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What we are covering in Topic 3:

- Topic 3: Internal Flow
  - Introduction
  - Flow Rates and Velocity Profiles
  - Viscosity, Laminar and Turbulent Flow (revision)
  - Flow in a Circular Pipe

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3. Internal Flow

(Massey §6, White §1 & §6, Potter §7, first year notes)

3.1. Introduction

In this part of the course we will be looking at the flow inside structures and how the fluid properties and duct shape affects the flow. The most common example is flow in a pipe. The flow may be laminar or turbulent. For turbulent flows the relationships between flow rate, pressure drop and pipe characteristics are obtained from empirical data (the Moody Chart and Los Factors). This was covered in 1st year. Here, we will be finding analytical relationships for the special case of laminar flow.

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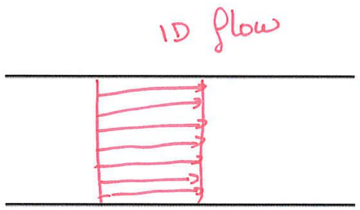
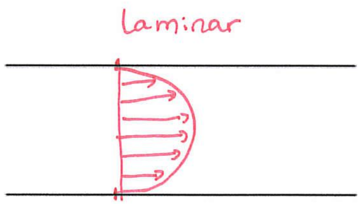
3.2. Flow Rates and Velocity Profiles

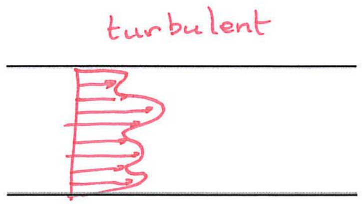
(a) Velocity Profiles across a Duct. Until now we have always assumed that the flow in a pipe is one dimensional (1D) i.e. all the fluid is travelling at the same velocity across the pipe section.



NO SLIP BOUNDARY CONDITION

This cannot be the case, because the fluid touching the pipe wall must be stationary. And if the fluid is viscous this will slow down the fluid next to it. A fluid distribution results (with the maximum at the centre). In this course we will be finding out what that distribution looks like.





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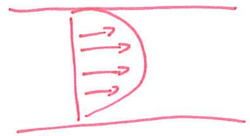
(b) Flow Rate from a Velocity Profile. If the velocity profile through a duct is known then the volume flow rate can be determined by integration.

$f(r)$

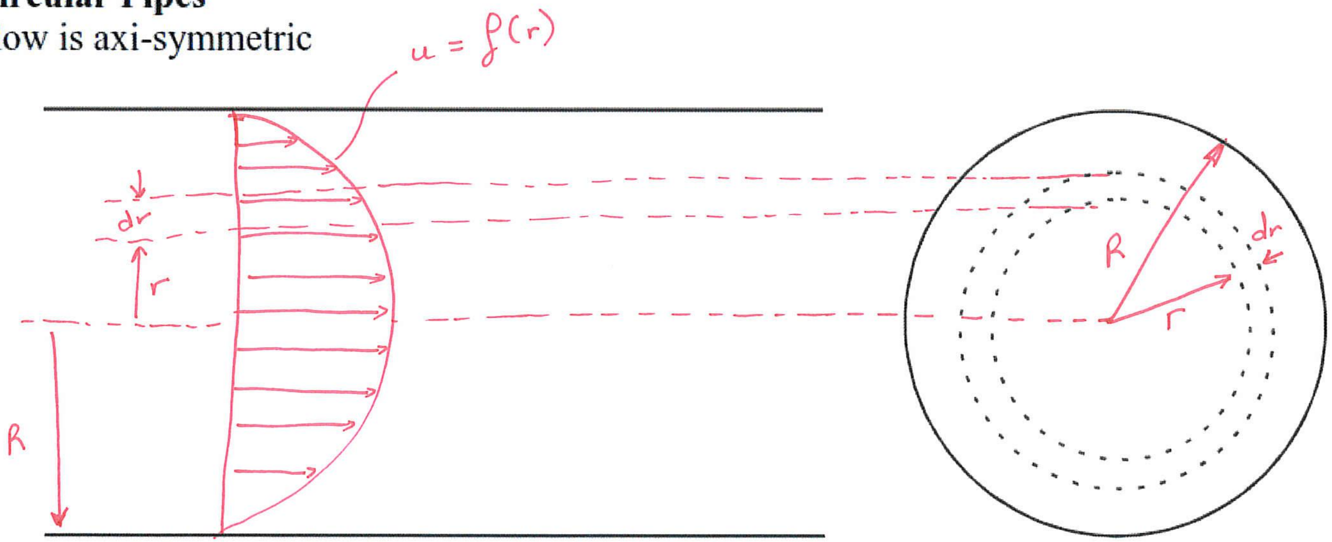
$u = \cancel{f(x, r)}$



- Choose an element over which the velocity can be assumed to be constant
- Set up an equation for the flow rate through that element in terms of the velocity, u and element area, A.
- Replace u with the velocity profile and A with the duct dimensions
- Integrate this relationship over the whole area.
- The mean velocity, u_m is then the flow rate divided by the area

Two common geometries:

Circular Pipes
Flow is axi-symmetric



Flow through small element $d\dot{q} = u\,dA$

Area of element $dA = 2\pi r\,dr$

Total flow through pipe $\dot{q} = \int_0^R u \; 2\pi r\,dr$ $u = f(r)$

Mean velocity $u_m = \frac{\dot{q}}{A}$

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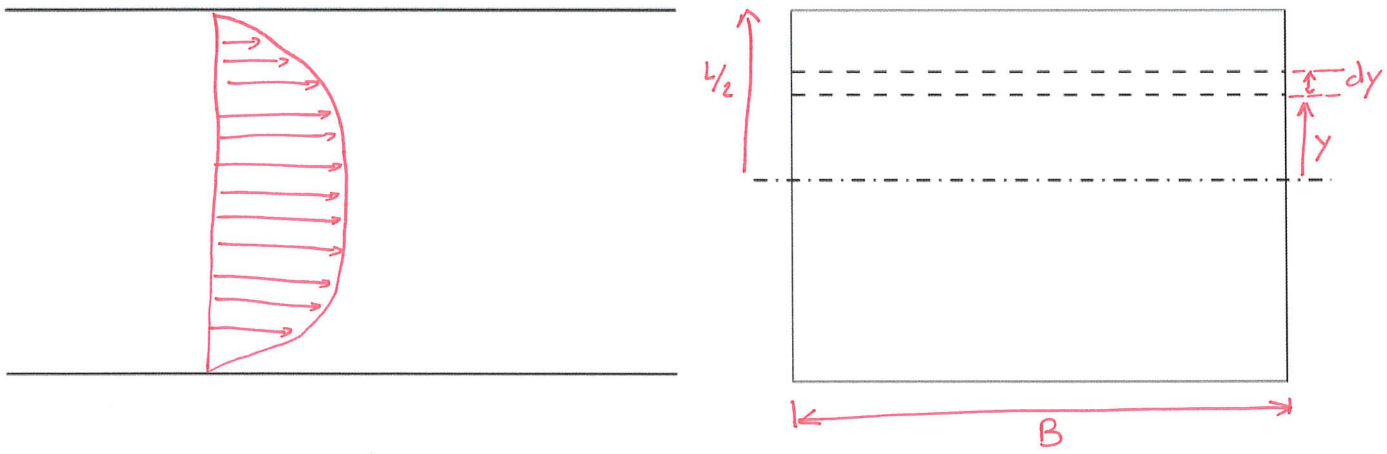
Rectangular Channels

Here we assume the width is much greater than the height (B>>L) so that the velocity profile is 2D.



Flow through small element $d\dot{q} = u\, dA$

Area of element $dA = B\, dy$

Total flow through pipe $\dot{q} = \int_{-L/2}^{L/2} u B\, dy$

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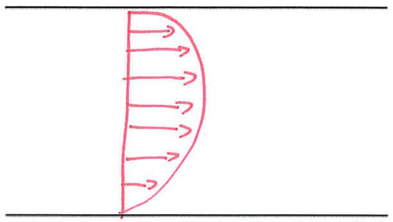
Example - Flow Rate through a Pipe

Oil of density 890 kg/m^3 flows though a pipe of radius R=10 cm. The resulting velocity profile has the form

$$u = u_{\max}\left(1 - \frac{r^2}{R^2}\right) \qquad \text{where } u_{max} = 0.2 \text{ m/s}$$

Determine the volume and mass flow rates down the pipe and the mean pipe velocity. $\dot{q}$? $\dot{m}$? u_m ?



$$d\dot{q} = u \, dA = u_{max}\left(1 - \frac{r^2}{R^2}\right)(2\pi r \, dr)$$

$$\dot{q} = 2\pi \, u_{max} \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$= 2\pi \, u_{max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 2\pi \, u_{max}\left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right) = 2\pi \, u_{max}\left(\frac{R^2}{2} - \frac{R^2}{4}\right) = 2\pi \, u_{max}\left(\frac{R^2}{4}\right)$$

$$= \pi \, u_{max}\left(\frac{R^2}{2}\right) = \pi \, (0.2 \text{ m/s})\left(\frac{0.1^2}{2}\right) = 3.14 \times 10^{-3} \text{ m}^3/\text{s}$$

$$\dot{m} = \rho \dot{q} = 2.79 \text{ kg/s}$$

The mean velocity is then:

$$u_m = \frac{\dot{q}}{A} = \frac{3.14 \times 10^{-3}}{\pi (0.1)^2} = 0.1 \text{ m/s}$$

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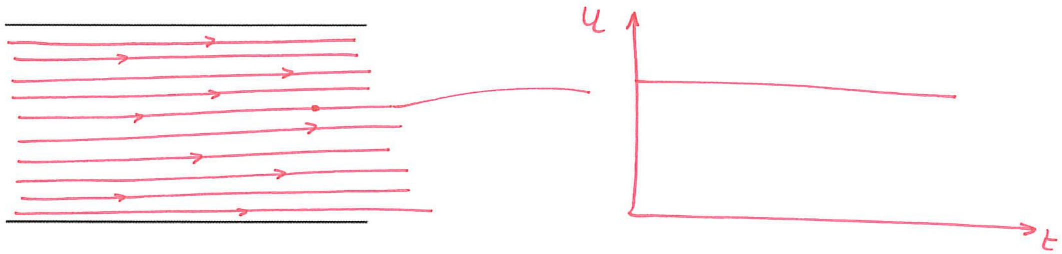
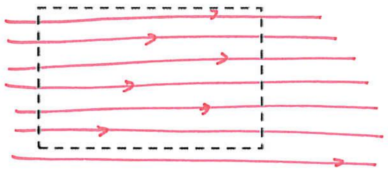
3.3. Viscosity, Laminar and Turbulent Flow (revision)

(a) **Laminar flow**. Particles within the fluid move in straight lines - steady flow. So at any given point the fluid velocity does not vary with time). But the velocity of particles on one line is not necessarily the same as that on another line. The fluid is moving in non-mixing layers - or *laminae*.



Consider a small control volume within the flow;



The FME states;

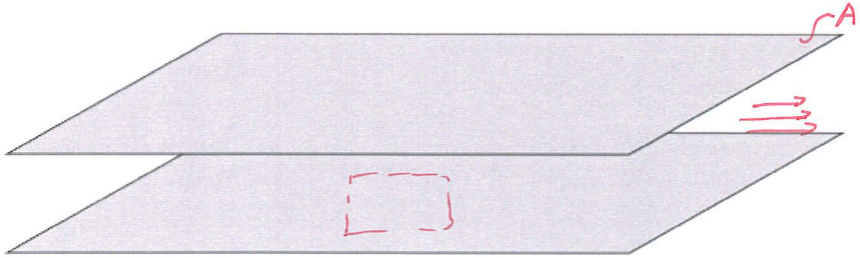
$$\sum F_x = \left(\dot{M}_{out} - \dot{M}_{in}\right)_x$$

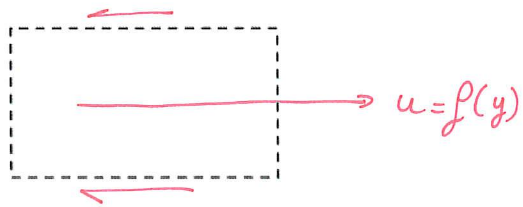
Since the fluid is flowing in straight lines (and velocity variation along a line is small) the momentum term is small compared to the force term. So;

$$\dot{M}_{OUT} = \dot{M}_{IN} \qquad \sum F_x = 0 \qquad (3.1)$$

The forces on the CV arise only from *viscous shear* of the fluid. This balance of shear forces is the basis for analysis of laminar internal flows.

(b) **Newton's Law of Viscosity**. Relates the shear force in a fluid with the fluid motion.





$$\tau = \mu \frac{du}{dy} \qquad (3.2)$$

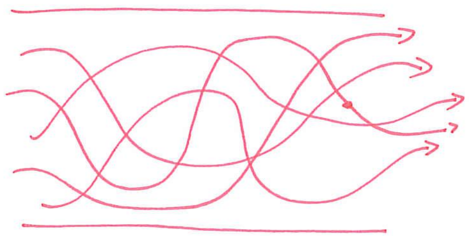
Remember μ the *viscosity*

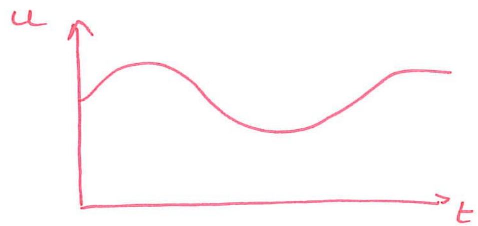
$\delta u/\delta y$ the *velocity gradient*.

τ the *shear stress* (force applied perpendicular to the area, F/A).

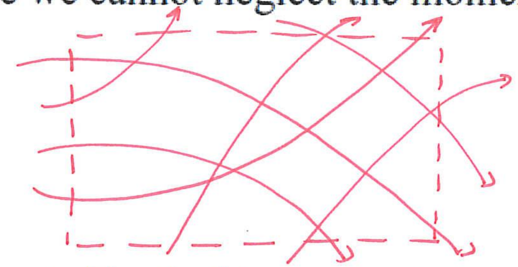
So if we know the velocity distribution, i.e. $u=f(y)$ we can differentiate to get $\delta u/\delta y$ and the shear stress, τ. And knowing the area over which this shear stress acts we can determine the shear force ($F_x = \tau_x.A$).

(c) **Turbulent Flow**. The paths of individual particles are no longer straight lines. The average fluid flow is down the pipe, but there are countless secondary fluid motions superimposed.





There is therefore significant mixing of fluid streams. i.e. lots of fluid motion into and out any CV occurs. In this case we cannot neglect the momentum term in the FME.



$$\dot{M}_{OUT} \neq \dot{M}_{IN}$$

The flow is non-steady (i.e. fluid velocities at a point vary with time). However, frequently the average velocity (at a point) is constant with time. We call this *mean steady flow*.

Analysis is difficult and fluids problems are usually solved by experiment (e.g. the minor and major loss coefficients) or by using CFD techniques (as in the FLUENT lab).

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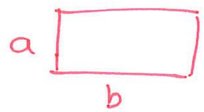


(b) **Reynolds Number** - we use this to determine whether the flow is laminar or turbulent.

$$\mathrm{Re} = \frac{\rho u d}{\mu} \tag{3.3}$$

ρ = density
u = velocity
d = characteristic diameter
μ = viscosity

for rectangular duct

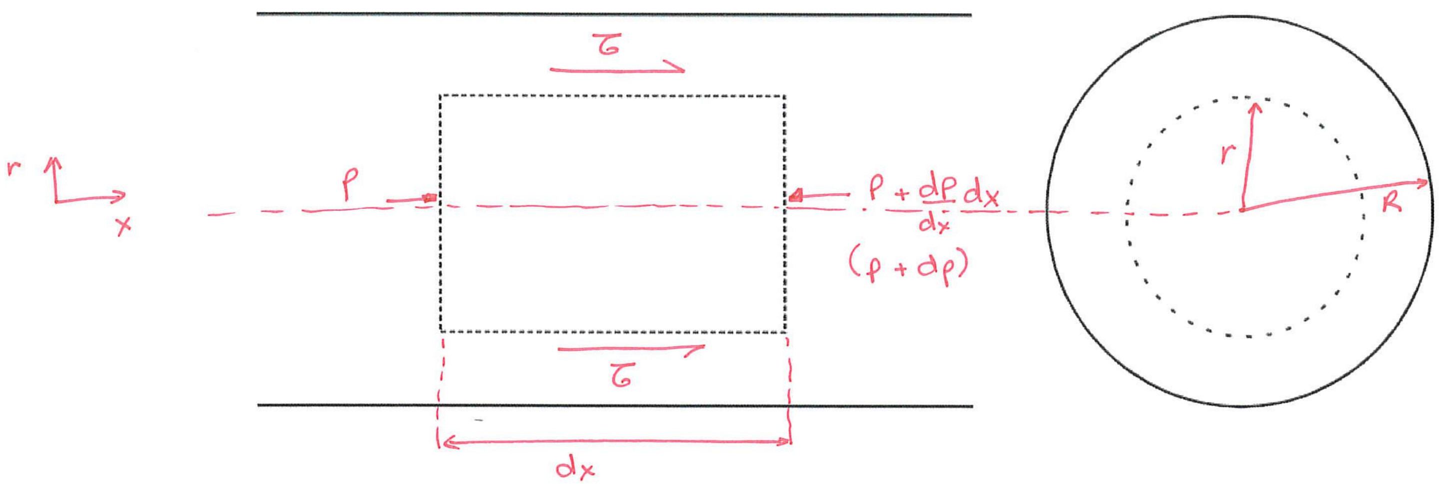
$d = \frac{4A}{P} = \frac{2ab}{a+b}$

Low Reynolds number (<2000 for pipe flow). High viscosity, low speed flows (e.g. gold syrup flowing slowly down a pipe) = laminar flow.

High Reynolds number (>2500 for pipe flow). Low viscosity, high speed flows (e.g. cold water tap full on) = turbulent flow.

3.4. Flow in a Circular Pipe

The method for analysing laminar flow is to consider the equilibrium (i.e. $\Sigma F=0$) of a suitable control volume. Consider a fluid flow along a pipe (there is a pressure gradient applied to the pipe). We choose a cylindrical element (note the choice of co-ordinate axes);



Forces on the element:

$$\rightarrow p\,(\pi r^2) \qquad \leftarrow (p+dp)(\pi r^2) \qquad \rightarrow \tau\,(2\pi r\, dx) \tag{3.4}$$

FME on the CV contents in the x-direction. $\sum F_x = 0$

$$p(\pi r^2) - (p+dp)(\pi r^2) + \tau(2\pi r\, dx) = 0$$

$$-dp\,\pi r^2 + \tau\,(2\pi r\, dx) = 0$$

$$\tau = \frac{r}{2}\frac{dp}{dx}$$

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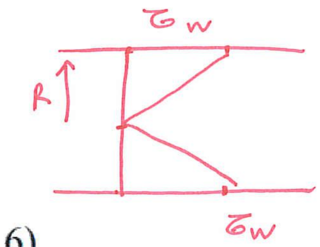
putting this into the above and rearranging;

$$\tau = \frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.5}$$

This gives us the distribution of shear stress at radius, r in the pipe in terms of the pressure variation down the pipe.



So if the pipe has a radius, R then the shear stress at the wall τ_w (i.e. r=R) is;

$$\tau_w = \frac{R}{2}\frac{dp}{dx} \qquad \tau_{Axis} = 0 \tag{3.6}$$

Now substituting (3.2) into (3.5) gives;

$$\tau = \mu\frac{du}{dy} = \frac{r}{2}\frac{dp}{dx}$$

Integrating w.r.t. r:

$$u = \frac{1}{2\mu}\frac{r^2}{2}\frac{dp}{dx} + K$$

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$$u = \frac{1}{2\mu}\frac{r^2}{2}\frac{dp}{dx} + K$$

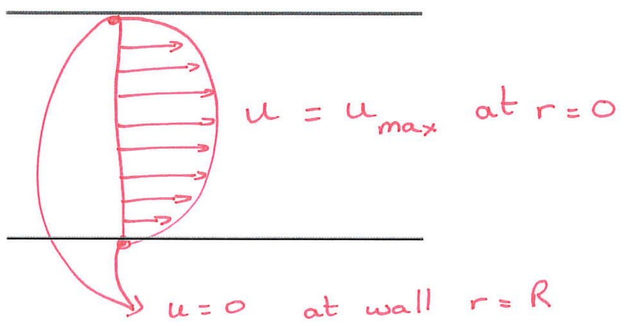
Boundary condition - when r=R, u=0 at pipe wall

gives $K = -\frac{1}{2\mu}\frac{r^2}{2}\frac{dp}{dx}$

$$u = \frac{1}{2\mu}\frac{\left(r^2 - R^2\right)}{2}\frac{\mathrm{d}\,p}{\mathrm{d}\,x} \tag{3.7}$$

This is called Poiseuille's equation.

From this we can see the velocity profile is parabolic u=f(r^2).



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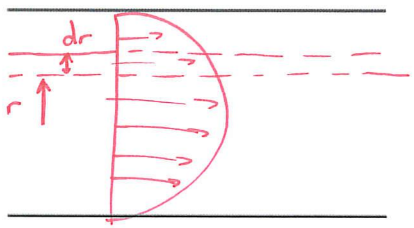

To determine the volumetric flow rate. Consider a small annular element, radius r and width δr;

$$d\dot{q} = u\,dA = u\,2\pi r\,dr$$

$$\dot{q} = \int_0^R u\,2\pi r\,dr$$

We can replace u from equation ~~(3.5)~~ [3.7]

$$\dot{q} = \int_0^R \frac{1}{2\mu}\left(\frac{r^2 - R^2}{2}\right)\cdot\frac{dp}{dx}\cdot 2\pi r\,dr$$

Integrating gives;

$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}\,p}{\mathrm{d}\,x} \qquad (3.8)$$

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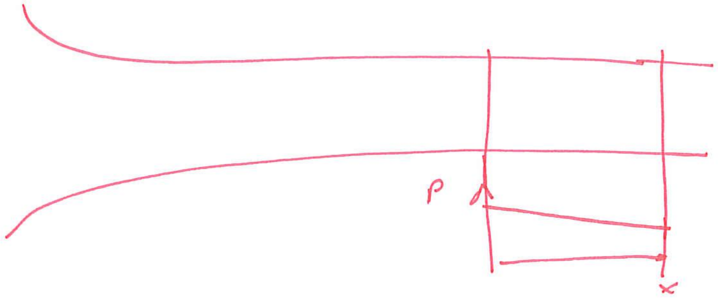

If the flow is fully developed (i.e. far enough away from the ends of the pipe) then the pressure, p falls uniformly over the length of the pipe, l. So we can write,

$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{l} \tag{3.9}$$

We can get the average fluid velocity down the pipe from;

$$u_m = \frac{\dot{q}}{A}$$

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Solving the Navier-Stokes Equation

For fluid flow in a cylindrical tube, we need to take the Navier-Stokes equation in cylindrical coordinates:

$$\rho\left(\cancelto{\text{steady}}{\frac{\partial u}{\partial t}}+\cancelto{\text{no swirl}}{v_r\frac{\partial u}{\partial r}}+\cancelto{\text{no swirl}}{\frac{v_\theta}{r}\frac{\partial u}{\partial \theta}}+\cancelto{\text{fully developed}}{u\frac{\partial u}{\partial x}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\cancelto{\text{no swirl}}{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}+\cancelto{\text{fully developed}}{\frac{\partial^2 u}{\partial x^2}}\right)$$

For developed flow in a circular pipe, the streamlines are parallel to the wall with no swirl, so that $v_r = v_\Theta = 0$ and $u = u(r)$ only.

We are left with

$$0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

Since x and r can be varied independently , we must have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \lambda \qquad \text{where } \lambda \text{ is a constant}$$

Multiply both sides with r and integrate:

$$r\frac{\partial u}{\partial r} = \frac{\lambda}{2}r^2 + A$$

Divide both sides by r and integrate:

$$u(r) = \frac{\lambda}{4}r^2 + A\ln r + B$$

The velocity must remain finite at r = 0 (so A = 0) and u = 0 at r = R, so

$$u(r) = \frac{\lambda}{4}\left(r^2 - R^2\right)$$

$$u = \frac{1}{4\mu}\left(r^2 - R^2\right)\frac{\partial p}{\partial x}$$

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from Poiseuille

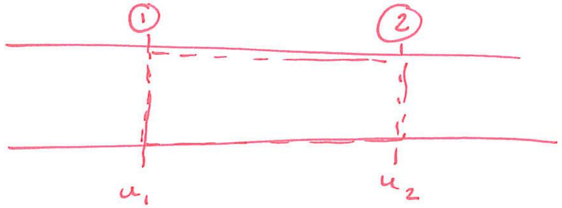
$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{dp}{dx}$$

Uses of the Poiseuille Equation

(a) Measurement of **fluid viscosity.** By measuring the fluid flow rate through a tube for a given applied pressure. We could then use (3.8) to determine the viscosity, μ. But in practice usually requires a very small bore tube - difficult to manufacture.

< 0.2 mm

(b) Head loss in laminar pipe flow. Look again at the CV above. Applying the SFEE between 1 & 2. Since $\dot{W} = \dot{Q} = 0$.



$$\left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2 + e_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1 + e_1\right) = 0$$

In this case of laminar pipe flow (by the MCE) $u_1 = u_2$.

The loss term (e_2-e_1) arises from friction losses in the pipe. These loses are then manifested as a temperature rise of the fluid. Rewrite them in terms of a head loss or a pressure loss;

$$h_f = \frac{e_2 - e_1}{g}$$

this gives; $h_f = \left(\frac{P_1}{\rho g} + z_1\right) - \left(\frac{P_2}{\rho g} + z_2\right)$

or $h_f = \frac{\Delta P}{\rho g}$ when $z_1 = z_2$ (3.10)

We can put Poiseuille's equation (3.9) into this to determine the head loss h_f.

$$h_f = \frac{8\dot{q}l\mu}{\pi R^4 \rho g} \tag{3.11}$$

$$u_m = \frac{\dot{q}}{A} = \frac{\dot{q}}{\frac{\pi d^2}{4}} = \frac{4\dot{q}}{\pi d^2}$$

or $$h_f = \frac{32 u_m l \mu}{d^2 \rho g}$$

$$h_f = \left(\frac{4\dot{q}}{\pi d^2}\right)\left(\frac{32 L\mu}{d^2 \rho g}\right)$$

The pressure loss is then:

$$\Delta p_L = h_f \rho g = \frac{32 u_m L \mu}{d^2}$$

Remember the definition of fiction factor:

$$f = \frac{\Delta p_L}{\frac{1}{2}\rho u_m^2}\frac{d}{l} = \frac{32 u_m L\mu}{d^2} \cdot \frac{1}{\frac{1}{2}\rho u_m^2} \cdot \frac{d}{L} = \frac{64\mu}{d\rho u_m} \tag{3.12}$$

Combining (3.11) and (3.12) gives:

$$f = \frac{64}{\text{Re}}$$

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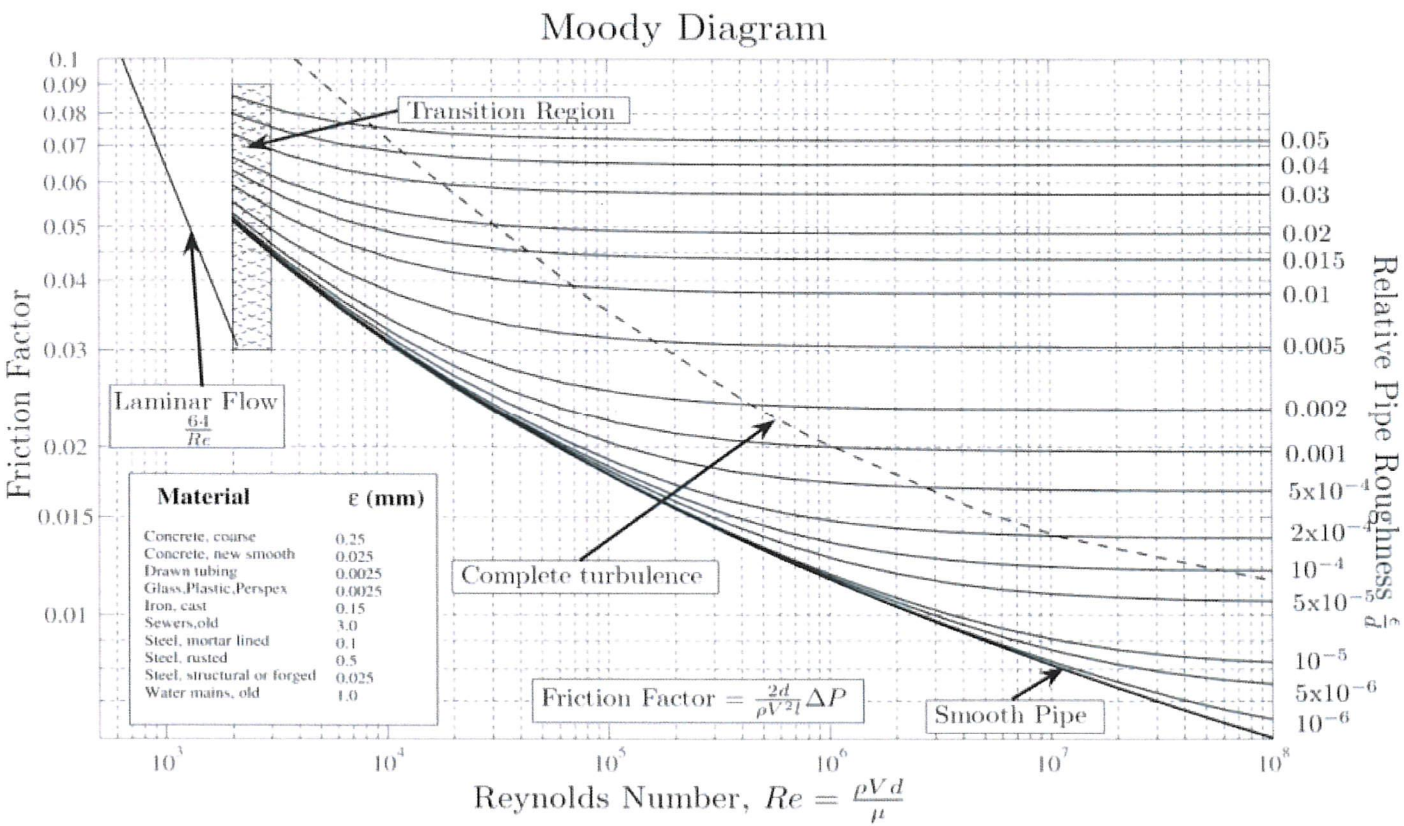
This gives us the ‘laminar flow’ line on the Moody Chart (which relates pipe losses to roughness and Reynolds number).



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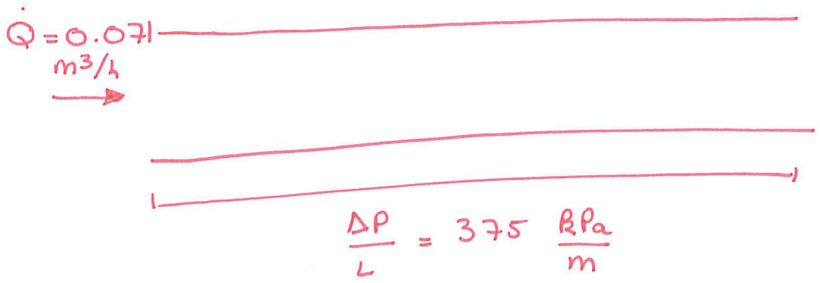
Example - Flow through a capillary tube.

A 5 mm diameter horizontal tube is used as a viscometer. When the flow rate is $0.071 m^3/h$ the pressure drop per unit length is 375 kPa/m. Density of the fluid is 900 kg/m^3. Estimate the viscosity of the liquid.



From Poiseuille's equation;

$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{\Delta P}{L}$$

$$\frac{0.071}{60.60} = -\frac{3.14 \times 0.0025^4}{8\mu} \times 375 \times 10^3$$

assumptions

! LAMINAR FLOW !

! FULLY DEVELOPED !

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Now we know the viscosity we should check to make sure that the flow is laminar. Using Reynolds number, equation (3.3). We first need the mean pipe flow velocity;

$$u_m = \frac{\dot{q}}{A} = \frac{0.071}{60 \cdot 60} \times \frac{1}{\pi\,(.0025)^2} = 1 \text{ m/s}$$

Re= $\dfrac{\rho\, u_m\, d}{\mu} = \dfrac{900 \cdot 1 \cdot (.005)}{.292} = 15$ $\ll 2500$

$\Rightarrow$ laminar

Example - Drinking beer through a straw.

pint in 3 mins

A straw is 20 cm long with a diameter of 2 mm, and you want to drink at a rate of 3 cm^3/s. Find (a) the head loss across the straw. Hence determine the pressure required to suck up the beer if the straw is held (b) horizontally, or (c) vertically. Take the viscosity of beer as 1.302×10^{-3} kg/ms at this temperature.

$\rho = 1000 \text{ kg/m}^3$

$$u_m = \frac{\dot{q}}{A} = \frac{3\times10^{-6}}{\pi(.001)^2} = 0.955 \text{ m/s}$$

First check if the flow is laminar;

$$Re = \frac{\rho u d}{\mu} = \frac{1000 \cdot (0.955)\, 2\times10^{-3}}{1.302\times10^{-3}} = 1468$$

< 2500

$\Rightarrow$ laminar (just)

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(a) Since the flow is laminar we can use Poiseuille & the expression for head loss in a laminar pipe flow (3.11);

$$h_f = \frac{8 \dot{q} L \mu}{\pi R^4 \rho g} = \frac{8\,(3\times10^{-6})\,(.2)\,(1.302\times10^{-3})}{3.14\,(0.001)^4\,(1000)\,(9.81)}$$

Head loss down the straw, $h_f =$ 0.203 m

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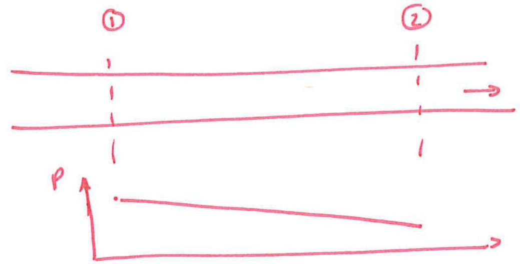


$\Delta p = p_1 - p_2$

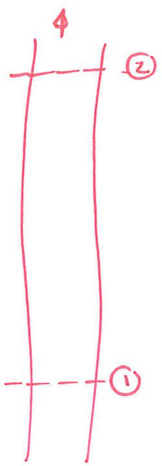
(b) For the horizontal straw;

$p^* = p + \rho g z$ but $z_1 = z_2$

so; $h_f = \dfrac{\Delta p}{\rho g}$

$$0.203 = \frac{\Delta p}{1000 \cdot 9.81} \Longrightarrow \Delta p = \underline{\underline{1991 \text{ Pa}}}$$

(b) For the vertical straw;



$$h_f = \frac{\Delta p}{\rho g} + (z_1 - z_2)$$

$$0.203 = \frac{\Delta p}{\rho g} - \Delta z = \frac{\Delta p}{1000 \cdot 9.81} - 0.2$$

$$\Delta p = 3953 \text{ Pa}$$

human lung

~ 2000 Pa