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MEC 208 Fluids Engineering

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Before we start

- Slides from lecture 4-5 are online
- Tutorials:
  - Wednesday 10-11pm and Thursday 2-3pm
  - Diamond Workroom 1, 2 and 3
- Next computer sessions (19th and 26th of October) will be in the Diamond

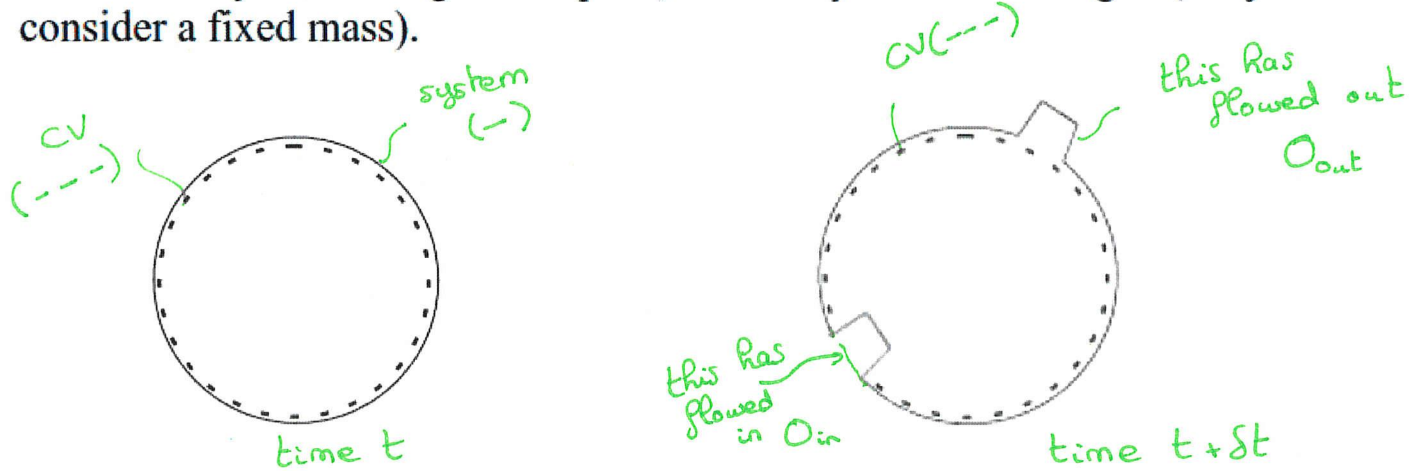
What we are covering in Topic 2:

2. Control Volume Analysis

2.1	**Introduction – Fluids in motion**
2.2	**Mass Conservation Equation (MCE)**
2.3.	**Force Momentum Equation (FME)**
2.4	**Torque Angular Momentum Equation (TAME)**
2.5	**Steady Flow Energy Equation (SFEE)**
2.6	**Differential Analysis of the equations**

2.4.2. Derivation of the TAME

Consider a CV coincident with a system at time t. At time $t+\delta t$ some fluid has flowed into the system whilst some has flowed out. The CV remains the same (remember in CV analysis we consider only a fixed region in space) but the system has changed (in system analysis we consider a fixed mass).



At time t the angular momenta of the CV and the system about some point A are identical.

$$(O_{cv})_t = (O_{system})_t$$

At time $t+\delta t$ the system has changed shape so its angular momentum about A has changed. Some angular momentum has flowed in whilst some has flowed out. So now;

$$(O_{cv})_{t+\delta t} = (O_{system})_{t+\delta t} + O_{in} - O_{out}$$

dividing by dt;

$$\frac{\partial (O_{cv})_{\cancel{t+\delta t}}}{\partial t} = \frac{\partial (O_{system})}{\partial t} + \dot{O}_{out} - \dot{O}_{in} \qquad (2.11)$$

We know that for a system equation (2.3) applies so;

$$T = \frac{\partial O_{system}}{\partial t} \qquad (2.12)$$

Combining (2.11) and (2.12) gives;

$$\left[\frac{\partial O_{cv}}{\partial t} = \frac{\partial O_{system}}{\partial t} + \dot{O}_{in} - \dot{O}_{out} \right]$$

$$T = \frac{\partial O_{cv}}{\partial t} + \dot{O}_{out} - \dot{O}_{in}$$

Now suppose we have a number of torques acting on the CV, and also a number of inflows and outflows of angular momentum, then;

$$\sum T_A = \left(\frac{\partial O_{CV}}{\partial t}\right)_A + \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A \qquad (2.13)$$

This is the Torque Angular Momentum Equation (TAME).

where $\sum T_A$ the sum of the torques acting on the CV contents

$\left(\sum \dot{O}_{in}\right)_A$ the sum of the angular momentum fluxes into the CV

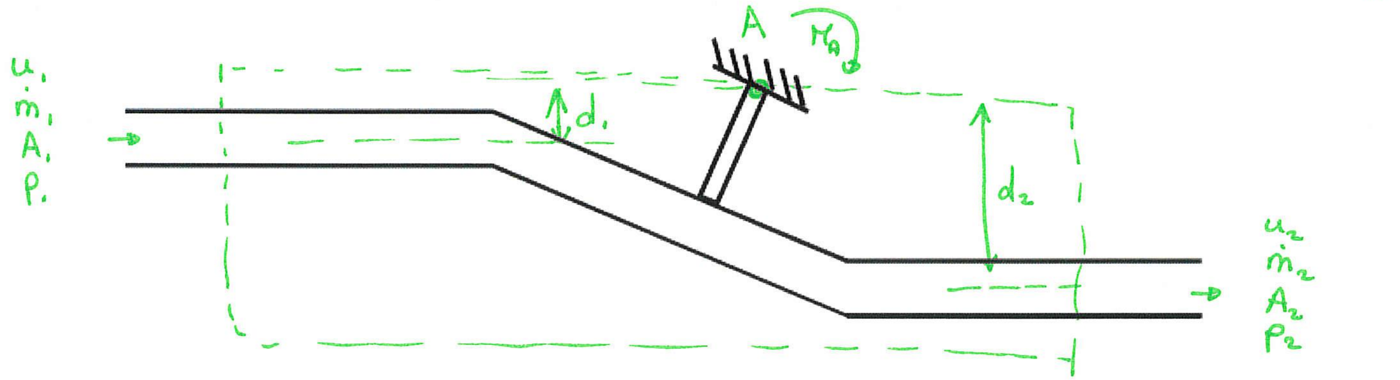
$\left(\frac{\partial O_{CV}}{\partial t}\right)_A$ the rate of change of angular momentum of the CV contents

Again torque and angular momentum are vector quantities, so we must specify about which point they act (in this case they act about the point A) and also their direction. A similar equation exists for torques about other points B, C and so on.

Note how similar this is to the force momentum equation (FME).

Example 2.2.   Torque on a pipe bend.

Determine the torque which must be resisted by the support. Assume steady state, incompressible, 1D flow.



⇒ torques : $+M_A$

$-p_1 A_1 d_1$ } pressure torques

$+p_2 A_2 d_2$

⇒ angular momentum : $-\dot m_1 u_1 d_1$

$-\dot m_2 u_2 d_2$

⇒ conservation of mass : $\dot m_1 = \dot m_2 = \dot m$

$\rho u_1 A_1 = \rho u_2 A_2$

TAME

$$\Sigma T_A = (\Sigma \dot O_{out} - \Sigma \dot O_{in}) + \cancel{\frac{\partial O}{\partial t}}^{\,0}$$

from incompressible steady 1D flow

$$M_A - p_1 A_1 d_1 + p_2 A_2 d_2 = (-\dot m_2 u_2 d_2) - (-\dot m_1 u_1 d_1)$$

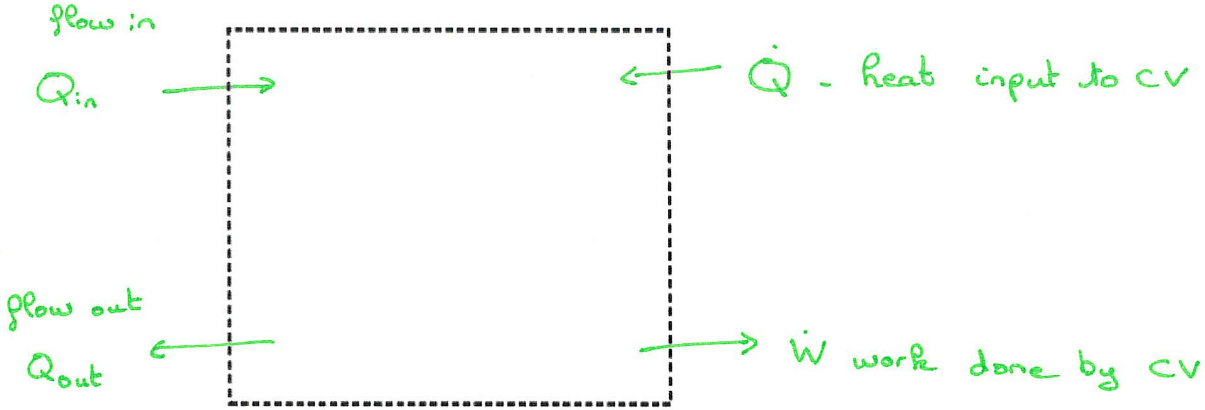
$$= \rho\,(-u_2^2 A_2 d_2 + u_1^2 A_1 d_1)$$

2.5. The Steady Flow Energy Equation (SFEE)

So far in this analysis of fluid flow using CV's we have not considered the effect of energy input or output from the fluid.

Clearly if a fluid flows past the vanes of a turbine, it loses some of its energy and its flow properites are changed as a result. We need some method for determining how energy transfer effects flow properties.

! NOTE
Q = flow rate
$\dot{Q}$ = heat flux !



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In thermodynamics we use the first law (*Q-W=ΔE)* to study the effects of energy transfer to and from systems. In this section we will use a CV form of the first law called the *steady flow energy equation*, SFEE.

(derivations in massey)

$$\frac{\dot{Q}-\dot{W}}{\dot{m}}=\left(e_2+\frac{p_2}{\rho_2}+\frac{u_2^2}{2}+gz_2\right)-\left(e_1+\frac{p_1}{\rho_1}+\frac{u_1^2}{2}+gz_1\right) \qquad (2.16)$$

e = specific internal energy

In deriving it we imply a number of assumptions;

steady continuous flow

heat and shaft work transfer at a constant rate

velocities are uniform over cross-sections 1 and 2 (i.e. 1D flow at 1 and 2)

no energy transfer due to electricity, magnetism, surface tension, nuclear reaction

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2.5.1. Different Forms of the SFEE

$\rho = \rho_1 = \rho_2$

(a) No heat or work transfer, $\dot{Q} = \dot{W} = 0$. If the fluid is incompressible and inviscid (i.e. frictionless) then there are no losses so $e_1 = e_2$; and the SFEE reduces to Bernou~~l~~lli’s equation;

$$\left(\frac{p_2}{\rho} + \frac{u_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{u_1^2}{2} + gz_1\right) = 0 \qquad (2.17)$$

(b) No heat or work transfer, $\dot{Q} = \dot{W} = 0$. E.g. flow of fluid through pipes and fittings.

$$\left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1\right) + (e_2 - e_1) = 0 \qquad (2.18)$$

The $(e_1\text{-}e_2)$ represents a loss of useful energy resulting from fluid friction. Often expressed as a head loss (i.e. by dividing by g);

$$h_f = \frac{e_2 - e_1}{g}$$

Frequently these head losses are determined by experiment. For pipe networks they are called *minor and major loss coefficients*. These are presented in tables or charts (remember the Moody chart).

(c) Compressible flow. $\rho_1 \neq \rho_2$

From thermodynamics remember enthalpy;

$$H = E + pV \tag{2.19}$$

or in its specific form (dividing 2.19 by mass, m);

$$h = e + \frac{p}{\rho} \tag{2.20}$$

replacing e in the SFEE gives;

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(h_2 + \frac{u_2^2}{2} + gz_2 \right) - \left(h_1 + \frac{u_1^2}{2} + gz_1 \right) \tag{2.21}$$

For gases the gravity terms are negligible (neglect gz_1 and gz_2) and for flow of a perfect gas;

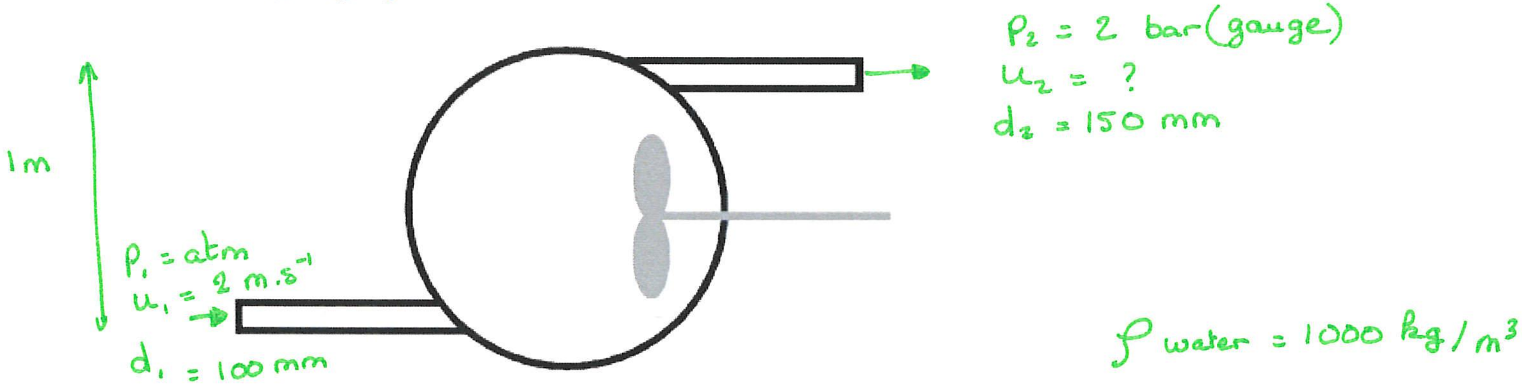
$$h = c_p T \tag{2.22}$$

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(c_p T_2 + \frac{u_2^2}{2} \right) - \left(c_p T_1 + \frac{u_1^2}{2} \right) \tag{2.23}$$

we will be using this form of the SFEE in the study of compressible fluid flows (i.e. gases) later in the course.

Example 2.3. - A Water Pump.

In the water pump shown below, determine the power required to draw the water from atmosphere and deliver it at a gauge pressure of 2 bar. Assume the fluid flow is frictionless.



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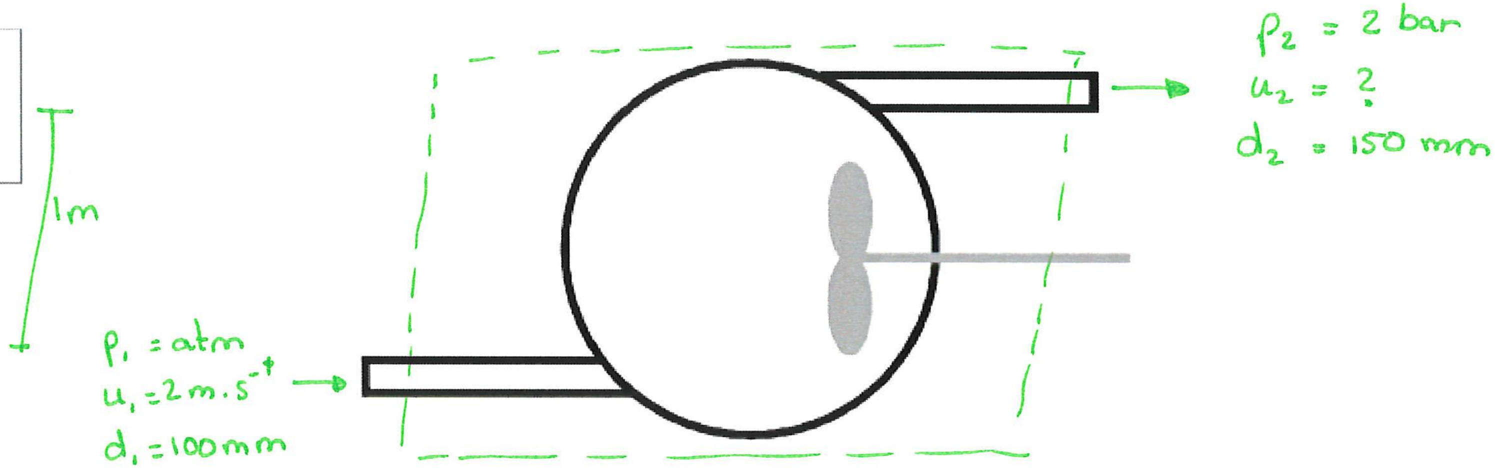

Draw a CV boundary around the pump.

by the MCE (assume incompressible, steady, 1D flow);

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow u_2 = \frac{\rho_1 u_1 A_1}{\rho_2 A_2} = u_1 \frac{A_1}{A_2} = u_1 \frac{d_1^2}{d_2^2}$$

$$\dot{m} = 1000\,(2)\,\left(\pi \left(\frac{.1}{2}\right)^2\right) = 15.7 \text{ kg.s}^{-1}$$

$$u_2 = u_1 \frac{d_1^2}{d_2^2} = 0.89 \text{ m.s}^{-1}$$

SFEE 1 to 2 - equation (2.16)

$$\frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2\right) - \left(e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1\right)$$

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We can assume that the heat flow to/from the fluid is negligible (if the fluid is at a similar temperature to the surrounding, or if the pump is insulated, this will be valid) $\dot{Q} = 0$

The flow is frictionless so $e_1 = e_2$ then;

$$-\frac{\dot{W}}{\dot{m}} = \left(\frac{P_2}{\rho} + \frac{U_2^2}{2} + g z_2\right) - \left(\frac{P_1}{\rho} + \frac{U_1^2}{2} + g z_1\right)$$

rearranging,

$$-\frac{\dot{W}}{\dot{m}} = \left(\frac{P_2 - P_1}{\rho}\right) + \left(\frac{U_2^2 - U_1^2}{2}\right) + g(z_2 - z_1)$$

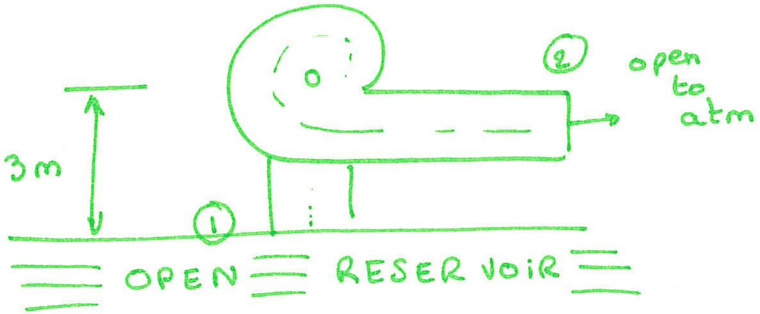
Now $p_1 = p_{atm}$, so since the discharge pressure is quoted as 2 bar *gauge*.

$p_2 - p_1 =$ $2 \times 10^5 \ N/m^2$

$$-\frac{\dot{W}}{\dot{m}} = \left(\frac{2 \times 10^5}{1000}\right) + \left(\frac{.89^2 - 2^2}{2}\right) + 9.81(1)$$

W=-3.3 kW The result turns out to be negative. So work is being done **on** the fluid **by** the pump.

EXTRA EXAMPLE



$\phi = 50\ mm$
$u_2 = 35\ m.s^{-1}$
$\rho = 998\ kg.m^{-3}$

The total head losses in the pump & pipework is 2m

If the pump is 75% efficient, what power is needed?

SFEE between 1 & 2 $\quad \frac{\dot{Q} - \dot{W}}{\dot{m}} = \left(e_2 + \frac{p_2}{\rho_2} + \frac{U_2^2}{2} + g z_2\right) - \left(e_1 + \frac{p_1}{\rho_1} + \frac{U_1^2}{2} + g z_1\right)$

assume no heat transfer $\quad \dot{Q} = 0$

pump is discharging to atmosphere $\quad p_1 = p_2 = atm$

re-write e_1 & e_2 : $\quad h_f = \frac{e_2 - e_1}{g}$

water from reservoir : $\quad u_1 = 0$

$\Rightarrow$ SFEE : $\quad -\frac{\dot{W}}{\dot{m}} = \frac{U_2^2}{2} + g(z_1 - z_2) + g h_f$

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By the MCE $\quad \dot{m} = \rho u A = 998 \times 35 \times \pi \times \dfrac{0.05^2}{4}$

$$= 68.6 \text{ kg. s}^{-1}$$

$$-\frac{\dot{W}}{\dot{m}} = \frac{U_2^2}{2} + g(z_1 - z_2) + g\,h_f$$

$$\dot{W} = -68.6\left[\frac{35^2}{2} + 9.81(3) + 9.81(2)\right]$$

$$= -68.6\,[612.5 + 49.1] = -45 \text{ kW}$$

! NEGATIVE !

⇒ pump is doing work on fluid

$$\dot{W}_p = \frac{\dot{W}}{\eta_p} = \frac{45}{.75} = 60 \text{ kW}$$

2.6 Differential forms of the Fundamental Laws

2.6.1 Introduction - Integral & Differential Relations

All the CV equations we have been using (MCE, FME, TAME, and SFEE) were derived by considering a finite region in space and balancing inflow and outflow properties. This gives us equations relating the effects of the sum of forces, *torques*, or *energy transfers* to *fluid flow*. These are known as *Integral Relations for a Control Volume.*

Another approach in fluid mechanics is to try to describe the detailed flow path at every point in a fluid flow field (x, y, z). Doing this gives us *Differential Relations for a Fluid Particle.*

We generate these differential equations by considering in infinitesimal region of the flow field.

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2.6.2 The Differential Equation of Mass Conservation

Imagine a fluid flow where the velocity (and all the other properties are varying in all directions. Fluid velocity in the x-direction is *u*, fluid velocity on the y-direction is *v*, and in the z-direction *w*.

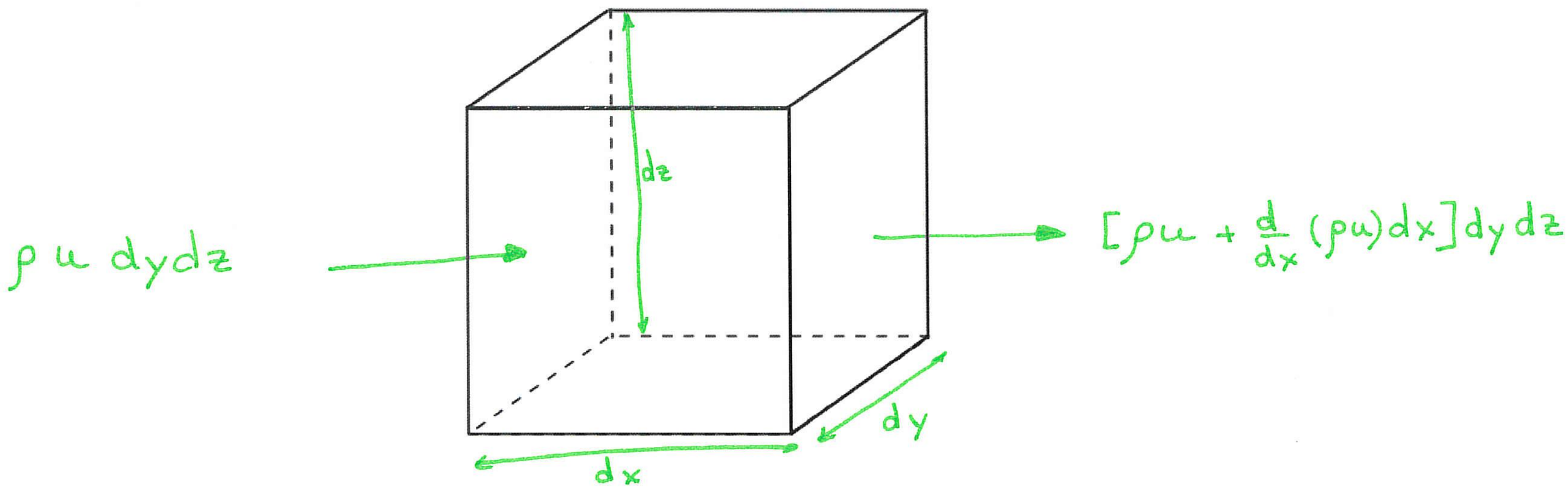
$\rho = f(x, y, z)$

$u, v, w = f(x, y, z)$

Consider a small element of the fluid flow field.

mass flow into element

$\dot{m} = \rho u A$



We treat the cube as a miniature CV. For the general case of a 3D flow fluid could be entering or leaving from any one of the six faces (on the figure only mass entering and leaving the x-face has been shown).

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Looking at the mass flows in/out at each direction in turn;

direction	inlet mass flow	outlet mass flow
x	$\rho u dy dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dy dz$
y	$\rho v dz dx$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)dy\right]dz dx$
z	$\rho w dx dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)dz\right]dx dy$

From the mass continuity equation (2.5);

$$\sum \dot{m}_{in} = \sum \dot{m}_{out} + \frac{\partial m_{cv}}{\partial t}$$

The rate of mass increase of the CV is caused by the rate of change of the density (since the volume is fixed - *dx.dy.dz*);

$$\frac{\partial m_{cv}}{\partial t} = \frac{\partial \rho}{\partial t} dxdydz$$

$M_{cv} = \rho\, dxdydz$

So including this with the data in the table, the MCE gives;

$$\frac{\partial \rho}{\partial t} dxdydz + \frac{\partial}{\partial x}(\rho u) dxdydz + \frac{\partial}{\partial y}(\rho v) dxdydz + \frac{\partial}{\partial z}(\rho w) dxdydz = 0$$

which simplifies to;

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.24}$$

This is the differential equation of mass conservation.

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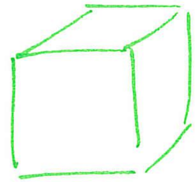
It is much more compact in vector form;

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0$$

If the fluid is incompressible this simplifies to;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



2.6.3 The Differential Equation of Force Momentum

A similar analysis to the above can be performed by considering the Force-Momentum balance of a flow element. We carry out the FME in each direction on the CV cube *dx.dy.dz*. This gives the differential equations of momentum.

There are three - one for each direction;

$$\rho F_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \tag{2.25}$$

$$\rho F_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \tag{2.26}$$

$$\rho F_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \tag{2.27}$$

These are also known as the *Navier-Stokes* equations.

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2.6.4 Physical Significance of the Navier Stokes Equations

Take for example the x-direction equation;

$$\rho F_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$\rho\frac{\partial u}{\partial t}$ a *transient term* - represents the acceleration of the fluid in a flow field. For steady flow this will be zero.

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ the *convection term* - represents the transfer of fluid from one place in space to another.

$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ the *diffusion term* - represents the effect of viscosity, transport of fluid from high to low gradients.

$\frac{\partial p}{\partial x}$ the *source term* - the pressure gradient which provides energy for the flow.

ρF_x the *body force term*, e.g. gravity, centrifugal forces, externally applied forces.

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The Navier-Stokes equations form a coupled set of highly non-linear partial differential equations.

There are four unknowns, p, u, v, and w; and there are four equations (2.24), (2.25), (2.26), and (2.27).

Usually it is not possible to solve these equations for all but the simplest problems (e.g. simple laminar flows).

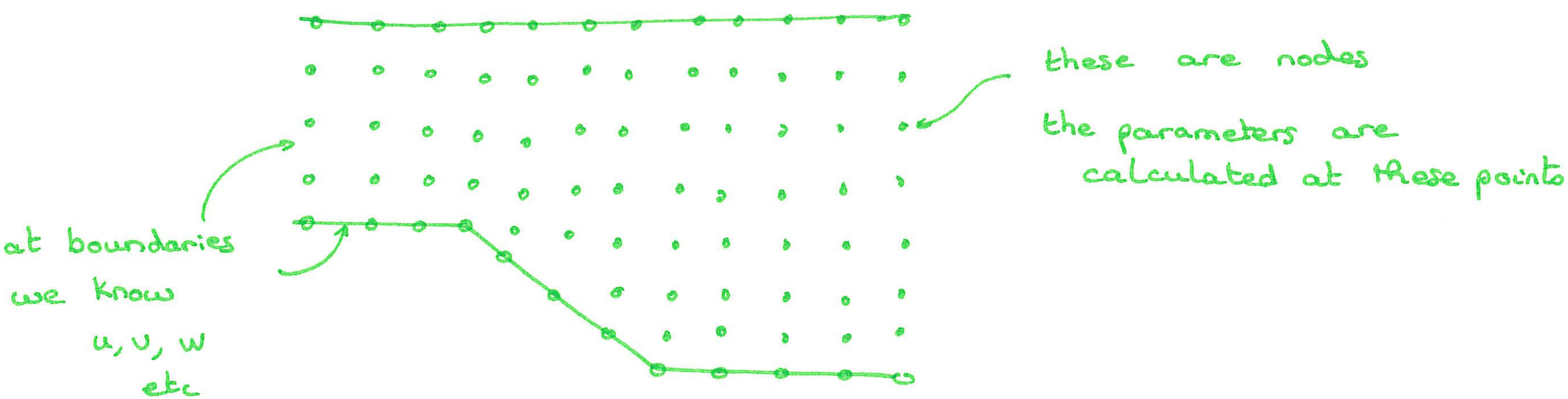
We have to resort to numerical methods - known as Computational Fluid Dynamics, CFD.

2.6.5 Numerical Analysis by the Finite Difference Method

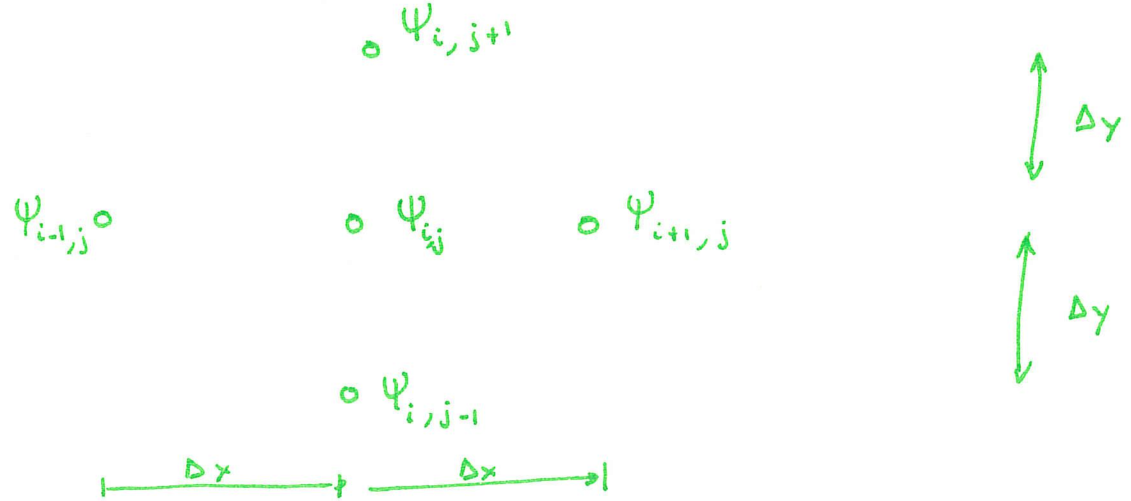
The Navier-Stokes equations are usually too complicated to solve analytically (i.e. there is no general solution). We use numerical techniques to solve them - a common way is by using the *finite difference method.*

The partial differential equations (Navier-Stokes) express how p, u, v, and w vary over the fluid flow. The idea behind the finite difference method is to approximate this variation by an equation relating the 'differences' between values at node points.

So the fluid flow problem is divided into a grid of nodes. E.g flow through a 2D expansion;



We want to find how the parameter ψ (where ψ could be either p, u, v, or w) varies over the grid. Lets take four adjacent nodes out of the grid. $\psi_{i,j}$ represents the value of ψ at the point (i,j);



So an approximation for the derivative; $\dfrac{\partial \psi}{\partial x}$ is;

$$\frac{\partial \psi}{\partial x} \approx \frac{1}{\Delta x}\left(\psi_{i+1,j} - \psi_{i,j}\right) \tag{2.28}$$

and similarly for the second derivative;

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\frac{1}{\Delta x}\left(\psi_{i+1,j} - \psi_{i,j}\right) - \frac{1}{\Delta x}\left(\psi_{i,j} - \psi_{i-1,j}\right)}{\Delta x}$$

$$\frac{\partial^2 \psi}{\partial x^2} \approx \frac{1}{\Delta x^2}\left(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}\right) \tag{2.29}$$

And there are a similar set of differentials for the y-direction.

$$\frac{\partial \psi}{\partial y} \approx \frac{1}{\Delta y}\left(\psi_{i,j+1} - \psi_{i,j}\right)$$

$$\frac{\partial^2 \psi}{\partial y^2} \approx \frac{1}{\Delta y^2}\left(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}\right) \qquad (2.30)$$

We then put these expressions for the first and second derivatives into the partial differential equations (2.25) - (2.27) for each and every node for all the variables *p*, *u*, *v*, and *w*.

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To demonstrate how this works we will use a much simpler partial differential equation called Laplace's equation;

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Substituting (2.29) and (2.30) gives;

$$2(1+\beta)\psi_{i,j} \approx \psi_{i-1,j} + \psi_{i,j+1} + \beta(\psi_{i,j+1} + \psi_{i,j-1}) \qquad (2.31)$$

where $\beta = \left(\frac{\Delta x}{\Delta y}\right)^2$

So we have an expression for ψ at a point in terms of its four nearest neighbours.

At the boundaries we know all the values of ψ. We guess initial values at all the other points. And we iterate by continually changing these values until equation (2.31) is satisfied.

The finer the mesh used the more accurate the result (but also the longer it takes).

There are lots of mathematical tricks for performing the iteration rapidly - beyond the scope of this course. However, the major commercial CFD codes use this basic method.

2.6.6 Commercial CFD Programs

The Navier-Stokes equations have been around since the early 19th century. But it has only been the development of computing power which has allowed their solution (in the 1950's and 1960's).

The major commercial CFD codes were developed in the 1970's and 1980's (PHOENIX at Imperial College, FLUENT at University of Sheffield, and CFX at the Atomic Energy Authority, etc.). They all work using this basic finite difference method.