead of $\delta\phi$:

$$\delta f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots \tag{2.18}$$

The off-shell structure function (2.12) is then included directly in the 3D convolution formula:

$$F_{2D}(x_D, Q^2) = \int dy_D dp_T^2 \, f(y_D, p_T^2; \gamma) \, \widetilde{F}_{2N}(x_N, Q^2, p^2) \; . \tag{2.19}$$

Fits (with also HT and TMCs included) have been tried with up to a third order polynomial, but the final result obtained with a second order polynomial [7]. **[Sergey: please confirm I understood well this.]**

2.4 CJ: 1D off-shell smearing function

The 3D smearing integral (2.19) can be explicitly spearted into an on-shell and off-shell contribution:

$$\begin{aligned} F_{2D}(x_D, Q^2) = & \int dy_D dp_T^2 \, f(y_D, p_T^2; \gamma) \, F_{2N}(x_N, Q^2) \\ & + \int dy_D dp_T^2 \, \frac{p^2 - m_N^2}{m_N^2} \, f(y_D, p_T^2; \gamma) \, F_{2N}(x_N, Q^2) \delta f(x_N) \; . \end{aligned} \tag{2.20}$$

Performing the dp_T^2 integrals, we then obtain a 1D convolution formula in terms of 2 spectral functions:

$$F_{2D}(x_D, Q^2) = \int dy_D \, \mathcal{S}(y_D; \gamma) \, F_{2N}\Big(\frac{x_D}{y_D}, Q^2\Big) + \int dy_D \, \mathcal{S}^{(1)}(y_D; \gamma) \, F_{2N}\Big(\frac{x_D}{y_D}, Q^2\Big) \, \delta f\Big(\frac{x_D}{y_D}\Big) \; , \tag{2.21}$$

where

$$\mathcal{S}(y_D; \gamma) \equiv \int dp_T^2 \, f(y_D, p_T^2; \gamma) \tag{2.22}$$

as for the on-shell case discussed in Chapter 1, see Eq. (1.27), and

$$\mathcal{S}^{(1)}(y_D; \gamma) \equiv \int dp_T^2 \, \frac{p^2 - m_N^2}{m_N^2} \, f(y_D, p_T^2; \gamma) \; . \tag{2.23}$$

Note that:

- These 2 functions can be considered, respectively, the 0th and 1st moments of f in the virtuality $v = (p^2 - m_N^2)/m_N^2$ variable; in other words, $\mathcal{S}^{(1)} = \langle v \rangle \mathcal{S}$, with the average $\langle v \rangle$ a function of y_D and γ defined by:

$$\langle v \rangle(y_D; \gamma) \equiv \frac{\mathcal{S}^{(1)}(y_D; \gamma)}{\mathcal{S}(y_D; \gamma)} \; . \tag{2.24}$$

- In the on-shell nucleon approximation, $\Delta\omega = 0$, so that $\mathcal{S}^{(1)} = 0$ and one recovers the on-shell smearing formula (1.30).

2.4.1 Coding notes

Offshell PDFs

The offshell PDF is calculated in the `fsupdf` routine from `theory/altpfit15.f`. The routine returns the on-shell PDF, or it's off-shell piece depending on the value of the onoff flag that can be retrieved calling `get_offshell_on(onoff)`:

$$\begin{cases} \phi(x, Q^2) & \text{if } \texttt{onoff} = \texttt{false} \\ \phi(x, Q^2)\delta\phi(x) & \text{if } \texttt{onoff} = \texttt{true} \end{cases} \tag{2.25}$$

Deuteron structure function with offshell corrections

The Deuteron F_2 and other structure functions are calculated in the `smearF2` routine found in `nucl/deuteronSF15.f`.

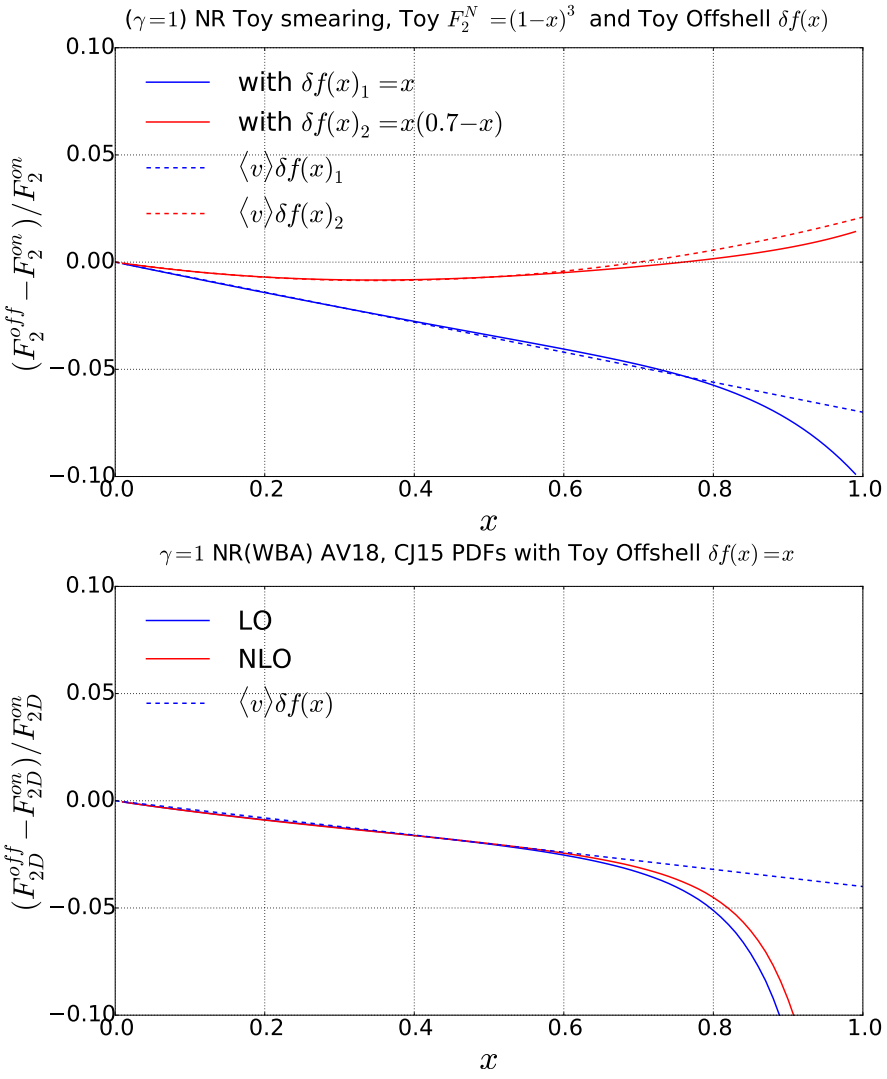


Figure 2.1: Off-shell correction to F_{2D} compared to the approximate $\langle v \rangle \delta f(x)$ with different conditions.

Chapter 3

Adding TMCs

We will discuss Target Mass Correction, with a focus on the models implemented in the CJ and AKP fits, and how these implementations affect the extraction of the offshels function. As in previous chapters we confine our discussion to teh F_2 structure function.

3.1 TMCs: general formalism

Target Mass Corrections for on-shell nucleons come in a variety of prescriptions [8], that have been reviewed side by side in [9]. Here we set up a general notation that allows us to treat these together for as long as possible.

The original TMC formulations are for on-shell nucleons. Here we consider **a simple generalization to off-shell nucleons consisting in replacing m_N^2 with p^2 everywhere**. Given an off-shell structure function $\widetilde{F}_N = \widetilde{F}_N(x_N, Q^2, p^2)$, see Chapter 2, we obtain the Target-Mass-Corrected structure function by applying a TMC operator $\mathcal{T}$ (that will be explicitly defined in Sections 3.1.2 and 3.1.3) as follows:

$$\widetilde{F}^{\rm TMC} \equiv \mathcal{T}\widetilde{F}_N\big(x_N, Q^2, p^2; \widetilde{\gamma}(p^2)\big) \ , \tag{3.1}$$

$$\equiv \mathcal{T}\widetilde{F}_N\big(\widetilde{\xi}_N(p^2), Q^2, p^2; \widetilde{\gamma}(p^2)\big) \tag{3.2}$$

where

$$\widetilde{\gamma} \overset{\text{r.f.}}{\equiv} \frac{q_z}{p_z} = \sqrt{1 + 4x_N^2 p^2/Q^2} \tag{3.3}$$

$$\widetilde{\xi}_N = \frac{2x_N}{1+\widetilde{\gamma}} \ . \tag{3.4}$$

Note:

- As usual, the tilde sign indicates off-shell variables depending on p^2, that tend to their on-shell counterpart as $p^2 \to m_N^2$: $\widetilde{\gamma} \to \gamma$,and $\widetilde{\xi}_N \to \xi_N$.
- All sources of p^2 dependence in Eqs. (3.2) and (3.1) are explitly marked; an explicit dependence on p^2 originating from the uncorrected $\widetilde{F}_N$ of Eq. (2.12), and an implicit dependence from $\widetilde{\xi}_N = \widetilde{\xi}_N(p^2)$ and $\widetilde{\gamma} = \widetilde{\gamma}(p^2)$.

In Eqs. (3.2) and (3.1), We have explicitly written the variables that the TMC corrected $F_N^{\rm TMC}$ depends on. In particular, the $\widetilde{\gamma}$ variable controls the limit in which TMCs can be neglected:

$$\mathcal{T}\widetilde{F}_N \xrightarrow{\widetilde{\gamma}\to 1} F_N \ . \tag{3.5}$$

As we will exploit later, $\widetilde{\gamma} \to 1$ can also be used to control the limit in which $Q \to \infty$ at fixed ξ_N (which we will call "*Nachtmann limit*"), or at fixed x_N (the well-known Bjorken limit). The Nachtmann limit will be instrumental in comparing the Georgi-Politzer and Ellis-Furmanski-Poetronzio TMC prescriptions. Given Eq. 3.4, there is, however, arbitrariness in chosing x_N or $\widetilde{\xi}_N$ to indicate the dependence of $\top\widetilde{F}_N$ on the invariant photon momentum fraction. Correspondingly, we have indicated two possibilities in Eqs. (3.2) and (3.1). One

can also use a mixed representation, as in some of the prescriptions for $\mathcal{T}$ to be discussed shortly. It is a matter of convenience what representation to chose - for example, Eq. (3.2) is best suited to relate the OPE and EFP prescriptions, but reducing the number of variables implicitly dependent on p^2 will be more convenient when studying the variation of $\mathcal{T}\widetilde{F}_N$ with the nucleon's offshellness.

3.1.1 Overview of TMC presriptions

As an overview, the TMC prescriptions considered in [9] and discussed in some detail in the next two subsections, are:

- ξ-scaling (ξ), also known as Aivazis-Olness-Tung (AOT);
- Accardi-Qiu (AQ) – a variant of $\xi - scaling$ that respects the $x_N \leq 1$ threshold;
- **Georgi-Politzer (OPE)** – derived in the OPE formalism, suffers from "threshold problem";
- Ellis-Furmanski-Petronzio (EFP) – a diagrammatic version of Georgi-Politzer's prescription, equivalent to OPE up to $O(\widetilde{\gamma}^2 - 1)$ in the Nachtmann limit (see later);
- **Approximated Georgi-Politzer ([OPE])** – simple, but precise, ξ-scaling-like approximation to OPE;
- Kulagin-Petti ($1/Q^2$) – power expansion of OPE up to ($1/Q^2$), that respects the $x_N \leq 1$ threshold.

The prescriptions in boldface, are utilized as default by the CJ15 ([OPE]) and AKP (OPE) fits. Some of these essentially only perform a rescaling of the massless $\widetilde{F}_N$, others are integral operators. Most of the original formulas depend on both x_N and $\widetilde{\xi}_N$, some only on $\widetilde{\xi}_N$, some only on x_N (but, as discussed above, one can in fact trade the one for the other at will):

	rescaling	integral	$\widetilde{\xi}_N$	x_N	CJ	AKP
ξ	✓		✓		✓	
AQ	✓		✓	✓	✓	
OPE		✓	✓	✓	(✓)	default
EFP		✓	✓	✓	(✓)	
[OPE]	✓		✓		default	
$1/Q^2$		✓		✓	(✓)	

Table 3.1: Rescaling operators overview, and availability in the CJ and AKP codes (the latter to verify with them). Notes: (✓) Code is available in a different (and old) `fitpack` branch: it can be resurrected and included in the main branch if deemed necessary or worthwhile.

3.1.2 Rescaling operators

These have similar structure and thier properties can be discussed together:

$$\mathcal{T}^{sc} = \left(\mathcal{T}^{\xi}, \mathcal{T}^{\mathrm{AQ}}, \mathcal{T}^{[\mathrm{OPE}]}\right) , \tag{3.6}$$

where [9]

$$\mathcal{T}^{\xi}\widetilde{F}_{2N} = \frac{1+\widetilde{\gamma}}{2\widetilde{\gamma}^2}\widetilde{F}_{2N}(\widetilde{\xi}_N, Q^2, p^2) \tag{3.7}$$

$$\mathcal{T}^{\mathrm{AQ}}\widetilde{F}_{2N} = \frac{1+\widetilde{\gamma}}{2\widetilde{\gamma}^2}\int_{x_N}^{1}\frac{dz}{z}\sum_i C_2^i(z)\phi_i\Big(\frac{\widetilde{\xi}_N}{z}, Q^2\Big) \tag{3.8}$$

$$\mathcal{T}^{[\mathrm{OPE}]}\widetilde{F}_{2N} = \frac{(1+\widetilde{\gamma})^2}{2\widetilde{\gamma}^3}\Big(1 + \frac{3(\widetilde{\gamma}^2-1)}{2\widetilde{\gamma}(1+\widetilde{\gamma})}(1-\widetilde{\xi}_N)^3\Big)\widetilde{F}_{2N}(\widetilde{\xi}_N, Q^2, p^2) \ . \tag{3.9}$$

- Equation 61 from arxiv:0709.1775

$$\frac{F_{2N}^{TMC}(x,Q^2)}{F_{2N}^{(0)}(\xi)} = \frac{(1+\gamma)^2}{4\gamma^3}\left(1+\frac{12\mu x^2(1+\gamma-2x)^2}{\gamma(1+\gamma)^3}\right) = \frac{(1+\gamma)^2}{4\gamma^3}\left(1+\frac{3(\gamma^2-1)(1+\gamma-2x)^2}{\gamma(1+\gamma)^3}\right) \tag{3.10}$$

  where, $\mu = M_N^2/Q^2 = \frac{(\gamma^1-1)}{4x^2}$

- Equation 9a from arxiv:1108.4734
  (this equation is different by a factor of 2 compare to Schienbein et al)

$$\frac{F_{2N}^{TMC}(x,Q^2)}{F_{2N}^{(0)}(\xi)} = \frac{(1+\gamma)^2}{2\gamma^3}\left(1+\frac{3(\gamma^2-1)(1+\gamma-2x)^2}{\gamma(1+\gamma)^3}\right) \tag{3.11}$$

- Equation 3.9 from our recent notes
  (this equation is very different compare to Schienbein et al)

$$\frac{F_{2N}^{TMC}(x,Q^2)}{F_{2N}^{(0)}(\xi)} = \frac{(1+\gamma)^2}{2\gamma^3}\left(1+\frac{3(\gamma^2-1)(1+\gamma-2x)^3}{2\gamma(1+\gamma)^4}\right) \tag{3.12}$$

- Modified version (from Alberto)

$$\frac{F_{2N}^{TMC}(x,Q^2)}{F_{2N}^{(0)}(\xi)} = \frac{(1+\gamma)^2}{4\gamma^3}\left[1+3\frac{(\gamma-1)}{\gamma}(1-\xi)^2\right] \tag{3.13}$$

It is not difficult to verify the following properties:

(a) $\mathcal{T}^{sc}$ are linear operators

(b) $\partial_{p^2}\mathcal{T}^{sc} = \mathcal{T}^{sc}\partial_{p^2}$

(c) $\partial_{\widetilde{\xi}_N}\mathcal{T}^{sc} = \mathcal{T}^{sc}\partial_{x_N}$

Property (c) for the case of AQ corrections is actually a trickier than the others, but not overmuch.

3.1.3 Integral operators

The integral operators are

$$\mathcal{T}^{int} = \left(\mathcal{T}^{\rm OPE}, \mathcal{T}^{\rm EFP}, \mathcal{T}^{1/Q^2}\right) , \tag{3.14}$$

and are actually a combination of scaling and integrals of $\widetilde{F}_N$ [9]:

$$\mathcal{T}^{\rm OPE}\widetilde{F}_{2N} = \frac{(1+\widetilde{\gamma})^2}{2\widetilde{\gamma}^3}\widetilde{F}_{2N}(\widetilde{\xi}_N,Q^2,p^2) + \frac{3x_N(\widetilde{\gamma}^2-1)}{2\widetilde{\gamma}^4}\tilde{h}_2(\widetilde{\xi}_N,Q^2,p^2) + \frac{3(\widetilde{\gamma}^2-1)^2}{4\widetilde{\gamma}^5}\tilde{g}_2(\widetilde{\xi}_N,Q^2,p^2) \tag{3.15}$$

$$\mathcal{T}^{\rm EFP}\widetilde{F}_{2N} = \frac{1}{\widetilde{\gamma}^2}\widetilde{F}_{2N}(\widetilde{\xi}_N,Q^2,p^2) + \frac{3\widetilde{\xi}_N(\widetilde{\gamma}^2-1)}{\widetilde{\gamma}^2(1+\widetilde{\gamma})}\tilde{h}_2(\widetilde{\xi}_N,Q^2,p^2) \tag{3.16}$$

$$\mathcal{T}^{1/Q^2}\widetilde{F}_{2N} = (2-\widetilde{\gamma}^2)\widetilde{F}_{2N}(x_N,Q^2,p^2) - \frac{1}{4}(\widetilde{\gamma}^2-1)\left[x\partial_{\widetilde{\xi}_N}\widetilde{F}_2(\widetilde{\xi}_N,Q^2,p^2)\big|_{\widetilde{\xi}_N=x_N} - 6x_N\tilde{h}_2(x_N,Q^2,p^2)\right] , \tag{3.17}$$

where

$$\tilde{h}_2(\xi,Q^2,p^2) = \int_\xi^1 \frac{du}{u^2}\widetilde{F}_2(u,Q^2,p^2) \tag{3.18}$$

$$\tilde{g}_2(\xi,Q^2,p^2) = \int_\xi^1 du\frac{dv}{v^2}\widetilde{F}_2(v,Q^2,p^2) = \int_\xi^1 \frac{du}{u^2}(u-\xi)\widetilde{F}_2(u,Q^2,p^2) \tag{3.19}$$

These operators also satisfy the first two properties of the scaling operators:

(d) $\mathcal{T}^{int}$ are linear operators

(e) $\partial_{p^2}\mathcal{T}^{int} = \mathcal{T}^{int}\partial_{p^2}$

It is interesting to note that fully writing $\mathcal{T}^{\rm OPE}$ fully in terms of $\widetilde{\xi}_N$, and using

$$\widetilde{\gamma} = 1 + \frac{1}{2}\Big(4x_N^2\frac{p^2}{m_N^2}\Big) - \frac{1}{4}\Big(4x_N^2\frac{p^2}{m_N^2}\Big) + \dots \tag{3.20}$$

$$= 1 + \frac{1}{2}\big(\widetilde{\gamma}^2-1\big) - \frac{1}{4}\big(\widetilde{\gamma}^2-1\big)^2 + \dots \ , \tag{3.21}$$

one can show that

$$\mathcal{T}^{\rm OPE}\widetilde{F}_{2N} = \mathcal{T}^{\rm EFP}\widetilde{F}_{2N} + O\big(\widetilde{\gamma}^2-1\big)^2 \ . \tag{3.22}$$

In other words, the EFP prescription reproduces the first 2 terms in the expansion of the OPE formula around the Nachtmann limit $\gamma = 1$. This property, hinted at in Ref. [9] and clarified here, is not obvious when making a "Bjorken limit" expansion in powers of $1/Q^2$ at fixed x_N. (Going to higher orders would require an analysis of 6-legs diagrams in DIS, which to my knowledge has not been attempted by anybody, yet, although it would be nice to see this if this would push the validity of Eq. (3.22) to $O(\widetilde{\gamma}^2-1)^3$.)

3.2 Combining TMCs and offshell corrections

Combining TMC and offshell corrections is straightforward in a 3D convolutions formalism:

$$F_{2D}^{\rm TMC}(x_D,Q^2) = \int dy_D dp_T^2\, f(y_D,p_T^2;\gamma)\,\mathcal{T}\widetilde{F}_{2N}(\xi_N,Q^2,p^2) \ . \tag{3.23}$$

This is what would be done, *e.g.* in AKP fits with HT switched off. The CJ fits adopt instead a 1D convolution formalism, in which the whole nucleon structure function is expanded to first order in the nucleon's virtuality, and then the first order coefficient is smeared separately from the on-shell structure function, see Eq. (2.21). With TMCs switched on, one needs to expand the mass corrected structure function:

$$\mathcal{T}\widetilde{F}_{2N}(x_N,Q^2,p^2;\widetilde{\gamma}(p^2)) = \mathcal{T}F_{2N}(x_N,Q^2;\gamma)\Big[1+\delta g(x_N,Q^2;\gamma)\,v\Big] + O(v^2) \tag{3.24}$$

where

$$\delta g(x_N,Q^2;\gamma) = \frac{m_N^2}{\mathcal{T}F_{2N}\big(x_N,Q^2;\gamma\big)}\frac{d}{dp^2}\mathcal{T}\widetilde{F}_{2N}\big(x_N,Q^2,p^2;\widetilde{\gamma}(p^2)\big)\Big|_{p^2=m_N^2} \tag{3.25}$$

Note that, for simplicity of formalism, we have used the x_N representation (3.1) for the mass corrected function, and that all the implicit p^2 dependence is absorbed into $\widetilde{\gamma} = \widetilde{\gamma}(p^2)$. The total p^2 derivative can be calculated explicitly using the properties of the TMC operators studied in the previous Section. Suppressing the function variables for notational compactness, we find:

$$\begin{aligned}\frac{d}{dp^2}\widetilde{F} &= \partial_{p^2}\mathcal{T}\widetilde{F} + \frac{d\widetilde{\gamma}}{dp^2}\,\partial_{\widetilde{\gamma}}\mathcal{T}\widetilde{F} \\ &= \mathcal{T}\big[\partial_p^2\widetilde{F}\big] + \frac{d\widetilde{\gamma}}{dp^2}\,\partial_{\widetilde{\gamma}}\mathcal{T}\widetilde{F} \\ &= \frac{1}{m_N^2}\mathcal{T}\big[F\delta f\big] + \frac{1}{p^2}\frac{\widetilde{\gamma}^2-1}{2\,\widetilde{\gamma}}\,\partial_{\widetilde{\gamma}}\mathcal{T}\widetilde{F} \ .\end{aligned} \tag{3.26}$$

Therefore, the off-shell function δg has 2 components:

$$\boxed{\delta g = \frac{\mathcal{T}\big[F_{2N}\delta f\big]}{\mathcal{T}F_{2N}} + \delta t \ ,} \tag{3.27}$$

that is, a target-mass-corrected version of the proper nucleon's off-shell δf, which is nonperturbative and needs to be fitted, and a **calculable δt component**:

$$\delta t(x,Q^2) \equiv \frac{\gamma^2-1}{2\,\gamma}\frac{1}{\mathcal{T}F_{2N}}\left[\partial_{\widetilde{\gamma}}\mathcal{T}\widetilde{F}_{2N}\right]_{p^2=m_N^2} \tag{3.28}$$

$$\equiv \frac{\gamma^2-1}{2\,\gamma}\frac{1}{\mathcal{T}F_{2N}}\left[\partial_{\gamma}\mathcal{T}F_{2N}\right]\,, \tag{3.29}$$

where we used $\partial_{\widetilde{\gamma}} \to \partial_\gamma$ in the $p^2 \to m_N^2$ limit. Unfortunately, δf cannot be further simplified without specifying the TMC prescription for the $\mathcal{T}$ operator, see Section 3.3.1. Note that without TMCs, $\mathcal{T} = \mathbb{I}$ and $\partial_{\widetilde{\gamma}} = 0$, so that $\delta g = \delta f$, as we obtained in Chapter 2. Note that:

- The 1D and 3D smearing formalisms with offshell TMCs are equivalent, up to corrections of order $O(v)$, if the δt term is explicitly included in the former.
- This was not done in the CJ15 fits, with the consequence of distorting the fitted offshell function compared to the "true" one defined in Chapter 2. How to relate these two is discussed next.
- However, $\delta t \propto 1/Q^2$. Therefore, in a full fit with also a fitted HT term, part of the offshell δf deformation can be absorbed in the HT term.

3.3 Off-shell vs. on-shell TMCs

In the CJ15 fits, so far (I think, need to verify), the TMCs have been effectively applied using m_N^2 as a correction parameter instead of p^2, namely, using ξ and γ in Eqs. (3.1)-(3.2), and in the explicit definitions of the TMC operators. This can be formally represented by an "on-shell TMC operator" $\mathcal{T}_*$:

$$\widetilde{F}_*^{\rm TMC} \equiv \mathcal{T}_*\widetilde{F}_N\left(x_N, Q^2, p^2; \gamma\right)\,. \tag{3.30}$$

The only dependence on p^2 is the explicit one, so that the offshell expansion function reads

$$\delta g_* = \frac{\mathcal{T}\left[F_{2N}\delta f\right]}{\mathcal{T}F_{2N}}\,, \tag{3.31}$$

without the additional component δt generated by the implicit p^2 dependence of $\widetilde{\gamma}$.

Let us now consider 2 different fits but withthe same dataset: one fit is performed using the $\mathcal{T}_*$ on-shell TMC operator (such as used in CJ15), and teh other is performed using the $\mathcal{T}$ offshell TMC operator. Both fits are done to to the same data set, but will find different best fit parameters; let us call these $\vec{a}_*$ and $\vec{a}$, respectively. Correspondingly, teh best fit structure functions and off-shell functions are

$$F_{N*} \equiv F_N(\vec{a}_*) \qquad \delta f_* \equiv \delta f(\vec{a}_*) \tag{3.32}$$

$$F_N \equiv F_N(\vec{a}) \qquad \delta f \equiv \delta f(\vec{a}) \tag{3.33}$$

In a fit performed to the same data set, and assuming to have sufficient proton target data to constrain the on-shell structure function (such as in CJ15), we can assume $F_{2N*} \approx F_{2N}$, and the Deuteron data will fit $\delta g_* \approx \delta g$. Equating formulas (3.27) and (3.31), we therefore obtain

$$\frac{\mathcal{T}\left[F_{2N*}\,\delta f_*\right]}{\mathcal{T}F_{2N*}} \approx \frac{\mathcal{T}\left[F_{2N}\,\delta f\right]}{\mathcal{T}F_{2N}} + \delta t\,. \tag{3.34}$$

In other words, the fit is distorting the parametrized off-shell function δf_* in order to accommodate the missed implicit p^2 dependence of $\widetilde{\gamma}$.

Extracting the "true" off-shell function δf from the on-shell-corrected δf_* is formally possible by inverting Eq. (3.34), and remarking that δt only depends on the well fitted on-shell quantities, so that δt is approximately independent of the fit:

$$\frac{\mathcal{T}\left[F_{2N}\,\delta f\right]}{\mathcal{T}F_{2N}} = \frac{\mathcal{T}\left[F_{2N*}\,\delta f_*\right]}{\mathcal{T}F_{2N*}} - \delta t \tag{3.35}$$

$$\approx \frac{\mathcal{T}\left[F_{2N}\,\delta f_*\right]}{\mathcal{T}F_{2N}} - \frac{\gamma^2-1}{2\,\gamma}\frac{1}{\mathcal{T}F_{2N}}\left[\partial_{\widetilde{\gamma}}\mathcal{T}\widetilde{F}_{2N}\right]_{p^2=m_N^2}\,. \tag{3.36}$$

Therefore,

$$\mathcal{T}\big[F_{2N}\,\delta f\big] = \mathcal{T}\big[F_{2N}\,\delta f_*\big] \underbrace{-\frac{\gamma^2-1}{2\,\gamma}\,\partial_\gamma \mathcal{T} F_{2N}}_{\Delta_{\rm off}} \ . \tag{3.37}$$

The $\Delta_{\rm off}$ term is expresses the corrections needed in the 1D smearing formalism to account for the p^2 dependence of $\widetilde{\xi}_N$ and $\widetilde{\gamma}$ when including TMCs.

3.3.1 Calculation of the true δf from $\delta f_{\rm CJ}$

For TMCs, the CJ fit adopts the "approximated OPE" operator $\mathcal{T}^{[\rm OPE]}$ of (3.9). Given that its operation simply consists in evaluating a function at a modified argument, and multiplying this by a factor,

$$\mathcal{T}^{[\rm OPE]} F = c(\gamma) F(\xi) \ , \tag{3.38}$$

with

$$c(\gamma) = \frac{(1+\gamma)^2}{2\gamma^3}\Big(1 + \frac{3(\gamma^2-1)}{2\gamma(1+\gamma)}(1-\xi)^3\Big) \tag{3.39}$$

and $\xi = 2x_N/(1+\gamma)$, the offshell function correction formula (3.34) becomes explicit:

$$\boxed{\delta f(\xi) = \delta f_{\rm CJ} - \frac{\gamma^2-1}{2\gamma}\left[\frac{\partial_\gamma F_2(\xi)}{F_2(\xi)} + \frac{\partial_\gamma c(\gamma)}{c(\gamma)}\right]} \tag{3.40}$$

Note:

- This formula can be used to compare the CJ15 fitted offshell function $\delta f_{\rm CJ}$ with that would be fitted by AKP in a TMC only (no HT) fit.

Chapter 4

Adding Higher Twist corrections

Here we discuss the implementation of Higher Twist (HT) corrections to DIS structure functions, and how these combine and interfere with off-shell corrections. Note: we do not yet consider TMCs; this will be done in a later chapter.

4.1 Multiplicative and additive corrections

Higher twist terms can be parametrized as multiplicative (MULT) or additive (ADD) corrections. These originate from multi-parton photon-nucleon scattering diagrams, and are implemented at the nucleon structure function level. Up to order $O(\Lambda^4_{\rm QCD}/Q^4)$,

$$\widetilde{F}_N^{HT}(x_N,Q^2,p^2) = \begin{cases} \widetilde{F}_N(x_N,Q^2,p^2)\left[1+\dfrac{C(x)}{Q^2}\right] & \text{MULT} \\ \widetilde{F}_N(x_N,Q^2,p^2)+\dfrac{H(x)}{Q^2} & \text{ADD} \end{cases} \tag{4.1}$$

with the *HT functions* $C(x)$ and $H(x)$ to be fitted to the DIS data. Note that:

- $\boldsymbol{\widetilde{F}_N}$ **is the leading twst (LT) structure function** (without any TMC, which will be discussed later).
- The HT functions are phenomenologically chosen independent of Q^2 due to not enough precision in the data (perhaps also lack of twist-4 evolution equations), and of p^2. The latter is to keep the formalism simple, or perhaps noting that the corrections would be of order $O(v\frac{1}{Q^2}$, and therefore parametrically doubly suppressed. **Any missed p^2 dependence of C and H will be effectively fitted inside δf, that may therefore differ in an multiplicative and additive fit**.
- The formalism is the same for any structure function, just add suitable subscripts to teh C and H higher twist functions.
- In the AKP fit, the HT=ADD form is chosen [7]. In the CJ15 fit, the multiplicative form HT=MULT is used [3], but in the more recent studies both have been used and compared [5].

4.2 Off-shell smearing with HT corrections

Multiplicative case

If we perform the offshell expansion of $\widetilde{F}_N^{\rm MULT}$, in powers of the nucleon virtuality $v = (p^2 - m_N^2)/m_N^2$, see Eq. (2.12), we obtain

$$\widetilde{F}_N^{\rm MULT}(x_N, Q^2, p^2) = F_N(x, Q^2)\big[1 + \delta f(x_N)\, v\big]\left[1 + \frac{C(x)}{Q^2}\right] = \underbrace{F_N(x, Q^2)\left[1 + \frac{C(x)}{Q^2}\right]}_{F_N^{\rm MULT}}\big[1 + \delta f(x_N)\, v\big] \tag{4.2}$$

Therefore,

$$\widetilde{F}_N^{\rm MULT}(x_N, Q^2, p^2) = F_N^{\rm MULT}(x_N, Q^2)\big[1 + \delta f^{\rm MULT}(x_N)\, v\big]\ , \tag{4.3}$$

where

$$F_N^{\rm MULT} \equiv F_N[1 + C/Q^2] \tag{4.4}$$

is the on-shell, multiplicatively corrected structure function, and

$$\boxed{\delta_f^{\rm MULT} = \delta f}\ . \tag{4.5}$$

its first order offshell expansion coefficient.

Additive case

Turning to the additive case and working in an analogous way,

$$\widetilde{F}_N^{\rm ADD}(x_N, Q^2, p^2) = F_N(x, Q^2)\big[1 + \delta f(x_N)\, v\big] + \frac{H(x)}{Q^2} = \underbrace{\left[F_N(x, Q^2) + \frac{H(x)}{Q^2}\right]}_{F_N^{\rm ADD}} + F_N(x, Q^2)\delta f(x_N)\, v \tag{4.6}$$

Therefore,

$$\widetilde{F}_N^{\rm ADD}(x_N, Q^2, p^2) = F_N^{\rm ADD}(x_N, Q^2)\big[1 + \delta f^{\rm ADD}(x_N)\, v\big]\ , \tag{4.7}$$

where

$$F_N^{\rm ADD} \equiv F_N + H/Q^2 \tag{4.8}$$

is the on-shell, additively corrected structure funtion, and

$$\boxed{\delta_f^{\rm ADD} = \frac{F_N(x_N, Q^2)}{F_N^{\rm ADD}(x_N, Q^2)}\,\delta f(x)} \tag{4.9}$$

its first order offshell expansion coefficient.

4.2.1 Comparison of HT functions

The higher-twist C and H functions are entirely subsumed in the on-shell corrected structure functions (4.4) and (4.8). Note, however, that this is actually due to the assumed independence of C and H on p^2, and is not a generic feature of off-shell structure functions.

By multiplying $\widetilde{F}_N$ through the square brackets in Eq. (4.4) or factorizing $F_N^{\rm ADD}$ in Eq. (4.8), one can compare the additive and multiplicative models, and define a *equivalent additive* function $\overline{H}$, and an *equivalent multiplicative* function $\overline{C}$, that convert the HT function fitted in one scheme into the other scheme:

$$\overline{C}(x, Q^2) \equiv \frac{H(x)}{\widetilde{F}_N(x_N, Q^2)} \tag{4.10}$$

$$\overline{H}(x, Q^2) \equiv C(x)\, \widetilde{F}_N(x_N, Q^2) \tag{4.11}$$

These, however acquire a Q^2 dependence that makes a direct comparison difficult. In a QCD fit that contains data over a sufficiently large Q^2 range, the LT and HT components of $F_N^{\rm MULT}$ and $F_N^{\rm ADD}$ can be separately determined by the data, so that one can assume, or rather define,

$$\left\langle \overline{C}(x,Q^2)\right\rangle_{Q^2} \equiv C(X) \tag{4.12}$$

$$\left\langle \overline{H}(x,Q^2)\right\rangle_{Q^2} \equiv H(X) \tag{4.13}$$

where C and H are fitted to the same data set. The average is to be understood as an average over Q^2 determined by the available data points and their experimental uncertainties. In this sense, the above equations can be actually considered definition of $\langle \ldots \rangle$.

4.2.2 Comparison of off-shell functions

In a QCD fit, the HT corrected, on-shell structure functions $F_N^{\rm MULT}$ and $F_N^{\rm ADD}$ appearing in Eqs. (4.3) and (4.7) are largely determined by the fitted free proton target experimental data. Thus we can assume that

$$F_N^{\rm ADD} \approx F_N^{\rm MULT} \ , \tag{4.14}$$

i.e., the fitted on-shell functions are a good representation of the free proton data. Then, in order to accommodate the Deuteron data, the fit is free to adjust $\delta_f^{\rm ADD}$ and $\delta_f^{\rm MULT}$; at the end, assuming we have a good fit also of the Dueteron data, we obtain

$$\delta f^{\rm ADD} \approx \delta f^{\rm MULT} \ . \tag{4.15}$$

Now, the question is: How do the fitted quantities $\delta_f^{\rm ADD}$ relate to the off-shell function δf? The answer is provided By Eqs. (4.5) and (4.9):

- The multiplicative $\boldsymbol{\delta f^{\rm MULT}}$ **provides a direct extraction of** $\boldsymbol{\delta f}$.
- However, $\boldsymbol{\delta f^{\rm ADD}}$ **does not directly access the offshell** $\boldsymbol{\delta f}$ **function**. Indeed, from Eq. (4.9) we can see that

$$\boxed{\delta f^{\rm ADD} = \frac{1}{1+\overline{C}(x,Q^2)/Q^2}\delta f \ ,} \tag{4.16}$$

  with δf and $\delta f^{\rm ADD}$ related by a (multiplicative) HT correction factor $1+\overline{C}/Q^2$. In words, the data actually constrain the quantity $\delta f^{\rm ADD}$, which is fitted by varying the parameter used for δf. As a consequence, the latter is distorted with respect to the "true" off-shell function.

4.3 CJ vs. AKP offshell extraction with HT corrections (no TMC)

The CJ fits adopt a multiplicative HT scheme, so that the fitted offshell function is given by reading Eq. (4.5) from left to right:

$$\delta f_{\rm CJ} = \delta f^{\rm MULT} \ . \tag{4.17}$$

The AKP fits utilize an additive scheme, so that reading Eq. (4.16) from left to right one obtains

$$\delta f_{\rm AKP} = \left[1+\frac{H(x)/F_N(x,Q^2)}{Q^2}\right]\delta f^{\rm ADD} \ . \tag{4.18}$$

Since, as argued, the fitted data constrain $\delta_f^{\rm MULT} \approx \delta_f^{\rm ADD}$, we obtain

$$\boxed{\delta f_{\rm CJ}(x) = \left\langle \frac{1}{1+\frac{\overline{C}(x,Q^2)}{Q^2}}\right\rangle_{Q^2} \delta f_{\rm AKP}(x) \approx \frac{1}{1+\frac{C_{\rm CJ}(x)}{Q^2_{\rm min}(x)}}\delta f_{\rm AKP}} \ . \tag{4.19}$$

The right hand side also gives one an approximated formula for the "true" offshell function in the AKP:

$$\boxed{\delta f_{\rm AKP}^{\rm true}(x) = \frac{1}{1+\frac{C_{\rm CJ}(x)}{Q^2_{\rm min}(x)}}\delta f_{\rm AKP}} \,. \tag{4.20}$$

Note that the in the first equality of Eq. (4.19), the average over Q^2 is determined by the available data points and their experimental uncertainties: it is thus difficult to precisely relate the CJ and AKP functions, or equivalently to extract the true AKP function (see also the discussion of Eqs. (4.12)-(4.12)). The approximation in the r.h.s. is obtained using

$$\left\langle \overline{C}(x,Q^2)\right\rangle_{Q^2} \approx C_{\rm CJ}(x) \tag{4.21}$$

$$\left\langle Q^2\right\rangle \gtrsim Q^2_{\rm min}(x) \ , \tag{4.22}$$

where $Q^2_{\rm min}(x)$ is the minimum Q^2 value compatible with the kinematic cuts used to select the experimental data, i.e/,

$$Q^2_{\rm min}(x) = \max\left[Q^2_{\rm cut}, (W^2_{\rm cut}-m_N^2)x/(1-x)\right]. \tag{4.23}$$

In the CJ15 fits, for example, $Q^2_{\rm cut} = 1.69$ GeV^2 and $W^2_{\rm cut} = 3.0$ GeV^2 [3]. Such an approximation should provide a ball park comparison of the "true" off-shell functions extracted in (non-TMC-corrected) CJ and AKP fits, as well as in any pair of multiplicative and additive fits that utilize the same DIS data set, for example those presented in [5].

Bibliography

[1] S. Kulagin. Notes on the nuclear smearing in the Deuteron. *LaTeX notes*, https://github.com/JeffersonLab/offshell-benchmark/files/4755033/AKP_convolution_pty_200609.pdf, June 2020.

[2] A. Accardi et al. Collinear factorization in deep inelastic lepton-nucleus collisions. *Paper Draft*, Jan 2010.

[3] A. Accardi, L.T. Brady, W. Melnitchouk, J.F. Owens, and N. Sato. Constraints on large-x parton distributions from new weak boson production and deep-inelastic scattering data. *Phys. Rev. D*, 93(11):114017, 2016.

[4] J.F. Owens, A. Accardi, and W. Melnitchouk. Global parton distributions with nuclear and finite-Q^2 corrections. *Phys. Rev. D*, 87(9):094012, 2013.

[5] A. Accardi. Benchmarking CJ vs AKP: recent progress. *Slides*, https://github.com/JeffersonLab/offshell-benchmark/issues/6, 12 Feb 2020.

[6] S. Kulagin. Discussion on CJ vs. AKP results. *Slides*, https://github.com/JeffersonLab/offshell-benchmark/issues/6, 12 Feb 2020.

[7] S.I. Alekhin, S.A. Kulagin, and R. Petti. Nuclear Effects in the Deuteron and Constraints on the d/u Ratio. *Phys. Rev. D*, 96(5):054005, 2017.

[8] Ingo Schienbein et al. A Review of Target Mass Corrections. *J. Phys. G*, 35:053101, 2008.

[9] L.T. Brady, A. Accardi, T.J. Hobbs, and W. Melnitchouk. Next-to leading order analysis of target mass corrections to structure functions and asymmetries. *Phys. Rev. D*, 84:074008, 2011. [Erratum: Phys.Rev.D 85, 039902 (2012)].