lectures copy

ROJ.

1. Fluids in Equilibrium

1.1. Introduction

Fluids in equilibrium - means the particles of the fluid are not moving with respect to one another.

There are three cases we will consider:

the fluid is **stationary** - *fluid statics*
the fluid is **accelerating uniformly**
fluid is **rotating as a body** } *relative equilibrium*

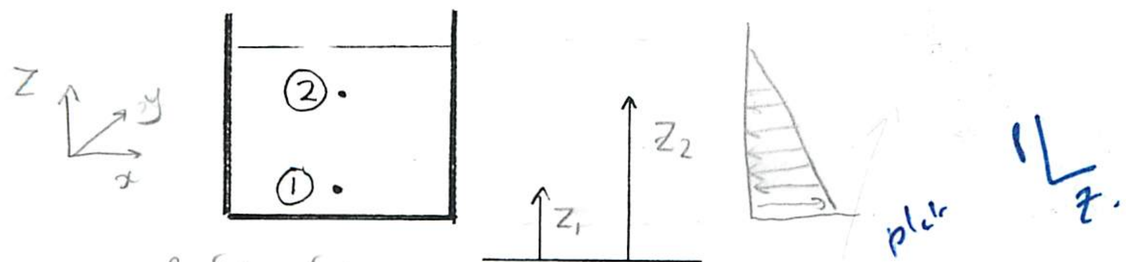
Because there is no flow, we can treat the fluid as a system like we do in solid mechanics. We can then apply 'system' equations - in this case Newton's 2nd Law.

1.2. Fluid Statics (revision)

(Massey §2.1-2.4)

This subject was covered in 1st year mechanics as forces on submerged bodies. The bodies are subjected to distributed loads caused by the pressure variation.

1.2.1. Pressure variation within a fluid;



$$\frac{\partial p}{\partial z} = -\rho g \qquad \& \ \frac{\delta p}{\delta x} = \frac{\delta p}{\delta y} = 0 \tag{1.1}$$

$$\int_1^2 \delta p = -\int_1^2 \rho g \, dz. \qquad p_2 - p_1 = \rho g(z_1 - z_2)$$

so if both the density and gravity, g are constant with z then;

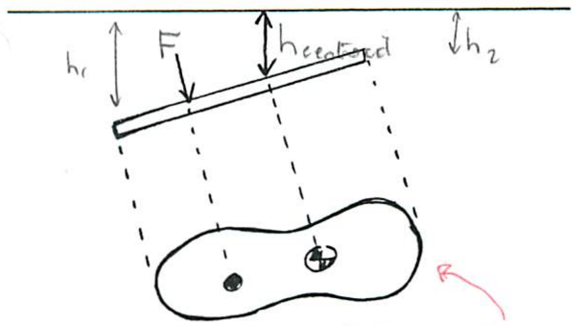
$$p + \rho g z = \text{constant} \qquad \text{law of manometry.} \tag{1.2}$$

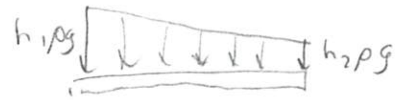
We use this expression in the measurement of pressure - manometry and to determine forces on submerged bodies.

1.2.2. Hydrostatic Forces on Submerged Surfaces

(Massey §2.4, White §2.5)

If we have a submerged body;





Consider moments square

The pressure distribution results in a distributed force on the body. We solve this like a solids problem i.e. resolving forces and taking moments.

The resultant force on the surface is given by;

$$F = p_{centroid} A = h_{centroid} \rho g A \tag{1.3}$$

and this force acts at the centre of pressure.

1.3. Fluids in Rigid Body Motion

If the fluid is moving as a rigid body (e.g. carrying a glass of water), there is no relative motion between the particles. We can then treat the body of fluid as a system (i.e. a fixed amount of mass). We then use system equations (Newton's 2nd Law) to solve relationships between forces and motion.

Since the particles of fluid are not moving with respect to one another - there is no viscous interaction between the fluid particles.

Need not consider viscous forces $F = \tau . A$

i.e. Newton's law of viscosity, $\tau = \mu \frac{du}{dy}$

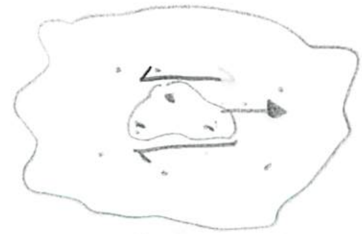
Only pressure forces $F = p.A$

body forces. $F_x, F_y, F_z,$ and mg

} resolve these like a solids problem

When we have fluids in motion we can no longer do this. Instead we use 'control volume analysis' - i.e. we look at a fixed volume in space rather than a fixed mass. This is the subject of later lectures.

fluid particles all moving together

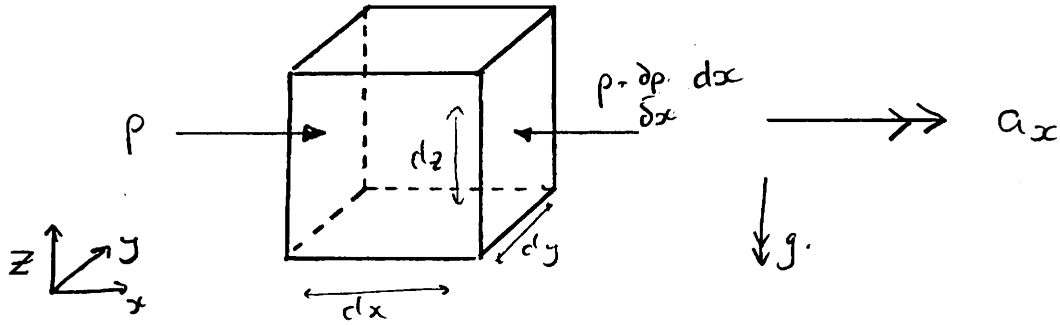


some fluid particles moving - shear stress present

Objective: determine the way the pressure varies thro' the fluid

1.3.1. Forces on a Fluid Element

Consider an element of fluid *dx*, *dy*, *dz* which is accelerating as a rigid body in the x-direction;



PTO

Since the fluid element is at rest (with respect to the fluid around it) there are no shear stresses along the element sides.

Treating the element as a system, applying Newton's 2nd in the x-direction;

$$p\,dy\,dz - \left(p + \frac{\partial p}{\partial x}\,dx\right)dy\,dz = \rho\,dx\,dy\,dz\,a_x \tag{1.4}$$

which simplifies to;

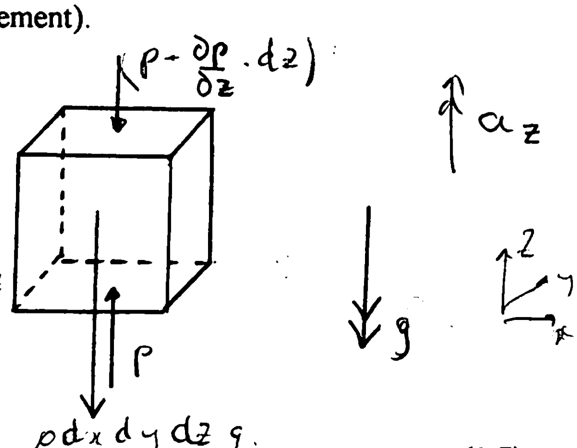
$$-\frac{\partial p}{\partial x}\,dx\,dy\,dz = \rho\,dx\,dy\,dz\,a_x$$

$$\boxed{\frac{\partial p}{\partial x} = -\rho a_x} \tag{1.5}$$

If the element had been accelerating in the y-direction we would have a similar expression;

$$\boxed{\frac{\partial p}{\partial y} = -\rho a_y} \tag{1.6}$$

But now consider the element accelerating in the z-direction (remember upwards is positive z). We must now include a body force (i.e. mass of the element).



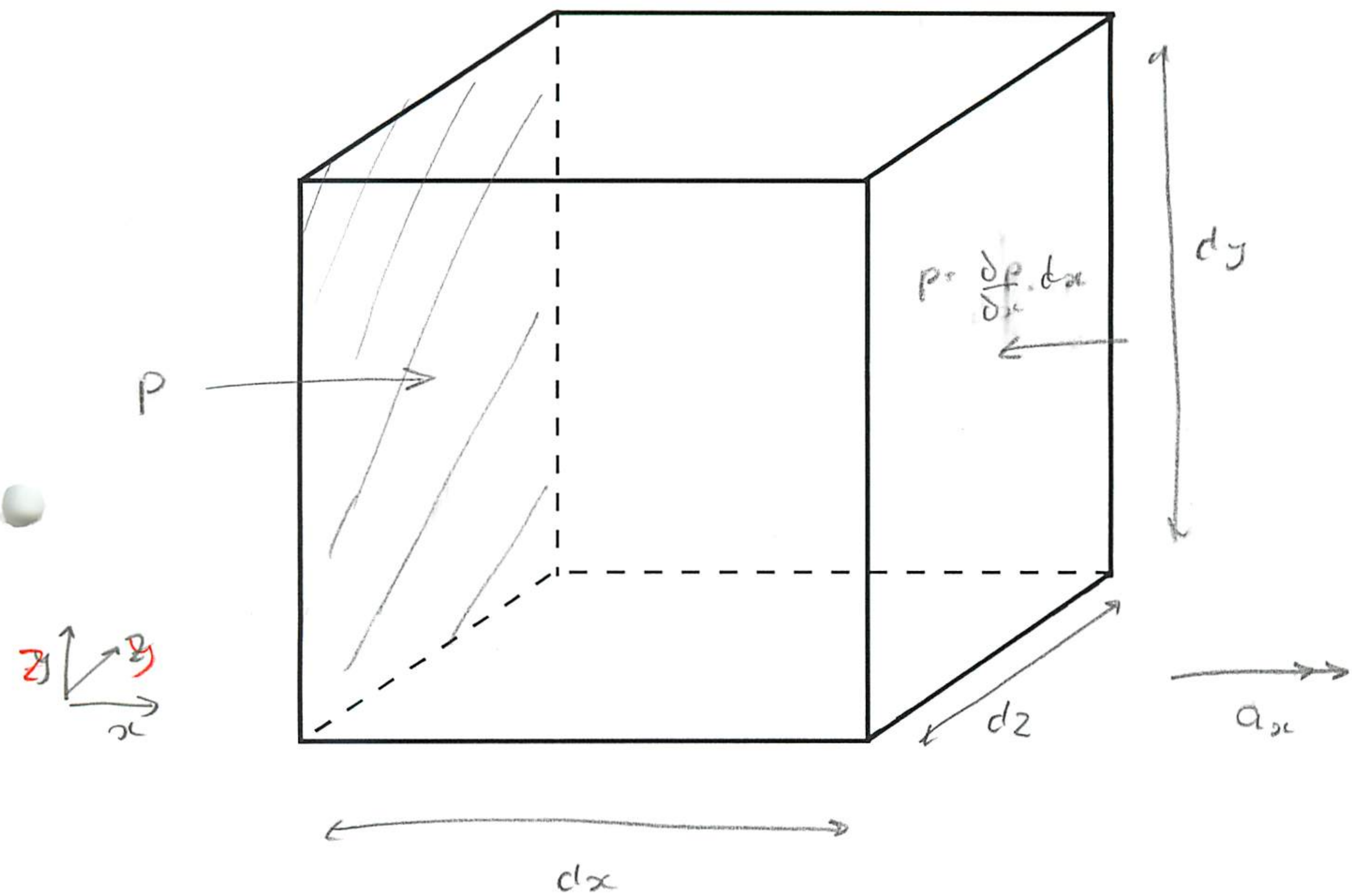
$$p\,dx\,dy - \left(p + \frac{\partial p}{\partial z}\,dz\right)dx\,dy - \rho\,dx\,dy\,dz\,g = \rho\,dx\,dy\,dz\,a_z$$

which simplifies to:

$$\boxed{\frac{\partial p}{\partial z} = -\rho(g + a_z)} \tag{1.7}$$

We can use these expressions to determine pressure distributions in fluids undergoing rigid body motion.

(Special case: $a_x = a_y = a_z = 0$ $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ $\frac{\partial p}{\partial z} = -\rho g$)



Newtons 2nd law $F = m.a.$ in x-direction.

$\longrightarrow$ +ve

Forces $p . dy\, dz \rightarrow$

$\left(p + \frac{\partial p}{\partial x}.dx\right) dy\, dz \leftarrow$

mass $\rho . dx . dy . dz$

$$p. dy\,dz - \left(p + \frac{\partial p}{\partial x}.dx\right) dy\,dz = \rho\, dx\,dy\,dz . a_x$$

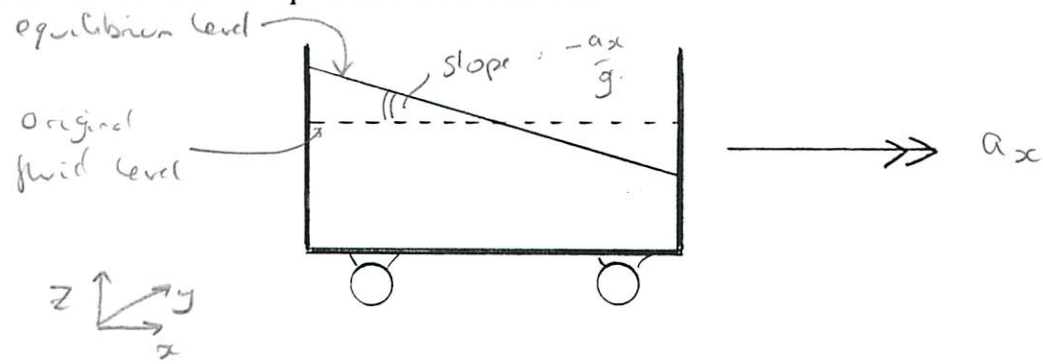
The expressions (1.5), (1.6), and (1.7) are most conveniently expressed in vector form;

$$\nabla \mathbf{p} = \rho(\mathbf{g} - \mathbf{a}) \tag{1.8}$$

$$\nabla p = i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y} + k\frac{\partial p}{\partial z}$$

1.3.2. Fluids in Uniform Linear Acceleration

Consider a fluid bath, resting on a trolley which is accelerating uniformly in the x-direction, a_x. What is the distribution of pressure within the fluid.



Once equilibrium has been reached (i.e. when the fluid has reached a stable position), the particles within the fluid are at rest with respect to one another there. There are therefore no shear stresses and equations (1.5), (1.6), & (1.7) apply.

In this case where $a_y = a_z = 0$;

$$\frac{\delta p}{\delta x} = -\rho a_x \qquad \frac{\delta p}{\delta y} = 0 \qquad \frac{\delta p}{\delta z} = -\rho g \tag{1.9}$$

we want to know the way the pressure varies through the fluid, i.e. ~~δp~~ dp. We do this using the total differential.

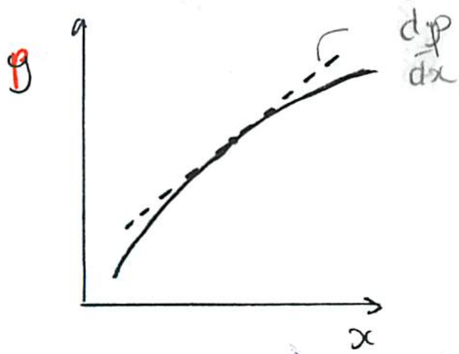
Expanding the partial differential, ~~δp~~; dp

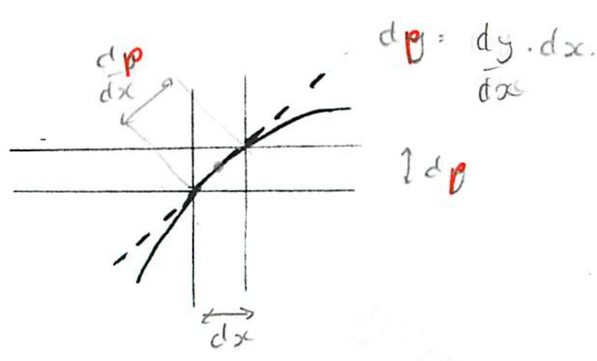
$$\boxed{\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z} \tag{1.10}$$

this is called the **total differential**.

If p=f(x,y,z) then we can express a small change in p (i.e. dp) as small changes in x, y, and z multiplied by the gradient of the function.

To see that this is true, consider the simpler case where some variable y=f(x).





& in 3D

$$dp = \frac{\delta p}{\delta x} \cdot dx + \frac{\delta p}{\delta y} \cdot dy + \frac{\delta p}{\delta z} \cdot dz$$

Then you can see from the graph that $d\cancel{y}p = dx\frac{dp}{dx}$.

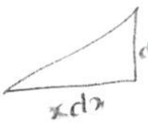
Returning to the total differential (1.10) and substituting (1.9) gives;

$$dp = -\rho a_x \, dx - \rho g \, dz.$$

We want to know the pressure distribution. Lines of constant pressure are given by $dp=0$. Hence;

$$dp = 0 = -\rho a_x dx - \rho g dz. \quad \Bigg\} \text{ for constant pressure line } dp = 0$$

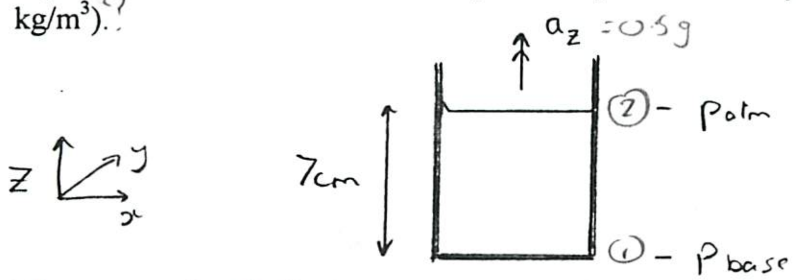
$$\left(\frac{dz}{dx}\right)_{p=0} = -\frac{a_x}{g} \tag{1.11}$$

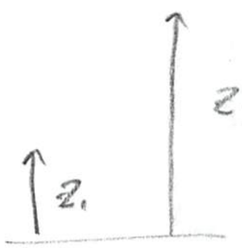


Constant pressure planes have a slope of $-\frac{a_x}{g}$. We know that the surface of the liquid must be a constant pressure plane (where $p=p_{atm}$). So the equilibrium position of the fluid will be as shown in the figure above.

Example 1 - Drinking Coffee in a Lift

A coffee cup is carried in a lift which accelerates upwards at 0.5g. What pressure is experienced at the bottom of the cup if the liquid is 7cm deep (the density of coffee is 1010 kg/m^3).?





We use equation (1.7)

$$\frac{\delta p}{\delta z} = -\rho(g + a_z)$$

$$= -\rho(g + 0.5g)$$

$$\delta p = -1.5\rho g \, \delta z \qquad \int_1^2 \delta p = -\int_1^2 1.5 \rho g \, \delta z$$

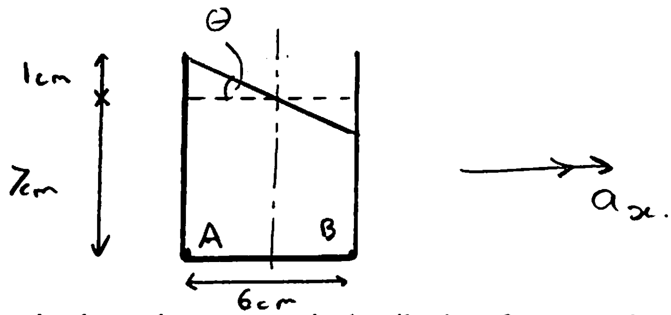
$$(p_1 - p_2) = 1.5\rho g (z_2 - z_1) = -1.5 \cdot 1010 \cdot 9.81 \, (0 - 0.07).$$

$$= 1040 \text{ N/m}^2 = 0.0104 \text{ bar}$$

this is gauge pressure for absolute pressure add p_{atm}. $= 0.0104$ bar

Example 2 - Drinking Coffee in a Racing Car

The coffee cup is now resting on a horizontal surface in a racing car. At what rate can the car accelerate if the coffee is not to spill? Sketch the gauge pressure distribution acting at the bottom of the cup.



If the mug is symmetric about the centre axis the tilted surface must intersect the ~~the~~ axis as shown (the volume of coffee must be conserved)

The slope of the surface is given by (1.11);

$$\frac{dz}{dx} = -\frac{a_x}{g} \qquad f_{/.}\ dp = 0.$$

so; $\tan\theta = \frac{a_x}{g}$

From the geometry of the cup; θ=18.4° for the liquid just to spill over. Hence;

$$\tan 18.4 = \frac{a_x}{9.81} \qquad\qquad \frac{\partial p}{\partial z} = -\rho(g+a_z)$$

maximum acceleration, a_x= $3.26\ m/s^2$

The pressure at the bottom of the cup may be determined from (1.7) noting that a_z=0;

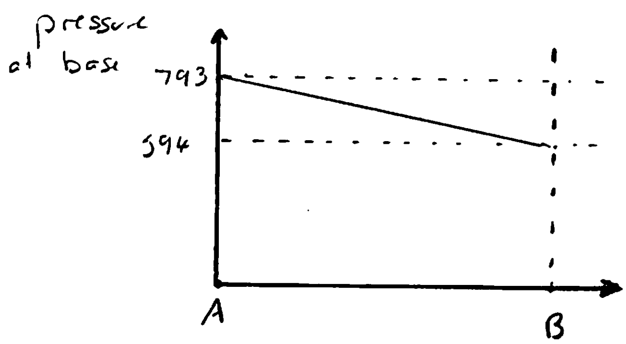
$$\frac{\partial p}{\partial z} = -\rho g \qquad \int_1^2 \partial p = -\int_1^2 \rho g\, \partial z$$

Integrating gives;

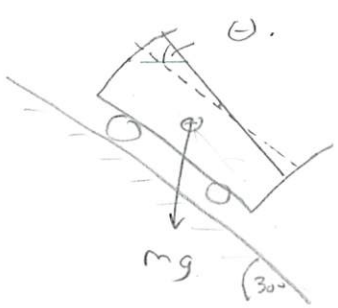
$$p_2 - p_1 = \rho g(z_1 - z_2)$$

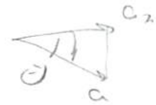
So gauge pressure at A= $1010 \cdot (0.08) \cdot 9.81 = 793$ Pa

So gauge pressure at B= $1010 \cdot (0.06) \cdot 9.81 \cdot 594$ Pa



Extra Example





First determine the accelerations in the x & z directions

$$mg \sin 30 = ma \qquad a = 0.5g$$

$$a_x = a \cos 30$$

$$a_z = -a \sin 30$$

$$\frac{\partial p}{\partial x} = -\rho a_x \qquad \frac{\partial p}{\partial y} = -\rho a_y \qquad \frac{\partial p}{\partial z} = -\rho(g + a_z)$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz = -\rho a_x\, dx - \rho(g + a_z)\, dz$$

for $dp = 0$ $$\frac{dz}{dx} = -\frac{a_x}{g + a_z}$$

$$\tan\Theta = -\frac{0.5g \cos 30}{g - 0.5g \sin 30}$$

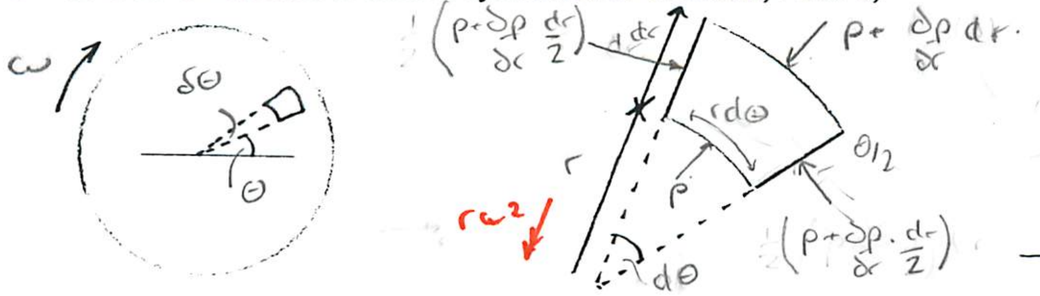
gives $\Theta = 30°$.

The slope is independent of the density of the fluid.

1.3.3. Fluids in Rigid Body Rotation

e.g. glass of water on a record player turntable

Now consider an element of fluid in cylindrical co-ordinates, r and θ,



The fluid is rotating at a constant angular velocity, ω. So, there is an acceleration towards the centre of the circle of $r\omega^2$.

Massey p 320-321

Newton's 2nd Law for the element (towards the centre);

$$\left(p+\frac{\partial p}{\partial r}\cdot dr\right)(r+dr)\,d\theta dz - p\cdot r\,d\theta dz - 2\left(p+\frac{\partial p}{\partial r}\cdot\frac{dr}{2}\right)dr\cdot dz\sin\frac{d\theta}{2} \tag{1.12}$$

simplifying and neglecting higher order terms, gives;

as $d\theta \to 0$ $\sin\frac{d\theta}{2} \to \frac{d\theta}{2}$ $\quad = \rho r\,d\theta\,dr\,dz\cdot r\omega^2$

$$\boxed{\frac{\partial p}{\partial r} = \rho r\omega^2} \tag{1.13}$$

We can repeat this analysis for forces in the z-direction - it is the same as the derivation of (1.7) above);

$$\boxed{\frac{\partial p}{\partial z} = -\rho g} \tag{1.14}$$

and also in the circumferential direction (θ–direction). Noting that since the fluid is rotating at constant velocity it is not accelerating circumferentially, $a_\theta=0$.

$$\boxed{\frac{\partial p}{\partial \theta} = 0} \tag{1.15}$$

To find the contours of equal pressure, we use the total differential, (1.10), this time $p=f(r,\theta,z)$;

(1.16) $\quad dp = \frac{\delta p}{\delta r}dr + \frac{\delta p}{\delta\theta}d\theta + \frac{\delta p}{\delta z}\cdot dz$

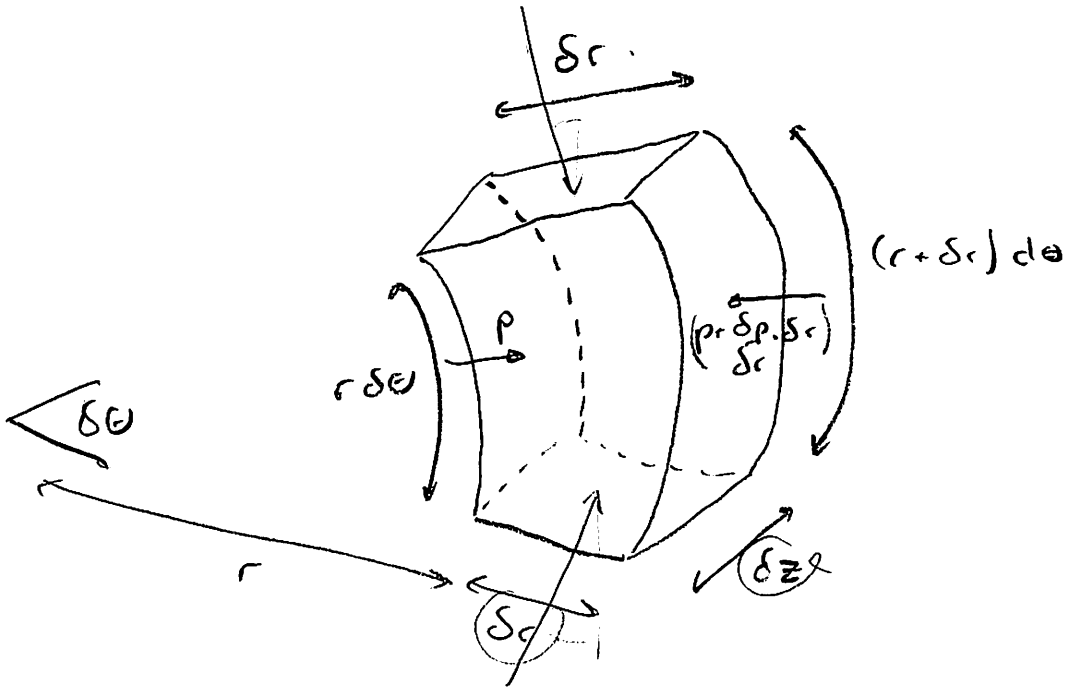
substituting (1.13), (1.14), and (1.15) gives;

$dp = \rho r\omega^2\,dr - \rho g\,dz$

integrating between two points 1 and 2;

$$p_2 - p_1 = \frac{\rho\omega^2(r_2^2-r_1^2)}{2} - \rho g(z_2-z_1) \tag{1.17}$$

so the pressure varies linearly with z and parabolically with r.



N2L towards centre of circle ←

$F = mr\omega^2$

Forces $\quad p . r\delta\theta . \delta z \rightarrow$

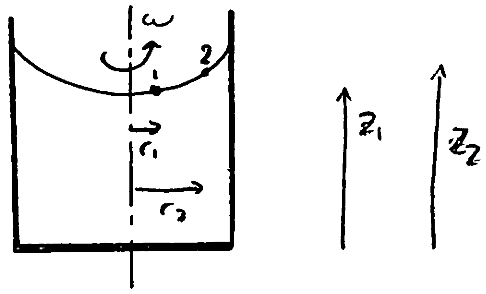
$$\left(p + \frac{\delta p}{\delta r} . \delta r\right)(r+\delta r)\,\delta\theta . \delta z \leftarrow$$

mass of the element $\quad \rho . r\delta\theta . \delta r . \delta z$

$$-2 \times \left(p + \frac{\partial p}{\partial r}\frac{dr}{2}\right) dr\,dz \sin\frac{d\theta}{2}$$

For a free surface $dp=0$ so if 1 and 2 are both on a free surface $p_1 = p_2 = p_{atm}$

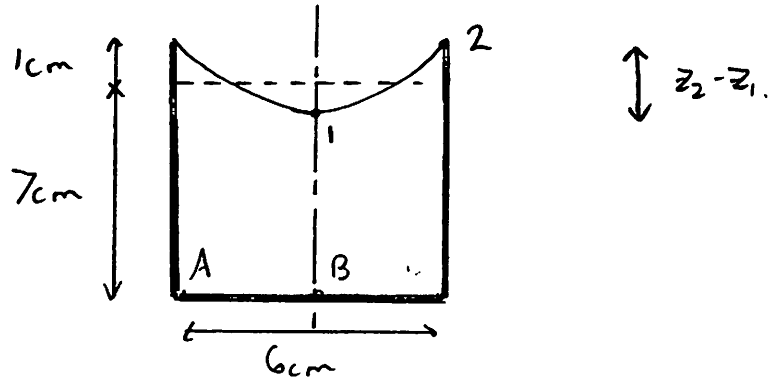
$$(z_2 - z_1)_{surface} = \frac{\omega^2 (r_2^2 - r_1^2)}{2g} \qquad (1.18)$$



The free fluid surface is parabolic.

Example 3 - A Spinning Coffee Cup

The coffee cup is now rotated about its central axis. At what angular velocity will the contents just spill out? Sketch the gauge pressure distribution acting at the bottom of the cup.



at the surface of the rotating fluid from (1.18);

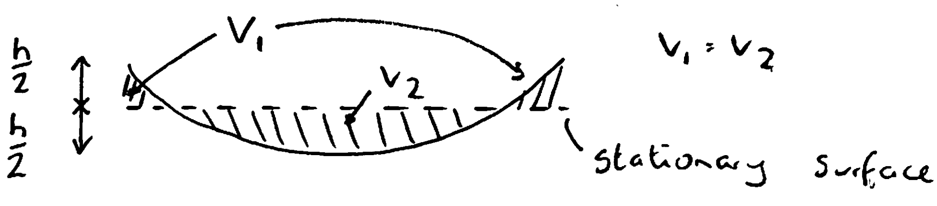
$$(z_2 - z_1)_{surface} = \frac{\omega^2 (r_2^2 - r_1^2)}{2g}$$

so if we choose points 1 and 2 to be at the axis and the periphery respectively.

Then $r_1=0$, $r_2=R=$ cm and $h=(z_2-z_1)$ and therefore;

$$h = \frac{\omega^2 R^2}{2g}.$$

The volume of a parabaloid is one half the base area times the height ($=\pi R^2 h/2$). So the still water level is halfway between the high and low points of the free surface;



So for the water to just reach the lip of the cup; $\frac{h}{2} = 0.01$.

$$2 \cdot 0.01 = \frac{\omega^2 \cdot 0.03^2}{2 \cdot 9.81}$$

This gives a maximum angular velocity of. $\omega = 20.9$ rad/S $= 200$ rpm.

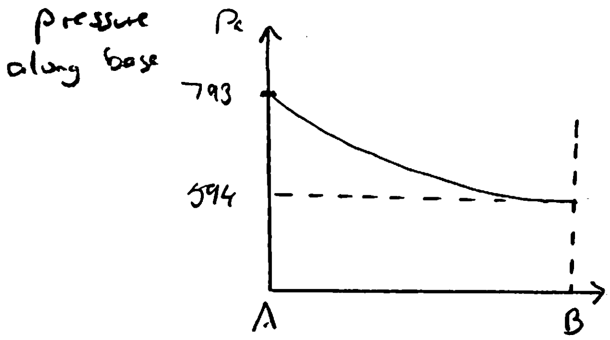
To determine the pressure distribution at the base we use;

$$p_2 - p_1 = \rho(z_1 - z_2)g$$

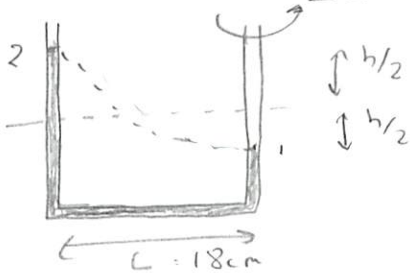
So gauge pressure at A= $1010 \cdot 0.08 \cdot 9.81 = 793$ Pa

So gauge pressure at B= $1010 \cdot 0.06 \cdot 9.81 = 594$ Pa

This time the distribution is parabolic;



$\Omega = 95\, rpm.$



$$(z_2 - z_1) = \frac{\omega^2 (r_2^2 - r_1^2)}{2g}$$

$z_2 - z_1 = h.$

assume $d \ll L.$

at ① $r = 0$

at ② $r = L$

$\omega = 95\, rpm \cdot \frac{2\pi}{60} = 9.945\ rad\ s^{-1}$

$$h = \frac{\omega^2 R^2}{2g} = \frac{9.95^2 \cdot .018^2}{2 \cdot 9.81} \quad .18$$

$= 0.164\ m$