3. Internal Flow

3.1. Introduction

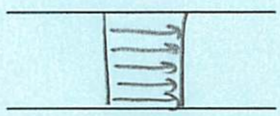
(Massey §6, White §1 & §6, first year notes)

In this part of the course we will be looking at the flow inside structures and how the fluid properties and duct shape affects the flow. The most common example is flow in a pipe. The flow may be laminar or turbulent. For turbulent flows the relationships between flow rate, pressure drop and pipe characteristics are obtained from empirical data (the Moody Chart and Los Factors). This was covered in 1st year. Here, we will be finding analytical relationships for the special case of laminar flow.

3.2. Flow Rates and Velocity Profiles

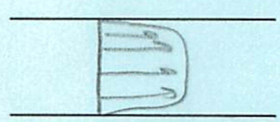
(a) Velocity Profiles across a Duct. Until now we have always assumed that the flow in a pipe is one dimensional (1D) i.e. all the fluid is travelling at the same velocity across the pipe section.

1 $u = u_1$



This cannot be the case, because the fluid touching the pipe wall must be stationary. And if the fluid is viscous this will slow down the fluid next to it. A fluid distribution results (with the maximum at the centre). In this course we will be finding out what that distribution looks like.

lamina turbulent

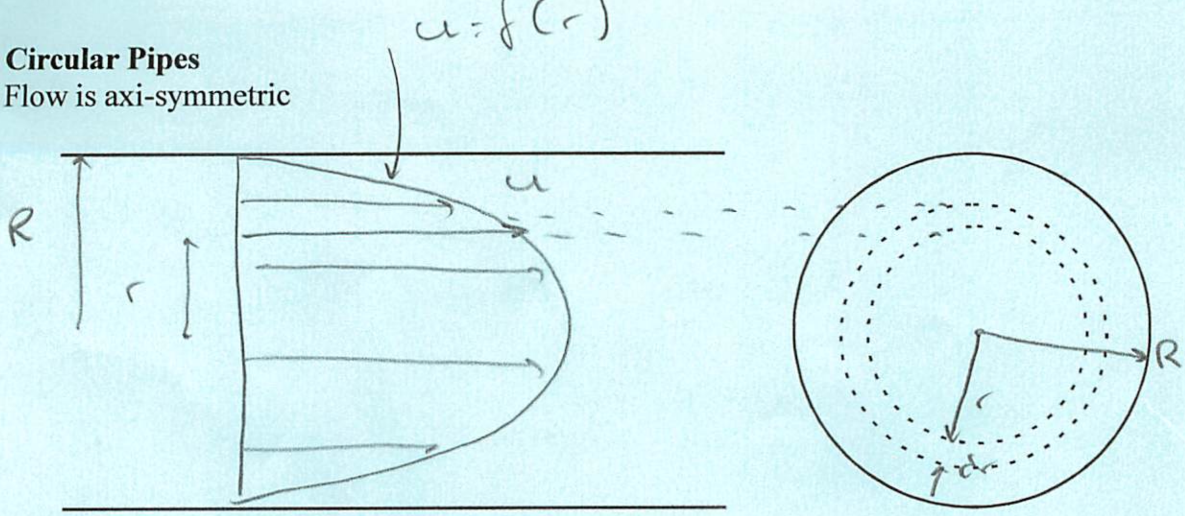


(b) Flow Rate from a Velocity Profile. If the velocity profile through a duct is known then the volume flow rate can be determined by integration.

- Choose an element over which the velocity can be assumed to be constant
- Set up an equation for the flow rate through that element in terms of the velocity, u and element area, A.
- Replace u with the velocity profile and A with the duct dimensions
- Integrate this relationship over the whole area.
- The mean velocity, u_m is then the flow rate divided by the area

Two common geometries:

Circular Pipes
Flow is axi-symmetric



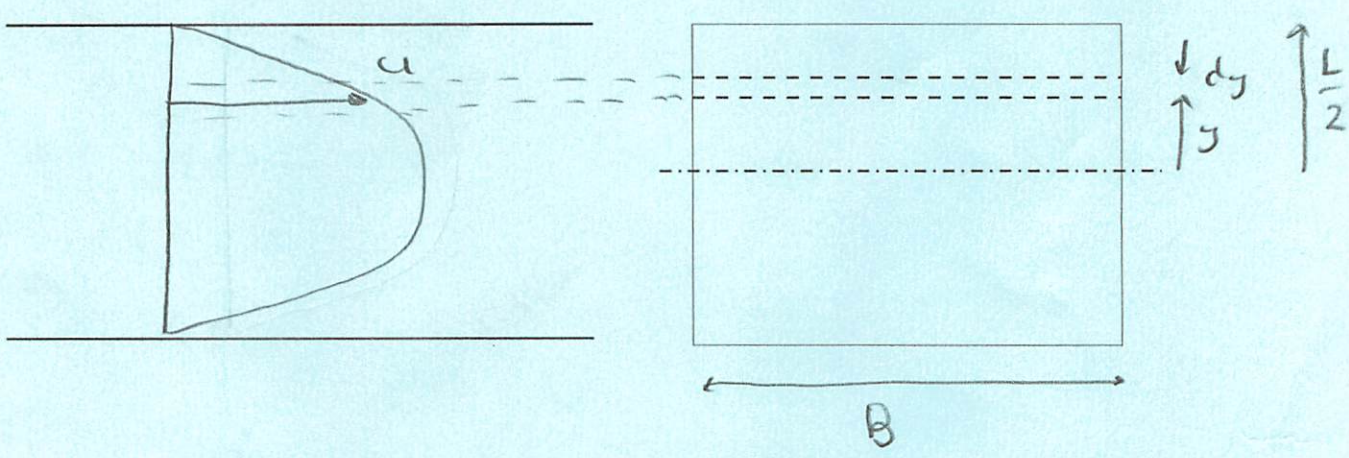
Flow through small element $d\dot{q} = u \, . \, dA$

Area of element $dA = 2\pi r \, dr$

Total flow through pipe $\dot{q} = \int_0^R u \, 2\pi r \, dr$ $\quad u = f(r)$

Mean velocity $u_m = \dfrac{\dot{q}}{A}$

Rectangular Channels
Here we assume the width is much greater than the height (B>>L) so that the velocity profile is 2D.



Flow through small element $d\dot{q} = u \, . \, dA$

Area of element $dA = B \, . \, dy$

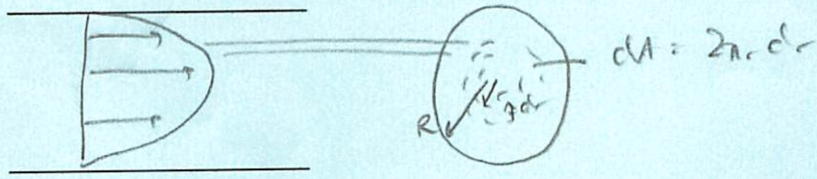
Total flow through pipe $\dot{q} = \int_{-\frac{L}{2}}^{L/2} u \, B \, dy$

Example - Flow Rate through a Pipe

Oil of density 890 kg/m^3 flows though a pipe of radius R=10 cm. The resulting velocity profile has the form

$$u = u_{\max}\left(1 - \frac{r^2}{R^2}\right) \qquad \text{where } u_{max}=0.2 \text{ m/s}$$

Determine the volume and mass flow rates down the pipe and the mean pipe velocity.



$$d\dot{q} = u \,.\, 2\pi r\, dr = u_{max}\left(1 - \frac{r^2}{R^2}\right) . 2\pi r\, dr$$

$$\dot{q} = 2\pi u_{max} \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$= 2\pi u_{max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{R^2 \pi u_{max}}{2} = \frac{.1^2 .\, 3.14 .\, 0.2}{2}$$

$$= 3.14 \times 10^{-3} \text{ m}^3/\text{s}$$

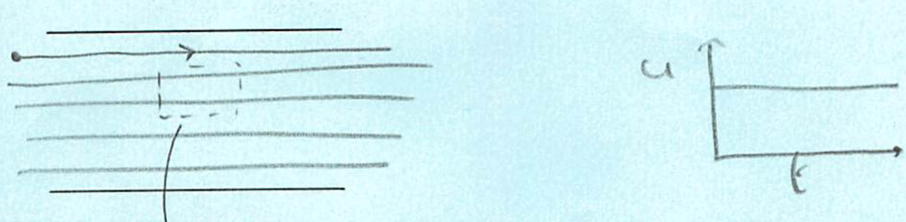
$$\dot{m} = \rho \dot{q} = 2.79 \text{ kg/s}$$

The mean velocity is then:

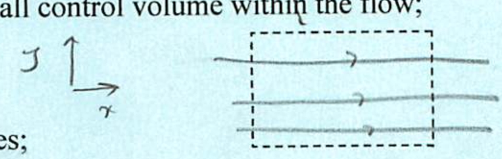
$$u_m = \frac{\dot{q}}{A} = \frac{3.14\times10^{-3}}{\pi \,.1^2} = 0.1 \text{ m/s.}$$

3.3. Viscosity, Laminar and Turbulent Flow (revision)

(a) **Laminar flow**. Particles within the fluid move in straight lines - steady flow. So at any given point the fluid velocity does not vary with time). But the velocity of particles on one line is not necessarily the same as that on another line. The fluid is moving in non-mixing layers - or *laminae*.



Consider a small control volume within the flow;



The FME states;

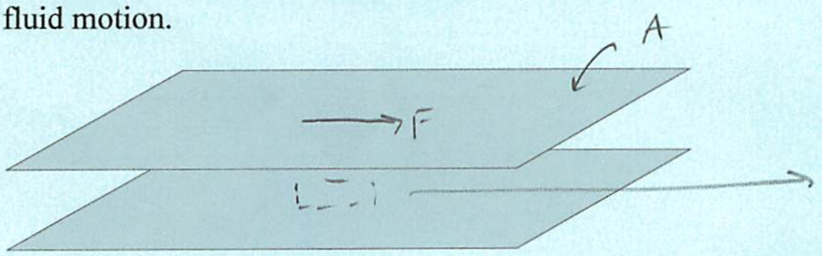
$$\sum F_x = \left(\dot{M}_{out} - \dot{M}_{in}\right)_x$$

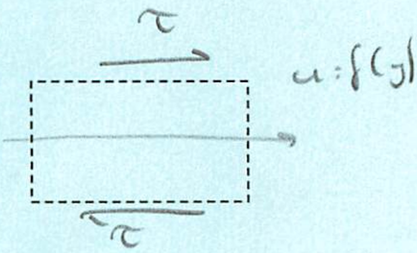
Since the fluid is flowing in straight lines (and velocity variation along a line is small) the momentum term is small compared to the force term. So;

$$\dot{M}_{out} \approx \dot{M}_{in} \qquad \Sigma F_x \approx 0 \tag{3.1}$$

The forces on the CV arise only from *viscous shear* of the fluid. This balance of shear forces is the basis for analysis of laminar internal flows. no forces for momentum change

(b) **Newton's Law of Viscosity** (Massey §1.5). Relates the shear force in a fluid with the fluid motion.



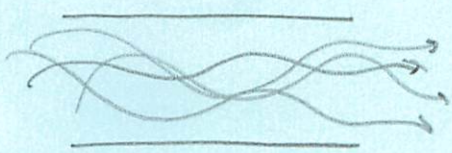


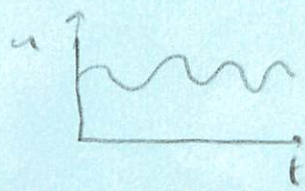
$$\tau = \mu \frac{du}{dy} \tag{3.2}$$

Remember	μ	the *viscosity* $ML^{-1}T^{-1}$ (Pa s) kg m^{-1} s^{-1}
	$\delta u/\delta y$	the *velocity gradient*.
	τ	the *shear stress* (force applied perpendicular to the area, *F/A*).

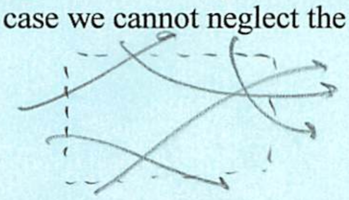
So if we know the velocity distribution, i.e. $u=f(y)$ we can differentiate to get $\delta u/\delta y$ and the shear stress, τ. And knowing the area over which this shear stress acts we can determine the shear force ($F_x=\tau_x.A$).

(c) **Turbulent Flow**. The paths of individual particles are no longer straight lines. The average fluid flow is down the pipe, but there are countless secondary fluid motions superimposed.





There is therefore significant mixing of fluid streams. i.e. lots of fluid motion into and out any CV occurs. In this case we cannot neglect the momentum term in the FME.



The flow is non-steady (i.e. fluid velocities at a point vary with time). However, frequently the average velocity (at a point) is constant with time. We call this *mean steady flow*.

Analysis is difficult and fluids problems are usually solved by experiment (e.g. the minor and major loss coefficients) or by using CFD techniques (as in the FLUENT lab).

(b) **Reynolds Number** - we use this to determine whether the flow is laminar or turbulent.

density
velocity

$$Re = \frac{\rho u d}{\mu} \tag{3.3}$$

characteristic length
eg pipe diameter

viscosity

Low Reynolds number (<2000 for pipe flow). High viscosity, low speed flows (e.g. golden syrup flowing slowly down a pipe) = laminar flow.

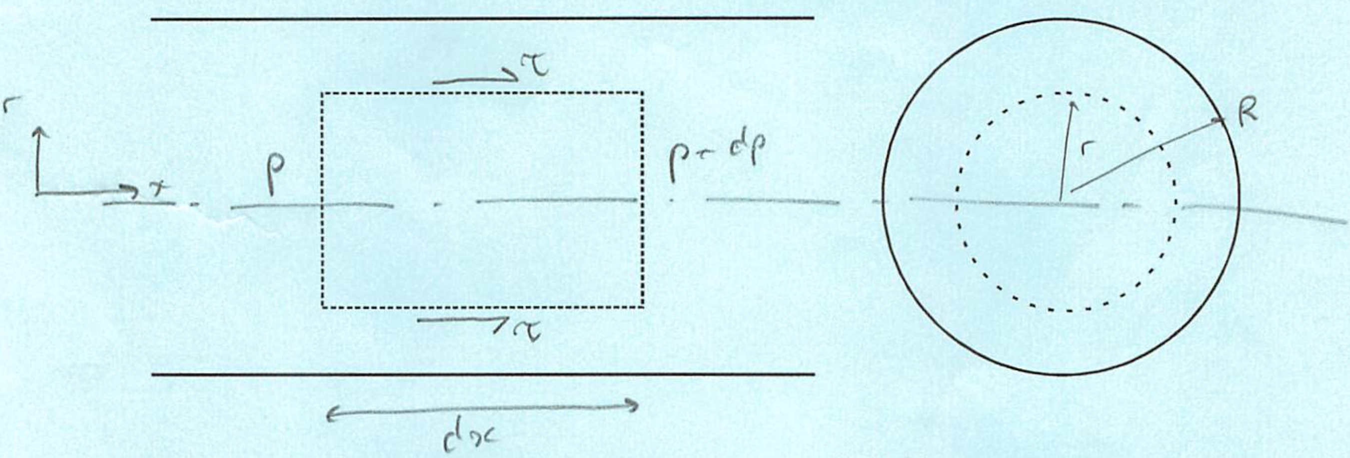
High Reynolds number (>2500 for pipe flow). Low viscosity, high speed flows (e.g. cold water tap full on) = turbulent flow.

Re = ratio of $\dfrac{\text{momentum}}{\text{shear force}}$

3.4. Flow in a Circular Pipe

(Massey §6.2,)

The method for analysing laminar flow is to consider the equilibrium (i.e. $\Sigma F=0$) of a suitable control volume. Consider a fluid flow along a pipe (there is a pressure gradient applied to the pipe). We choose a cylindrical element (note the choice of co-ordinate axes);



Forces on the element:

$$\rightarrow p \cdot \pi r^2 \qquad (p+dp)\,\pi r^2 \leftarrow \qquad \overrightarrow{\tau\, 2\pi r\, dx} \tag{3.4}$$

FME on the CV contents in the x-direction. $\sum F_x = 0$

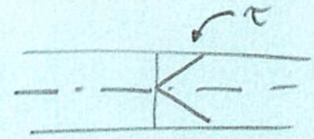
$$p\,\pi r^2 - (p+dp)\,\pi r^2 + \tau \cdot 2\pi r\, dx = 0$$

$$-dp\,\pi r^2 + \tau \cdot 2\pi r\, dx = 0$$

putting this into the above and rearranging;

$$\tau = \frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.5}$$

This gives us the distribution of shear stress at radius, r in the pipe in terms of the pressure variation down the pipe.



So if the pipe has a radius, R then the shear stress at the wall τ_w (i.e. $r=R$) is;

$$\tau_w = \frac{R}{2}\frac{dp}{dx} \qquad \tau_{axis} = 0 \tag{3.6}$$

Now substituting (3.2) into (3.5) gives;

$$\mu \frac{du}{dr} = \frac{r}{2}\frac{dp}{dx}$$

Integrating w.r.t. r:

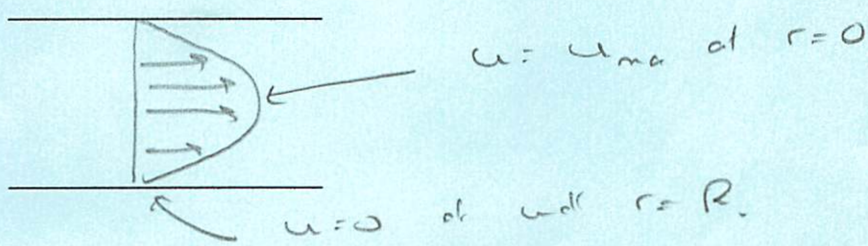
$$u = \frac{1}{2\mu}\frac{r^2}{2}\frac{dp}{dx} + K$$

Boundary condition - when $r=R$, $u=0$ i.e. at pipe wall

gives $K = -\frac{1}{2\mu}\frac{R^2}{2}\frac{dp}{dx}$

$$u = \frac{1}{2\mu}\frac{\left(r^2 - R^2\right)}{2}\frac{\mathrm{d}\,p}{\mathrm{d}\,x} \tag{3.7}$$

from this we can see the velocity profile is parabolic u=f(r^2).



To determine the volumetric flow rate. Consider a small annular element, radius r and width δr;

$$d\dot{q} = 2\pi r\, u\, dr$$

$$\dot{q} = \int_0^R u\, 2\pi r\, dr$$

We can replace u from equation (3.5)

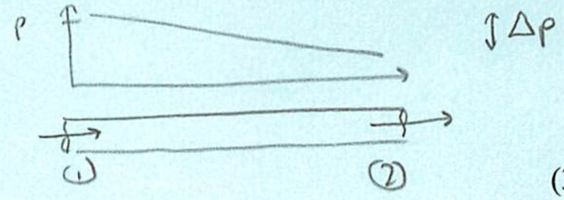
$$\dot{q} = \int_0^R \frac{1}{2\mu}\frac{(r^2 - R^2)}{2} \cdot \frac{dp}{dx} \cdot 2\pi r\, dr$$

Integrating gives;

$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}\,p}{\mathrm{d}\,x} \tag{3.8}$$

This is called Poiseuille's equation. If the flow is fully developed (i.e. far enough away from the ends of the pipe) then the pressure, p falls uniformly over the length of the pipe, l. So we can write,

$$\frac{dp}{dx} = -\frac{(p_1-p_2)}{L} = -\frac{\Delta p}{l}$$



$$\dot{q} = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{l} \tag{3.9}$$

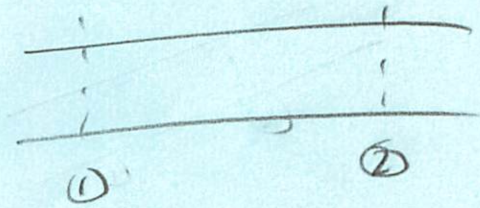
We can get the average fluid velocity down the pipe from;

$$u_m = \frac{\dot{q}}{A}$$

Uses of the Poiseuille Equation

(a) Measurement of **fluid viscosity.** By measuring the fluid flow rate through a tube for a given applied pressure. We could then use (3.8) to determine the viscosity, μ. But in practice usually requires a very small bore tube - difficult to manufacture.

(b) Head loss in laminar pipe flow. Look again at the CV above. Applying the SFEE between 1 & 2. Since $\dot{W} = \dot{Q} = 0$.



$$\left(\frac{p_2}{\rho_2} + \frac{u_2^2}{2} + gz_2 + e_2\right) - \left(\frac{p_1}{\rho_1} + \frac{u_1^2}{2} + gz_1 + e_1\right) = 0$$

In this case of laminar pipe flow (by the MCE) $u_1 = u_2$.

The loss term $(e_2 - e_1)$ arises from friction losses in the pipe. These loses are then manifested as a temperature rise of the fluid. Rewrite them in terms of a head loss or a pressure loss;

$$h_f = \frac{e_2 - e_1}{g}$$

this gives;

$$h_f = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$$

or $h_f = \frac{\Delta p}{\rho g}$ when $z_1 = z_2$ (3.10)

We can put Poiseuille's equation (3.9) into this to determine the head loss h_f.

$$h_f = \frac{8\dot{q}l\mu}{\pi R^4 \rho g} \tag{3.11}$$

or $$h_f = \frac{32 u_m l \mu}{d^2 \rho g}$$

The pressure loss is then:

$$\Delta p_L = h_f \rho g = \frac{32\, u_m L \mu}{d^2}$$

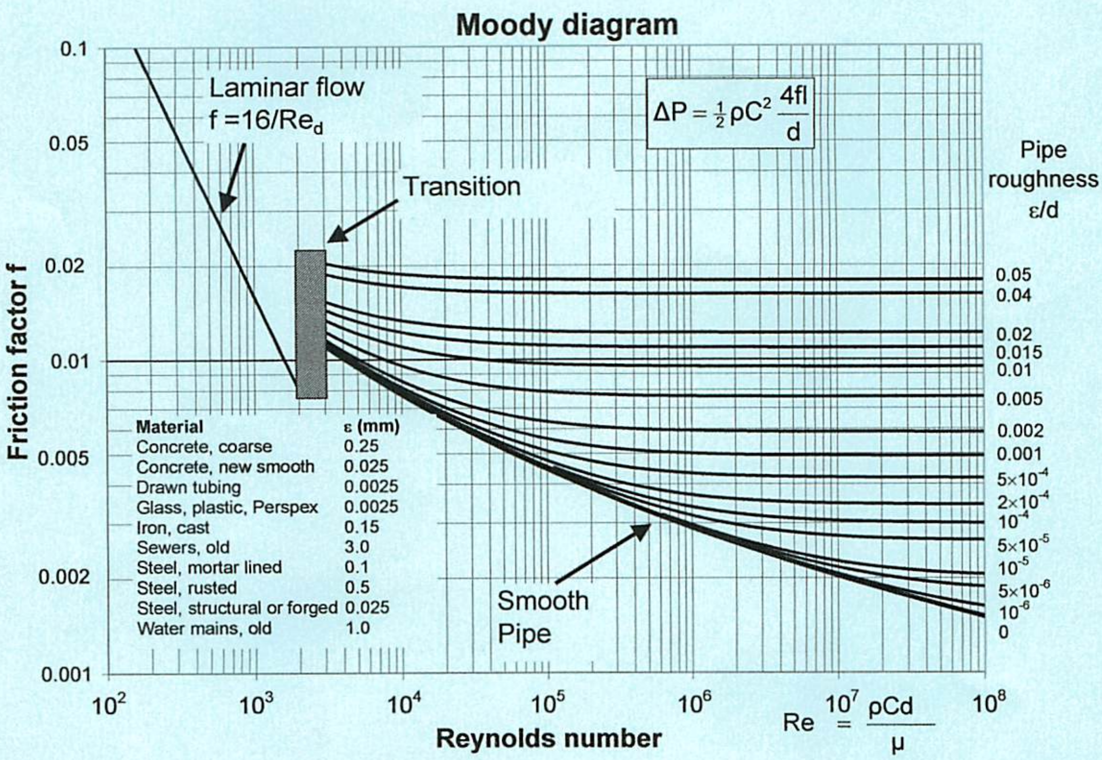
Remember the definition of fiction factor:

$$f = \frac{\Delta p_L}{\frac{1}{2}\rho u_m^2}\frac{d}{4l} = \frac{32\, u_m L \mu}{d^2} \cdot \frac{1}{\frac{1}{2}\rho u_m^2} \cdot \frac{d}{4L} = \frac{16\,\mu}{\rho u_m d} \qquad (3.12)$$

Combining (3.11) and (3.12) gives:

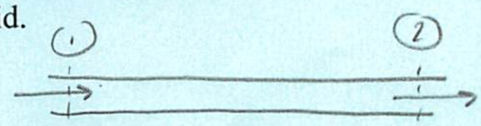
$$f = \frac{16}{\mathrm{Re}}$$

This gives us the 'laminar flow' line on the Moody Chart (which relates pipe losses to roughness and Reynolds number).



Example - Flow through a capillary tube.

A 5 mm diameter horizontal tube is used as a viscometer. When the flow rate is $0.071\,m^3/h$ the pressure drop per unit length is 375 kPa/m. Density of the fluid is 900 kg/m^3. Estimate the viscosity of the liquid.



$$\frac{\Delta p}{L} = -375\,kPa/m$$

$$\dot{q} = 0.071\,m^3/h$$

From Poiseuille's equation;

$$\dot{q} = \frac{-\pi R^4}{8\mu}\frac{\Delta p}{L}$$

laminar flow
fully developed

$$\frac{0.071}{60.60} = \frac{3.14 \cdot 0.0025^4}{8 \cdot \mu} \cdot 375 \times 10^3$$

$$\mu = 0.292 \text{ kg/ms}$$

Now we know the viscosity we should check to make sure that the flow is laminar. Using Reynolds number, equation (3.3). We first need the mean pipe flow velocity;

$$u_m = \frac{\dot{q}}{A} = \frac{0.071}{60.60} \cdot \frac{1}{\pi \cdot .0025^2} = 1 \text{ m/s}$$

Re= $\dfrac{\rho u_m d}{\mu} = \dfrac{900 \cdot 1 \cdot .005}{.292} = 15 \quad \ll 2000.$

Example - Drinking beer through a straw.

pint in 3 mins

A straw is 20 cm long with a diameter of 2 mm, and you want to drink at a rate of 3 cm^3/s. Find (a) the head loss across the straw. Hence determine the pressure required to suck up the beer if the straw is held (b) horizontally, or (c) vertically. Take the viscosity of beer as 1.302x10^{-3} kg/ms at this temperature. $\rho = 1000 \text{ kg/m}^3$

$$u_m = \frac{\dot{q}}{A} = \frac{3\times10^{-6}}{\pi \cdot .001^2} = 0.955 \text{ m/s}$$

First check if the flow is laminar;

$$Re = \frac{\rho u d}{\mu} = \frac{1000 \cdot 0.955 \cdot 2\times10^{-3}}{1.302 \times 10^{-3}} = 1460 \quad < 2000$$

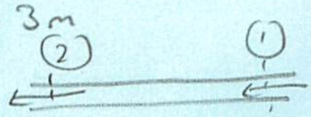
ie laminar (just)

(a) Since the flow is laminar we can use Poiseuille & the expression for head loss in a laminar pipe flow (3.11);

$$h_f = \frac{8 \dot{q} l \mu}{\pi R^4 \rho g} = \frac{8 \cdot 3\times10^{-6} \cdot .2 \cdot 1.302\times10^{-3}}{3.14 \cdot .001^4 \cdot 1000 \cdot 9.81}$$

Head loss down the straw, h_f= 0.203m

(b) For the horizontal straw;

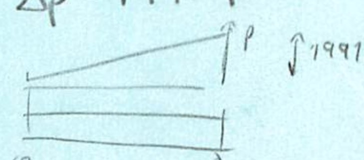


$\Delta p = p_1 - p_2$

$p^* = p + \rho g z$ but $z_1 = z_2$ $\quad p = p^*$

so; $h_f = \dfrac{\Delta p}{\rho g} \qquad 0.203 = \dfrac{\Delta p}{1000 \cdot 9.81}$

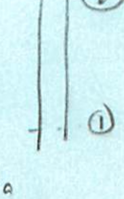
$\Delta p = 1991$ Pa



(b) For the vertical straw;

$$h_f = \frac{\Delta p^*}{\rho g} = \frac{(p_1 + \rho g z_1) - (p_2 + \rho g z_2)}{\rho g}$$

$$= \frac{\Delta p}{\rho g} + (z_1 - z_2)$$



$$.203 = \frac{\Delta p}{1000 \cdot 9.81} - 0.2$$

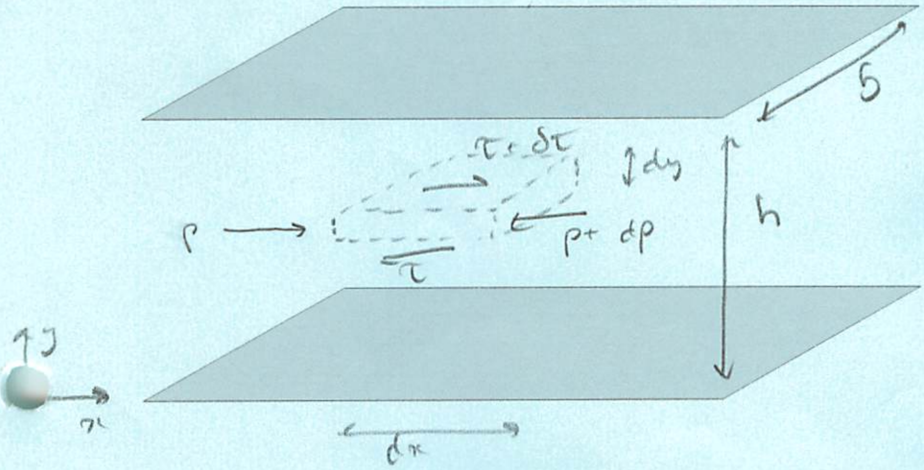
$\Delta p = 3953$ Pa

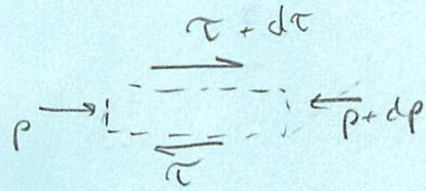
human lung ~ 2000Pa

3.5. Flow Between Infinitely Wide Parallel Plates

(Massey §6.3, White §6.6 p327-330)

We will now apply a similar analysis (FME and Newton's law of viscosity) to study the flow between two parallel planes.





Force on the element in the x-direction

$$p\,dy\,b - (p+dp)\,dy\,b - \tau\,dx\,b + (\tau+d\tau)\,dx\,b = 0$$

simplifying gives;

$$-dp\,dy + d\tau\,dx = 0$$

$$\frac{d\tau}{dy} = \frac{dp}{dx} \qquad (3.12)$$

From Newton's law of viscosity (equation 3.2);

$$\tau = \mu \frac{du}{dy}$$

differentiating w.r.t. y;

$$\frac{d\tau}{dy} = \mu \frac{d^2 u}{dy^2} \qquad (3.13)$$

combining (3.12) and (3.13) gives;

$$\frac{d^2 u}{dy^2} = \left[\frac{1}{\mu}\frac{dp}{dx}\right] \qquad (3.14)$$

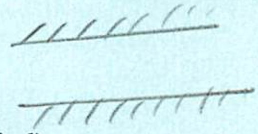
Now, we know that $\frac{dp}{dx} \neq f(y)$. i.e. the pressure does not vary in the y-direction. So we can integrate (3.14) twice;

$$\frac{du}{dy} = \left(\frac{1}{\mu}\frac{dp}{dx}\right) y + K_1$$

$$u = \left(\frac{1}{\mu}\frac{dp}{dx}\right)\frac{y^2}{2} + K_1 y + K_2$$

(3.15)

This gives the velocity distribution between the plates. We use the *boundary conditions* to determine the constants, K_1 and K_2.



(a) **Both plates fixed**. Boundary conditions: u=0 when y=0,
u=0 when y=h
dp/dx≠0 (i.e. a pressure is applied)

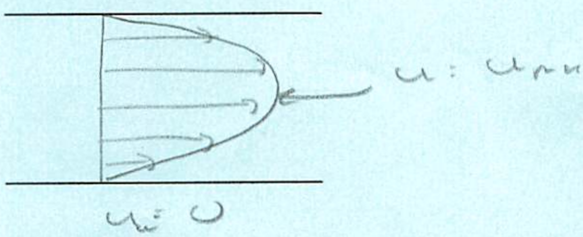
gives K_2=0

$$K_1 = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{h}{2}$$

and;
$$u = \left[-\frac{1}{2\mu}\frac{dp}{dx}\right](hy - y^2)$$

(3.16)

This is the velocity distribution of the fluid between the plates;



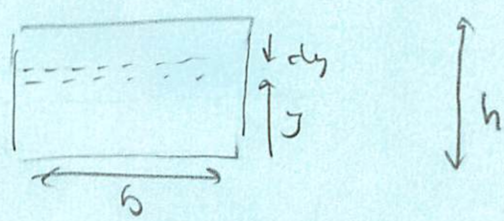
The maximum velocity occurs at the centre y=h/2

$$u_{max} = \frac{-h^2}{8\mu}\frac{dp}{dx}$$

We can get the flow rate by integrating the velocity distribution. Consider the flow through a small rectangular element;

$$d\dot{q} = u b\, dy$$

$$\dot{q} = \int_0^h u b\, dy$$



we replace the variable, *u* with equation (3.16)

$$\frac{\dot{q}}{b} = \int_0^h \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy)\,dy$$

$$\frac{\dot{q}}{b} = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}$$ (3.17)

We can also obtain the shear stress distribution;

$$\tau = \mu\frac{du}{dy}$$

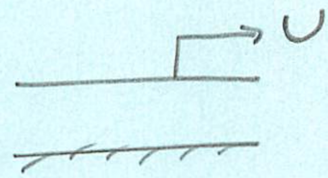
but we know (3.16) that;

$$u = \left[-\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right](hy - y^2)$$

$$\frac{du}{dy} = \left[-\frac{1}{2\mu}\frac{dp}{dx}\right](h-2y)$$

$$\tau = \mu\frac{du}{dy}$$

$$\tau = \left[-\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}x}\right](h-2y) \qquad (3.18)$$



(b) **One plate moving no applied pressure.**

Boundary conditions: u=0 when y=0,
u=U when y=h
dp/dx=0

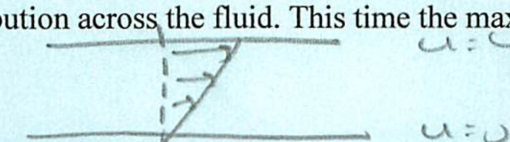
This type of flow is known as *Couette flow*. It is flow induced by the movement of a boundary. Whereas *Poiseuille flow* is induced by a pressure gradient.

Putting the boundary conditions into (3.15) gives;

$$K_2 = 0 \qquad \text{and} \qquad K_1 = \frac{U}{h} \qquad \text{and} \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = 0$$

So $u = \dfrac{Uy}{h}$ (3.19)

This is the velocity distribution across the fluid. This time the maximum is at a wall.



The volume flow rate, again is obtained from;

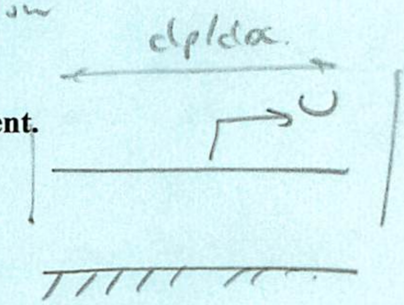
$$\dot{q} = \int_0^h u b\,dy = \int_0^h \frac{Uy}{h} b\,dy = \left[\frac{Uy^2 b}{2h}\right]_0^h$$

substituting (3.19) gives $\dfrac{\dot{q}}{b} = \dfrac{Uh}{2}$

to get the shear stress profile;

$$\tau = \mu\frac{du}{dy} = \mu\frac{U}{h}$$

∴ constant shear stress across the flow



(c) Combinations of one moving plate and an applied pressure gradient.

Boundary conditions; u=0 when y=0
u=U when y=h
dp/dx≠0

Applying these boundary conditions to (3.15) gives;

$$K_2 = 0 \qquad \text{and} \qquad K_1 = \frac{U}{h} - \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{h}{2}$$

So in this case the velocity distribution is;

$$u = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\left(y^2 - hy\right) + \frac{Uy}{h} \tag{3.20}$$

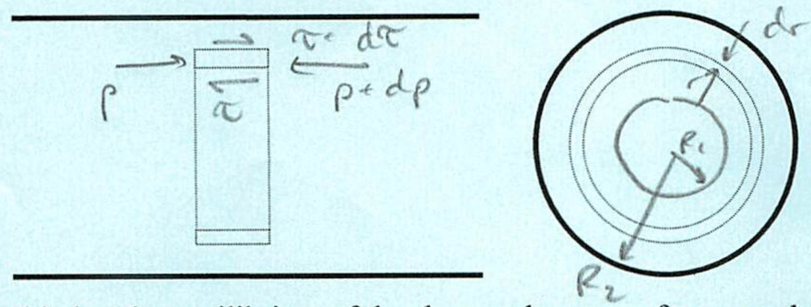
Again the flow rate and shear stress may be determined from this velocity distribution (see tutorial sheet 3 Q4), we find,

$$\frac{\dot{q}}{b} = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{Uh}{2} \tag{3.21}$$

3.6. Flow Between Concentric Cylinders

(Massey §6.3 p164, White §6.6 p330-333)

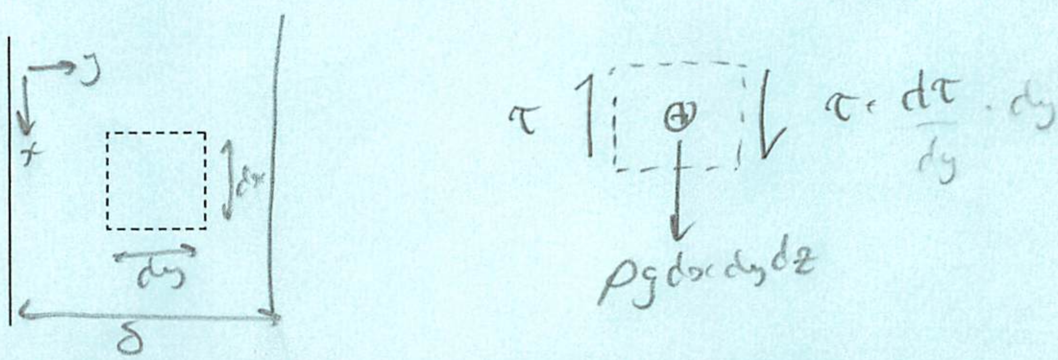
Consider the axial flow between two cylinders;



By considering the equilibrium of the shear and pressure forces on the annular element, relationships for the velocity distribution can be determined (see tutorial sheet 3 Q6).

3.7 Laminar Film on a Vertical Wall

A viscous layer of Newtonian fluid is flowing down a wall under the effect of gravity. There is no pressure gradient since the free surface of the fluid is exposed to atmosphere. The thickness of the fluid layer is assumed to be constant. We will carry out a laminar analysis to determine the velocity distribution in the thin layer.



Forces on the fluid element in the x-direction (assuming no momentum change)

$$\left(\tau + \frac{d\tau}{dy}dy\right)dx\,dz - \tau\,dx\,dz + \rho g\,dx\,dy\,dz = 0$$

Simplifying gives:

$$\frac{d\tau}{dy} = -\rho g$$

Substituting Newton's law of viscosity:

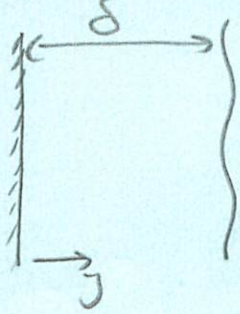
$$\mu \frac{d^2u}{dy^2} = -\rho g$$

Integrating twice:

$$\frac{du}{dy} = -\frac{\rho g}{\mu} y + K_1$$

$$u = -\frac{\rho g}{2\mu} y^2 + K_1 y + K_2$$

Boundary conditions



When $y=0$ $u=0$

$$K_2 = 0$$

When $y=\delta$ $\tau = 0$ $\frac{du}{dy} = 0$

$$K_1 = \frac{\rho g \delta}{\mu}$$

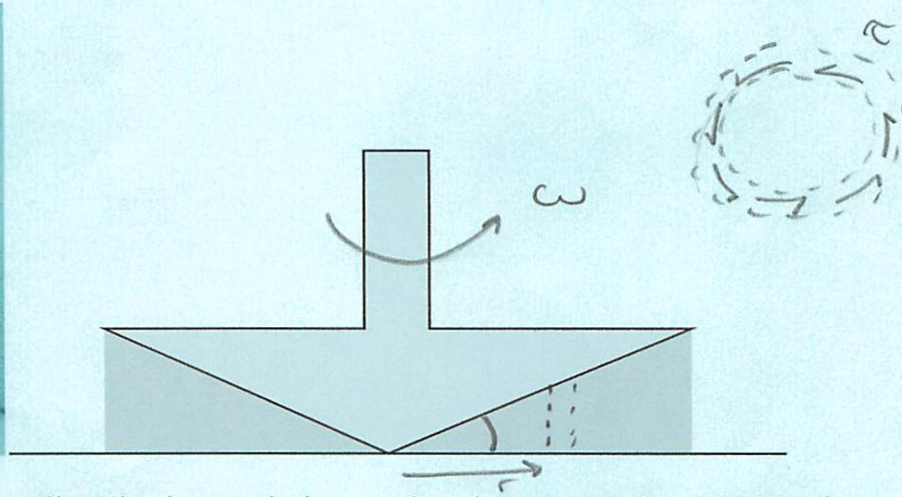
Gives the velocity profile:

$$u = -\frac{\rho g}{\mu}\frac{y^2}{2} + \frac{\rho g}{\mu}\delta y = \frac{\rho g}{\mu}\left(\delta y - \frac{y^2}{2}\right)$$

3.8 The Cone on Plate Viscometer

(Massey §6.6)

Laminar flow theory can be used as a way of measuring viscosity of liquids. Some designs are based on measuring the flow rate through thin tubes (capillary viscometers). Another design uses a cone that rotates against a flat.



Provided the cone angle is small – a laminar analysis can relate the torque to the viscosity of the liquid and the geometry of the cone.

Consider an annular element

dT= $\tau \, dA \cdot r$

Assume the velocity distribution between the base and the cone is linear. Then substitute expressions for the wetted area of the element and the torque. See Q7 Sheet 3.

3.9 Concluding Remarks

This laminar flow analysis required the use of the FME (momentum terms are negligible), the MCE, and the SFEE. The end result being velocity distributions, shear stress distributions, and volume flow rates for a range of geometries.

The following assumptions were made

1. The flow is fully developed and steady.

2. Laminar flow, Re is low (less than 2000 for pipe flow) – i.e. high viscosity fluid slopw moving.

3. The fluid is Newtonian (i.e. obeys Newton's law of viscosity)

4. The viscosity is constant throughout the flow. $\dot{Q} = 0$. If heat had been added the temperature would rise and the viscosity would fall (dramatically). This is a serious limitation and difficult to analyse. Analysis must include energy equation, continuity, fluid properties - frequently becomes complex - use CFD.

5. The fluid was assumed incompressible. If the fluid is compressible the velocity change from layer to layer will cause a density change. More complex analysis.

Generally laminar flow analysis is suitable for engineering situations where flow is slow, viscous and passageways are small e.g. flow past pistons, flow through bearings.