The University Of Sheffield.

MEC 208 Fluids Engineering

Dr. Cécile M. Perrault

Responses 50 | Polling Closed

The University Of Sheffield.

SESSION 338827

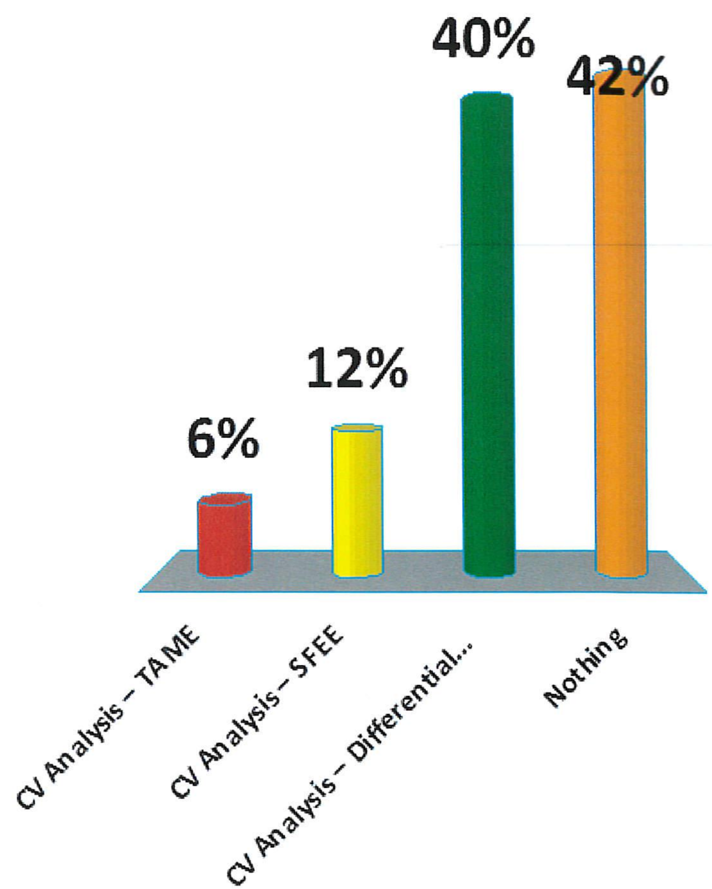
Of the material covered in this lecture, which did you understand least ?

A. ~~CV Analysis – TAME~~ Flow & velocity profile

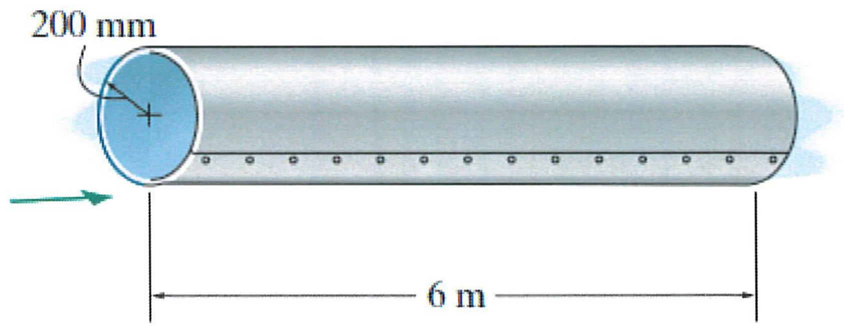
B. ~~CV Analysis – SFEE~~ revisions

C. ~~CV Analysis – Differential forms of equations~~ flow in circular pipe

D. Nothing



The University Of Sheffield.



Air is forced through the circular duct. If the flow is 0.3 m^3/s and the pressure drops 0.5 Pa for every 1 m of length, determine the friction factor for the duct. Take $\rho_a = 1.202\ kg/m^3$.

from equation 3.12

$$f = \frac{\Delta P}{\frac{1}{2} \rho u_m^2} \frac{d}{L}$$

$$f = \frac{(0.5 \frac{N}{m^2})}{0.5 (1.202)(2.387)^2} \quad \frac{0.4}{1 m}$$

$$= 0.0584$$

$$Q = VA$$

$$V = \frac{Q}{A} = \frac{0.3\ m^3/s}{\pi (0.2)^2} = 2.387\ m/s$$

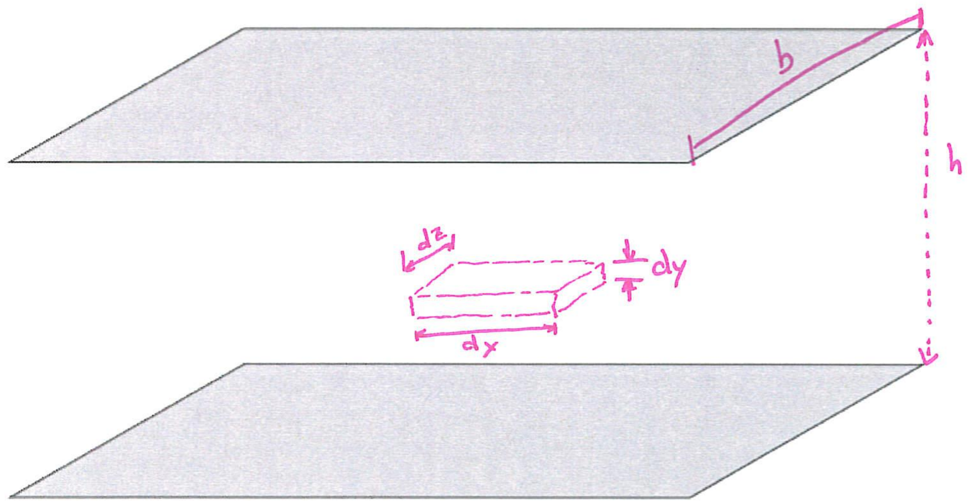
What we are covering in Topic 3:

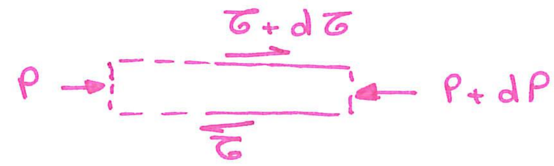
- Topic 3: Internal Flow
  - Flow between Infinitely Wide Parallel Plates
  - Flow between Concentric Cylinders
  - Laminar Film on a Vertical Wall
  - The Cone on Plate Viscometer

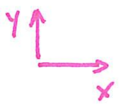
3.5. Flow Between Infinitely Wide Parallel Plates

width channel > 8 height channel

We will now apply a similar analysis (FME and Newton's law of viscosity) to study the flow between two parallel planes.







Force on the element in the x-direction

$$p\,dy\,dz - (P + dP)\,dy\,dz + (\tau + d\tau)(dx\,dz) - \tau\,dx\,dz = 0$$

simplifying gives; $-dP\,dy\,dz + d\tau\,dx\,dz = 0$

$$-dP\,dy + d\tau\,dx = 0$$

$$d\tau\,dx = dP\,dy$$

$$\frac{d\tau}{dy} = \frac{dp}{dx} \qquad (3.12)$$

From Newton's law of viscosity (equation 3.2);

$$\tau = \mu \frac{du}{dy}$$

differentiating w.r.t. y;

$$\frac{d\tau}{dy} = \mu \frac{d^2 u}{dy^2} \tag{3.13}$$

combining (3.12) and (3.13) gives;

$$\frac{d\tau}{dy} = \frac{dP}{dx} = \mu \frac{d^2 u}{dy^2} \tag{3.14}$$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dP}{dx}$$

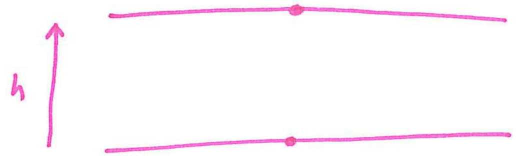
Now, we know that $\frac{dp}{dx} \neq f(y)$. i.e. the pressure does not vary in the y-direction. So we can integrate (3.14) twice;

$$\frac{du}{dy} = \frac{1}{\mu} \frac{dP}{dx} y + K_1$$

$$u(y) = \frac{1}{\mu} \frac{dP}{dx} \frac{y^2}{2} + K_1 y + K_2 \tag{3.15}$$

This gives the velocity distribution between the plates. We use the *boundary conditions* to determine the constants, K_1 and K_2.

$$u(y) = \frac{1}{2\mu}\frac{dP}{dx}y^2 + K_1 y + K_2$$



(a) **Both plates fixed**. Boundary conditions: u=0 when y=0,
u=0 when y=h
dp/dx≠0 (i.e. a pressure is applied)

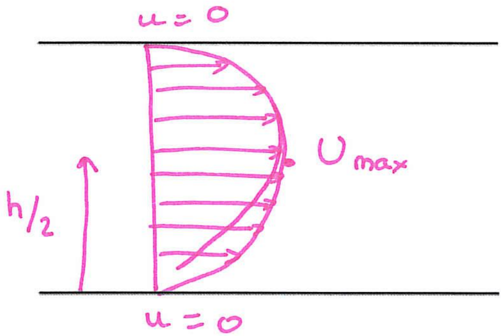
gives K_2=0

$$K_1 = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{h}{2}$$

and;

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 - \frac{1}{\mu}\frac{dp}{dx}\frac{h}{2}y = \left[-\frac{1}{2\mu}\frac{dP}{dx}\right](hy - y^2) \qquad (3.16)$$

This is the velocity distribution of the fluid between the plates;



The maximum velocity occurs at the centre y=h/2

$$U_{max} = -\frac{1}{2\mu}\frac{dP}{dx}\left(h\left(\frac{h}{2}\right) - \left(\frac{h}{2}\right)^2\right) = -\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{h^2}{2} - \frac{h^2}{4}\right) = -\frac{h^2}{8\mu}\frac{dP}{dx}$$

We can get the flow rate by integrating the velocity distribution. Consider the flow through a small rectangular element;

$$\dot{q} = u\,b\,dy$$

$$\dot{q} = \int_0^h u\,b\,dy$$

we replace the variable, *u* with equation (3.16)

$$\dot{q} = \int_0^h -\frac{1}{2\mu}\frac{dp}{dx}(hy - y^2)\,b\,dy = -\frac{b}{2\mu}\frac{dp}{dx}\left[\frac{3hy^2 - 2y^3}{6}\right]_0^h = -\frac{b}{2\mu}\frac{dp}{dx}\frac{h^3}{6}$$

$$\frac{\dot{q}}{b} = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad (3.17)$$

We can also obtain the shear stress distribution;

$$\tau = \mu\frac{du}{dy}$$

The University Of Sheffield.

but we know (3.16) that;

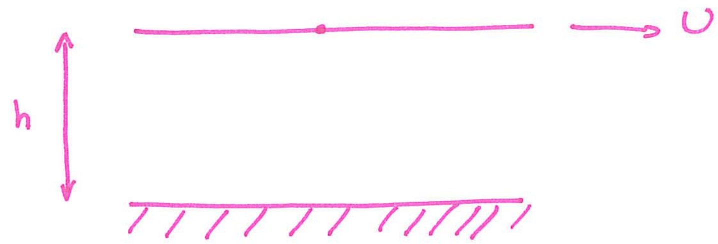
$$u = \left[-\frac{1}{2\mu} \frac{\mathrm{d}\, p}{\mathrm{d}\, x} \right] \left(hy - y^2 \right)$$

$$\frac{du}{dy} = \left[-\frac{1}{2\mu} \frac{dP}{dx} \right] [h - 2y]$$

$$\tau = \mu \frac{du}{dy} = \mu \left[-\frac{1}{2\mu} \frac{dP}{dx} \right] [h - 2y]$$

$$\tau = \left[-\frac{1}{2} \frac{\mathrm{d}\, p}{\mathrm{d}\, x} \right] (h - 2y) \quad (3.18)$$

$$u = +\frac{1}{2\mu}\frac{dP}{dx}y^2 + K_1 y + K_2$$



(b) **One plate moving no applied pressure.**

Boundary conditions: u=0 when y=0,
u=U when y=h
dp/dx=0

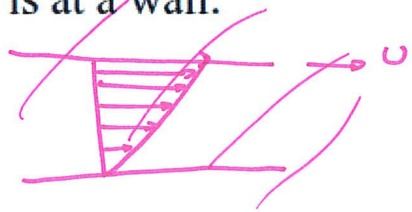
This type of flow is known as *Couette flow*. It is flow induced by the movement of a boundary. Whereas *Poiseuille flow* is induced by a pressure gradient.

Putting the boundary conditions into (3.15) gives;

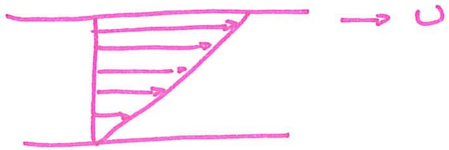
$$K_2 = 0 \qquad \text{and} \qquad K_1 = \frac{U}{h} \qquad \text{and} \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = 0$$

So $$u = \frac{U}{h}y \qquad (3.19)$$

This is the velocity distribution across the fluid. This time the maximum is at a wall.



This is the velocity distribution across the fluid. This time the maximum is at a wall.



The volume flow rate, again is obtained from;

$\dot{q} = u A = u b \, dy$ $\qquad$ $\dot{q} = \int_0^h \frac{U}{h} y \, b \, dy = \left[\frac{bU}{h} \frac{y^2}{2} \right]_0^h$

substituting (3.19) gives $\qquad \dfrac{\dot{q}}{b} = \dfrac{Uh}{2}$

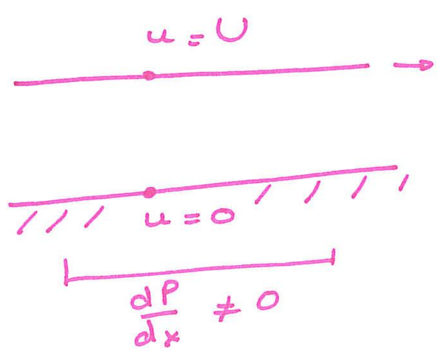
to get the shear stress profile;

$\tau = \mu \frac{du}{dy} = \mu \frac{U}{h}$ $\Leftarrow$ shear stress is constant across the flow

The University Of Sheffield.

$$u = \frac{1}{2\mu}\frac{dP}{dx} y^2 + K_1 y + K_2$$

(c) Combinations of one moving plate and an applied pressure gradient.



Boundary conditions; u=0 when y=0
u=U when y=h
dp/dx≠0

Applying these boundary conditions to (3.15) gives;

$$K_2 = 0 \qquad \text{and} \qquad K_1 = \frac{U}{h} - \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{h}{2}$$

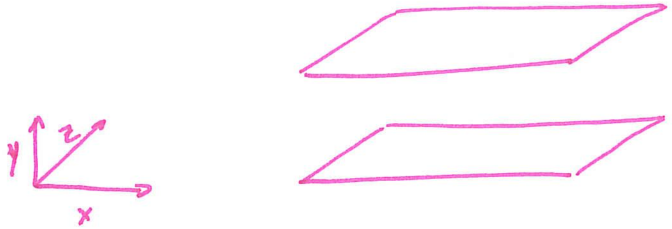
So in this case the velocity distribution is;

$$u = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\left(y^2 - hy\right) + \frac{Uy}{h} \qquad (3.20)$$

Again the flow rate and shear stress may be determined from this velocity distribution (see tutorial sheet 3 Q4), we find,

$$\frac{\dot{q}}{b} = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{Uh}{2} \qquad (3.21)$$

The University Of Sheffield.



Solving the Navier-Stokes Equations

$$\rho\left(\overset{\text{steady}}{\cancel{\frac{\partial u}{\partial t}}}+\overset{\text{fully developed}}{\cancel{u\frac{\partial u}{\partial x}}}+\overset{0}{\cancel{v\frac{\partial u}{\partial y}}}+\overset{0}{\cancel{w\frac{\partial u}{\partial z}}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\overset{\text{fully developed}}{\cancel{\frac{\partial^2 u}{\partial x^2}}}+\frac{\partial^2 u}{\partial y^2}+\overset{\text{wide channel}}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right)$$

For developed flow between parallel plates, the streamlines are parallel to the plates so that $u = u(y)$ only and $v = w = 0$.

$$0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

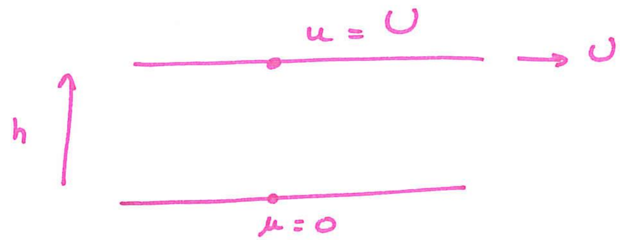
$$\left(\frac{\partial^2 u}{\partial y^2}\right)=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

We can define

$$\left(\frac{\partial^2 u}{\partial y^2}\right)=\lambda \qquad \text{where } \lambda \text{ is a constant}$$

This can be integrated twice

$$u(y)=\frac{\lambda}{2}y^2+Ay+B$$



Since at y = 0, u = 0 and at y = h, u = U, it gives

$$A = \frac{U}{h} - \frac{\lambda h}{2} \qquad \text{and} \qquad B = 0$$

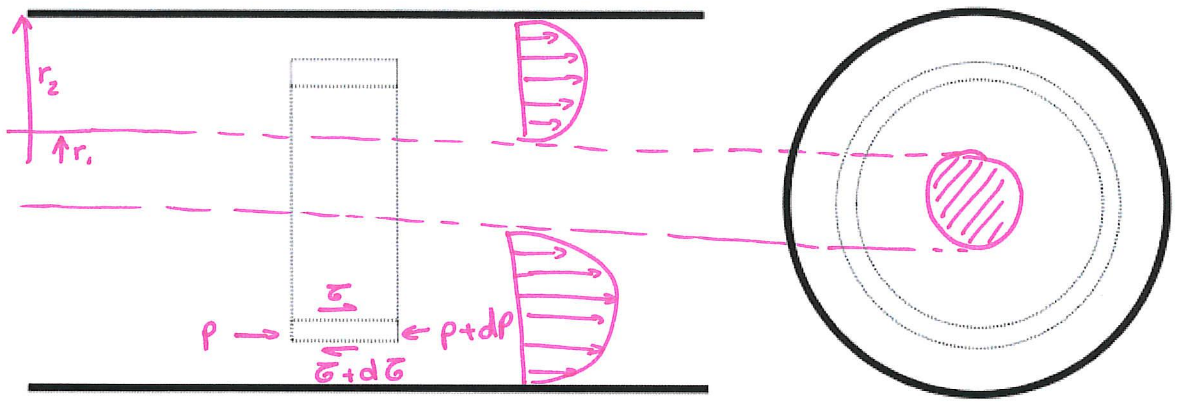
$$u(y) = \frac{\lambda}{2}(y^2 - hy) + \frac{U}{h}y$$

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy) + \frac{U}{\underset{h}{\cancel{a}}}y$$

The University Of Sheffield.

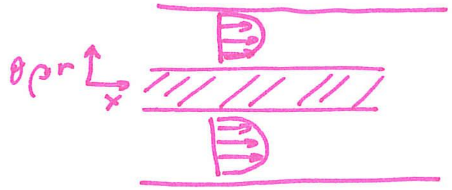
3.6. Flow Between Concentric Cylinders

Consider the axial flow between two cylinders;



By considering the equilibrium of the shear and pressure forces on the annular element, relationships for the velocity distribution can be determined (see tutorial sheet 3 Q6).

The University Of Sheffield.



Solving the Navier-Stokes Equation

For fluid flow in an annulus, we need to take the Navier-Stokes equation in cylindrical coordinates:

$$\rho\left(\cancelto{\text{steady}}{\frac{\partial u}{\partial t}}+\cancelto{0}{v_r\frac{\partial u}{\partial r}}+\cancelto{0}{\frac{v_\theta}{r}\frac{\partial u}{\partial \theta}}+\cancelto{\text{fully developed}}{u\frac{\partial u}{\partial x}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}+\cancelto{\text{fully developed}}{\frac{\partial^2 u}{\partial x^2}}\right)$$

For developed flow in an annulus, the streamlines are parallel to the wall with no swirl, so that $v_r = v_\theta = 0$ and $u = u(r)$ only.

We are left with

$$0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

The University Of Sheffield.

Since x and r can be varied independently, we must have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\lambda$$ where λ is a constant

$\lambda = \frac{1}{\mu}\frac{dP}{dx}$

Multiply both sides with r and integrate:

$$r\frac{\partial u}{\partial r}=\frac{\lambda}{2}r^2+A$$

Divide both sides by r and integrate:

$$u(r)=\frac{\lambda}{4}r^2+A\ln r+B$$

The velocity is u = 0 at r = r_1 and r_2, so

$$0=\frac{1}{\mu}\frac{dp}{dx}\frac{R_2^2}{4}+A\,ln\,R_2+B$$

$$0=\frac{1}{\mu}\frac{dp}{dx}\frac{R_1^2}{4}+A\,ln\,R_1+B$$

The University Of Sheffield.

Subtracting

$$A\ln R_2 - A\ln R_1 = \frac{1}{4\,\mu}\frac{dp}{dx}(R_2^2 - R_1^2)$$

$$A = \frac{\frac{1}{4\mu}\frac{dp}{dx}(R_2^2 - R_1^2)}{\ln(\frac{R_2}{R_1})}$$

Solving for B

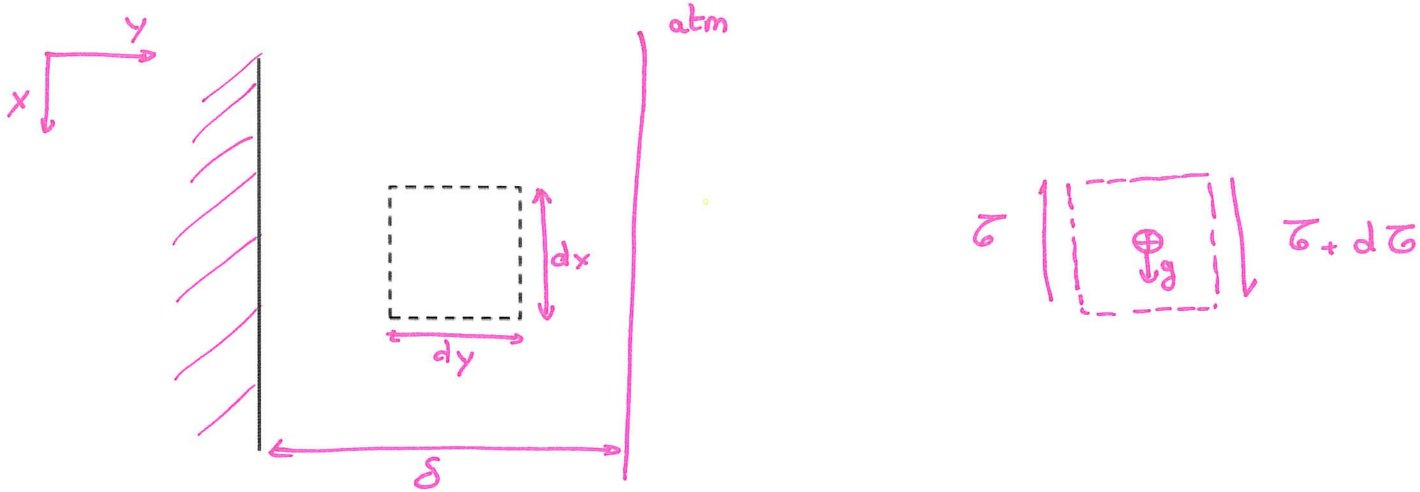
$$B = -\frac{1}{\mu}\frac{dp}{dx}\left[\frac{R_1^2}{4} - \frac{\ln R_1(R_2^2 - R_1^2)}{\ln\left(\frac{R_2}{R_1}\right)}\right] = -\frac{dp}{dx}\frac{R_1^2}{4\mu} - A\,ln\,R_1$$

$$u = \frac{1}{4\mu}\left[-\frac{dp}{dx}\right]\left[R_2^2 - r^2 + \frac{R_2^2 - R_1^2}{ln\,(R_2/R_1)}\ln\frac{R_2}{r}\right]$$

The University Of Sheffield.

3.7 Laminar Film on a Vertical Wall

A viscous layer of Newtonian fluid is flowing down a wall under the effect of gravity. There is no pressure gradient since the free surface of the fluid is exposed to atmosphere. The thickness of the fluid layer is assumed to be constant. We will carry out a laminar analysis to determine the velocity distribution in the thin layer.



Forces on the fluid element in the x-direction (assuming no momentum change)

$$g(\rho\, dx\, dy\, dz) + (\tau + d\tau)(dx\, dz) - \tau\, dx\, dz = 0$$

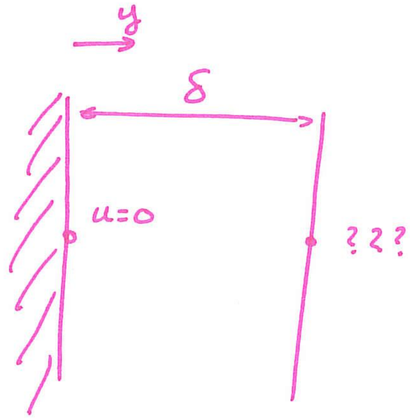
Simplifying gives: $\rho g\, dy + d\tau = 0$

$$\frac{d\tau}{dy} = -\rho g$$

Substituting Newton's law of viscosity: $\tau = \mu \frac{du}{dy}$

$$\frac{d\tau}{dy} = \mu \frac{d^2u}{dy^2} = -\rho g$$

Integrating twice:



$$\frac{du}{dy} = \frac{-\rho g}{\mu} y + K_1$$

$$u = -\frac{\rho g}{2\mu} y^2 + K_1 y + K_2$$

Boundary conditions

When $y=0$ $\quad u = 0$

$$K_2 = 0$$

When $y=\delta$ $\quad \frac{du}{dy} = 0$

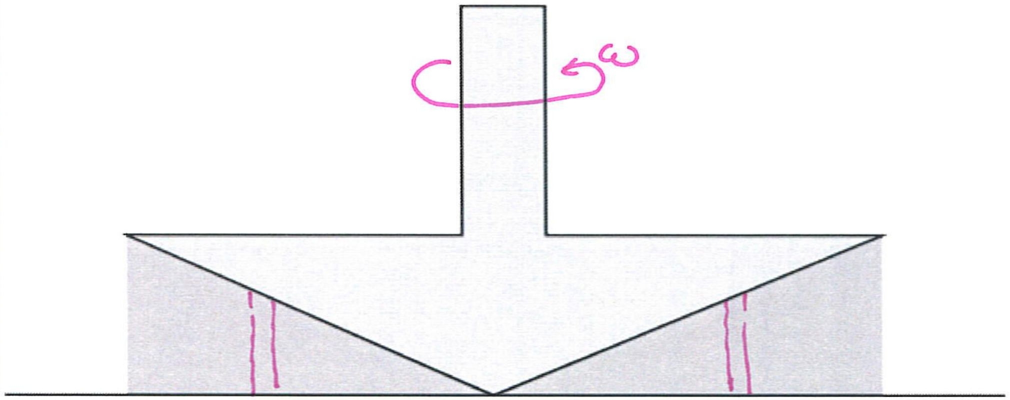
$$0 = -\frac{\rho g}{\mu} \delta + K_1 \Rightarrow K_1 = \frac{\rho g}{\mu} \delta$$

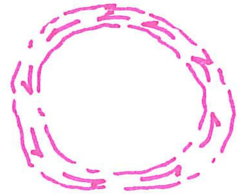
Gives the velocity profile:

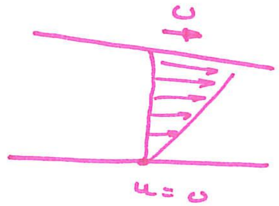
$$u = -\frac{\rho g}{\mu} \frac{y^2}{2} + \frac{\rho g \delta}{\mu} y = \frac{\rho g}{\mu} \left(\delta y - \frac{y^2}{2}\right)$$

3.8 The Cone on Plate Viscometer

Laminar flow theory can be used as a way of measuring viscosity of liquids. Some designs are based on measuring the flow rate through thin tubes (capillary viscometers). Another design uses a cone that rotates against a flat.







couette flow

Provided the cone angle is small – a laminar analysis can relate the torque to the viscosity of the liquid and the geometry of the cone.

$\tau = \mu \frac{du}{dy}$

Consider an annular element

dT= $\tau \, r \, dA$

Assume the velocity distribution between the base and the cone is linear. Then substitute expressions for the wetted area of the element and the torque. See Q7 Sheet 3.

3.9 Concluding Remarks

This laminar flow analysis required the use of the FME (momentum terms are negligible), the MCE, and the SFEE. The end result being velocity distributions, shear stress distributions, and volume flow rates for a range of geometries.

The following assumptions were made

1. The flow is fully developed and steady.

2. Laminar flow, Re is low (less than 2000 for pipe flow) – i.e. high viscosity fluid slow moving.

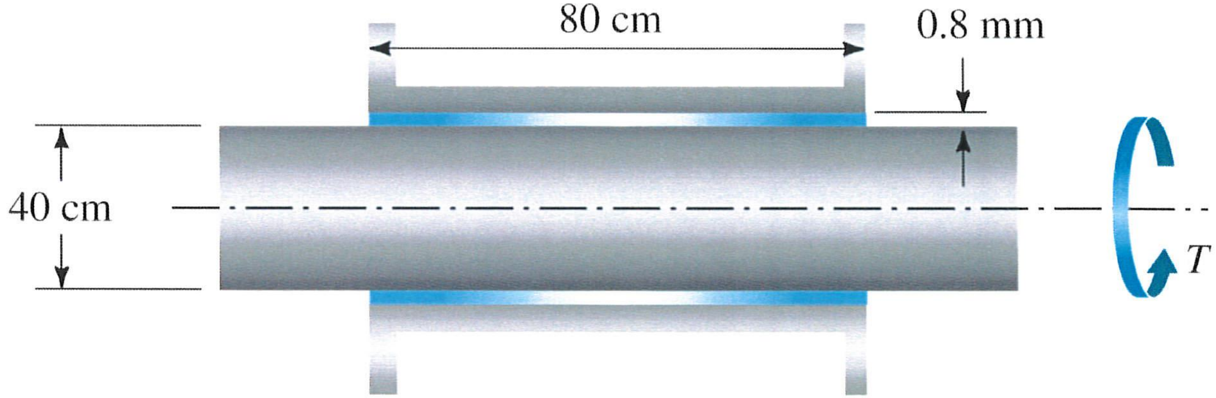
3. The fluid is Newtonian (i.e. obeys Newton's law of viscosity)

4. The viscosity is constant throughout the flow. $\dot{Q} = 0$. If heat had been added the temperature would rise and the viscosity would fall (dramatically). This is a serious limitation and difficult to analyse. Analysis must include energy equation, continuity, fluid properties - frequently becomes complex - use CFD.

5. The fluid was assumed incompressible. If the fluid is compressible the velocity change from layer to layer will cause a density change. More complex analysis.

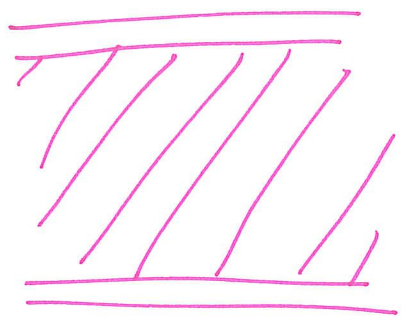
Generally laminar flow analysis is suitable for engineering situations where flow is slow, viscous and passageways are small e.g. flow past pistons, flow through bearings.

The University Of Sheffield.



$\mu = 0.1 \frac{N \cdot s}{m^2}$

Calculate the torque T necessary to rotate the rod shown at 30 rad/s if the fluid filling the gap is SAE-10W oil at 20C. Assume a linear velocity profile.



$T = \tau r dA$

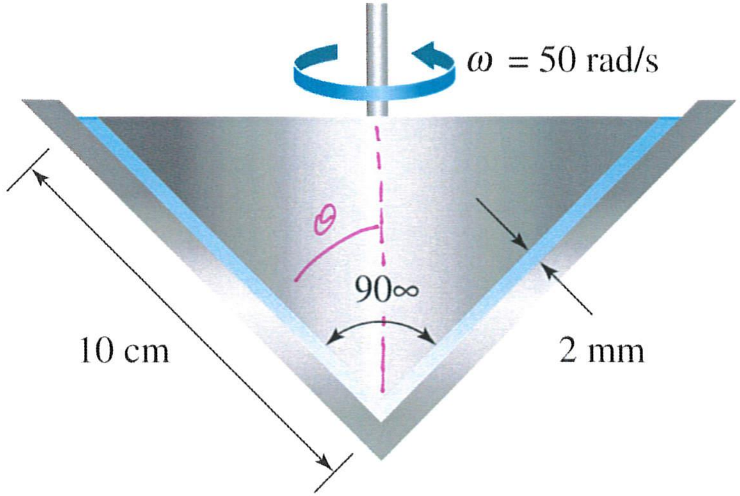
Couette flow $\Rightarrow$ $u = \frac{U}{a} y$

$\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = 0.1 \frac{(0.2 \times 30)}{0.0008}$

$= 750 \ Pa$

$T = \tau r dA = \tau r (2\pi(\frac{D}{2})) L = 750 \times 0.2 \times \pi \times 0.4 \times 0.8 = 151 \ N.m$

The University Of Sheffield.



Find the torque T needed to rotate the cone shown if the fluid filling the gap is oil with $\mu = 0.01$ N.s/m^2. Assume a linear velocity profile.

$\Rightarrow$ couette flow $\quad u = \frac{U}{a} y = \frac{r\omega}{a} y = \frac{50r}{0.02} y$

$$\tau = \mu \frac{du}{dy} = \mu \left(\frac{50r}{0.02}\right) = 25\,000 r$$

$$dA = 2\pi r \frac{dr}{\sin\theta}$$

$$T = \int_A \tau r \, dA = \int_0^{0.1 \sin 45} (25\,000 r)(r)\left(2\pi r \frac{dr}{\sin 45}\right) = 25000 \times \frac{2\pi}{0.707} \int_0^{0.0707} r^3 \, dr$$

$$= \frac{50\,000\pi}{0.707} \times \frac{0.0707^4}{4} = 1.388 \text{ N.m}$$