2

Some Maximum Principles for Elliptic Equations

2.1 Linear elliptic operators of order two

As always, Ω denotes an open non-empty subset of $\mathbb{R}^N$.

Definition 2.1 (i) The operator L, defined by

$$Lu(x) := \left\{ \sum_{i,j=1}^{N} a_{ij}(x)\, \partial_i \partial_j + \sum_{j=1}^{N} b_j(x)\, \partial_j + c(x) \right\} u(x) \tag{2.1}$$

whenever $u \in C^2(\Omega)$ and $x \in \Omega$, is a *linear partial differential operator*, of *order two*. Here

$$a = (a_{ij}) : \Omega \to \mathbb{R}^{N^2}, \quad b = (b_j) : \Omega \to \mathbb{R}^N, \quad c : \Omega \to \mathbb{R}$$

are given measurable functions. The $N \times N$ matrix a is *symmetric*: $a_{ji}(x) = a_{ij}(x)$ for all $i, j \in \{1, \ldots, N\}$ and all $x \in \Omega$. [This involves no loss of generality because $\partial_j \partial_i u = \partial_i \partial_j u$.]

(ii) We say that L is *elliptic at* $x \in \Omega$ iff there is a number $\lambda(x) > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j \geq \lambda(x) |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^N; \tag{2.2}$$

that L is *elliptic in* Ω iff it is elliptic at every $x \in \Omega$; and that L is *uniformly* elliptic in Ω iff there is a constant $\lambda_0 > 0$ such that $\lambda(x) \geq \lambda_0$ for all $x \in \Omega$. The best (largest) values $\lambda(x)$ and λ_0 are, respectively, the pointwise and uniform *moduli of ellipticity* of L. □

Here are three examples to which we can apply the definition with almost no calculation.

1. If $L = \Delta +$ lower order terms, then $a_{ij}(x) = \delta_{ij}$ [the Kronecker delta, Chapter 0, (v)], so that L is uniformly elliptic in every Ω, with $\lambda_0 = 1$.

2. Let $x_1, \ldots, x_{N-1}$ be space variables, while x_N denotes time. Then the operators $\partial_1^2 + \cdots + \partial_{N-1}^2 - \partial_N^2$ of the wave equation, and $\partial_1^2 + \cdots + \partial_{N-1}^2 - \partial_N$ of the heat equation, are not elliptic: choose $\xi_i = \delta_{Ni}$ in (2.2).

3. The Tricomi operator $\partial_1^2 + x_1 \partial_2^2$ is elliptic in the half-plane $\{ x \in \mathbb{R}^2 \mid x_1 > 0 \}$ but not uniformly so; the pointwise modulus of ellipticity is

$$\lambda(x) = \begin{cases} x_1 & \text{if } 0 < x_1 \le 1, \\ 1 & \text{if } x_1 > 1. \end{cases}$$

Exercise 2.2 Given that $a(x)$ is symmetric and satisfies (2.2), prove the following.

(a) Ellipticity is invariant under rotation of co-ordinate axes. That is, if R is a constant, orthogonal $N \times N$ matrix, $y := Rx$ and $h(y) := Ra(x)R^{-1}$, then

$$\sum_{i,j} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{p,q} h_{pq}(y) \frac{\partial}{\partial y_p} \frac{\partial}{\partial y_q}$$

and

$$\sum_{p,q} h_{pq}(Rx) \eta_p \eta_q \ge \lambda(x) |\eta|^2 \quad \text{for all} \quad \eta \in \mathbb{R}^N.$$

(b) The pointwise modulus of ellipticity is the smallest eigenvalue of $a(x)$.

(c) Let $g(x)$ be a non-positive $N \times N$ matrix; we write $g(x) \le 0$, meaning that $\xi g(x) \xi \le 0$ for all $\xi \in \mathbb{R}^N$. Then

$$\operatorname{trace}\big(a(x)\, g(x)\big) := \sum_{i,j} a_{ij}(x)\, g_{ji}(x) \le 0.$$

(Here the rule for matrix multiplication is summation over adjacent subscripts:

$$g(x)\, \zeta := \left(\sum_j g_{ij}(x)\, \zeta_j \right)_{i=1}^{N}, \quad \xi\, g(x)\, \zeta := \sum_{i,j} \xi_i\, g_{ij}(x)\, \zeta_j,$$

so that row and column vectors need not be distinguished in such expressions.) □

2.2 The weak maximum principle

Definition 2.3 The operators to be considered in this section and the next two are

$$\begin{aligned} L_0 &:= \sum_{i,j=1}^{N} a_{ij}(x)\,\partial_i\partial_j \;+\; \sum_{j=1}^{N} b_j(x)\,\partial_j, \\ L &:= L_0 + c(x), \qquad \text{with } \; c(x) \le 0 \; \text{ for all } \; x \in \Omega, \\ L_1 &:= \sum_{i,j=1}^{N} a_{ij}\,\partial_i\partial_j \;+\; \sum_{j=1}^{N} b_j\,\partial_j \;+\; c, \quad \text{with } \; c \le 0; \end{aligned}$$

in L_1 all coefficients a_{ij}, b_j and c are constants. Thus L_0 is the particular L with $c = 0$ (the zero function), while L_1 is the particular L with constant coefficients.

All three are *uniformly elliptic*: for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \;\ge\; \lambda_0|\xi|^2, \quad \lambda_0 = \text{const.} > 0. \tag{2.3}$$

All coefficients are bounded and measurable: in L_0 and L, for all i and j,

$$\sup_{x\in\Omega}\, |a_{ij}(x)| < \infty, \quad \sup_{x\in\Omega}\, |b_j(x)| < \infty, \quad \sup_{x\in\Omega}\, |c(x)| < \infty.$$

□

Definition 2.4 We shall say that u is a *C^2-subsolution relative to L and Ω* iff $u \in C^2(\Omega)$ and $Lu \ge 0$ in Ω. (Here L may be replaced by L_0 or L_1.) □

We distinguish L_0 from L because stronger conclusions are possible when $c = 0$, and L_1 from L because a different kind of subsolution will be used for L_1. However, in the following three versions of the weak maximum principle (which is not to be despised, relative to the strong maximum principle), hypothesis (a) is always the same; it ensures, as was noted in Exercise 1.4, that $\sup_\Omega\, u = \max_{\overline{\Omega}}\, u$.

Theorem 2.5 (the weak maximum principle for L_0). *Suppose that*

(a) *Ω is bounded, $u \in C(\overline{\Omega})$;*

(b) *u is a C^2-subsolution relative to L_0 and Ω.*

Then the supremum of u is attained on the boundary:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof (i) Define, for arbitrary $\varepsilon > 0$ and for a constant K to be chosen presently,

$$v(x) := u(x) + \varepsilon e^{Kx_1}, \quad x \in \overline{\Omega}.$$

Now, for all $x \in \Omega$,

$$\begin{aligned} L_0\left(e^{Kx_1}\right) &= \left\{a_{11}(x)K^2 + b_1(x)K\right\} e^{Kx_1} \\ &\geq \left(\lambda_0 K^2 - \{\sup_\Omega |b_1|\} K\right) e^{Kx_1} \quad [\text{in } (2.3),\ \xi = (K, 0, \dots, 0)] \\ &> 0 \qquad \text{if we choose } K > \frac{1}{\lambda_0} \sup_\Omega |b_1|. \end{aligned}$$

Hence $L_0 v > 0$ in Ω.

(ii) Assume (for contradiction) that $\sup_\Omega v$ is attained at $x_0 \in \Omega$. Then $(\partial_j v)(x_0) = 0$ for all $j \in \{1, \dots, N\}$, and the Hessian matrix

$$H(x_0) := \left((\partial_i \partial_j v)(x_0)\right) \leq 0.$$

[Otherwise $\zeta H(x_0)\zeta = \alpha > 0$, say, for some $\zeta \in \mathbb{R}^N$ with $|\zeta| = 1$, and the Taylor formula

$$v(x_0 + h) = v(x_0) + 0 + \tfrac{1}{2} \sum_{i,j} (\partial_i \partial_j v)(x_0)\ h_i h_j + o\left(|h|^2\right)$$

leads to a contradiction, because we can choose $h = \beta\zeta$ with $\beta > 0$ so small that $v(x_0 + h) > v(x_0)$.] The result of Exercise 2.2, (c), now shows that

$$\begin{aligned} (L_0 v)(x_0) &= \sum_{i,j} a_{ij}(x_0)\ (\partial_j \partial_i v)(x_0) + 0 \\ &= \operatorname{trace}\left(a(x_0)\ H(x_0)\right) \leq 0, \end{aligned}$$

which contradicts the result of step (i).

(iii) Accordingly, for every $\varepsilon > 0$ and all $x \in \overline{\Omega}$,

$$u(x) < v(x) \leq \max_{\partial\Omega} v \leq \max_{\partial\Omega} u + \varepsilon K_1,$$

where

$$K_1 := \max_{x \in \partial\Omega} e^{Kx_1}.$$

It follows that $u(x) \leq \max_{\partial\Omega} u$ for all $x \in \overline{\Omega}$. [Otherwise $u(x_0) = \max_{\partial\Omega} u + \delta$ for some $x_0 \in \Omega$ and some $\delta > 0$; we obtain a contradiction by choosing $\varepsilon = \delta / 2K_1$.] □

The weak maximum principle for L involves the non-negative part u^+ of u [see Chapter 0, (v)] and states less than the theorem for L_0 when

$\max_{\partial\Omega} u < 0$. *However, if* $\max_{\partial\Omega} u \geq 0$, *then* $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ *exactly as before*, because in that case $\max_{\partial\Omega} u^+ = \max_{\partial\Omega} u$, so that strict inequality in (2.4) is impossible.

Theorem 2.6 (the weak maximum principle for L). *Suppose that*

(a) Ω *is bounded,* $u \in C(\overline{\Omega})$;

(b) u *is a* C^2*-subsolution relative to* L *and* Ω.

Then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+. \tag{2.4}$$

Proof Let $\Omega^+ := \{ x \in \Omega \mid u(x) > 0 \}$. This set is open in $\mathbb{R}^N$: if $y \in \Omega^+$, say $u(y) = \alpha > 0$, then there is a number $\delta > 0$ such that both $\mathscr{B}(y, \delta) \subset \Omega$ [since Ω is open] and $u(x) > \alpha/2$ whenever $x \in \mathscr{B}(y, \delta)$ [since u is continuous], so that $\mathscr{B}(y, \delta)$ is in Ω^+.

If Ω^+ is empty, then $\max_{\overline{\Omega}} u \leq 0$ and the theorem is true.

Suppose then that Ω^+ is not empty. The hypotheses $L_0 u \geq -c(x)u$ in Ω and $c(x) \leq 0$ in Ω imply that $L_0 u \geq 0$ in Ω^+; by Theorem 2.5, the maximum of u over $\overline{\Omega^+}$ equals that over $\partial\Omega^+$; hence there is a point

$$x_0 \in \partial\Omega^+ \text{ such that } u(x_0) = \max_{\overline{\Omega^+}} u > 0.$$

If $x_0 \in \Omega$ (Figure 2.1) we have a contradiction: by continuity, $u > 0$ in $\mathscr{B}(x_0, \rho)$ for some $\rho > 0$; on the other hand, $\mathscr{B}(x_0, \rho)$ contains points of $\Omega \setminus \Omega^+$, because $x_0 \in \partial\Omega^+$, and $u \leq 0$ at such points. Therefore $x_0 \in \partial\Omega$. □

Remark 2.7 If u is a C^2*-supersolution* relative to L and Ω, which means that $u \in C^2(\Omega)$ and $Lu \leq 0$ in Ω, then $-u$ is a C^2-subsolution. If also condition (a) holds, then

$$\max_{\overline{\Omega}}(-u) \ \leq \ \max_{\partial\Omega}(-u)^+,$$

where

$$\begin{aligned}(-u)^+(x) \ = \ \max_{[x \text{ fixed}]} \{ -u(x), 0 \} \ &= \ -\min_{[x \text{ fixed}]} \{ u(x), 0 \} \\ &= \ -u^-(x),\end{aligned}$$

so that

$$\max_{\overline{\Omega}}(-u) \ \leq \ \max_{\partial\Omega}(-u^-).$$

Equivalently,

$$\min_{\overline{\Omega}} u \ \geq \ \min_{\partial\Omega} u^-. \tag{2.5}$$

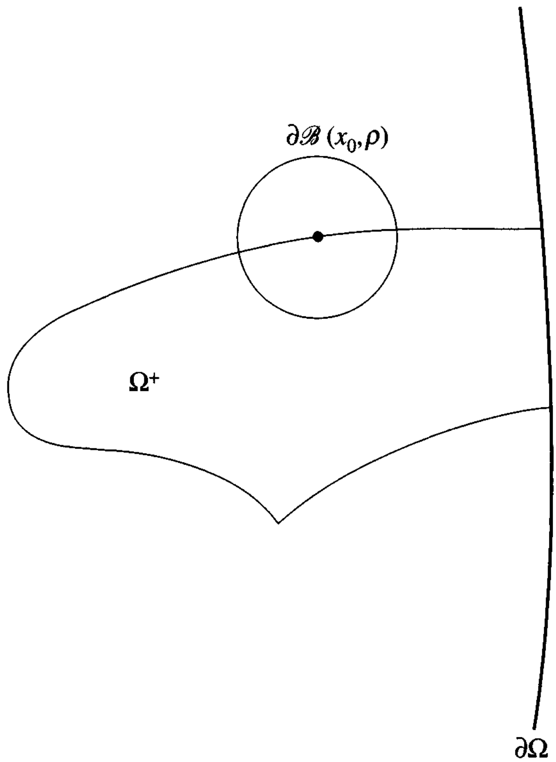

Fig. 2.1.

If u is a C^2-*solution* relative to L and Ω, which means that $u \in C^2(\Omega)$ and $Lu = 0$ in Ω, and condition (a) holds, then (2.4) and (2.5) imply that

$$\min_{\partial\Omega} u^- \le u(x) \le \max_{\partial\Omega} u^+ \quad \text{for all} \ \ x \in \overline{\Omega}. \tag{2.6}$$

Similarly, all our results for subsolutions have implications for super-solutions and solutions. □

Remark 2.8 The *Dirichlet problem* for L in a bounded set Ω is to find v such that

$$\left.\begin{array}{c} Lv = f \ \text{ in } \ \Omega, \\ v\big|_{\partial\Omega} = g, \quad v \in C(\overline{\Omega}) \cap C^2(\Omega), \end{array}\right\} \tag{2.7}$$

where f and g are given functions. This problem has *at most one solution*, because the difference $u := v_1 - v_2$ of two solutions satisfies $Lu = 0$ in Ω,

$u = 0$ on $\partial\Omega$, and has the smoothness required for (2.6); therefore $u = 0$ on $\overline{\Omega}$. □

Remark 2.9 (i) *The condition* $c \le 0$ *in* Ω (Definition 2.3) *cannot be omitted from Theorem 2.6.* Once again this is illustrated by eigenfunctions of the Laplace operator. For example, let Ω be the rectangle $(0,\alpha) \times (0,\beta)$ in $\mathbb{R}^2$, and let

$$u(x) = \sin\frac{m\pi x_1}{\alpha}\sin\frac{n\pi x_2}{\beta}, \qquad m, n \in \mathbb{N}. \tag{2.8}$$

Calculating $\triangle u$, we see that

$$\triangle u + cu = 0 \ \text{ in } \ \Omega, \quad \text{where } \ c = \left(\frac{m\pi}{\alpha}\right)^2 + \left(\frac{n\pi}{\beta}\right)^2 > 0,$$

and, in contrast to (2.4), $\max_{\overline{\Omega}} u = 1$ while $\max_{\partial\Omega} u^+ = 0$.

(ii) *We cannot replace* u^+ *by* u *in* (2.4). [As was noted earlier, this would give $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.] For, let Ω be the unit ball $\mathscr{B}(0,1)$ in $\mathbb{R}^N$, let $L = \triangle - 1$, and let $u(x) = -3N - |x|^2$ on $\overline{\Omega}$. Then $\triangle u = -2N$, so that

$$Lu(x) = \triangle u(x) - u(x) = N + |x|^2 > 0 \ \text{ in } \Omega,$$

and

$$\max_{\overline{\Omega}} u = -3N > -3N - 1 = \max_{\partial\Omega} u.$$

□

The requirement in Definition 2.4 that subsolutions be in $C^2(\Omega)$ can cause embarrassment. For example, the Newtonian potential of constant density in a bounded open set G is not twice differentiable at points of ∂G; when ∂G is unknown *a priori* and may be unpleasant, a need to consider second derivatives of the potential would be a source of difficulty. We now define subsolutions for which membership of $C^1(\Omega)$ is ample smoothness. However, we do this only for the operator L_1, because a proof of something like Theorem 2.11 for an operator with variable coefficients requires (I believe) considerably more machinery.

Definition 2.10 We shall say

(a) that u is a *generalized subsolution relative to* L_1 *and* Ω iff $u \in C^1(\Omega)$ and

$$\begin{aligned}\Lambda_1(\varphi, u; \Omega) \quad &:= \quad \int_\Omega \left\{ -\sum_{i,j=1}^N a_{ij}\,(\partial_i\varphi)\,(\partial_j u) + \sum_{j=1}^N b_j\,\varphi\,\partial_j u + c\varphi u \right\} \\ &\ge \quad 0 \ \text{ whenever } \ \varphi \in C_c^\infty(\Omega) \ \text{ and } \ \varphi \ge 0;\end{aligned}$$

(b) that u is a *distributional subsolution relative to* L_1 *and* Ω iff u is locally integrable in Ω (integrable on each compact subset of Ω) and

$$\begin{aligned}\Lambda_{1d}(\varphi,u;\Omega) \quad &:= \quad \int_\Omega \left\{ \sum_{i,j=1}^N a_{ij}\,(\partial_j\partial_i\varphi)\,u - \sum_{j=1}^N b_j\,(\partial_j\varphi)\,u + c\varphi u \right\} \\ &\geq \quad 0 \;\text{ whenever }\; \varphi \in C_c^\infty(\Omega) \;\text{ and }\; \varphi \geq 0.\end{aligned}$$

Then u is a *generalized supersolution* iff $-u$ is a generalized subsolution; u is a *generalized solution* iff it is both a generalized subsolution and a generalized supersolution (cf. Remark 2.7). *Distributional supersolutions* and *distributional solutions* are defined similarly. □

Evidently the key to this definition is integration by parts:

$$\Lambda_1(\varphi,u;\Omega) = \int_\Omega \varphi L_1 u \;\text{ if }\; \varphi \in C_c^\infty(\Omega) \;\text{ and }\; u \in C^2(\Omega); \tag{2.9}$$

$$\Lambda_{1d}(\varphi,u;\Omega) = \Lambda_1(\varphi,u;\Omega) \;\text{ if }\; \varphi \in C_c^\infty(\Omega) \;\text{ and }\; u \in C^1(\Omega). \tag{2.10}$$

Since a C^2-subsolution u satisfies $L_1 u \geq 0$ in Ω, we see from (2.9) that *a* C^2*-subsolution* (relative to L_1 and Ω) *is a generalized subsolution*, and from (2.10) that *a generalized subsolution is a distributional subsolution*. On the other hand, a distributional subsolution is a generalized subsolution only if it is also in $C^1(\Omega)$, and a generalized subsolution is a C^2-subsolution only if it is also in $C^2(\Omega)$. [In this last case, we use (2.9) and Exercise 1.16 to deduce that $L_1 u \geq 0$ in Ω.]

Note that, in the following theorem, hypothesis (a) swamps the condition of local integrability demanded in Definition 2.10, (b).

Theorem 2.11 (the weak maximum principle for L_1). *Suppose that*

(a) Ω *is bounded,* $u \in C(\overline{\Omega})$;

(b) u *is a distributional subsolution relative to* L_1 *and* Ω.

Then the previous conclusions hold:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad \textit{if} \;\; c = 0, \tag{2.11a}$$

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ \quad \textit{if} \;\; c < 0. \tag{2.11b}$$

Proof (i) Let an arbitrary point $\xi \in \Omega$ be given; we shall prove the theorem by showing that

$$u(\xi) \leq \begin{cases} \max_{\partial\Omega} u & \text{if } \; c = 0, \qquad (2.12a) \\ \max_{\partial\Omega} u^+ & \text{if } \; c < 0. \qquad (2.12b) \end{cases}$$

Adopting a standard trick, we choose the following test function φ in the definition of distributional subsolution.

$$\varphi(y) = k_\rho(x-y) \quad \text{for all } y \in \Omega, \tag{2.13a}$$

where k_ρ is a smoothing kernel as in Exercise 1.23; ρ and x are parameters satisfying

$$0 < \rho \le \tfrac{1}{3}\,\mathrm{dist}(\xi, \partial\Omega), \tag{2.13b}$$

$$x \in \overline{G(\rho)}, \qquad \text{where } \; G(\rho) := \left\{ z \in \Omega \;\middle|\; \mathrm{dist}(z, \partial\Omega) > 2\rho \right\}, \tag{2.13c}$$

as is illustrated in Figure 2.2. This choice of φ is legitimate because $k_\rho(x-y) = 0$ when $|y-x| \ge \rho$, so that $\operatorname{supp} k_\rho(x-.) \subset \Omega$, and certainly $k_\rho(x-.)$ is infinitely differentiable and non-negative in Ω.

(ii) Now let

$$u_\rho(x) = \int_\Omega k_\rho(x-y)\, u(y)\, \mathrm{d}y, \quad x \in \overline{G(\rho)}, \tag{2.14}$$

where, equally well, the integral could be written as one over $\mathscr{B}(x,\rho)$. Then $u_\rho \in C^\infty(\overline{G(\rho)})$ by Exercise 1.23; the present boundary $\partial\Omega$ plays no part when $x \in \overline{G(\rho)}$. The definition of distributional subsolution states that

$$\begin{aligned}
0 \;&\le\; \Lambda_{1d}(k_\rho(x-.),\; u;\; \Omega) \\
&=\; \int_\Omega \left\{ \sum_{i,j=1}^N a_{ij} \left[\frac{\partial^2}{\partial y_i \partial y_j} k_\rho(x-y) \right] u(y) \right. \\
&\qquad \left. - \sum_{j=1}^N b_j \left[\frac{\partial}{\partial y_j} k_\rho(x-y) \right] u(y) + c k_\rho(x-y)\, u(y) \right\} \mathrm{d}y \\
&=\; \sum_{i,j=1}^N a_{ij} \int_\Omega \left[\frac{\partial^2}{\partial x_i \partial x_j} k_\rho(x-y) \right] u(y)\, \mathrm{d}y \\
&\qquad + \sum_{j=1}^N b_j \int_\Omega \left[\frac{\partial}{\partial x_j} k_\rho(x-y) \right] u(y)\, \mathrm{d}y + c \int_\Omega k_\rho(x-y)\, u(y)\, \mathrm{d}y \\
&=\; L_1 u_\rho(x).
\end{aligned}$$

Thus u_ρ is a C^2-subsolution relative to L_1 and $G(\rho)$; by the weak maximum principle for L_0 and for L,

$$u_\rho(\xi) \le \begin{cases} \max_{\partial G(\rho)} u_\rho & \text{if } \; c = 0, \\ \max_{\partial G(\rho)} (u_\rho)^+ & \text{if } \; c < 0. \end{cases}$$

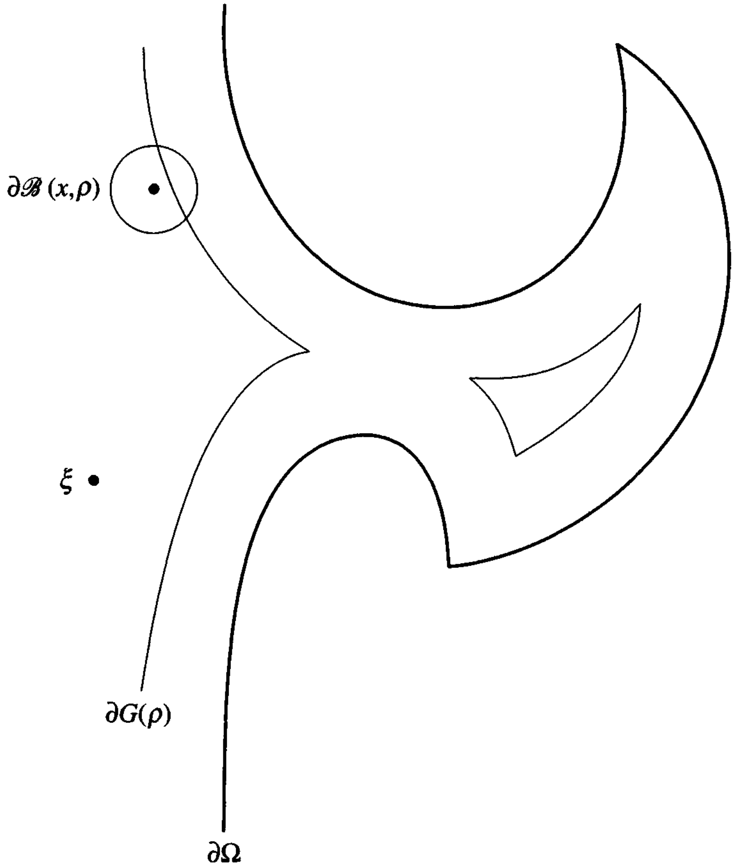

Fig. 2.2.

Consequently, if

$$\lim_{\rho\to 0} u_\rho(\xi) = u(\xi), \tag{2.15}$$

$$\limsup_{\rho\to 0}\left\{\max_{\partial G(\rho)} u_\rho\right\} \le \max_{\partial\Omega} u, \tag{2.16a}$$

$$\limsup_{\rho\to 0}\left\{\max_{\partial G(\rho)} (u_\rho)^+\right\} \le \max_{\partial\Omega} u^+, \tag{2.16b}$$

then (2.12) will follow [because for $c = 0$ we shall be able to contradict $u(\xi) = \max_{\partial\Omega} u + \mu$, $\mu > 0$, by choosing ρ sufficiently small; similarly for $c < 0$].

(iii) Consider in passing the statements

$$\lim_{\rho\to 0}\left\{\max_{\partial G(\rho)} u_\rho\right\} = \max_{\partial\Omega} u, \tag{2.17a}$$

$$\lim_{\rho\to 0}\left\{\max_{\partial G(\rho)} (u_\rho)^+\right\} = \max_{\partial\Omega} u^+. \tag{2.17b}$$

These may seem simpler than (2.16a,b) and, with (2.15), they certainly imply (2.12). Moreover, (2.17a,b) are true. However, their proof is longer, and slightly harder, than that of (2.16a,b) because a *lower* bound for $\max_{\partial G(\rho)} u_\rho$ emerges less easily than the upper bound that we shall find.

(iv) Since $u \in C(\overline{\Omega})$ and $\overline{\Omega}$ is compact, u is uniformly continuous: for every $\varepsilon > 0$ there is a number $\delta_\varepsilon > 0$ such that

$$y, z \in \overline{\Omega} \text{ and } |y - z| < \delta_\varepsilon \Rightarrow |u(y) - u(z)| < \varepsilon; \tag{2.18}$$

we reduce δ_ε, if necessary, in order that $\delta_\varepsilon \le \frac{1}{3}\operatorname{dist}(\xi, \partial\Omega)$.

To prove (2.15), we observe that, for every $\varepsilon > 0$,

$$\begin{aligned} |u(\xi) - u_\rho(\xi)| &= \left|\int_{\mathscr{B}(\xi,\rho)} k_\rho(\xi - y)\{u(\xi) - u(y)\}\, \mathrm{d}y\right| \\ &< \int_{\mathscr{B}(\xi,\rho)} k_\rho(\xi - y)\, \varepsilon\, \mathrm{d}y \quad \text{if } \rho < \delta_\varepsilon \\ &= \varepsilon. \end{aligned}$$

To prove (2.16a), we write

$$M := \max_{\partial\Omega} u, \quad v_\rho := u_\rho\big|_{\partial G(\rho)}.$$

Now, if $x \in \partial G(\rho)$ and $y \in \mathscr{B}(x, \rho)$, then $\operatorname{dist}(y, \partial\Omega) < 3\rho$ [because $\operatorname{dist}(x, \partial\Omega) = 2\rho$ and $|y - x| < \rho$]; if also $3\rho < \delta_\varepsilon$, then $u(y) < M + \varepsilon$ [because $\operatorname{dist}(y, \partial\Omega) < \delta_\varepsilon$ and by (2.18)]. Accordingly, for all $x \in \partial G(\rho)$ and every $\varepsilon > 0$,

$$\begin{aligned} v_\rho(x) &= \int_{\mathscr{B}(x,\rho)} k_\rho(x - y)\, u(y)\, \mathrm{d}y \\ &< \int_{\mathscr{B}(x,\rho)} k_\rho(x - y)\, (M + \varepsilon)\, \mathrm{d}y \quad \text{if } 3\rho < \delta_\varepsilon \\ &= M + \varepsilon, \end{aligned} \tag{2.19}$$

which proves (2.16a).

It remains to prove (2.16b). If $M < 0$, then (2.19) shows that, for $3\rho < \delta_{-M}$ and for all $x \in \partial G(\rho)$, we have $v_\rho(x) < 0$ and hence $(v_\rho)^+(x) = 0$; therefore, both sides of (2.16b) are zero. If $M = 0$, then (2.19) shows that $v_\rho(x) < \varepsilon$ for every $\varepsilon > 0$ and for all $x \in \partial G(\rho)$, if $3\rho < \delta_\varepsilon$; again

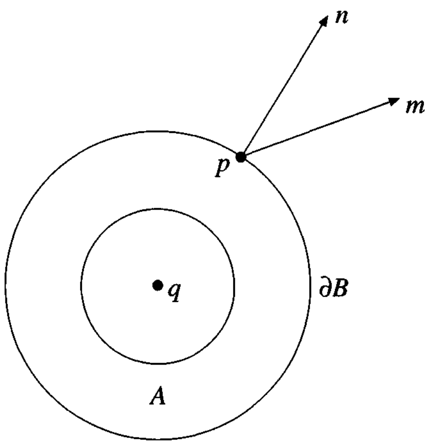

Fig. 2.3.

both sides of (2.16b) are zero. If $M > 0$, then $\max_{\partial\Omega} u^+ = M$, and (2.19) implies (2.16b) once more. $\square$

2.3 The boundary-point lemma and the strong maximum principle

Lemma 2.12 (the boundary-point lemma for balls). *Suppose that*

(a) *$B \subset \Omega$ is a ball, $u \in C(\overline{B})$;*

(b) *u is a C^2-subsolution relative to L_0 or L and B, or a distributional subsolution relative to L_1 and B;*

(c) *$u(x) < u(p)$ for all $x \in B$ and some $p \in \partial B$, with $u(p) \geq 0$ when the coefficient c is not the zero function.*

Let m be an outward unit vector at p ($m \cdot n > 0$ and $|m| = 1$, where n denotes the outward unit normal to ∂B at p). Then

$$\liminf_{t \downarrow 0} \frac{u(p) - u(p - tm)}{t} > 0, \tag{2.20}$$

which implies that

$$\frac{\partial u}{\partial m}(p) := \lim_{t \downarrow 0} \frac{u(p) - u(p - tm)}{t} > 0 \tag{2.21}$$

whenever this one-sided directional derivative exists.

Proof (i) As in Figure 2.3, let $B =: \mathscr{B}(q, \rho)$ and $A := \mathscr{B}(q, \rho) \setminus \overline{\mathscr{B}(q, \frac{1}{2}\rho)}$; it will suffice to consider the annular set $\overline{A}$. Also, let $M := u(p) = \sup_B u$.

If we can find a function $v \in C^2(\overline{A})$ such that

$$v(p) = 0, \tag{I}$$

$$\frac{\partial v}{\partial m}(p) < 0, \tag{II}$$

$$u + v \le M \quad \text{on} \quad \overline{A}, \tag{III}$$

then we can prove (2.20) as follows. Let $w := u + v$. For $0 < t < \frac{1}{2}\rho$,

$$\frac{w(p) - w(p - tm)}{t} = \frac{M - w(p - tm)}{t} \ge 0 \qquad \text{[by (I) and (III)]},$$

whence

$$\begin{aligned} &\liminf_{t\downarrow 0} \frac{u(p) - u(p - tm)}{t} \\ &\quad = \liminf_{t\downarrow 0} \frac{\{w(p) - w(p - tm)\} - \{v(p) - v(p - tm)\}}{t} \\ &\quad \ge \liminf_{t\downarrow 0} \frac{-v(p) + v(p - tm)}{t} \\ &\quad = -\frac{\partial v}{\partial m}(p) > 0 \qquad \text{[by (II)]}. \end{aligned}$$

(ii) Consider the function defined on $\overline{A}$ by

$$v(x) := \delta \left(\mathrm{e}^{-Kr^2} - \mathrm{e}^{-K\rho^2} \right), \qquad r := |x - q|,$$

and shown in Figure 2.4; both positive constants δ and K are still to be chosen.

Certainly $v \in C^2(\overline{A})$; also (I) and (II) hold, since

$$\frac{\partial v}{\partial m}(p) = (m \cdot n) \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r=\rho} < 0.$$

For (III), we shall use the weak maximum principle, first considering the values of $u + v$ on ∂A. For $r = \rho$ we have $u \le M$, $v = 0$ and hence $u + v \le M$, with equality at p. For $r = \frac{1}{2}\rho$, we have $u < M$ by hypothesis (c); if $M - \alpha$ denotes the maximum of u for $r = \frac{1}{2}\rho$ [the supremum of a continuous function on a compact set is attained], then $\alpha > 0$. Choose $\delta = \alpha$; then $u \le M - \alpha$ and $v < \alpha$ for $r = \frac{1}{2}\rho$. Accordingly,

$$\max\nolimits_{\partial A}\ (u + v) = M.$$

(iii) If we can choose K so that $Lv \ge 0$ in A (hence so that $L_0 v \ge 0$ in A, or $L_1 v \ge 0$ in A), then condition (III) will follow from one of

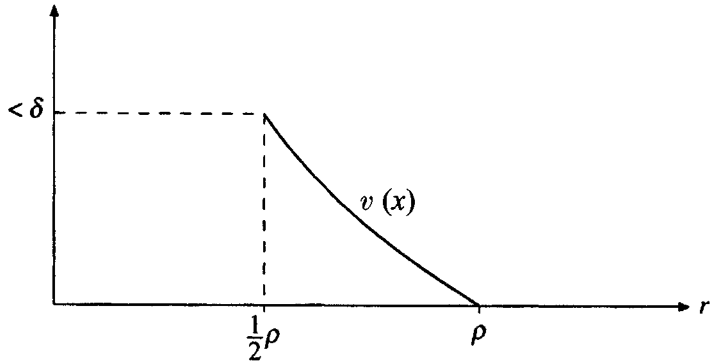

Fig. 2.4.

our three versions of the weak maximum principle, applied to $u+v$ and A. The condition $u(p) \geq 0$ when $c \neq 0$ banishes one difference in these versions. To deal with a distributional subsolution u, we add to the given condition,

$$\Lambda_{1d}(\varphi, u; A) \geq 0 \quad \text{whenever} \quad \varphi \in C_c^\infty(A) \quad \text{and} \quad \varphi \geq 0,$$

the condition $\Lambda_{1d}(\varphi, v; A) \geq 0$ for the same φ; this property of v will be implied by integration by parts [as in (2.9) and (2.10)] once we have $L_1 v \geq 0$ in A. Then, with $u+v \in C(\overline{A})$ and $\Lambda_{1d}(\varphi, u+v; A) \geq 0$, condition (III) will follow from Theorem 2.11.

(iv) It remains to calculate Lv and choose K. Since

$$\begin{aligned} \partial_j \mathrm{e}^{-Kr^2} &= \mathrm{e}^{-Kr^2}(-K)2(x_j - q_j), \\ \partial_i \partial_j \mathrm{e}^{-Kr^2} &= \mathrm{e}^{-Kr^2}\left\{4K^2(x_i - q_i)(x_j - q_j) - 2K\delta_{ij}\right\}, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{\delta} Lv(x) = \mathrm{e}^{-Kr^2} \Bigg\{ & 4K^2 \sum_{i,j} a_{ij}(x)\,(x_i - q_i)(x_j - q_j) \\ & - 2K \sum_j a_{jj}(x) - 2K \sum_j b_j(x)\,(x_j - q_j) \Bigg\} \\ & + c(x)\left\{\mathrm{e}^{-Kr^2} - \mathrm{e}^{-K\rho^2}\right\}. \end{aligned}$$

By the condition (2.3) of uniform ellipticity,

$$\begin{aligned}\frac{1}{\delta}Lv(x) &\geq \mathrm{e}^{-Kr^2}\left\{4K^2\lambda_0 r^2 - 2K\sup_A\left(\sum_j |a_{jj}| + |b|\rho\right) - \sup_A |c|\right\}\\ &> 0 \qquad \text{for } x \in \overline{A}\end{aligned}$$

if we choose K sufficiently large, because $r^2 \geq (\frac{1}{2}\rho)^2$. □

Note a change of direction in the statement of the next theorem: there is no mention of $\overline{\Omega}$ or of $\partial\Omega$.

Theorem 2.13 (the strong maximum principle). *Suppose that*

(a) Ω *is a region (open and connected, possibly unbounded);*

(b) u *is a* C^2*-subsolution relative to* L_0 *or* L *and* Ω*, or a generalized subsolution relative to* L_1 *and* Ω*;*

(c) $\sup_\Omega u \geq 0$ *when the coefficient* c *is not the zero function.*

Under these hypotheses, if $\sup_\Omega u$ *is attained at a point of* Ω*, then* u *is constant in* Ω.

Proof Let $M := \sup_\Omega u$, and assume that this supremum is attained at $\hat{x} \in \Omega$. Define

$$F := \left\{x \in \Omega \;\middle|\; u(x) = M\right\}, \qquad G := \left\{x \in \Omega \;\middle|\; u(x) < M\right\};$$

then F is closed in the metric space Ω, and not empty because $\hat{x} \in F$; the set G is open in the metric space Ω. If G is empty, the theorem is true.

Suppose then that there is a point $x_0 \in G$. We shall obtain a contradiction by means of Lemma 2.12, first using the result that, because Ω is open and connected in $\mathbb{R}^N$, it is pathwise connected (Burkill & Burkill 1970, p.44; Cartan 1971, p.42). This implies existence of a continuous arc

$$\gamma := \left\{\, \xi(t) \;\middle|\; 0 \leq t \leq 1 \,\right\} \subset \Omega \;\text{ with } \xi(0) = x_0,\; \xi(1) = \hat{x},$$

as shown in Figure 2.5. Here $\xi \in C([0,1], \mathbb{R}^N)$, so that γ is compact; if Ω has a boundary, then $\operatorname{dist}(\gamma, \partial\Omega) > 0$ because $\partial\Omega$ is closed in $\mathbb{R}^N$ and disjoint from γ.

Let $\tilde{x}$ be the first point of γ at which $u(x) = M$; here 'first' means 'with smallest t'. Possibly $\tilde{x} = \hat{x}$. Let q be any point of γ that is strictly between x_0 and $\tilde{x}$, and is such that $|q - \tilde{x}| < \operatorname{dist}(\gamma, \partial\Omega)$ when Ω has a boundary. Now consider the ball $B := \mathscr{B}(q, \rho)$ with $\rho := \operatorname{dist}(q, F)$. Then $\rho \leq |q - \tilde{x}| < \operatorname{dist}(\gamma, \partial\Omega)$, so that $B \subset \Omega$; also, $B \subset G$ by construction. There exists a point $p \in F \cap \partial B$ because F is closed (possibly $p = \tilde{x}$). All

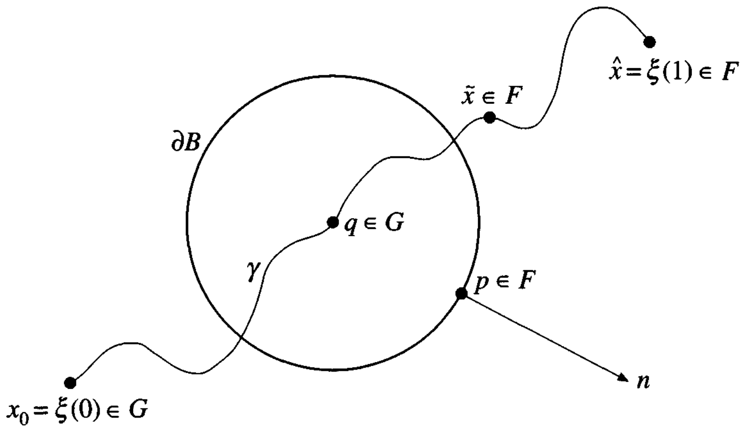

Fig. 2.5.

the hypotheses of Lemma 2.12 hold, so that at p the outward normal derivative

$$\frac{\partial u}{\partial n}(p) = n \cdot (\nabla u)(p) > 0.$$

But since $p \in F$, it is an interior maximum point of $u \in C^1(\Omega)$. Hence $(\nabla u)(p) = 0$ and we have our contradiction. □

There are many boundary-point lemmas for elliptic operators and sets other than balls, but Lemma 2.12 is probably the heart of the matter. Theorem 2.15 is a consequence of that lemma, seasoned by a touch of the strong maximum principle. First, we need a definition.

Definition 2.14 A set Ω has the *interior-ball property* at a point $p \in \partial\Omega$ iff there exists a ball $B_0 \subset \Omega$ such that $p \in \partial B_0$; it has the *exterior-ball property* at p iff there exists a ball $B_1 \subset \mathbb{R}^N \setminus \overline{\Omega}$ such that $p \in \partial B_1$. □

Figure 2.6 shows two cases of the interior-ball property for Ω, and therefore two cases of the exterior-ball property for $\mathbb{R}^N \setminus \overline{\Omega}$. Note that a unit vector m at p, outward from an interior ball B_0, need not be outward from Ω.

Theorem 2.15 (a boundary-point theorem for Ω). *Suppose that*

(a) Ω *is a region;*

(b) u *is a C^2-subsolution relative to L_0 or L and Ω, or a generalized subsolution relative to L_1 and Ω;*

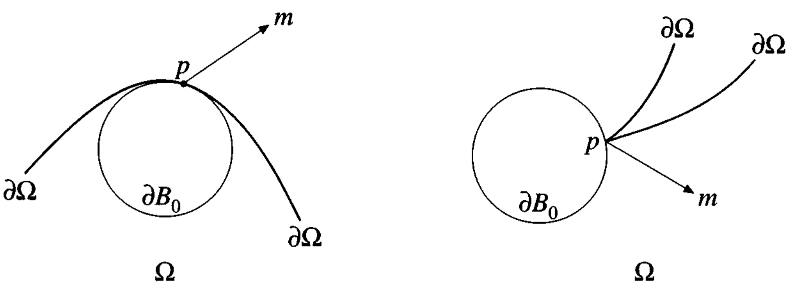

Fig. 2.6.

(c) *there is a point* $p \in \partial\Omega$ *such that* $u \in C(\Omega \cup \{p\})$ *and* $u(p) = \sup_\Omega u$, *with* $u(p) \geq 0$ *when the coefficient* c *is not the zero function*;

(d) Ω *has the interior-ball property at* p.

Let m *be a unit vector at* p, *outward from an interior ball* B_0 *at* p. *Then either*

$$\liminf_{t\downarrow 0} \frac{u(p) - u(p - tm)}{t} > 0 \tag{2.22}$$

(*which implies that* $(\partial u/\partial m)(p) > 0$ *whenever this derivative exists*), *or* u *is constant in* Ω.

Proof Let x_0 be the centre of the ball B_0 and let $\rho_0 := |p - x_0|$, so that $B_0 = \mathscr{B}(x_0, \rho_0)$. Now consider the smaller ball $B := \mathscr{B}(q, \frac{1}{2}\rho_0)$ with $q := \frac{1}{2}(p + x_0)$. Since $\overline{B} \subset B_0 \cup \{p\}$, we have $\overline{B} \subset \Omega \cup \{p\}$ and hence $u \in C(\overline{B})$.

If $u(x) < u(p)$ for all $x \in B$, then Lemma 2.12 implies (2.22). If $u(\hat{x}) = u(p)$ for some $\hat{x} \in B$, then $u(\hat{x}) = \sup_\Omega u$ and the strong maximum principle implies that u is constant in Ω. □

Suppose that p is what may be called an *edge point*; for example, $\Omega = (0,1)^2 \subset \mathbb{R}^2$ and $p = (0,0)$, or $\Omega = (0,1)^3 \subset \mathbb{R}^3$ and $p = (\frac{1}{2}, 0, 0)$. Then Ω lacks the interior-ball property at p, but something can still be said, for a subsolution, about an outward derivative or difference quotient at p. This is the subject of Appendix E.

Remark 2.16 (on the condition $c \leq 0$ in Ω). *For a subsolution* u, *if* $\sup_\Omega u = 0$ *in an application of the weak or strong maximum principle, or if* $\sup_B u = 0$ *in an application of the boundary-point lemma for a ball* B, *then the condition* $c \leq 0$ *in* Ω (*imposed in Definition* 2.3) *can be omitted.*

Proof We use the decomposition $c(x) = c^+(x)+c^-(x)$ [defined in Chapter 0, (v)]. The foregoing theorems and lemma are valid for the operator L^- and bilinear form Λ_1^- defined by

$$L^- u := Lu - c^+(x)u \quad \left(= L_0 u + c^-(x)u\right),$$

$$\Lambda_1^-(\varphi, u; \Omega) := \Lambda_1(\varphi, u; \Omega) - c^+ \int_\Omega \varphi u \qquad (c^+ = c > 0).$$

When $\sup_\Omega u = 0$, we can use L^- in place of L because $Lu \geq 0$ and $u \leq 0$ imply that $L^- u \geq 0$. Again, when $\sup_\Omega u = 0$, we can use Λ_1^- in place of Λ_1 because $\Lambda_1(\varphi, u; \Omega) \geq 0$, $\varphi \geq 0$ and $u \leq 0$ imply that $\Lambda_1^-(\varphi, u; \Omega) \geq 0$. □

2.4 A maximum principle for thin sets Ω

All our maximum principles so far have required that the coefficient $c(x) \leq 0$ for all $x \in \Omega$, unless it happens to be known for a subsolution u that $\sup_\Omega u = 0$ (Remark 2.16), or for a supersolution v that $\inf_\Omega v = 0$. In this section we proceed to a weak maximum principle for thin sets Ω in which both $c(x)$ and $u(x)$ are unrestricted in sign *a priori*. By a *thin set* Ω we mean one of specified diameter and small volume: $|\Omega| < \delta$, where the positive number δ depends only on $\operatorname{diam}\Omega$ and on constants independent of Ω. To derive this maximum principle, we need some form of the basic estimate for elliptic equations that is presented here as Theorem 2.18. This estimate, in turn, is a consequence of elementary results in Appendix A for the Newtonian potential and of the weak maximum principle in Theorem 2.11.

Given a bounded open subset G of $\mathbb{R}^N$, we define a *modified Newtonian kernel* $\tilde{K}$ by

$$\tilde{K}(x) := \begin{cases} -\frac{1}{2}|x| + \frac{1}{2}\operatorname{diam} G & \text{if } N = 1, \\ \dfrac{1}{2\pi}\log\dfrac{\operatorname{diam} G}{|x|} & \text{if } N = 2, \\ \kappa_N \dfrac{1}{|x|^{N-2}} & \text{if } N \geq 3, \end{cases} \tag{2.23}$$

where $x \neq 0$ if $N \geq 2$ and where κ_N is as in (A.18b) of Appendix A. This function differs from the Newtonian kernel K introduced by (A.18) only for $N = 1$ or 2, and then only by the addition of a constant which ensures that $\tilde{K}(x_0 - x) \geq 0$ whenever $x_0, x \in \overline{G}$. The corresponding *modified Newtonian potential* of a suitable density function $g : G \to \mathbb{R}$ is defined by

$$v(x_0) := \int_G \tilde{K}(x_0 - x)\, g(x)\, \mathrm{d}x, \quad x_0 \in \mathbb{R}^N, \tag{2.24}$$

but here we restrict attention to field points $x_0 \in \overline{G}$.

Lemma 2.17 *Let G be a bounded open subset of $\mathbb{R}^N$ and let v be the modified Newtonian potential defined by* (2.23) *and* (2.24). *If $g \in L_p(G)$ with $1 \le p < \infty$ for $N = 1$, or with $N/2 < p < \infty$ for $N \ge 2$, then $v \in C(\overline{G})$ and v is a distributional solution* (*Definition* A.7) *of $-\Delta v = g$ in G. Moreover,*

$$\left|v(x_0)\right| \le \Gamma(N,p)\,(\operatorname{diam} G)^{2-(N/p)}\, \| g \,|L_p(G)\| \quad \textit{for all} \quad x_0 \in \overline{G}, \tag{2.25}$$

where, with the notation $1/p + 1/q = 1$ and $\sigma_N := |\partial \mathscr{B}_N(0,1)|$,

$$\left.\begin{aligned}
&\Gamma(1,1) = \frac{1}{2}, \quad \Gamma(1,p) = \left(\frac{1}{2}\right)^{1/p} \left(\frac{1}{q+1}\right)^{1/q} \quad \textit{for} \quad 1 < p < \infty, \\
&\Gamma(2,p) = \left(\frac{1}{2\pi}\right)^{1/p} \left\{ \int_0^1 \left(\log \frac{1}{\rho}\right)^q \rho \,\mathrm{d}\rho \right\}^{1/q} \quad (1 < p < \infty), \\
&\textit{for } N \ge 3, \quad \Gamma(N,p) = \frac{1}{N-2} \left(\frac{1}{\sigma_N}\right)^{1/p} \left(\frac{1}{N - Nq + 2q}\right)^{1/q} \\
&\qquad\qquad \left(\frac{N}{2} < p < \infty\right).
\end{aligned}\right\} \tag{2.26}$$

Proof For $N \ge 2$, it follows from Theorem A.6 that $v \in C(\overline{G})$, and from Theorem A.8 that v is a distributional solution of $-\Delta v = g$ not merely in G but in $\mathbb{R}^N$; for $N = 1$, the proofs are similar in strategy but much easier in detail. The bound (2.25) results from the Hölder inequality; we integrate $\tilde{K}(x_0 - x)^q$ over the ball $\mathscr{B}(x_0, \operatorname{diam} G)$, using $R := |x - x_0|$ as variable of integration. □

Notation and terminology The next theorem involves both the constant-coefficient operator L_1 (Definition 2.3) and the Lebesgue space $L_1(\Omega)$. Confusion will be avoided by unfailing display of Ω in the symbol $L_p(\Omega)$. Extending slightly the terminology in Definition 2.10, we shall say that $L_1 u + f \ge 0$ in Ω *in the distributional sense* iff u and f are locally integrable in Ω and

$$\int_{\Omega}\left\{\sum_{i,j=1}^{N} a_{ij}\left(\partial_j\partial_i\varphi\right)u-\sum_{j=1}^{N} b_j\left(\partial_j\varphi\right)u+c\varphi u+\varphi f\right\}\geq 0$$
$$\text{whenever}\quad \varphi\in C_c^{\infty}(\Omega)\quad\text{and}\quad \varphi\geq 0. \tag{2.27}$$

Theorem 2.18 (a basic estimate for the operator L_1). *Suppose that*

(a) *Ω is bounded, $u\in C(\overline{\Omega})$;*
(b) *$L_1u+f\geq 0$ in Ω in the distributional sense, where $f\in L_p(\Omega)$ with $1\leq p<\infty$ if $N=1$, or with $N/2<p<\infty$ if $N\geq 2$;*
(c) *$u\big|_{\partial\Omega}\leq 0$.*

Then

$$\max_{\overline{\Omega}} u\leq A\left\| f \;\middle|\; L_p(\Omega)\right\|, \tag{2.28}$$

where A is independent of u and $|\Omega|$ (but depends on $\operatorname{diam}\Omega$*). In fact, coarse inequalities give*

$$A=\frac{\Gamma(N,p)}{\lambda_0}\exp\left(\frac{|b|\operatorname{diam}\Omega}{\lambda_0}\right)(\operatorname{diam}\Omega)^{2-(N/p)}, \tag{2.29}$$

where $\Gamma(N,p)$ is as in Lemma 2.17*, λ_0 is the (positive) smallest eigenvalue of the matrix (a_{ij}) and $b=(b_1,\dots,b_N)$ is the vector of coefficients in the term $b\cdot\nabla$ of L_1.*

Proof (i) We make two co-ordinate transformations. First, let P be an orthogonal $N\times N$ matrix such that the transformation $y=Px$ makes the y_j-axes principal axes of the matrix a; say $\left(PaP^{-1}\right)_{ij}=:\lambda_i\delta_{ij}$ for $i,j\in\{1,\dots,N\}$, where $\lambda_0:=\min_j\lambda_j>0$. Second, we make a dilatation $z=Ey$, where $E_{ij}=\lambda_i^{-1/2}\delta_{ij}$, in order to transform L_1 to $\triangle+\cdots$. Writing

$$b^*:=EPb,\quad G:=EP(\Omega),\quad u^*(z):=u\left(P^{-1}E^{-1}z\right)=u(x),$$

and transforming f and φ like u, we obtain from hypothesis (b) that

$$\left\{\triangle+b^*\cdot\nabla+c\right\}u^*(z)+f^*(z)\geq 0\qquad\text{in } G \text{ in the d.s.},$$

where $\triangle$ and ∇ are with respect to z, and d.s. means ‘distributional sense’. More explicitly,

$$\int_G\left\{\left(\triangle\varphi^*-b^*\cdot\nabla\varphi^*+c\varphi^*\right)u^*+\varphi^*f^*\right\}\mathrm{d}z\geq 0$$
$$\text{whenever}\quad \varphi^*\in C_c^{\infty}(G)\quad\text{and}\quad \varphi^*\geq 0.$$

(ii) Next, first derivatives are removed by the transformation

$$\left.\begin{array}{c} u^*(z) =: \eta(z)\,\hat{u}(z), \quad f^*(z) =: \eta(z)\,\hat{f}(z), \quad \varphi^*(z) =: \dfrac{1}{\eta(z)}\check{\varphi}(z), \\ \text{where} \;\; \eta(z) := \exp\left(-\tfrac{1}{2}b^* \cdot z\right). \end{array}\right\} \tag{2.30}$$

Hypothesis (b) now becomes

$$\left(\triangle - k^2\right)\hat{u}(z) + \hat{f}(z) \geq 0 \;\text{ in } G \;\text{ in the d.s.,} \tag{2.31a}$$

where $-k^2 := c - \frac{1}{4}|b^*|^2 \leq 0$; more explicitly,

$$\int_G \left\{ \left(\triangle\check{\varphi} - k^2\check{\varphi}\right)\hat{u} + \check{\varphi}\hat{f} \right\} \mathrm{d}z \geq 0 \;\text{ whenever }\; \check{\varphi} \in C_c^\infty(G) \;\text{ and }\; \check{\varphi} \geq 0.$$

Conditions (a) to (c) also imply that $\hat{u} \in C(\overline{G})$, that $\hat{f} \in L_p(G)$ for the same p as in (b), and that

$$\hat{u}\big|_{\partial G} \leq 0. \tag{2.31b}$$

(iii) We compare the function $\hat{u}$ with the modified Newtonian potential v of $\left(\hat{f}\right)^+$. (Note that $\left(\hat{f}\right)^+ = \left(f^+\right)^\wedge$.) In other words,

$$v(z) := \int_G \widetilde{K}(z-\zeta)\,\left(\hat{f}\right)^+(\zeta)\,\mathrm{d}\zeta, \qquad z \in \overline{G},$$

whence $v \in C(\overline{G})$ and

$$\left(\triangle - k^2\right)v(z) + \left(\hat{f}\right)^+(z) \;=\; -k^2 v(z) \;\text{ in } G \text{ in the d.s.,} \tag{2.32a}$$

$$v(z) \;\geq\; 0 \;\text{ on } \overline{G}, \tag{2.32b}$$

by Lemma 2.17 and because $\widetilde{K}(z-\zeta) \geq 0$ and $\left(\hat{f}\right)^+(\zeta) \geq 0$.

Let $w := \hat{u} - v$; then (2.31a), (2.31b) and (2.32a), (2.32b) imply that

$$\left(\triangle - k^2\right)w \geq k^2 v - \left(\hat{f}\right)^- \geq 0 \;\text{ in }\; G \;\text{ in the d.s.,}$$

$$w\big|_{\partial G} \leq 0,$$

and the weak maximum principle (Theorem 2.11) ensures that

$$\max_{\overline{G}} w \;\leq\; \max_{\partial G} w^+ \;=\; 0.$$

The inequality (2.25) now yields

$$\max_{\overline{G}} \hat{u} \;\leq\; \max_{\overline{G}} v \;\leq\; \Gamma(N,p)\,(\operatorname{diam} G)^{2-(N/p)}\,\|\hat{f} \mid L_p(G)\|. \tag{2.33}$$

Returning to Ω, u and f, we note first that

$$\left|b^*\right| \leq \lambda_0^{-1/2}\,|b|, \qquad \operatorname{diam} G \leq \lambda_0^{-1/2}\operatorname{diam}\Omega.$$

We may suppose that $0 \in \Omega$, because the operator L_1 and the norm $\| f \mid L_p(\Omega) \|$ are invariant under translation of co-ordinate axes; then (2.30) implies that

$$\max_{\overline{\Omega}} u \;\leq\; \exp\left(\frac{1}{2}\frac{|b| \operatorname{diam}\Omega}{\lambda_0}\right) \max_{\overline{G}} \hat{u},$$

$$\| \hat{f} \mid L_p(G) \| \;\leq\; \exp\left(\frac{1}{2}\frac{|b| \operatorname{diam}\Omega}{\lambda_0}\right) \lambda_0^{-N/2p}\, \| f \mid L_p(\Omega) \|.$$

The result (2.28) now follows from (2.33) and these inequalities. $\square$

As was mentioned earlier, the virtue of the following maximum principle is that the signs of the coefficient $\gamma(x)$ and of the subsolution u are both unrestricted in Ω.

Theorem 2.19 (a maximum principle for thin sets Ω). *Suppose that*

(a) *Ω is bounded, $u \in C(\overline{\Omega})$;*
(b) *$L_{10}u + \gamma(x)u \geq 0$ in Ω in the distributional sense, where L_{10} is the operator L_1 with coefficient $c = 0$ and $\gamma \in L_\infty(\Omega)$;*
(c) *$u\big|_{\partial\Omega} \leq 0$.*

Then

$$u \leq 0 \quad \text{on} \quad \overline{\Omega} \qquad \text{whenever} \quad |\Omega| < \delta, \tag{2.34}$$

where the positive number δ is independent of u and $|\Omega|$ (but depends on $\operatorname{diam}\Omega$*). In fact, we may take*

$$\delta^{1/N} = \frac{\lambda_0}{2\Gamma(N,N)\,\operatorname{diam}\Omega\;\|\gamma \mid L_\infty(\Omega)\|} \quad \exp\left(-\frac{|b|\,\operatorname{diam}\Omega}{\lambda_0}\right), \tag{2.35}$$

where the notation is that explained after (2.29).

Proof (i) The first step is to write $L_{10} + \gamma(x) \geq 0$ in a more tractable form. We introduce a constant $c \leq 0$ with $|c|$ so large that

$$g(x) := -c + \gamma(x) \geq 0 \quad \text{almost everywhere in} \quad \Omega;$$

this can be done with $\| g \mid L_\infty(\Omega) \| \leq 2\| \gamma \mid L_\infty(\Omega) \|$. Then

$$L_1 u + g(x)u = L_{10}u + \gamma(x)u \geq 0 \quad \text{in} \quad \Omega \quad \text{in the d.s.},$$

where d.s. means 'distributional sense', as before.

(ii) The second step is to decompose $g(x)u$:

$$L_1 u + g(x)u^+ \geq -g(x)u^- \geq 0 \quad \text{in } \Omega \text{ in the d.s.},$$

to recall that $u\big|_{\partial\Omega} \leq 0$, and to apply Theorem 2.18 with $f = gu^+$; the choice $p = N$ is admissible for all $N \in \mathbb{N}$. This yields

$$\begin{aligned} \max_{\overline{\Omega}} u &\leq A \, \| gu^+ \mid L_N(\Omega) \| \\ &\leq A \, \| g \mid L_\infty(\Omega) \| \; \max_{\overline{\Omega}} u^+ \, |\Omega|^{1/N}. \end{aligned} \tag{2.36}$$

If $\max_{\overline{\Omega}} u < 0$, then (2.34) holds. If $\max_{\overline{\Omega}} u \geq 0$, then (2.36) and our bound for $\| g \mid L_\infty(\Omega) \|$ imply that

$$\left(\max_{\overline{\Omega}} u^+\right) \left\{ 1 - 2A \, \| \gamma \mid L_\infty(\Omega) \| \, |\Omega|^{1/N} \right\} \leq 0,$$

from which (2.34) and (2.35) follow if we choose $|\Omega|$ to be so small that the expression in braces is positive. □

2.5 Steps towards Phragmén–Lindelöf theory

All three versions of the weak maximum principle in §2.2 require Ω to be bounded and u to be continuous on $\overline{\Omega}$. If we relax one or other of these conditions, what other hypotheses will ensure that a subsolution, relative to L and Ω, can be bounded above in terms of its values on $\partial\Omega$? This is the theme of the remainder of this chapter, but, as it stands, the question is much too wide; we narrow it as follows.

(a) Among the many unbounded, proper subsets of $\mathbb{R}^N$ that might be considered, our favourite will be the half-space $D := \left\{ x \in \mathbb{R}^N \;\middle|\; x_N > 0 \right\}$.

(b) In the rest of this chapter, the dimension $N \geq 2$.

(c) The condition $u \in C(\overline{\Omega})$ will be relaxed at only one or two boundary points; typically to $u \in C(\overline{\Omega} \setminus \{p\})$, where p is a specified point of $\partial\Omega$.

(d) Only the Laplace operator $\triangle$ will be considered. There is no disgrace in this restriction; good answers to our question are sensitive to details of the differential operator L, and each proof involves a comparison function tailored rather closely to the task in hand. To launch here into the more general theory initiated by Gilbarg (1952) and E. Hopf (1952a) would be a catastrophic attempt to run before we have learned to walk.

Definition 2.20 We shall say that u is *subharmonic in* Ω iff it is a distributional subsolution relative to $\triangle$ and Ω; that is, iff u is locally integrable in Ω and

$$\int_\Omega (\triangle\varphi)u \geq 0 \quad \text{whenever} \quad \varphi \in C_c^\infty(\Omega) \quad \text{and} \quad \varphi \geq 0.$$

Then u is *superharmonic in* Ω iff $-u$ is subharmonic there; u is *harmonic in* Ω iff it is both subharmonic and superharmonic in Ω. □

This definition of 'harmonic' (which follows inevitably from the useful definition of 'subharmonic' that we have adopted) scarcely does justice to harmonic functions. Theorems B.6 and B.10 show that, if u is harmonic in Ω according to Definition 2.20, then, after re-definition on a set of measure zero, not only is u a C^2-solution of $\triangle u = 0$, but also u is real-analytic in Ω.

Our opening question can now be replaced by the following. If u is continuous on $\overline{D}$ and subharmonic in D, to what rate of growth, as $|x| \to \infty$, must $u(x)$ be restricted in order that $\sup_D u = \sup_{\partial D} u$? If u is continuous merely on $\overline{\Omega} \setminus \{p\}$ and subharmonic in Ω, to what rate of growth, as $x \to p$, must $u(x)$ be restricted in order that $\sup_\Omega u = \sup_{\partial\Omega\setminus\{p\}} u$?

We begin by inspecting some simple and explicit harmonic functions that *vanish* on the boundary of the half-space D or on a punctured boundary $\partial\Omega \setminus \{p\}$; these functions indicate rates of growth that are too large in the context of our questions.

Examples 1. The harmonic polynomials

$$p_1(x) = x_N,\ p_2(x) = 2x_1x_N,\ p_3(x) = 3x_1^2x_N - x_N^3, \ldots,$$
$$p_m(x) = \operatorname{Im}(x_1 + \mathrm{i}x_N)^m, \ldots \tag{2.37}$$

all vanish on ∂D; if $N \geq 3$, there are many more such polynomials.

But, apart from the zero function, no function springs to mind that is continuous on $\overline{D}$, is harmonic in D, vanishes on ∂D and is $o(r)$ as $r := |x| \to \infty$. This is significant: the critical rate of growth for the result $\sup_D u = \sup_{\partial D} u$ will turn out to be close to growth like r as $r \to \infty$.

2. If we seek functions that are continuous on $\overline{D} \setminus \{0\}$, tend to zero at infinity, are harmonic in D and vanish on $\partial D \setminus \{0\}$, then the prototype is

$$q_1(x) = x_N / r^N, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad r := |x|. \tag{2.38}$$

This is the potential of a particular *dipole* (§A.4); more precisely, the potential of a multipole of type $(0,\dots,0,1)$. Differentiating this repeatedly in horizontal directions (with respect to x_j, $j \le N-1$), we generate multipole potentials like

$$\left.\begin{aligned} q_2(x) &= \partial_1 q_1(x) = -Nx_1x_Nr^{-N-2}, \\ q_3(x) &= \partial_1^2 q_1(x) = Nx_N\{\,(N+2)x_1^2 - r^2\,\}r^{-N-4}, \end{aligned}\right\} \qquad (2.39)$$

which retain the properties listed before (2.38), but have a stronger singularity at the origin, relative to q_1, and a more rapid decay at infinity.

However, no non-trivial function springs to mind that has the properties listed before (2.38) and is $o(r^{-N+1})$ as $x \to 0$ with $x \in D$. Again this is significant: the critical rate of growth for the result $\sup_\Omega u = \sup_{\partial\Omega\setminus\{p\}} u$ will turn out to be close to growth like $|x-p|^{-N+1}$ as $x \to p$, when $\partial\Omega$ is smooth.

3. We now allow a singularity at the south pole $p := (0,\dots,0,-a)$ of the ball $B := \mathscr{B}(0,a)$ in $\mathbb{R}^N$. The *Poisson kernel* (§B.5) gives an example of a function continuous on $\overline{B}\setminus\{p\}$, harmonic in B and equal to zero on $\partial B\setminus\{p\}$:

$$P(x,p) = \text{const.}\,(a^2-r^2)|x-p|^{-N}, \quad x \in \overline{B}\setminus\{p\},\ r := |x|. \qquad (2.40)$$

Again the singularity at p is of dipole type, and again appropriate differentiation generates a stronger singularity at p, while conserving the value zero on $\partial B\setminus\{p\}$. Thus the harmonic function

$$\begin{aligned} Q(x) &= (x_N\partial_1 - x_1\partial_N)P(x,p) \\ &= \text{const.}\ x_1(a^2-r^2)|x-p|^{-N-2}, \quad x \in \overline{B}\setminus\{p\}, \end{aligned} \qquad (2.41)$$

has a quadrupole singularity at p.

Rather as in Example 2, no non-trivial function springs to mind that is continuous on $\overline{B}\setminus\{p\}$, harmonic in B, equal to zero on $\partial B\setminus\{p\}$ and is $o(|x-p|^{-N+1})$ as $x \to p$; this is significant in the same way as before.

The Phragmén–Lindelöf theory that follows *must be distinguished from Phragmén–Lindelöf theory for holomorphic functions* (complex-analytic functions). In that theory one supposes that, for example, $\sup_{\partial D}|u+iv|$ is known, where D is the upper half of the complex plane $\mathbb{C}$; the analogous situation for us, when $N=2$, is that only $\sup_{\partial D}|u|$ (or only $\sup_{\partial D}|v|$) is known. In the case of holomorphic functions (Hille 1973, Chapter 18; Titchmarsh 1932, §§5.6–5.8) much more can be inferred because much more is given.

Definition 2.21 Let $D := \{\,x \in \mathbb{R}^N \mid x_N > 0\,\}$, $N \ge 2$; let $D_a = D \cap \mathscr{B}(0,a)$

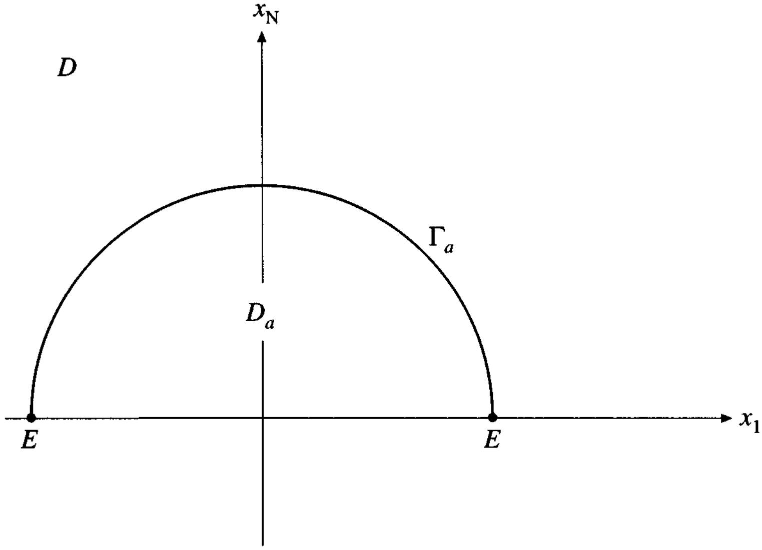

Fig. 2.7.

and $\Gamma_a := D \cap \partial\mathscr{B}(0,a)$ (Figure 2.7); denote the equator of $\mathscr{B}(0,a)$ by $E := \partial D \cap \partial\mathscr{B}(0,a)$.

A function with values $V(x,a)$ will be called a *comparison function of the first kind* iff

(a) for each $a \in (0,\infty)$,

$$V(.,a) \in C(\overline{D}_a \setminus E) \cap C^2(D_a) \quad \text{and} \quad V(.,a) \geq 0 \quad \text{on} \quad \overline{D}_a \setminus E, \tag{2.42}$$

$$(\triangle V)(.,a) = 0 \quad \text{in} \quad D_a, \tag{2.43}$$

$$V(.,a) > 0 \quad \text{on} \quad \Gamma_a; \tag{2.44}$$

(b) $\inf \{ V(x,a) \mid x \in \Gamma_a \} \to \infty$ as $a \to \infty$;

(c) there is a function $\lambda : D \to (0,\infty)$ such that $V(x_0,a) \leq \lambda(x_0)$ whenever $x_0 \in D$ and $a \geq 2|x_0|$. □

Functions having these properties will be displayed in due course; first, we prove the lemma for which the definition has been designed. This lemma shows that, if a function u is continuous on $\overline{D}$, is subharmonic in D and is smaller, in order of magnitude, than $V(.,a(n))$ on some sequence $(\Gamma_{a(n)})$ of hemispheres marching to infinity, then we retain the result $\sup_D u = \sup_{\partial D} u$.

The proof will show that the important case of the growth condition (2.45) is that in which the limit inferior *equals* zero.

Lemma 2.22 *Let V be a comparison function of the first kind. If $u \in C(\overline{D})$, if u is subharmonic in D and if*

$$\liminf_{a\to\infty} \sup\left\{ \frac{u(x)}{V(x,a)} \;\middle|\; x \in \Gamma_a \right\} \le 0, \tag{2.45}$$

then

$$\sup_D u = \sup_{\partial D} u.$$

Proof (i) We may suppose that $\sup_{\partial D} u < \infty$, otherwise the result is trivial. Let $\tilde{u} := u - \sup_{\partial D} u$. Then $\sup_{\partial D} \tilde{u} = 0$ and $\tilde{u}$ also satisfies the growth condition (2.45), because the definition of $\tilde{u}$ and hypothesis (b) imply that

$$\sup\left\{ \frac{|u(x) - \tilde{u}(x)|}{V(x,a)} \;\middle|\; x \in \Gamma_a \right\} \to 0 \quad \text{as } a \to \infty.$$

Let both $x_0 \in D$ and $\varepsilon > 0$ be given; we shall prove the lemma by showing that $\tilde{u}(x_0) < \varepsilon$.

(ii) If

$$\liminf_{a\to\infty} \; \sup\left\{ \tilde{u}(x) \;\middle|\; x \in \Gamma_a \right\} \le 0,$$

then no comparison function is needed. For, there is a sequence $\big(a(n)\big)$ tending to infinity for which the supremum of $\tilde{u}(x)$ over $\Gamma_{a(n)}$ tends to a non-positive limit. We choose $a(k)$ so large that $x_0 \in D_{a(k)}$ and so large that $\tilde{u}(x) < \varepsilon$ on $\Gamma_{a(k)}$. Then the weak maximum principle, Theorem D.11, applied to $\tilde{u}$ on $\overline{D}_{a(k)}$ shows that $\tilde{u}(x_0) < \varepsilon$ as desired.

(iii) It remains to consider the following case: there is a number $A \ge 0$ such that

$$\sup\left\{ \tilde{u}(x) \;\middle|\; x \in \Gamma_a \right\} > 0 \quad \text{whenever} \quad a > A,$$

and

$$\liminf_{a\to\infty} \; \sup\left\{ \frac{\tilde{u}(x)}{V(x,a)} \;\middle|\; x \in \Gamma_a \right\} = 0. \tag{2.46}$$

Let

$$s(a) := \sup\left\{ \frac{\tilde{u}(x)}{V(x,a)} \;\middle|\; x \in \Gamma_a \right\} \quad \text{for} \quad a > A;$$

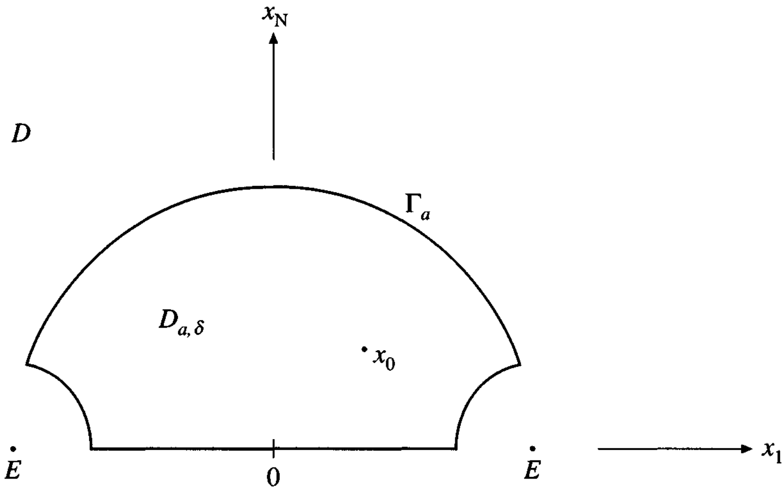

Fig. 2.8.

then $s(a) > 0$. We so choose a that, at the given point x_0,

$$s(a)\ V(x_0, a) < \tfrac{1}{2}\varepsilon; \tag{2.47}$$

this can be done because (2.46) states that there is a sequence $\big(a(n)\big)$ for which $s\big(a(n)\big) \to 0$ as $n \to \infty$ and $a(n) \to \infty$, while hypothesis (c) ensures that $V\big(x_0, a(n)\big) \le \lambda(x_0)$ whenever $a(n) \ge 2|x_0|$. With a now fixed at this value, define

$$\begin{aligned} \varphi(x) &:= \tilde{u}(x) - s(a)\ V(x, a) \qquad \text{for } x \in \overline{D}_a \setminus E, \\ D_{a,\delta} &:= \big\{x \in D \ \big|\ |x| < a,\ \operatorname{dist}(x, E) > \delta > 0\big\} \end{aligned}$$

(see Figure 2.8). Choose δ so small that $x_0 \in D_{a,\delta}$ and so small that

$$\operatorname{dist}(x, E) = \delta \ \text{ and } \ x \in \overline{D} \ \Rightarrow\ \tilde{u}(x) < \tfrac{1}{2}\varepsilon;$$

this choice is possible because $\tilde{u}\big|_E \le 0$ and $\tilde{u} \in C(\overline{D})$.

(iv) Finally, we apply the weak maximum principle (Theorem 2.11) to φ on $\overline{D}_{a,\delta}$. Certainly $\varphi \in C(\overline{D}_{a,\delta})$, and φ is subharmonic in $D_{a,\delta}$, because $\tilde{u}$ is subharmonic there and $V(., a)$ is harmonic.

The boundary values of φ are as follows. On $\partial D_{a,\delta} \cap \partial D$ we have $\tilde{u}(x) \le 0$ and $V(x, a) \ge 0$, hence $\varphi(x) \le 0$. On the part of $\partial D_{a,\delta}$ distant δ from E we have $\tilde{u}(x) < \tfrac{1}{2}\varepsilon$ and $V(x, a) \ge 0$, hence $\varphi(x) < \tfrac{1}{2}\varepsilon$. On

$\partial D_{a,\delta} \cap \Gamma_a$ we have $\varphi(x) \leq 0$ by the definition of $s(a)$:

$$x \in \Gamma_a \Rightarrow \varphi(x) = V(x,a) \left\{ \frac{\tilde{u}(x)}{V(x,a)} - \sup_{y \in \Gamma_a} \frac{\tilde{u}(y)}{V(y,a)} \right\} \leq 0.$$

Therefore the weak maximum principle implies that $\varphi(x) < \frac{1}{2}\varepsilon$ on $\overline{D}_{a,\delta}$; it follows from (2.47) that

$$\tilde{u}(x_0) = \varphi(x_0) + s(a)V(x_0, a) < \varepsilon,$$

as desired. □

The next item is a naive application of Lemma 2.22, based on a simple comparison function and intended to make Definition 2.21 less mysterious. In this example, V is independent of a, and does not have a discontinuity on the equator E. The full force of Lemma 2.22 will emerge only in §2.7, after more elaborate comparison functions have been constructed.

Example 2.23 *Let $D := \{ x \in \mathbb{R}^2 \mid x_2 > 0 \}$. If $u \in C(\overline{D})$, if u is subharmonic in D and if, for some constant $\beta \in (0,1)$,*

$$\liminf_{a \to \infty} \sup \{ a^{-\beta} u(x) \mid x \in \Gamma_a \} \leq 0 \tag{2.48}$$

(in particular, if $u(x) = o(r^\beta)$ for some $\beta \in (0,1)$ as $r := |x| \to \infty$), then

$$\sup_D u = \sup_{\partial D} u.$$

Proof Denote points of $\overline{D}$ by $x = (r\cos\theta, r\sin\theta)$, $0 \leq \theta \leq \pi$. We claim that the formula

$$V(x) := r^\beta \sin(\beta\theta + k), \qquad k := (1-\beta)\frac{\pi}{2}, \qquad x \in \overline{D},$$

defines a comparison function of the first kind. For, referring to Definition 2.21, we observe that $V \in C(\overline{D}) \cap C^2(D)$; that $V \geq 0$ on $\overline{D}$, with $V > 0$ on $\overline{D} \setminus \{0\}$, because

$$\sin(\beta\theta + k) \geq \sin k \quad \text{for} \quad 0 \leq \theta \leq \pi;$$

and that $\triangle V = 0$ in D because

$$V(x) = \operatorname{Im} e^{ik} z^\beta \qquad (z := x_1 + ix_2 = re^{i\theta}).$$

Thus V satisfies condition (a) of Definition 2.21; it satisfies (b) because $V(x) > a^\beta \sin k$ when $x \in \Gamma_a$; for (c), we may choose $\lambda(x_0) := V(x_0)$ or $\lambda(x_0) := r_0^\beta$.

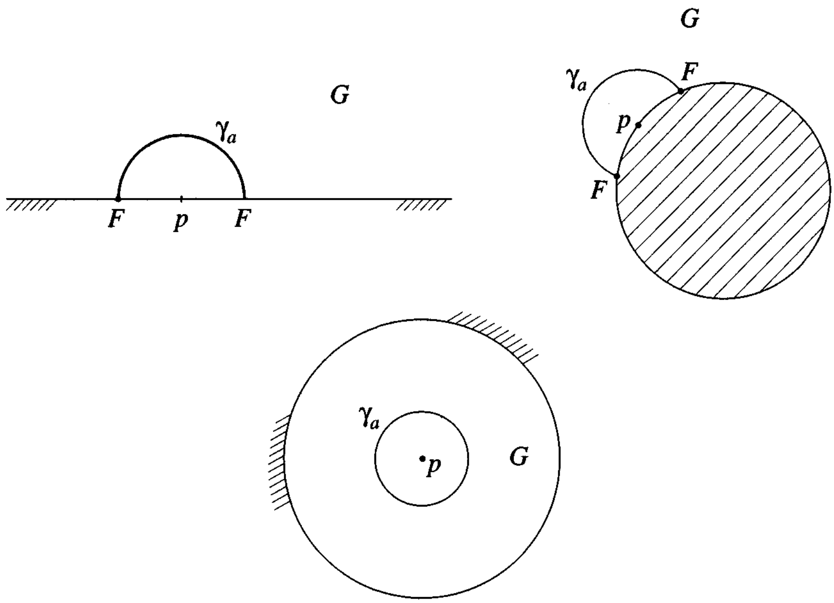

Fig. 2.9.

If the growth condition (2.48) implies (2.45) for the present function V, then Lemma 2.22 implies the present result. Now,

$$\sin k < \frac{V(x)}{a^\beta} \leq 1 \quad \text{when } x \in \Gamma_a,$$

so that the two growth conditions are equivalent. $\square$

Definition 2.24 Let G be a connected open set in $\mathbb{R}^N$, $N \geq 2$, let $p \in \partial G$ be given, let $G_a := G \backslash \overline{\mathscr{B}(p,a)}$, let $\gamma_a := G \cap \partial \mathscr{B}(p,a)$ and let $F := \partial G \cap \partial \mathscr{B}(p,a)$. Here $a \in (0, a_0)$ and a_0 is a positive constant depending only on G. (Three possible configurations are shown in Figure 2.9; in the third, $G := \mathscr{B}(p, a_0) \setminus \{p\}$ and F is empty.)

A function with values $W(x,a)$ will be called a *comparison function of the second kind* iff

(a) for each $a \in (0, a_0)$,

$$W(.,a) \in C(\overline{G}_a \setminus F) \cap C^2(G_a) \quad \text{and} \quad W(.,a) \geq 0 \quad \text{on } \overline{G}_a \setminus F, \tag{2.49}$$

$$(\triangle W)(.,a) = 0 \quad \text{in } G_a, \tag{2.50}$$

$$W(.,a) > 0 \quad \text{on } \gamma_a; \tag{2.51}$$

(b) $\inf\{\, W(x,a) \mid x \in \gamma_a \,\} \to \infty$ as $a \to 0$;

(c) there is a function $\lambda : G \to (0,\infty)$ such that $W(x_0,a) \leq \lambda(x_0)$ whenever $x_0 \in G$, $a \leq \frac{1}{2}|x_0 - p|$ and $a < a_0$. □

Lemma 2.25 *Let G be as in Definition 2.24 and let W be a comparison function of the second kind. Let Ω be a bounded open subset of G such that $p \in \partial\Omega \cap \partial G$. If $u \in C(\overline{\Omega} \setminus \{p\})$, if u is subharmonic in Ω and if*

$$\liminf_{a\to 0}\ \sup\left\{ \frac{u(x)}{W(x,a)} \,\middle|\, x \in \Omega \cap \partial\mathscr{B}(p,a) \right\} \leq 0, \tag{2.52}$$

then

$$\sup_{\Omega} u = \sup_{\partial\Omega\setminus\{p\}} u.$$

Proof The proof resembles that of Lemma 2.22, but to condense it ruthlessly would be a false economy.

(i) We may suppose that $\sup_{\partial\Omega\setminus\{p\}} u < \infty$, otherwise the result is trivial. Let $\tilde{u} := u - \sup_{\partial\Omega\setminus\{p\}} u$. Then $\sup_{\partial\Omega\setminus\{p\}} \tilde{u} = 0$ and $\tilde{u}$ also satisfies (2.52), because $u - \tilde{u}$ is a (finite) constant and by condition (b) in Definition 2.24.

Let both $x_0 \in \Omega$ and $\varepsilon > 0$ be given; we shall prove the lemma by showing that $\tilde{u}(x_0) < \varepsilon$. To this end, write

$$\Omega_a := \Omega \setminus \overline{\mathscr{B}(p,a)} \quad \text{and} \quad \sigma_a := \Omega \cap \partial\mathscr{B}(p,a).$$

(ii) If

$$\liminf_{a\to 0}\ \sup\{\, \tilde{u}(x) \mid x \in \sigma_a \,\} \leq 0,$$

then no comparison function is needed. We argue as in the proof of Lemma 2.22, using small surfaces $\sigma_{a(n)}$ with $a(n) \to 0$ instead of large hemispheres $\Gamma_{a(n)}$ with $a(n) \to \infty$.

(iii) It remains to consider the following case: there is a number $\alpha > 0$ such that

$$\sup\{\, \tilde{u}(x) \mid x \in \sigma_a \,\} > 0 \quad \text{whenever} \quad a < \alpha,$$

and

$$\liminf_{a\to 0}\ \sup\left\{ \frac{\tilde{u}(x)}{W(x,a)} \,\middle|\, x \in \sigma_a \right\} = 0. \tag{2.53}$$

Let

$$s(a) := \sup\left\{ \frac{\tilde{u}(x)}{W(x,a)} \,\middle|\, x \in \sigma_a \right\} \quad \text{for} \quad a < \alpha;$$

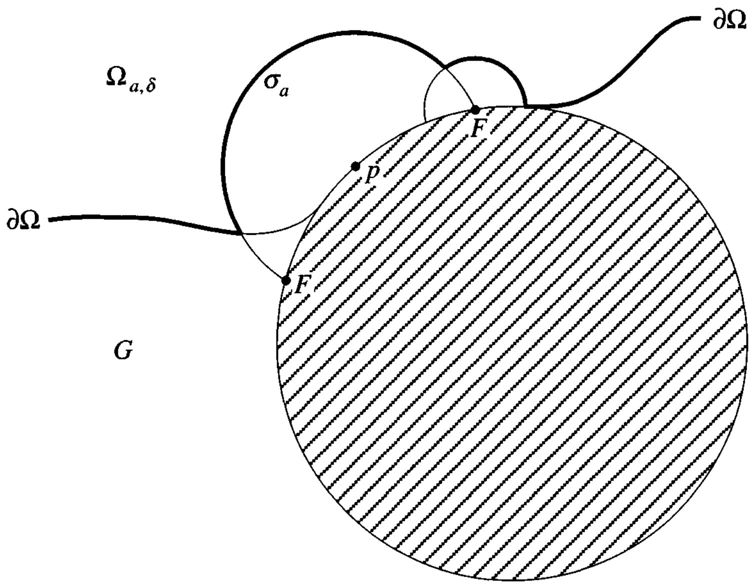

Fig. 2.10.

then $s(a) > 0$. Choose a to be such that, at the given point x_0,

$$s(a)\ W(x_0, a) < \tfrac{1}{2}\varepsilon; \tag{2.54}$$

this can be done because of (2.53) and hypothesis (c) in Definition 2.24. With a now fixed at this value, define

$$\psi(x) := \tilde{u}(x) - s(a)\, W(x, a) \qquad \text{for } \ x \in \overline{\Omega}_a \setminus F.$$

If $\partial\Omega$ intersects F, define

$$\Omega_{a,\delta} := \left\{ x \in \Omega \ \middle|\ |x - p| > a,\ \operatorname{dist}(x, F) > \delta > 0 \right\}$$

(Figure 2.10); choose δ so small that $x_0 \in \Omega_{a,\delta}$ and so small that

$$\operatorname{dist}(x, F) = \delta \ \text{ and } \ x \in \overline{\Omega} \ \Rightarrow\ \tilde{u}(x) < \tfrac{1}{2}\varepsilon;$$

this choice is possible because $\tilde{u}\big|_{\partial\Omega\cap F} \le 0$ and $\tilde{u} \in C\left(\overline{\Omega} \setminus \{p\}\right)$. If $\partial\Omega$ does not intersect F (in particular, if F is empty), define $\Omega_{a,\delta} := \Omega_a$. In either case, $W(.,a)$ is continuous on $\overline{\Omega}_{a,\delta}$ and $\tilde{u}(x) < \frac{1}{2}\varepsilon$ on $\partial\Omega_{a,\delta} \setminus \sigma_a$.

(iv) Now apply the weak maximum principle, Theorem 2.11, to ψ on $\overline{\Omega}_{a,\delta}$, observing that $\psi \in C(\overline{\Omega}_{a,\delta})$ and that ψ is subharmonic in $\Omega_{a,\delta}$. On $\partial\Omega_{a,\delta} \setminus \sigma_a$ we have $\tilde{u}(x) < \frac{1}{2}\varepsilon$ and $W(x,a) \ge 0$, hence $\psi(x) < \frac{1}{2}\varepsilon$. On

$\partial\Omega_{a,\delta} \cap \sigma_a$ we have $\psi(x) \le 0$ by the definition of $s(a)$. The maximum principle ensures that $\psi(x) < \frac{1}{2}\varepsilon$ on $\overline{\Omega}_{a,\delta}$; it follows from (2.54) that

$$\tilde{u}(x_0) = \psi(x_0) + s(a)\ W(x_0, a) < \varepsilon,$$

as desired. □

Like Example 2.23, our first application of Lemma 2.25 will involve only simple comparison functions. But the result is better than that of Example 2.23; it is not restricted to $\mathbb{R}^2$ and it is best possible in a certain sense (Exercise 2.44) when no smoothness is demanded of $\partial\Omega$ at p.

Theorem 2.26 *Let Ω be bounded in $\mathbb{R}^N$, $N \ge 2$, let $p \in \partial\Omega$ and write $\sigma_a := \Omega \cap \partial\mathscr{B}(p,a)$. Let b be so large that $\Omega \subset \mathscr{B}(p,b)$.*

If $u \in C(\overline{\Omega} \setminus \{p\})$, if u is subharmonic in Ω and if

$$\liminf_{a\to 0}\ \sup\left\{ \frac{u(x)}{\log(b/a)}\ \middle|\ x \in \sigma_a \right\} \le 0 \quad \textit{when} \quad N = 2, \tag{2.55}$$

$$\liminf_{a\to 0}\ \sup\left\{ a^{N-2}u(x)\ \middle|\ x \in \sigma_a \right\} \le 0 \quad \textit{when} \quad N \ge 3, \tag{2.56}$$

then

$$\sup_\Omega u = \sup_{\partial\Omega\setminus\{p\}} u.$$

Proof The comparison functions are potentials of point sources (multiples of Newtonian kernels), discussed at some length in Appendix A.

(i) For $N = 2$, we choose $G := \mathscr{B}(p,b) \setminus \{p\}$ for the set in Definition 2.24 and define

$$W(x) := \log\frac{b}{|x-p|} \quad \text{for} \quad x \in \overline{\mathscr{B}(p,b)} \setminus \{p\}.$$

Then $G_a = \mathscr{B}(p,b) \setminus \overline{\mathscr{B}(p,a)}$ and $\gamma_a = \partial\mathscr{B}(p,a)$; condition (a) of Definition 2.24 is satisfied. Since $W(x) = \log(b/a)$ when $x \in \gamma_a$, condition (b) holds. For (c), we choose $\lambda(x_0) := W(x_0)$. Thus W is a comparison function of the second kind. The growth conditions (2.52) and (2.55) coincide for this function W; therefore Lemma 2.25 implies the present result.

(ii) For $N \ge 3$, we choose $G := \mathbb{R}^N \setminus \{p\}$ for the set in Definition 2.24 and define

$$W(x) := |x-p|^{-N+2} \qquad \text{for} \quad x \in \mathbb{R}^N \setminus \{p\}.$$

One checks without difficulty, very much as in (i), that this function W is a comparison function of the second kind. The growth conditions (2.52) and (2.56) coincide for this W; again Lemma 2.25 implies the present result. □

2.6 Comparison functions of Siegel type

This section concerns functions $g(.\,;a)$, $g_e(.\,;a)$ and $g_2(.\,;a,b)$ with the property that $ag(.\,;a)$ is a useful comparison function of the first kind, while $a^{-N+1}g_e(.\,;a)$ and $a^{-N+1}g_2(.\,;a,b)$ are corresponding comparison functions of the second kind; $g_e(.\,;a)$ and $g_2(.\,;a,b)$ are defined on different domains. These functions will enable us to extend Example 2.23 to half-spaces in $\mathbb{R}^N$ for all $N \geq 2$; to improve the rate of growth allowed in Example 2.23 from approximately $o(r^\beta)$, where $r := |x|$ and $\beta < 1$, to approximately $o(r^2/x_N)$ as $r \to \infty$; and to improve the rate of growth allowed in Theorem 2.26 from approximately $o(|x-p|^{-N+2})$, for $N \geq 3$ and $x \to p$, to something slightly bigger than $o(|x-p|^{-N+1})$, provided that Ω has the exterior-ball property at p (Definition 2.14).

The functions g, g_e and g_2 will be called *of Siegel type* because g for $N = 2$, displayed here in (2.59), was introduced into Phragmén–Lindelöf theory by D. Siegel (1988). The construction of g for all $N \geq 2$, from the Poisson integral formula for functions harmonic in a ball (§B.5), is the subject of Appendix C. The functions g_e and g_2 result from applications to g of the Kelvin transformation (§B.3).

It will be convenient to use the *signum function*, defined by

$$\operatorname{sgn} t := \begin{cases} -1 & \text{if} \quad t < 0, \\ 0 & \text{if} \quad t = 0, \\ 1 & \text{if} \quad t > 0. \end{cases} \tag{2.57}$$

Theorem 2.27 *Let $B := \mathscr{B}(0,a)$ in $\mathbb{R}^N$, $N \geq 2$, and let $E := \{ x \in \partial B \mid x_N = 0 \}$ denote the equator of B.*

(a) *There exists a function $g = g(.\,;a)$, which we call the primary function of Siegel type, such that $g \in C(\overline{B} \setminus E) \cap C^\infty(B)$ and*

$$\begin{aligned} \triangle g &= 0 \quad \textit{in} \quad B, && (2.58a) \\ g(x) &= a/x_N \quad \textit{on} \quad \partial B \setminus E, && (2.58b) \\ |g(x)| &\leq \text{const.}\, |x_N|/a \quad \textit{if} \quad r := |x| \leq a/2, && (2.58c) \end{aligned}$$

where the constant depends only on N. Also, $\operatorname{sgn} g(x) = \operatorname{sgn} x_N$ *on $\overline{B} \setminus E$, and $g(x;a)$ depends only on x/a.*

(b) *For $N = 2$, let $(x,y) \in \mathbb{R}^2$ and $z = x + iy \in \mathbb{C}$. Then, on $\overline{B} \setminus E \subset \mathbb{R}^2$,*

$$g(x,y;a) = \operatorname{Im}\left(\frac{a}{a-z} - \frac{a}{a+z}\right) = \frac{ay}{(a-x)^2+y^2} + \frac{ay}{(a+x)^2+y^2}. \quad (2.59)$$

Proof See Appendix C. □

Corollary 2.28 *The exterior function $g_e = g_e(.\,;a)$ of Siegel type is defined by*

$$g_e(x;a) := \left(\frac{a}{r}\right)^{N-2} g\left(\frac{a^2}{r^2}x;a\right), \quad x \in \mathbb{R}^N \setminus (B \cup E), \quad (2.60)$$

where again $r := |x|$ and B, E are as in Theorem 2.27. It follows that $g_e \in C\left(\mathbb{R}^N \setminus \{B \cup E\}\right) \cap C^\infty(\mathbb{R}^N \setminus \overline{B})$ and that

$$\begin{aligned} \triangle g_e &= 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}, && (2.61a) \\ g_e(x) &= a/x_N \quad \text{on } \partial B \setminus E, && (2.61b) \\ |g_e(x)| &\le \text{const.}\, a^{N-1} r^{-N} |x_N| \quad \text{if } r \ge 2a, && (2.61c) \end{aligned}$$

where the constant depends only on N. Also, $\operatorname{sgn} g_e(x) = \operatorname{sgn} x_N$ on $\mathbb{R}^N \setminus (B \cup E)$, and $g_e(x;a)$ depends only on x/a.

Proof A calculation, done fully in Theorem B.15, shows that

$$\triangle g_e(x) = \left(\frac{a}{r}\right)^{N+2} (\triangle g)\left(\frac{a^2}{r^2}x\right) = 0$$

if $r > a$ and hence $|a^2 x/r^2| = a^2/r < a$. The remaining properties of g_e are immediate consequences of the definition (2.60) and the corresponding properties of g. □

Inspection of Corollary 2.28 and Definition 2.24 shows that $a^{-N+1} g_e$, restricted to $\overline{D} \setminus (B \cup E)$ (where D is our usual half-space), is a comparison function of the second kind, with $G = D$ and $p = 0$ in the notation of Definition 2.24. The restriction to $\overline{D} \setminus (B \cup E)$ is needed in order that $g_e \ge 0$. Therefore $a^{-N+1} g_e$ can be used in Lemma 2.25 for sets Ω that are on one side of a hyperplane containing the point p of $\partial\Omega$ at which u may be discontinuous; this is illustrated in Figure 2.11. In particular, $a^{-N+1} g_e$ can be used for convex sets Ω.

Suppose now that Ω has merely the exterior-ball property (Definition 2.14) at the specified point $p \in \partial\Omega$. Then a suitable comparison function $a^{-N+1} g_2(.\,;a,b)$ is found by inversion relative to a sphere as follows.

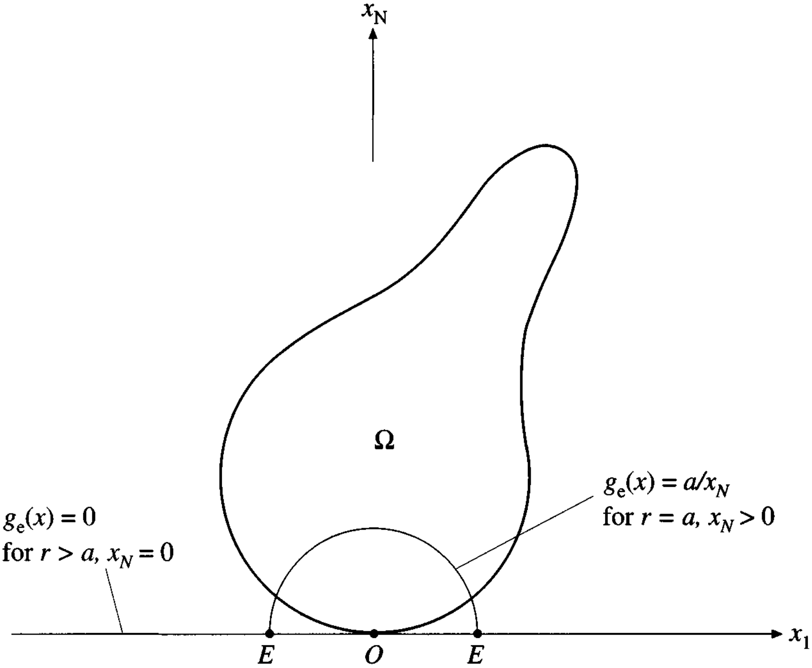

Fig. 2.11.

Choose co-ordinates so that p and an exterior ball B_q at p are given by

$$p = (0, \dots, 0, 2b), \quad q = (0, \dots, 0, b), \quad B_q = \mathscr{B}(q, b) \tag{2.62}$$

for some $b > 0$, as shown in the right half of Figure 2.12. Define

$$V := \left\{ \xi \in \mathbb{R}^N \;\middle|\; \xi_N < 2b \right\}, \quad V_a := V \setminus \overline{\mathscr{B}(p,a)} \text{ with } 0 < a < b;$$

we shall use $g_e(p - \xi; a)$ for $\xi \in \overline{V}_a \setminus E$, where E now denotes the equator of $\mathscr{B}(p, a)$. Under the transformation

$$\left.\begin{aligned} &\xi = \frac{4b^2}{r^2}x \quad (r := |x| > 0), \\ \text{equivalently} \qquad &x = \frac{4b^2}{\rho^2}\xi \quad (\rho := |\xi| > 0), \end{aligned}\right\} \tag{2.63}$$

which is inversion relative to the sphere $\partial\mathscr{B}(0, 2b)$,

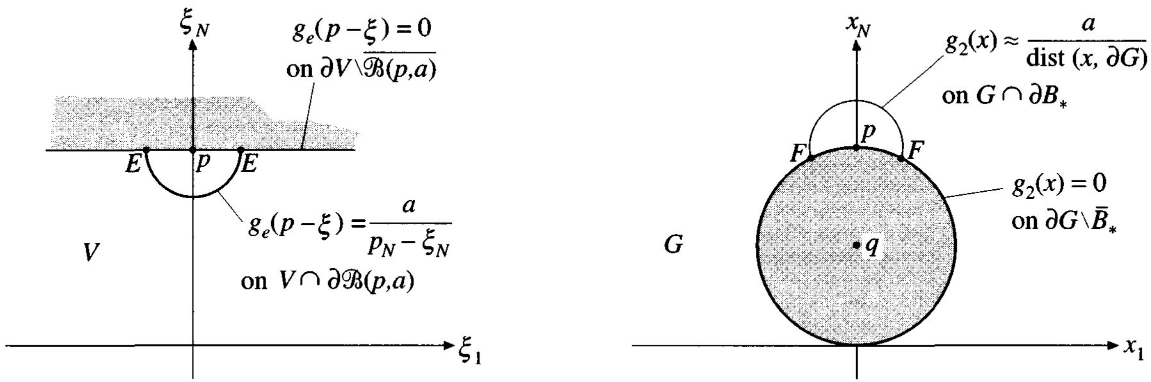
ξ_N
$g_e(p-\xi)=0$
on $\partial V \setminus \overline{\mathcal{B}(p,a)}$
E p E
$g_e(p-\xi)=\dfrac{a}{p_N-\xi_N}$
on $V \cap \partial\mathcal{B}(p,a)$
V
ξ_1
x_N
$g_2(x) \approx \dfrac{a}{\text{dist}\,(x, \partial G)}$
on $G \cap \partial B_*$
F p F
$g_2(x)=0$
on $\partial G \setminus \bar{B}_*$
G
q
x_1

Fig. 2.12.

$$\text{the punctured half-space } V \setminus \{0\} \text{ has image } G := \mathbb{R}^N \setminus \overline{B}_q; \tag{2.64a}$$

$$\left.\begin{array}{l} \text{the ball } \mathscr{B}(p,a) \text{ has image } B_* := \mathscr{B}(p_*, a_*), \\ \text{where} \quad p_* := \dfrac{1}{1-(a/2b)^2} p, \qquad a_* := \dfrac{1}{1-(a/2b)^2} a; \end{array}\right\} \tag{2.64b}$$

$$\text{the equator } E \text{ has image } F := \partial G \cap \partial B_*. \tag{2.64c}$$

Note that p is a fixed point of the map (2.63) and that, near p, this transformation is approximately reflection in the hyperplane ∂V. Therefore ∂B_* is close to $\partial \mathscr{B}(p,a)$ for small radii a. A more precise statement is that

$$p_N - \xi_N = \frac{4b^2}{r^2} \left(x_N - p_N + \frac{|x-p|^2}{2b} \right). \tag{2.65}$$

Here the factor $4b^2/r^2$ will be unimportant when g_2 comes to be used in Theorem 2.36 [because $4b^2/r^2 \to 1$ as $x \to p$], but we shall need

$$z(x) := x_N - p_N + \frac{|x-p|^2}{2b}, \tag{2.66}$$

which is almost $\operatorname{dist}(x, \partial G)$ for points x near ∂G. In fact, a calculation shows that

$$z\big|_{\partial G} = 0, \quad 1 - \frac{1}{2}\frac{z(x)}{b} \le \frac{\operatorname{dist}(x, \partial G)}{z(x)} \le 1 \quad \text{for all } x \in G, \tag{2.67a,b}$$

and

$$\max_{x \in \partial B_*} z(x) = a\left(1 - \frac{a}{2b}\right)^{-2}. \tag{2.67c}$$

Corollary 2.29 *Let g_e be as in Corollary 2.28. We adopt the notation (2.62), (2.64a,b,c) and (2.66) for any $b > 0$ and any $a \in (0,b)$. Then the two-ball function $g_2 = g_2(.\,; a,b)$ of Siegel type is defined by*

$$g_2(x;a,b) := \left(\frac{2b}{r}\right)^{N-2} g_e\left(p - \frac{4b^2}{r^2}x; a\right), \quad x \in \overline{G} \setminus \left(B_* \cup F \cup \{0\}\right) \tag{2.68}$$

and by $g_2(0) := \lim_{r \downarrow 0} g_2(x) = 0$. It follows that $g_2 \in C\left(\overline{G} \setminus \left\{B_* \cup F\right\}\right) \cap C^\infty(G \setminus \overline{B}_*)$ and that

$$\Delta g_2 = 0 \quad \textit{in} \quad G \setminus \overline{B}_*, \tag{2.69a}$$

$$g_2(x) = \left(\frac{2b}{r}\right)^{N-4} \frac{a}{z(x)} \quad \textit{on} \quad G \cap \partial B_*, \tag{2.69b}$$

$$\left.\begin{aligned} &\textit{if } x \in \overline{G} \setminus \mathscr{B}(p', 2a'), \quad \textit{where} \quad p' := \frac{1}{1-(a/b)^2}p, \ a' := \frac{1}{1-(a/b)^2}a, \\ &\textit{then } |g_2(x)| \le \text{const.}\, a^{N-1}\left(\frac{2b}{r}\right)^{N-2}\left|p - \frac{4b^2}{r^2}x\right|^{-N}\left|p_N - \frac{4b^2}{r^2}x_N\right|, \end{aligned}\right\} \tag{2.69c}$$

where the constant depends only on N. Also, $g_2(x) = 0$ on $\partial G \setminus \overline{B}_*$, and $g_2(x) > 0$ in $G \setminus \overline{B}_*$.

Proof We have $\triangle g_2 = 0$ in $G \setminus \overline{B}_*$ by Theorem B.15, already cited in the proof of Corollary 2.28. The remaining properties of g_2 follow from those of g_e by direct calculation. □

2.7 Some Phragmén–Lindelöf theory for subharmonic functions

We return to the half-space $D := \left\{ x \in \mathbb{R}^N \;\middle|\; x_N > 0 \right\}$, $N \ge 2$.

Theorem 2.30 *If $u \in C(\overline{D})$, if u is subharmonic in D and if*

$$\liminf_{a\to\infty} \max\left\{ \frac{x_N\ u(x)}{a^2} \;\middle|\; x \in \overline{D}, \quad |x| = a \right\} = 0, \tag{2.70}$$

then

$$\sup_D u = \sup_{\partial D} u.$$

Proof Let g continue to denote the primary function of Siegel type (Theorem 2.27), and let $V(.,a) := ag(.\,;a)$ on $\overline{D}_a \setminus E$, where $D_a := D \cap \mathscr{B}(0,a)$ and $E := \partial D \cap \partial\mathscr{B}(0,a)$. Theorem 2.27 shows that this V is a comparison function of the first kind (Definition 2.21); in particular,

$$V(x,a) = \frac{a^2}{x_N} \ge a \qquad \text{for} \;\; x \in \Gamma_a := D \cap \partial\mathscr{B}(0,a),$$

and

$$V(x_0,a) \le \;\text{const.}\, x_{0N} \qquad \text{for} \;\; x_0 \in D \;\; \text{and} \;\; a \ge 2|x_0|,$$

where the constant depends only on N.

Therefore the present theorem is implied by Lemma 2.22. The supremum over Γ_a in (2.45) of that lemma can now be written as a maximum over $\overline{\Gamma}_a$ because the function with values $x_N\, u(x)/a^2$ is continuous on $\overline{D}$; the maximum cannot be negative, because of values for $x_N = 0$. □

Corollary 2.31 *Let G be an unbounded open subset of the half-space D. If $u \in C(\overline{G})$, if u is subharmonic in G and if*

$$\liminf_{a\to\infty} \max\left\{ \frac{x_N\, u(x)}{a^2} \;\middle|\; x \in \overline{G},\quad |x| = a \right\} \le 0, \tag{2.71}$$

then

$$\sup_G u = \sup_{\partial G} u.$$

Proof In the proof of Lemma 2.22 we replace D by G, keeping the same comparison function V of the first kind. Thus D_a is replaced by $G_a := G \cap \mathscr{B}(0,a)$ and Γ_a now means $G \cap \partial\mathscr{B}(0,a)$. We define $s(a)$ and choose the radius a exactly as before. If ∂G does not intersect the equator E, we need not remove a neighbourood of E from $\overline{G_a}$. The shape of ∂G is unimportant; what matters is that $\tilde{u}(x) \le 0$ on ∂G by the definition of $\tilde{u}$, and that $\varphi(x) \le 0$ on Γ_a by the definition of $s(a)$.

After this extension of Lemma 2.22, the present corollary results from the choice $V = ag$ made in the proof of Theorem 2.30. The maximum in the growth condition (2.71) may be negative once more, because x_N need not descend to zero when $x \in \overline{G}$ and $|x| = a$. □

Remark 2.32 (i) *If we add to Corollary* 2.31 *the hypotheses: G is connected and $u \in C^1(G)$, then*

$$u(x) < \sup_{\partial G} u \quad \textit{for all } x \in G,$$

unless u is constant on $\overline{G}$. This is an immmediate consequence of the strong maximum principle (Theorem 2.13).

(ii) Let G be as in Corollary 2.31. *If $u \in C(\overline{G})$, if u is superharmonic in G and if*

$$\limsup_{a\to\infty} \min\left\{ \frac{x_N\, u(x)}{a^2} \;\middle|\; x \in \overline{G},\quad |x| = a \right\} \ge 0, \tag{2.72}$$

then

$$\inf_G u = \inf_{\partial G} u.$$

This follows from Corollary 2.31 by an argument like that in Remark 2.7.

(iii) For a function u that is continuous on $\overline{G}$ and *harmonic in G* (hence is in $C^\infty(G)$, by Theorem B.6), we wish to conclude that

$$\inf_{\partial G} u \le u(x) \le \sup_{\partial G} u \quad \text{for all } x \in \overline{G}$$

(with strict inequality for $x \in G$ if G is connected and u is not a constant). It may be worthwhile to retain both (2.71) and (2.72) as hypotheses, but the simpler condition

$$\liminf_{a \to \infty} \max \left\{ \frac{x_N |u(x)|}{a^2} \,\middle|\, x \in \overline{G}, \quad |x| = a \right\} = 0 \tag{2.73}$$

is sufficient. □

When a subset of D is significantly narrower near infinity than is D itself, much larger rates of growth are permissible. Corollary 2.31 is then far from sharp. We illustrate this by two examples; observe that, just as Theorem 2.30 extends to unbounded open subsets of D, so Examples 2.33 and 2.34 extend to unbounded open subsets of S and of H, respectively.

Example 2.33 *Consider the sector*

$$S := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,\middle|\, r > 0, \quad 0 < \theta < \beta \right\}, \qquad \beta \in (0, 2\pi).$$

If $u \in C(\overline{S})$, if u is subharmonic in S and if

$$\liminf_{R \to \infty} \max \left\{ \frac{\sin(\pi\theta/\beta)\ u(x,y)}{R^{\pi/\beta}} \,\middle|\, (x,y) \in \overline{S}, \quad |(x,y)| = R \right\} = 0, \tag{2.74}$$

then

$$\sup_S u = \sup_{\partial S} u.$$

Proof This statement is merely a transcription of Theorem 2.30, for $N = 2$, under the conformal map of S onto D. Write $z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$ for points of $\overline{S}$, and $\zeta = \xi + \mathrm{i}\eta = \rho\mathrm{e}^{\mathrm{i}t}$ for points of $\overline{D}$; the appropriate mapping is

$$\left.\begin{aligned} \zeta &= z^{\pi/\beta}, \\ \text{equivalently} \quad \rho &= r^{\pi/\beta}, \quad t = \pi\theta/\beta, \end{aligned}\right\} \quad r \geq 0,\ 0 \leq \theta \leq \beta. \tag{2.75}$$

This is a homeomorphism of the closed sector $\overline{S}$ onto the closed half-plane $\overline{D}$; it is also a C^∞ map, with C^∞ inverse, of $\overline{S} \setminus \{0\}$ onto $\overline{D} \setminus \{0\}$.

Let $\hat{u}(\xi,\eta) := u(x(\xi,\eta), y(\xi,\eta))$ under the mapping (2.75). Then $\hat{u} \in C(\overline{D})$ because $u \in C(\overline{S})$. Also, $\hat{u}$ satisfies the growth condition (2.70) because u satisfies (2.74). We now show that $\hat{u}$ is subharmonic in D. Given $\hat{\varphi} \in C_c^\infty(D)$ satisfying $\hat{\varphi} \geq 0$, define $\varphi(x,y) := \hat{\varphi}(\xi(x,y), \eta(x,y))$. Then $\varphi \in C_c^\infty(S)$, $\varphi \geq 0$ and

$$\hat{\varphi}_{\xi\xi} + \hat{\varphi}_{\eta\eta} = \frac{\varphi_{xx} + \varphi_{yy}}{|\ \mathrm{d}\zeta / \mathrm{d}z\ |^2}, \qquad \mathrm{d}\xi\ \mathrm{d}\eta = \left| \frac{\mathrm{d}\zeta}{\mathrm{d}z} \right|^2 \mathrm{d}x\ \mathrm{d}y,$$

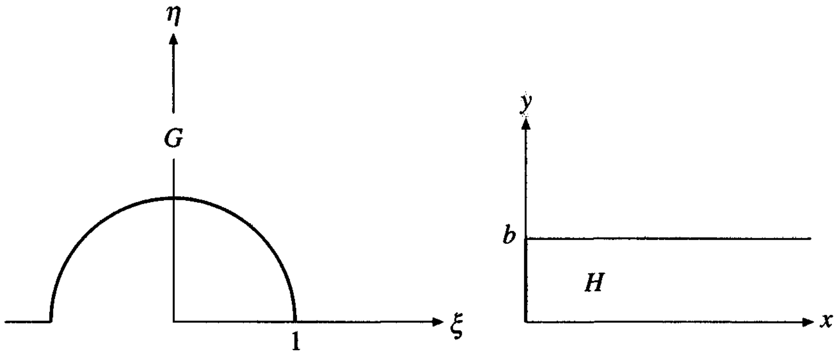

Fig. 2.13.

so that

$$\int\!\!\int_D (\hat{\varphi}_{\xi\xi} + \hat{\varphi}_{\eta\eta})\, \hat{u}\; d\xi\; d\eta = \int\!\!\int_S (\varphi_{xx} + \varphi_{yy})\, u\; dx\; dy \geq 0.$$

The result $\sup_D \hat{u} = \sup_{\partial D} \hat{u}$ now implies that $\sup_S u = \sup_{\partial S} u$. □

If we allow $\beta = 2\pi$ in the definition of S, then the foregoing result remains true even though $\overline{S} = \mathbb{R}^2$ and (2.75) is no longer a homeomorphism of $\overline{S}$ onto $\overline{D}$. Indeed, the weaker condition $\hat{u} \in C(\overline{D})$ can replace $u \in C(\overline{S})$, provided that (2.74) is replaced by

$$\liminf_{R\to\infty} \sup \left\{ \frac{\sin(\theta/2)\; u(x,y)}{R^{1/2}} \;\middle|\; (x,y) \in S, \quad |(x,y)| = R \right\} = 0 \quad (2.76)$$

for $\beta = 2\pi$. The condition $\hat{u} \in C(\overline{D})$ is weaker for $\beta = 2\pi$ in that it allows limiting values $u(x, 0+)$ as $y \downarrow 0$ and $u(x, 0-)$ as $y \uparrow 0$ such that $u(x, 0+) \neq u(x, 0-)$ for $x > 0$. When contemplating $\sup_{\partial S} u$, we must then regard the upper and lower sides of ∂S as distinct.

Example 2.34 *Consider the half-strip $H := (0, \infty) \times (0, b)$ in $\mathbb{R}^2$. If $u \in C(\overline{H})$, if u is subharmonic in H and if*

$$\liminf_{c\to\infty} \max \left\{ \frac{\sin(\pi y/b)\; u(c,y)}{\exp(\pi c/b)} \;\middle|\; 0 \leq y \leq b \right\} = 0, \qquad (2.77)$$

then

$$\sup_H u = \sup_{\partial H} u.$$

Proof This result is implied by Corollary 2.31 and the conformal map of H onto $G := D \setminus \overline{\mathscr{B}(0,1)}$; as in Theorem 2.30, the maximum in the growth condition cannot be negative [because the function in question is zero for $y = 0, b$]. Again let $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta = \rho e^{\mathrm{i}t}$; the relevant map is now (Figure 2.13)

$$\left.\begin{aligned} &\zeta = \exp\frac{\pi z}{b}, \\ \text{equivalently} \quad &\rho = \exp\frac{\pi x}{b}, \quad t = \frac{\pi y}{b}, \end{aligned}\right\} \quad x \geq 0,\ 0 \leq y \leq b.$$

The rest is essentially as in Example 2.33. □

Finally, we derive two more results for subharmonic functions that may be discontinuous at $p \in \partial\Omega$; as was promised earlier, these theorems allow a rate of growth larger than that in Theorem 2.26, for certain boundaries $\partial\Omega$.

Theorem 2.35 *Let Ω be bounded in $\mathbb{R}^N$, $N \geq 2$, let $p \in \partial\Omega$ and assume that Ω is on one side of a hyperplane A containing p. (Figure 2.11 shows a case with $p = 0$ and $A = \{ x \mid x_N = 0 \}$.) Let $d_A(x) := \operatorname{dist}(x, A)$.*

If $u \in C(\overline{\Omega} \setminus \{p\})$, if u is subharmonic in Ω and if

$$\liminf_{a\to 0} \max\left\{ a^{N-2} d_A(x)\, u(x) \;\middle|\; x \in \overline{\Omega} \cap \partial\mathscr{B}(p,a) \right\} \leq 0, \tag{2.78}$$

then

$$\sup_\Omega u = \sup_{\partial\Omega\setminus\{p\}} u.$$

Proof Choose co-ordinates so that $p = 0$, $A = \{ x \mid x_N = 0 \}$ and Ω lies in the half-space D; then $d_A(x) = x_N$. With g_e denoting the exterior function of Siegel type (Corollary 2.28), define $W(.\,, a) := a^{-N+1} g_e(.\,; a)$ on $\overline{D} \setminus (B \cup E)$, where $B := \mathscr{B}(0, a)$ and $E := \partial D \cap \partial B$. Then Corollary 2.28 shows W to be a comparison function of the second kind (Definition 2.24); in particular

$$W(x,a) = \frac{a^{-N+2}}{x_N} \geq a^{-N+1} \qquad \text{for} \quad x \in \gamma_a := D \cap \partial B,$$

and

$$W(x_0, a) \;\leq\; \text{const.}\, |x_0|^{-N} x_{0N} \qquad \text{for} \;\; x_0 \in D \;\; \text{and} \;\; a \leq \tfrac{1}{2}|x_0|,$$

where the constant depends only on N.

Accordingly, the theorem follows from Lemma 2.25. The supremum over $\Omega \cap \partial\mathscr{B}(p,a)$ in (2.52) of that lemma can now be written as a maximum over $\overline{\Omega} \cap \partial\mathscr{B}(p,a)$ because the function with values $a^{N-2}d_A(x)\,u(x)$ is continuous on $\overline{\Omega} \setminus \{p\}$. □

In the next theorem, the growth condition (2.79) may seem absurd because of the elaborate p_*, a_* and because of the detailed knowledge of u that seems to be assumed. However, as with other growth conditions that we have met, there are simpler statements that imply (2.79). For example, if $u(x) = o(|x-p|^{-N+1})$ as $x \to p$, or if d_B denotes distance to an exterior ball at p and $u(x) = o\left(|x-p|^{-N+2}/d_B(x)\right)$ as $x \to p$, then (2.79) is amply satisfied.

Theorem 2.36 *Let Ω be bounded in $\mathbb{R}^N$, $N \geq 2$, and assume that Ω has the exterior-ball property at $p \in \partial\Omega$ (Definition 2.14). Let $\mathscr{B}(q,b)$, with $b = |p-q|$, be an exterior ball at p such that $2q - p \notin \overline{\Omega}$ (Figure 2.14). Let $d_B(x) := \operatorname{dist}\big(x, \mathscr{B}(q,b)\big)$.*

If $u \in C(\overline{\Omega} \setminus \{p\})$, if u is subharmonic in Ω and if

$$\liminf_{a\to 0} \max\left\{ a^{N-2}d_B(x)\,u(x) \;\middle|\; x \in \overline{\Omega} \cap \partial\mathscr{B}(p_*, a_*) \right\} \leq 0, \qquad (2.79)$$

where

$$p_* := p + \frac{1}{2}\,\frac{(a/b)^2}{1-(a/2b)^2}(p-q), \quad a_* := \frac{a}{1-(a/2b)^2}, \qquad (2.80)$$

then

$$\sup_\Omega u = \sup_{\partial\Omega\setminus\{p\}} u.$$

Proof (i) We make two changes in Definition 2.24 and Lemma 2.25. (Readers who distrust such tinkering with previous results may prefer to prove the theorem by means of Exercise 2.45.) First, the ball $\mathscr{B}(p,a)$ is replaced by $\mathscr{B}(p_*, a_*)$, where p_* and a_* are as in (2.80). Second, the condition $a \leq \frac{1}{2}|x_0 - p|$, which accompanies the inequality $W(x_0, a) \leq \lambda(x_0)$ in (c) of Definition 2.24, is replaced by

$$\left.\begin{aligned} a' &\leq \tfrac{1}{2}\,|x_0 - p'|, \\ \text{where} \quad p' &:= p + 2\frac{(a/b)^2}{1-(a/b)^2}(p-q), \\ a' &:= \frac{a}{1-(a/b)^2}. \end{aligned}\right\} \qquad (2.81)$$

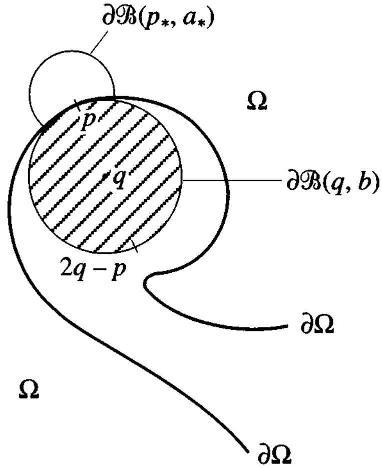

Fig. 2.14.

Here b is fixed and, in the proof of Lemma 2.25, the radius a is always chosen to be so small that certain inequalities hold. Such choices are not thwarted by the new perturbation terms in p_*, a_*, p' and a', so that Lemma 2.25 remains valid [with $\partial\mathscr{B}(p_*, a_*)$ replacing $\partial\mathscr{B}(p, a)$ in (2.52)].

(ii) Choose co-ordinates so that $p = (0, \ldots, 0, 2b)$ and $q = (0, \ldots, 0, b)$, as in Figure 2.12; Let $G := \mathbb{R}^N \setminus \overline{\mathscr{B}(q, b)}$ and $B_* := \mathscr{B}(p_*, a_*)$. Referring to Corollary 2.29, define

$$W_*(x, a) := a^{-N+1} g_2(x; a, b) \qquad \text{for } \; x \in \overline{G} \setminus \left(B_* \cup F\right).$$

Then W_* is a comparison function of the second kind, according to the modified Definition 2.24; from (2.69b) we obtain

$$W_*(x, a) = \left(\frac{2b}{|x|}\right)^{N-4} \frac{a^{-N+2}}{z(x)} \geq a^{-N+1} \left\{1 - \, O\left(\frac{a}{b}\right)\right\} \text{ for } \; x \in G \cap \partial B_*,$$

and (2.69c) implies a bound

$$W_*(x_0, a) \leq \lambda(x_0) \qquad \text{for } \; x_0 \in G \text{ and } \; a' \leq \tfrac{1}{2}\,|x_0 - p'|.$$

The growth conditions

$$\liminf_{a \to 0} \; \sup \left\{ \frac{u(x)}{W_*(x, a)} \;\middle|\; x \in \Omega \cap \partial B_* \right\} \leq 0 \qquad (2.82)$$

and (2.79) are equivalent because our choice of co-ordinates and (2.67) imply that

$$\frac{|x|}{2b} = 1 + O\left(\frac{a}{b}\right) \quad \text{and} \quad \frac{z(x)}{d_B(x)} = 1 + O\left(\frac{a}{b}\right) \quad \text{for } x \in G \cap \partial B_*,$$

and because the function with values $a^{N-2} d_B(x) u(x)$ is continuous on $\overline{\Omega} \setminus \{p\}$. Therefore the modified Lemma 2.25 implies the present theorem. □

2.8 Exercises

Exercise 2.37 Suppose that $\mathscr{B}(p,a) \subset \Omega \subset \mathscr{B}(q,b)$ in $\mathbb{R}^N$, and that $f := \Omega \to \mathbb{R}$ satisfies $0 \le k \le f(x) \le l$ for all $x \in \Omega$. Prove that, if it exists, the solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem $-\Delta u = f$ in Ω, $u\big|_{\partial\Omega} = 0$ is bounded by

$$\frac{k}{2N}\left(a^2 - |x-p|^2\right) \le u(x) \le \frac{l}{2N}\left(b^2 - |x-q|^2\right) \quad \text{for all } x \in \overline{\Omega}.$$

If only the foregoing information is given, can this estimate be improved?

Exercise 2.38 Assume that Ω is a bounded region, that u is subharmonic in Ω (Definition 2.20) and v superharmonic in Ω, that $u, v \in C(\overline{\Omega})$ and that $u\big|_{\partial\Omega} \le v\big|_{\partial\Omega}$. Prove that either $u(x) < v(x)$ for all $x \in \Omega$, or $u = v$.

Exercise 2.39 Let u be harmonic and continuous in a region Ω; then $u \in C^\infty(\Omega)$, by Theorem B.6. Prove that if $\sup_\Omega |\nabla u|$ is attained in Ω, then ∇u is a constant vector. Show by an example that, if $\inf_\Omega |\nabla u|$ is attained in Ω, then ∇u need not be a constant vector.

Exercise 2.40 Prove that, if $\partial\Omega$ is of class C^2, then Ω has the interior-ball property (Definition 2.14) at every boundary point.

Exercise 2.41 Writing $x = (r\cos\theta, r\sin\theta)$ for points of $\mathbb{R}^2$, consider the region $\Omega := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;\middle|\; r > 0,\ 0 < \theta < \beta \right\}$ for $\beta \in (0, 2\pi]$, and the function u defined by

$$u(x) := \begin{cases} -r^{\pi/\beta} \sin\dfrac{\pi\theta}{\beta} & \text{if } x \in \overline{\Omega} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that (a) $\triangle u = 0$ in Ω; (b) for $\beta \in (0, \pi)$ the boundary-point lemma cannot be applied at the origin, and its conclusion does not hold there; (c) for $\beta \in [\pi, 2\pi]$ the boundary-point lemma does indeed describe the behaviour of u near the origin.

What distinguishes the case $\beta = \pi$ in (c)?

Exercise 2.42 Let Ω be a bounded region with $\partial\Omega$ of class C^1 and with the interior-ball property (Definition 2.14) at every boundary point. The *Neumann problem* for L in Ω (where L is as in Definition 2.3) is to find v such that

$$Lv = f \text{ in } \Omega, \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = g, \qquad v \in C^1(\overline{\Omega}) \cap C^2(\Omega), \tag{2.83}$$

where f, g are given functions and $\partial v/\partial n$ denotes the outward normal derivative.

Prove that, if they exist, any two solutions of (2.83) differ only by a constant, and that this constant is zero when the coefficient c is not the zero function.

Exercise 2.43 The *Lebesgue spine.* Write $x = (x_1, x_2, z)$ for points of $\mathbb{R}^3$, let $s := (x_1^2 + x_2^2)^{1/2}$ and define

$$\Omega := \left\{ x \in \mathbb{R}^3 \;\middle|\; 0 < |x| < 1;\; s > \exp(-1/z) \text{ if } z > 0, \;\; s > 0 \text{ if } z = 0 \right\}.$$

Show that the function v defined by

$$v(x) := \int_0^1 \frac{\zeta \, d\zeta}{\left\{ s^2 + (z - \zeta)^2 \right\}^{1/2}}, \qquad x \in \overline{\Omega} \setminus \{0\},$$

belongs to $C^\infty(\Omega)$ and satisfies $\triangle v = 0$ in Ω; that

$$v(x) = (z + |z|) \log \frac{1}{s} + \varphi(x) \quad \text{with} \quad \varphi \in C(\overline{\Omega}) \quad \text{and} \quad \varphi(0) = 1;$$

and that v has no extension in $C(\overline{\Omega})$.

Define $g \in C(\partial\Omega)$ by $g(0) := 3$ and $g(x) := v(x)$ for $x \in \partial\Omega \setminus \{0\}$. Prove that the Dirichlet problem of finding u such that

$$\triangle u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = g, \qquad u \in C(\overline{\Omega}) \cap C^2(\Omega)$$

has no solution.

[Assume that u exists and apply Theorem 2.26 to $u - v$ and to $-u + v$.]

Exercise 2.44 (i) Let Ω be bounded in $\mathbb{R}^N$, $N \geq 2$, and let $\partial\Omega$ have an isolated point (for example, $\Omega = \mathscr{B}(0,1) \setminus \{0\}$). Use Theorem 2.26 to prove that the Dirichlet problem of finding u such that

$$\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = g, \quad u \in C(\overline{\Omega}) \cap C^2(\Omega)$$

has no solution for certain functions $g \in C(\partial\Omega)$.

(ii) Prove that Theorem 2.26 is best possible in the following sense. If the hypotheses (2.55) and (2.56) are changed to

$$\limsup_{a\to 0} \sup\left\{ \frac{u(x)}{\log(b/a)} \,\middle|\, x \in \sigma_a \right\} < \infty \quad \text{when } N = 2,$$
$$\limsup_{a\to 0} \sup\left\{ a^{N-2}u(x) \,\middle|\, x \in \sigma_a \right\} < \infty \quad \text{when } N \geq 3,$$

and the other hypotheses remain unchanged, then the conclusion is false.

Exercise 2.45 Prove Theorem 2.36, for the case when $\sup_{\partial\Omega\setminus\{p\}} u < \infty$, by inversion relative to the sphere $\partial\mathscr{B}(2q-p, 2b)$, by use of the corresponding Kelvin transform (§B.3) of the function $\tilde{u} := u - \sup_{\partial\Omega\setminus\{p\}} u$ and by application of Theorem 2.35.

[Use convenient co-ordinates, as in (2.63).]

Exercise 2.46 Let Ω be an unbounded open subset of the half-space $D := \{ x \in \mathbb{R}^N \mid x_N > 0 \}$, $N \geq 2$, and let Ω have the exterior-ball property (Definition 2.14) at each point of the set $P := \{ p^1, \ldots, p^k \} \subset \partial\Omega$.

Suppose that $u \in C(\overline{\Omega} \setminus P)$; that u is subharmonic in Ω; that u satisfies both

$$\liminf_{R\to\infty} \max\left\{ \frac{x_N\, u(x)}{R^2} \,\middle|\, x \in \overline{\Omega},\ |x| = R \right\} \leq 0$$

(cf. Corollary 2.31) and the growth condition (2.79) on small surfaces $\overline{\Omega} \cap \partial\mathscr{B}(p_*^m, a_*)$ for each $m \in \{1, \ldots, k\}$. (The radius b of exterior balls at the points p^m can be chosen to be independent of m.)

Prove that $\sup_\Omega u = \sup_{\partial\Omega\setminus P} u$.