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MEC 208 Fluids Engineering

Dr. Cécile M. Perrault

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Before we start

- Slides from lecture 3 are online
- Tutorials:
  - Wednesday 10-11pm and Thursday 2-3pm
  - Diamond Workroom 1, 2 and 3
- Next computer sessions (19^{th} and 26^{th} of October) will be in the Diamond

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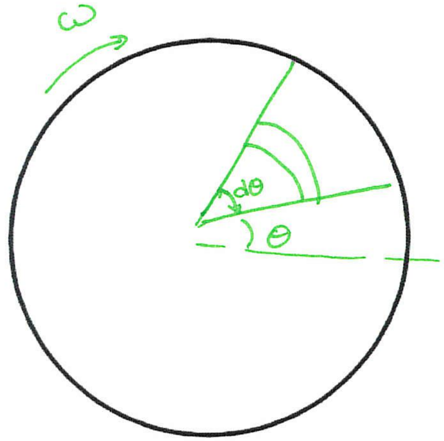
1. FLUIDS IN EQUILIBRIUM

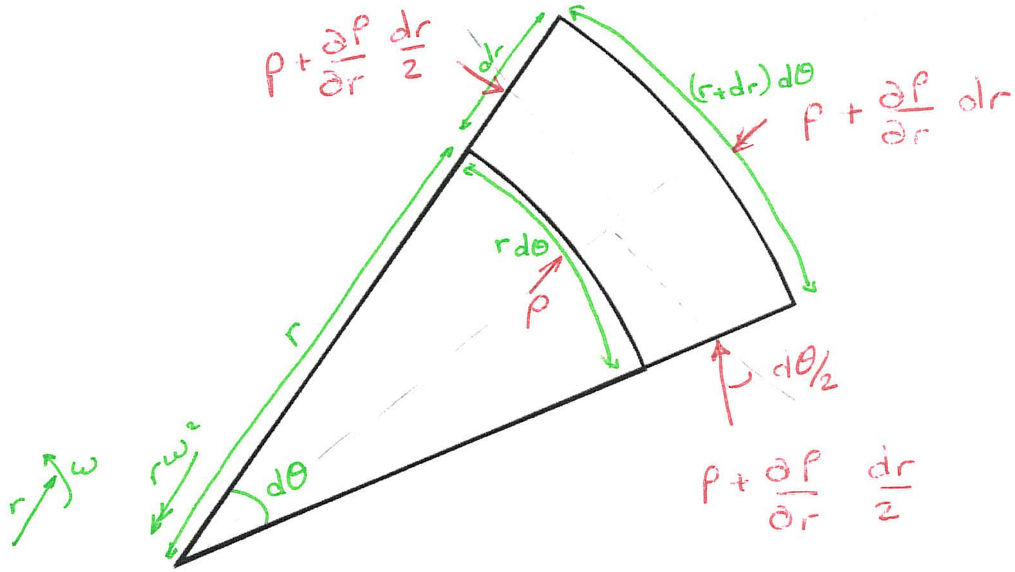
What we are covering in Topic 1:

- **1. Fluids in Equilibrium**
  - **1.1 Introduction**
  - **1.2 Fluid Statics**
    - **1.2.1. Pressure variation within a fluid**
    - **1.2.2. Hydrostatic Forces on Submerged Surfaces**
  - **1.3. Fluids in Rigid Body Motion**
    - **1.3.1. Forces on a Fluid Element**
    - **1.3.2. Fluids in Uniform Linear Acceleration**
    - **1.3.3. Fluids in Rigid Body Rotation**

1.3.3. Fluids in Rigid Body Rotation

Now consider an element of fluid in cylindrical co-ordinates, r and θ;





The fluid is rotating at a constant angular velocity, ω. So, there is an acceleration towards the centre of the circle of $r\omega^2$.

Newton's 2nd Law for the element (towards the centre);

$$\left(p+\frac{\partial p}{\partial r}dr\right)(r+dr)\,d\theta\, dz - p\,r\,d\theta\, dz - 2\left(p+\frac{\partial p}{\partial r}\frac{dr}{2}\right)dr\,dz\sin\frac{d\theta}{2} = \rho\, r\, d\theta\, dr\, dz \cdot r\omega^2 \quad (1.12)$$

simplifying and neglecting higher order terms, gives;

as $d\theta \to 0$ $\quad \sin\frac{d\theta}{2} \to \frac{d\theta}{2}$

$$\boxed{\frac{\partial p}{\partial r} = \rho r \omega^2} \quad (1.13)$$

We can repeat this analysis for forces in the z-direction - it is the same as the derivation of (1.7) above);

$$\boxed{\frac{\partial p}{\partial z} = -\rho g} \qquad (1.14)$$

and also in the circumferential direction (θ–direction). Noting that since the fluid is rotating at constant velocity it is not accelerating circumferentially, $a_\theta=0$.

$$\boxed{\frac{\partial p}{\partial \theta} = 0} \qquad (1.15)$$

To find the contours of equal pressure, we use the total differential, (1.10), this time $p=f(r,\theta,z)$;

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta + \frac{\partial p}{\partial z} dz \qquad (1.16)$$

substituting (1.13), (1.14), and (1.15) gives;

$$dp = \rho r \omega^2 dr + (-\rho g) dz$$

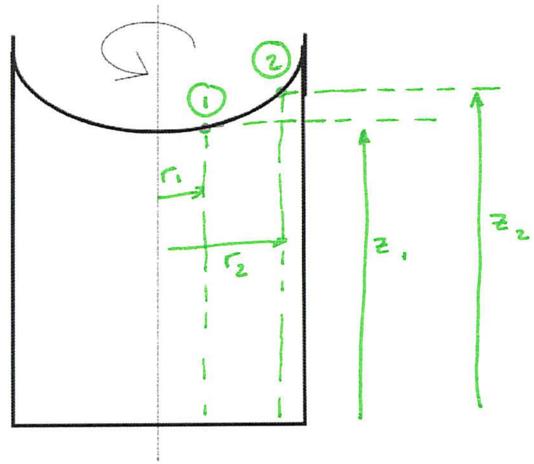
integrating between two points 1 and 2;

$$(p_2 - p_1) = \rho \omega^2 \left(\frac{r_2^2 - r_1^2}{2} \right) - \rho g (z_2 - z_1) \qquad (1.17)$$

so the pressure varies linearly with z and parabolically with r.

For a free surface dp=0 so if 1 and 2 are both on a free surface $p_1 = p_2 = p_{atm}$

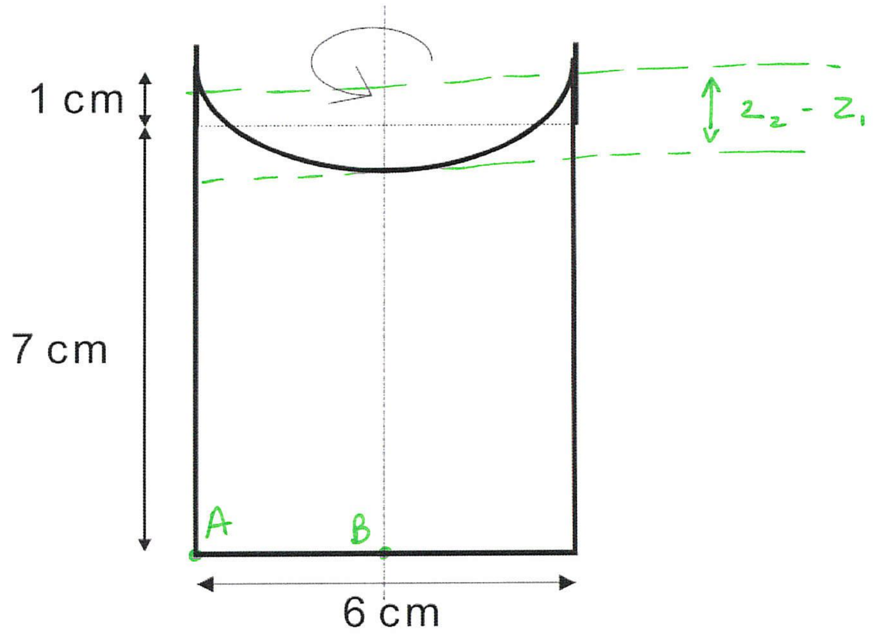
$$(z_2 - z_1)_{surface} = \frac{\omega^2 (r_2^2 - r_1^2)}{2g} \qquad (1.18)$$

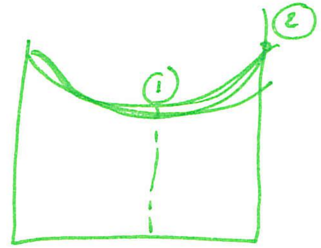


The free fluid surface is parabolic.

Example 3 - A Spinning Coffee Cup

The coffee cup is now rotated about its central axis. At what angular velocity will the contents just spill out? Sketch the gauge pressure distribution acting at the bottom of the cup.





at the surface of the rotating fluid from (1.18); $dp = 0$

$$(z_2 - z_1)_{surface} = \frac{\omega^2 (r_2^2 - r_1^2)}{2g}$$

so if we choose points 1 and 2 to be at the axis and the periphery respectively.

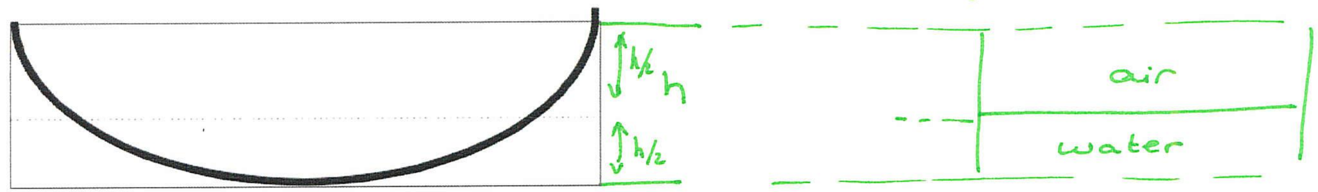
Then $r_1=0$, $r_2=R=3$ cm and $h=(z_2-z_1)$ and therefore;

$$h = \frac{\omega^2 R^2}{2g}$$

The volume of a parabaloid is one half the base area times the height ($=\pi R^2 h/2$). So the still water level is halfway between the high and low points of the free surface;

$$V_{air\ after} = \frac{\pi R^2 h}{2}$$

$$V_{air\ before} = \pi R^2 \left(\frac{h}{2}\right)$$



So for the water to just reach the lip of the cup; $\frac{h}{2} = 0.01$ m

$$2(0.01) = \frac{\omega^2 (0.03)^2}{2(9.81)}$$

This gives a maximum angular velocity of . $\omega = 20.9$ rad/s $= 200$ rpm

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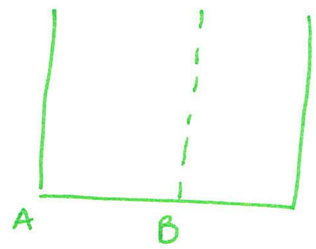
To determine the pressure distribution at the base we use;

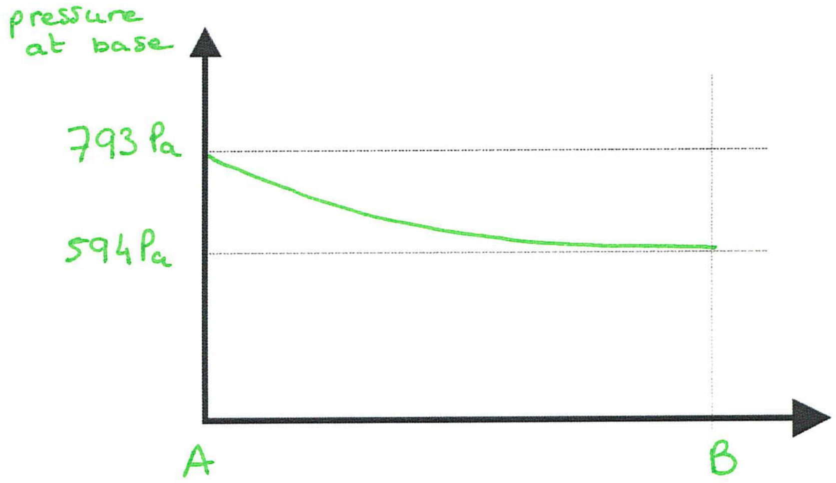
$$p_2 - p_1 = \rho (z_1 - z_2) g$$

So gauge pressure at A= $1010 \cdot 0.08 \cdot 9.81 = 793 \text{ Pa}$

So gauge pressure at B= $1010 \cdot 0.06 \cdot 9.81 = 594 \text{ Pa}$



This time the distribution is parabolic;



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CV ANALYSIS

2.1. Introduction- Fluids in Motion

In solid mechanics we use the following laws

Conservation of mass $m_{system} = \text{constant} \qquad \frac{dm}{dt} = 0$ (2.1)

Newton's 2nd law $F = ma = \frac{d}{dt}(mv)$ (2.2)

Torque- angular momentum $T = I\omega \qquad T = \frac{dO}{dt} \qquad O = \text{angular momentum}$ (2.3)

Conservation of energy $Q - W = dE$ (2.4)

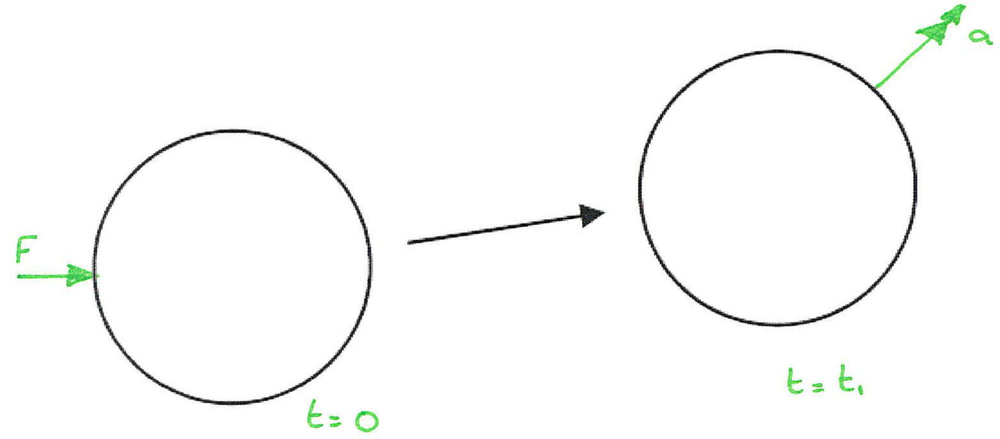
We apply these laws to fixed quantities of mass (or systems). They are known as *system equations*.

When we have a fluid in motion it is usually not convenient to follow fixed quantities of mass. Instead it is more likely that the fluid is the environment to our engineering component; and we want to know the effect of the fluid on our component.

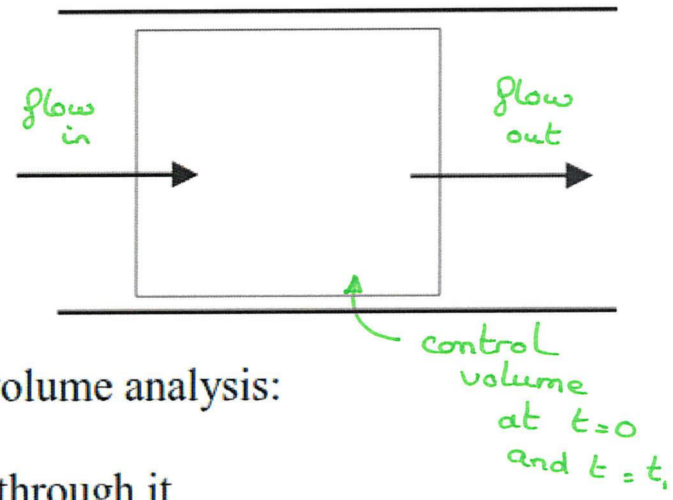
It is more useful to consider a specific region in space i.e. a *control volume*. So we must therefore rewrite the basic equations (2.1), (2.2), (2.3), and (2.4) for control volumes rather than systems.

These are
- the mass continuity equation, MCE
- the force momentum equation, FME
- the torque angular momentum equation, TAME
- the steady flow energy equation, SFEE

Solids



Fluids



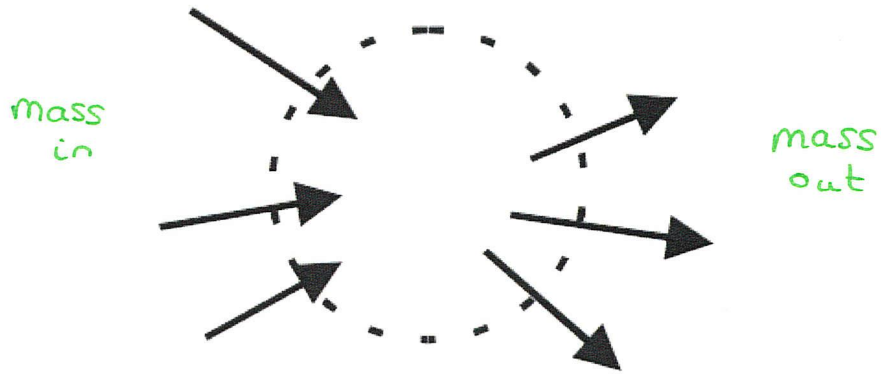
Some types of problem that we can solve using control volume analysis:

the mass flow through a pipe network
the force on a pipe bend caused by fluid flowing through it
the speed of rotation of a garden sprinkler
the power generated by a water turbine

in solids
$\frac{dm}{dt} = 0$

2.2. The Mass Conservation Equation, MCE

For the control volume apply the law of conservation of mass.



rate at which mass enters CV =

rate at which mass leaves CV
+
rate of mass accumulation in CV

$$\boxed{\sum \dot{m}_{in} = \sum \dot{m}_{out} + \frac{\partial m_{cv}}{\partial t}} \quad (2.5)$$

this is the CV form of equation (2.1)

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Special cases:

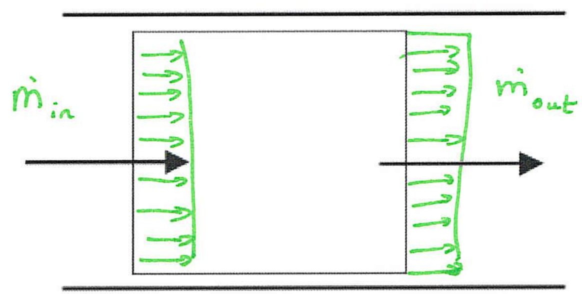
(a) steady flow (i.e. the fluid parameters (e.g. *u*, *p*, *T*) do not vary with time)

$$\sum \dot{m}_{in} = \sum \dot{m}_{out} \tag{2.6}$$

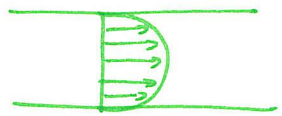
(b) 1D incompressible flow (the flow is a function of one co-ordinate axis only)

$$\dot{m}_{in} = \rho u_{in} A_{in} \qquad \dot{m}_{out} = \rho u_{out} A_{out} \tag{2.7}$$

e.g. flow through a large bore pipe;



∇ in small pipe



$u = f(r, x)$

NOT <u>1D</u>

2.3. The Force Momentum Equation, FME

For a control volume;

$$\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x + \left(\frac{\partial M_{CV}}{\partial t}\right)_x \qquad (2.8)$$

where $\sum F_x$ sum of forces on CV contents

$\sum \dot{M}$ the momentum flow rate (flux)

$\left(\frac{\partial M_{CV}}{\partial t}\right)$ rate of change of momentum of the CV

This is the CV form of equation (2.2) which we use in solids. Sometimes it is called the *linear momentum equation*.

Momentum and force are both vector quantities. So equation (2.8) must be used in each direction in turn. There is a similar equation for the y-direction etc,

i.e. $$\sum F_y = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_y + \left(\frac{\partial M_{CV}}{\partial t}\right)_y$$

2.3.1. Different Forms of the FME

(a) steady flow, $\left(\frac{\partial M_{CV}}{\partial t}\right) = 0$

$$\sum F_x = \left(\sum \dot{M}_{out} - \sum \dot{M}_{in}\right)_x \qquad (2.9)$$

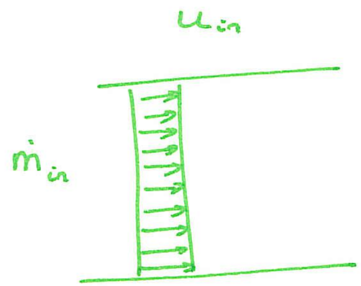
does not vary with time

(b) steady 1D incompressible flow

the velocity at inlet/outlet can be considered constant across the section. So

$\dot{M}_{in} = \dot{m}_{in} u_{in}$

$$\sum F_x = \dot{m}\left(u_{out} - u_{in}\right)_x \qquad (2.10)$$

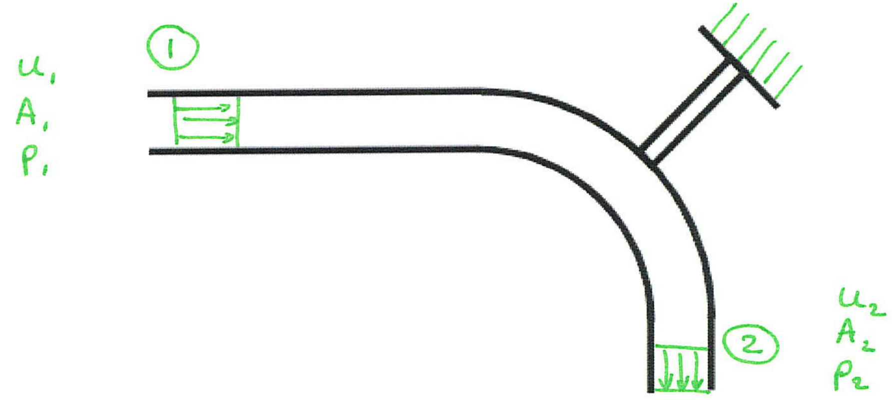


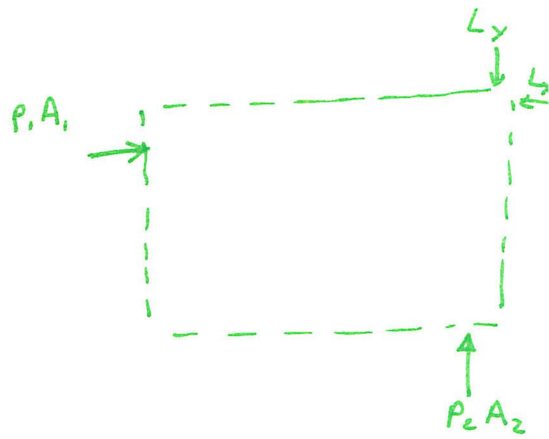
$\dot{M} = \dot{m}_{in} u_{in}$

1D, steady, incompressible

Example 2.1 - Force on a pipe bend

Find reaction force from ~~a~~ support





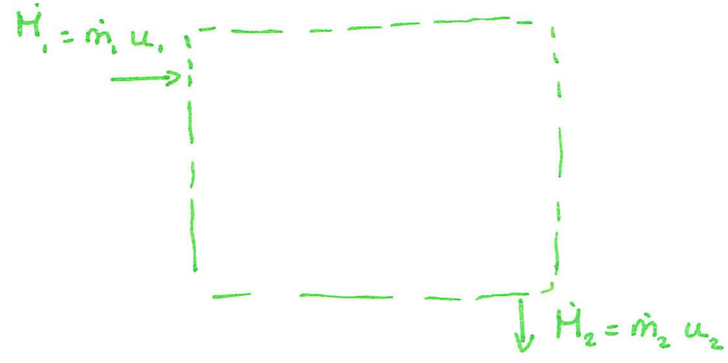
We will apply the the FME in the x-direction initially;

$$\Sigma F_x = (\Sigma \dot{M}_{out} - \Sigma \dot{M}_{in})_x + \left(\frac{\partial M_{cv}}{\partial t}\right)_x$$

Firstly determine the forces on the CV in x-direction

Force caused by pressure acting at station 1 $= p_1 A_1$

Force on CV contents exerted by pipe support $= -L_x$



Now determine the momentum fluxes;

$$(\dot{M}_{in}) = (\dot{M}_1) = \dot{m}_1 u_1$$

$$(\dot{M}_{out}) = (\dot{M}_2) = \dot{m}_2 u_2$$

in x-direction : $(\dot{M}_{out})_x = 0$

so applying the FME equation (2.9)

$$(\Sigma F)_{cv_x} = (\Sigma \dot{M}_{out} - \Sigma \dot{M}_{in})_x + \cancel{\left(\frac{\partial M_{cv}}{\partial t}\right)_x}^{0}$$

$$p_1 A_1 - L_x = -u_1 \dot{m}_1 \quad \Rightarrow \quad L_x = \dot{m} u_1 + p_1 A_1$$

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If we repeat the analysis in the y-direction we find;

$$\left(\Sigma F\right)_{cv_y} = \left(\Sigma \dot{M}_{out} - \Sigma \dot{M}_{in}\right)_y + \left(\cancel{\frac{\partial M_{cv}}{\partial t}}^{\,0}\right)_y$$

$$p_2 A_2 - L_y = -u_2 \dot{m}_2 \implies L_y = p_2 A_2 + \dot{m}_2 u_2$$

These forces (L_x and L_y) are exerted **on** the CV contents **by** the surroundings. Thus the force exerted **by** the fluid **on** the pipe bend is equal and opposite.

2.4. The Torque Angular Momentum Equation, TAME

There is a similar expression to the Force-Momentum Equation which relates Torque to Angular Momentum.

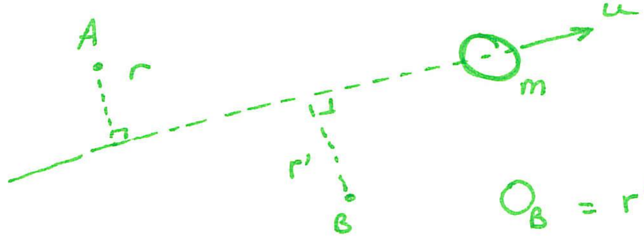
The FME is used to determine *forces* caused by the *momentum flux* of a fluid. The TAME is useful for determining the *torque*, T (or moment) on a body caused by *angular momentum flux*, $\dot{O}$ of a fluid.

2.4.1 Introduction - Angular Momentum

In solids we ~~we~~ know that applying a torque to a system causes a rate of increase of angular momentum; i.e. equation (2.3).

Angular momentum in solids. Consider a mass, m is travelling at u . The angular momentum, O_A is defined as the moment of linear momentum about the point A;

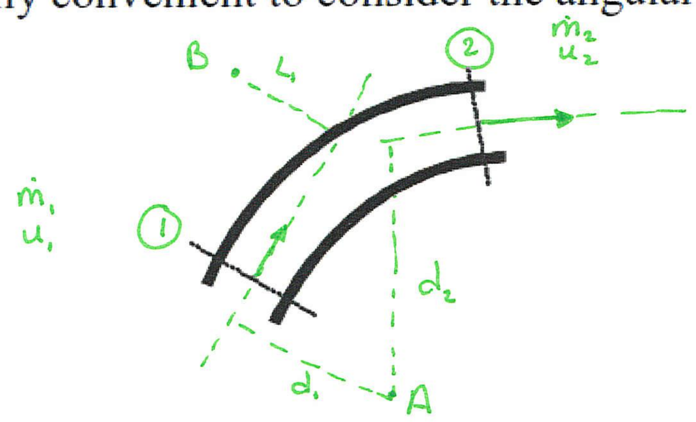
$$O_A = rmu$$



$O_B = r'mu$

where r is the perpendicular distance from the mass to the point A.

Angular momentum in fluids. A fluid particle can also have angular momentum. It is usually convenient to consider the angular momentum *flow rate* or *flux* of a fluid stream.



So; $\left(\dot{O}_{in}\right)_A = d_1\dot{m}_1u_1$ ↻ $\left(\dot{O}_{out}\right)_A = d_2\dot{m}_2u_2$ ↻

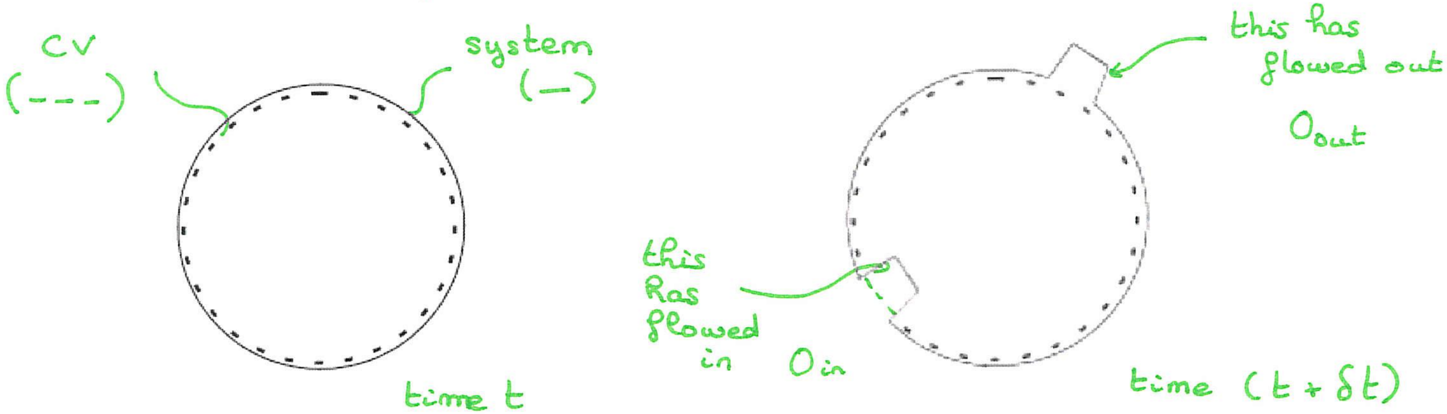
Note: Angular momentum is a vector quantity. So we must state about which point it is being determined. There would be similar expressions for the angular momentum flux about B.

$\left(\dot{O}_{in}\right)_B = L_1\dot{m}_1u_1$ ↺

In fluids we can derive a relationship between the torque, T and rate of change of angular momentum, $\dot{O}$ for a control volume.

2.4.2. Derivation of the TAME

Consider a CV coincident with a system at time t. At time $t+\delta t$ some fluid has flowed into the system whilst some has flowed out. The CV remains the same (remember in CV analysis we consider only a fixed region in space) but the system has changed (in system analysis we consider a fixed mass).



At time t the angular momenta of the CV and the system about some point A are identical.

$$(O_{cv})_t = (O_{system})_t$$

At time $t+\delta t$ the system has changed shape so its angular momentum about A has changed. Some angular momentum has flowed in whilst some has flowed out. So now;

$$(O_{cv})_{t+\delta t} = (O_{system})_{t+\delta t} + O_{in} - O_{out}$$

dividing by dt;

$$\frac{\partial O_{cv}}{\partial t} = \frac{\partial O_{system}}{\partial t} + \dot{O}_{in} - \dot{O}_{out} \qquad (2.11)$$

We know that for a system equation (2.3) applies so;

$$T = \frac{\partial O_{system}}{\partial t} \qquad (2.12)$$

Combining (2.11) and (2.12) gives;

$$T = \frac{\partial O_{cv}}{\partial t} + \dot{O}_{out} - \dot{O}_{in}$$

Now suppose we have a number of torques acting on the CV, and also a number of inflows and outflows of angular momentum, then;

$$\sum T_A = \left(\frac{\partial O_{CV}}{\partial t}\right)_A + \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A \qquad (2.13)$$

This is the Torque Angular Momentum Equation (TAME).

where $\sum T_A$ the sum of the torques acting on the CV contents

$\left(\sum \dot{O}_{in}\right)_A$ the sum of the angular momentum fluxes into the CV

$\left(\frac{\partial O_{CV}}{\partial t}\right)_A$ the rate of change of angular momentum of the CV contents

Again torque and angular momentum are vector quantities, so we must specify about which point they act (in this case they act about the point A) and also their direction. A similar equation exists for torques about other points B, C and so on.

Note how similar this is to the force momentum equation (FME).

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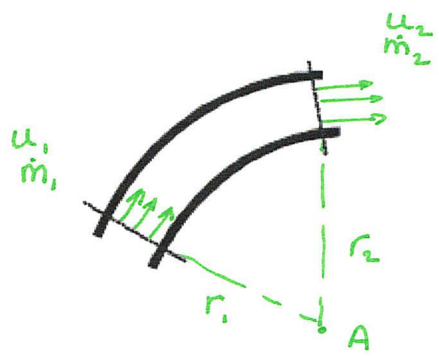
2.4.3 Different Forms of the TAME

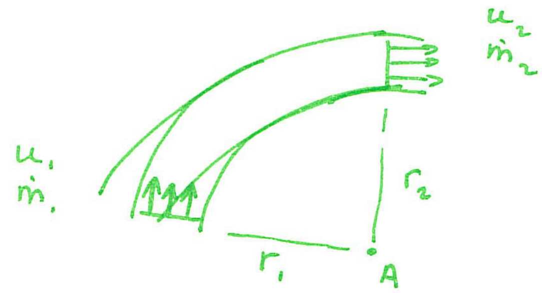
(a) Steady flow, the angular momentum of the CV does not change with time;

$\left(\frac{\partial O}{\partial t}\right)_{cv} = 0$

$$\sum T_A = \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A \quad (2.14)$$

(b) Steady 1D incompressible flow. If we consider 1D flows into and out of a control volume;





All the fluid flowing into the CV has a velocity of u_1 whilst all the fluid flowing out of the CV has a velocity of u_2.

So $\qquad \dot{O}_{in} = \dot{m}_1 u_1 r_1$

$\dot{O}_{out} = \dot{m}_2 u_2 r_2$

but also by the mass conservation equation for 1D steady state flow - equation (2.7)

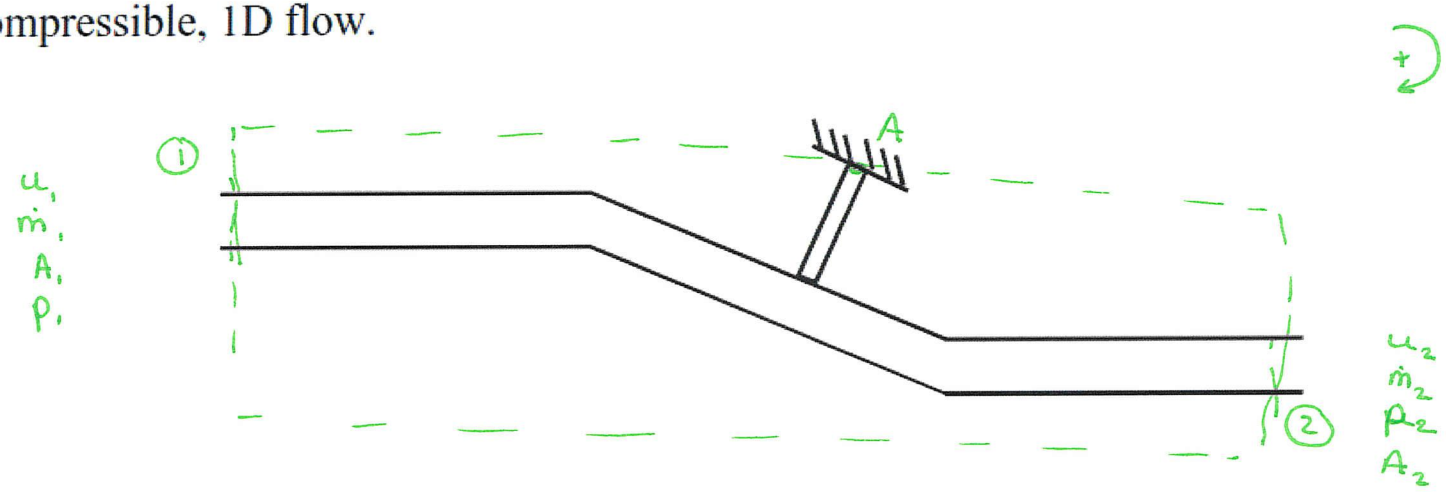
$$\dot{m}_1 = \dot{m}_2 = \dot{m}$$

Putting these into the TAME above (2.14)

$$\sum T_A = \dot{m}\left(r_2 u_2 - r_1 u_1\right) \qquad (2.15)$$

Example 2.2. Torque on a pipe bend.

Determine the torque which must be resisted by the support. Assume steady state, incompressible, 1D flow.



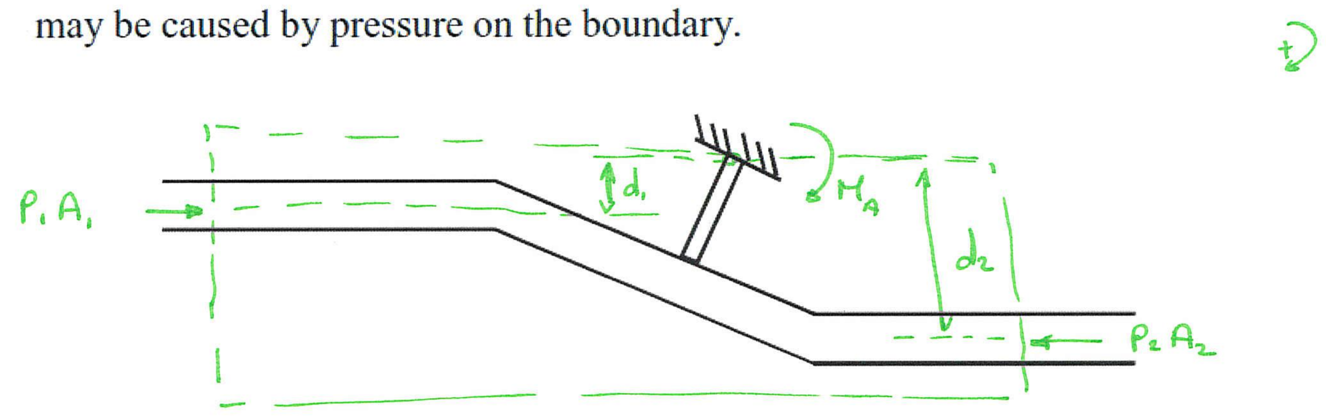
Step 1 Draw a CV - choose the location so that at the boundaries we know the fluid properties (i.e. the CV should pass through stations 1 and 2 and point A.

Step 2 Choose a point about which we determine torques and angular momentum. Choose a direction for positive torque.

Since we want the torque at the pipe support, choose that point, A. Arbitrarily lets make clockwise positive.

Step 3 Determine the torques acting on the CV.

These may be externally applied (e.g. the torque exerted by the pipe support) or they may be caused by pressure on the boundary.



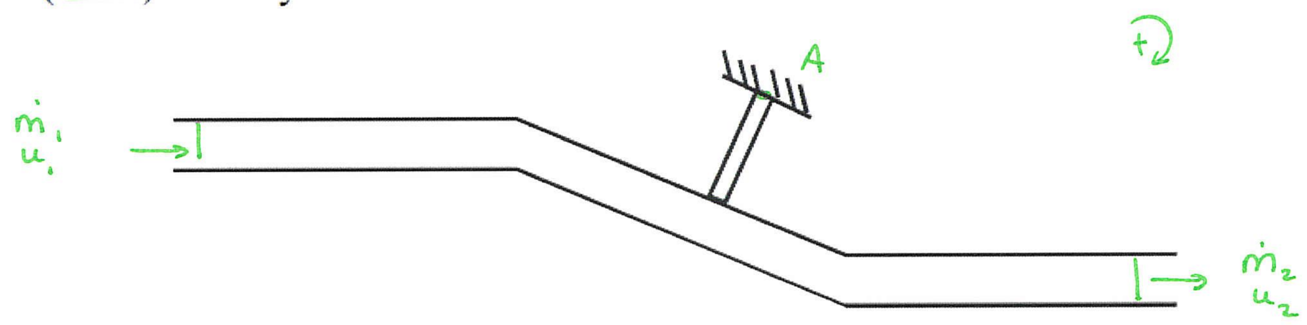
In this case we have the following torques;

$+M_A$ (this is want we want to find)

$$\left.\begin{array}{l} -p_1 A_1 d_1 \\ p_2 A_2 d_2 \end{array}\right\} \text{pressure torques}$$

Step 4. Determine the angular momentum flux into and out of the CV.

For angular momentum flux about the point A, we must determine the tangential (~~whirl~~) velocity and the radius about which it acts.



So angular momentum flux in: $-\dot{m}_1 u_1 d_1$

and angular momentum flux out: $-\dot{m}_2 u_2 d_2$

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Step 5. Use the mass conservation equation (2.6) to relate inflows to outflows

$$\dot{m}_1 = \dot{m}_2 = \dot{m}$$

In this case we have 1D flow (i.e. across pipe section 1 we can assume the velocity is constant, u_1). So;

$$\dot{m}_1 = \dot{m}_2$$

$$\rho u_1 A_1 = \rho u_2 A_2$$

Step 6. Assemble the torque angular momentum equation (TAME).

$$\sum T_A = \left(\sum \dot{O}_{out} - \sum \dot{O}_{in}\right)_A \qquad \left(\frac{\partial O}{\partial t}\right)_{cv} = 0$$

from 1D steady incompressible

Substituting the torques and momentum flux

$$M_A + p_2 A_2 d_2 - p_1 A_1 d_1 = (-\dot{m}_2 u_2 d_2) - (-\dot{m}_1 d_1 u_1)$$

and replacing the mass flow rate,

$$M_A - p_1 A_1 d_1 + p_2 A_2 d_2 = \rho (u_1^2 A_1 d_1 - u_2^2 A_2 d_2)$$

Step 7. This will give the torque, M_A exerted **on** the CV **by** the surroundings. Then the torque exerted **by** the fluid (i.e. the CV contents) **on** the pipe support will be equal and opposite to this.