Harmonic analysis notes

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Contents

Chapter 1

Muckenhoupt weights

This chapter is based on the Informal Analysis seminar at Brown as well as Chapter 5 of Stein's book on harmonic analysis.

Throughout we work in $\mathbb{R}^d$ and let $|\cdot|$ denote Lebesgue measure on $\mathbb{R}^d$.

Recall the definition of the maximal operator:

$$Mf(x) = \sup_{r>0} ⨍_{B(x,r)} f(y)\ dy$$

whenever $f \geq 0$ is locally integrable and the dashed integral denotes a mean. Thus $Mf(x)$ dominates the means of f taken on balls centered at x. Then one has:

Theorem 1.1 (Hardy-Littlewood maximal inequality)**.** Let $1 < p \leq \infty$. Then M satisfies the strongtype (p,p) inequality; that is,

$$||Mf||_{L^p} \lesssim_{p,d} ||f||_{L^p}.$$

Proof. The strongtype (∞,∞) inequality is trivial, so we may assume $p < \infty$. We first prove the weaktype $(1,1)$ inequality

$$\lambda|\{Mf > \lambda\}| \lesssim_d ||f||_{L^1} \tag{1.1}$$

uniformly in $\lambda > 0$. If $Mf(x) > \lambda$ then by definition there is a ball B_x centered at x such that

$$⨍_{B_x} |f(y)|\ dy > \lambda.$$

After running the Vitali covering algorithm, one can find a countable set $\mathcal{B}$ of balls such that

$$\{Mf > \lambda\} \subseteq \bigcup_{B \in \mathcal{B}} 5B.$$

Therefore

$$|\{Mf > \lambda\}| \leq 5^d \sum_{B \in \mathcal{B}} |B| \leq \frac{5^d}{\lambda} ||f||_{L^1}.$$

This proves (1.1). The general claim follows from Marcinkiewicz interpolation when we note that M is obviously bounded on L^∞. $\square$

This motivates the following definition.

Definition 1.2. Fix $1 < p < \infty$. A *Muckenhoupt measure* μ is a positive Borel measure such that

$$||Mf||_{L^p(\mu)} \lesssim_p ||f||_{L^p(\mu)}.$$

If $d\mu(x) = \omega(x)\ dx$, we call ω a *Muckenhoupt weight*. ♢

In fact, it will turn out that every Muckenhoupt measure is absolutely continuous, so by the Radon-Nikodym theorem, there is a canonical isomorphism between the spaces of Muckenhoupt weights and Muckenhoupt measures. It is immediately true that 1 is a Muckenhoupt weight.

1.1 The space of weights

We now consider the space of A_p weights. This section follows Chapter V of Stein.

Throughout, fix $1 < p < \infty$ and let p' denote the Hölder dual of p.

Definition 1.3. Let $\omega \in L^1_{loc}$ be nonnegative and nonzero. We define

$$A_p(\omega) = \sup_B \fint_B \omega(x)\ dx \left(\fint_B \omega(x)^{-p'/p}\ dx\right)^{p/p'}$$

where the supremum is taken over all balls B. We let A_p be the space of ω for which $A_p(\omega) < \infty$. ♢

Notice that if $A_p(\omega) < \infty$ then $\omega > 0$ almost everywhere. Indeed, in that case

$$\fint_B \omega(x)^{-p'/p}\ dx < \infty$$

which forces $\{\omega = 0\}$ to have measure zero. It also follows straight from the definitions that

$$A_{p'}(\omega^{-p'/p})^{1/p'} = A_p(\omega)^{1/p}.$$

Lemma 1.4. If $p_1 \le p_2$ then $A_{p_1} \subseteq A_{p_2}$.

Proof. Let $q_i = p_i'/p_i$; then $q_2 \le q_1$. So

$$A_{p_2}(\omega) = \sup_B \fint_B \omega \left(\fint_B \omega^{-q_2}\right)^{q_2} \le \sup_B \fint_B \omega \left(\fint_B \omega^{-q_1}\right)^{q_1} = A_{p_1}(\omega)$$

which was desired. □

For this reason we define $A_\infty = \bigcup_p A_p$, setting

$$A_\infty(\omega) = \inf_p A_p(\omega). \tag{1.2}$$

1.2 The reverse Hölder inequality

This section follows Chapter V of Stein.

To prove the maximal inequality weighted by A_p we first need:

Theorem 1.5 (reverse Hölder inequality)**.** For every $\omega \in A^\infty$ there is a $r > 1$ such that for all balls B,

$$\left(\fint_B \omega(x)^r \, dx\right)^{1/r} \lesssim \fint_B \omega(x) \, dx$$

where the constant depends on ω but not B. $\diamondsuit$

Before we prove the reverse Hölder inequality, we need a characterization of A_∞.

We view ω as a measure whenever ω is a weight, thus

$$\omega(E) = \int_E \omega(x) \, dx$$

whenever $\omega \in L^1_{loc}$ is nonnegative and E is a Borel set.

Lemma 1.6. $A_p(\omega)$ is the infimum of all constants $C > 0$ such that for every nonnegative $f \in L^1_{loc}$ and every ball B,

$$\left(\fint_B f(x) \, dx\right)^p \leq \frac{C}{\omega(B)} \int_B f^p \, d\omega.$$

Proof. Suppose that $\omega \in A_p$ and f, B are given. Then

$$\fint_B f(x) \, dx = \fint_B f(x)\omega(x)^{1/p}\omega(x)^{-1/p} \, dx,$$

so by Hölder's inequality,

$$\left(\fint_B f(x) \, dx\right)^p \leq |B|^{-p} \int_B f^p \, d\omega \left(\omega(x)^{-p'/p} \, dx\right)^{p/p'}.$$

On the other hand,

$$\left(\int_B \omega(x)^{-p'/p} \, dx\right)^{p/p'} \leq \frac{A_p(\omega)}{\omega(B)},$$

so the claim follows with $C = A_p(\omega)$.

Conversely, supppose that there is a finite choice of C, let $\varepsilon > 0$, and let

$$f = (\omega + \varepsilon)^{-p'/p}.$$

Now $p' - 1 = p'/p$ by definition of p', so

$$f \leq (\omega + \varepsilon)^{-p'}\omega.$$

Since we can take C finite, we then have

$$\int_B f(x)\ dx \leq \int_B (\omega+\varepsilon)^{-p'}\omega \lesssim_\varepsilon \int_B \omega(x)\ dx < \infty$$

since $\omega \in L^1_{loc}$. Now

$$f^p\omega = (\omega+\varepsilon)^{-p'}\omega$$

which gives the bound

$$\fint_B \omega(x)\ dx \left(\fint_B (\omega+\varepsilon)^{-p'}\omega\right)^{p/p'} \leq C.$$

This bound is uniform in ε, so taking $\varepsilon \to 0$ we get $A_p(\omega) \leq C$. □

Write kB for the dilation of a ball B by a factor of $k > 0$.

Definition 1.7. A *doubling measure* is a Radon measure μ such that

$$\mu(2B) \lesssim \mu(B)$$

uniformly in balls B. ♢

Lemma 1.8. If $\omega \in A_p$, then ω is a doubling measure with doubling constant $A_p(\omega)2^{dp}$.

Proof. Apply Lemma 1.6 on the ball $2B$ with $f = 1_B$, thus

$$\left(\fint_{2B} 1_B(x)\ dx\right)^p \leq \frac{A_p(\omega)}{\omega(2B)}\int_{2B} 1_B\ d\omega.$$

This simplifies to

$$2^{dp} = \left(\frac{|B|}{|2B|}\right)^p \leq \frac{A_p(\omega)\omega(B)}{\omega(2B)}.$$

Solving for $\omega(B)/\omega(2B)$ we get the claim. □

Definition 1.9. Let ω be a weight such that for every $\alpha \in (0,1)$ there is a $\beta \in (0,1)$ such that for every ball B and Borel set $F \subseteq B$ such that $|F| \geq \alpha|B|$,

$$\omega(F) \geq \beta\omega(B).$$

Then we say that ω is *fair*. ♢

The intuition is that ω does not favor any one region of a ball more than another.

Lemma 1.10. Suppose that $\omega \in A_\infty$. Then ω is fair.

Proof. Suppose that $\omega \in A_p$. Then by Lemma 1.6 with $f = 1_F$ we have

$$\frac{|F|^p}{|B|^p} \leq A_p(\omega)\frac{\omega(F)}{\omega(B)}$$

which gives the claim with $\beta = \alpha^p/A_p(\omega)$. □

Lemma 1.11. If ω is a fair weight, then for every $1 < r \ll 2$,

$$\left(\fint_Q \omega(x)^r \, dx\right)^{1/r} \lesssim \fint_Q \omega(x) \, dx \tag{1.3}$$

uniformly in cubes Q.

Proof. Let $Q_0 = Q$. After rescaling everything we may assume that Q_0 is a dyadic cube and

$$\omega(Q_0) = |Q_0| = 1.$$

In that case, we must obtain a uniform bound

$$\int_{Q_0} \omega(x)^r \, dx \lesssim 1 \tag{1.4}$$

to deduce (1.3). Let $f = \omega 1_{Q_0}$.

We need a dyadic version of the Hardy-Littlewood maximal operator. Namely, we set

$$M^\Delta g(x) = \sup_{x \in Q} \fint_Q g(y) \, dy$$

whenever $g \in L^1_{loc}$ is nonnegative, and Q ranges over all dyadic cubes. Note that by convention we take dyadic cubes as open, so the dyadic cubes of a given volume only tile almost all of $\mathbb{R}^d$.

Let

$$E_k(N) = \{x \in Q_0 : M^\Delta f(x) > 2^{Nk}\}.$$

Then, if Q is a dyadic cube in $E_{k-1}(N)$, then

$$|E_k(N) \cap Q| \le 2^{d-N} |Q|. \tag{1.5}$$

To see this, let $R \subseteq Q$ be a maximal dyadic cube in $E_k(N)$. By the Calderón-Zygmund decomposition of f (with 2^{Nk} viewed as the cutoff for "f large"), we have

$$|R| \le 2^{-Nk} \int_R f(x) \, dx$$

so, summing over all such R,

$$|E_k(N) \cap Q| = \sum_R |R| \le 2^{-Nk} \int_Q f(x) \, dx.$$

On the other hand, the Calderón-Zygmund decomposition of f gives

$$\int_Q f(x) \, dx \le 2^{d+N(k-1)}$$

so (1.5) holds.

Since ω is a fair weight, there is a $\beta \in (0,1)$ such that if $|F| \leq |Q|/2$, then $\omega(F) \leq \beta\omega(Q)$. By (1.5), if N is so large that $2^{d-N} \leq 1/2$, then

$$\omega(E_k(N) \cap Q) \leq \beta\omega(Q).$$

Taking the union over all dyadic cubes Q in $E_{k-1}(N)$, we get

$$\omega(E_k(N)) \leq \beta\omega(E_{k-1}(N)).$$

But $\omega(E_1(N)) \leq \omega(Q_0) = 1$, so by induction,

$$\omega(E_k(N)) \leq \beta^k.$$

Since $\omega = f$ on Q_0 and $f \leq M^\Delta f$ we have $\omega^r \leq (M^\Delta f)^{r-1}\omega$, which implies

$$\int_{Q_0} \omega(x)^r \, dx \leq \int_{Q_0 \cap \{M^\Delta f \leq 1\}} (M^\Delta f)^{r-1} \, d\omega + \sum_{k=0}^\infty \int_{E_k(N) \setminus E_{k-1}(N)} (M^\Delta f)^{r-1} \, d\omega.$$

Trivially,

$$\int_{Q_0 \cap \{M^\Delta f \leq 1\}} (M^\Delta f)^{r-1} \, d\omega \leq 1.$$

By definition of $E_k(N)$,

$$\int_{E_k(N) \setminus E_{k-1}(N)} (M^\Delta f)^{r-1} \, d\omega \leq 2^{N(k+1)(r-1)}\omega(E_k(N)) \leq 2^{N(k+1)(r-1)}\beta^k.$$

Thus we get a bound

$$\int_{Q_0} \omega(x)^r \, dx \leq 1 + \sum_{k=0}^\infty 2^{N(k+1)(r-1)}\beta^k$$

which does not depend on Q_0, at least as long as $1 < r \ll 2$. □

Now we are ready to prove the reverse Hölder inequality. Since ω is a doubling measure and for any ball B we can find a cube Q with $B \subseteq Q \subseteq 2B$, we may replace balls with cubes in the statement of the reverse Hölder inequality, thus for every cube Q, we must bound

$$\left(\fint_Q \omega(x)^r \, dx\right)^{1/r} \lesssim \fint_Q \omega(x) \, dx$$

uniformly in Q. This is just the content of (1.3), which holds whenever ω is fair and $1 < r \ll 2$.

1.3 Characterization of Muckenhoupt weights

In this section we prove a characterization of Muckenhoupt weights. This follows Chapter V of Stein still.

Definition 1.12. A *radially decreasing convolution kernel* Φ is a nonnegative radial function which is radially decreasing and satisfies $\int_{\mathbb{R}^d} \Phi = 1$. $\diamondsuit$

Now if Φ is a radially decreasing kernel we write $\Phi_\varepsilon(x) = \varepsilon^{-d}\Phi(x/\varepsilon)$. Thus Φ_ε is also a radially decreasing kernel, and as $\varepsilon \to 0$, Φ_ε approximates the convolution identity.

Theorem 1.13. Let μ be a Radon measure and $1 < p < \infty$. Then the following are equivalent:

1. μ is a Muckenhoupt measure.
2. The inequality
$$||\Phi_\varepsilon * f||_{L^p(\mu)} \lesssim ||f||_{L^p(\mu)} \tag{1.6}$$
is uniform in $\varepsilon > 0$, Φ a radially decreasing kernel, and $f \in L^1_{loc}$ nonnegative.
3. μ is absolutely continuous, and its Radon-Nikodym derivative ω satisfies $A_p(\omega) < \infty$.
4. μ is absolutely continuous and the inequality
$$\left(\fint_B f(x)\ dx\right)^p \lesssim \frac{1}{\mu(B)} \int_B f^p\ d\mu \tag{1.7}$$
is uniform in $f \in L^1_{loc}$ nonnegative and B a ball.

$\diamondsuit$

We recall that we already showed that (3) and (4) are equivalent with implied constant $A_p(\omega)$. The equivalence of (1) and (3) is what we really want, since it shows that A_p is canonically isomorphic to the space of Muckenhoupt measures.

We first show that $\{p : A_p(\omega) < \infty\}$ is open.

Lemma 1.14. Suppose $A_p(\omega) < \infty$. Then there is a $q < p$ with $A_q(\omega) < \infty$.

Proof. Let $\sigma = \omega^{-p'/p}$. Then $\sigma \in A_{p'}$, so σ satisfies the reverse Hölder inequality
$$\left(\fint_B \sigma(x)^r\ dx\right)^{1/r} \lesssim \fint_B \sigma(x)\ dx$$
for some $1 < r \lll 2$. Then there is a $1 < q < p$ such that $rp'/p = q'/q$, and by definition of σ it follows that $A_q(\omega) < \infty$. $\square$

Lemma 1.15. One has
$$Mf(x) = \sup_\Phi |f| * \Phi(x)$$
where the supremum ranges over all radially decreasing kernels.

Proof. If Φ is a radially decreasing kernel, it is easy to check that Φ is a limit in L^1 of weighted averages of functions of the form $|B_r|^{-1}1_{B_r}$ where B_r is the ball of radius $r > 0$ at 0. Thus we may restrict the supremum to be over $\Phi = |B_r|^{-1}1_{B_r}$, thus the claim is
$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|}|f| * 1_{B_r} = \sup_{r>0} \frac{1}{|B_r|} \int_{\mathbb{R}^d} f(y)1_{B_r}(x-y)\ dy$$
which is obvious. $\square$

Lemma 1.16. If (1.6) holds uniformly in ε for $\Phi = |B_1|^{-1} 1_{B_1}$, then μ is absolutely continuous.

Proof. Let K be a null compact set. We must show that $\mu(K) = 0$, so that $\mu = 0$.

Let $U_n = \{x : d(x, K) < 1/n\}$ and $f_n = 1_{U_n \setminus K}$. Then (f_n) is a decreasing sequence and $f_n \to 0$ everywhere. Since μ is a Radon measure, it is finite on compact sets, thus $f_n \in L^p(\mu)$. By dominated convergence, $||f_n||_{L^p(\mu)} \to 0$. So by (1.6),

$$\lim_{n\to\infty} ||\Phi_\varepsilon * f_n||_{L^p(\mu)} = 0$$

uniformly in ε, and in particular

$$\lim_{n\to\infty} ||\Phi_{1/n} * f_n||_{L^p(\mu)} = 0. \tag{1.8}$$

On the other hand, if $x \in K$, then

$$\Phi_{1/n} * f_n(x) = 1. \tag{1.9}$$

Indeed,

$$\Phi_{1/n} * f_n(x) = \frac{1}{|B_{1/n}|} \int_{\mathbb{R}^d} f_n(y) g_n(x-y)\ dy$$

where $g_n = 1_{B_{1/n}}$. Then

$$\int_{\mathbb{R}^d} f_n(y) g_n(x-y)\ dy = \int_{U_n \setminus K} g_n(x-y)\ dy = \int_{\mathbb{R}^d \setminus K} g_n(x-y)\ dy$$

by definition of U_n. Since K is a null compact set, it follows that

$$\int_{\mathbb{R}^d} f_n(y) g_n(x-y)\ dy = |B_{1/n}|.$$

Plugging this back in we get (1.9). The only way to avoid a contradiction between (1.8) and (1.9) is if $\mu(K) = 0$. $\square$

Let us prove that (1) implies (3). Suppose that μ is a Muckenhoupt measure. Applying Lemmata 1.15 and 1.16, and the Radon-Nikodym theorem, we see that μ has a Radon-Nikodym derivative ω. Let $f \in L^p(\mu)$ be nonnegative, thus

$$||Mf||_{L^p(\mu)} \lesssim ||f||_{L^p(\mu)}.$$

But

$$\fint_B f(y)\ dy \lesssim Mf(x)$$

if $x \in B$, so

$$\left(\fint_B f(y)\ dy\right)^p \lesssim Mf(x)^p,$$

we can integrate both sides in ω over B to get

$$\left(\fint_B f(y)\,dy\right)^p \lesssim \frac{1}{\omega(B)}\int_B Mf^p\,d\omega.$$

Since all integrals are taken over B we may assume that f is supported in B, whence

$$\int_B Mf^p\,d\omega \lesssim \int_B f^p\,d\omega$$

since ω is a Muckenhoupt weight. Therefore $A_p(\omega) < \infty$.

Let us now prove that (2) implies (4). By Lemma 1.16, μ is absolutely continuous, so by the Radon-Nikodym theorem it is weighted by some ω. Let $f \geq 0$ and B be given, and let $\delta = \operatorname{rad} B$. If $\Phi = 1_{B_1}$, B_1 the ball centered at 0 of radius 1, then

$$2^{-d}\fint_B f \leq \fint_{B(x,2\delta)} f = T_\varepsilon f(x)$$

whenever $x \in B$. Taking L^p norms of both sides and using (1.6), we deduce (4). We can drop the assumption that $\Phi = 1_{B_1}$ by first rescaling Φ so $\Phi = 1_{B_r}$ for some $r > 0$ and then replacing a general Φ with $\alpha 1_{B_r} \leq \Phi$ for some $\alpha > 0$ and $r > 0$, which exists since Φ is radially decreasing. That proves (4) in the general case.

Now we show that (4) implies (2). Again we first consider $\Phi = 1_{B_1}$. In this case it suffices to show that

$$||\Phi * f||_{L^p(\omega)} \lesssim ||f||_{L^p(\omega)} \tag{1.10}$$

where the implied constant is only allowed to depend on $A_p(\omega)$. Indeed, by dilation invariance, this will then prove the result for Φ_ε as well. As usual we may assume that f is nonnegative. If $x \in B_1$, then $B(x,1) \subset B_2$, so

$$\Phi * f(x) \leq 2^d \fint_{B_2} f.$$

Therefore

$$\int_{B_1} |\Phi * f|^p\,d\mu \leq 2^{dp}\left(\fint_{B_2} f\right)^p \omega(B_1) \lesssim \int_{B_2} f^p\,d\mu$$

by (4). In other words,

$$\int_{\mathbb{R}^d} |\Phi * f|^p 1_{B_1}\,d\mu \lesssim \int_{\mathbb{R}^d} f^p 1_{B_2}\,d\mu.$$

By translation invariance of convolution with Φ and $A_p(\omega)$, we get

$$\int_{\mathbb{R}^d} |\Phi * f(x)|^p 1_{B_1}(x-y)\,d\mu(x) \lesssim \int_{\mathbb{R}^d} f(x)^p 1_{B_2}(x-y)\,d\mu(x)$$

uniformly in $y \in \mathbb{R}^d$. Integrating in y and using Fubini's theorem, we get (1.10). Using a limiting argument, we can drop the assumption that $\Phi = 1_{B_1}$ to get (2).

Lemma 1.17. The Hardy-Littlewood maximal operator M is weaktype $(L^p(\mu), L^p(\mu))$ iff $d\mu(x) = \omega(x)\,dx$ for some ω with $A_p(\omega) < \infty$.

Proof. Suppose that M is weaktype $(L^p(\mu), L^p(\mu))$, thus

$$\mu\{Mf > \alpha\} \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^d} |f|^p \, d\mu \tag{1.11}$$

uniformly in $\alpha > 0$. Since M is bounded on $L^\infty(\mu)$, by Marcinkiewicz interpolation, it follows that if $q > p$ then M is bounded on $L^q(\mu)$. In other words, (1.6) holds with p replaced by q, so by Lemma 1.16 and the Radon-Nikodym theorem we may write $d\mu(x) = \omega(x) \, dx$. If f is nonnegative and supported in a ball B, $x \in B$, then as usual we have

$$⨍_B f(y) \, dy \leq 2^d Mf(x).$$

Plugging $\alpha = 2^{-d-1} ⨍_B f$ into (1.11), we get

$$\left(⨍_B f(y) \, dy\right)^p \omega(B) \lesssim 2^{(d+1)p} \int_B |f|^p \, d\mu$$

which implies that ω satisfies (4). Since (4) implies (3), it follows that $A_p(\omega) < \infty$.

Conversely, if $A_p(\omega) < \infty$, set

$$M_\omega f(x) = \sup_{\delta > 0} ⨍_{B(x,\delta)} |f| \, d\omega,$$

thus M_ω is the Hardy-Littlewood maximal operator induced by ω. Since ω is a doubling measure, the proof that the Hardy-Littlewood maximal operator is weaktype $(L^1(\omega), L^1(\omega))$ goes through, but with possibly a worse constant than 5 in the Vitali covering algorithm. That is,

$$\omega\{M_\omega g > \alpha\} \lesssim \frac{1}{\alpha} \int_{\mathbb{R}^d} g \, d\omega$$

uniformly in $\alpha > 0$. On the other hand, since (3) holds, so does (4), and thus

$$(Mf)^p \lesssim M_\omega |f|^p.$$

Plugging in $g = |f|^p$ and replacing α with α^p we deduce (1.11). □

Let us finally show that (3) implies (1). Suppose that $A_p(\omega) < \infty$, and let $1 < q < p$ be such that $A_q(\omega) < \infty$. By Lemma 1.17, M is weaktype $(L^q(\omega), L^q(\omega))$, and M is clearly bounded on $L^\infty(\omega)$. Therefore M is bounded on $L^p(\omega)$, which implies (1).