NORMS ON VECTOR SPACES

Recall that a **norm** on a real or complex vector space V is a function $F : V \to \mathbf{R}$ such that

$$\begin{aligned} F(x+y) &\leq F(x) + F(y) \\ F(cx) &= |c|F(x) \\ F(x) &> 0 \quad \text{if } x \neq 0 \end{aligned}$$

for all vectors x and y and for all scalars c.

We say that two norms F and G on V are **equivalent** if there is a $\lambda < \infty$ such that

$$\lambda^{-1} G(x) \leq F(x) \leq \lambda G(x)$$

for all $x \in V$.

Theorem 1. *Any two norms on a finite-dimensional vector space are equivalent.*

Proof. It suffices to prove it in case the vector space is $\mathbf{R}^n$ or $\mathbf{C}^n$ and G is the Euclidean norm $\|\cdot\|$. Let $\mu = \max_{1 \leq i \leq n} F(\mathbf{e}_i)$. Then $\mu < \infty$ and

$$F(x) = F\left(\sum_i x_i \mathbf{e}_i\right) \leq \sum_i |x_i| F(\mathbf{e}_i) \leq \mu \sum_i |x_i| \leq \mu\sqrt{n}\,\|x\| \tag{1}$$

by the Cauchy-Schwartz Inequality (applied to the vectors x and $\sum_i \mathbf{e}_i$). Thus

$$|F(x) - F(y)| \leq |F(x-y)| \leq \mu\sqrt{n}\|x-y\|$$

so F is continuous. Hence F restricted to the unit sphere $\{x : \|x\| = 1\}$ attains its minimum at some point p. Note that $F(p) > 0$. We claim that

$$F(x) \geq F(p)\,\|x\|. \tag{2}$$

If $x = 0$, this is trivially true. If $x \neq 0$, then then

$$F(x) = \|x\|\, F\left(\frac{x}{\|x\|}\right) \geq \|x\|\, F(p).$$

By (1) and (2), we are done. $\square$

In particular, if V is the space of $n \times n$ complex matrices, then the operator norm $\|\cdot\|_{\text{op}}$ is equivalent to any other norm, e.g., to the norm $(\sum_{i,j} |a_{ij}|^2)^{1/2}$.

Recall that the operator norm of the $n \times n$ complex matrix A is

$$\begin{aligned} \|A\|_{\text{op}} &= \sup_{x \in \mathbf{C}^n,\, \|x\| \leq 1} \|Ax\| \\ &= \sup_{x \in \mathbf{C}^n,\, \|x\| = 1} \|Ax\| \\ &= \sup_{x \in \mathbf{C}^n,\, x \neq 0} \frac{\|Ax\|}{\|x\|}, \end{aligned}$$

where $\|\cdot\|$ denotes the standard Euclidean norm: $\|x\| = (\sum_i |x_i|^2)^{1/2}$.