5. Compressible Flow

5.1. Introduction

(Massey §12.1-12.3, White §9.1)

So far we have considered incompressible flow (i.e. liquid or low speed gas flows). If fluid is moving close to the speed of sound density changes become significant.

The Mach Number, M - ratio of the fluid velocity to the speed of sound in the fluid;

$$M = \frac{u}{a} \qquad a = \text{velocity of sound} \tag{5.1}$$

For liquids it is difficult to cause a large density change (need pressures ~1000 atm). Liquids are virtually incompressible.

For gases even very low pressure ratios (2:1) can result in significant density changes. Typically for $M >$ we must consider compressibility effects.

In this course we will simplify the analysis by making three assumptions about the flow;

reversible - no losses due to friction

adiabatic - no heat transfer to/from fluid

perfect gas - use the perfect gas equation of state

5.2. Thermodynamics Concepts

(a) **Perfect gas** (i.e. ideal gas). Ideal gas equation of state applies

$$p = \rho RT \qquad pV = mRT \tag{5.2}$$

(b) **Enthalpy**. The combination of properties (E+pV) occurs so frequently that we define a new property; enthalpy, H, and specific enthalpy, $h = \frac{H}{m}$

$$H = E + pV \qquad h = e + \frac{p}{\rho} \quad \text{units J/kg} \tag{5.3}$$

$$\tag{5.4}$$

(c) **Specific Heat Capacities.**

$$de = c_v dT \quad \& \quad dh = c_p dT \tag{5.5}$$

For a perfect gas the specific heats are constant so;

$$e_2 - e_1 = c_v (T_2 - T_1) \quad \& \quad h_2 - h_1 = c_p (T_2 - T_1) \quad \text{units J/kg K} \tag{5.6}$$

$$\boxed{\gamma = \frac{c_p}{c_v}} \quad \& \quad \boxed{R = c_p - c_v} \qquad R = \frac{\bar{R}}{M} \leftarrow \text{molecular mass} \tag{5.7}$$

(d) **Reversible Process.** If a process can be reversed by returning all the heat or work lost/gained during the process - known as a reversible process. Viscous (fluid friction) effects are a source of irreversibility.

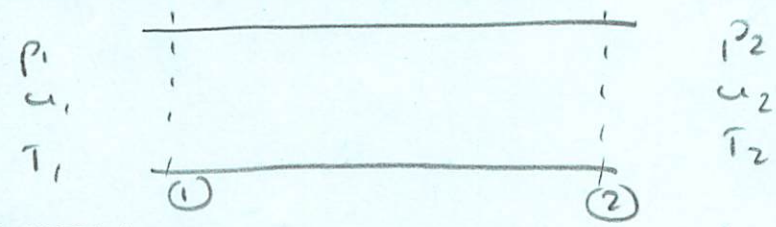
(e) **Isentropic Process.** A process where the entropy is constant. For this to happen $\Delta Q=0$ so the process must be adiabatic and frictionless (i.e. reversible). For an isentropic process from 1 to 2 (i.e. $s_1=s_2$). For proofs see 1st year thermodynamics.

$$\frac{p_1}{p_2} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \qquad \text{for a isentropic process} \tag{5.8}$$

The analysis of compressible flow is greatly simplified if we assume isentropic flow.

5.3. Stagnation Conditions

Consider a compressible flow between states 1 and 2;



The SFEE gives;

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2 + e_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1 + e_1\right)$$

If the process is adiabatic and we have no shaft work $\dot{Q} = \dot{W} = 0$. For a gas since the density is low then we can neglect the gravitational effects (neglect gz_1 and gz_2). Then the SFEE becomes;

$$\left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + e_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + e_1\right) = 0$$

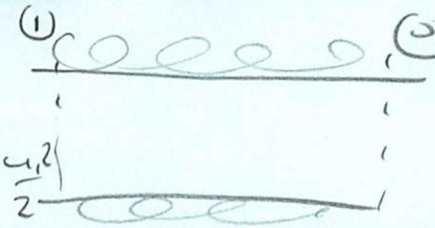
or substituting $h = \dfrac{p}{\rho} + e$ gives

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{5.9}$$

thus at any point in the flow the sum $h + \dfrac{u^2}{2}$ is constant.

We define a new term the stagnation enthalpy, h_0.

$$\boxed{h_0 = h + \frac{u^2}{2}}$$



u_1

h_1

$h_{01} = h_1 + \frac{u_1^2}{2}$ $u_2 = 0$

$h_{02} = h_2$ (5.10)

Thus if a fluid flow is reduced to rest adiabatically it will have an enthalpy equal to, h_0 the stagnation enthalpy.

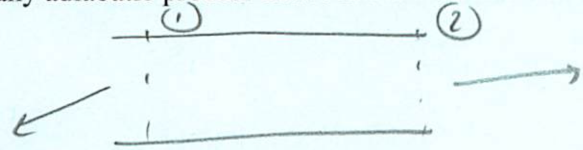
If the compressible fluid is also a perfect gas then we can include equation (5.5) $dh = c_p\,dT$

So, $c_p T_0 = c_p T + \dfrac{u^2}{2}$ or $T_0 = T + \dfrac{u^2}{2c_p}$

Where T_0 is the *stagnation temperature*.

$$\boxed{\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}} \qquad (5.11)$$

Thus for any adiabatic process between state 1 and 2 we can say;



stagnation state

imaginary process where fluid is brought to rest isentropically

01

02

$h_{01} = h_{02}$ and for a perfect gas $T_{01} = T_{02}$ (5.12)

5.4. Isentropic Pressure and Density Relations

The expressions (5.10) and (5.11) requires the flow to be adiabatic. If the flow is also isentropic, then we can use the equation (5.8) to find relations for pressure and density;

For an isentropic process we can relate the temperature change to the pressure change (or density change using equation (5.8);

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{T_1}{T_2}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \qquad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1}$$

So for combining this with (5.11) gives a relationship between the static pressure and the stagnation pressure.

$$\boxed{\frac{p_0}{p} = \left(1 + \frac{u^2}{2c_p T}\right)^{\frac{\gamma}{\gamma-1}}} \qquad (5.13)$$

$$\boxed{\frac{\rho_0}{\rho} = \left(1 + \frac{u^2}{2c_p T}\right)^{\frac{1}{\gamma-1}}} \qquad (5.14)$$

The quantities p_0 and ρ_0 are the *stagnation pressure and density*. They represent the pressure and density which would be achieved if the flow were brought isentropically to rest.

The stagnation values (p_0, T_0, h_0 and ρ_0) are useful reference conditions in a flow.

Note: Don't get confused between static and stagnation. Static pressure & static temperature are the actual fluid properties (i.e. ordinary pressure and temperature). Symbols, p and T.

For any fluid state given by h, p, T, ρ there is a corresponding stagnation state h_0, p_0, T_0, ρ_0 which is achieved by an imaginary process where the fluid is reduced to rest isentropically.

The stagnation properties are simply another way of writing the static properties to incorporate the flow velocity.

Example 5.1 Determine the Stagnation Conditions

Air flowing at 75 m/s is at a pressure of 140 kPa and 260°C. Determine the stagnation conditions.

air $R: 287\ J/kg\,K$ $\quad c_p: 997\ J/kg\,K$ $\quad \gamma: 1.4$

The static conditions are;

$p = 140\,kPa$ $\quad T = 260°C = 533\,K$ $\quad u = 75\,m/s$

For the stagnation conditions p_0, T_0, and ρ_0. Imagine an *isentropic* process from static to stagnation conditions;

$$T_0 = T + \frac{u^2}{2c_p} = 533 + \frac{75^2}{2.997} = 535.8\,K$$

$$p_0 = p\left(1 + \frac{u^2}{2c_pT}\right)^{\gamma/\gamma-1} = 140\times10^3\left(1 + \frac{75^2}{2.997.533}\right)^{\frac{1.4}{1.4-1}}$$

For the stagnation density;

$$= 142.6\,kPa$$

EITHER: determine static density

$$\rho = \frac{p}{RT} = \frac{140\times10^3}{287.533} = 0.915\,kg/m^3$$

Then $\rho/\rho_0 = \left(1 + \frac{u^2}{2c_pT}\right)^{1/\gamma-1} = 1.013$

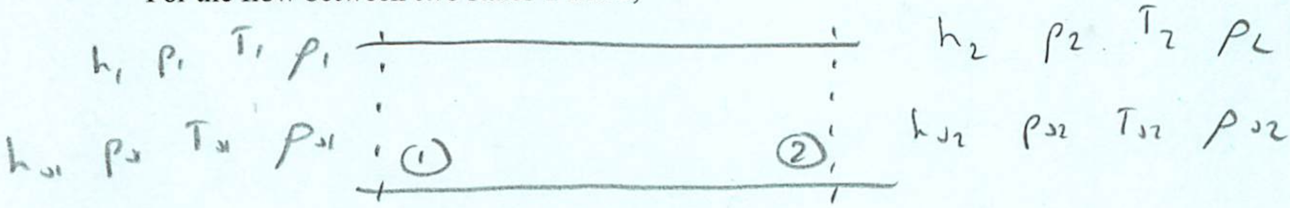
$\rho_0 =$ $0.915 . 1.013 = 0.927\,kg/m^3$

OR: quicker to determine the stagnation density by using the equation of state for stagnation conditions i.e.;

$$\rho_0 = \frac{p_0}{RT_0} = \frac{142.6\times10^3}{287.535.8} = 0.927\,kg/m^3$$

Summary - Static and Stagnation Conditions

For the flow between two states 1 and 2;



(i) For any *adiabatic* compressible process, with no shaft work, Stagnation enthalpy is constant. The SFEE gives; $h_{01} = h_{02}$ $\therefore h_{01} = h_1 + \frac{u_1^2}{2}$

(ii) If the fluid is a *perfect gas* then, $dh = c_p dT$, and stagnation temperature is constant $T_{01} = T_{02}$.

(iii) If the process from 1 to 2 is *reversible* (i.e. it is also *isentropic*), and both 01 to 1 and 2 to 02 are reversible, then;

$$\frac{T_{01}}{T_{02}} = \left(\frac{p_{01}}{p_{02}}\right)^{\frac{\gamma-1}{\gamma}}$$

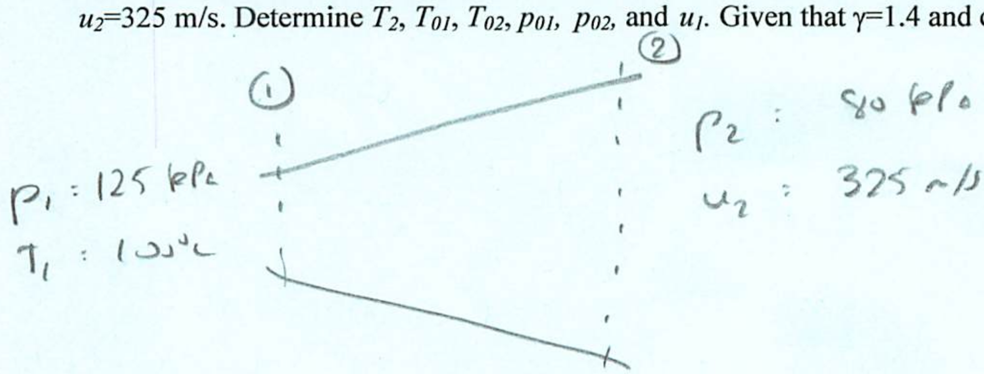
state 01 = state 02

but since $T_{01} = T_{02}$ then $p_{01} = p_{02}$. So for a isentropic process the stagnation state does not change.

(iv) If the process from 1 to 2 is *irreversible* then p_0 and ρ_0 do not remain constant but vary throughout the flow as energy is lost to friction.

Example 5.2 Isentropic expansion of air

Air expands isentropically through a duct from p_1=125 kPa and T_1=100°C to p_2=80 kPa and u_2=325 m/s. Determine T_2, T_{01}, T_{02}, p_{01}, p_{02}, and u_1. Given that γ=1.4 and c_p=993 J/kg K.



For the *isentropic* process 1 to 2 we can use (5.8)

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}} \qquad \frac{373}{T_2} = \left(\frac{125}{80}\right)^{\frac{0.4}{1.4}} \qquad T_2 = 328\text{K}$$

For the isentropic process 2 to 02 we can use (5.11)

$$\frac{T_{02}}{T_2} = 1 + \frac{u_2^2}{2c_pT_2} \qquad \frac{T_{02}}{328} = 1 + \frac{325^2}{2.993.328} \qquad T_{02} = 381\text{ K}$$

Since 1 to 2 is isentropic T_{01}=T_{02}= 381 K

For the isentropic process 2 to 02 we can use (5.13)

$$\frac{p_{02}}{p_2} = \left(1 + \frac{u_2^2}{2c_pT_2}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{p_{02}}{80\times10^3} = \left(1 + \frac{325^2}{2.993.328}\right)^{\frac{1.4}{1.4-1}} \qquad p_{02} = 135.4\text{K}$$

Since 1 to 2 is isentropic p_{01}=p_{02}= 135.4 kPa

For the isentropic process 1 to 01 we can use (5.11)

$$\frac{T_{01}}{T_1} = 1 + \frac{u_1^2}{2c_pT_1} \qquad \frac{381}{373} = 1 + \frac{u_1^2}{2.993.373} \qquad u_1 = 126\text{ m/s.}$$

5.5 Compressible Flow and Bernouilli's Equation

If we thought our gas was incompressible we could use Bernouilli's equation (remember this is the version of the SFEE assuming incompressible adiabatic, frictionless/reversible flow).

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{u_2^2}{2} + gz_2 \qquad \text{assume } \dot{W} = \dot{Q} = 0,\ \rho \text{ const},\ e_2 = e_1$$

Again neglecting gravity effects. And if states 1 and 2 are static and stagnation;

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho} + \frac{u_2^2}{2} \qquad \frac{p}{\rho} + \frac{u^2}{2} = \frac{p_0}{\rho}$$

rearranging $\quad \frac{p_0}{p} = 1 + \rho\frac{u_2^2}{2p}$

and for a perfect gas including (5.2) $\quad p/\rho = RT$

$$\frac{p_0}{p} = 1 + \frac{u^2}{2RT} \qquad (5.15)$$

This is the relation between static and stagnation conditions if we assume our gas is *incompressible*. We can compare the results from this with those from equation (5.13) for *compressible* flow.

If the fluid is air at 1 bar and 300K;

gas velocity	$\frac{p_0}{p} = 1 + \frac{u^2}{2RT}$	$\frac{p_0}{p} = \left(1 + \frac{u^2}{2c_pT}\right)^{\frac{\gamma}{\gamma-1}}$
20 m/s	1.0023	1.0023
200 m/s	1.232	1.252
500 m/s	2.45	3.4

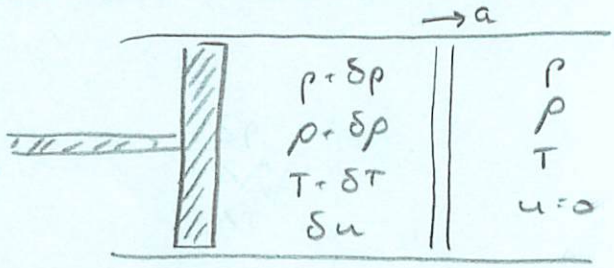
As the velocity of the gas increases the assumption becomes less valid. Typically for M<0.3 we can assume incompressibility.

5.6. Sonic Velocity and Mach Number

The velocity of sound is the rate of propagation of a pressure pulse through a still fluid. Pressure changes do not occur instantaneously. Sound waves are pressure waves through the gas (note: a faint sound corresponds to pressure fluctuations of 3 x 10^{-5} Pa, whilst the threshold of pain is 100 Pa).

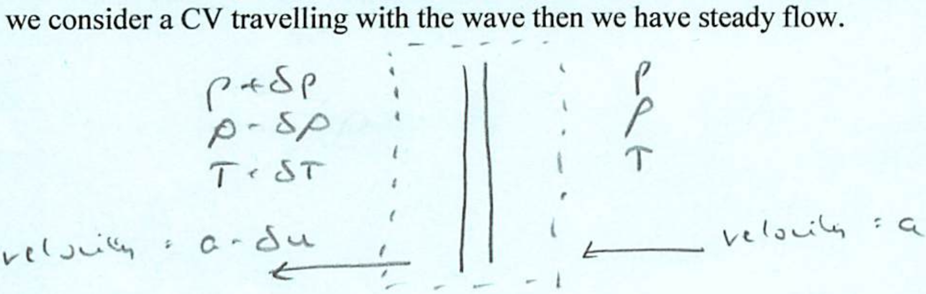
We need to find the speed of the waves (i.e. the speed of sound) and how it depends on the state of the gas.

Consider a tube full of stationary gas. A sudden small piston movement causes a wave of pressure to travel down the tube. The wave travels at a speed *a* down the tube;



tube of cross section A.

This is a non-steady flow (i.e. the speed and pressure at a point vary with time). However if we consider a CV travelling with the wave then we have steady flow.



Applying the MCE to the CV;

$$\rho a A = (\rho + \delta\rho)(a - \delta u) . A$$

Neglecting second order terms;

$$a\partial\rho - \rho\partial u = 0 \qquad (5.16)$$

Applying the FME to the CV; x-direction +ve $\longrightarrow$

$$\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x \quad = \dot{m}\left(u_{out} - u_{in}\right)_x$$

$$(p + \delta p)A - pA = \rho a A\left[(-a + \delta u) - (-a)\right]$$

$$\delta p - \rho a \,\delta u = 0 \tag{5.17}$$

Eliminating δu by combining (5.16) and (5.17) gives;

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right) \tag{5.18}$$

If the pressure wave has a very small amplitude then only a small amount of heat flow and friction losses occur. The passage of the wave may therefore be assumed to be reversible and adiabatic (and hence isentropic). So for a perfect gas (5.8) gives;

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

$$\frac{p}{\rho^\gamma} = \text{constant} \qquad p = K\rho^\gamma$$

Differentiating w.r.t. ρ

$$\frac{\delta p}{\delta \rho} = K\gamma\rho^{\gamma-1}$$

substituting for K gives;

$$\frac{\delta p}{\delta \rho} = \frac{p}{\rho^\gamma}\gamma\rho^{\gamma-1} \qquad \frac{\delta p}{\delta \rho} = \frac{\gamma p}{\rho}$$

so combining this with (5.18) gives;

$$a = \sqrt{\gamma R T} \tag{5.19}$$

Thus the speed of sound in a gas is obtained easily from the state of the gas.

Example 5.3 The Speed of Sound in Air

Determine the speed of sound in air at 15°C. (R=287 J/kgK, γ=1.4).

$$a = \sqrt{\gamma R T}$$

$$= \sqrt{1.4 \cdot 287 \cdot (273 + 15)}$$

$$= 340 \text{ m/s}$$

5.7 Compressible Flow Relations for a Perfect Gas

It is useful to rewrite equations (5.11), (5.13), and (5.14) in terms of M;

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_pT} \quad \text{and} \quad u = Ma$$

$c_p = \frac{R\gamma}{\gamma - 1}$

$\frac{T_0}{T} = 1 + \frac{M^2 a^2}{2c_pT} = 1 + \frac{M^2 \gamma R T}{2\left(\frac{R\gamma}{\gamma-1}\right)T}$

gives;

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 \qquad (5.20)$$

— adiabatic

$$\frac{p_0}{p} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma - 1}} \qquad (5.21)$$

adiabatic, reversible
= isentropic flow

$$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{1}{\gamma - 1}} \qquad (5.22)$$

and since $a = \sqrt{\gamma R T}$

$$\frac{a_0}{a} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{1}{2}} \qquad (5.23)$$

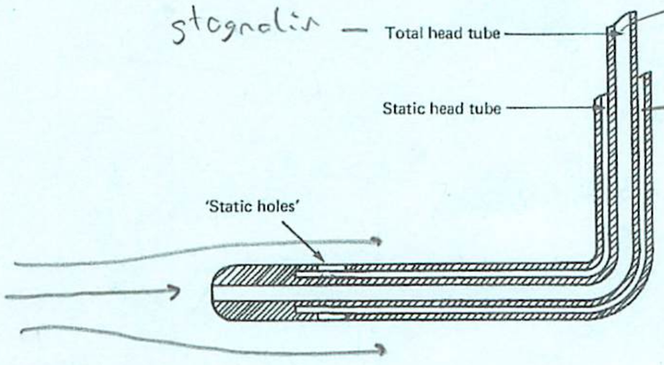
These relations apply to a compressible perfect gas flowing adiabatically for (5.20) and adiabatically and reversibly (i.e. isentropically) for (5.21) & (5.22)

Example 5.4 A Pitot Tube in Compressible Flow

A pitot-static tube is used to measure the static (1 bar) and stagnation (or total) pressures (1.5 bar) of air flowing in a wind tunnel. Find the Mach number.

stagnation —

$p_0 \; \rho_0 \; T_0$



$p \; \rho \; T$

We assume air is a perfect gas and the process by which the air is brought to rest at the nose of the pitot tube is frictionless and adiababtic. We can then use the isentropic relation (5.21);

$$\frac{p_0}{p} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]^{\gamma/\gamma-1} \qquad \frac{1.5}{1} = \left[1+\frac{1.4-1}{2}M^2\right]^{\frac{1.4}{1.4-1}}$$

$M=$ 0.78

i.e. M<1 subsonic flow. In practice we cannot use a pitot tube in supersonic flow. A shock wave forms ahead of the nose so fluid is not brought to rest isentropically.

5.8. Critical Conditions

Just as the stagnation conditions are a useful reference, it is also useful to describe the fluid properties when the flow is sonic. These sonic properties are know as critical conditions; Putting M=1 (i.e. sonic flow) into (5.20)-(5.23) gives;

$$\frac{T_c}{T_0} = \frac{2}{\gamma+1} \tag{5.24}$$

$$\frac{p_c}{p_0} = \left[\frac{2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} \tag{5.25}$$

$$\frac{\rho_c}{\rho_0} = \left[\frac{2}{\gamma+1}\right]^{\frac{1}{\gamma-1}} \tag{5.26}$$

& $p_{c1} = p_{c2}$
$T_{c1} = T_{c2}$
$\rho_{c1} = \rho_{c2}$
for isentropic flow

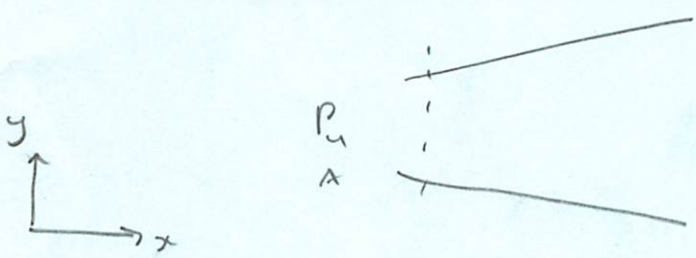
$$\frac{a_c}{a_0} = \left[\frac{2}{\gamma+1}\right]^{\frac{1}{2}} \tag{5.27}$$

As before, in isentropic flow all the critical properties are constant. In adiabatic non-isentropic flow a_c and T_c are constant, but p_c and ρ_c may vary.

5.9. Effect of Area Variation in a Duct

(Massey §12.8, White §9.4)

Consider the flow in a duct with area which varies in the x-direction;



how does u vary with x ?

The MCE tells us

$\rho(x)u(x)A(x) = \dot{m} = \text{constant}$

since ρ, u, and A are all functions of x. Differentiating with respect to x

$$\frac{d\dot{m}}{dx} = \frac{d}{dx}(\rho u A) = Au\frac{d\rho}{dx} + \rho A\frac{du}{dx} + \rho u\frac{dA}{dx} = 0$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \tag{5.28}$$

This is the differential form of the MCE.

Also for a perfect gas; $\dfrac{p}{\rho^{\gamma}} = K$

differentiating w.r.t. x and substituting for K (as we did for 3.18)

$$\frac{dp}{dx} = \frac{\gamma p}{\rho}\frac{d\rho}{dx} \qquad \frac{p}{\rho} + \frac{u^2}{2} + gz = K \tag{5.29}$$

We can also differentiate Bernouilli's equation in the same way. This gives Euler's equation;

$$\frac{1}{\rho}\frac{dp}{dx} + u\frac{du}{dx} + g\frac{dz}{dx} = 0 \qquad \frac{1}{\rho}\frac{dp}{dx} + u\frac{du}{dx} = 0$$

Neglecting the gravity terms and substituting dp/dx (5.29);

$$\frac{\gamma p}{\rho^2}\frac{d\rho}{dx} = -u\frac{du}{dx} \qquad \text{rearranging} \qquad \frac{1}{\rho}\frac{d\rho}{dx} = -\frac{\rho u}{\gamma p}\frac{du}{dx}$$

putting this into (5.28);

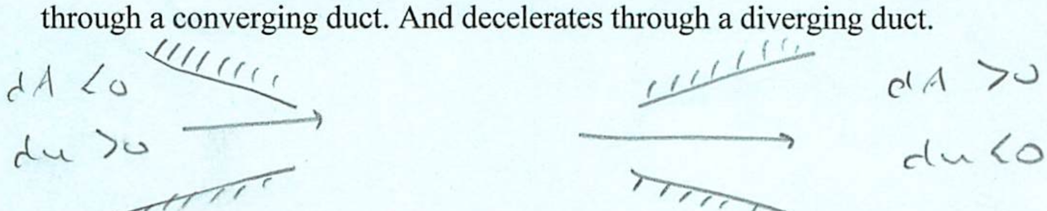
$$-\frac{\rho u}{\gamma p}\frac{du}{dx} + \frac{1}{u}\frac{du}{dx} + \frac{1}{A}\frac{dA}{dx} = 0$$

Substituting for M; $\quad M = \dfrac{u}{\sqrt{\gamma RT}} = u\sqrt{\dfrac{\rho}{\gamma p}}$

$$\boxed{\frac{A}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\left(M^2 - 1\right) = \frac{\mathrm{d}A}{\mathrm{d}x}} \tag{5.30}$$

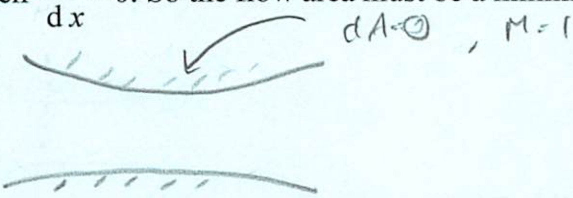
This equation describes how the flow velocity varies when the cross section area changes. Inspection tells us some unusual aspects;

(i) If M<1 (subsonic) then $\dfrac{\mathrm{d}A}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ must have opposite signs. So the fluid accelerates through a converging duct. And decelerates through a diverging duct.

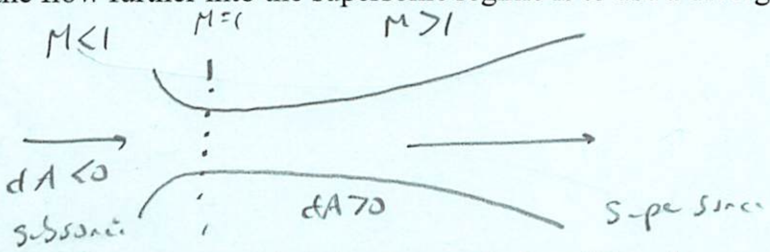


(ii) If M>1 (supersonic) then $\dfrac{\mathrm{d}A}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ must have the same sign. So the fluid accelerates through a diverging duct. And decelerates through a converging duct.

(iii) If M=1 (sonic) then $\dfrac{dA}{dx}$=0. So the flow area must be a minimum i.e. a throat.



So, a converging section will accelerate subsonic flow until M=1. Then the only way to accelerate the flow further into the supersonic regime is to use a diverging section.

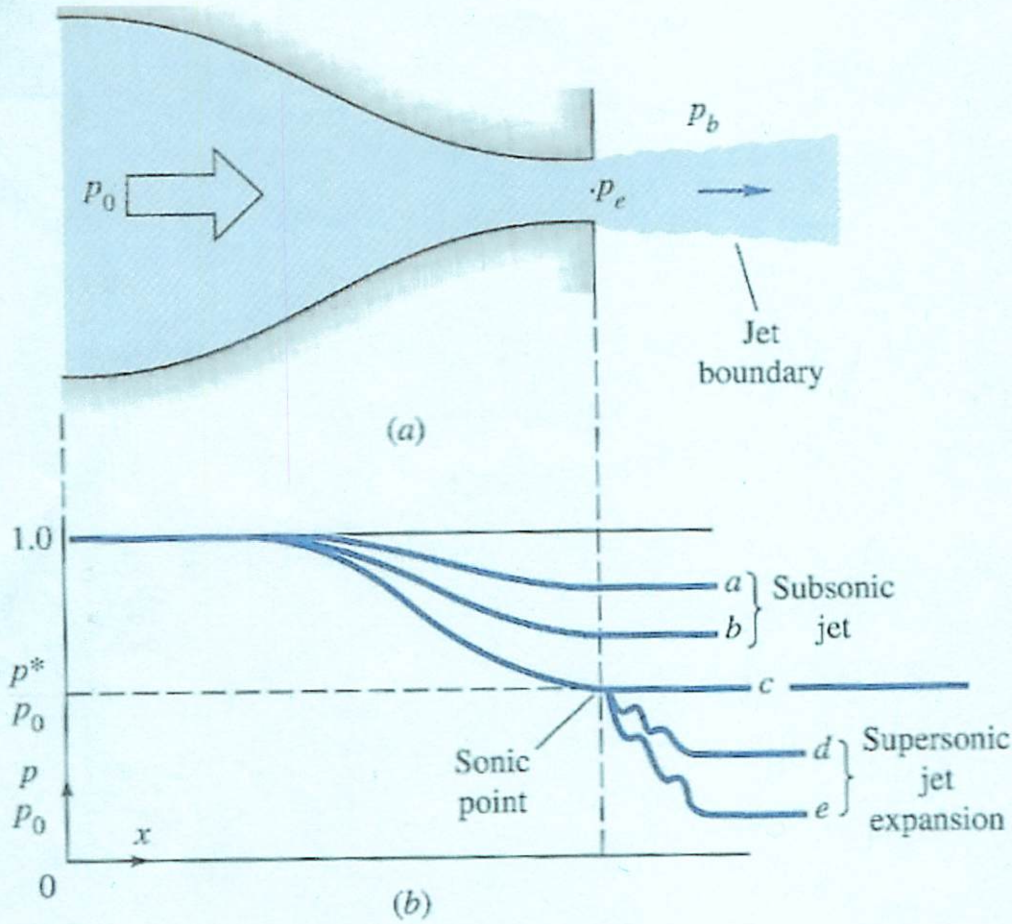


This converging-diverging nozzle is known as a DeLaval Nozzle.

5.10. Choking

(Massey §12.8, White §9.4)

Consider isentropic flow from a through a convergent nozzle. If the back pressure p_b is reduced, what happens?



Curve (0) no flow (p_b=p_0)

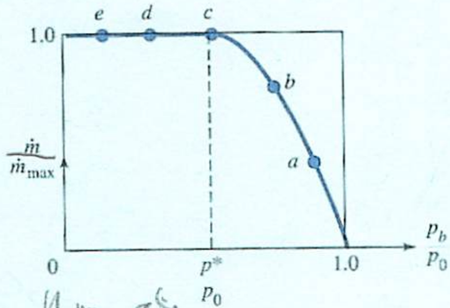
Curve (a, b) $\dfrac{\mathrm{d}A}{\mathrm{d}x}$ is negative from 1 to t. So for M<1, u and M are both increasing.
dA <0 so du >0 flow accelerates

Curve (c) The velocity has reached the point that at the throat, M_t=1 (just equal). At this point the throat reached critical conditions. i.e. p_t=p_c and T_t=T_c throat is Sonic

Curve (d & e) If p_b is reduced further, we cannot make M>1 inside the nozzle (since we must have $\dfrac{\mathrm{d}A}{\mathrm{d}x}$>0 to accelerate a supersonic flow). So the drop in pressure must occur outside the nozzle (between t and b) while the flow between 1 and t remains the same. Pressure drops outside the nozzle in a supersonic jet.

Physically: when the fluid has reached sonic velocity (u=a and M=1) then when the back pressure is lowered, the expansion wave travels upstream at the same speed that the flow is travelling downstream. i.e. the 'information' that the p_b is being reduced never reaches the gas inside the nozzle.

The mass flow rate cannot be increased beyond its value for curve (c). The nozzle is said to be *choked.*



What is the max flow rate.

By the MCE;

$$\dot m = \rho_t u_t A_t$$

If the flow is choked (i.e. sonic at the throat) then M_t=1 and at the throat the gas is at critical conditions so $\rho_t = \rho_c$ & $u_t = a_c$

Remember the sonic velocity a is not a fixed constant but varies with the gas temperature $a_c = \sqrt{\gamma R T_c}$. Then

$$\dot m_{max} = \rho_c a_c A_t$$

refer the conditions at the throat ρ_c to stagnation conditions, ρ_0

$$\dot m_{max} = \left(\frac{\rho_c}{\rho_0}\rho_0\right)\left(\frac{a_c}{a_0}a_0\right)A_t$$

Now $a_c = \sqrt{\gamma R T_c}$ & $a_0 = \sqrt{\gamma R T_0}$

so substituting $\frac{a_c}{a_0} = \sqrt{\frac{T_c}{T_0}}$ and $\rho_0 = \frac{p_0}{RT_0}$

$$\dot m_{max} = \left(\frac{\rho_c}{\rho_0}\frac{p_0}{RT_0}\right)\cdot\left(\sqrt{\frac{T_c}{T_0}}\cdot\sqrt{\gamma R T_0}\right)\cdot A_t$$

substituting the relations (5.24) and (5.26)

$$\dot m_{max} = \frac{p_0}{RT_0}\cdot A_t\cdot\sqrt{\gamma R T_0}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{2}{\gamma+1}\right)^{1/2}$$

$$\dot m_{max} = \frac{p_0}{\sqrt{T_0}}A_t\left[\sqrt{\frac{\gamma}{R}}\left(1+\frac{2}{\gamma+1}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}}\right] \quad \text{(5.31)}$$

~ fixed for a particular gas

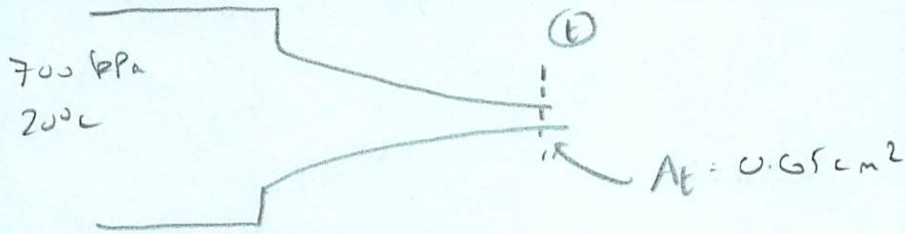
So, the only way to increase the mass flow through a choked nozzle is to;

increase the throat area, A_t
increase the inlet stagnation pressure, p_0
decrease the inlet stagnation temperature, T_0

$$= \frac{p_0}{\sqrt{T_0}} A_t \left[\sqrt{\frac{\gamma}{R}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2\gamma-2}}\right]$$

Example 5.5. Flow through a Convergent Nozzle

Air in a tank at 700 kPa and 20°C exhausts through a converging nozzle of throat area 0.65 cm^2 to atmosphere. Determine the initial mass flow assuming isentropic flow.



The reservoir will be at stagnation conditions;

$p_o =$ 700 kPa

$T_0 =$ 293K

and the back pressure at atmospheric, $p_b =$ 1 bar = 100kPa

Firstly we determine whether the nozzle is choked from the critical conditions for sonic flow;

$$\frac{p_c}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1} = \left(\frac{2}{1.4+1}\right)^{1.4/1.4-1} = 0.528$$

$p_c =$ 0.528 . 700 = 369.6 kPa

But our back pressure is well below this (we are following a curve like (d) in the above diagram). So the flow is choked and the throat is at critical conditions. The mass flow rate is then found from these conditions; $\dot{m} = \rho_t u_t A_t = \rho_c u_c A_t$

$$\frac{T_c}{T_0} = \frac{2}{\gamma+1}$$

$\cancel{T_0} =$ T_c = 244.2

$u_c =$ $a_t = a_c = \sqrt{\gamma R T_c} = \sqrt{1.4 . 287 . 244.2} = 313$ m/s

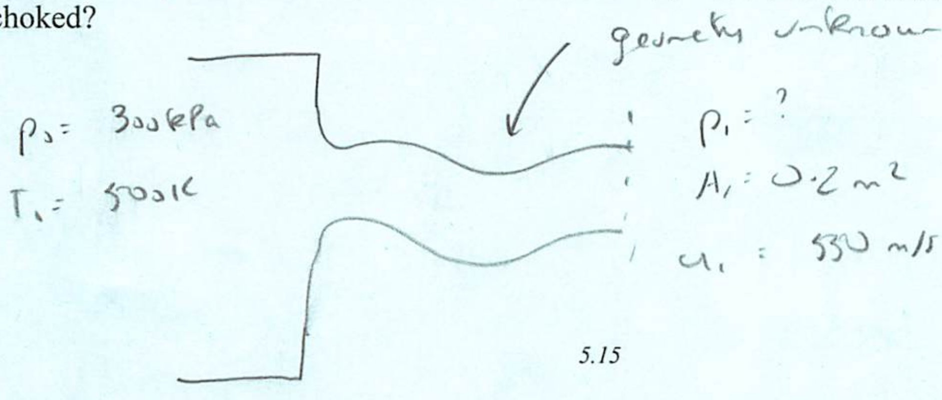
$$\rho_c = \frac{p_c}{RT_c} = \frac{369.6 \times 10^3}{287 . 244.2} = 5.27 \text{ kg/m}^3$$

$$\dot{m}_{max} = \rho_c u_c A_t$$

$= 5.27 . 313 . 0.65 \times 10^{-4} = 0.107$ kg/s

Example 5.6. Supersonic Expansion

Air is flows isentropically from a reservoir, where p=300 kPa and T=500K to a section 1 in a duct where A_1=0.2 m^2 and u_1=550 m/s. Determine (a) M_1, (b)T_1, (c) p_1, (d) $\dot{m}$. Is the flow choked?



The air in the reservoir is at stagnation conditions (velocity is zero). So p_0=300 kPa and T_0=500K. For an adiabatic process $T_0 = T_{01}$

So applying

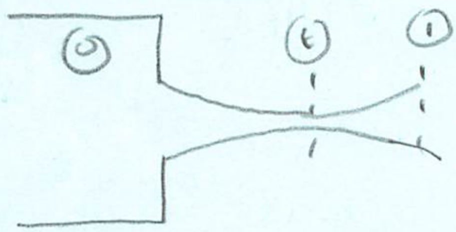
$$\frac{T_{01}}{T_1} = 1 + \frac{u_1^2}{2c_pT_1} \qquad \frac{500}{T_1} = 1 + \frac{550^2}{2.993.T_1} \qquad \text{gives} \quad T_1 = 348\text{K}$$

Now we can determine a_1;

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4 . 287 . 348)} = 374 \text{ m/s}$$

So $M_1 = \frac{u_1}{a_1} = \frac{550}{374} = 1.47$

Therefore at 1 the flow is supersonic. Since in the reservoir the flow was at stagnation, there must be a throat somewhere between the reservoir and 1. The flow at that throat must then be sonic so the nozzle is choked.



Now for the isentropic process 0 to 1

$$\frac{p_0}{p_1} = \left[1 + \left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}} \qquad \frac{300\times10^3}{p_1} = \left[1 + \left(\frac{1.4-1}{2}\right).1.47^2\right]^{\frac{1.4}{0.4}}$$

We use the perfect gas equation to find the density; $\quad p_1 = 86$ kPa

$$\rho_1 = \frac{p_1}{RT_1} = \frac{86\times10^3}{287.348} = 0.861 \text{ kg/m}^3$$

and hence the mass flow rate;

$$\dot{m} = \rho_1 u_1 A_1 = 0.861 . 550 . 0.2 = 94.7 \text{ kg/s}$$

5.11. Normal Shock Waves

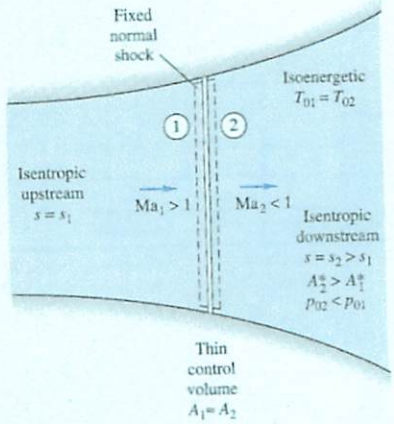
A shock wave is a rapid discontinuity of flow properties. The ~~pressure~~ velocity suddenly drops over a few micrometers. Upstream M>1 of the shock whilst downstream M<1. Because it occurs over a very short distance there is no time for heat transfer to occur – so the shock wave is adiabatic. This means that (from the SFEE):

$$h_{01} = h_{02}$$

And so:

$$T_{01} = T_{02}.$$

However it is not reversible (there are friction losses). So it is not isentropic (i.e. equations 5.13 and 5.14 don't apply). So how do the properties change across a shock wave? Consider a CV around a stationary shock wave.



Applying the MCE:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{5.32}$$

Applying the FME:

Forces on the CV $\quad p_1 A_1 - p_2 A_2$

$$\sum \dot{M}_{in} = \dot{m} u_1$$

$$\sum \dot{M}_{out} = \dot{m} u_2 \qquad \& \; A_1 = A_2$$

$$p_1 A_1 - p_2 A_2 = \dot{m}(u_2 - u_1) \tag{5.33}$$

Combining (5.32) and (5.33) gives:

$$u_2^2 - u_1^2 = (p_2 - p_1)\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$$

Since it is adiabatic:

$$h_1 - h_2 = \frac{1}{2}(u_2^2 - u_1^2)$$

Gives

$$h_2 - h_1 = \frac{(p_2 - p_1)}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \tag{5.34}$$

This is called the Rankin-Hugoniot relation. Note that it is independent of the equation of state for the gas.If we assume the gas is perfect then:

$$h = c_p T = \frac{\gamma p}{(\gamma - 1)\rho}$$

Using this to replace h in equation (5.34) gives:

$$\frac{\rho_2}{\rho_1} = \frac{1 + \beta\left(\dfrac{p_2}{p_1}\right)}{\beta + \left(\dfrac{p_2}{p_1}\right)} \quad \text{where } \beta = \left(\frac{\gamma+1}{\gamma-1}\right) \tag{5.35}$$

Compare this with an isentropic change in pressure (e.g. a weak pressure wave):

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} \tag{5.36}$$

Entropy change across the shock wave (a measure of irreversibility)

Remember:

$$T ds = dh - \frac{dp}{\rho}$$

$\int ds = \int \frac{dh}{T} - R \int \frac{dp}{p}$ $\qquad \rho T = \frac{p}{R}$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R \ln\frac{p_2}{p_1} = c_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1}$$

$$s_2 - s_1 = c_v \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\right] \tag{5.37}$$

The table below (for the case where γ=1.4) uses the non-isentropic relationship for the pressure change across the shock wave compared with an isentropic change.

	$\frac{\rho_2}{\rho_1}$		
$\frac{p_2}{p_1}$	Irreversible Equation (5.28)	Isentropic Equation (5.29)	$\frac{s_2 - s_1}{c_v}$ Equation (5.30)
0.5	0.6154	0.6095	-0.0134
.9	0.9275	0.9275	-0.00005
1.0	1.01	1.0	0
1.1	1.00704	1.00705	0.00004
1.5	1.3333	1.3359	0.0027
2.0	1.6250	1.6407	0.0134

- A decrease in enthalpy i.e. $(s_2-s_1)<0$ is not possible – violates the 2nd law of thermodynamics. This means that the pressure must always increase across a shock wave.
- For the shock wave the pressure ratio is greater than it isentropic value
- For small pressure ratios the difference is negligible and the shock may be considered as reversible and isentropic.

5.12. Property Changes Across a Shock Wave & Shock Tables

(a) Temperature Ratio

We know $T_{01} = T_{02}$ since across the shock wave it is adiabatic.
Then:

$$T_{01} = T_{02}$$

$$\frac{T_2}{T_1} = \frac{T_2}{T_{02}} \cdot \frac{T_{02}}{T_{01}} \cdot \frac{T_{01}}{T_1}$$

But remember the definition of static and stagnation temperature $\frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right)M^2$

$$\frac{T_2}{T_1} = \frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2} \tag{5.38}$$

(b) Density Ratio

From MCE $\quad \rho_1 u_1 = \rho_2 u_2$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{M_1 a_1}{M_2 a_2}$$

$c = \sqrt{\gamma R T}$ $\qquad \frac{a_1}{a_2} = \sqrt{\frac{T_1}{T_2}}$

Substituting equation ~~(5.31)~~ 5.38):

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2}\right]^{0.5} \tag{5.39}$$

(c) Static Pressure Ratio

From the FME equation (5.26):

$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

Replacing ρ in this equation $\quad \rho = \frac{p}{RT}$

And u=Ma

Gives: $\quad p_1 + \frac{p_1}{RT_1} u_1^2 = p_2 + \frac{p_2}{RT_2} u_2^2 \qquad u_1 = M_1\sqrt{\gamma R T_1} \quad u_2 = M_2\sqrt{\gamma R T_2}$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \tag{5.40}$$

Notice how this is always greater than one. The static pressure increases across a shock wave.

(d) Stagnation Pressure Ratio

Lets write:

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2} \cdot \frac{p_2}{p_1} \cdot \frac{p_1}{p_{01}}$$

Remember the relationship between static and stagnation conditions $\quad \frac{p_0}{p} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\gamma/\gamma-1}$

Gives:

$$\frac{p_{02}}{p_{01}} = \frac{p_2}{p_1}\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_2^2}{1+\left(\frac{\gamma-1}{2}\right)M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{5.41}$$

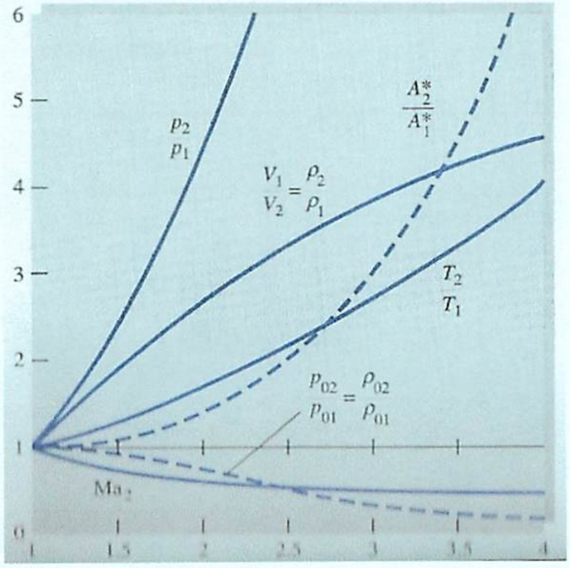
(e) Mach Number Ratio

For a perfect gas $\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$

Combining (5.32), (5.32), (5.33) gives:

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\left(\frac{2\gamma}{\gamma-1}\right)M_1^2 - 1} \tag{5.42}$$

These equations show how the properties of the gas change before and after a shock wave. The following picture shows this graphically.

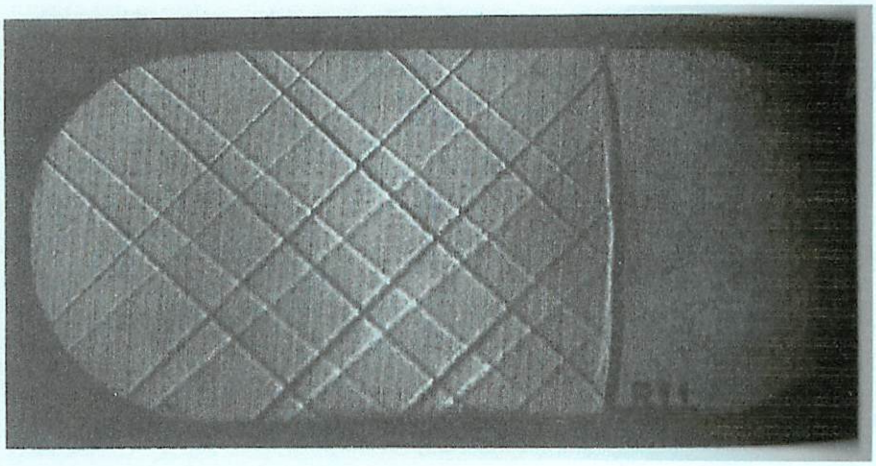


These relationships are also expressed in table form – known as *shock tables*. See page 68 & 69 of the Little Book of Thermodynamics. The tables are presented for increasing values of M_1 (for a gas where γ=1.4). The downstream properties can then be calculated.

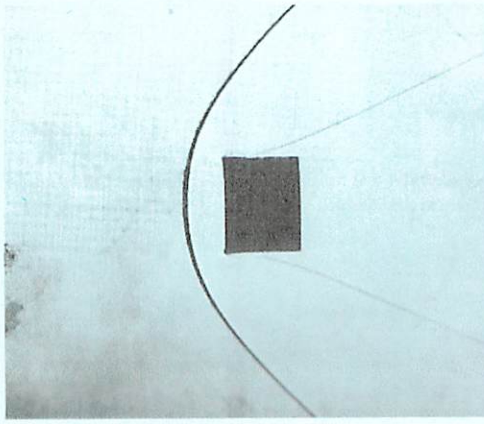
For shock waves we find:

- The upstream flow is supersonic – the downstream flow is subsonic
- Rarefaction (expansion) shocks are not possible. Only compression shocks (where the pressure increase) occur.
- The entropy increases across a shock. Stagnation pressure and stagnation density decrease.

The figure shows supersonic flow (left) and a shock wave forming in a tube. The criss-cross lines are caused by roughness and effects on the pipe wall – they are called Mach waves (they are not shock waves). The shock wave forms and past this the flow is sub-sonic – no Mach waves.

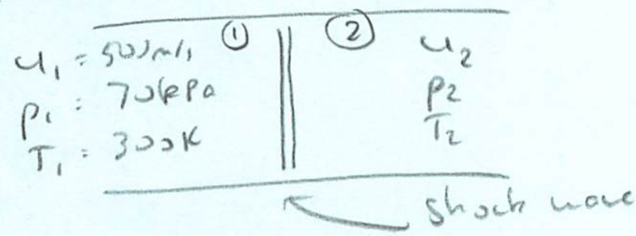


Shock waves also form in external flow. The figure shows a shock forming ahead of a bluff body in a supersonic air stream. The flow inside the shock is then sub-sonic. As the flow passes the corners of the body it speeds up to supersonic again and another shock forms on the sides of the body (this is called an oblique shock).



Example 5.7 – Property change across a shock

Air flowing at a velocity of 500 ms^{-1}, a static pressure of 70 kPa and a static temperature of 300 K undergoes a normal shock. For the location after the normal shock wave determine (a) Mach number, (b) velocity, (c) static pressure and temperature, and (d) stagnation pressure and temperature.



Answer:

We are being asked to find M_2 , V_2, P_2, T_2, P_{02}, T_{02}

First find M_1 is required, hence find a_1

$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot 287 \cdot 300} = 347$ $\qquad M_1 = \frac{500}{347} = 1.44$

We can procede in two ways, **either** using equations for property changes across a shock (5.38) to (5.42) – **or** use the shock tables.

We'll do this by using the shock tables

Looking in the tables for when M_1=1.44 gives:

$$\frac{p_2}{p_1} = 2.253 \quad \frac{\rho_2}{\rho_1} = 1.759 \quad \frac{T_2}{T_1} = 1.281 \quad \frac{p_{02}}{p_{01}} = 0.948$$

We can use mass continuity to find the velocity, u_2

$$\rho_1 u_1 = \rho_2 u_2 \qquad \frac{u_1}{u_2} = \frac{\rho_2}{\rho_1}$$

$$u_2 = \frac{500}{1.759} = 284\ m/s$$

For the static temperature

$$\frac{T_2}{T_1} = 1.281 \qquad T_2 = 300 \cdot 1.281 = 384.3\text{K}$$

For the static pressure

$$\frac{p_2}{p_1} = 2.253 \qquad p_2 = 70 \cdot 2.253 = 157.7 \text{ kPa}$$

We now need the stagnation pressure and temperature. This is easily obtained from the static pressure and temperature and the Mach number using the *isentropic* equations (5.20) and (5.21) but this time lets use the isentropic flow tables.

Looking in the tables for when M_1=1.44 gives:

$$\frac{p_0}{p} = 3.368 \qquad \frac{T_0}{T} = 1.415$$

Starting with the temperatures:

$$\frac{T_{01}}{T_1} = 1.415 \qquad T_{01} = 300 \cdot 1.415 = 424.5 \text{ K}$$

But remember that flow across a shock is adiabatic so;

$$T_{02} = T_{01} = 424.5 \text{ K}$$

Now to find p_{02}. For the isentropic flow 01 to 1:

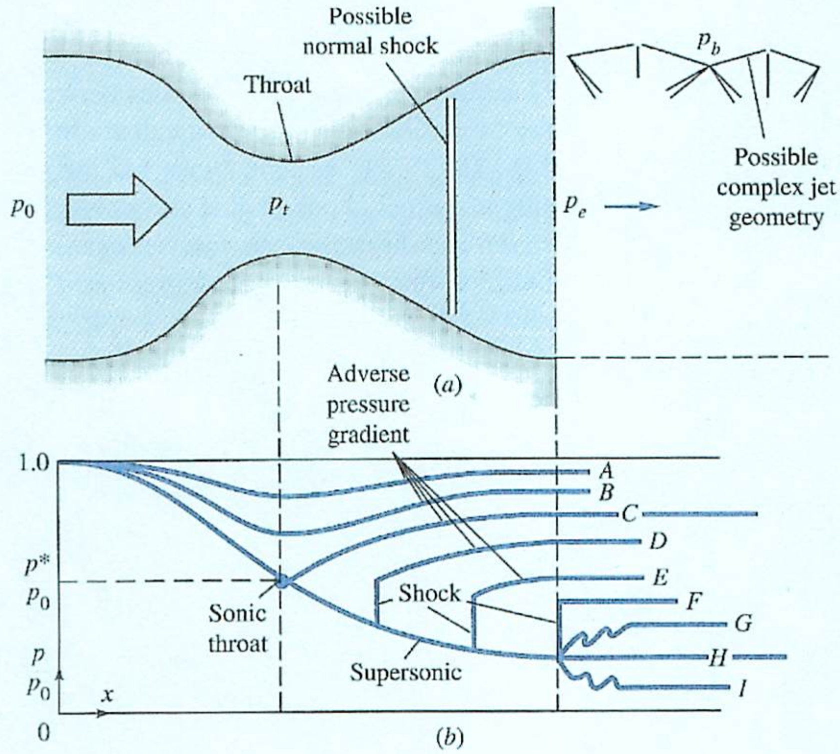
$$\frac{p_{01}}{p_1} = 3.368 \qquad p_{01} = 3.368 \cdot 70 = 235.8 \text{ kPa}$$

And for the flow across the shock:

$$\frac{p_{02}}{p_{01}} = 0.948 \qquad p_{02} = 0.948 \cdot 235.8 = 223.4 \text{ kPa}$$

5.13. Converging-Diverging Nozzles

We know that if we want to accelerate flow from sub-sonic to supersonic we have to have a converging-diverging duct. If the back pressure is low enough we will have super-sonic flow in the diverging part and a variety of shock wave conditions can occur.



Curves (0) no flow (p_b=p_0)

Curve (A, B) The back pressure is not low enough to cause sonic flow at the throat.

Curve (C) The velocity has reached the point that at the throat, M_t=1. The flow in the nozzle is sub-sonic throughout.

Curve (H) Here the back pressure p_b is such that the pressure ratio p_o/p_b exactly corresponds to the critical area ratio A_e/A_c for a supersonic M_e at the exit. The velocity has reached the point that at the throat, M_t=1. The flow in the diverging part of the nozzle is sonic throughout (and isentropic in the entire nozzle). This is called the *design pressure ratio* of the nozzle.

According to the isentropic flow equations it would not be possible for the p_b to lie between curves C & H.

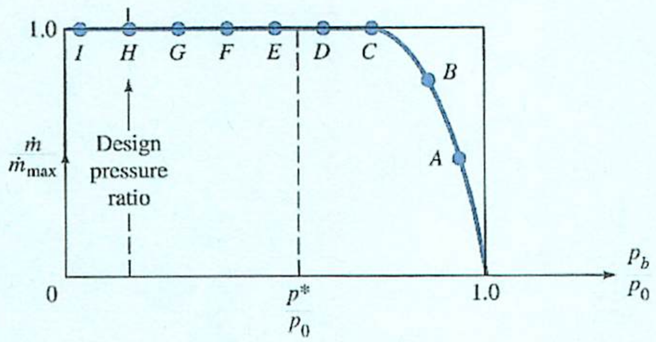
Curve (D,E) The throat remains choked at the sonic value, and a shock wave forms in the diverging part – this makes $p_e = p_b$. This means there is a sub-sonic diffuser flow to the back-pressure condition.

Curve (F) The shock wave stands right at the exit.

Curve (G) There is no place for a shock wave to form in the duct. Instead a series of oblique shocks forms outside the nozzle in the supersonic jet. The flow then compresses back up to p_b.

Curve (I) The back pressure is dropped further but the nozzle cannot respond. The exit flow expands in a complex pattern in the jet until it falls to p_b.

For all pressures below case C, the flow is choked and the mass flow has reached its maximum value.



The air in the reservoir is at stagnation conditions (velocity is zero). So p_0=300 kPa and T_0=500°K. For an adiabatic process $T_0 = T_{01}$

So applying

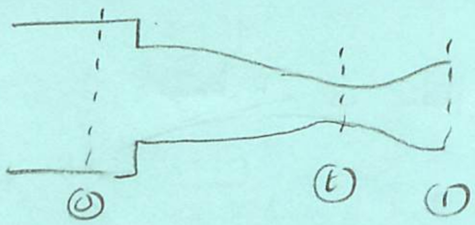
$$\frac{T_{01}}{T_1} = 1 + \frac{u_1^2}{2c_p T_1} \qquad \frac{500}{T_1} = 1 + \frac{550^2}{2.993 . T_1} \qquad \text{gives } \underline{T_1 = 348\ K}$$

Now we can determine a_1;

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4 . 287 . 348)} = 374\ m/s.$$

So $M_1 = \frac{u_1}{a_1} = \frac{550}{374} = \underline{1.47.}$

Therefore at 1 the flow is supersonic. Since in the reservoir the flow was at stagnation, there must be a throat somewhere between the reservoir and 1. The flow at that throat must then be sonic so the nozzle is choked.



Now for the isentropic process 0 to 1

$$\frac{p_0}{p_1} = \left[1 + \left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}} \qquad \frac{300\times10^3}{p_1} = \left[1 + \left(\frac{1.4-1}{2}\right) . 1.47^2\right]^{\frac{1.4}{0.4}}$$

$$\underline{p_1 = 86\ kPa}$$

We use the perfect gas equation to find the density;

$$\rho_1 = \frac{p_1}{RT_1} = \frac{86\times10^3}{287 . 348} = 0.861\ kg/m^3$$

and hence the mass flow rate;

$$\dot{m} = \rho_1 u_1 A_1$$

$$= 0.861 . 550 . 0.2 = 94.7\ kg/s.$$

At the exit the flow is choked

$\Rightarrow$ $T_2 = T_c$

$p_2 = p_c$

At the inlet the flow is at stagnation

$p_1 = p_0$

$T_1 = T_0$

applying $\frac{T_c}{T_0} = \frac{2}{\gamma+1}$ $\quad T_c = 750 \cdot \frac{2}{1.37+1} = 633\,K$

Then $u_2 = a_2$ since flow is choked

$= \sqrt{\gamma R T_2}$

$R = c_p - c_v = c_p - \frac{c_p}{\gamma} = 1.06\times10^3 - \frac{1.06\times10^3}{1.37}$

$= 286\ J/kg\,K$

$\therefore \quad u_2 = \sqrt{1.37 \cdot 286 \cdot 633} = \underline{498\ m/s}$