Lecturers copy RDJ

2. Control Volume Analysis

2.1. Introduction- Fluids in Motion

(White §3.1, Massey §3.1-3.3, first year notes)

In solid mechanics we use the following laws

Conservation of mass $m_{system} = \text{constant} \quad \frac{dm}{dt} = 0$ (2.1)

Newton's 2nd law $F = m a = \frac{d}{dt}(mv)$ ✓ (2.2)

Torque- angular momentum $T = I \dot{\omega} \quad T = \frac{d\Theta}{dt}$ ✓ Θ = angular momentum. (2.3)

Conservation of energy $Q - W = dE$ (2.4)

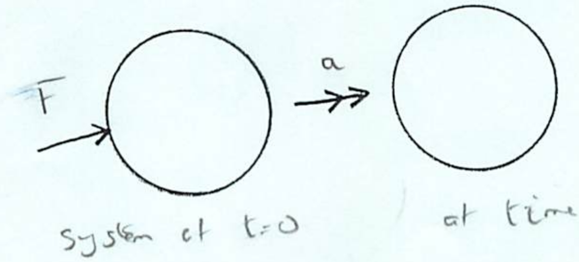
We apply these laws to fixed quantities of mass (or systems). They are known as *system equations*.

When we have a fluid in motion it is usually not convenient to follow fixed quantities of mass. Instead it is more likely that the fluid is the environment to our engineering component; and we want to know the effect of the fluid on our component.

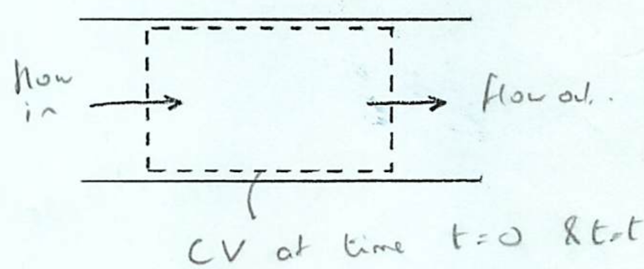
It is more useful to consider a specific region in space i.e. a *control volume*. So we must therefore rewrite the basic equations (2.1), (2.2), (2.3), and (2.4) for control volumes rather than systems.

These are
- the mass continuity equation, MCE
- the force momentum equation, FME
- the torque angular momentum equation, TAME
- the steady flow energy equation, SFEE

Solids



Fluids

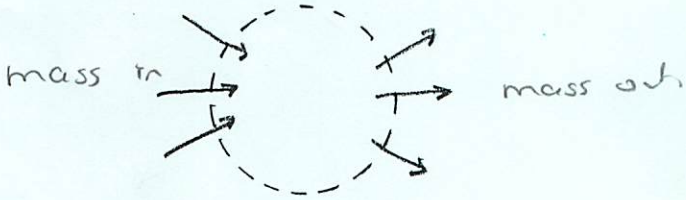


Some types of problem that we can solve using control volume analysis:
- the mass flow through a pipe network
- the force on a pipe bend caused by fluid flowing through it
- the speed of rotation of a garden sprinkler
- the power generated by a water turbine
- the power required by a pump

2.2. The Mass Conservation Equation, MCE

(Massey §3.4, first year notes)

For the control volume apply the law of conservation of mass.



rate at which mass enters CV =

rate at which mass leaves CV
+
rate of mass accumulation in CV

$$\boxed{\sum \dot{m}_{in} = \sum \dot{m}_{out} + \frac{\partial m_{cv}}{\partial t}} \tag{2.5}$$

this is the CV format of equation (2.1)

Special cases:

(a) steady flow (i.e. the fluid parameters (e.g. u, p, T) do not vary with time)

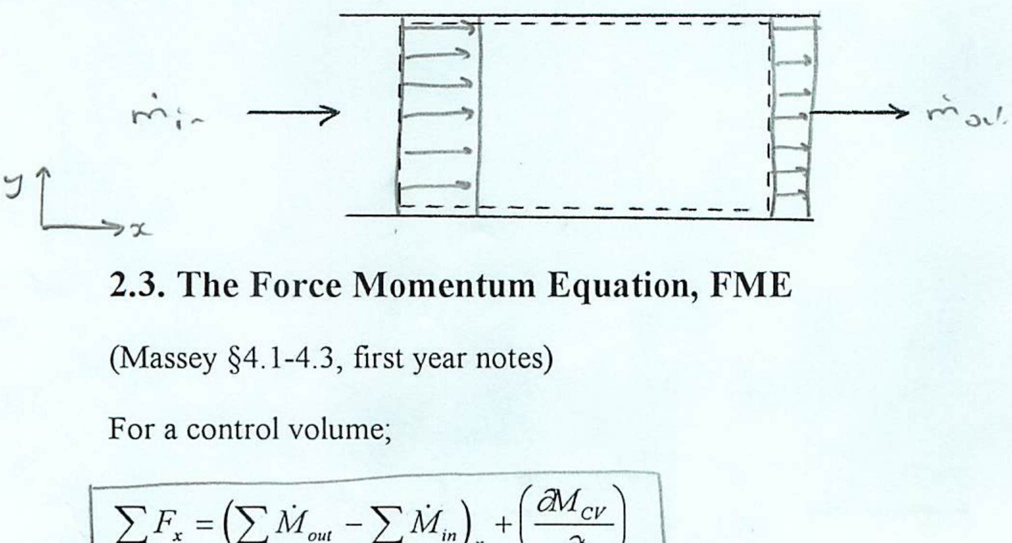
$$\sum \dot{m}_{in} = \sum \dot{m}_{out} \tag{2.6}$$

(b) 1D incompressible flow (the flow is a function of one co-ordinate axis only)

$\dot{m}_{in} = \rho_{in}\, u_{in}\, A_{in}$ $\qquad$ $\dot{m}_{out} = \rho\, u_{out}\, A_{out}$

(2.7)

e.g. flow through a large bore pipe;



1D flow
$u = f(x)$ only
$u \neq f(y,z)$

2.3. The Force Momentum Equation, FME

(Massey §4.1-4.3, first year notes)

For a control volume;

$$\boxed{\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x + \left(\frac{\partial M_{CV}}{\partial t}\right)_x} \tag{2.8}$$

where $\sum F_x$ sum of forces on CV contents

$\sum \dot{M}$ the momentum flow rate (or flux)

$\left(\frac{\partial M_{CV}}{\partial t}\right)$ rate of change of momentum of the CV.

This is the CV form of equation (2.2) which we use in solids. Sometimes it is called the *linear momentum equation.*

Momentum and force are both vector quantities. So equation (2.8) must be used in each direction in turn. There is a similar equation for the y-direction etc,

i.e. $$\sum F_y = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_y + \left(\frac{\partial M_{CV}}{\partial t}\right)_y$$

2.3.1. Different Forms of the FME

(a) steady flow, $\left(\frac{\partial M_{CV}}{\partial t}\right) = 0$ does not vary with time

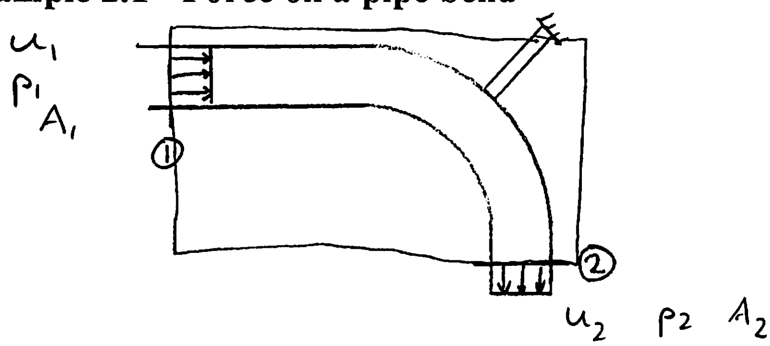
$$\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x \tag{2.9}$$

(b) steady 1D incompressible flow $\dot{m}_{in}$ u_{in} $\dot{M}_{in} = \dot{m}_{in} u_{in}$.

the velocity at inlet/outlet can be considered constant across the section.
So $\dot{M}_{in} = \dot{m}_{in} u_{in}$

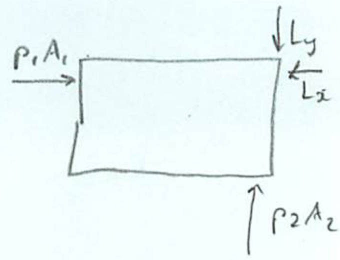
$$\sum F_x = \dot{m}\left(u_{out} - u_{in}\right)_x \tag{2.10}$$

Example 2.1 - Force on a pipe bend



We will apply the the FME in the x-direction initially;

Firstly determine the forces on the CV in x-direction $\xrightarrow{+ve}$



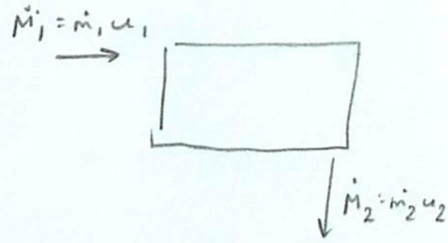
Force caused by pressure acting at station 1 $= p_1 A_1$

Force on CV contents exerted by pipe support $= -L_x$

Now determine the momentum fluxes;

$$(\dot{M}_{in})_x = \dot{m}_1 u_1$$

$$(\dot{M}_{out})_x = 0$$



so applying the FME equation (2.9)

$$-L_x + p_1 A_1 = -\dot{m}_1 u_1$$

$$L_x = p_1 A_1 + \dot{m}_1 u_1$$

If we repeat the analysis in the y-direction we find; $\uparrow$ +ve

$$-L_y + p_2 A_2 = -\dot{m}_2 u_2$$

$$L_y = p_2 A_2 + \dot{m}_2 u_2$$

These forces (L_x and L_y) are exerted **on** the CV contents **by** the surroundings. Thus the force exerted **by** the fluid **on** the pipe bend is equal and opposite.

2.4. The Torque Angular Momentum Equation, TAME

(White §3.5)

There is a similar expression to the Force-Momentum Equation which relates Torque to Angular Momentum.

The FME is used to determine *forces* caused by the *momentum flux* of a fluid. The TAME is useful for determining the *torque, T* (or moment) on a body caused by *angular momentum flux,* $\dot{O}$ of a fluid.

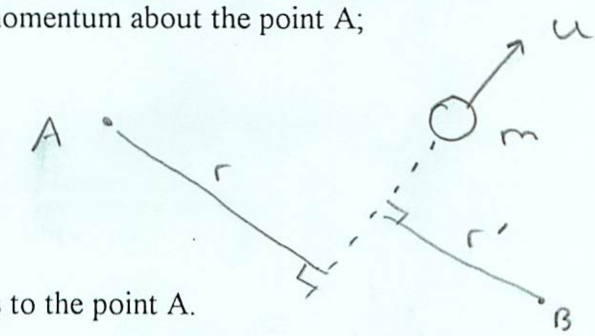
2.4.1 Introduction - Angular Momentum

In solids we we know that applying a torque to a system causes a rate of increase of angular momentum; i.e. equation (2.3).

Angular momentum in solids. Consider a mass, m is travelling at u . The angular momentum,O_A is defined as the moment of linear momentum about the point A;

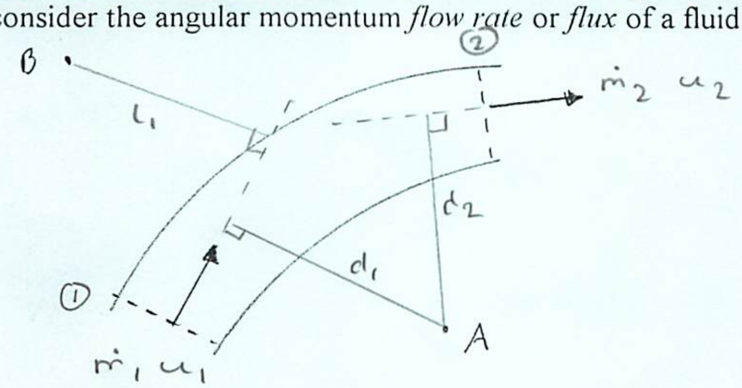
$$O_A = rmu$$

$$O_B = r'mu$$



X where r is the perpendicualr distance from the mass to the point A.

Angular momentum in fluids. A fluid particle can also have angular momentum. It is usually convenient to consider the angular momentum *flow rate* or *flux* of a fluid stream.



So; $\left(\dot{O}_{in}\right)_A = d_1 \dot{m}_1 u_1$ $\left(\dot{O}_{out}\right)_A = d_2 \dot{m}_2 u_2$

& also the direction

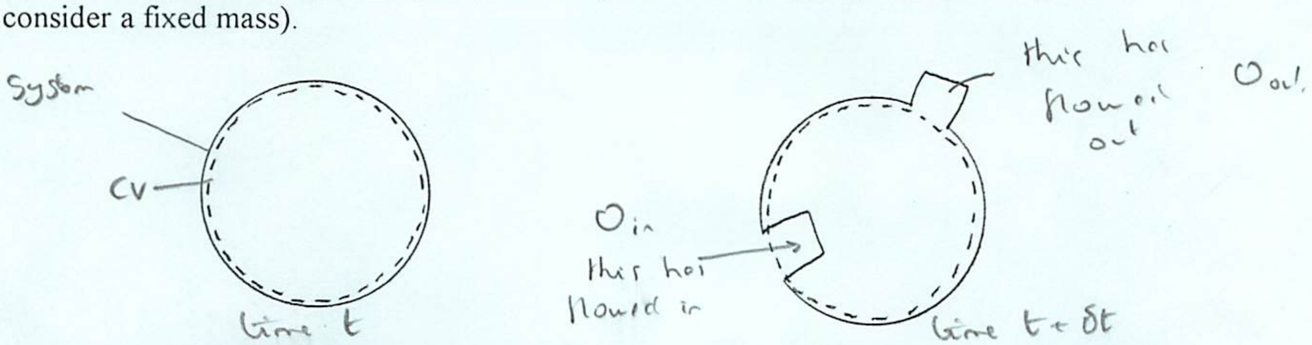
Note: Angular momentum is a vector quantity. So we must state about which point it is being determined. There would be similar expressions for the angular momentum flux about point B.

$$\left(\dot{O}_{in}\right)_B = l_1 \dot{m}_1 u_1$$

In fluids we can derive a relationship between the torque, T and rate of change of angular momentum, $\dot{O}$ for a control volume.

2.4.2. Derivation of the TAME

Consider a CV coincident with a system at time t. At time $t+\delta t$ some fluid has flowed into the system whilst some has flowed out. The CV remains the same (remember in CV analysis we consider only a fixed region in space) but the system has changed (in system analysis we consider a fixed mass).



At time t the angular momenta of the CV and the system about some point A are identical.

$$(O_{cv})_t = (O_{system})_t$$

At time $t+\delta t$ the system has changed shape so its angular momentum about A has changed. Some angular momentum has flowed in whilst some has flowed out. So now;

$$(O_{cv})_{t+\delta t} = (O_{system})_{t+\delta t} + O_{in} - O_{out}$$

dividing by dt;

$$\frac{\delta O_{cv}}{\delta t} = \frac{\delta O_{system}}{\delta t} + \dot{O}_{in} - \dot{O}_{out}$$

(2.11)

We know that for a system equation (2.3) applies so;

$$T = \frac{\delta O_{system}}{\delta t} \qquad (2.12)$$

Combining (2.11) and (2.12) gives;

$$T = \frac{\delta O_{cv}}{\delta t} + \dot{O}_{out} - \dot{O}_{in}$$

Now suppose we have a number of torques acting on the CV, and also a number of inflows and outflows of angular momentum, then;

$$\boxed{\sum T_A = \left(\frac{\partial O_{CV}}{\partial t}\right)_A + \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A} \qquad (2.13)$$

This is the Torque Angular Momentum Equation (TAME).

where $\sum T_A$ the sum of the torques acting on the CV contents

$\left(\sum \dot{O}_{in}\right)_A$ the sum of the angular momentum fluxes into the CV

$\left(\frac{\partial O_{CV}}{\partial t}\right)_A$ the rate of change of angular momentum of the CV contents

Again torque and angular momentum are vector quantities, so we must specify about which point they act (in this case they act about the point A) and also their direction. A similar equation exists for torques about other points B, C and so on.

Note how similar this is to the force momentum equation (FME).

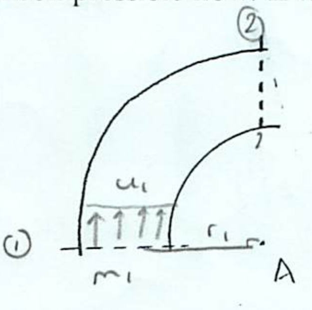
2.4.3 Different Forms of the TAME

(a) Steady flow, the angular momentum of the CV does not change with time;

$$\frac{\delta O_{cv}}{\delta t} = 0$$

$$\sum T_A = \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A \qquad (2.14)$$

(b) Steady 1D incompressible flow. If we consider 1D flows into and out of a control volume;



T not good drawing

• A

strictly r is not constant across the fluid section but usually d<<r

All the fluid flowing into the CV has a velocity of u_1 whilst all the fluid flowing out of the CV has a velocity of u_2.

So $\quad \dot{O}_{in} = \dot{m}_1 u_1 r_1$

$\quad\quad \dot{O}_{out} = \dot{m}_2 u_2 r_2$

but also by the mass conservation equation for 1D steady state flow - equation (2.7)

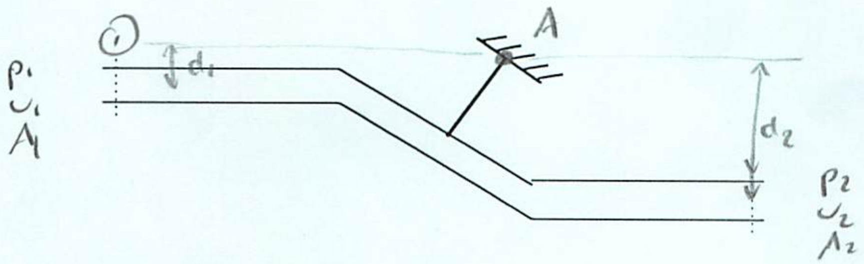
$$\dot{m}_1 = \dot{m}_2 = \dot{m}$$

Putting these into the TAME above (2.14)

$$\sum T_A = \dot{m}\left(r_2 u_2 - r_1 u_1\right) \qquad (2.15)$$

Example 2.2. Torque on a pipe bend.

Determine the torque which must be resisted by the support. Assume steady state, incompressible, 1D flow.



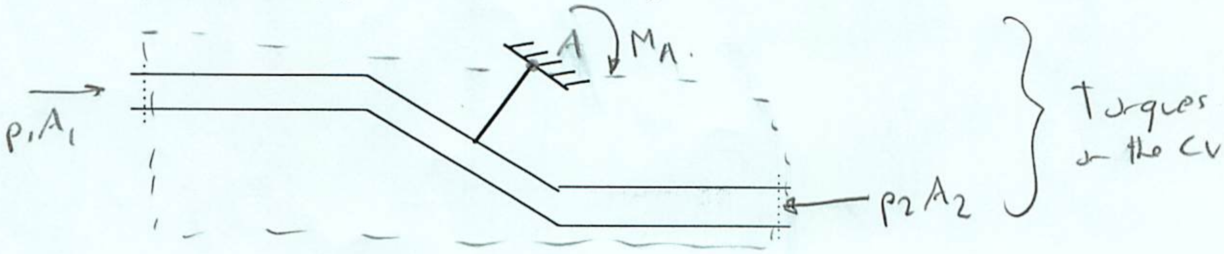
Step 1 Draw a CV - choose the location so that at the boundaries we know the fluid properites (i.e. the CV should pass through stations 1 and 2 and point A.

Step 2 Choose a point about which we determine torques and angular momentum. Choose a direction for positive torque. (+ve A)

Since we want the torque at the pipe support, choose that point, A. Arbitrarily lets make clockwise positive.

Step 3 Determine the torques acting on the CV.

These may be externally applied (e.g. the torque exerted by the pipe support) or they may be caused by pressure on the boundary.



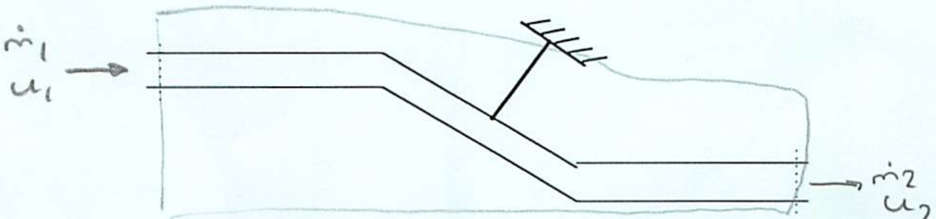
In this case we have the following torques;

$+M_A$ (this is want we want to find)

$$\left.\begin{array}{l}-p_1 A_1 d_1 \\ +p_2 A_2 d_2\end{array}\right\} \text{pressure torques acting on the CV.}$$

Step 4. Determine the angular momentum flux into and out of the CV.

For angular momentum flux about the point A, we must determine the tangential (whirl) velocity and the radius about which it acts.



So angular momentum flux in: $(\dot{O}_{in})_A = -\dot{m}_1 u_1 d_1$

and angular momentum flux out: $(\dot{O}_{out})_A = -\dot{m}_2 u_2 d_2$

Step 5. Use the mass conservation equation (2.6) to relate inflows to outflows

$$\dot{m}_1 = \dot{m}_2$$

In this case we have 1D flow (i.e. across pipe section 1 we can assume the velocity is constant, u_1). So;

$$\rho u_1 A_1 = \rho u_2 A_2$$

Step 6. Assemble the torque angular momentum equation (TAME).

$$\sum T_A = \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A$$

Substituting the torques and momentum flux

$$M_A - p_1 A_1 d_1 + p_2 A_2 d_2 = (-\dot{m}_2 u_2 d_2) - (-\dot{m}_1 u_1 d_1)$$

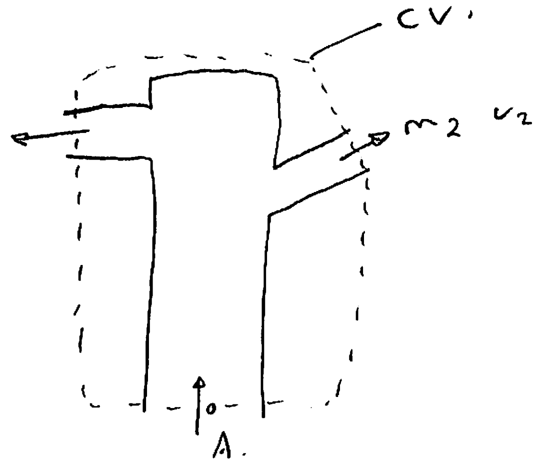
and replacing the mass flow rate,

$$M_A - p_1 A_1 d_1 + p_2 A_2 d_2 = \rho\left[u_1^2 A_1 d_1 - u_2^2 A_2 d_2\right]$$

Step 7. This will give the torque, M_A exerted **on** the CV **by** the surroundings. Then the torque exerted **by** the fluid (i.e. the CV contents) **on** the pipe support will be equal and opposite to this.

Torque on CV contents exerted by support $= M_A$

Torque on pipe support $= -M_A$.



Angular momentum fluxes about A $\circlearrowright$ +ve.

$$(\dot{O}_{in})_A = 0$$

$$(\dot{O}_{out})_A = -\dot{m}_1 v_1 d_1 \quad \text{(negative since } \circlearrowright \text{)}$$
$$+ \dot{m}_2 v_2 d_2 \quad \text{(positive since } \circlearrowright \text{)}.$$

Torques on the CV $\circlearrowright$ +ve

$$\sum T = M_a$$

no pressure torques since $p_1 = p_2 = p_{atm}$.

By the MCE

$$\dot{m}_3 = \dot{m}_1 + \dot{m}_2 = \rho v_1 A_1 + \rho v_2 A_2.$$

$$280 = (1000 \cdot 36.8 \cdot 3800 \times 10^{-6}) + (1000 \cdot v_2 \cdot 3800 \times 10^{-6})$$

$$v_2 = 36.8 \text{ m/s}.$$

$$\dot{m}_1 = \rho v_1 A_1$$
$$= 1000 \cdot 36.8 \cdot 3800 \times 10^{-6} = 140 \text{ kg s}^{-1}$$

$$\dot{m}_2 = \rho v_2 A_2$$
$$= 1000 \cdot 36.8 \cdot 3800 \times 10^{-6} = 140 \text{ kg s}^{-1}$$

Assemble the TAME

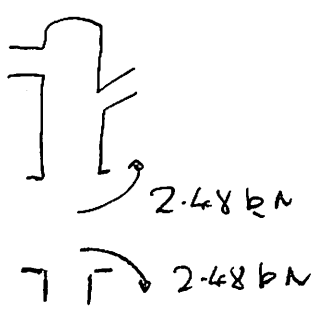
$$\sum T_A = \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A$$

$$M_A = (-\dot{m}_1 u_1 d_1 + \dot{m}_2 u_2 d_2) - 0$$

$$= (-140 \cdot 36.8 \cdot .75 + 140 \cdot 36.8 \cdot .27)$$

$$= -2480 \text{ N m} = -2.48 \text{ kN m}$$

i.e. the torque on the CV contents acts



So torque on pipe support is 2.48 kNm

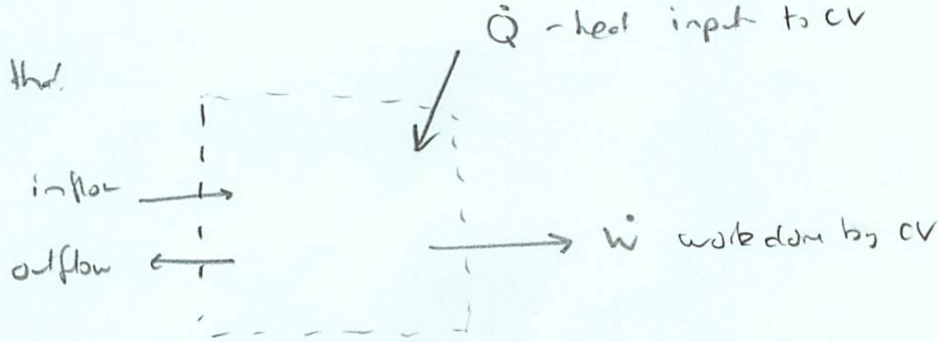
2.5. The Steady Flow Energy Equation (SFEE)

(Massey §3.6, first year notes)

So far in this analysis of fluid flow using CV's we have not considered the effect of energy input or output from the fluid.

Clearly if a fluid flows past the vanes of a turbine, it loses some of its energy and its flow properites are changed as a result. We need some method for determining how energy transfer effects flow properties.

Derivation similar to that.



In thermodynamics we use the first law (Q-W=ΔE) to study the effects of energy transfer to and from systems. In this section we will derive and use a CV form of the first law called the *steady flow energy equation*, SFEE. Derivation see Massey p. 81.

$$\boxed{\frac{\dot{Q}-\dot{W}}{\dot{m}} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2\right) - \left(e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1\right)} \tag{2.16}$$

e - specific internal energy $= \frac{E}{\dot{m}}$

In deriving it we imply a number of assumptions;

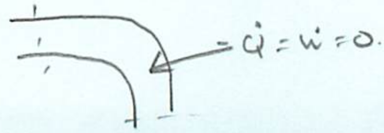
- steady continuous flow
- heat and shaft work transfer at a constant rate
- velocities are uniform over cross-sections 1 and 2 (i.e. 1D flow at 1 and 2)
- no energy trafer due to electricity, magnetism, surface tension, nuclear reaction

2.5.1. Different Forms of the SFEE

(a) No heat or work transfer, $\dot{Q} = \dot{W} = 0$. If the fluid is incompressible and inviscid (i.e. frictionless) then there are no losses so e_1=e_2; and the SFEE reduces to Bernouilli's equation;

$$\left(\frac{p_2}{\rho} + \frac{u_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{u_1^2}{2} + gz_1\right) = 0 \tag{2.17}$$

$\left.\begin{array}{l}\text{inviscid} \\ \text{ideal fluid} \\ \text{frictionless}\end{array}\right\} \mu = 0$



(b) No heat or work transfer, $\dot{Q} = \dot{W} = 0$. E.g. flow of fluid through pipes and fittings.

$$\left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1\right) + (e_2 - e_1) = 0 \qquad (2.18)$$

The (e_1-e_2) represents a loss of useful energy resulting from fluid friction. Often expressed as a head loss (i.e. by dividing by g);

$$h_f = \frac{e_2 - e_1}{g}$$

Frequently these head losses are determined by experiment. For pipe networks they are called *minor and major loss coefficients*. These are presented in tables or charts (remember the Moody chart).

(c) Compressible flow. i.e. $\rho_1 \neq \rho_2$

From thermodynamics remember enthalpy;

$$H = E + pV \qquad (2.19)$$

or in its specific form (dividing 2.19 by mass, m);

$$h = e + \frac{p}{\rho} \qquad (2.20)$$

replacing e in the SFEE gives;

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(h_2 + \frac{u_2^2}{2} + gz_2\right) - \left(h_1 + \frac{u_1^2}{2} + gz_1\right) \qquad (2.21)$$

For gases the gravity terms are negligable (neglect gz_1 and gz_2) and for flow of a perfect gas;

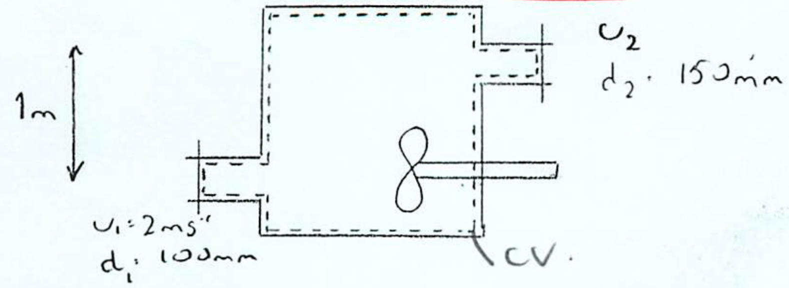
$$h = c_p T \qquad (2.22)$$

$$\boxed{\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(c_p T_2 + \frac{u_2^2}{2}\right) - \left(c_p T_1 + \frac{u_1^2}{2}\right)} \qquad (2.23)$$

we will be using this form of the SFEE in the study of compressible fluid flows (i.e. gases) later in the course.

Example 2.3. - A Water Pump.

In the water pump shown below, determine the power required to deliver the water at a gauge pressure of 2 bar. Assume the fluid flow is frictionless. $p_1 =$ atm



Draw a CV boundary around the pump.

by the MCE (assume incompressible, steady, 1D flow);

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$\dot{m} = 1000 \cdot 2 \cdot \pi \frac{.1^2}{4} = 15.7 kg\, s^{-1}$$

$$u_2 = u_1 \cdot \frac{d_1^2}{d_2^2} = 0.89\ m\, s^{-1}$$

SFEE 1 to 2 - equation (2.16)

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2\right) - \left(e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1\right)$$

We can assume that the heat flow to/from the fluid is negligable (if the fluid is at a similar temperature to the surrounding, or if the pump is insulated, this will be valid) $\dot{Q} = 0.$

The flow is frictionless so $e_1 = e_2$ then;

$$-\frac{\dot{W}}{\dot{m}} = \left(\frac{p_2}{\rho} + \frac{u_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{u_1^2}{2} + gz_1\right)$$

rearranging,

$$-\frac{\dot{W}}{\dot{m}} = \left(\frac{p_2 - p_1}{\rho}\right) + \left(\frac{u_2^2 - u_1^2}{2}\right) + g(z_2 - z_1)$$

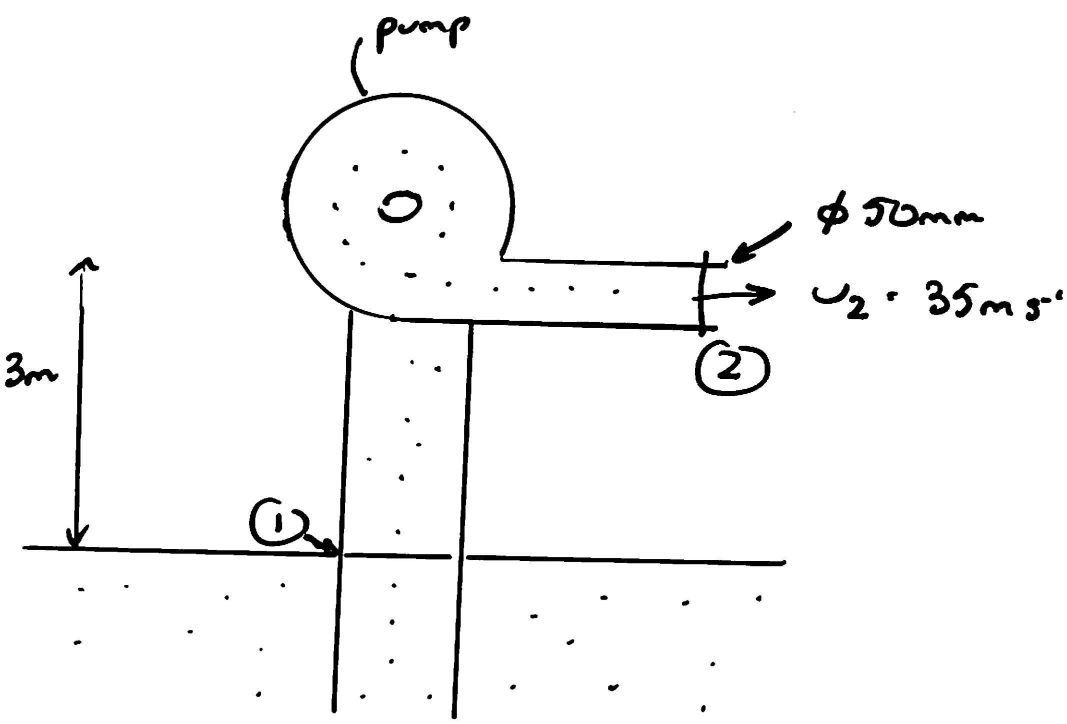
Now $p_1 = p_{atm}$, so since the discharge pressure is quoted as 2 bar *guage*.

$p_2 - p_1 =$ $2 \times 10^5\ N/m^2$

$$-\frac{\dot{W}}{15.7} = \left(\frac{2 \times 10^5}{1000}\right) + \left(\frac{.89^2 - 2^2}{2}\right) + 9.81 \cdot 1$$

$\dot{W} = -3.3$ kW   The result turns out to be negative. So work is being done **on** the fluid **by** the pump.

Example - Power supply to a pump



The total head loss in the pump & pipework is 2m. If the pump is 75% efficient what power is needed?

SFEE between ① & ②

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(\frac{p_2}{\rho} + \frac{u_2^2}{2} + g z_2 + e_2 \right) - \left(\frac{p_1}{\rho} + \frac{u_1^2}{2} + g z_1 + e_1 \right)$$

assume no heat transfer $\dot{Q} = 0$

pump discharges to atmosphere $p_1 = p_2 = p_{atm}$

rewrite e_1 & e_2

$$h_f = \frac{e_2 - e_1}{g}$$

water from reservoir so $u_1 = 0$

$$-\frac{\dot{W}}{\dot{m}} = + \frac{u_2^2}{2} + g(z_2 - z_1) + g h_f$$

~~Vo~~

by the MCE

$$\dot{m} = \rho u A \;=\; 998 \cdot 35 \cdot \pi \frac{.05^2}{4}$$

$$= 68.6 \text{ kg/s}$$

$$\dot{W} = -68.6 \left[+ \frac{35^2}{2} + 9.81(+3) + 9.81 \cdot 2 \right]$$

$$= -68.6 \left(-612.5 + 49.1 \right) = -45 \text{ kW}$$

$$= -45 \text{ kW}$$

−ve ie pump is doing work on the fluid