Homework 2

1a) $P(\text{Disease} \mid \text{Positive}) = \dfrac{P(\text{Positive} \mid \text{Disease}) \cdot P(\text{Disease})}{P(\text{Positive})}$

$P(\text{Disease}) = 0.001$

$P(\text{Positive}) = (0.95)(0.001) + (0.1 \cdot 0.999)$

$P(\text{Positive} \mid \text{Disease}) = 0.95$

$P(\text{positive}) = P(\text{positive} \mid \text{Disease}) \cdot P(\text{Disease}) + P(\text{Positive} \mid \text{No Disease}) \cdot P(\text{No Disease}) = 0.999$

$P(\text{Positive} \mid \text{No Disease}) = 1 - 0.9 = 0.1$

$P(\text{Disease} \mid \text{positive}) = \dfrac{(0.95) \cdot (0.001)}{(0.95) \cdot (0.001) + (0.1)(0.999)}$

$= \boxed{0.00942}$

b) Bayes' theorem combines old knowledge and new info to update our beliefs about an event. The prior probability represented the probability of Tom having the disease based on the population. The new data we figured out was P(Positive | Disease) which represents the possibility of observing a positive result given Tom actually has the disease. Bayes' theorem takes both these parts and calculates the posterior probability - P(Disease | Positive) which represents the possibility of Tom having the disease given the positive result.

c) $P(\text{Disease} \mid \text{Positive}_2) = \dfrac{P(\text{Positive}_2 \mid \text{Disease}) \cdot P(\text{Disease} \mid \text{Positive}_1)}{P(\text{Positive}_2)}$

$P(\text{Positive}_2 \mid \text{Disease}) = 0.95$

$P(\text{Disease} \mid \text{Positive}_1) = 0.00942$

$P(\text{Positive}_2) = (0.95)(0.00942) + (0.1)(0.991)$

$0.008949 + 0.0991 = \boxed{0.108049}$

2a) Bayes' Theorem: $P(\theta | X=x) \propto P(X=x|\theta) \cdot \times P(\theta)$

$P(X=0|\theta) = 1-\theta$

$P(X=1|\theta) = \theta$

$P(\theta) = 1/3$ when θ is $\{0, \frac{1}{2}, 1\}$

$$P(\theta = \cdot | X=0) = \frac{(1-\theta)\,1/3}{\frac{1}{3}(1-0)+\frac{1}{3}(1-\frac{1}{2})+\frac{1}{3}(1-1)}$$

$$= \frac{\frac{1}{3}(1-\theta)}{\frac{1}{2}} = \boxed{\frac{2}{3}(1-\theta)}$$

$$P(\theta = \cdot | X=1) = \frac{\theta \cdot \frac{1}{3}}{\frac{1}{3}(1)+\frac{1}{3}(1-\frac{1}{2})+\frac{1}{3}(1-1)} = \frac{\frac{1}{3}\theta}{\frac{1}{2}} = \frac{2}{3}\theta$$

$P(\theta | X=0) = \frac{2}{3}(1-\theta)$

$P(\theta | X=1) = \frac{2}{3}\theta$

2b) $P(X_1=0, X_2=0|\theta) = (1-\theta)^2$

$P(X_1=0, X_2=1|\theta) = \theta(1-\theta)$

$P(X_1=1, X_2=0|\theta) = \theta(1-\theta)$

$P(X_1=1, X_2=1|\theta) = \theta^2$

$$P(\theta = 1, X_1=0, X_2=0) = \frac{\frac{1}{3}(1-\theta)^2}{\frac{1}{3}(1-0)^2+\frac{1}{3}(1-\frac{1}{2})^2+\frac{1}{3}(1-1)^2}$$

$$= \frac{\frac{1}{3}\theta(1-\theta)}{\frac{1}{12}} = 4\theta(1-\theta)$$

$P(\theta) = \cdot) X_1=1, X_2=0) = 4\theta(1-\theta)$

$$P(\theta = \cdot | X_1=1, X_2=1) = \frac{\frac{1}{3}\theta^2}{\frac{1}{3}(1-0)^2+\frac{1}{3}(1-\frac{1}{2})^2+\frac{1}{3}(1-\theta)^2}$$

$$\frac{\frac{1}{3}\theta^2}{\frac{5}{12}} = \boxed{\frac{4}{5}\theta^2}$$