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5.5 Compressible Flow and Bernoulli's Equation

If we thought our gas was incompressible we could use Bernoulli's equation (remember this is the version of the SFEE assuming incompressible adiabatic, frictionless/reversible flow).

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{u_2^2}{2} + g z_2$$

assume ρ constant $\dot{Q} = \dot{W} = 0$ $e_1 = e_2$

Again neglecting gravity effects. And if states 1 and 2 are static and stagnation;

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho} + \frac{u_2^2}{2} \qquad \frac{p}{\rho} + \frac{u^2}{2} = \frac{p_0}{\rho}$$

rearranging $\quad \dfrac{p_0}{p} = 1 + \rho \dfrac{u^2}{2p}$

and for a perfect gas including (5.2) $\longrightarrow \quad \dfrac{p}{\rho} = RT$

$$\frac{p_0}{p} = 1 + \frac{u^2}{2RT} \tag{5.15}$$

This is the relation between static and stagnation conditions if we assume our gas is *incompressible*. We can compare the results from this with those from equation (5.13) for *compressible* flow.

If the fluid is air at 1 bar and 300K;

gas velocity	$\frac{p_0}{p} = 1 + \frac{u^2}{2RT}$	$\frac{p_0}{p} = \left(1 + \frac{u^2}{2c_pT}\right)^{\frac{\gamma}{\gamma-1}}$
20 m/s	1.0023	1.0023
200 m/s	1.232	1.252
500 m/s	2.45	3.4

As the velocity of the gas increases the assumption becomes less valid. Typically for M<0.3 we can assume incompressibility.

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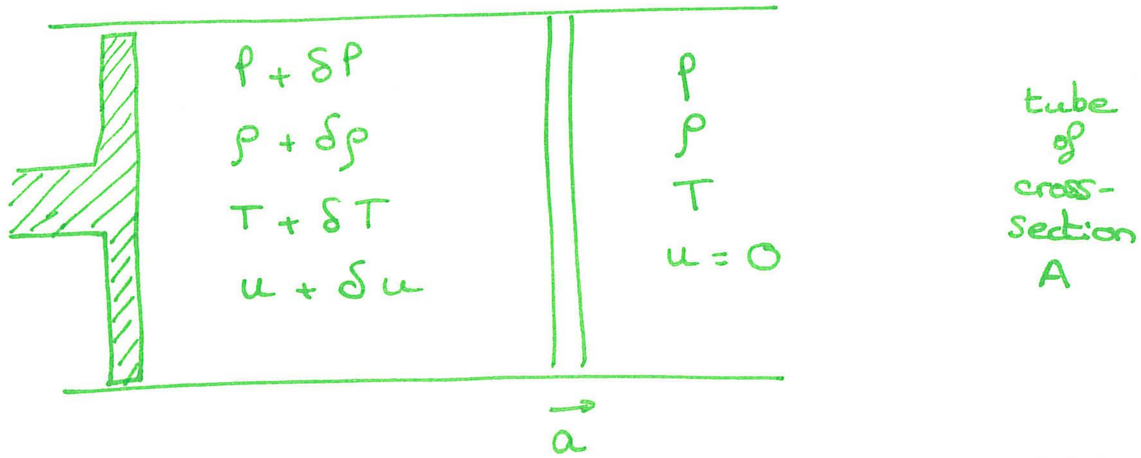
$M = \frac{u}{a}$

5.6. Sonic Velocity and Mach Number

The velocity of sound is the rate of propagation of a pressure pulse through a still fluid. Pressure changes do not occur instantaneously. Sound waves are pressure waves through the gas (note: a faint sound corresponds to pressure fluctuations of 3×10^{-5} Pa, whilst the threshold of pain is 100 Pa).

We need to find the speed of the waves (i.e. the speed of sound) and how it depends on the state of the gas.

Consider a tube full of stationary gas. A sudden small piston movement causes a wave of pressure to travel down the tube. The wave travels at a speed *a* down the tube;



This is a non-steady flow (i.e. the speed and pressure at a point vary with time). However if we consider a CV travelling with the wave then we have steady flow.

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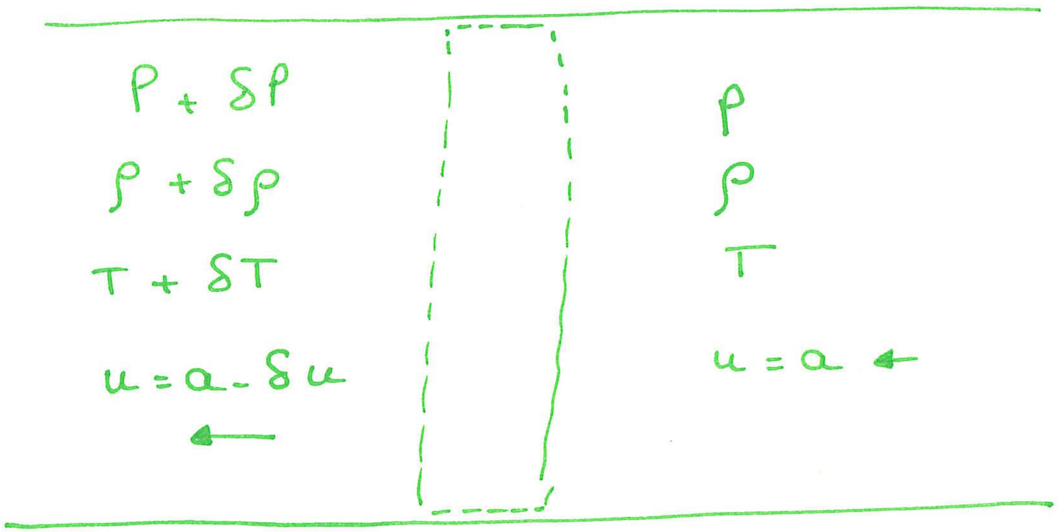
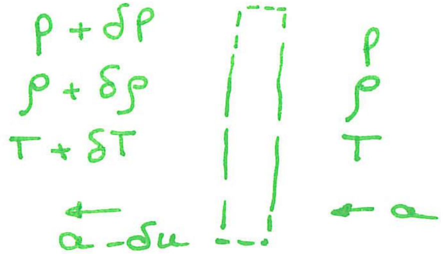


Applying the MCE to the CV;

$$\rho a A = (\rho + \delta\rho)(a - \delta u)A$$

$$\rho \cancel{a} A = \rho \cancel{a} A + \delta\rho\, a A - \rho\, \delta u A - \cancel{\delta\rho\, \delta u} A$$

$$a\delta\rho - \rho\delta u = 0$$



Neglecting second order terms;

$$a\delta\rho - \rho\delta u = 0 \tag{5.16}$$

Applying the FME to the CV;

$$\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x \quad = \dot{m}\,(u_{out} - u_{in})_x$$

$$(p + \delta p)A - pA = \rho a A\,[(-a + \delta u) - (-a)]$$

$$\delta p - \rho a \delta u = 0 \tag{5.17}$$

Eliminating δu by combining (5.16) and (5.17) gives;

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right) \tag{5.18}$$

If the pressure wave has a very small amplitude then only a small amount of heat flow and friction losses occur. The passage of the wave may therefore be assumed to be reversible and adiabatic (and hence isentropic). So for a perfect gas (5.8) gives;

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

$$\frac{p_1}{\rho_1^{\gamma}} = \frac{p_2}{\rho_2^{\gamma}}$$

$$\frac{p}{\rho^{\gamma}} = \text{constant} = K$$

$$p = K\rho^{\gamma}$$

Differentiating w.r.t. ρ

$$\frac{\partial p}{\partial \rho} = K\gamma\rho^{\gamma-1}$$

substituting for K gives;

$$\frac{\partial p}{\partial \rho} = \left(\frac{p}{\rho^{\gamma}}\right)\gamma\rho^{\gamma-1} = \frac{p}{\rho}\gamma$$

so combining this with (5.18) gives;

$$\boxed{a = \sqrt{\gamma RT}} \qquad (5.19)$$

Thus the speed of sound in a gas is obtained easily from the state of the gas.

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Example 5.3 The Speed of Sound in Air

Determine the speed of sound in air at 15°C. (R=287 J/kgK, γ=1.4).

$$a = \sqrt{\gamma R T}$$

$$= \sqrt{1.4\,(287)(273+15)}$$

$$= 340 \text{ m/s}$$

5.7 Compressible Flow Relations for a Perfect Gas

It is useful to rewrite equations (5.11), (5.13), and (5.14) in terms of M;

remember $C_p = \frac{R\gamma}{\gamma - 1}$

$a^2 = \gamma RT$

(5.11) $\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}$ and $u = Ma$

$$\frac{T_0}{T} = 1 + \frac{(Ma)^2}{2\left(\frac{R\gamma}{\gamma-1}\right)T} = 1 + \frac{M^2(\gamma RT)}{\frac{2R\gamma T}{\gamma-1}} = 1 + \left(\frac{\gamma-1}{2}\right)M^2$$

gives;

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma - 1}{2}\right)M^2 \quad \text{— adiabatic} \tag{5.20}$$

$$\frac{p_0}{p} = \left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma - 1}} \tag{5.21}$$

adiabatic reversible & isentropic flow

$$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]^{\frac{1}{\gamma - 1}} \tag{5.22}$$

and since $a = \sqrt{\gamma RT}$

$$\frac{a_0}{a} = \left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]^{\frac{1}{2}} \tag{5.23}$$

These relations apply to a compressible perfect gas flowing adiabatically for (5.20) and adiabatically and reversibly (i.e. isentropically) for (5.21) & (5.22)

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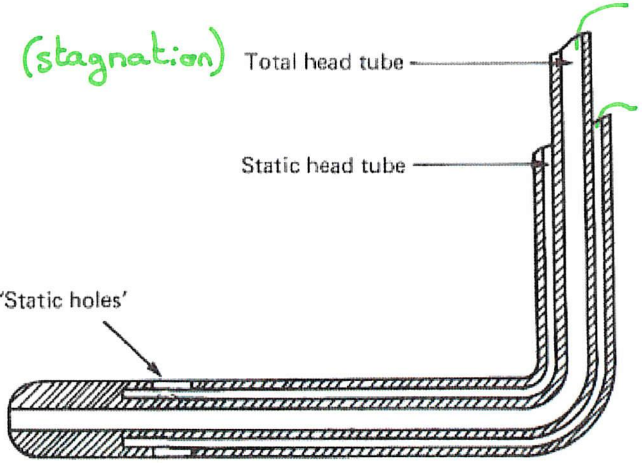
Example 5.4 A Pitot Tube in Compressible Flow

A pitot-static tube is used to measure the static (1 bar) and stagnation (or total) pressures (1.5 bar) of air flowing in a wind tunnel. Find the Mach number.



We assume air is a perfect gas and the process by which the air is brought to rest at the nose of the pitot tube is frictionless and adiababtic. We can then use the isentropic relation (5.21);

$$\frac{P_o}{P} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma - 1}} \qquad \frac{1.5}{1} = \left[1 + \frac{(1.4-1)}{2} M^2\right]^{\frac{1.4}{0.4}}$$

$M =$ 0.78

i.e. M<1 subsonic flow. In practice we cannot use a pitot tube in supersonic flow. A shock wave forms ahead of the nose so fluid is not brought to rest isentropically.

5.8. Critical Conditions

Just as the stagnation conditions are a useful reference, it is also useful to describe the fluid properties when the flow is sonic. These sonic properties are know as critical conditions; Putting M=1 (i.e. sonic flow) into (5.20)-(5.23) gives;

$$\frac{T_c}{T_0} = \frac{2}{\gamma + 1} \tag{5.24}$$

$$\frac{p_c}{p_0} = \left[\frac{2}{\gamma + 1}\right]^{\frac{\gamma}{\gamma - 1}} \tag{5.25}$$

$$\frac{\rho_c}{\rho_0} = \left[\frac{2}{\gamma + 1}\right]^{\frac{1}{\gamma - 1}} \tag{5.26}$$

$$\frac{a_c}{a_0} = \left[\frac{2}{\gamma + 1}\right]^{\frac{1}{2}} \tag{5.27}$$

As before, in isentropic flow all the critical properties are constant. In adiabatic non-isentropic flow a_c and T_c are constant, but p_c and ρ_c may vary.

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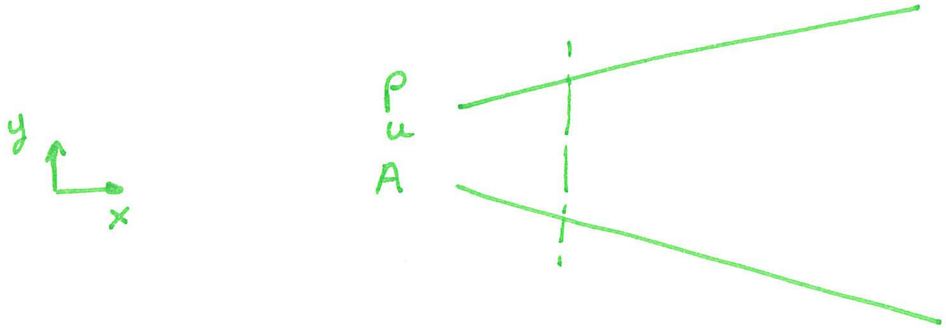
5.9. Effect of Area Variation in a Duct

Consider the flow in a duct with area which varies in the x-direction;



how does u change with x ?

The MCE tells us

$$\rho(x)u(x)A(x) = \dot{m} = \text{constant}$$

since ρ, u, and A are all functions of x. Differentiating with respect to x

$$\frac{\partial \dot{m}}{\partial x} = \frac{\partial}{\partial x}(\rho u A) = A u \frac{\partial \rho}{\partial x} + \rho A \frac{\partial u}{\partial x} + \rho u \frac{\partial A}{\partial x} = 0$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \tag{5.28}$$

This is the differential form of the MCE.

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Also for a perfect gas; $\frac{p}{\rho^{\gamma}} = K$

differentiating w.r.t. x and substituting for K (as we did for 3.18)

$$\frac{p}{\rho} + \frac{u^2}{2} + gz = K$$

$$\frac{\partial p}{\partial x} = \frac{\gamma p}{\rho} \frac{\partial \rho}{\partial x} \qquad (5.29)$$

We can also differentiate Bernoulli's equation in the same way. This gives Euler's equation;

$$\frac{1}{\rho}\frac{dp}{dx} + u\frac{\partial u}{\partial x} + g\frac{\partial z}{\partial x} = 0 \qquad \frac{1}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial u}{\partial x} = 0$$

Neglecting the gravity terms and substituting dp/dx (5.29);

$$\frac{\gamma p}{\rho^2}\frac{\partial \rho}{\partial x} = -u\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial \rho}{\partial x} = -\frac{\rho u}{\gamma p}\frac{\partial u}{\partial x}$$

putting this into (5.28);

$$-\frac{\rho u}{\gamma p}\frac{\partial u}{\partial x}+\frac{1}{u}\frac{\partial u}{\partial x}+\frac{1}{A}\frac{\partial A}{\partial x}=0$$

Substituting for *M*;

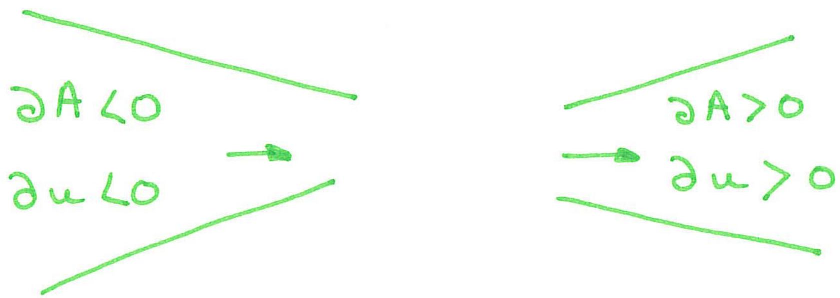
remember $M=\frac{u}{a}=\frac{u}{\sqrt{\gamma RT}}=u\sqrt{\frac{\rho}{\gamma p}}$

$$\boxed{\frac{A}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\left(M^2-1\right)=\frac{\mathrm{d}A}{\mathrm{d}x}} \tag{5.30}$$

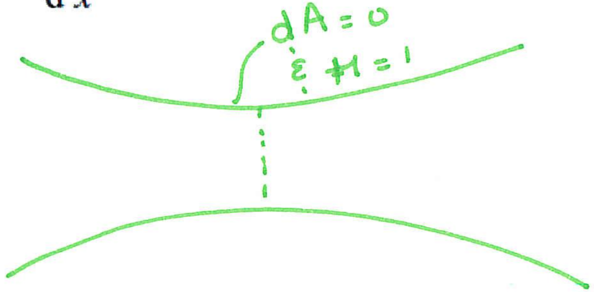
This equation describes how the flow velocity varies when the cross section area changes. Inspection tells us some unusual aspects;

(i) If M<1 (subsonic) then $\frac{\mathrm{d}A}{\mathrm{d}x}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ must have opposite signs. So the fluid accelerates through a converging duct. And decelerates through a diverging duct.

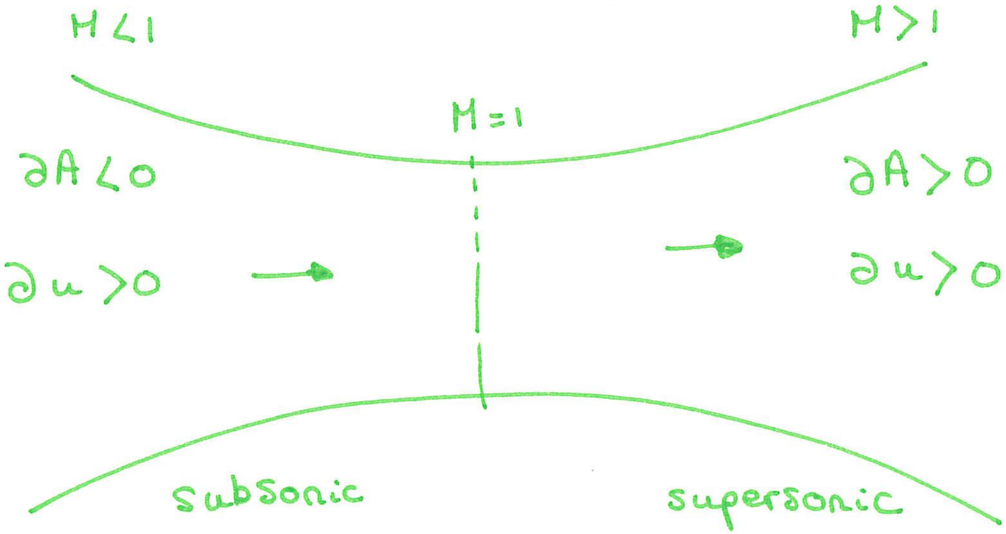
(ii) If M>1 (supersonic) then $\frac{\mathrm{d}A}{\mathrm{d}x}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ must have the same sign. So the fluid accelerates through a diverging duct. And decelerates through a converging duct.



(iii) If M=1 (sonic) then $\frac{dA}{dx}$=0. So the flow area must be a minimum i.e. a throat.



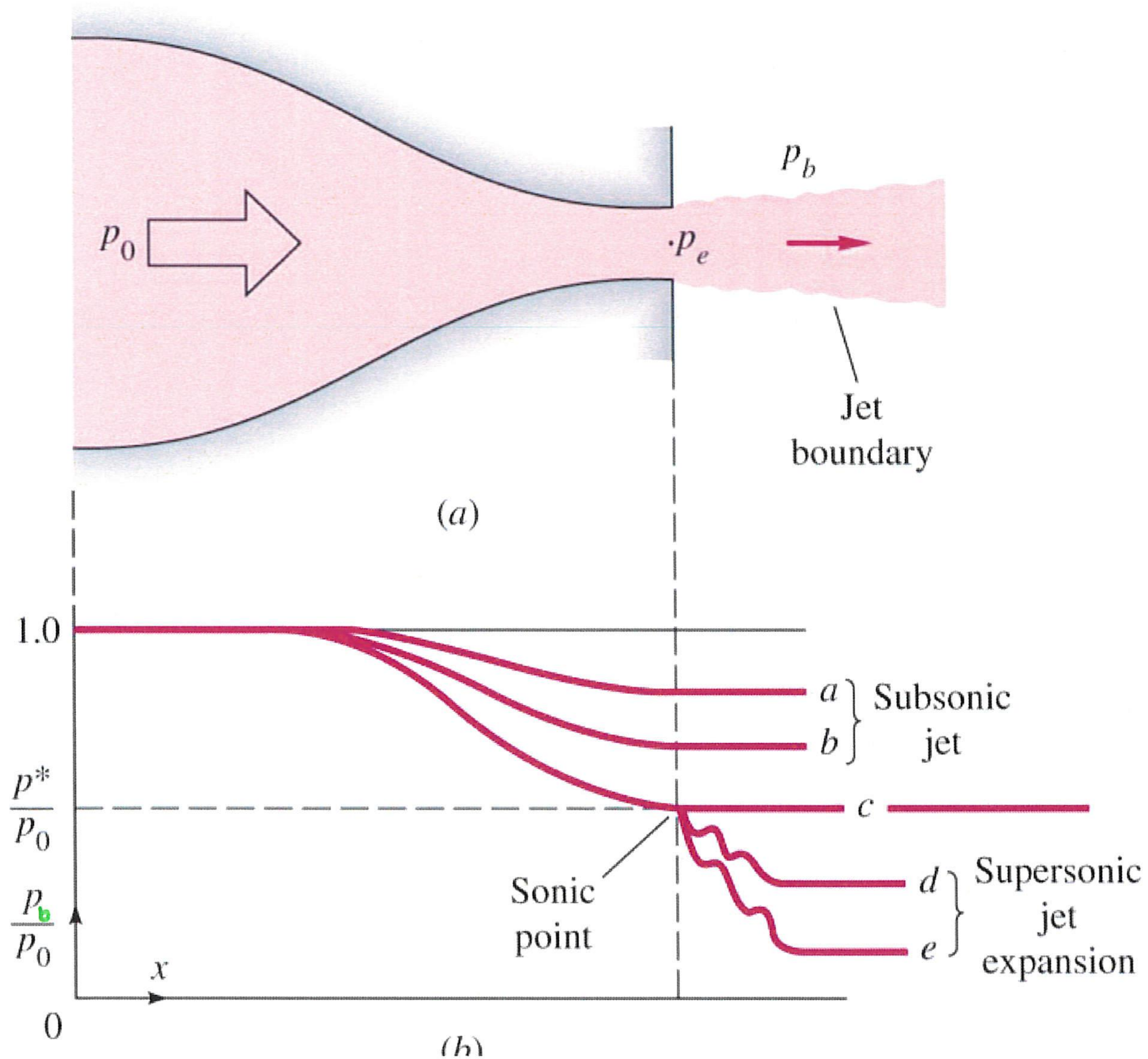
So, a converging section will accelerate subsonic flow until M=1. Then the only way to accelerate the flow further into the supersonic regime is to use a diverging section.



This converging-diverging nozzle is known as a DeLaval Nozzle.

5.10. Choking

Consider isentropic flow through a convergent nozzle. If the back pressure p_b is reduced, what happens?



Curve (0) no flow ($p_b = p_0$)

Curve (a, b) $\dfrac{dA}{dx}$ is negative from 1 to t. So for M<1, u and M are both increasing.

Curve (c) The velocity has reached the point that at the throat, $M_t=1$ (just equal). At this point the throat reached critical conditions. i.e. $p_t = p_c$ and $T_t = T_c$

Curve (d & e) If p_b is reduced further, we cannot make M>1 inside the nozzle (since we must have $\dfrac{dA}{dx}>0$ to accelerate a supersonic flow). So the drop in pressure must occur outside the nozzle (between t and b) while the flow between 1 and t remains the same. Pressure drops outside the nozzle in a supersonic jet.

Physically: when the fluid has reached sonic velocity (u=a and M=1) then when the back pressure is lowered, the expansion wave travels upstream at the same speed that the flow is travelling downstream. i.e. the 'information' that the p_b is being reduced never reaches the gas inside the nozzle.

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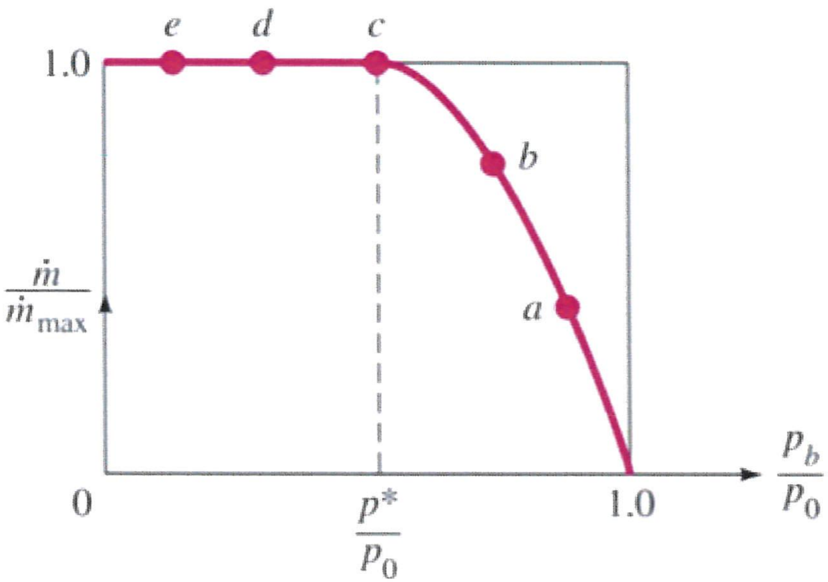
The mass flow rate cannot be increased beyond its value for curve (c). The nozzle is said to be *choked*.



By the MCE;

$$\dot{m} = \rho_t u_t A_t$$

If the flow is choked (i.e. sonic at the throat) then M_t=1 and at the throat the gas is at critical conditions so

$\rho_t = \rho_c$; $u_t = a_c$

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Remember the sonic velocity a is not a fixed constant but varies with the gas temperature $a_c = \sqrt{\gamma R T_c}$. Then

$$\dot{m}_{\max} = \rho_c a_c A_t$$

refer the conditions at the throat ρ_c to stagnation conditions, ρ_0

$$\dot{m}_{\max} = \left(\frac{\rho_c}{\rho_0}\rho_0\right)\left(\frac{a_c}{a_0}a_0\right)A_t$$

Now $a_c = \sqrt{\gamma R T_c}$ & $a_o = \sqrt{\gamma R T_o}$

so substituting $\frac{a_c}{a_0} = \sqrt{\frac{T_c}{T_0}}$ and $\rho_0 = \frac{p_0}{RT_0}$ $\dot{m}_{max} = \left(\frac{\rho_c}{\rho_o}\frac{P_o}{RT_o}\right)\left(\sqrt{\frac{T_c}{T_o}}\sqrt{\gamma R T_o}\right)A_t$

$$\dot{m}_{max} = \frac{P_o}{RT_o} A_t \sqrt{\gamma R T_o} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{1/2}$$

substituting the relations (5.24) and (5.26)

$$\dot{m}_{max} = \frac{P_0}{R T_0} A_t \sqrt{\gamma R T_0} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{1/2}$$

$$\dot{m}_{\max} = \frac{p_0}{\sqrt{T_0}} A_t \left[\sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma+1} \right)^{\frac{(\gamma+1)}{2(\gamma-1)}}} \right] \qquad (5.31)$$

So, the only way to increase the mass flow through a choked nozzle is to;

increase the throat area, A_t

increase the inlet stagnation pressure, p_0

decrease the inlet stagnation temperature, T_0

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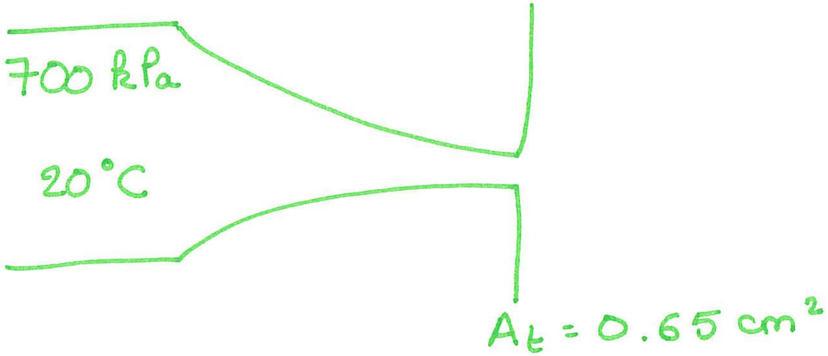
Example 5.5. Flow through a Convergent Nozzle

Air in a tank at 700 kPa and 20°C exhausts through a converging nozzle of throat area 0.65 cm^2 to atmosphere. Determine the initial mass flow assuming isentropic flow.



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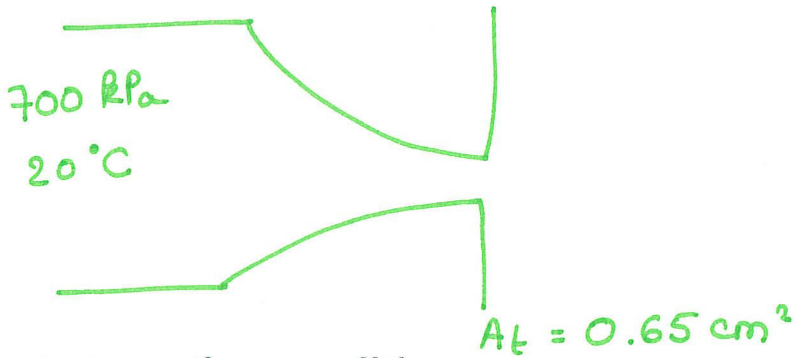

The reservoir will be at stagnation conditions;

p_o= 700 kPa

T_0= 20°C

and the back pressure at atmospheric, p_b= 1 bar = 100 kPa

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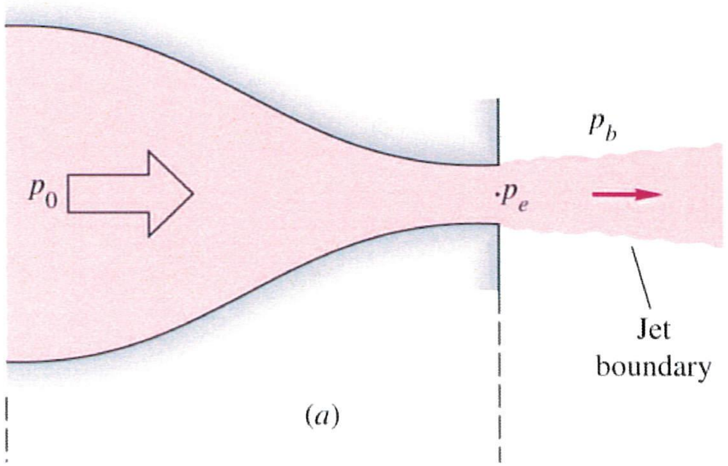
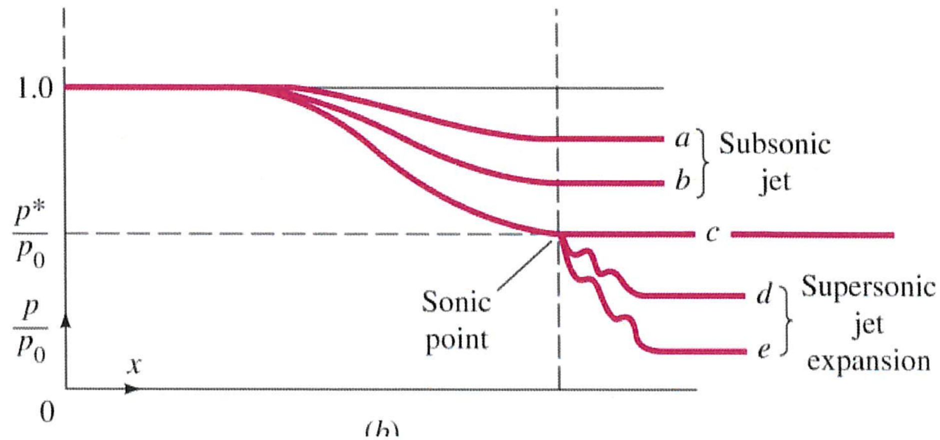
Firstly we determine whether the nozzle is choked from the critical conditions for sonic flow;

$$\frac{P_c}{P_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{2}{1.4+1}\right)^{\frac{1.4}{1.4-1}} = 0.528$$

$p_c = 0.528 \times 700 = 369.6$ kPa



(a)



(b)

But our back pressure is well below this (we are following a curve like (d) in the above diagram). So the flow is choked and the throat is at critical conditions. The mass flow rate is then found from these conditions;

$$\dot{m} = \rho_t u_t A_t = \rho_c u_c A_t$$

$$\frac{T_c}{T_0} = \frac{2}{\gamma+1}$$

$T_c =$ 244.2 K

$u_c = a_t = a_c = \sqrt{\gamma R T_c} = \sqrt{1.4 \times 287 \times 244.2} = 313 \text{ m/s}$

$$\rho_c = \frac{P_c}{R T_c} = \frac{369.6 \times 10^3}{287 \times 244.2} = 5.27 \text{ kg/m}^3$$

$$\dot{m}_{\max} = \rho_c u_c A_t$$

$$= 5.27 \times 313 \times 0.65 \times 10^{-4} = 0.107 \text{ kg/s}$$

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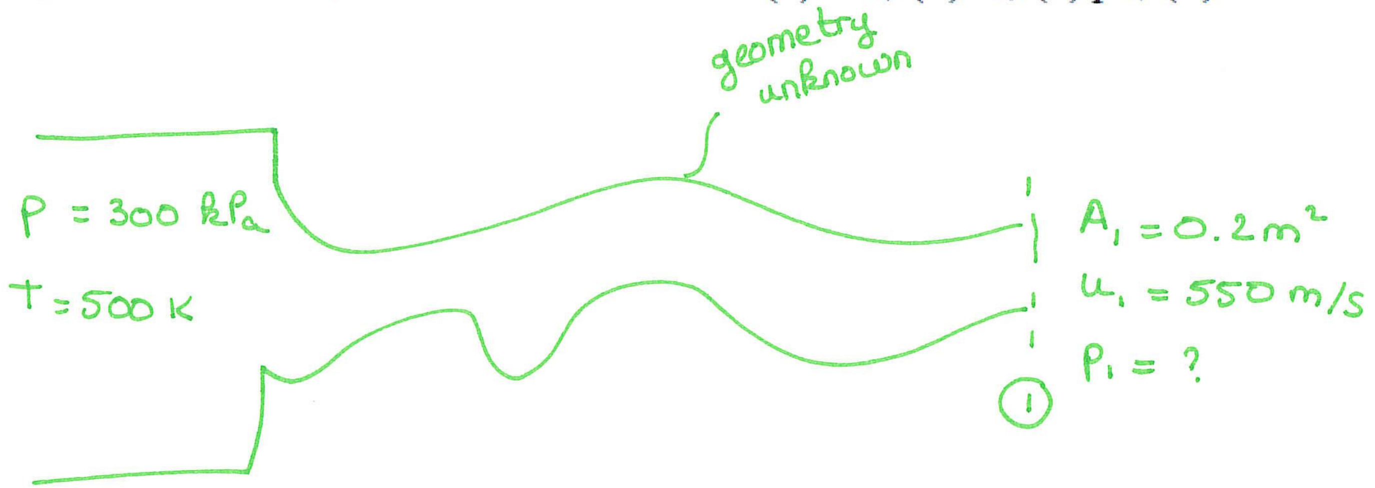
Example 5.6. Supersonic Expansion

Air is flows isentropically from a reservoir, where p=300 kPa and T=500K to a section 1 in a duct where A_1=0.2 m^2 and u_1=550 m/s. Determine (a) M_1, (b)T_1, (c) p_1, (d) $\dot{m}$. Is the flow choked?



The air in the reservoir is at stagnation conditions (velocity is zero). So p_0=300 kPa and T_0=500K. For an adiabatic process $T_0 = T_{01}$

So applying

$$\frac{T_{01}}{T_1} = 1 + \frac{u_1^2}{2c_p T_1} \qquad \frac{500}{T_1} = 1 + \frac{550^2}{2 \times 993 \times T_1} \quad \text{gives} \quad T_1 = 368\,K$$

Now we can determine a_1;

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \times 287 \times 348} = 374\ m/s$$

So $M_1 = \frac{u_1}{a_1} = \frac{550}{374} = 1.47$

Therefore at 1 the flow is supersonic. Since in the reservoir the flow was at stagnation, there must be a throat somewhere between the reservoir and 1. The flow at that throat must then be sonic so the nozzle is choked.

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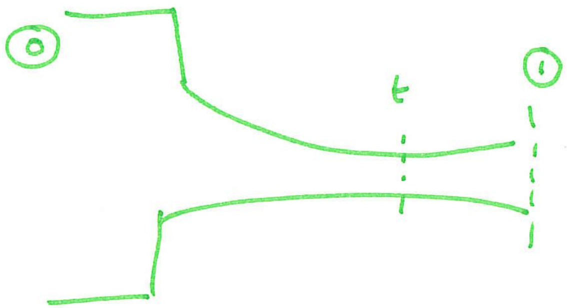


Now for the isentropic process 0 to 1

$$\frac{P_0}{P_1} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M_1^2\right]^{\frac{\gamma}{\gamma-1}} \qquad \frac{300 \times 10^3}{P_1} = \left[1 + \left(\frac{1.4-1}{2}\right) \times 1.47^2\right]^{\frac{1.4}{0.4}}$$

We use the perfect gas equation to find the density;

$$\rho_1 = \frac{P_1}{R\,T_1} = \frac{86 \times 10^3}{287 \times 368} = 0.861 \text{ kg/m}^3 \qquad P_1 = 86 \text{ kPa}$$

and hence the mass flow rate;

$$\dot{m} = \rho_1 u_1 A_1 = 0.861 \times 550 \times 0.2 = 94.7 \text{ kg/s}$$

5.11. Normal Shock Waves

A shock wave is a rapid discontinuity of flow properties. The velocity suddenly drops over a few micrometers. Upstream $M>1$ of the shock whilst downstream $M<1$. Because it occurs over a very short distance there is no time for heat transfer to occur – so the shock wave is adiabatic. This means that (from the SFEE):

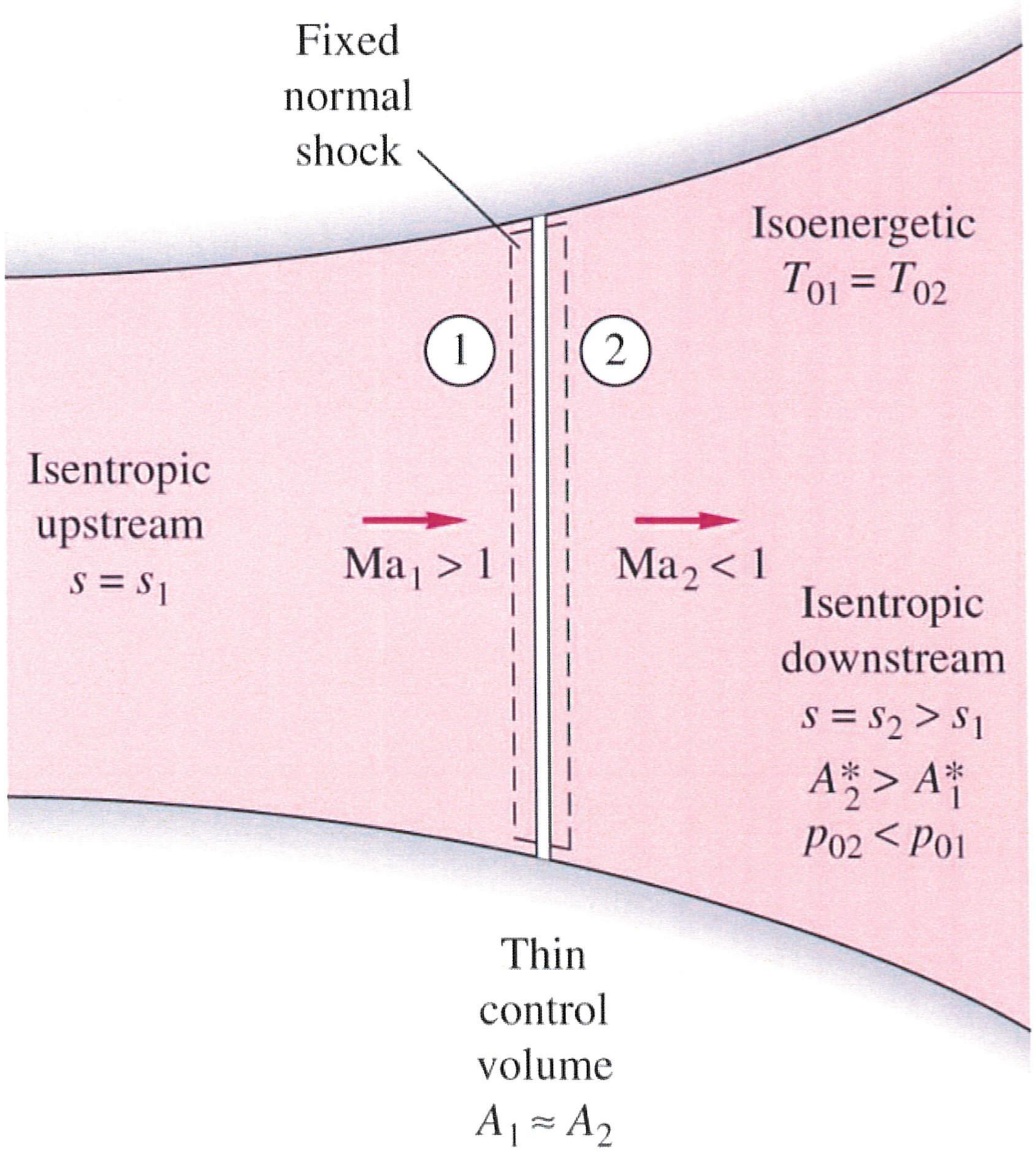
$$h_{01} = h_{02}$$

And so:

$$T_{01} = T_{02}\ .$$

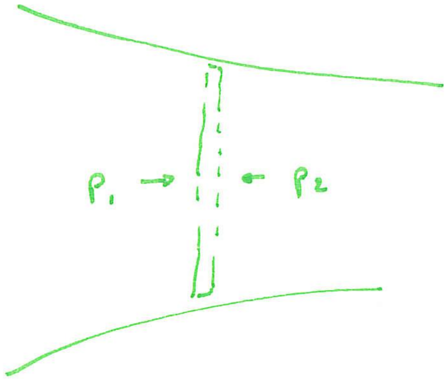
p_0, ρ_0

However it is not reversible (there are friction losses). So it is not isentropic (i.e. equations 5.13 and 5.14 don't apply). So how do the properties change across a shock wave? Consider a CV around a stationary shock wave.



Applying the MCE:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{5.32}$$



Applying the FME:

Forces on the CV

$$p_1 A_1 - p_2 A_2$$

$$\sum \dot{M}_{in} = \dot{m} u_1$$

$$\sum \dot{M}_{\cancel{in}\ out} = \dot{m} u_2$$

$$p_1 A_1 - p_2 A_2 = \dot{m}(u_2 - u_1) \qquad \therefore A_1 = A_2 \tag{5.33}$$

Combining (5.32) and (5.33) gives:

$$u_2^2 - u_1^2 = (p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Since it is adiabatic:

$$h_1 - h_2 = \frac{1}{2}\left(u_2^2 - u_1^2\right)$$ [see p5.2 of notes]

Gives

$$h_2 - h_1 = \frac{(p_2 - p_1)}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \tag{5.34}$$

This is called the Rankin-Hugoniot relation. Note that it is independent of the equation of state for the gas. If we assume the gas is perfect then:

$$h = c_p T = \frac{\gamma p}{(\gamma - 1)\rho}$$

Using this to replace h in equation (5.34) gives:

$$\frac{\rho_2}{\rho_1} = \frac{1+\beta\left(\dfrac{p_2}{p_1}\right)}{\beta+\left(\dfrac{p_2}{p_1}\right)} \quad \text{where } \beta = \left(\frac{\gamma+1}{\gamma-1}\right) \tag{5.35}$$

Compare this with an isentropic change in pressure (e.g. a weak pressure wave):

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} \tag{5.36}$$

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Entropy change across the shock wave (a measure of irreversibility)

Remember:

$$Tds = dh - \frac{dp}{\rho}$$

$$\int ds = \int \frac{dh}{T} - R \int \frac{dp}{p} \qquad \rho T = \frac{p}{R}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

$$s_2 - s_1 = c_v \ln \left[\frac{p_2}{p_1} \left(\frac{\rho_1}{\rho_2} \right)^{\gamma} \right] \tag{5.37}$$

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The table below (for the case where γ=1.4) uses the non-isentropic relationship for the pressure change across the shock wave compared with an isentropic change.

	$\frac{\rho_2}{\rho_1}$		
$\frac{p_2}{p_1}$	Irreversible Equation (5.28)	Isentropic Equation (5.29)	$\frac{s_2 - s_1}{c_v}$ Equation (5.30)
0.5	0.6154	0.6095	-0.0134
.9	0.9275	0.9275	-0.00005
1.0	1.01	1.0	0
1.1	1.00704	1.00705	0.00004
1.5	1.3333	1.3359	0.0027
2.0	1.6250	1.6407	0.0134

- A decrease in enthalpy i.e. $(s_2 - s_1) < 0$ is not possible – violates the 2nd law of thermodynamics. This means that the pressure must always increase across a shock wave.

- For the shock wave the pressure ratio is greater than it isentropic value

- For small pressure ratios the difference is negligible and the shock may be considered as reversible and isentropic.

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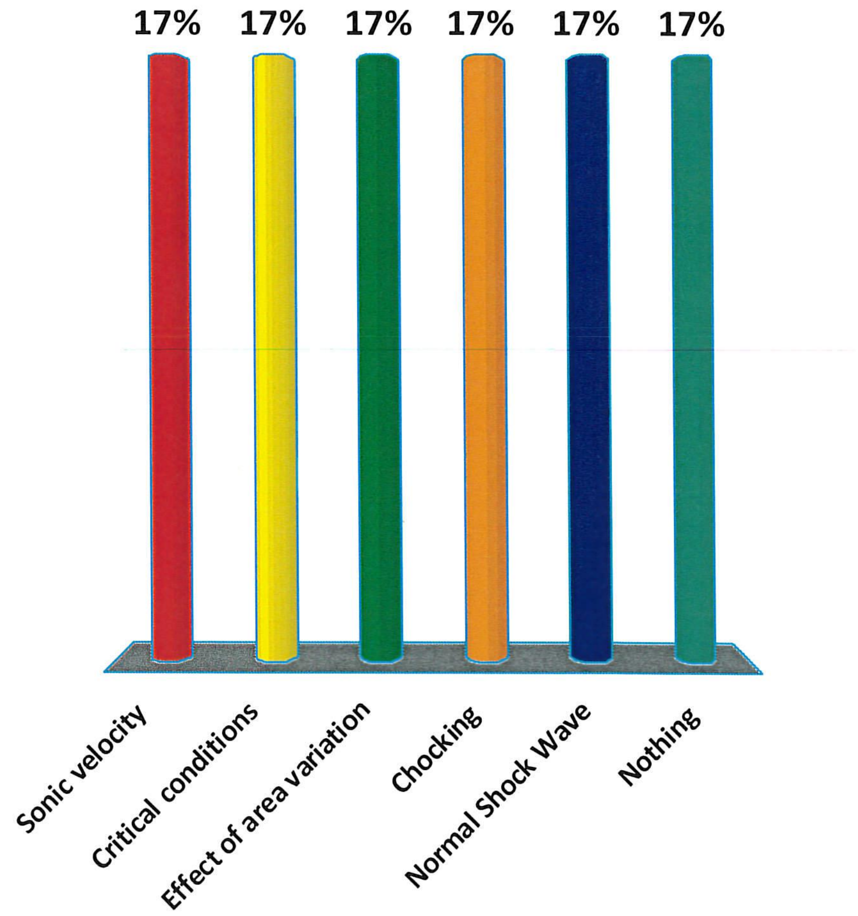
SESSION #: XXXXX

Least understood part of the lecture

A. Sonic velocity
B. Critical conditions
C. Effect of area variation
D. Chocking
E. Normal Shock Wave
F. Nothing



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THAT'S IT !!