Collaborative Notes - Math 185

Mathematics Undergraduate Student Association, UC Berkeley

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Contents

The following notes are collaborative notes meant for use by undergraduates at UC Berkeley, taking Math 185 (complex analysis). However, they may be read or edited by anyone interested in the material.

The original draft of these notes is loosely based on the Spring 2018 section taught by Charles Hadfield, and was written by Aidan Backus. We do not claim that any of the proofs given are original work.

As these notes are a "perpetual draft", they may contain errors. As such, they can be edited on ShareLaTeX; however, for particularly egregious errors, you are encouraged to contact Aidan at aidan-backus@berkeley.edu.

Things I'd like to do with these notes still:

1. Add a lot more examples and exercises. Really the only chapter where I'm satisfied with the current exercises and examples are the preliminaries.

2. Add lots of pretty pictures. Unfortunately I'm not equipped to do this myself.

3. Possibly add a chapter on Riemann surfaces. I'm not sure of the value of this – we sure spent a lot of time on this topic in my 185, introducing the notions of sheaves of holomorphic maps and complex structures, and setting up the machinery needed to state Riemann-Roch, but I don't think most 185 classes even mention this topic.

Chapter 1

Topological preliminaries

$\mathbf{C}$ is a vector space over $\mathbf{R}$ of dimension 2. We can think of an element $z \in \mathbf{C}$ as a vector $(x, y) \in \mathbf{R}^2$ by the identification $z = x + iy$, where $i^2 = -1$ is the imaginary unit.

Definition 1.1. If $z = x + iy \in \mathbf{C}$, we write

$$x = \operatorname{Re} z$$

for the *real part* of z and

$$y = \operatorname{Im} z$$

for the *imaginary part*.

Of course, we can reflect an imaginary number across the real line:

Definition 1.2. If $z = x + iy \in \mathbf{C}$, we write

$$\overline{z} = x - iy$$

for the *complex conjugate* of z.

$\mathbf{C}$ has a norm, and its norm satisfies the identities $|x+iy| = \sqrt{x^2 + y^2}$, $|zw| = |z||w|$, and $|z+w| \leq |z|+|w|$ for each $z = x + iy$ and w.

The last estimate is the triangle inequality, which gives $\mathbf{C}$ a metric (and therefore topological) structure: its metric $d : \mathbf{C}^2 \to [0, \infty)$ is $d(z, w) = |z - w|$. Thus, open sets in $\mathbf{C}$ are those which can be written as a union of $B_r z$, where $B_r z$ denotes the ball around z of radius $r > 0$. The closed sets in $\mathbf{C}$ are complements of open sets.

Definition 1.3. We will write $\Phi : \mathbf{R}^2 \to \mathbf{C}$ to mean

$$\Phi(x, y) = x + iy$$

and $\Psi = \Phi^{-1}$.

Φ and Ψ are vector space isomorphisms, but stronger than that, they are isometries: they preserve distance and angle as well as vector space structure. In particular, they are homeomorphisms: they preserve topological structure. So, as a normed real vector space, a metric space, or as a topological space, $\mathbf{R}^2$ and $\mathbf{C}$ are identical.

But make no mistake: $\mathbf{C}$ is a field (it has multiplication) while $\mathbf{R}^2$ is not. Moreover, $\mathbf{C}$ has a very different differential structure than $\mathbf{R}^2$, and it is this difference, given by the Cauchy-Riemann equations that we'll consider later, that will make complex analysis worthy of study.

1.1 Notation

Throughout these notes, U will denote an arbitrary open subset of $\mathbf{C}$. We'll usually think of it as the domain of whichever function we're studying (and usually $\mathbf{C}$ will be the codomain). Occasionally we'll need to identify U with its counterpart in $\mathbf{R}^2$, in which case we'll write $\tilde{U} = \Psi(U)$.

A particularly useful open set will be the unit disk $\mathbf{D} = B_1 0$. Clearly any ball can be deformed into $\mathbf{D}$ after a dilation and a translation, and U locally consists of balls (this is the definition of an open set!), so understanding the behavior of $\mathbf{D}$ allows us to locally understand U. On the other hand, we'll later learn that up to a deformation which locally preserves angles, *any* simply connected open set in $\mathbf{C}$ is isomorphic either $\mathbf{D}$ or $\mathbf{C}$.

We will also have a use for the boundary of $\mathbf{D}$, the unit sphere

$$S^1 = \{z \in \mathbf{C} : |z| = 1\}.$$

Definition 1.4. Elements $\omega \in S^1$ are called *phases.*

1.2 Review of 104

As proven in Math 104 for $\mathbf{R}^n$, the following topological results hold. To prove them for $\mathbf{C}$, just identify $\mathbf{R}^2$ with $\mathbf{C}$ by the homeomorphism Φ.

We'll start by examining the properties of sequences:

Lemma 1.5. *Let $\{a_n : n \in \mathbf{N}\}$ be a sequence in $\mathbf{C}$ and $L \in \mathbf{C}$. Then*

$$\lim_{n\to\infty} a_n = L$$

if and only if both

$$\lim_{n\to\infty} \operatorname{Re} a_n = \operatorname{Re} L$$

and

$$\lim_{n\to\infty} \operatorname{Im} a_n = \operatorname{Im} L.$$

Lemma 1.6. *$F \subseteq \mathbf{C}$ is closed iff each sequence in F which converges in $\mathbf{C}$ converges in F.*

Theorem 1.7. *Cauchy sequences converge in $\mathbf{C}$.*

Definition 1.8. If a_n is a sequence in $\mathbf{C}$ and $|a_n|$ is convergent, we say that a_n is *absolutely convergent.*

Lemma 1.9. *Absolutely convergent sequences converge.*

However, we don't just want to talk about convergence of sequence of points. Sequences of functions are also useful.

Definition 1.10. Let $D \subseteq \mathbf{C}$ and let f_n be a sequence of functions $D \to \mathbf{C}$. Further, let $z \in D$ and k, m, n be natural numbers.

We say $f_n \to f$ *uniformly* if

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall x \in D \; \forall n > N \; |f_n(x) - f(x)| < \epsilon.$$

Furthermore, $\sum f_n$ is *uniformly convergent* if its sequence of partial sums is.

Theorem 1.11 (Weierstrass M-test)**.** *Suppose $\{a_i\}$ is a sequence in $(0, \infty)$. If $\forall k \; \forall z \; |f_k(z)| < a_k$ and $\sum a_k$ converges, then $\sum f_k$ is uniformly convergent.*

Now we consider the notion of compactness.

Definition 1.12. Let $\mathcal{U}$ be a family of open sets of some topological space X. If $\bigcup \mathcal{U} = X$, we say that $\mathcal{U}$ is an *open cover* of X. The elements of $\mathcal{U}$ are called *scraps.*

Definition 1.13. Let K be a topological space. K is *compact* if for each open cover $\mathcal{U}$, there exists an open subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ only has finitely many scraps.

Theorem 1.14 (Heine-Borel)**.** *$K \subset \mathbf{C}$ is compact if and only if it is closed and bounded.*

Lemma 1.15. *If $\mathbf{C} \supset K_1 \supseteq K_2 \supseteq \dots$ and the K_is are compact, then their intersection is nonempty.*

Lemma 1.16. *Let K be a metric space. K is compact if and only each sequence in K has a convergent subsequence in K.*

Open covers won't be as important as sequences to us. The actual importance of open covers for us is that they allow us to define the notion of a Lebesgue number, which we'll discuss later in the preliminaries.

One of the most useful notions that sequences give rise to is that of a cluster point. Some topologists call cluster points *accumulation points*, *limit points*, or *close points*.

Definition 1.17. Let $Y \subseteq X$, where X is a metric space. Y *clusters* at $x \in X$ if there is a sequence in $X \setminus \{x\}$ which converges to x.

Theorem 1.18 (Bolzano-Weierstrass)**.** *Let K be a compact metric space and $A \subseteq K$ be infinite. There is a point $x \in A$ such that A clusters at a.*

Sequences also allow us to define what it means for a function to be continuous.

Lemma 1.19. *Suppose X is a metric space, $f : X \to \mathbf{C}$, $L \in \mathbf{C}$, and $z_0 \in \mathbf{C}$.*

If $z_0 \in X$, then the following are equivalent:

1. *If $a_n \to a$ is a convergent sequence, then $f(a_n) \to f(a)$.*
2. *$\forall \varepsilon > 0\ \exists \delta > 0\ \forall d \in D\ |d - z_0| < \delta \implies |f(d) - f(z_0)| < \varepsilon$.*
3. *$f^{-1}(U)$ is open.*

Definition 1.20. If $f : X \to \mathbf{C}$ satisfies the hypotheses of 1.19, then f is *continuous*.

Lemma 1.21. *Suppose $K \subset \mathbf{C}$ is compact and $f : K \to \mathbf{C}$ is continuous. Then $f(K)$ is compact.*

Definition 1.22. Suppose $D \subseteq \mathbf{C}$ clusters at $z_0 \in \mathbf{C}$. $f : D \to \mathbf{C}$ has a *limit* $L \in \mathbf{C}$ at z_0 if

$$\forall \varepsilon > 0\ \ \exists \delta > 0 \text{ s.t. } \forall d \in D\ 0 < |d - z_0| < \delta \implies |f(d) - L| < \varepsilon.$$

Continuous functions are especially well-behaved when their domains are compact.

Lemma 1.23. *Let K be compact and $f : K \to \mathbf{C}$ be continuous. Then $f(K)$ is compact.*

Continuity is also a means for giving a topological definition of connectivity.

Definition 1.24. Suppose that X is a topological space, and there does not exist a surjective, continuous function $X \to \{0, 1\}$. Then we say that X is *connected*.

Lemma 1.25. *X is connected if and only if the only subsets of X which are both open and closed are X itself and $\emptyset$.*

Lemma 1.26. *Intervals $[a, b]$ and their Cartesian products $[a, b] \times [c, d] \times \cdots \times [e, f]$ are connected and compact.*

Moreover, $\mathbf{C}$ is connected but not compact.

Now we examine some "special" subsets of $\mathbf{C}$.

Definition 1.27. A topological space X is *discrete* if every subset of X is open.

Definition 1.28. Let $D \subseteq \mathbf{C}$. D is *convex* if for each line segment $[a, b]$ where $a, b \in D$, $[a, b] \subseteq D$.

Lemma 1.29. *Let $a \in U$ and $r > 0$. Then $\overline{B_r a} \subset U$ iff $\exists R > r$ with $B_R a \subseteq U$.*

None of this should be new material. If it's unfamiliar, refer to any 104 text, such as Rudin, Pugh, or Ross. It would be fruitful to come up with a few explicit examples of each type of space and set presented above.

1.3 Curves

Often in complex analysis, we want to compute the integral of a function not along an interval, but along a much more general path through $\mathbf{C}$. Curves allow us to do this.

Recall that U is assumed to be an open set in $\mathbf{C}$.

Definition 1.30. Let X be a topological space and $a < b$ be real numbers. A *curve* in X is a continuous map $\gamma : [a, b] \to X$.

$\gamma(a)$ is called the *initial point* and $\gamma(b)$ is called the *final point.* We write $\gamma^* = \gamma([a, b])$.

If $\gamma(a) = \gamma(b)$ we say that γ is *closed.*

γ is *constant* if γ^* consists of a point.

If $\mu : [c, d] \to X$ is also a curve and $\gamma(b) = \mu(c)$ then we define the curve

$$\gamma \oplus \mu : [a, b + d - c] \to \mathbf{C}$$

by $\gamma \oplus \mu(t) = \gamma(t)$ if $t \leq b$ or $\mu(t)$ otherwise.

Notice that this definition only really has teeth when the space X "looks like" $\mathbf{R}^n$ $(n \geq 2)$, for example if $X = U$. Consider the case when X is discrete, for example. Because $[a, b]$ is connected, the only curves in X are constant!

On the other hand, when the codomain X has a notion of differentiability, we can define especially nice curves:

Definition 1.31. A curve γ in $\mathbf{C}$ is *smooth* if γ' exists and is continuous.

If each γ_i is a smooth curve and

$$\gamma = \bigoplus_{i \leq N} \gamma_i = \gamma_1 \oplus \cdots \oplus \gamma_N$$

then we say γ is *piecewise smooth.*

Notice that these definitions conflict with the usual usage of the words closed and smooth! Any curve is necessarily closed (in fact, compact) by 1.21, and a smooth function is C^∞, not C^1.

Also notice that curves aren't just their image: they come with a parametrization, which equips them with an orientation (a "direction") and a speed. As we will see, the parametrization won't matter much, only the orientation.

Definition 1.32. If $\gamma : [0, 1] \to X$ is a curve, then $-\gamma$ is given by $-\gamma(t) = \gamma(1 - t)$.

The following is just the usual arc length formula from calculus.

Definition 1.33. If γ is a curve in $\mathbf{C}$ then the *length* of γ is

$$\ell(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Curves which wind around a point once will prove themselves to be especially useful.

Definition 1.34. If $r > 0$, define $\Gamma_r(z_0)$ by the curve $t \mapsto z_0 + re^{it}$ for $t \in [0, 2\pi]$.

Notice that if γ is a curve, then $\gamma'(t)$ is the tangent vector at t, multiplied by the speed. Dividing out by the speed will give us the direction the curve is facing. Since multiplying by an element of the form $e^{i\theta}$ corresponds by a rotation by θ, we have the following definition:

Definition 1.35. Suppose γ_1 and $\gamma_2 : [-1, 1] \to \mathbf{C}$ are curves, $\gamma_1(0) = \gamma_2(0) = z$, and $\gamma_1'(0) \neq 0$ and $\gamma_2'(0) \neq 0$.

The *angle* between γ_1 and γ_2 at 0 is the unique $\theta \in (-\pi, \pi]$ such that

$$\frac{\gamma_1'(0)}{|\gamma_1'(0)|} = \frac{\gamma_2'(0)}{|\gamma_2'(0)|} e^{i\theta}.$$

That this θ is in fact unique won't become clear until we've studied properties of the exponential function exp in greater detail. That's fine; we won't need the notion of angle until much later on.

Often we will need to draw a curve γ between two points in $\mathbf{C}$, so that we can integrate a function along γ. Path-connectivity allows us to describe when this is possible.

Definition 1.36. Suppose that X is a topological space, and for each $x, y \in X$, there is a curve $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then we say that X is *path-connected.*

There is a close relationship between path-connectivity and connectivity, which we'll exploit to its full potential.

Lemma 1.37. *If X is a path-connected space, then X is connected.*

Proof. Suppose that X is not connected. Then there is a surjective continuous function $f : X \to \{0,1\}$. Suppose that $f(x) = 0$ and $f(y) = 1$. Let $\gamma : [0,1] \to X$ be a curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Now the function $f \circ \gamma : [0,1] \to \{0,1\}$ is continuous and surjective, which is a contradiction. $\square$

Lemma 1.38. *If U is connected then U is path-connected.*

Proof. Let $z \in U$ and let $V \subseteq U$ be the set of points $w \in U$ such that there is a curve from z to w. V is nonempty, since $z \in U$. Let $W = U \setminus V$; we'll prove that V and W are both open, so $W = \emptyset$ by 1.25.

Since U is open, there is an open ball $B_r z \subseteq U$. A straight line connects z to any point in $B_r z$, so $B_r z$ is path-connected. So $B_r z \subseteq V$, and this works for any point in V. So V is open. On the other hand, if $w \in W$, then there is an open ball $B_s w \subseteq W$ (for if not, then by the same argument, there's a path from a point in V to w, and thus a path from z to w). So W is open, so V is closed. $\square$

The above proof was an example of an "open and closed argument", an idea we'll see again whenever we have to apply connectivity.

The hypothesis that U is open in a Euclidean space in the above lemma is essential; the lemma fails spectacularly in worse spaces. However, we won't ever need them. As such, we'll be happy to assume that U is connected, when we actually want U to be path-connected.

We often want to talk about ways to deform a curve into another. Homotopy makes this rigorous, furnishing continuous functions which morph a curve into another.

Definition 1.39. Suppose that X is a topological space and

$$\begin{cases} \gamma_0 : [0,1] \to X \\ \gamma_1 : [0,1] \to X \end{cases}$$

are curves in X. They are *X-homotopic* if there exists a continuous $H : [0,1]^2 \to X$ such that

$$\begin{cases} H(t,0) = \gamma_0(t) \\ H(t,1) = \gamma_1(t) \end{cases}$$

If γ_0 is closed, then γ_0 is *X-nullhomotopic* (or *X-contractible*) if γ_1 is constant. The function H is called a *homotopy*, and if, if γ_1 is constant, a *nullhomotopy.*

Notice that X-homotopy is an equivalence relation. In particular, if two closed curves are X-nullhomotopic then they are X-homotopic.

The hypothesis that H is continuous is critical to the definition: if H was not continuous, we could break up the connected set $[0,1]$ into multiple pieces, allowing us to go around "obstacles" in X. For this reason, homotopy can be used to detect "holes" in X, as the following example demonstrates:

Example 1.40. Consider the curves $\gamma_1, \gamma_2 : [0,\pi]$ given by

$$\gamma_1(\theta) = e^{i\theta}$$

and

$$\gamma_2(\theta) = e^{-i\theta}.$$

They are **C**-homotopic, as witnessed by the homotopy

$$H(\theta, s) = e^{f(s)i\theta}$$

where $f : [0,1] \to [-1,1]$ is a continuous function such that $f(0) = 1$ and $f(1) = -1$.

On the other hand, they are not $\mathbf{C} \setminus \{0\}$-homotopic, as any homotopy must pass through 0.

Definition 1.41. If X is path-connected and each closed curve in X is X-nullhomotopic, then we say that X is *simply connected.*

If you've taken algebraic topology, this definition is equivalent to the statement that the fundamental group is trivial:

$$\pi_1(X) = \{0\}.$$

If not, that's fine; you won't need the fundamental group in 185.

Example 1.42. **C** is simply connected, so any two closed curves in **C** are **C**-homotopic.

On the other hand, $\mathbf{C} \setminus \{0\}$ is not simply connected, as $\Gamma_1 0$ is not nullhomotopic.

The above example is critical: anything interesting that happens in complex analysis happens when we integrate along a circle around a point that causes a space to fail to be simply connected.

1.4 Lebesgue numbers

We'll use the notion of a Lebesgue number to prove the homotopy theorem. If you're not familiar with Lebesgue numbers, there's no reason to peruse this section until you're trying to understand the proof.

Definition 1.43. Let $\mathcal{U}$ be an open cover of a metric space X and $\mu > 0$. We say that μ is a *Lebesgue number* for $\mathcal{U}$ if, for each $x \in X$, there is a scrap $U \in \mathcal{U}$ such that $B_\mu(x) \subseteq U$.

In other words, while $\mathcal{U}$ might have scraps which are arbitrarily small, as long as the Lebesgue number is positive, most of them are irrelevant: we can fit any sufficiently small ball into a scrap anyways.

Lemma 1.44. *If K is a compact metric space with open cover $\mathcal{U}$, then $\mathcal{U}$ has a Lebesgue number > 0.*

Proof. If $K = \mathcal{U}$ then we're done. Otherwise, since K is compact, there is a finite subcover $A_1, A_2, \dots, A_n$. For each $i \leq n$, let $C_i = K \setminus A_i$. Since C_i is nonempty and compact, the function $f : K \to \mathbf{R}$ given by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} \min_{y \in C_i} d(x, y)$$

is well-defined and continuous, and > 0 since no point is in every scrap.

Since K is compact, f attains a minimum. So let

$$\mu = \frac{1}{2} \min_K f > 0.$$

For each $B_\mu x \subset K$, $B_\mu x \subseteq A_i$ for at least one A_i, so μ is a Lebesgue number. $\square$

1.5 Topologies

For completeness we state the definition of a topological space. You can safely skip this section unless you're trying to learn the proof of Montel's theorem, because otherwise the only topological space that is relevant to this class is U.

Definition 1.45. Let X be a set, whose elements we will call *points*, and let τ be a set whose elements are subsets of X, which we will call *open sets.* τ is called a *topology* in X if:

1. $\emptyset \in \tau$.

2. $X \in \tau$.
3. For each set of open sets $\mathcal{V} \subset \tau$, their union $\bigcup \mathcal{V} \in \tau$.
4. For each finite set of open sets $V_1, V_2, \dots, V_n$, their intersection $\bigcap V_j \in \tau$.

Once we have selected a topology, X is called a *topological space*. Furthermore, if $K \subseteq X$ and $X \setminus K \in \tau$, then K is said to be a *closed set*. If $V \in \tau$ and $x \in V$, we say that V is an *open neighborhood* of x.

It should be clear that a metric space is also topological.

The usual notions of curves, compactness, and so on carry over to topological spaces, because their definitions do not depend on the metric itself, only the topology. On the other hand, we need to reformulate continuity and convergence.

Definition 1.46. Let X, Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if, for each $V \subseteq Y$ open, $f^{-1}(V)$ is also open.

Definition 1.47. Let X be a topological space. A sequence x_n in X *converges* to $x \in X$ if, for each open neighborhood $V \ni x$, there exists $N \in \mathbf{N}$ such that for each $n > N$, $x_n \in V$.

Finally we give a necessary condition for the notion of sequential compactness to make sense.

Definition 1.48. Let X be a topological space. X is said to be a *sequential Hausdorff space* if for each two points $x_1, x_2 \in X$, there are disjoint open neighborhoods $V_1 \ni x_1$, $V_2 \ni x_2$, and if for each closed set $K \subseteq X$, and each convergent sequence x_n in K, $\lim x_n \in K$.

Theorem 1.49. *If K is a compact, sequential Hausdorff space, then for each sequence x_n in K, there is a convergent subsequence.*

Proof. Suppose not. Then there is a sequence x_n, such that no subsequence converges. Thus for each point $x \in K$, there is an open set V_x such that $x_n \in V_x$ only finitely often. Then the V_x form an open cover so it has a finite subcover. So x_n fits in finitely many open neighborhoods finitely many times; but x_n has infinitely many terms so this is a contradiction. $\square$

1.6 The Arzela-Ascoli theorem

The Arzela-Ascoli theorem is a mechanism for us to generalize the notion of "compactness" to function spaces, where being simply closed and bounded is not enough: another necessary hypothesis is equicontinuity. We'll use this theorem to prove Montel's theorem on normal families, and ultimately the fabled Riemann mapping theorem.

Definition 1.50. Let K be a compact space. $C^0(K)$ denotes the space of continuous functions $K \to \mathbf{C}$.

Notice that the 0 in C^0 denotes the number of times $f \in C^0$ is differentiable. The reason that we assume K to be compact is so that $f(K)$ is compact, and thus bounded, so that the following definition makes sense.

Definition 1.51. Let $f \in C^0(K)$. The sup-*norm* of f is

$$||f|| = \sup_K |f|.$$

As with any norm, the sup-norm defines a metric, and thus a topology on $C^0(K)$. Topologies on spaces of functions are one of the main areas of concern in functional analysis, but we won't need much functional analysis.

As you should verify if it was not proven in 104, if K is a complete metric space, then the sup-norm turns $C^0(K)$ into a complete metric space as well.

Definition 1.52. Let $\mathcal{F} \subseteq C^0(K)$. $\mathcal{F}$ is *equicontinuous* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $f \in \mathcal{F}$ and $x, y \in K$, if $d(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The key thing about equicontinuity is that δ is not allowed to depend on the individual function, or on the point $x \in K$. Once $\mathcal{F}$ has been chosen, δ is purely a function of ε. One can think of δ as a measure on how rapidly f oscillates, so equicontinuity is morally the statement that $\mathcal{F}$ does not contain elements which oscillate arbitrarily rapidly.

Theorem 1.53 (Arzela-Ascoli)**.** *Let $K \subset \mathbf{C}$ be compact. A subset $\mathcal{F} \subset C^0(K)$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.*

Proof. Suppose that $\mathcal{F}$ is closed, bounded, and equicontinuous and let f_n be a sequence in $\mathcal{F}$. Then f_n is bounded and if it converges, it does so in $\mathcal{F}$. Now let

$$D = \{x + yi \in K : (x, y) \in \mathbf{Q}^2\}$$

be the set of rational points of K. Then D is dense and countable in K, and there is a sequence d_n whose image is D.

$f_n(d_1)$ is bounded, so a subsequence converges, say $f_{1;n}(d_1) \to y_1$. Moreover, $f_{1;n}(d_2)$ is bounded, so a subsequence converges, say $f_{2;n}(d_2) \to y_2$. On the other, hand $f_{2;n}(d_1) \to y_1$ still.

Iterating this construction countably many times, one has a sequence of sequences $f_{m;k}$ such that if $j \leq m$ then $f_{m;k}$ is a subsequence of $f_{j;k}$ and $f_{m;k}(d_j) \to y_j$. So given m we can choose ℓ large enough such that if $j \leq m$ and $k \geq \ell$ then $|f_{m;k}(d_j) - y_j| < m^{-1}$.

Now let $g_m = f_{m;\ell}$. Then g_m is a subsequence of f_m, so to prove that $\mathcal{F}$ is compact we just need to prove that g_m converges. We'll show that g_m is Cauchy in $\mathcal{F}$. For $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x - y| < \delta$, $|g_m(x) - g_m(y)| < \varepsilon/3$, since $\mathcal{F}$ is equicontinuous.

Fix x. By density of D and compactness of K, we can find a J large enough that for each x, there is a d_j with $j \leq J$ such that $|x - d_j| < \delta$. So $|f(x) - f(d_j)| < \varepsilon/3$. On the other hand, $\{d_1, \dots, d_J\}$ is finite, so there is M such that if $\ell, m > M$ and $j \leq J$ then $|g_m(d_j) - g_\ell(d_j)| < \varepsilon/3$. Now apply the triangle inequality.

On the other hand, if $\mathcal{F}$ is compact, let $\varepsilon > 0$. Then there is a finite open cover $\mathcal{U}$ of $\mathcal{F}$ by balls of radius $\varepsilon/3$, say centered on the functions f_k. Given k, there is some $\delta_k > 0$ such that for $|x - y| < \delta_k$, $|f_k(x) - f_k(y)| < \varepsilon/3$. Let δ be the max of the δs (which is finite, since $\mathcal{U}$ is finite). Given any $f \in \mathcal{F}$ and $|x - y| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \varepsilon.$$

Thus $\mathcal{F}$ is equicontinuous with oscillation measure δ. ☐

Exercise 1.54. Let a_n be a sequence in $[1, 2]$. Prove that the sequence

$$f_n(x) = \frac{1}{a_n} \sin(a_n x) + \cos(x + a_n)$$

has a convergent subsequence.

1.7 Green's theorem

Green's theorem is an equation from multivariable calculus which relates a line integral to the area integral it encloses. We'll use it to prove the mean value property of harmonic functions.

Theorem 1.55 (Green)**.** *Let $V \subset \mathbf{R}^2$ be open, simply connected, and bounded, and $\gamma : [0, 1] \to \mathbf{R}^2$ be a smooth closed curve, injective except at $\gamma(0)$, such that $\gamma^* = \partial V$ and that γ is positively oriented: if one travels along γ and faces left, they will be looking into the interior of V. Let $W \supset \overline{V}$ be open.*

If $f : W \to \mathbf{R}$, $f = (u, v)$, and $\gamma = (x, y)$, then

$$\int_\gamma v(x(t), y(t)) \frac{dx}{dt} - u(x(t), y(t)) \frac{dy}{dt}\ dt = \iint_V \partial_x u + \partial_y v\ dA.$$

The proof of Green's theorem uses Stokes' theorem from differential geometry, so it's rather difficult. Luckily we only need a special case: when $V = \mathbf{D} = \{(x, y) \in \mathbf{R}^2 : |x^2 + y^2| < 1\}$. Then $\gamma = \Gamma_1 0$.

Proof. We'll prove

$$\int_\gamma u(x(t), y(t))\frac{dx}{dt}\ dt = \iint_{\mathbf{D}} -\frac{\partial u}{\partial y}\ dA.$$

The proof of

$$\int_\gamma v(x(t), y(t))\frac{dy}{dt}\ dt = \iint_{\mathbf{D}} \frac{\partial v}{\partial x}\ dA$$

is similar, and then Green's theorem follows by linearity.

The smooth function $g : (-1, 1) \to \mathbf{R}^2$ given by

$$g(x) = \sqrt{1-x^2}$$

parameterizes $\mathbf{D}$ in the sense that

$$\mathbf{D} = \{(x, y) \in \mathbf{R}^2 : -1 < x < 1, -g(x) < y < g(x)\}.$$

So we can integrate

$$\iint_{\mathbf{D}} \frac{\partial u}{\partial y}\ dy\, dx = \int_{-1}^1 \int_{-g(x)}^{g(x)} \frac{\partial u}{\partial y}\ dA = \int_0^1 u(x, g(x)) - u(x, -g(x))\ dx.$$

On the other hand, γ can be rewritten using the parametric equations $(x, -g(x))$ for the lower half circle, and $(x, g(x))$ for the upper half circle (though the orientation here is reversed). Then

$$\int_\gamma u(x(t), y(t))\frac{dx}{dt}\ dt = \int_0^1 u(x, -g(x)) - u(x, g(x))\ dx$$

as desired. □

The following corollary is actually the form of Green's theorem we actually need. Recall that ∇f is the *gradient* of f; that is,

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}).$$

The *Laplacian* of f, written Δf, is

$$\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

We'll be a bit hard-and-fast with the notation, "cancelling" $dx/dt\ dt = dx$.

Corollary 1.56. *Let $V \subset \mathbf{R}^2$ be open, simply connected, and bounded, and $\gamma : [0, 1] \to \mathbf{R}^2$ be a smooth, positively oriented, closed curve, injective except at $\gamma(0)$, such that $\gamma^* = \partial V$. Let $W \supset \overline{V}$ be open, and let n be the outward unit normal vector for ∂V.*

If $f : W \to \mathbf{R}$ is smooth, then

$$\int_\gamma \nabla f \cdot dn = \iint_V \Delta f\ dA.$$

Proof. Let $\nabla f = (v, -u)$. Then on the one hand,

$$\int_\gamma u\, dx + v\, dy = \int_\gamma (v, -u) \cdot (dy, -dx) = \int_\gamma \nabla f \cdot dn.$$

On the other,

$$\iint_V \Delta f\ dA = \iint_V \nabla \cdot (v, -u)\ dA = \iint_V \partial_x v - \partial_y u\ dA = \int_\gamma u\, dx + v\, dy$$

by Green's theorem. □

Chapter 2

Complex calculus

We're ready to generalize calculus to the complex setting.

Recall that $U \subseteq \mathbf{C}$ is assumed to be an open set.

2.1 Cauchy-Riemann equations

Definition 2.1. Suppose $D \subseteq \mathbf{C}$ clusters at $z_0 \in \mathbf{C}$.

$f : D \to \mathbf{C}$ is *differentiable* at z_0 if, for some $L \in \mathbf{C}$, the function

$$f'(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0 \\ L, & z = z_0 \end{cases}$$

is continuous.

We say that the function f' is the *derivative* of f.

Multiplying through by the denominator $z - z_0$ shows that the following lemma characterizes differentiability.

Lemma 2.2. *Suppose $D \subseteq \mathbf{C}$ clusters at $z_0 \in \mathbf{C}$. The following are equivalent:*

1. *f is differentiable at z_0 and $f'(z_0) = L$.*
2. *$\exists \phi : D \to \mathbf{C}$ continuous at z_0 with $\phi(z_0) = L$ and*
$$\forall z \in D \;\; f(z) = f(z_0) + (z - z_0)\phi(z).$$
3. *$\exists \psi : D \to \mathbf{C}$ continuous at z_0 with $\psi(z_0) = 0$ and*
$$\forall z \in D \;\; f(z) = f(z_0) + (z - z_0)(L + \psi(z)).$$

We think of ψ as the sublinear Taylor remainder of f at z_0 and $\phi = L + \psi$ where we are viewing L as a linear operator on $\mathbf{C}$.

Recall from 104:

Lemma 2.3. *Differentiability implies continuity.*

Lemma 2.4. *If $f, g : U \to \mathbf{C}$ are differentiable, $\lambda \in \mathbf{C}$, and $z \in U$ then:*

1. *$(f + g)' = f' + g'$,*
2. *$(fg)' = f'g + g'f$,*
3. *$(\lambda f)' = \lambda f'$,*

4. $(f \circ g)'(z) = f'(g(z))g'(z)$, *and*

5. *if* $|f| > 0$ *then* $1/f = -f'/f^2$.

Definition 2.5. A *holomorphic* function on an open set U is one which is differentiable on U. The space of holomorphic functions on U is written $\mathcal{O}(U)$.

Note that holomorphicity is stronger than differentiability: a function which is differentiable only at a point, or on a set with an empty interior, is not holomorphic.

The $\mathcal{O}$ derives from the Italian *olomorfa* (holomorphic); in a more abstract formulation, $\mathcal{O}$ is what is called a "sheaf", and the notation $\mathcal{O}$ has been stolen by algebraic geometers to mean sheaves in general.

Recall that a function is analytic if it has a Taylor series. Our goal is to prove that a function is holomorphic iff it is analytic. This is much stronger than differentiability on an open set in $\mathbf{R}^2$ – which doesn't even imply second-differentiability, let alone analyticity!

To see how this could be possible, let's start by identifying functions in $\mathbf{R}^2$ with their counterparts in $\mathbf{C}$ and seeing why differentiability in $\mathbf{C}$ is so special.

Recall the homeomorphisms $\Phi : \mathbf{R}^2 \to \mathbf{C}$ and $\Psi = \Phi^{-1}$. Let $\tilde{U} = \Psi(U)$. Then for each map $f : U \to \mathbf{C}$ we have a natural $\tilde{f} : \tilde{U} \to \mathbf{R}^2$ given by

$$\tilde{f} = \Psi \circ f \circ \Phi.$$

In 105, it is defined that $\tilde{f} = (u, v)$ is differentiable at $\Phi(z_0)$ if there is a linear map $M : \mathbf{R}^2 \to \mathbf{R}^2$ (the derivative) and sublinear map $r : \tilde{U} \to \mathbf{R}^2$ satisfying the analogue of 2.2, namely

$$\forall z \in \tilde{U}\ \tilde{f}(\Psi(z)) = \tilde{f}(\Psi(z_0)) + M(\Psi(z - z_0)) + r(\Psi(z)).$$

Now let $z \in \mathbf{C}$, and $\Psi(z) = (x, y)$. If we multiply z by $w = a + ib$, this corresponds to multiplying $\Psi(z)$ by the *matrix*

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

for

$$\Psi(wz) = \Psi((a + ib)(x + iy)) = \Psi(ax - by + i(ay + bx)) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = Mz.$$

In the language of linear algebra, complex numbers are nothing more than antisymmetric linear operators!

Theorem 2.6 (Cauchy-Riemann equations)**.** *Let* Ψ *and* Φ *be as above. Then if* $f : U \to \mathbf{C}$, $z_0 \in U$, *and* $\tilde{f} = (u, v)$, *the following are equivalent:*

1. f *is differentiable at* z_0.

2. u *and* v *are differentiable at* $\Psi(z_0)$ *and*

$$\begin{cases} \partial_1 u(\Psi(z_0)) = \partial_2 v(\Psi(z_0)) \\ \partial_2 u(\Psi(z_0)) = -\partial_1 v(\Psi(z_0)). \end{cases}$$

Proof. By 2.2, the derivative M of $\tilde{f}$ is given by $r(\Psi(z)) = \Psi(z - z_0)\psi(\Psi(z))$ and $(a, b) = (\operatorname{Re} f'(z_0), \operatorname{Im} f'(z_0))$. Then we use the argument above, taking close note of the fact that the partial derivatives of $\tilde{f}$ are precisely the entries in M, and thus must satisfy certain relations. □

Your first line of attack in proving that a function is not holomorphic is to show that the Cauchy-Riemann equations fail.

Example 2.7. $z \mapsto \operatorname{Re} z$, $z \mapsto \operatorname{Im} z$, $z \mapsto |z|$, and $z \mapsto \overline{z}$ are not holomorphic *anywhere*! This may be shocking, because two of those are differentiable (in fact, linear) in $\mathbf{R}$ and the third is differentiable off of 0 in $\mathbf{R}$.

We'll prove that $f(z) = \operatorname{Re} z$ is not holomorphic. To see this, observe that $u(x, y) = x$ and $v(x, y) = 0$. Then $\partial_1 u(x, y) = 1$ while $\partial_2 v(x, y) = 0$.

Exercise 2.8. Prove that in fact $z \mapsto \operatorname{Im} z$, $z \mapsto |z|$, and $z \mapsto \overline{z}$ are not holomorphic anywhere.

The Cauchy-Riemann equations are the most important equations in the class. (In fact, when Aidan took 185, the entire first midterm came down to proving consequences of them.) As we will later see, the Cauchy-Riemann equations imply if f is holomorphic, then u and v are "harmonic conjugates", which will have important consequences in PDE.

Exercise 2.9. Say that $f : U \to \mathbf{C}$ is *antiholomorphic* if there is a holomorphic function $g : U \to \mathbf{C}$ such that for each $z \in U$, $f(z) = g(\overline{z})$.

Prove that if f is both holomorphic and antiholomorphic, then $f(U)$ is countable. What happens if U is connected?

2.2 Integration on curves

Just as holomorphic functions in complex analysis correspond to harmonic functions in multivariable calculus, complex integrals will correspond to line integrals. We'll start by defining a "preintegral" that we can use to define the actual integral we care about.

Definition 2.10. If $f : [a, b] \to \mathbf{C}$ and $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable, then

$$\int_a^b f = \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

We might also write

$$\int_a^b f(x)\, dx$$

to indicate the variable of integration, when it's helpful.

Recall from 104:

Lemma 2.11. *Integration is linear.*

More verbosely, if $c \in \mathbf{C}$ and $f, g : [a, b] \to \mathbf{C}$ are integrable, then

$$\int_a^b cf + g = c \int_a^b f + \int_a^b g.$$

The following integral estimate is quite useful.

Lemma 2.12. *If $f : [a, b] \to \mathbf{C}$ is continuous, then $|f|$ is integrable and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. $|f|$ is continuous, so there exist $r \geq 0$ and $\theta \in [0, 2\pi)$ such that

$$\int_a^b f = r(\cos\theta + i \sin\theta).$$

Put $\zeta = \cos\theta - i\sin\theta$. Then $|\zeta| = 1$ and

$$\begin{aligned} \left| \int_a^b f \right| &= \zeta \int f = \operatorname{Re} \zeta \int_a^b f = \int_a^b \operatorname{Re} \zeta f \\ &\leq \int_a^b |\operatorname{Re} \zeta f| \leq \int_a^b |\zeta f| = \int_a^b |f|. \end{aligned}$$

□

Now we can define the true integral:

Definition 2.13. Let $\gamma : [a, b] \to \mathbf{C}$ be piecewise smooth, $\gamma^* \subseteq D \subseteq \mathbf{C}$, and $f : D \to \mathbf{C}$ be continuous.

If γ is smooth, we define the *contour integral* to be

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

If not, then $\gamma = \oplus\gamma_i$ and each γ_i is smooth, so we write

$$\int_\gamma f(z)\ dz = \sum_{i=1}^{N} \int_{\gamma_i} f(z)\ dz.$$

The whole point of the class is to study the properties of holomorphic functions defined on open sets by taking contour integrals around those open sets.

As an example, which is quite important in its own right, let's integrate $1/z$ around the origin, where it blows up:

Lemma 2.14. *If $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ is defined by $f(z) = z^{-1}$ then*

$$\int_{\Gamma_1 0} f(z)\ dz = 2\pi i.$$

Proof.

$$\int_0^{2\pi} f(\Gamma_1 0(t))\Gamma_1 0'(t)\ dt = \int_0^{2\pi} e^{-it} i e^{it}\ dt = 2\pi i.$$

□

Perhaps unsurprisingly, this is possible precisely because $\mathbf{C} \setminus \{0\}$ is not simply connected. As a result of this lemma, we'll often see $2\pi i$s pop up in theorems whenever we integrate around a function which blows up.

Lemma 2.15. *Let $\gamma : [a, b] \to \mathbf{C}$ be piecewise smooth, $f, \tilde{f} : \gamma^* \to \mathbf{C}$, and $\lambda \in \mathbf{C}$. We have:*

1. $$\int_\gamma \lambda f + \tilde{f} = \lambda \int_\gamma f + \int_\gamma \tilde{f}.$$

2. $$\left|\int_\gamma f\right| \leq \ell(\gamma) \max_{z \in \gamma^*} |f(z)|.$$

3. *If $\phi : [c, d] \to [a, b]$ is smooth and strictly increasing then $\gamma \circ \phi$ is piecewise smooth and*

$$\int_\gamma f = \int_{\gamma \circ \phi} f.$$

4. *If $f_n : \gamma^* \to \mathbf{C}$ and $f_n \to f$ uniformly then*

$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. As an example, we prove (3). You should try to prove the others for practice!

We can assume γ is smooth, for if not, we can just break up γ into finitely many smooth curves. Then

$$\begin{aligned}
\int_{\gamma\circ\phi} f &= \int_c^d (f\circ\gamma\circ\phi)(\gamma\circ\phi)' \\
&= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\ dt \\
&= \int_a^b f(\gamma(s))\gamma'(s)\ ds \\
&= \int_\gamma f.
\end{aligned}$$

□

So far, nothing too crazy has happened. In fact, the fundamental theorem of calculus holds just fine:

Theorem 2.16 (fundamental theorem of calculus, part I)**.** *If f is holomorphic on U and γ is contained in U, then*

$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)).$$

Proof.

$$\int_\gamma f' = \int_a^b f'\circ\gamma\cdot\gamma' = \int_a^b (f\circ\gamma)' = f(\gamma(b)) - f(\gamma(a)).$$

□

Just like in calculus, *if* a function has an antiderivative, then its integral is determined by the value of the antiderivative on the boundary of the integration domain.

Moreover, if f' vanishes on a connected set U, we can draw a curve γ inside U to reach any point in U, and by the fundamental theorem we have:

Corollary 2.17. *If $f \in \mathcal{O}(U)$ and U is connected, with $f' \equiv 0$, then f is constant.*

2.3 Power series

Recall from 104 that a power series is a function of the form

$$f(z) = \sum_{n=k}^{\infty} \alpha_k (z-a)^k.$$

We can and do, without loss of generality, assume $a = 0$; this simplifies notation greatly without affecting convergence.

Definition 2.18. If $\sum \alpha_n z^n$ is a power series and if R is the supremum of all values r such that $\sum |\alpha_n| r^n$ converges, then R is its *radius of convergence.*

This seems kind of sketchy; what if the series only converged on a set which was not circular? Fortunately, this never happens!

Lemma 2.19. *Let R be the radius of convergence of $f(z) = \sum \alpha_k z^k$. Then:*

1. *If $|z| < R$, then $f(z)$ converges.*
2. *If $|z| > R$, then $f(z)$ diverges.*
3. *If $0 < r < R$ and $|z| \leq r$ then $\sum \alpha_n z^n$ converges absolutely uniformly.*

Proof. Take $z \neq 0$.

Suppose that $\sum \alpha_n z^n$ converges. Then $\alpha_n z^n$ is bounded, say $|\alpha_n z^n| \leq M$. If $0 < r < |z|$, then $|\alpha_n| r^n| \leq M(r/|z|)^n$, and therefore $\sum |\alpha_n z^n|$ converges. Therefore $r < R$, so $|z| \leq R$. This proves the contrapositive of (2).

Now, if $0 < r < R$, $\sum |\alpha_n| r^n$ converges whenever $|z| \leq r$. Then $|\alpha_n z^n| \leq |\alpha_n| r^n$ and so (3) follows by the Weierstrass M-test, implying (1). $\square$

We can use the root test and the ratio test, developed in 104, to compute radii of convergence. For example, the radius of convergence of $\sum z^n$ is 1. You should practice this on a few power series of your own devision.

Functions which are expressible by power series are known as analytic functions:

Definition 2.20. $f : U \to \mathbf{C}$ is *analytic* at $a \in U$ if $\exists R > 0$ and a power series $\sum \alpha_n (z-a)^n$ with a radius of convergence $r \geq R$ such that $B_R(a) \subseteq U$ and $\sum \alpha_n (z-a)^n = f(z)$ for each $z \in B_R(a)$.

Note, an analytic function has a power series which is locally valid. If the topology of U isn't too nice, a power series may diverge in parts of U but a different power series will hold there. It's okay; the function is still analytic in that case.

Polynomials are clearly analytic (all but finitely $\alpha_n = 0$), but most functions which we care about turn out to be analytic. Our goal, recall, is to prove every holomorphic function is analytic! Let's begin.

Lemma 2.21. *The power series $\sum \alpha_n z^n$ and $\sum n\alpha_n z^{n-1}$ have the same radius of convergence.*

Proof. Let R and $\hat{R}$ be the respective radii of convergence and $0 < r < R$. Then we can find a $\rho \in (r, R)$. For each such ρ, $\sum |\alpha_n| \rho^n$ converges, but $n(r/\rho)^n \to 0$ and is therefore bounded by a constant $M \geq |n(r\rho)^n|$. Therefore

$$n|\alpha_n| r^n = n \left(\frac{r}{\rho}\right)^n |\alpha_n| \rho^n \leq M |\alpha_n| \rho^n$$

and since the constant M is irrelevant, $\sum n|\alpha_n| r^n$ converges. So $\hat{R} > r$, implying that $\hat{R} \geq R$.

But we can repeat the same argument with R and $\hat{R}$ reversed. So $R \geq \hat{R}$. $\square$

This is what we needed to prove Taylor's theorem, which characterizes analytic functions.

Theorem 2.22 (Taylor)**.** *If $R \in (0, \infty)$ and $\sum \alpha_n z^n$ has a radius of convergence $\geq R > 0$, define $f : B_R(0) \to \mathbf{C}$ by*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$$

Then f is smooth in $B_R(0)$ and for each $z \in B_R(0)$,

$$f'(z) = \sum_{n=1}^{\infty} n\alpha_n z^{n-1}.$$

Moreover,

$$\alpha_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Fix $z_0 \in B_R(0)$, $\epsilon > 0$, and $r \in (|z_0|, R)$. By the above lemma, we can find an N such that

$$\sum_{n=N+1}^{\infty} n|\alpha_n| r^{n-1} < \frac{\epsilon}{4}.$$

Therefore, $\forall z \in B_r(0) \setminus \{z_0\}$, the remainder

$$\begin{aligned}
R(z) &= \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n\alpha_n z_0^{n-1} \right| \\
&= \left| \sum_{n=0}^{\infty} \alpha_n \left(\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right) \right| \\
&= \left| \sum_{n=1}^{\infty} \alpha_n \left(\left(\sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right) - n z_0^{n-1} \right) \right| \\
&\leq \left| \sum_{n=1}^{N} \alpha_n \sum_{k=0}^{n-1} z^k z^{n-1-k} - z_0^{n-1} \right| + \left| \sum_{n=N+1}^{\infty} 2n|\alpha_n| r^{n-1} \right| \\
&= A(z) + B(z)
\end{aligned}$$

and A is continuous, so there exists a δ such that if $z \in B_\delta(z_0) \setminus \{z_0\}$ then $A(z) < \epsilon/2$. We already estimated $B(z) < \epsilon/2$. So $R(z) < \epsilon$.

Therefore f is differentiable and α_1 is as desired. Furthermore, this argument applies to the derivatives of f as well, so by induction f is smooth and each α_n is as desired. ☐

Corollary 2.23. *Analytic functions are holomorphic.*

By Taylor's theorem, if a function is analytic in U, U is connected, and we know its behavior on a small open set in U, we already know its behavior everywhere.

Corollary 2.24. *Suppose $\sum \alpha_n z^n$ and $\sum \beta_n z^n$ are power series with radii of convergence R_1, R_2. If $0 < \epsilon \leq \min R_i$ and $\forall z \in B_\epsilon(0)$ we have $\alpha_n z^n = \beta_n z^n$, then $\forall n$ $\alpha_n = \beta_n$.*

Proof. Look at $\sum(\alpha_n - \beta_n)z^n$. The derivatives of this are all 0. ☐

As an aside, here's a way to compute products of analytic functions:

Theorem 2.25. *Let $r > 0$ and $f(z) = \sum \alpha_n z^n$, $g(z) = \sum \beta_n z^n$ be power series with radius of convergence $\geq r$. Put*

$$\gamma_n = \sum_{j \leq n} \alpha_j \beta_{n-j}.$$

Then $\sum \gamma_n z^n$ has radius of convergence $\geq r$ and $\sum \gamma_n z^n = f(z)g(z)$.

Proof. fg is holomorphic, and the proof follows by telescoping and using the binomial formula. ☐

2.4 Trigonometry, logarithms, and Euler's formula

By 2.24, we are justified in taking the power series definitions of exp, sin, and cos – if we wanted them to be analytic, we have no choice but to accept them!

Definition 2.26.

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

and

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

Recall the following formula from calculus:

Theorem 2.27 (Euler's formula)**.** *If $\theta \in \mathbf{R}$ then*

$$\exp i\theta = \cos\theta + i\sin\theta.$$

If you haven't seen the proof of this, you should derive it for practice from the power series definitions.

Corollary 2.28.

$$e^z = e^w \iff z - w \in 2\pi i\mathbf{Z}.$$

Proof. Let $z = x + iy$. Then $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$. So exp is periodic in its imaginary part with period $2\pi i$, but injective in its real part. ☐

In particular, log is *not* well-defined as the inverse of exp, because exp is no longer a bijection.
But there was another definition of log that was developed in 104.

Definition 2.29. Let $z \in \mathbf{C} \setminus (-\infty, 0]$. The *natural logarithm* of z is given by

$$\log z = \int_1^z \frac{dt}{t}.$$

This integral only makes sense if there's a straight line segment from 1 to z, which is why we exclude $(-\infty, 0]$ (since $1/t$ isn't a well-defined function at 0). In spite of this pathology, log is fairly well-behaved.

Lemma 2.30. log *is holomorphic on its domain and its derivative is $z \mapsto z^{-1}$.*

Proof. Apply the fundamental theorem of calculus to the definition of the logarithm. ☐

Since we can apply calculus, we can mimic the usual proofs from 104:

Corollary 2.31. $\exp \circ \log = \mathrm{id}$. *Moreover,* $\log ab = \log a + \log b$.

Sadly, the opposite relation ($\log \circ \exp = \mathrm{id}$) does not hold, because log's image is not all of $\mathbf{C}$:

Corollary 2.32. $\operatorname{Im} \log z \in (-\pi, \pi]$.

Proof. log is continuous because it is holomorphic, and its domain is connected, so its image

$$V = \log(\mathbf{C} \setminus \{0\})$$

must be connected. Moreover, $\log((0, \infty]) = \mathbf{R}$ so $\mathbf{R} \subseteq V$.

On the other hand, we have already seen that $y \mapsto \exp iy$ is periodic if $y \in \mathbf{R}$ and its period is 2π. Since the trig functions are symmetric about 0 it must be that $V = (-\pi, \pi]$. ☐

Finally, recall from calculus the definition for exponentiation with arbitrary base:

Definition 2.33. Let $a, z \in \mathbf{C}$. Then

$$a^z = \exp(z \log a).$$

Example 2.34. Let's compute 2^{1+i}. We have

$$2^{1+i} = \exp(1\log 2 + i\log 2) = \exp(\log 2)(\cos\log 2 + i\sin\log 2) = 2(\cos\log 2 + i\sin\log 2).$$

Chapter 3

Integration in a convex set

Before we can get to the real meat of the course – the residue calculus – we'll prove preliminary versions of the theorems we'll see later under a very strong assumption: that the domain U is *convex*. This will allow us to draw line segments and integrate along them using the fundamental theorem of calculus. Later we'll see that convex sets are rather boring, and the theorems generalize and become much more powerful and natural when U is merely simply connected (or sometimes just connected). However, to generalize these theorems, we'll need the weak versions; we'll exploit the fact that U is *locally* convex because it is open.

Recall that U is assumed open.

We're going to start off proving a rather technical representation formula which will allow us to construct analytic functions. The rest of the chapter, and indeed the class, is going to be fallout from it.

Lemma 3.1. *Let γ be a curve, $g : \gamma^* \to \mathbf{C}$ be continuous, $z_0 \in U = \mathbf{C} \setminus \gamma^*$, and*

$$f(z) = \int_\gamma \frac{g(w)}{w - z}\, dw.$$

Then f is analytic, and for each $n \in \mathbf{N}$, its coefficients are

$$\alpha_n = \int_\gamma \frac{g(w)}{(w - z_0)^{n+1}}\, dw.$$

Moreover, the radius of convergence of f is at least

$$\inf_{w \in \gamma^*} |w - z_0| > 0.$$

Proof. Let $z \in B_R z_0$ and $w \in \gamma^*$. Then

$$\left|\frac{z - z_0}{w - z_0}\right| \leq \left|\frac{z - z_0}{R}\right| < 1$$

so, using a geometric series,

$$\begin{aligned}
\frac{1}{w - z} &= \frac{1}{w - z_0 - (z - z_0)} \\
&= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\
&= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n
\end{aligned}$$

converges uniformly.

Define $h, h_1, h_2, \cdots : \gamma^* \to \mathbf{C}$ by

$$h_n(w) = \frac{g(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

and

$$h(w) = \frac{g(w)}{w - z_0}.$$

These are continuous on γ^* and $\sum h_i = h$ by another geometric series computation; the limit converges uniformly on γ^*. By compactness of γ^*, we can swap limits, so

$$f(z) = \int_\gamma h = \sum_{n=0}^{\infty} \int_\gamma h_n = \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \frac{g(w)}{w - z_0}\ dw = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

on $B_R z_0$. ☐

3.1 Winding numbers

A special case occurs when $g \equiv 1$.

Definition 3.2. If γ is closed, then define $\operatorname{Ind}_\gamma : \mathbf{C} \setminus \gamma^* \to \mathbf{C}$ by

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z},$$

the *index function* or *winding number* of γ.

By 3.1, $\operatorname{Ind}_\gamma$ is analytic. As we shall see, the winding number of γ at z tells us how many times γ winds around z counterclockwise (minus the times it winds around z clockwise).

Lemma 3.3. *If γ is closed, then $\operatorname{Ind}_\gamma(\mathbf{C} \setminus \gamma^*) \subseteq \mathbf{Z}$, and $\operatorname{Ind}_\gamma(z)$ is identically 0 outside of the region encircled by γ.*

Proof. Put $f : [a, b] \to \mathbf{C}$ by

$$f(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\ ds.$$

Then f is differentiable and

$$f'(t) = \frac{\gamma'(t)}{\gamma(t) - z}.$$

Define $g : [a, b] \to \mathbf{C}$ by

$$g(t) = e^{-f(t)}(\gamma(t) - z).$$

Then $g' \equiv 0$, so g is constant. In particular,

$$e^{f(t) - f(a)} = \frac{\gamma(t) - z}{\gamma(a) - z}$$

but $f(a) = 0$ and $\gamma(b) = \gamma(a)$. Therefore,

$$e^{f(b)} = e^{f(b) - f(a)} = \frac{\gamma(a) - z}{\gamma(a) - z} = 1.$$

So $f(b) \in 2\pi i\mathbf{Z}$, but $f(b) = 2\pi i \operatorname{Ind}_\gamma(z)$.

A computation verifies that

$$\lim_{|z| \to \infty} \operatorname{Ind}_\gamma(z) = 0$$

but since $\operatorname{Ind}_\gamma$ takes values in the integers it must be identically 0 if $|z|$ is sufficiently large. But then by analyticity, $\operatorname{Ind}_\gamma$ is 0 on any connected components which include such sufficiently large $|z|$. ☐

So why does the winding number behave the way it does? Let's start by looking at an example:

Example 3.4. Let $U = \mathbf{D}$ and $\gamma = \Gamma_1 0$. By 2.14,

$$\operatorname{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w} = 1$$

and, by connectivity, $\operatorname{Ind}_\gamma$ is identically 1 inside of U. But on the other hand, 3.3 implies that $\operatorname{Ind}_\gamma$ is identically 0 outside of U.

Now suppose that instead of wrapping around U 1 time,

$$\gamma = \bigoplus_{j=1}^{n} \Gamma_1 0;$$

that is, γ winds around U n times. Then, $\operatorname{Ind}_\gamma = n \operatorname{Ind}_{\Gamma_1 0} = n$ inside of U.

On the other hand, $-\Gamma_1 0$ has winding number -1 inside of U. So, for $U = \mathbf{D}$, $\operatorname{Ind}_\gamma$ does count the number of times γ winds around U counterclockwise, minus the times it winds around clockwise.

It shouldn't be hard to believe that this result generalizes when γ is a less simple curve, or when U has a less obvious geometry. The homotopy theorem (4.17) makes this rigorous: as long as the deformation of γ isn't "too bad", the integral, and thus the winding number, is unchanged.

Of course, we haven't proven the homotopy theorem, so we can't use this trick in any proofs yet. But it does provide a nice way to visualize the winding number, as well as foreshadowing some of the big ideas from later on.

3.2 Cauchy-Goursat in a convex set

In multivariable calculus one often worries about conservative vector fields, those for which line integrals around closed curves vanish. You might intuit that holomorphic functions $U \to \mathbf{C}$ are "conservative", and if the topology of U isn't too bad, you'd be right.

First, a notational convenience, which we won't need after its use to prove the next few theorems.

Definition 3.5. For $z_1, z_2, z_3 \in \mathbf{C}$, write Δ to mean the triangle with those endpoints, including the interior, and $\partial\Delta$ to indicate the path around the triangle, oriented counterclockwise.

If the z_is are collinear then this is a degenerate triangle.

Theorem 3.6 (Cauchy-Goursat)**.** *If $p \in U$, $\Delta_0 \subseteq U$, f is continuous, and f is holomorphic on $U \setminus \{p\}$ then*

$$\int_{\partial\Delta} f = 0.$$

Proof. There are three cases, depending on where p is in relation to $\partial\Delta_0$.

If $p \notin \Delta_0$, then define z_i' to be the point opposite z_i on $\partial\Delta_0$. Then we have four triangles:

1. Δ_0^1, determined by z_1, z_2', z_3',
2. Δ_0^2, determined by z_1', z_2, z_3',
3. Δ_0^3, determined by z_1', z_2', z_3, and
4. Δ_0^4, determined by z_1', z_2', z_3'.

Then

$$\left|\int_{\partial\Delta_0} f\right| \le \sum_{i=1}^{4}\left|\int_{\partial\Delta_0^i} f\right| \le 4\left|\int_{\partial\Delta_0^k} f\right|$$

for some index k. Set $\Delta_1 = \Delta_0^k$.

By induction we get a nested sequence of triangles

$$\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \dots$$

and

$$\left|\int_{\partial\Delta_k} f\right| \leq 4\left|\int_{\partial\Delta_{k+1}} f\right|$$

with $\ell(\partial\Delta_k) = \ell(\partial\Delta_{k+1})/2$, so

$$\left|\int_{\partial\Delta_0} f\right| \leq 4^n\left|\int_{\partial\Delta_n} f\right|.$$

Since the Δ_n are compact there is a point in their intersection, say $z_0 \neq p$ and so in a sufficiently small ball around that point,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)$$

for a sublinear ψ. Integrating the affine part, we get a map

$$z \mapsto f(z_0)z + \frac{1}{2}f'(z_0)(z - z_0)^2$$

and so for each n,

$$\int_{\partial\Delta_n} f(z_0) + f'(z_0)(z - z_0)\ dz = 0.$$

Fix an $\epsilon > 0$. We can find $\delta > 0$ such that

$$||\psi||_{C^0(B_\delta z_0 \cap U)} < \epsilon$$

and $\Delta_n \subseteq B_\delta z_0$. From there,

$$\begin{aligned}
\left|\int_{\partial\Delta_n} f\right| &= \left|\int_{\partial\Delta_n} f(z) - (f(z_0 + f'(z_0)(z - z_0))\ dz\right| \\
&= \left|\int_{\partial\Delta_n} (z - z_0)\psi(z)\ dz\right| \\
&\leq \ell(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |z - z_0||\psi(z)| \\
&\leq \ell(\partial\Delta_n)^2 ||\psi||_{C^0} \\
&< \epsilon\ell(\partial\Delta_n)^2.
\end{aligned}$$

In the second case, p is a vertex of Δ_0, say $p = z_1$. For $\epsilon > 0$ choose p_2, p_3 along the segments $[z_1, z_2]$ and $[z_2, z_3]$ with $|z_1, p_i| < \epsilon$. This determines a triangle Δ_1 by (z_1, p_2, p_3), and $\Delta_0 \setminus \Delta_1$ is a polygon which can be triangulated, say by $\Delta_2, \ldots$.

Then if $k \geq 2$, f is holomorphic on Δ_k and so its integral along $\partial\Delta_k$ vanishes. In particular,

$$\left|\int_{\partial\Delta_0} f\right| \leq \int_{\partial\Delta_1} |f| < 3\epsilon||f||_{C^0}$$

so its integral vanishes.

Finally if $p \in \Delta_0$ and it is not a vertex, then we can triangulate Δ_0 such that p is a vertex and the theorem follows. $\square$

We need continuity at p because if not, then f might be horribly behaved. If we define

$$f(z) = \begin{cases} z^{-1}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

then Cauchy-Goursat fails by 2.14.

We now give a sufficient condition for a function to have an antiderivative.

Lemma 3.7. *Suppose that U is convex, and $f : U \to \mathbf{C}$ is continuous.*

If for each triangle $\Delta \subseteq U$,

$$\int_{\partial\Delta} f = 0$$

then for each $a \in U$, the function

$$F(z) = \int_a^z f(t)\ dt$$

is holomorphic on U and $F' = f$.

Proof. Fix $z_0 \in U$ and allow z to range over U. Then if Δ is the triangle given by (z, z_0, a) we have

$$0 = \int_{\partial\Delta} f = \int_a^z f + \int_z^{z_0} f + \int_{z_0}^a f = F(z) - F(z_0) + \int_z^{z_0} f$$

while

$$\int_{z_0}^z f(z_0)\ dw = f(z_0)(z - z_0)$$

and

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{z_0}^z f(w) - f(z_0)\ dw.$$

Fix $\epsilon > 0$. By continuity we can find a $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &\leq \frac{1}{|z - z_0|} \int_z^{z_0} |f(w) - f(z_0)|\ dw \\ &< \frac{1}{|z - z_0|} \ell(z, z_0)\epsilon C = \epsilon C\end{aligned}$$

for some constant C. Convexity guarantees that all these segments are in fact contained in U. □

It will be useful to adopt some nonstandard terminology for dealing with the corollaries of Cauchy-Goursat. The true power of Cauchy-Goursat is that it holds even if we don't know that f is holomorphic everywhere; we only need f to satisfy these conditions:

Definition 3.8. Suppose $f : U \to \mathbf{C}$ is continuous. If f is holomorphic on all but finitely many points of U, we say that f is *cofinitely holomorphic*.

As it turns out, a cofinitely holomorphic function will end up being holomorphic, which is why nobody actually uses this terminology.

Theorem 3.9 (fundamental theorem of calculus, part II)**.** *Suppose γ is closed in U convex, $a \in U$ and $f : U \to \mathbf{C}$ is cofinitely holomorphic.*

Then the function

$$F(z) = \int_a^z f$$

has $F' = f$ and is holomorphic and moreover

$$\int_\gamma f = 0.$$

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be the points at which f is not holomorphic. If $A = \emptyset$ then the theorem is immediate by Cauchy-Goursat and 3.7.

Otherwise, Break up U into subsets $U_1, U_2, \ldots, U_n$ such that $a_j \in U_j$ and for each $k \neq j$, $a_k \neq U_j$. Apply Cauchy-Goursat to each one, and notice that since open sets overlap (since U, being convex, is necessarily connected), any triangle can be broken up into triangles small enough to fit into at least one U_j. Thus we can apply 3.7. □

3.3 Cauchy's integral formula

One consequence of Cauchy-Goursat and the fundamental theorem of calculus is that if we know the value of a cofinitely holomorphic function on a curve around a region, we know its value at every point on that region. This will be a theme we see over and over; for example, Poisson's integral formula, 7.21, says that we know the value of a harmonic function if we know its values on a curve.

Theorem 3.10 (Cauchy's integral formula)**.** *Suppose γ is closed in U convex and $f : U \to \mathbf{C}$ is cofinitely holomorphic. Then*

$$f(z)\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\, dw.$$

Proof. Define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(w), & w = z. \end{cases}$$

g is continuous and holomorphic away from z so

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma g \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\, dw - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\, dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\, dw - f(z)\operatorname{Ind}_\gamma(z). \end{aligned}$$

□

Now we're ready for the sockdolager:

Corollary 3.11. *Suppose U is convex and $f : U \to \mathbf{C}$.*

If f is cofinitely holomorphic, then it is analytic; in particular, f is holomorphic.

Furthermore, if $z_0 \in U$, $r > 0$ and $B_r(z_0) \subseteq U$, we can find α_n such that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

with radius of convergence $\geq r$, and there exists a unique holomorphic function g with the maximal radius of convergence afforded by this theorem, such that when g is restricted to the appropriate ball, $g = f$.

Proof. Fix $\rho \in (0, r)$. Then $\Gamma_\rho z_0 \subseteq U$ and if $z \in B_\rho z_0$, then $\operatorname{Ind}_{\Gamma_\rho} z = 1$. Moreover, if $\eta \in (\rho, r)$, then the ball $B_\eta z_0$ is convex.

So,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho z_0} \frac{f(w)}{w-z}\, dw.$$

This is analytic by 3.1 with the desired power series. □

Definition 3.12. With hypotheses as in 3.11, g is the *analytic continuation* of f.

In summary:

Theorem 3.13 (Morera)**.** *Let $f : U \to \mathbf{C}$. The following are equivalent:*

1. *f is holomorphic.*
2. *For each $\Delta \subseteq U$, $\int_{\partial\Delta} f = 0$.*
3. *$f^{(n)}$ is holomorphic.*

4. f is analytic.

5. $f^{(n)}$ is analytic.

Proof. By restricting ourselves to a small ball and then patching the functions together, we can assume that U is convex. If f is holomorphic, then f and each $f^{(n)}$ are analytic and by Cauchy-Goursat, for each $\Delta \subseteq U$, $\int_{\partial\Delta} f = 0$. On the other hand, if f is analytic, then f and each $f^{(n)}$ is holomorphic. If for each $\Delta \subseteq U$, $\int_{\partial\Delta} f = 0$, then there exists a holomorphic function F by 3.7 such that $F' = f$; in particular f is holomorphic. ☐

This is huge; it's definitely not true in $\mathbf{R}^n$!

Corollary 3.14 (Cauchy's integral formula for derivatives)**.** *Suppose $f : U \to \mathbf{C}$ is holomorphic, $a \in U$, and $r > 0$ with $\overline{B_r a} \subset U$. Then*

$$|f^{(n)}(z)| = \frac{n!}{2\pi i} \int_{\Gamma_r a} \frac{f(w)}{(w-z)^{n+1}}\ dw.$$

Proof. Apply Cauchy's formula and induct on the Taylor coefficients, using 1.29 to ensure all the balls make sense. ☐

Exercise 3.15. Let $R > 0$ and $B_R 0 \subset U$. If $f \in \mathcal{O}$ prove that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta.$$

(Hint: If $w \in B_R 0$ then

$$z \mapsto \frac{f(z)}{R^2 \overline{w} z}$$

is holomorphic on $B_{R^2/|w|} 0 \cap U$.)

3.4 Cauchy's estimate

By using the estimate 2.12 on Cauchy's integral formula for derivatives, the following, very useful estimate follows.

Corollary 3.16 (Cauchy's estimate)**.** *With hypotheses as in 3.14,*

$$|f^{(n)}(a)| \le \frac{C}{r^n} \max_{w \in \partial B_r a} |f(w)|$$

for some constant C which only depends on n.

This has important implications in algebra, of all places.

Definition 3.17. A holomorphic function is *entire* if its domain is $\mathbf{C}$.

Theorem 3.18 (Liouville)**.** *An entire function is constant or unbounded.*

Proof. Suppose $f : \mathbf{C} \to \mathbf{C}$ and $|f(z)| \le M$. Then for each $r > 0$, the estimate

$$|f'(z)| \le \frac{M}{r}$$

holds. In particular $f' \equiv 0$. ☐

The following estimate is true clearly for linear polynomials, and then by induction and iterated differentiation one has:

Lemma 3.19. *Let $f \neq 0$ be a polynomial. Then we have $\mu > 0$ and $R \geq 1$ such that for all z with $|z| > R$,*

$$|f(z)| \geq \mu|z|.$$

So polynomials necessarily grow without bound.

The punchline is that polynomials must be zero *somewhere* in $\mathbf{C}$, though this is clearly not true in $\mathbf{R}$.

Theorem 3.20 (fundamental theorem of algebra)**.** *A nonconstant polynomial has a root.*

Proof. If not, then $1/f$ is bounded on $\mathbf{C} \setminus B_r 0$ for some $r > 0$. But $1/f$ is bounded on the compact set $\overline{B_r 0}$, because the lack of roots guarantees its continuity. So $1/f$ is constant. ☐

Thus $\mathbf{C}$ is an algebraically closed field: that is, one can completely factor a polynomial in $\mathbf{C}$ into linear factors. One proves this by inducting on the roots of f.

Liouville's theorem greatly restricts what sort of functions can be entire, as you should explore.

Exercise 3.21. Say that $f : U \to \mathbf{C}$ is *exponentially bounded* by $C \in \mathbf{R}$ if there exists $R > 0$ such that if $|z| > R$, then $|f(z)| < \exp(Cz)$.

Characterize all entire functions which are exponentially bounded by $C \leq 0$, and give an example of a function which is not exponentially bounded by any $C > 0$.

Exercise 3.22. Let f, g be entire. Say that f *dominates* g if for each $z \in \mathbf{C}$, $|f(z)| \leq |g(z)|$.

Show that if f dominates g, then there is a complex number w such that for each $z \in \mathbf{C}$, $f(z) = wg(z)$.

3.5 Locally uniform convergence

Another consequence of Cauchy-Goursat is that we don't actually care that much about uniform convergence in $\mathbf{C}$: it's locally uniform convergence that's interesting.

Definition 3.23. Suppose $f_n \to f$ is a convergent sequence of functions in U. If, for each $z \in U$, there is a neighborhood $V \ni z$ on which $f_n \to f$ uniformly, then $f_n \to f$ *locally uniformly.*

Theorem 3.24 (Weierstrass)**.** *Suppose $f_n \to f$ locally uniformly and each f_n is holomorphic. Then f is holomorphic and convergence of its derivatives is locally uniform.*

Proof. For $a \in U$ there exists $r > 0$ with $B_r a \subseteq U$ and $f_n \to f$ uniformly on $B_r a$. So f is continuous on $B_r a$. By Morera's theorem, if $\Delta \subset B_r a$, then

$$\oint_{\partial \Delta} f = 0$$

whence f is holomorphic.

Suppose now that $f_n \to f$ uniformly on $B_{2r} a$, then $f_n \to f$ uniformly on $\partial B_r a$. By Cauchy's estimate, we can bound differences in f' by differences in $\partial B_r a$. ☐

Corollary 3.25. *Suppose that $r > 0$ and $f_n \to f$ locally uniformly on $B_r 0$, with each f_n holomorphic. Suppose further that*

$$f_m(z) = \sum_{n=0}^{\infty} \alpha_n^{(m)} z^n$$

and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n;$$

then $\alpha_n^{(m)} \to \alpha_n$.

Proof. By induction on Weierstrass's theorem. ☐

The strategy to show that a limit function f is holomorphic is as follows: use the M-test to show uniform convergence on each open set, which is enough to apply Weierstrass's theorem.

Chapter 4

Integration in a simply connected set

Nothing too exciting has happened yet, because we've been hampered by the assumption that U is convex, and that f is continuous even when it fails to be holomorphic. As it turns out, local convexity is all we really need, and complex analysis becomes more interesting when f behaves more spectacularly at points where it's not holomorphic.

The residue calculus is a powerful technique for dealing with poles, points where f fails to be holomorphic or continuous, but not "too badly". It will allow us to compute integrals around closed curves just by summing up a few easy limits – and to compute integrals around curves which aren't closed, including integrals in $\mathbf{R}$, by extending them to a closed curve without affecting their value. It will also allow us to characterize holomorphic functions further as geometric transformations which "preserve angle".

Recall that U is assumed open. Furthermore we will assume $0 \le R_1 < R_2 \le \infty$.

4.1 Zeroes of holomorphic functions

Definition 4.1. If $a \in U$, $f \in \mathcal{O}$ with $f(a) = 0$, we say that a is an *isolated zero* if it is contained in a punctured ball on which f is nonzero. We further say that a is of *order* $N > 0$ if each $f^{(n)}(a) = 0$ for $n < N$ and $f^{(N)} \neq 0$. If, for each $n > 0$, $f^{(n)}(a) = 0$, then a is a *zero of infinite order.*

We'll soon see that the only interesting holomorphic functions are those which have only finite-order, isolated zeroes; if not, they'll just be identically zero!

We can "factor" out the zeroes of a holomorphic function, just like a polynomial.

Theorem 4.2. *If $a \in U$, $f \in \mathcal{O}$ with $f(a) = 0$, then $\exists r > 0$ with $B_r a \subseteq U$ such that exactly one of the following is true:*

1. *$f \equiv 0$ on $B_r a$ and a is a zero of infinite order.*
2. *a is an isolated zero and there exists a unique $g : B_r a \to \mathbf{C}$ holomorphic with $|g| > 0$ and $f(z) = (z-a)^N g(z)$ where N is the order of a.*

Proof. We can pick $B_R a \subseteq U$ on which $f(z) = \sum \alpha_n (z-a)^n$ uniformly and

$$\alpha_n = \frac{f^{(n)}(a)}{n!}.$$

Then if for each n, $f^{(n)}(a) = 0$, $\alpha_n = 0$, so $f \equiv 0$.

Otherwise it's of order $N < \infty$ and

$$f(z) = \sum_{n=N}^{\infty} \alpha_n (z-a)^n = (z-a)^N \sum_{k=0}^{\infty} \alpha_{N+k} (z-a)^k$$

and this last series is g. Moreover, for R sufficiently small, $g \neq 0$, because a is isolated by continuity of g. $\square$

Because locally, holomorphic functions act like polynomials, we can rattle off several corollaries which resemble properties of polynomials.

Corollary 4.3. *If $a \in U$, $f \in \mathcal{O}$ then $f(a) = 0$ of order N iff*

$$\lim_{z \to a} \frac{f(z)}{(z-a)^N}$$

exists and is nonzero.

Proof. One direction is obvious. Otherwise we have $f(a) = 0$, and the order is finite. So we can factor out a g, which is the limit, which is continuous and nonzero. $\square$

The following proof is an example of an "open and closed" argument, which will be a powerful tool for proving theorems about holomorphic functions on connected sets.

Corollary 4.4. *If U is connected, $B_r a \subseteq U$, and f is holomorphic with $f|_{B_r a} \equiv 0$ then $f \equiv 0$.*

Proof. The set

$$V = \{p \in U : \exists s > 0 \ f|B_s p \equiv 0\}$$

is a union of open balls, thus open. On the other hand, the set

$$W = \{p \in U : \exists s > 0 \ \forall z \in B_s p \setminus \{0\} \ f(z) \neq 0\}$$

is also a union of open balls, thus also open, and $U = V \cup W$, $\emptyset = V \cap W$. Since U is connected and $a \in V$, $W = \emptyset$. $\square$

Corollary 4.5. *Consider U connected $\supseteq V \neq \emptyset$ open, and f, g holomorphic, if $f|_V = g|_V$ then $f = g$.*

Proof. $(f - g)|_V \equiv 0$ so $f - g \equiv 0$. $\square$

In other words, if U is connected, then the behavior of f is completely determined by its behavior on any open set in U. This isn't too surprising, as an open set is all we need to construct all the Taylor coefficients of f.

In more extreme cases, we can characterize f by its behavior on a *sequence* in U.

Corollary 4.6. *For U connected, if there exists a sequence $z_n \to z$ of distinct points with $z \in U$, and $f(z_n) = 0$ for each z_n then $f \equiv 0$.*

Proof. The zeroes of f cluster at z, so z is not an isolated zero; thus, z must be a zero of infinite order. $\square$

Notice that we need $z \in U$; the counterexample is $z \mapsto \sin \pi/z$, $z_n = 1/n$, whose zeroes cluster at 0, where the map is not holomorphic.

4.2 Essential singularities

Definition 4.7. If f is holomorphic except at $a \in U$, then a is an *isolated singularity* of f. Moreover, if there is a holomorphic function $g : U \to \mathbf{C}$ such that $f \equiv g$ off of a, then a is *removable*.

Recall the notion of an analytic continuation, Definition 3.12. In the definition above, the function g is an analytic continuation of f.

The classic example of a removable singularity is -1 for the function

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}$$

which morally is $g(x) = x + 1$. Removable singularities are mostly harmless.

Theorem 4.8 (Riemann)**.** *If f has an isolated singularity at $a \in U$ and there is a neighborhood $V \ni a$ on which f is bounded, then a is removable.*

Proof. Consider $h : U \to \mathbf{C}$ with $h(z) = (z-a)f(z)$ away from a and $h(a) = 0$. h is continuous because f is bounded, and is clearly holomorphic away from a, thus cofinitely holomorphic, so Morera's theorem applies and h is holomorphic.

So, by 2.2, there exists g continuous with $h(z) = (z-a)g(z)$ (since $h(a) = 0$). But then by Cauchy-Goursat, g is holomorphic. But $g \equiv f$ away from a. □

Since, in order for an isolated singularity a to fail to be removable, f must be unbounded near a, the behavior of f near a must be calamitous. In particular, case (3) of the following theorem is shocking.

Theorem 4.9 (Casorati-Weierstrass)**.** *If f has an isolated singularity at $a \in U$ then exactly one of the following is true.*

1. *a is removable.*
2. *There exists $m \in \mathbf{N}$ and $c_1, \dots, c_m \in \mathbf{C}$ such that $c_m \neq 0$ and the function*
$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$
has a removable singularity at a.
3. *For each neighborhood V of a, $f(V)$ is dense in $\mathbf{C}$.*

Before the proof, we define what's going on in (2) and (3).

Definition 4.10. If (2) holds, then a is a *pole* of order m. A pole of order 1 is *simple*.
If (3) holds, then a is an *essential singularity*.

Definition 4.11. Suppose that $A \subset U$ is discrete, $f : U \setminus A \to \mathbf{C}$ is holomorphic, and f has no essential singularities. Then we say that f is *meromorphic* and write $f \in \mathcal{M}$.

So, meromorphic functions have poles, but no essential or nonisolated singularities.

Proof of Casorati-Weierstrass. Suppose that (3) does not hold. Then $f(V)$ isn't dense somewhere in $\mathbf{C}$, so there exist $w \in \mathbf{C}$, $r > 0$, and $\mu > 0$ with $B_r a \subseteq U$ and for $z \in B_r a \setminus \{a\}$, $|f(z) - w| > \mu$.

Define $g : B_r a \setminus \{a\} \to \mathbf{C}$ by
$$g(z) = \frac{1}{f(z) - w}.$$
g is holomorphic, but also bounded because
$$\frac{1}{|f(z) - w|} < \frac{1}{\mu},$$
so by Riemann, g has an analytic continuation $h : B_r a \to \mathbf{C}$.

Either $h(a) = 0$ or not. If not, then $f(z) - w$ does not tend to ∞ at a, so f is bounded and therefore has a removable singularity. So (1) holds.

Otherwise, a is a zero of order $m < \infty$ for h. So $\exists b : B_r a \to \mathbf{C}$ holomorphic such that $b(a) \neq 0$ and $h(z) = (z-a)^m b(z)$ and
$$f(z) = w + \frac{1}{b(z)(z-a)^m}.$$
But $1/b \neq 0$ so $1/b$ is holomorphic and has a power series,
$$\frac{1}{b(z)} = \sum_{n=0}^{\infty} \alpha_n (z-a)^n$$

whence

$$f(z) = w + \sum_{n=0}^{\infty} \alpha_n (z-a)^{n-m}.$$

When $n > m$ these terms are benign and can be absorbed into the Taylor series for f. Otherwise, $c_i = \alpha_i$, implying (2). $\square$

As it turns out, if (3) holds, then something much stronger is true: except possibly for a single point in the codomain, the function is surjective on arbitrarily small balls.

Theorem 4.12 (Picard's great theorem)**.** *Suppose that $f : U \to \mathbf{C}$ is holomorphic and has an essential singularity at $a \in U$. Then for each neighborhood $V \ni a$, either $f(V \setminus \{a\}) = \mathbf{C}$ or there exists $b \in \mathbf{C}$ such that $f(V \setminus \{a\}) = \mathbf{C} \setminus \{b\}$.*

Sadly, we won't prove this theorem. (It follows from a difficult argument that depends on Harnack's inequality, 7.25.) But indeed, as promised, essential singularities are calamitous.

So, beyond the obvious "hideous", what do essential singularities look like?

Example 4.13. Let $f(z) = \exp(1/z)$. Away from 0, f is a composition of holomorphic functions, hence holomorphic. On the other hand, there doesn't seem to be a good way to define $f(0)$, so 0 is singular and nonremovable.

We'll examine the behavior of f close to the singularity. Let $z = Re^{i\theta}$. Then

$$|f(z)| = \left|\exp\left(\frac{\cos\theta - i\sin\theta}{R}\right)\right| = \exp(R^{-1}\cos\theta) = \begin{cases} \exp(R^{-1}), & \text{if } \cos\theta > 0 \\ 1/\exp(R^{-1}), & \text{if } \cos\theta < 0 \\ 1, & \text{if } \cos\theta = 0. \end{cases}$$

As $R \to 0$, $f \to \infty$, $f \to 0$, and $|f| \to 1$, all from different directions! Thus 0 could not possibly be a pole, so 0 is essential.

The moral is that essential singularities occur precisely when f approaches different points from different directions.

Exercise 4.14. Show that the f in the above example satisfies Picard's great theorem (in particular, it satisfies the Casorati-Weierstrass theorem): for each $z \neq 0$ and $\varepsilon > 0$, show that there exists w such that $|w| < \varepsilon$ and $f(w) = z$. (Hint: consider the curve $\gamma : [0, 2\pi) \to \mathbf{C}$ given by $\gamma(\theta) = e^{i\theta}\cos\theta$.)

Now we'll show that a function has to be singular *somewhere* if it's not a polynomial.

Exercise 4.15. Suppose that f is an entire function, and for each $R > 0$ we can find a $S > 0$ such that if $|z| > S$ then $|f(z)| > R$. (That is, f has a "pole" at ∞, rather than an "essential singularity".) Prove that f is a polynomial. (Hint: use Bolzano-Weierstrass and 4.6 to get bounds on the numbers of zeroes and numbers of poles.)

4.3 The homotopy theorem

So ends our examination of essential singularities. Most of the rest of the class will be spent dealing with effects of poles. We'll start off by characterizing the orders of poles. A quick glance at the series in case (2) of the Casorati-Weierstrass theorem confirms:

Corollary 4.16. *If $f : U \setminus \{a\} \to \mathbf{C}$ is holomorphic, then a is a pole of order m iff*

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

and is finite.

The next theorem will show why we care about poles so much: integrals around closed curves only are interesting if the curve encloses singularities.

Recall the notion of a homotopy of curves, Definition 1.39.

Theorem 4.17 (homotopy theorem)**.** *If $f : U \to \mathbf{C}$ is holomorphic and γ_0 and γ_1 are U-homotopic, closed curves then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

In particular, if γ_0 is U-nullhomotopic then

$$\int_{\gamma_0} f = 0.$$

Proof. Consider a homotopy $\Phi : [0,1]^2 \to U$ between γ_0 and γ_1. $[0,1]^2$ is compact, so there are $\varepsilon > 0$ and $N > 0$ such that for each (t_1, s_1) and $(t_2, s_2) \in [0,1]^2$, if $|(t_1, s_1) - (t_2, s_2)| < 2/N$ then $|\Phi(t_1, s_1) - \Phi(t_2, s_2)| < \varepsilon$. (That is, if N is large, then we can divide $[0,1]^2$ into tiles of size $\leq 2/N$ whose images will be within ε of each other.)

Put $z_{nm} = \Phi(n/N, m/N)$ for $n, m \leq N$. In particular if $k, j \leq 1$ then $|z_{nm} - z_{n-k,m-j}| < \varepsilon$ and therefore is contained in U. The ball around z_{nm} is *convex*, so

$$\int_{z_{n-1,m-1}}^{z_{n,m-1}} f + \int_{z_{n,m-1}}^{z_{nm}} f - \int_{z_{n-1,m-1}}^{z_{n-1,m}} f - \int_{z_{n-1,m}}^{z_{nm}} f = 0$$

because the curve generated when the last two integrals are reversed is closed.

The z_{nm} tile $[0,1]^2$ so

$$\sum_{j=1}^{N} \int_{z_{j-1,0}}^{z_{j,0}} f = \sum_{j=1}^{N} \int_{z_{j-1,N}}^{z_{j,N}} f.$$

Each of these is a discretization of γ_0 or γ_1, so the result holds. ☐

We needed to painstakingly prove Cauchy-Goursat and related theorems for convex sets precisely as lemmata to the homotopy theorem: we used that U was locally convex in its proof.

Now that we know the homotopy theorem, all those old results generalize trivially to simply connected sets.

Corollary 4.18 (Cauchy-Goursat)**.** *Suppose that U is simply connected and $f : U \to \mathbf{C}$ is holomorphic. Then if γ is closed,*

$$\int_{\gamma} f = 0.$$

Corollary 4.19 (fundamental theorem of calculus)**.** *If U is simply connected, and $f : U \to \mathbf{C}$ is holomorphic, then the function $F : U \to \mathbf{C}$ given by*

$$F(z) = \int_{\gamma} f$$

is holomorphic with $F'(z) = f(z)$, where $\gamma(0)$ is constant and $\gamma(1) = z$.

Corollary 4.20 (Cauchy's integral formula)**.** *Suppose $f : U \to \mathbf{C}$ is holomorphic, γ is a closed curve in U enclosing a simply connected set V such that for each $z \in V$, $\mathrm{Ind}_\gamma(z) = 1$. Then if $w, z \in V$,*

$$|f^{(n)}(z)| = \frac{n!}{2\pi i} \int_{\Gamma_r a} \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

4.4 Laurent series

A Laurent series is a "two-sided" Taylor series, analogous to the improper integral $\int_{-\infty}^{\infty}$. We'll need finitely many negative terms to deal with poles – and infinitely many to deal with essential singularities. They describe functions which are holomorphic away from a singularity.

Recall that $0 \leq R_1 < R_2 \leq \infty$. We made that assumption so that we could make the following definition:

Definition 4.21. If $a \in \mathbf{C}$ define the *annulus*

$$A(a, R_1, R_2) = \{z \in \mathbf{C} : R_1 < |z - a| < R_2\}.$$

The annulus $A(a, 0, \infty)$ is just the plane punctured at 0.

Definition 4.22. For each $k \in \mathbf{Z}$, let $z_k \in \mathbf{C}$. The series $\sum z_k$ *converges* if $\sum_{k\geq 0} z_k$ and $\sum_{k>0} z_{-k}$ converges. Moreover, its sum is

$$\sum_{k=-\infty}^{\infty} z_k = \sum_{k=0}^{\infty} z_k + \sum_{k=1}^{\infty} z_{-k}.$$

Definition 4.23. If $a \in \mathbf{C}$, a *Laurent series* about a is a series

$$\sum_{k=-\infty}^{\infty} \alpha_k (z-a)^k$$

with coefficients $\alpha_k \in \mathbf{C}$.

Just like when we developed Taylor series, it was convenient to assume that the "center" $a = 0$. Here the series will always be centered on 0. So, for the next theorem, we'll assume f is singular at the origin.

Lemma 4.24. *If f is holomorphic on $A(0, R_1, R_2)$ and $r \in (R_1, R_2)$ then define, for each $k \in \mathbf{Z}$,*

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_r 0} \frac{f(w)}{w^{k+1}}\, dw.$$

Then α_k is independent of r, and the series

$$\sum_{k=-\infty}^{\infty} \alpha_k z^k = f(z).$$

Proof. Independence of r follows by the homotopy theorem: we could choose any curve which winds around $B_{R_1} 0$.

For $z \in A(0, R_1, R_2)$, choose r_1, r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$, and define

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

on $A(0, R_1, R_2) \setminus \{z\}$. Then g has a removable singularity at z, since f is holomorphic and thus has

$$|f'(z)| = \lim_{w \to z} \left| \frac{f(w) - f(z)}{w - z} \right| < \infty.$$

We'll identify g with its analytic continuation.

$\Gamma_{r_1} 0$ and $\Gamma_{r_2} 0$ are homotopic so

$$\int_{\Gamma_{r_1} 0} g = \int_{\Gamma_{r_2} 0} g.$$

Splitting up g, we have

$$\int_{\Gamma_{r_2} 0} \frac{f(w)}{w-z}\, dw - \int_{\Gamma_{r_1} 0} \frac{f(w)}{w-z}\, dw = f(z) \left[\int_{\Gamma_{r_2} 0} \frac{dw}{w-z} - \int_{\Gamma_{r_1} 0} \frac{dw}{w-z} \right].$$

Notice that $(w-z)^{-1}$ is not holomorphic, but it is holomorphic away from z, and that $\Gamma_{r_1} 0$ does not loop around w. So the final integral vanishes, and a now familiar computation shows that

$$\left[\int_{\Gamma_{r_2} 0} - \int_{\Gamma_{r_1} 0} \right] \frac{f(w)}{w-z}\, dw = 2\pi i f(z).$$

On $\partial B_{r_1} 0$, $h_n(w) = (w/z)^n$ has $\sum h_n$ uniformly convergent.

Thus,

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{\Gamma_{r_1}0}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\int_{\Gamma_{r_1}0}\frac{f(w)}{1-w/z}\,dw \\
&= \frac{1}{2\pi i}\frac{1}{z}\int_{\Gamma_{r_1}0} f(w)\sum_{n=0}^{\infty}\frac{w^n}{z}\,dw \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{1}{z^{n+1}}\int_{\Gamma_{r_1}0} f(w)w^n\,dw \\
&= \sum_{n=1}^{\infty}\alpha_{-n}z^{-n}.
\end{aligned}
$$

On the other hand,

$$\int_{\Gamma_{r_2}0}\frac{f(w)}{w-z}\,dw = 2\pi i\sum_{n=0}^{\infty}\alpha_n z^n$$

as desired. □

In other words, 3.14 holds even when f has singularities, as long as we allow n to range over $\mathbf{Z}$.

Definition 4.25. With notation as in 4.24, $\sum \alpha_k(z-a)^k$ is the *Laurent series* of f about a. Define $h : A(a, R_1, \infty) \to \mathbf{C}$ by

$$h(z) = \sum_{k=-\infty}^{-1}\alpha_k(z-a)^k.$$

Then h is called the *principal part* of f about a. If $R_1 = 0$, we say that α_{-1} is the *residue* of f at a and write $\alpha_{-1} = \operatorname{Res}_a f$.

We can now rephrase the Casorati-Weierstrass theorem in terms of Laurent series.

Corollary 4.26 (Casorati-Weierstrass)**.** *Let f have an isolated singularity at $a \in U$ and*

$$f(z) = \sum_{k=-\infty}^{\infty}\alpha_k(z-a)^k.$$

Let $m \in \mathbf{N}$ and h be the principal part of f. Then:

1. *a is removable iff $\forall k \le -1$, $\alpha_k = 0$.*
2. *a is an isolated pole of order m iff $\forall k < -m$, $\alpha_k = 0$ and $\alpha_{-m} \neq 0$.*
3. *a is an essential singularity iff there are infinitely many nonzero α_k $(k < 0)$.*
4. *$f - h$ has a removable singularity at a.*

We can also rephrase some of the results about factoring out zeroes and poles from holomorphic functions. This will be our main tool for computing residues.

Corollary 4.27 (zero-factoring)**.** *Suppose that f has an isolated pole of order m at a. Then*

$$\operatorname{Res}_a f = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).$$

In particular, if f has a simple pole at a, then

$$\operatorname{Res}_a f = \lim_{z\to a}(z-a)f(z).$$

We'll now observe that zeroes and poles are in some sense dual to each other. This will be a major theme once we start considering the argument principle.

Exercise 4.28. Show that if f is meromorphic with a pole at a of order m, then $1/f$ has a removable singularity at a, and if we analytically continue $1/f$ over a, then $1/f$ has a zero of order m at a.

We'll close off by considering some ring-theoretic properties of $\mathcal{M}$. If you haven't taken 113 you should skip these.

Exercise 4.29. Show that the sheaf of holomorphic functions $\mathcal{O}(U)$ is a ring. When is it an integral domain?

Exercise 4.30. Suppose that U is connected. Show that $\mathcal{M}(U)$ is the field of fractions of the ring of holomorphic functions $\mathcal{O}(U)$.

Show that $\mathcal{M}(U)$ is not a field if U is not connected. (Hint: what are the units?)

Exercise 4.31. Suppose that U is connected, so $\mathcal{M}(U)$ is a field. Let $f \in \mathcal{M}(U)$ and define the *order* of f at $a \in U$ as follows: if $f(a) \neq 0$ and a is not a pole then $o_a(f) = 0$; if $f(a) = 0$ of order m, then $o_a(f) = m$; if $f(a) = \infty$ of order m, then $o_a(f) = -m$.

Show that o_a is a *valuation* on $\mathcal{M}(U)$. That is, if $f, g \in \mathcal{M}(U)$ then $o_a(fg) = o_a(f) + o_a(g)$, $o_a(f/g) = o_a(f) + o_a(g)$, $o_a(f+g) \leq \min(o_a(f), o_a(g))$, and if $o_a(f) \neq o_a(g)$ then $o_a(f+g) = \min(o_a(f), o_a(g))$.

4.5 The residue theorem

Notice that each of the terms in the Laurent series other than the residue have an antiderivative. In particular, if γ is closed and U-nullhomotopic, and f is holomorphic off of a, then each term of the Laurent series except the residue vanishes, and

$$\frac{1}{2\pi i}\int_\gamma f = \frac{\alpha_{-1}}{2\pi i}\int_\gamma \frac{dz}{w-z} = \alpha_{-1}\operatorname{Ind}_\gamma a = \operatorname{Res}_a f \operatorname{Ind}_\gamma a$$

where α_k are the terms of the Laurent series of f.

As a result, if f has no essential singularities, then the only interesting term of the principal part is the residue, which encodes valuable information about f. Moreover, the following theorem is motivated:

Theorem 4.32 (residue theorem)**.** *Let $A \subset U$ be discrete and $f : U \setminus A \to \mathbf{C}$ be holomorphic. Suppose that for each $a \in A$, f has an isolated singularity at a.*

If $\gamma^* \subset U \setminus A$ and γ is closed and U-nullhomotopic, then

$$A' = \{a \in A : |\operatorname{Ind}_\gamma a| > 0\}$$

is finite. Moreover,

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{a_k \in A'} \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

Proof. Let Φ be a nullhomotopy of γ with image K. Then $K = \Phi([0,1]^2)$, which is compact.

Moreover, if $a \in A \setminus K$, then $\operatorname{Ind}_\gamma a = 0$. Therefore $A' \subseteq A \cap K$. Moreover, $A \cap K$ is infinite it must cluster somewhere in K by the Bolzano-Weierstrass theorem – but we assumed that A was discrete, so $A \cap K$ is finite. In particular A' is finite.

$V = (U \setminus A) \cup A'$ is therefore open and γ is therefore V-nullhomotopic. Let h_k be the principal part of f centered on a_k. Then $g = f - \sum h_k$ is holomorphic on V, so its integral vanishes. Then

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_\gamma \sum_{k=1}^N h_k = \sum_{k=1}^N \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

∎

Chapter 5

Applications of the residue calculus

The residue calculus is the use of the residue theorem to reduce integrals – which are inheritely analytic objects – to finite sums, which are algebraic in nature. The residue theorem will now serve as a sledgehammer with which we can smash open several problems: integrals over $\mathbf{R}$, combinatorial problems involving counting zeroes and poles, and basic results about the geometry of $\mathbf{C}$, among others.

Recall that U is assumed open.

5.1 Improper integration

Before Mathematica, mathematicians would use the residue theorem to compute integrals from real analysis. Even now, we tend to want to use some concrete examples to illustrate the power of the residue theorem, rather than the way some algebraists learn new theorems, which is to rephrase everything in terms of category theory.

The following lemma will allow us to construct closed curves so we can actually apply the residue theorem. The idea is that we can start with a curve that is not closed and draw a harmless arc through the half-plane, on which the integral will vanish.

Lemma 5.1 (Jordan). *Let $f : \{z \in \mathbf{C} : \operatorname{Im} z \geq 0\} \to \mathbf{C}$ be continuous and*

$$\lim_{R\to\infty} \sup_{\theta\in[0,\pi]} |f(Re^{i\theta})| = 0.$$

Define $\gamma_R : [0, \pi] \to \{z \in \mathbf{C} : \operatorname{Im} z \geq 0\}$ by $\gamma_R(\theta) = Re^{i\theta}$. For $m > 0$,

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z)\ dz = 0.$$

Proof. Immediately we have

$$\int_{\gamma_R} e^{imz} f(z)\ dz = \int_0^\pi f(Re^{i\theta}) \exp(imRe^{i\theta}) iRe^{i\theta}\ d\theta = R\int_0^\pi f(Re^{i\theta}) \exp(mR(i\cos\theta - \sin\theta) ie^{i\theta}\ d\theta.$$

Thus

$$\left|\int_{\gamma_R} e^{imz} f(z)\ dz\right| \leq \int_{\gamma_R} |e^{imz} f(z)|\ dz \leq R\int_0^\pi |f(Re^{i\theta})| e^{-aR\sin\theta}\ d\theta \leq 2R \max_{\gamma_R} |f| \int_0^{\pi/2} e^{-aR\sin\theta}\ d\theta.$$

But sin is nonnegative on $[0, \pi/2]$, so the right hand integral vanishes as $R \to \infty$. □

We also have a general formula for solving improper integrals of rational functions on $\mathbf{R}$. The idea behind the proof is the same as Jordan's lemma: we draw an arc through the half-plane, on which the function vanishes at infinity.

Lemma 5.2. *Let $p, q : \mathbf{C} \to \mathbf{C}$ be polynomials. If the degree of q exceeds the degree of p by at least two, $A = \{\operatorname{Im} z > 0\}$ is the upper half plane, and if $x \in \mathbf{R}$ then $q(x) \neq 0$, then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{a \in A} \operatorname{Res}_a f.$$

Proof. Let $\gamma_R = [-R, R] \oplus \eta_R$ where $\eta_R : [0, \pi] \to \mathbf{C}$ is given by $\eta_R(\theta) = Re^{i\theta}$. Then γ_R is a simple closed curve.

q has finitely many zeroes, so $f = p/q$ has finitely many poles, so we may choose R so large that all of the poles in A are encircled by γ_R. Let d be the degree of f; then there is a constant $\mu > 0$ such that for each $|z| > 1$, $|p(z)| \leq \mu |z|^d$. Similarly, there are constants $\rho > 0$ and $r > 0$ such that if $|z| > r$ then $|q(z)| \geq \rho |z|^{d+2}$. Choose $R > r$. Then

$$\left| \int_{\eta_R} \frac{p(z)}{q(z)} dz \right| \leq R \int_0^{\pi} \frac{|p(Re^{i\theta})|}{|q(Re^{i\theta})|} d\theta \leq \frac{\mu}{\rho} R^{-1}$$

which vanishes at infinity. Notice that for each pole $a \in A$, $\operatorname{Ind}_{\gamma_R}(a) = 1$. Therefore

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = \lim_{R \to \infty} \left[\int_{\gamma_R} - \int_{\eta_R} \right] \frac{p(z)}{q(z)} dz = \int_{\gamma_r} \frac{p(z)}{q(z)} dz = 2\pi i \sum_{a \in A} \operatorname{Res}_a f$$

by the residue theorem. $\square$

Now let's make like we're high schoolers and compute some integrals.

Example 5.3. Let's compute

$$\int_0^{\infty} \frac{\sin x}{x}\ dx.$$

Put $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$, $f(z) = z^{-1} e^{iz}$. For $\theta \in [0, \pi]$, consider

$$\begin{cases} \mu_\varepsilon(\theta) & = \varepsilon e^{i(\pi - \theta)} \\ \gamma_R(\theta) & = Re^{i\theta}. \end{cases}$$

Then $\gamma_R \oplus [-R, -\varepsilon] \oplus \mu_\varepsilon \oplus [\varepsilon, R]$ is closed and $\mathbf{C} \setminus \{0\}$-nullhomotopic.

By Jordan's lemma applied to γ_R and the residue theorem,

$$\left[\int_\varepsilon^R + \int_{\gamma_R} + \int_{-R}^{\varepsilon} + \int_{\mu_\epsilon} \right] f = \left[\int_\varepsilon^R + \int_{-R}^{\varepsilon} + \int_{\mu_\epsilon} \right] f = 0.$$

But

$$\left[\int_\varepsilon^R + \int_{-R}^{-\varepsilon} \right] f = 2i \int_\varepsilon^R \frac{\sin x}{x}\ dx$$

and

$$\lim_{\varepsilon \to 0} \int_{\mu_\varepsilon} f = \lim_{\varepsilon \to 0} \int_0^{\pi} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\ d\theta = \lim_{\varepsilon \to 0} -i \int_0^{\pi} e^{i\varepsilon e^{i\theta}}\ d\theta = -i\pi.$$

So,

$$\int_0^R \frac{\sin x}{x}\ dx = \frac{i\pi}{2i} = \frac{\pi}{2}.$$

For $\operatorname{Re} z > 1$, the *Riemann ζ-function* is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Example 5.4. We will compute $\zeta(2)$. This works for any $2n$, $n \geq 1$.

Let $g : \mathbf{C} \setminus \mathbf{Z} \to \mathbf{C}$ be $g(z) = \cot \pi z$. Then g is holomorphic, and if $z \in (0,1) \times i\mathbf{R}$, $k \in \mathbf{Z}$, $g(z+k) = g(z)$. Moreover, for each compact $K \subseteq \mathbf{C} \setminus \mathbf{Z}$, g is bounded. In particular, for $\delta > 0$, g is bounded on $K_k = [k+\delta, k+1-\delta]$. g is also bounded outside of $\{z \in \mathbf{C} : |\operatorname{Im} z| < \delta\}$, as is clear if one rewrites g in terms of the complex exponential. So g has poles precisely on $\mathbf{Z}$.

If $f(z) = z^{-2} g(z)$ then

$$\lim_{N \to \infty} \int_{\Gamma_{N+\delta} 0} f = 0$$

since by Jordan's lemma applied twice. Moreover, close to 0, g satisfies

$$\frac{1 + o(z^2)}{\pi z + o(z^2)} = \frac{1}{z} \frac{1}{\pi}.$$

In particular, $\operatorname{Res}_0 g = \pi^{-1}$, so

$$\operatorname{Res}_n f = \frac{1}{\pi n^2}.$$

At 0, f has a pole of third order, so by the zero-factoring corollary, $\operatorname{Res}_0 f = -\pi/3$.

It follows that

$$0 = \lim_{N \to \infty} \lim_{\delta \to 0} \int_{\Gamma_{N+\delta} 0} f = \lim_{N \to \infty} 2 \sum_{k=1}^{N} \frac{1}{\pi k^2} - \frac{\pi}{3} = \lim_{N \to \infty} 2 \sum_{k=1}^{\infty} \frac{1}{\pi k^2} - \frac{\pi}{3} = 2\zeta(2) - \frac{\pi}{3}$$

whence

$$\zeta(2) = \frac{\pi^2}{6}.$$

Exercise 5.5. Let $U \subseteq \mathbf{C}$ be open, $\overline{\mathbf{D}} \subseteq U$, and $f : U \to \mathbf{C}$ holomorphic. Compute

$$\int_0^{2\pi} f(e^{i\theta}) \cos^2(\frac{\theta}{2})\, d\theta.$$

5.2 The argument principle

The residue theorem will allow us to prove the argument principle, a powerful combinatorial theorem with lots of consequences.

Definition 5.6. γ is *simple* if $\operatorname{Ind}_\gamma(\mathbf{C} \setminus \gamma^*) = \{0, 1\}$.

Simple curves are those which wind around exactly once, counterclockwise. They're deformations of counterclockwise circles.

Theorem 5.7 (argument principle)**.** *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \operatorname{Ind}_\gamma(a) = 1\},$$

$A \subset U$ is finite, $f : U \setminus A \to \mathbf{C}$ is holomorphic, and f has N zeroes and P poles in U_1 (repeated by order).

If, for each $a \in A$, f has a finite-order pole at a, and has no poles and zeroes on γ^*, then

$$N - P = \operatorname{Ind}_{f \circ \gamma}(0).$$

Proof. Since the poles are isolated, $U \setminus A$ is connected. Similarly

$$V = U \setminus (A \cup \{z \in U : f(z) = 0\})$$

is open, and $|f| > 0$ on V. Define $g : V \to \mathbf{C}$ by $g = f'/f$ which is holomorphic.

Suppose that f has a zero at a of order m. Then there exists a *locally* holomorphic h, nonzero at a, given by $f(z) = (z-a)^m h(z)$. Then, locally,

$$g(z) = \frac{mh(z)}{(z-a)h(z)} + \frac{h'(z)}{h(z)}$$

and the second term is holomorphic. The first term has a residue of m.

Repeating the argument for poles (with signs swapped), we can apply the residue theorem to the integral

$$\frac{1}{2\pi i}\int_\gamma g = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ dz.$$

□

The argument principle is a counting principle: if γ winds around U once, and f is meromorphic on U, then we can count the zeroes and poles, and their difference will be $\mathrm{Ind}_{f\circ\gamma}(0)$. Why 0? Because we're counting zeroes! If we were counting the points where $f = b$, we would have $\mathrm{Ind}_{f\circ\gamma}(b)$.

Lemma 5.8. *Suppose $f : U \to \mathbf{C}$ is holomorphic, γ closed in U, and $b \in U \setminus \gamma^*$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}\ dz = \mathrm{Ind}_{f\circ\gamma}(b).$$

Proof. We can assume γ is smooth, so $f\circ\gamma$ is holomorphic, and that the domain of γ is $[0,1]$. Then

$$\begin{aligned}\mathrm{Ind}_{f\circ\gamma}(b) &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dz}{z-b} = \frac{1}{2\pi i}\int_0^1 \frac{(f\circ\gamma)'(t)\ dt}{(f\circ\gamma)(t)-b}\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)\ dt}{f(\gamma(t))-b} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\ dz}{f(z)-b}.\end{aligned}$$

□

Corollary 5.9. *With hypotheses as in 5.7, $A = \emptyset$, and $b \in \mathbf{C}\setminus f(\gamma^*)$, $f - b$ has $\mathrm{Ind}_{f\circ\gamma}(b)$ zeroes in U_1.*

Exercise 5.10. Let

$$U = \{z \in \mathbf{C} : \mathrm{Im}\, z > 0\}$$

be the half space. Let $h : U \to \mathbf{C}$ be given by

$$h(z) = \int_0^z \frac{dw}{w^2-1}.$$

Show that h has an analytic continuation $\tilde{h} : \mathbf{C}\setminus[-1,1] \to \mathbf{C}$.

Exercise 5.11. Let $\gamma : [0, 2\pi] \to \mathbf{C}$ be the curve

$$\gamma(\theta) = \frac{6e^{i\theta} + 2e^{3\theta}}{10e^{2\theta} - 1}$$

Compute $\mathrm{Ind}_\gamma(0)$ and $\mathrm{Ind}_\gamma(1+i)$.

Once again we have an easy proof of the fundamental theorem of algebra.

Exercise 5.12. Let p be a polynomial of degree n and $\gamma_R(\theta) = p(Re^{i\theta})$ (where $\theta \in [0, 2\pi)$). Show that if R is large enough then $\mathrm{Ind}_{\gamma_R}(0) = n$, but that if, for each $z \in \overline{B_R 0}$, $p(z) \neq 0$, then $\mathrm{Ind}_{\gamma_R}(0) = 0$. Why does this imply the fundamental theorem of algebra?

5.3 Rouche's theorem

We can now see that adding a function g which is smaller than f to f won't affect its number of zeroes.

Theorem 5.13 (Rouche). *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \operatorname{Ind}_\gamma(a) = 1\},$$

$f, g : U \to \mathbf{C}$ are holomorphic, and for each $z \in \gamma^*$, $|g(z)| < |f(z)|$. Let N_h denote the number of zeroes of h repeated by multiplicity. Then $N_f = N_{f+g}$.

Proof. Let $\tau \in [0,1]$ and $f_\tau = f + \tau g$. Now $f_\tau \neq 0$ on γ^* because if not then $|f(z)| = |\tau g(z)| \leq |g(z)|$ somewhere on γ^*. So we can apply the argument principle.

Put $\phi : [0,1] \to \mathbf{C}$ by

$$\phi(\tau) = \frac{1}{2\pi i} \int_\gamma \frac{f_\tau'}{f_\tau} = \operatorname{Ind}_{f_\tau \circ \gamma}(0)$$

which is constant. In particular $\phi(0) = \phi(1)$ so $N_f = N_{f+\tau g}$. □

We can understand the behavior of zeroes under the effect of locally uniform convergence. if f is the locally uniform limit of a sequence of functions f_n, then each zero of f forms by zeroes of the f_ns "converging" together to a single point!

Theorem 5.14. *Suppose U is connected, $f_k : U \to \mathbf{C}$ are holomorphic, $f_k \to f$ locally uniformly, f is not identically zero, $a \in U$, $m \in \mathbf{N}$, and $f(a) = 0$.*

a is a zero of order m iff there is a neighborhood V such that $a \in V \subseteq U$ and for each $s > 0$, if $B_s a \subseteq V$ then cofinitely many of the f_n have m zeroes in $B_s a$, counted by multiplicity.

Proof. Choose $r > 0$ such that $f_k \to f$ uniformly on $B_r a$ and $\forall z \in B_r a \setminus \{a\} \subseteq U$, $f(z) \neq 0$. Let $s < r$ such that

$$\varepsilon = \min_{\partial B_s a} |f| > 0.$$

Then

$$\exists N \in \mathbf{N}\ \forall n > N |f_n - f| < \varepsilon$$

on $B_r a$. In particular, for $z \in \partial B_s a$, $|f_n(z) - f(z)| < |f(z)| \leq \varepsilon$. Let $\gamma = \Gamma_s a$ and apply Rouche. Then

$$N_{f_n} = N_{f_n - f + f} = N_f.$$

□

Corollary 5.15 (Hurwitz). *If U is connected, $f_k : U \to \mathbf{C}$ are holomorphic, and $f_k \to f$ locally uniformly, and $\forall k$ $|f_k| > 0$, then either $f \equiv 0$ or $|f| > 0$.*

As with the argument principle, there's nothing special about 0: we could translate f and do this for any $b \in \mathbf{C}$. Doing this for *every* such b, and requiring that each f_n hits b at most once, gives the following:

Corollary 5.16. *If U is connected, $f_k : U \to \mathbf{C}$ are holomorphic and injective, $f_k \to f$ locally uniformly, then f is constant or injective.*

Exercise 5.17. Let $a > e$. How many zeroes does

$$e^z = az^n$$

have on $\mathbf{D}$?

Exercise 5.18. Give a one-line proof of the fundamental theorem of algebra.

5.4 Open maps and maximum moduli

The higher-dimensional version of the next theorem, proven in functional analysis, is one of the most powerful theorems in mathematics, an analyst's Zorn's lemma.

Theorem 5.19 (open mapping theorem)**.** *Let $f : U \to \mathbf{C}$ be holomorphic and not constant. Then $f(U)$ is open.*

To prove it, let's rephrase the open mapping theorem as a statement about zeroes in neighborhoods, so we can apply the argument principle.

Lemma 5.20. *With hypotheses as in 5.19, suppose $a \in U$, $f(a) = b$, and $f - b$ has a zero at a of order $N < \infty$.*

Then there are open neighborhoods U_0 of a and V_0 of b such that $f(U_0) = V_0$. Moreover, if $w \in V_0 \setminus \{b\}$, then $f - w$ has N simple zeros in U_0 and no other zeroes.

Proof. $\exists r > 0$ such that $B_{2r}a \subseteq U$ and for $z \in B_{2r}a$, $f(z) - b \neq 0$ and $f'(z) \neq 0$. Now do the argument principle with $\gamma = \Gamma_r a$. Then $\mathrm{Ind}_{f\circ\gamma}(b) = N$.

Let $V_0 = \mathrm{Ind}^{-1}_{f\circ\gamma}(\{N\})$ which is open because $\{N\}$ is open in the topology of $\mathbf{Z}$. Similarly let $U_0 = B_r a \cap f^{-1}(V_0)$, which is clearly open. Then $f(U_0) \subseteq V_0$.

Let $w \in V_0$. If $w = b$ then $f(a) = w$. Otherwise, $\mathrm{Ind}_{f\circ\gamma}(w) = N$ so $f - w$ has N zeroes by multiplicity in $B_r a$. Let z be such a zero; then $f(z) = w$ so $V_0 \subseteq f(U_0)$. Also, $f'(z) \neq 0$, so

$$0 \neq \frac{d}{dz}(f'(z) - w)$$

so f has a simple zero there. □

Proof of 5.19. $U = \bigcup U_0$, so $V = \bigcup V_0$. Open sets remain open after arbitrary unions. □

The open mapping theorem has important corollaries for dealing with extrema.

Corollary 5.21 (maximum modulus principle)**.** *If U is connected, $f : U \to \mathbf{C}$ holomorphic, and $|f|$ attains its max on U, then f is constant.*

Proof. If $|f|$ has a maximum at $a \in U$ then there are U_0 and V_0, as in the open mapping theorem. But then $\exists c \in V_0$ with $|c| > |b|$. □

Similarly:

Corollary 5.22. *Suppose that U is connected, and $f : U \to \mathbf{C}$ is holomorphic.*

If $\mathrm{Re}\, f$ or $\mathrm{Im}\, f$ attains a max or min, or $|f|$ attains its min, then f is constant.

Moreover, if U is bounded and f is also defined and continuous on ∂U, then

$$\sup_U |f| = \max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

The maximum modulus principle allows us to characterize holomorphic functions from the ball to itself: up to a translation, they must be rotations or weak contractions.

Lemma 5.23 (Schwarz)**.** *Let $f : \mathbf{D} \to \mathbf{D}$ be holomorphic and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $a \neq 0$ and there exists $a \in \mathbf{D}$ with $f|(a)| = |a|$ then $f(z) = \lambda z$ on $\mathbf{D}$ where $\lambda \in S^1$.*

Proof. Let $g : \mathbf{D} \to \mathbf{C}$ be given by the analytic continuation of $g(z) = z^{-1}f(z)$. Then, for $r < 1$, $|z| = r$,

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

and in particular max modulus implies $|g|_{B_r 0} \leq r^{-1}$. It follows that $|g| \leq 1$. So $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

On the other hand, if $|f(z)| = |z|$ then g is constant by max modulus. So $\lambda = g(z)$. □

Later, we'll use a little conformal geometry to greatly strengthen Schwarz' lemma, bringing it to bear on much more general domains than $\mathbf{D}$. But first, let us demonstrate a typical application of Schwarz:

Theorem 5.24 (Schwarz-Pick)**.** *Let $f : \mathbf{D} \to \mathbf{D}$ be holomorphic. Then if $z_1, z_2 \in \mathbf{D}$,*

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|.$$

Proof. Fix z_1 and let $\varphi, \psi : \mathbf{D} \to \mathbf{D}$ be given by

$$\psi(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)} - z}$$

and

$$\varphi(z) = \frac{z_1 - z}{1 - \overline{z_1}z}.$$

As you should verify, $\tilde{f} = \varphi \circ f \circ \psi^{-1} : \mathbf{D} \to \mathbf{D}$ sends $0 \mapsto 0$. Thus by Schwarz' lemma, if $z \in \mathbf{D}$ then

$$|\tilde{f}(z)| = \left|\frac{f(z_1) - f(\psi^{-1}(z))}{1 - \overline{f(z_1)}f(\psi^{-1}(z))}\right| \leq |z|.$$

Now take $z = \psi(z_2)$. □

This is a typical problem, par for the course for Schwarz: deform f into a function $\tilde{f} : \mathbf{D} \to \mathbf{D}$ which preserves 0, apply Schwarz, and then deform $\tilde{f}$ back into f.

Exercise 5.25. Let $R, S > 0$, $a \in \mathbf{C}$, and $f : B_R a \to B_S a$ be holomorphic. Prove that for each $z \in B_R a$,

$$|f(z) - f(a)| \leq \frac{2S}{R}|z - a|.$$

Finally, let's see why it was that we required our domain U to be open.

Exercise 5.26. Let $K \subset \mathbf{C}$ be compact, and let $V(K)$ be the vector space of complex-differentiable functions $K \to \mathbf{C}$. Give a formula for the (complex) dimension of $V(K)$, in terms of the topology of K.

Chapter 6

Conformal geometry

We're in the position to give a geometric interpretation of holomorphicity. Holomorphic functions are those which, locally, preserve angles of intersections of curves. A subclass of functions (the so-called "Moebius transformations") are those which preserve angles globally. We'll be able to use holomorphic functions to completely classify open, simply connected sets in $\mathbf{C}$. Later on, we'll use this classification to solve PDE in $\mathbf{R}^2$.

Recall that U is assumed open.

6.1 Conformal maps

Recall the notion of an angle between two curves (Definition 1.35). We'll now see that a holomorphic function is one which locally preserves such angles. (Note that holomorphic functions do not, in general, send lines to lines and circles to circles.)

Lemma 6.1. *Let $\gamma_1(0) = \gamma_2(0) = z \in U$ and $f : U \to \mathbf{C}$ be holomorphic. Let θ be the angle between two curves at* 0.

If $|f'| > 0$ then $\theta(\gamma_1, \gamma_2) = \theta(f \circ \gamma_1, f \circ \gamma_2)$.

Proof. One computes

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{(f' \circ \gamma_1)(0)\gamma_1'(0)}{(f' \circ \gamma_2)(0)\gamma_2'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}.$$

Now take the arguments. □

If one plots a holomorphic function in Mathematica, it looks very pretty, with lots of symmetries and loops. This is why.

Definition 6.2. $f : U \to V$ is *biholomorphic*, *conformal*, *angle-preserving*, or a *complex diffeomorphism* if f is a bijection and f and f^{-1} are both holomorphic.

If such a conformal map exists then U and V are *conformally equivalent* or *conformal*.

Notice that some writers only assume that a conformal map is injective, not surjective!

By Liouville's theorem, $\mathbf{D}$ and $\mathbf{C}$ are not conformal, but on the other hand, this shocking result is true:

Theorem 6.3 (Riemann mapping theorem)**.** *If U is simply connected, then U is conformal with $\mathbf{D}$ or $\mathbf{C}$.*

We'll prove the Riemann mapping theorem once we have some more machinery.

But how shall we prove that a function is conformal? All we need is for F to be holomorphic and injective, and F is guaranteed to be conformal by the open mapping theorem:

Theorem 6.4 (inverse function theorem)**.** *Let $F : U \to \mathbf{C}$ be injective and holomorphic, $V = F(U)$. Then F is conformal from U to V and $|F'| > 0$.*

Proof. By the open mapping theorem, V is open. For $a \in U$, $b = F(a)$, $r > 0$, $F(U \cap B_r a)$ is open, and the $U \cap B_r a$ are a basis for the topology of V, so F^{-1} is continuous.

If $F'(a) = 0$ then at a, $F - b$ has an order-2 zero, so we have $r > 0$ such that if $w \in B_1 b \setminus \{b\}$ then $F - w$ has two simple zeroes. So we have $\beta \neq \gamma$ such that $F(\beta) = w = f(\gamma)$, which is a contradiction. So $|F'| > 0$.

If H is the continuous extension of $\frac{z-a}{F(z)-b}$ then H is continuous. So

$$f^{-1}(b) = \lim_{v \to b} \frac{F^{-1}(u) - F^{-1}(b)}{u - b} = \lim_{u \to b} (H \circ F^{-1})(u) = \frac{1}{F'(a)}.$$

So F^{-1} is holomorphic. $\square$

In special relativity, a particle P's equations of motion are preserved under a group known as the Lorentz group. However, if P's mass vanishes, then the equations of motion are preserved under *any* conformal mapping.

6.2 Automorphisms of the disk

The key step in problem-solving using conformal maps is to realize that there are not so many automorphisms $\mathbf{D} \to \mathbf{D}$. In fact, if an automorphism of the disk preserves 0, it's nothing more than a rotation.

Lemma 6.5. *Let $F : \mathbf{D} \to \mathbf{D}$ be conformal and $F(0) = 0$. Then $\exists \omega \in S^1$ such that for each $z \in \mathbf{D}$,*

$$F(z) = \omega z.$$

Proof. Apply Schwarz to F and F^{-1} so that

$$|z| \leq |F(z)| \leq |z|.$$

$\square$

Theorem 6.6. *$F : \mathbf{D} \to \mathbf{D}$ is conformal if and only if there exist unique $a \in \mathbf{D}$ and $\omega \in S^1$ such that*

$$F(z) = \omega \frac{z - a}{1 - \overline{a} z}.$$

Moreover, a can be chosen so that $F(a) = 0$, and

$$F^{-1}(z) = \omega^{-1} \frac{z + a}{1 + \overline{a} z}.$$

Proof. Suppose that F is conformal and let $G : \mathbf{D} \to \mathbf{D}$ be given by

$$G(z) = \frac{z - a}{1 - \overline{a} z}.$$

Then since $|\overline{a} z| < 1$, $G \circ F^{-1}$ is conformal and $G(a) = 0$, thus $G \circ F^{-1}(0) = 0$. So by 6.5, $G \circ F^{-1}(z) = \omega z$ for some $\omega \in S^1$. Existence of ω is immediate (how could one rotate the disk at two different speeds?) and since a is determined by the preimage of 0, a is unique as well.

On the other hand, if

$$F(z) = \omega \frac{z - a}{1 - \overline{a} z},$$

then since $|\overline{a} z| < 1$, F is holomorphic, as is its inverse (that it actually is an inverse is immediate). $\square$

Thanks to the inverse function theorem, once one's constructed a bijection it's not to show that the function is, in fact, conformal.

Example 6.7. The half space

$$U = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$$

is conformal with $\mathbf{D}$ as witnessed by the function

$$F(z) = \frac{z-i}{z+i}.$$

To see this, consider that

$$F'(z) = \frac{z+i+z-i}{(z+i)^2} = \frac{2z}{(z+i)^2} \neq 0,$$

so by the inverse function theorem it is injective. Moreover, if $|w| < 1$, then for

$$z = i\frac{1+w}{1-w}$$

it's clear that $F(z) = w$, so F is surjective.

You should generalize the above results to show that, up to some details, it completely characterizes conformal maps between the half space and the ball:

Exercise 6.8. Let U be the half space and $F : U \to \mathbf{D}$ be conformal. Show that

$$F(z) = \omega\frac{z-a}{z-\overline{a}}$$

for some $\omega \in S^1$ and $a \in U$.

Conformal maps are a useful tool for generalizing results about $\mathbf{D}$ to larger sets. The general idea is to show that there exists a conformal map $\varphi : U \to \mathbf{D}$ by the Riemann mapping theorem, and then, rather than studying a function $f : U \to U$, study the function $\tilde{f} = \varphi \circ f \circ \varphi^{-1} : \mathbf{D} \to \mathbf{D}$, which is much more easily understood.

Exercise 6.9. Let U be the half space and $f : U \to U$ be holomorphic. Prove that if $z, w \in U$ then

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} - f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{\overline{z_1} - z_2}\right|.$$

6.3 The Riemann mapping theorem

All we've done has been building up to this: a classification of all simply connected open sets in $\mathbf{C}$, up to conformality.

The below material critically on the Arzela-Ascoli theorem from functional analysis. The statement and proof of Arzela-Ascoli is given in the preliminaries chapter, since it's not directly related to complex analysis. It would also be worthwhile to review the definition of a topology, so that the compact-open topology feels less foreign. (In fact, many 185 classes skip over Montel's theorem and the Riemann mapping theorem entirely, to avoid covering Arzela-Ascoli. If this is the case, you can just skip this section, black box the Riemann mapping theorem – we won't need it to prove much, though it will serve as motivation – and carry on.)

First we need the notion of a normal family. Normal families are certain well-behaved spaces of holomorphic functions. These are of a similar flavor to the equicontinuous spaces of functions that you may have studied in 104. Recall that $\mathcal{O}$ is the sheaf of holomorphic functions on U.

Definition 6.10. Let $\mathcal{F} \subset \mathcal{O}$. $\mathcal{F}$ is said to be a *normal family* if for each sequence $f_1, f_2, \dots$ in $\mathcal{F}$, there is a subsequence f_{k_n} which converges locally uniformly.

Notice that the definition of normality feels very much like compactness. In fact, one can put a topology on $\mathcal{O}$ (the *compact-open topology*) in which a function converges iff it converges locally uniformly; then $\mathcal{F}$ is normal iff it is *precompact* in $\mathcal{O}$ under the compact-open topology (a subspace is precompact if its closure is compact).

We shall characterize normal families. The key hypothesis is local uniform bounding:

Definition 6.11. Let $\mathcal{F} \subset \mathcal{O}$. $\mathcal{F}$ is said to be a *locally uniformly bounded family* if for each point $z \in U$ there is an open set $V \ni z$ and a constant $M > 0$ such that for each $f \in \mathcal{F}$, $|f| < M$ on V.

Theorem 6.12 (Montel)**.** *$\mathcal{F}$ is locally uniformly bounded iff it is normal.*

Proof. Let $\mathcal{F}$ be a locally uniformly bounded family on U, $z \in U$ and $\varepsilon > 0$. Then there are $M > 0$ and $r > 0$ such that for each $f \in \mathcal{F}$, $|f| < M$ on $\overline{B_{2r}z_0} \subset U$. Moreover, if $z, w \in B_{2r}z_0$ then

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\Gamma_{2r}z_0} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta$$

by Cauchy's integral formula. Moreover for each ζ such that $|\zeta - z_0| = 2r$, and $z, w \in B_r z_0$,

$$|(\zeta - z)(\zeta - w)| > r^2.$$

Since $f(\zeta) < M$ anyways,

$$|f(z) - f(w)| \leq \frac{2|z - w|}{r} \sup_{\partial B_{2r}z_0} |f| < \frac{2M|z - w|}{r}.$$

Therefore if $\delta < r$ and $\delta < r\varepsilon M^{-1}/4$, then for $|z - w| < \delta$ one has $f(z) - f(w)| < \varepsilon.$

Since this δ only depended on ε and M and not f, $\mathcal{F}$ is equicontinuous on each $\overline{B_{2r}z_0}$, which are compact, and clearly $\mathcal{F}$ is bounded on each. Taking closures, $\mathcal{F}$ restricted to a sufficiently small compact set satisfies the hypotheses of the Arzela-Ascoli theorem and is thus compact in C^0, thus normal.

On the other hand, if $\mathcal{F}$ is normal, then $\mathcal{F}$ restricted to a sufficiently small compact set K is compact in C^0. On each such K, $\mathcal{F}$ is uniformly bounded. Therefore $\mathcal{F}$ is locally uniformly bounded. $\square$

Now we'll begin proving the lemmata that are necessary for the Riemann mapping theorem. Similar to the open mapping theorem, the statement of the Riemann mapping theorem is too weak for it to be clear how to prove it. We'll need to prove a stronger statement, using Montel's theorem, Hurwitz' theorem, and Schwarz' lemma as crucial lemmata, in order to recover the Riemann mapping theorem.

Our strategy will be as follows: construct a normal family $\mathcal{F}$ of "candidate functions", and exploit compactness to show that a candidate function which maximizes a certain quantity actually exists; this function will be the long-sought-after conformal map.

Lemma 6.13. *Suppose that $U \neq \mathbf{C}$ is simply connected and $z_0 \in U$. Then the space*

$$\mathcal{F} = \{f : U \to \mathbf{D} \,|\, f \text{ is holomorphic and injective, and } f(z_0) = 0\}$$

is a nonempty normal family.

Proof. For convenience, let V be the strip

$$V = \{z \in \mathbf{C} : |\operatorname{Im} z| < \pi\}.$$

Let $a \notin U$ and observe that $z - a \neq 0$. Since exp has a period of $2\pi i$, exp is a conformal map $V \to \mathbf{C} \setminus \{0\}$. In particular, its inverse log is conformal $\mathbf{C} \setminus \{0\} \to V$. Now let $\ell(z) = \log(z - a)$; ℓ is holomorphic and injective.

Suppose that there does not exist $\delta > 0$ such that for each $z \in U$,

$$|\ell(z) - \ell(z_0) - 2\pi i| < \delta.$$

Then there is a sequence z_n in U such that $\ell(z_n) \to \ell(z_0) + 2\pi i$. By continuity of exp, $z_n \to z_0 e^{2\pi i} = z_0$. So $\ell(z_n) \to \ell(z_0) \neq \ell(z_0) + 2\pi i$, which is a contradiction, so δ exists.

Let

$$g(z) = \frac{1}{\ell(z) - \ell(z_0) - 2\pi i}.$$

Then g is clearly injective and holomorphic and $|g| < \delta^{-1}$, so g is bounded, and

$$f(z) = \frac{g(z) - g(z_0)}{\delta^{-1} + |g(z_0)|}$$

satisfies $f \in \mathcal{F}$. Thus $\mathcal{F}$ is nonempty.

Moreover, each $f \in \mathcal{F}$ satisfies $|f| < 1$, so $\mathcal{F}$ is uniformly bounded by 1; therefore $\mathcal{F}$ is normal by Montel's theorem. □

Lemma 6.14. *With $\mathcal{F}$ and z_0 as in 6.13, there exists a conformal $F \in \mathcal{F}$ and if*

$$\lambda = \sup_{f \in \mathcal{F}} |f'(z_0)|$$

then $|F'(z_0)| = \lambda$.

$\mathcal{F}$ is precompact in $\mathcal{O}$, but this doesn't mean that $\mathcal{F}$ is compact! So, while we want to use compactness to prove the existence of a function F that satisfies the desired hypotheses, we'll have to prove that in fact $F \in \mathcal{F}$ rather than $F \in \partial\mathcal{F}$.

Proof of 6.14. Since elements of $\mathcal{F}$ are injective, their derivatives are nonzero by the argument principle, so $\lambda > 0$, and there is a sequence f_n such that $|f_n'(z_0)| \to \lambda$. By Montel's theorem, it has a convergent subsequence with a limit $F \in \mathcal{O}$, such that $F(z_0) = 0$ and $F'(z_0) = \lambda$ by Weierstrass' theorem. Moreover, $|F| \leq 1$. F is nonconstant since $\lambda > 0$, so by the open mapping theorem, $|F| < 1$, and by Hurwitz' theorem, F is injective. Therefore $F \in \mathcal{F}$.

Moreover, if F is not surjective, then there exists $w \in \mathbf{D} \setminus F(U)$. Let

$$\psi(z) = \frac{w - z}{1 - \overline{w}z}.$$

By 6.6, ψ is an automorphism of $\mathbf{D}$, $\psi(w) = 0$ and $|\psi \circ F| > 0$. Now define

$$g(z) = \exp\left(\frac{\log \psi(F(z))}{2}\right).$$

Then $g(z_0) = \sqrt{w}$. By 6.6 again,

$$\tilde{\psi}(z) = \frac{\sqrt{w} - z}{1 - \overline{\sqrt{w}}z}$$

is also an automorphism of $\mathbf{D}$ and $\tilde{\psi}(\sqrt{w}) = 0$. Let $G = \tilde{\psi} \circ g$. Then $G \in \mathcal{F}$.

Now let $s(z) = z^2$ and $\varphi = \psi^{-1} \circ s \circ \tilde{\psi}^{-1}$. Then $\varphi(0) = 0$ so by Schwarz $|\varphi(z)| \leq |z|$. Moreover, φ is not injective anywhere, so $|\varphi'| < 1$. But $F = \varphi \circ G$ so

$$\lambda = |F'(z_0)| = |\varphi'(0)G'(z_0)| < |G'(z_0)| \leq \lambda$$

which is a contradiction.

So F is surjective. Therefore the inverse function theorem implies that F is conformal. □

6.14 implies the Riemann mapping theorem, where F is the desired conformal map $U \to \mathbf{D}$. But we can do better, establishing that up to a rotation, F is the *unique* conformal map in $\mathcal{F}$.

Theorem 6.15 (Riemann mapping theorem, but stronger)**.** *Let $U \neq \mathbf{C}$ be simply connected and $z_0 \in U$. Then there exists a conformal map $F : U \to \mathbf{D}$ such that $F(z_0) = 0$. Moreover, if*

$$\omega = \frac{F'(z_0)}{|F'(z_0)|},$$

then F is the unique conformal map such that $F(z_0) = 0$ and $F'(z_0) = \omega|F'(z_0)|$.

Proof. Existence is immediate by 6.14. Now let $\mathcal{F}$ and λ be as in 6.14. Suppose that $\tilde{F} \in \mathcal{F}$ also is conformal with argument ω and let $f = F \circ \tilde{F}^{-1}$. Then f is an automorphism of $\mathbf{D}$, and $f(0) = 0$. So by 6.5, $f(z) = \eta z$ for some $\eta \in S^1$. But then

$$\eta = f'(z) = F'(z_0)(\tilde{F}')^{-1}(0) = \frac{\omega}{\omega}\frac{|F'(z_0)|}{|\tilde{F}'(z_0)|} = \frac{\lambda}{\lambda} = 1.$$

□

6.4 The Riemann sphere

The most basic conformal maps are those of the form $z \mapsto wz$, where $w = \lambda e^{i\theta}$ ($\lambda \in (0, \infty)$ and $\theta \in [0, 2\pi)$). This map dilates its domain by λ and then rotates it by θ; morally, it's just an injective linear map. We can readily generalize to affine maps $z \mapsto wz + z_0$ (where $z_0 \in \mathbf{C}$). These are an injective linear map followed by a translation by z_0. Since the derivative is constantly $\lambda > 0$, these maps are conformal.

However, most examples of conformal maps so far are *fractions* of affine maps. This presents trouble, but not too much trouble: in general, the denominator might be 0, in which case the conformal map blows up, but that's nothing a clever definition won't fix.

Definition 6.16. The *Riemann sphere* $\hat{\mathbf{C}} = \mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$ (also known as the *one-point compactification* of $\mathbf{C}$ or the *complex projective line*) is the plane $\mathbf{C}$ equipped with an additional *point at infinity* ∞. An open ball centered at infinity is

$$B_\delta \infty = \{z \in \mathbf{C} : z\delta > 1\}$$

and an open set in $\hat{\mathbf{C}}$ is the union of open sets in $\mathbf{C}$ with open balls centered at infinity.

We define arithmetic on $\hat{\mathbf{C}}$ in the expected way: for each $z \neq 0$, $z + \infty = z\infty = \infty$, while $z/0 = \infty$ and $z/\infty = 0$. We dare not define 0∞, however.

If a function f is meromorphic, then at each point z in a discrete set A, f has a pole. However, a pole is just a point $z \mapsto \infty$. So we can think of meromorphic functions as holomorphic maps, where the codomain is allowed to include ∞. That is, a meromorphic function is a holomorphic map $U \to \hat{\mathbf{C}}$. In particular, the sheaf of meromorphic functions $\mathcal{M}$ is a field (Exercise 4.30.)

The moral is that we have no reason to fear quotients of affine maps; they are still conformal, but map $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ rather than $\mathbf{C} \to \mathbf{C}$. We shall use this characterization extensively.

Theorem 6.17. *$\hat{\mathbf{C}}$ is homeomorphic to the sphere $S^2 \subset \mathbf{R}^3$.*

That is, there are continuous bijections $F : \hat{\mathbf{C}} \to S^2$ and $G : S^2 \to \hat{\mathbf{C}}$ such that $F^{-1} = G$.

Moreover, F can be chosen so that 0 is sent to the south pole, ∞ to the north pole, and the unit circle S^1 to the equator.

Proof. We shall construct "charts", homeomorphisms between pieces of S^2 and $\hat{\mathbf{C}}$, and patch them together. Let $\hat{x}, \hat{y}, \hat{z}$ be the usual basis vectors in $\mathbf{R}^3$. Observe that the map $\rho(z) = 1/z$ is a homeomorphism $\mathbf{C} \to \hat{\mathbf{C}} \setminus \{0\}$, and $\mathbf{C} \cup \rho(\mathbf{C})$ covers $\hat{\mathbf{C}}$. $\hat{z}$ is the "north pole" of S^2, $-\hat{z}$ its south pole.

Use the identification homeomorphism $\Psi : \mathbf{C} \to \operatorname{span}(\hat{x}, \hat{y})$, as in Definition 1.3: that is,

$$\Psi(a + ib) = a\hat{x} + b\hat{y}.$$

This maps $\mathbf{C}$ onto the plane $\operatorname{span}(\hat{x}, \hat{y})$.

If $\omega \in S^2 \setminus \{\hat{z}\}$, then there is a unique line that passes through ω and $\hat{z}$. This line is not horizontal, so it passes through a unique point $\varphi(\omega) \in \Psi(\mathbf{C})$. The map φ is clearly a continuous bijection. So φ identifies $S^2 \setminus \{\hat{z}\}$ with $\mathbf{C}$; it is our first chart.

The second chart ψ is constructed similarly. We start by inverting $\mathbf{C}$ via ρ, and then apply the same projection method, but centered on the south pole $-\hat{z}$. This identifies $S^2 \setminus \{-\hat{z}\}$ with $\rho(\mathbf{C})$.

Restricting ψ and φ to $\mathbf{C} \cap \rho(\mathbf{C})$, the "transition" $\psi \circ \varphi^{-1}(z) = 1/z$ is itself a homeomorphism. Thus the map

$$G(\omega) = \begin{cases} \infty, & \text{if } \omega = \hat{z} \\ \varphi(\omega), & \text{otherwise} \end{cases}$$

is a homeomorphism. □

The way to visualize the Riemann sphere is to imagine 0 at the south pole, ∞ at the north pole, and the rest of the plane "folded" over the sphere, with S^1 along the "equator".

The term "projective line" is a bit confusing, but essentially means that $\hat{\mathbf{C}}$ locally resembles the *one-dimensional* vector space $\mathbf{C}$. (Of course, this space is two-dimensional over $\mathbf{R}$; and of course $\hat{\mathbf{C}}$ cannot be embedded in any proper subspace of $\mathbf{R}^3$.) The term "one-point compactification" is justified by the fact that S^2 is compact and hence:

Corollary 6.18. $\hat{\mathbf{C}}$ *is compact.*

Exercise 6.19. Show that $\hat{\mathbf{C}}$ is simply connected. (Hint: use the charts from the proof of 6.17.)

Exercise 6.20. Show that $\hat{\mathbf{C}}$ is compact without appealing to 6.17.

Using the Riemann sphere, we can generalize the Riemann mapping theorem, generalizing from simply connected sets to connected sets whose topology isn't too bad.

Exercise 6.21. Let $U \subset \mathbf{C}$ be connected. Prove that, if $V = \mathbf{C} \setminus U$ has an infinite connected subset W, then U is conformal with a subset of $\mathbf{D}$.

Now that we have the Riemann sphere to work with, we have no problem with the following definition.

Definition 6.22. A *linear fractional transformation* or *Moebius transformation* is a map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an invertible matrix.

In particular, the determinant $ad - bc \neq 0$. For if $ad = bc$, then

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c(cz+d)} = \frac{a}{c}$$

is constant. We have no problem defining $z \mapsto \infty$ if $cz + d = 0$. By the quotient rule, the derivative exists and is nonzero everywhere. Therefore a Moebius transformation is a conformal map.

Be warned: multiple matrices give the same Moebius transformation:

Lemma 6.23. *Let A, B be invertible 2×2 matrices. They determine the same Moebius transformation iff there exists $\lambda \neq 0$ such that $A = \lambda B$.*

Proof.

$$\frac{\lambda az - \lambda b}{\lambda cz - \lambda d} = \frac{az-b}{cz-d}.$$

□

What does a Moebius transformation "look like"? Given an invertible 2×2 matrix, define the four functions

$$f_1(z) = z + \frac{d}{c}, \qquad f_2(z) = z^{-1}$$

$$f_3(z) = \frac{bc-ad}{c^2} z, \qquad f_4(z) = z + \frac{a}{c}.$$

f_1 and f_4 are translations, while f_2 sends balls centered at 0 to balls centered at ∞.

f_3 has two effects. It rescales z by $|bc - ad|/|c^2|$ (this action is said to be *homothetic*) and then rotates it by the phase of $(bc - ad)/c^2$.

Now,

$$f_4 \circ f_3 \circ f_2 \circ f_1 : z \mapsto \frac{az+b}{cz+d}$$

is a Moebius transformation, so this decomposition completely characterizes the geometric behavior of a Moebius transformation. Summarizing:

Theorem 6.24. *Moebius transformations can be decomposed into a translation, followed by a sphere inversion, followed by a homothetic transformation, followed by a rotation, followed by another translation.*

From the above characterization, it's not hard to see the following:

Corollary 6.25. *Moebius transformations send circles to circles and lines to lines. Moreover, the composition of Moebius transformations is a Moebius transformation.*

Exercise 6.26. Show that a Moebius transformation $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ has at most two fixed points (that is, points $\gamma \in \hat{\mathbf{C}}$ such that $f(\gamma) = \gamma$) and explicitly compute them.

Exercise 6.27. Characterize all Moebius transformations which send the half space to itself.

Now we'll give a group-theoretic characterization of Moebius transformations. If you haven't taken 113, you may want to skip the rest of this section.

Definition 6.28. The *Moebius group* $\mathrm{Aut}(\hat{\mathbf{C}})$ is the group of Moebius transformations of $\hat{\mathbf{C}}$ under composition.

Definition 6.29. $\mathrm{GL}(\mathbf{C}^2)$ is the *general linear group* on $\mathbf{C}^2$: the group of linear isomorphisms $\mathbf{C}^2 \to \mathbf{C}^2$ under composition. Let H be the normal subgroup of $\mathrm{GL}(\mathbf{C}^2)$

$$H = \{\lambda \,\mathrm{id} \in \mathrm{GL}(\mathbf{C}^2) : \lambda \in \mathbf{C} \setminus \{0\}\}.$$

The *projective linear group* is the quotient group $\mathrm{PGL}(\mathbf{C}^2) = \mathrm{GL}(\mathbf{C}^2)/H$.

As you should verify, H is in fact a normal subgroup, and elements of $\mathrm{PGL}(\mathbf{C}^2)$ are linear transformations modulo the equivalence relation of being multiples of each other by some $\lambda \neq 0$.

Theorem 6.30. $\mathrm{PGL}(\mathbf{C}^2) \cong \mathrm{Aut}(\hat{\mathbf{C}})$.

Proof. Choose a basis for $\mathbf{C}^2$, so that we can represent the elements of $\mathrm{GL}(\mathbf{C}^2)$ as matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is invertible, $ad - bc \neq 0$, so it determines a Moebius transformation $\phi(A)$, and any Moebius transformation can be written this way. Moreover, if $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is also a matrix, then their product is

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

while the composition of the corresponding Moebius transformations $\phi(A), \phi(B)$ is

$$\phi(A) \circ \phi(B) : z \mapsto \frac{a\frac{\alpha z + \beta}{\gamma z + \delta} + b}{c\frac{\alpha z + \beta}{\gamma z + \delta} + d} = \frac{(a\alpha + b\gamma)z + a\beta + b\delta}{(c\alpha + d\gamma)z + c\beta + d\delta} = \phi(AB)(z)$$

so the map $\phi : \mathrm{GL}(\mathbf{C}^2) \to \mathrm{Aut}(\hat{\mathbf{C}})$ is a homomorphism.

Moreover, for each $\lambda \neq 0$, $\lambda \,\mathrm{id} \mapsto \mathrm{id}$, so H is the kernel of ϕ. Now apply the first isomorphism theorem and recall $\mathrm{PGL}(\mathbf{C}^2) = \mathrm{GL}(\mathbf{C}^2)/H$. □

So operations on Moebius transformations are actually just operations on 2×2 matrices.

Corollary 6.31. *If $A, B \in \mathrm{GL}(\mathbf{C}^2)$, and $\phi : \mathrm{GL}(\mathbf{C}^2) \to \mathrm{Aut}(\hat{\mathbf{C}})$ is the map which sends a linear transformation to a Moebius transformation, then*

$$\phi(A)^{-1} = \phi(A^{-1})$$

and

$$\phi(A)\phi(B) = \phi(AB).$$

6.5 Glossary of conformal maps

For the purposes of Math 185, it is seldom necessary to invoke the full power of the Riemann mapping theorem. In particular, the Riemann mapping theorem is not constructive; it doesn't give an explicit formula for any particular conformal map $\mathbf{D} \to U$, only that one exists and is unique up to rotation. So, we'll consider some explicit examples of conformal maps, which can be used to reduce commonly-seen spaces to $\mathbf{D}$.

First we'll consider the Moebius transformations.

Recall that the half space

$$U = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$$

is conformal with $\mathbf{D}$, witnessed by the Moebius transformation

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

Note that this transformation preserves ∞.

There is also the "inversion" which sends $0 \mapsto \infty$, the Moebius transformation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note in particular that this transformation preserves the unit circle S^1.

Finally, the rotations $z \mapsto e^{i\theta} z$ and translations $z \mapsto \lambda z$ are useful conformal (actually Moebius) transformations.

Now for some more exotic spaces.

Recall that log is a bijection $\mathbf{C} \setminus (-\infty, 0] \to \{z \in \mathbf{C} : |\operatorname{Im} z| < \pi$ and $\log'(z) = 1/z \neq 0$, so log and hence exp are conformal. In particular, for $n \in \mathbf{N}$,

$$\exp : \{z \in \mathbf{C} : |\operatorname{Im} z - 2\pi n| < \pi\} \to \mathbf{C} \setminus (-\infty, 0]$$

serves as an isomorphism between an infinite horizontal strip and the plane minus a ray. Applying appropriate Moebius transformations allows one to identify any strip and any plane-minus-ray.

Now observe that on $\mathbf{D}$,

$$\tan'(z) = \frac{1}{1 + z^2} \neq 0$$

so tan is conformal. Moreover, arctan is well-defined precisely on the *vertical* trip $\{z \in \mathbf{C} : |\operatorname{Re} z| < \pi/4\}$. Thus tan maps $\mathbf{D}$ onto that strip; i.e.

$$\tan : \mathbf{D} \to \{z \in \mathbf{C} : |\operatorname{Re} z| < \pi/4\}$$

is conformal. An appropriate Moebius transformation makes it possible to deform the vertical strip into a horizontal strip, and so a composition will allow one to deform the disk (or the half space) into a plane minus ray.

Chapter 7

Harmonic functions

We'd like to apply some of the methods of complex analysis to study PDE in $\mathbf{R}^2$. Historically, this was one of the reasons complex analysis was invented in the first place.

Throughout this chapter, we'll assume $V \subseteq \mathbf{R}^2$ is open, and $\Psi(V) = U$, where $\Psi : \mathbf{C} \to \mathbf{R}^2$ is the identification homeomorphism 1.3 ($\Psi(x+iy) = (x,y)$). Thus we identify $\mathbf{R}^2$ with $\mathbf{C}$. Moreover, when it's convenient, we'll abuse notation and write $e^{i\theta}$ to mean the point $(\cos\theta, \sin\theta) \in \mathbf{R}^2$.

7.1 The Laplacian

Recall from multivariable calculus the notion of a Laplacian.

Definition 7.1. Let $f : V \to \mathbf{R}$ be a twice-differentiable function. The *Laplacian* of f is

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Physically, one can think of the Laplacian as measuring curvature or "gradient". One generally thinks of systems whose Laplacians are zero as being in "steady state" (and indeed a steady-state chemical gradient or heat gradient will have zero Laplacian).

In linear algebra or elsewhere you probably learned some of the following PDE. They justify

Definition 7.2. A twice-differentiable function $f : V \times [0, \infty) \to \mathbf{R}$ *solves the heat equation* if

$$(\Delta - \partial_t) f = 0.$$

Here we're taking Δ in the spatial variables V only.

Definition 7.3. A twice-differentiable function $f : V \times [0, \infty) \to \mathbf{R}$ *solves the wave equation* if

$$(\Delta - \partial_t^2) f = 0.$$

Definition 7.4. A twice-differentiable function $f : V \times [0, \infty) \to \mathbf{C}$ *solves Schroedinger's equation* if

$$(\Delta + i\partial_t) f = H,$$

where $H : V \to \mathbf{R}$ is called the *potential.*

In all of these equations, the time derivative is a function of curvature: as $\Delta f \to 0$, the function stabilizes in time. This justifies our previous claim that $\Delta f = 0$ means that f is in a "steady state". It also suggests that functions whose Laplacians vanish are of especial interest.

Definition 7.5. A twice-differentiable function $f : V \to \mathbf{R}$ is said to be *harmonic* or *solves Laplace's equation* if

$$\Delta f = 0.$$

If we have some way of studying harmonic functions, we understand the behavior of the above PDE. For example, we can solve the *one-dimensional* wave equation using harmonic functions:

Example 7.6 (Wick rotation)**.** In one dimension, the Laplacian is simply a second-derivative: $\Delta f = f^{(2)}$. Therefore the one-dimensional wave equation is

$$(\partial_x^2 - \partial_t^2) f(t, x) = 0.$$

Now we introduce the *Wick rotation*, a change of variables $\tau = it$. Then $\partial_\tau^2 = -\partial_t^2$, so if $g(\tau, x) = f(-i\tau, x)$ then

$$(\partial_x^2 + \partial_\tau^2) g(\tau, x) = 0$$

so g is harmonic in (x, τ). Thus, the Wick rotation is a bijection between harmonic functions in $\mathbf{R}^2$ (more generally, $\mathbf{R}^{n+1}$) and solutions to the wave equation in $\mathbf{R}$ (more generally, $\mathbf{R}$).

So what's the connection to complex analysis? Recall the Cauchy-Riemann equations: we identified a function $f : U \to \mathbf{C}$ with

$$\tilde{f} = \Psi \circ f \circ \Phi.$$

Then $\tilde{f} : V \to \mathbf{R}^2$ could be written as $\tilde{f} = (u, v)$, and f was holomorphic if and only if

$$\begin{cases} \partial_1 u = \partial_2 v \\ \partial_2 u = -\partial_1 v. \end{cases}$$

Lemma 7.7. *Let notation be as above. If f is holomorphic, then u and v are harmonic.*

Proof. It follows by Cauchy-Riemann and equality of mixed partials;

$$\partial_1^2 u = \partial_1 \partial_2 v = \partial_2 \partial_1 v = -\partial_2^2 u$$

and

$$\partial_1^2 v = -\partial_1 \partial_2 u = -\partial_2 \partial_1 u = -\partial_2^2 v.$$

$\square$

To turn this into an "if and only if", we need to see that u and v aren't any old harmonic functions: they are married by the Cauchy-Riemann equations.

Definition 7.8. Suppose that $u, v : V \to \mathbf{R}$ are harmonic and solve the Cauchy-Riemann equations. Then we say that u and v are *harmonic conjugates*.

Given u, we'll show that if the topology of V isn't too bad, then u has a harmonic conjugate v. Thus, given *any* harmonic function on V we'll be able to construct a natural holomorphic function $u + iv$ on U. That way, PDE problems on V reduce to complex analysis on U.

Theorem 7.9. *Let V be simply connected. If $u : V \to \mathbf{R}$ is harmonic then it has a harmonic conjugate.*

Proof. Let $g : U \to \mathbf{C}$ be given by

$$g(z) = \partial_1 u(\Psi(z)) - i\partial_2 u(\Psi(z)).$$

Rewrite $g = a + ib$; we need to show that (a, b) satisfies the Cauchy-Riemann equations, and then g will be holomorphic. Indeed, $a = \partial_1 u \circ \Psi$ and $b = -\partial_2 u \circ \Psi$. So, up to an identification,

$$\partial_1 a = \partial_1^2 u = -\partial_2^2 u = \partial_2 b.$$

Similarly,

$$\partial_2 a = \partial_1 \partial_2 u = -\partial_1 b.$$

Since U is simply connected, Cauchy-Goursat furnishes a holomorphic function $h : U \to \mathbf{C}$ where $h' = g$. Put $v = \operatorname{Im} h$. Now $\operatorname{Re} h = \operatorname{Re} \int g = u$ so $h = u + iv$ and (u, v) are harmonic conjugates. $\square$

We rattle off some corollaries, which use the fact that V is locally simply connected. This shows that we have solved two of the three basic PDE problems (existence, regularity, and uniqueness) for Laplace's equation.

Corollary 7.10 (regularity of Laplace's equation)**.** *Each harmonic function is smooth.*

Corollary 7.11 (maximum modulus principle)**.** *If V is connected and $u : V \to \mathbf{R}$ is harmonic and attains its maximum on V, then u is constant. Moreover, if V is bounded, then*

$$\sup_V u = \max_{\overline{V}} u = \max_{\partial V} u.$$

Harmonic functions are completely determined by their behavior on the boundary of an open set.

Corollary 7.12 (uniqueness of Laplace's equation)**.** *Let V be connected and $u, v : \overline{V} \to \mathbf{R}$ be harmonic. If $u = v$ on ∂V then $u = v$ on V.*

Proof. $u - v$ and $v - u$ are harmonic by linearity of Δ. But $u - v = v - u = 0$ on ∂V, so $u - v \leq 0$ and $v - u \leq 0$. Thus $u - v = v - u = 0$ on V. ☐

7.2 The mean value property

Harmonic functions satisfy the following miraculous equation, which says that harmonic functions are always equal to their averages on circles centered at each point.

Theorem 7.13 (mean value property)**.** *Let $u : V \to \mathbf{R}$ be harmonic and $z \in V$. If γ is a circle centered on z of radius $r > 0$ such that $B_r z \subset V$, then*

$$u(z) = \frac{1}{2\pi r} \int_\gamma u(w)\ dw.$$

Proof. Let $f = u + iv$, where v is the harmonic conjugate of u. Then f is holomorphic, and

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_r z} \frac{f(w)}{w - z} dw = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

and it follows when we take the real part of the integral. ☐

The above characterization of the mean value property is particularly useful for us, since complex analysis is particularly interested in integrals around curves. However, there is a *different* notion of a mean value property, which uses an area integral, as defined in multivariable calculus: why integrate around the ball? Why not integrate on the ball itself?

Lemma 7.14. *Let $u : V \to \mathbf{R}$. If u satisfies the mean value property, then, for each $z \in V$ and $r > 0$,*

$$u(z) = \frac{1}{\pi r^2} \int_{B_r z} u\ dA.$$

Proof. Without loss of generality, assume $z = 0$. Then define

$$\phi(r) = \frac{1}{2\pi r} \int_{\Gamma_r 0} u(w)\ dw.$$

Then ϕ is constant, so $\phi' = 0$. Switching to polar coordinates, one has

$$\frac{1}{\pi r^2} \int_{\Gamma_r 0} u(w)\ dw = \frac{1}{\pi r^2} \int_0^r \phi(s)\ ds$$

and $\phi(s)$ is independent of s, so we can pull out $\phi(s) = u(z)$, which completes the proof. ☐

From this, it follows that the mean value property completely characterizes harmonic functions in C^2: if the mean value property holds for u, then Δu necessarily is 0.

Theorem 7.15 (mean value property, converse)**.** *Let $u : V \to \mathbf{R}$ be C^2. If, for each $z \in V$ and $r > 0$,*

$$u(z) = \frac{1}{\pi r^2} \int_{B_r z} u(w)\ dA,$$

then u is harmonic.

Proof. Without loss of generality, we can assume $\Delta u > 0$ on an open $W \subseteq V$ for the sake of contradiction (and if not, just add a negative sign to the relevant equalities). Furthermore, since there were no hypotheses on V except that it is open, we may assume $W = V$. Let ϕ and z be as in the proof of 7.14. Then $\phi' = 0$ and by Green's theorem, if n is the outward unit normal vector for $\partial\mathbf{D}$, then

$$0 = \phi'(r) = \frac{1}{2\pi r} \int_{\Gamma_r 0} \nabla u \cdot dn = \frac{1}{\pi r^2} \int_{B_r 0} \Delta u\ dA > 0$$

which is a contradiction. □

All this should further the same intuition that Cauchy's estimate gave: a holomorphic (equivalently, harmonic) function cannot grow too fast, but it cannot also oscillate too wildly. Its growth is controlled by the mollifying effects of the mean value property, which is really just Cauchy's integral formula unmasked.

The mean value property also gives a nice way to prove that a function is harmonic.

Exercise 7.16. Show that away from 0, the function

$$h(z) = \log |z|$$

is harmonic, but it does not have a harmonic conjugate.

In fact, control over the mean value is so useful in analysis one introduces the following, much weaker notion.

Definition 7.17. Let $u : V \to \mathbf{R}$ be continuous and, for each $z \in V$ and $r > 0$,

$$u(z) \leq \frac{1}{2\pi r} \int_{\Gamma_r z} u(w)\ dw.$$

Then we say that u is *subharmonic*.

Of course, there is an analogous notion of superharmonic functions.

By mimicking the proofs of the usual claims about harmonic functions, you can recover quite a lot about subharmonic functions (and, analogously, superharmonic functions).

Exercise 7.18. Show that if u is subharmonic then it satisfies a maximum principle, and that if u is furthermore C^2, then $\Delta u \geq 0$.

7.3 Boundary value problems

A common problem in PDE theory is the boundary-value problem, or Dirichlet problem. Given a PDE, and the value of a function on the boundary of a space, we want to construct a solution on the whole space. If such a solution exists, then it is unique; and we'll be able to show that, for simply connected sets, a solution does in fact exist.

Definition 7.19. Let $R > 0$. The function $P : B_R 0 \times \partial B_R 0$ given by

$$P_R(re^{i\varphi}, e^{i\theta}) = \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\varphi}|^2}$$

is called *Poisson's kernel* for the ball of radius R.

Lemma 7.20. *Poisson's kernel satisfies*

$$2\pi = \int_0^{2\pi} P_1(re^{i\varphi}, e^{i\theta}) d\theta.$$

Moreover, $P_1 \geq 0$.

Proof. It follows by by 3.15 with $f = 1$, $R = 1$, and $z = re^{i\varphi}$. □

Our method for solving the boundary-value problem is to smash the boundary condition against Poisson's kernel. First we'll solve Laplace's equation in a ball.

Theorem 7.21 (Poisson's integral formula)**.** *Let* $R > 0$ *and* $h : \partial B_R 0 \to \mathbf{R}$ *be a continuous function. Let* $u : \overline{B_R 0} \to \mathbf{R}$ *be*

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

on $B_R 0$ *and* $u = h$ *on* $\partial B_R 0$.

Then u *is harmonic on* $B_R 0$ *and continuous on* $\overline{B_R 0}$.

Proof. Without loss of generality, assume $R = 1$.

Let $f : \mathbf{D} \to \mathbf{C}$ be given by

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1 0} \frac{h(w)}{w} \left(2\frac{w}{w - z} - 1 \right) dw.$$

Then f is holomorphic and, if $z = re^{i\varphi}$,

$$\begin{aligned}
\operatorname{Re} f(z) &= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} \frac{h(w)}{w} \left(2\frac{w}{w - z} - 1 \right) \\
&= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} h(w) \frac{e^{i\theta} + re^{i\varphi}}{e^{i\theta} - re^{i\varphi}} \, d\theta \\
&= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} h(w) \frac{1 - r^2}{|e^{i\theta} - re^{i\varphi}|^2} \, d\theta \\
&= u(re^{i\varphi})
\end{aligned}$$

so u is harmonic. However, there is the possibility that u fails to be continuous as $r \to 1$. (That is, u is continuous on the boundary and in the interior, but not as it jumps from the interior to the boundary.)

Let $\alpha \in [0, 2\pi)$. We'll show continuity at $e^{i\alpha}$. By compactness, h is bounded, say $|h| < M$. Then if $\varepsilon > 0$ we can find $\delta > 0$ such that whenever $\theta \in [\alpha - 4\delta, \alpha + 4\delta]$,

$$|h(e^{i\theta}) - h(e^{i\alpha})| < \varepsilon.$$

Then

$$\begin{aligned}
|u(re^{i\varphi}) - u(e^{i\alpha})| &= \frac{1}{2\pi} \int_0^{2\pi} \left| h(e^{i\theta}) - h(e^{i\alpha}) \right| P_1(re^{i\varphi}, e^{i\theta}) \, d\theta \\
&= \frac{1}{2\pi} \int_{\alpha - \pi}^{\alpha + \pi} \left| h(e^{i\theta}) - h(e^{i\alpha}) \right| P_1(re^{i\varphi}, e^{i\theta}) \, d\theta \\
&\leq \frac{1}{2\pi} \int_{\alpha - \pi}^{\alpha + \pi} |h(e^{i\theta}) - h(e^{i\alpha})| P_1(re^{i\theta}, e^{i\varphi}) \, d\theta \\
&= \frac{1}{2\pi} \left[\int_{\alpha - 4\delta}^{\alpha + 4\delta} + \int_{\alpha - \pi}^{\alpha - 4\delta} + \int_{\alpha + 4\delta}^{\alpha + \pi} \right] |h(e^{i\theta}) - h(e^{i\alpha})| P_1(re^{i\theta}, e^{i\varphi}) \, d\theta.
\end{aligned}$$

We'll estimate these integrals individually. First, by 7.20,

$$\int_{\alpha-4\delta}^{\alpha+4\delta} |h(e^{i\theta}) - h(e^{i\alpha})| P_1(re^{i\theta}, e^{i\varphi})\, d\theta < \int_{\alpha-4\delta}^{\alpha+4\delta} \varepsilon P_1(re^{i\theta}, e^{i\varphi})\, d\theta = 2\pi\varepsilon.$$

Second, $|\theta - \alpha| \geq 4\delta$, so if $|\theta - \varphi| < 2\delta$ then $2\delta \leq |\theta - \varphi|$. Since $|\theta| < \pi$ and $|\varphi| < \pi$ we have

$$|\sin 2\delta| \leq |e^{2i\delta}| \leq |e^{2i\theta} - e^{2i\varphi}| \leq |e^{i\theta} - re^{i\varphi}|.$$

$$\begin{aligned}\left[\int_{\alpha-\pi}^{\alpha-4\delta} + \int_{\alpha+4\delta}^{\alpha+\pi}\right] |h(e^{i\theta}) - h(e^{i\alpha})| P_1(re^{i\theta}, e^{i\varphi})\, d\theta &= 2M \left[\int_{\alpha-\pi}^{\alpha-4\delta} + \int_{\alpha+4\delta}^{\alpha+\pi}\right] \frac{1-r^2}{|re^{i\theta} - e^{i\varphi}|^2}\, d\theta \\ &\leq \frac{2M(1-r^2)}{\sin^2 2\delta} \left[\int_{\alpha-\pi}^{\alpha-4\delta} + \int_{\alpha+4\delta}^{\alpha+\pi}\right] d\theta \\ &< \frac{2\pi M}{\sin^2 2\delta}(1-r^2)\end{aligned}$$

which vanishes as $r \to 1$. □

Though we've only solved the problem on the ball, we'll be able to solve this problem on any simply connected V except for $\mathbf{C}$ itself, by the Riemann mapping theorem. As an example, we'll solve Laplace's equation in a half space.

Corollary 7.22 (Poisson's integral formula for a half space)**.** *Suppose that $\phi : \mathbf{R} \to \mathbf{R}$ is continuous and*

$$U = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$$

is the half space. Let $F : \overline{U} \to \overline{\mathbf{D}}$ be

$$F(z) = \frac{z-i}{z+i}$$

as in 6.7, $\tilde{\phi} = \phi \circ F^{-1}$, and

$$\tilde{u}(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\phi}(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

as in 7.21.

If $u = F^{-1} \circ \tilde{u} \circ F$, then u is harmonic on U and continuous on $\overline{U}$.

More generally, if V is open and simply connected with nonempty boundary and $F : V \to \mathbf{D}$ is angle-preserving then, given $\phi : \partial V \to \mathbf{R}$, one can construct $u : \overline{V} \to \mathbf{R}$ by $u = F^{-1} \circ \tilde{u} \circ F$ where $\tilde{u}$ solves the Dirichlet problem on $\overline{V}$ with boundary condition $\phi \circ F^{-1}$.

Exercise 7.23 (Poisson's integral formula for a quadrant)**.** Solve the Dirichlet problem on the quadrant

$$V = \{z \in \mathbf{C} : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}.$$

Poisson's kernel guarantees that locally uniform limits of harmonic functions are harmonic.

Theorem 7.24 (Harnack)**.** *Let $u_1, u_2, \cdots : V \to \mathbf{R}$ be harmonic and u be their locally uniform limit. Then u is harmonic.*

Proof. u is continuous because the limit is locally uniform. Let $p \in V$ and assume without loss of generality that $p = 0$. Then there exists $R > 0$ such that $B_R 0 \subseteq V$ and the limit is uniform on $\overline{B_R 0}$. So by Poisson's kernel, if $r < R$ then

$$u_n(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} u_n(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

and this equality holds even as $n \to \infty$. □

7.4 Harnack's inequality

The mean value property restricts the possible growth of a harmonic function. As a result, the following inequality, which describes the growth of *positive* harmonic functions, shouldn't be too much of a surprise.

Theorem 7.25 (Harnack's inequality)**.** *Suppose that $R > 0$, $u : \overline{B_R 0} \to \mathbf{R}$ is continuous on $\overline{B_R 0}$ and harmonic on $B_R 0$, and $u \geq 0$ on $B_R 0$. Then for each $r \in (0, 1)$ and $\theta \in [0, 2\pi)$,*

$$\frac{R-r}{R+r} u(0) \leq u(re^{i\theta}) \leq \frac{R+r}{R-r} u(0).$$

Proof. For each $\varphi \in [0, 2\pi)$,

$$R - r \leq |Re^{i\theta} - re^{i\varphi}| \leq R + r;$$

this follows by application of the triangle inequality to the ball. One then has

$$\frac{R-r}{R+r} = \frac{R^2 - r^2}{(R+r)^2} \leq \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\varphi}|^2} = P_R(re^{i\theta}, e^{i\varphi}) \leq \frac{R^2 - r^2}{(R-r)^2} = \frac{R+r}{R-r}.$$

By the mean value property,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi})\ d\varphi$$

but on the other hand, Poisson's kernel yields

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) P_R(re^{i\theta}, e^{i\varphi}) d\varphi \leq \frac{R+r}{R-r} \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi})\ d\varphi = \frac{R+r}{R-r} u(0).$$

Similarly for the other inequality. □

Thus, there are no nontrivial positive entire harmonic functions.

Corollary 7.26. *Suppose that $u : \mathbf{R}^2 \to (0, \infty)$ is harmonic. Then u is constant.*

Proof. u restricts to $\overline{B_R 0}$ for each $R > 0$. But also

$$\lim_{R \to \infty} \frac{R-r}{R+r} = \lim_{R \to \infty} \frac{R+r}{R-r} = 1.$$

So for each $z \in \mathbf{C}$, $u(z) \leq u(0) \leq u(z)$. □

Harnack's inequality also takes a more general form, which is clear by dividing up a space into overlapping open balls and applying Harnack's inequality to each one.

Corollary 7.27 (Harnack's inequality, generalized)**.** *Suppose that V is connected and bounded. Then there is a constant $C > 0$, which only depends on V, such that for each harmonic function $u : V \to (0, \infty)$ and each $z, w \in V$,*

$$C^{-1} u(z) \leq u(w) \leq C u(z).$$

Harnack's inequality gives us another convergence theorem for harmonic functions.

Theorem 7.28 (Harnack's monotone convergence theorem)**.** *Suppose $u_1, u_2, \cdots : V \to \mathbf{R}$ are harmonic and $u_1 \leq u_2 \leq \dots$. If $p_0 \in V$ and the sequence $u_n(p_0)$ converges in $\mathbf{R}$, then there is a function $u : V \to \mathbf{R}$ such that $u_n \to u$ locally uniformly. In particular, u is harmonic.*

Proof. Without loss of generality, we may assume that for each $n \geq 1$, $u_n \geq 0$; if not, just translate.

For each such harmonic function $v \geq 0$, Harnack's inequality gives, on a sufficiently small ball of radius R,

$$\frac{R - |p-q|}{R + |p-q|} v(p) \leq v(q) \leq \frac{R + |p-q|}{R - |p-q|} v(p)$$

whence, for p and q sufficiently close,

$$\frac{1}{4}v(p) \leq v(q) \leq 4v(p).$$

Suppose that W_1 is the set of points p in V such that the sequence $u_n(p)$ is bounded. Because of the above estimates, W_1 is open. Similarly, the set W_2 of points p such that $u_n(p)$ is unbounded is open. Since W_1 is nonempty and V is connected, $V = W_1$. Thus we can put

$$u(z) = \sup_{n \in \mathbf{N}} u_n(z).$$

Thus $u_n \leq u$. If $m \geq n$, then $u_m - u_n \geq 0$ is harmonic and if $|p-q|$ is sufficiently small (say an element of a set X), then

$$(u_m - u_n)(q) \leq 4(u_m - u_n)(p)$$

which is uniformly Cauchy on X, thus converges locally uniformly on V.

By Harnack's theorem, then, u is harmonic. $\square$

Harnack's inequality also gives us a metric which we can apply to any bounded connected open subset of $\mathbf{R}^2$, which is, from the point of a hyperbolic geometer, the natural metric on $\mathbf{D}$.

Definition 7.29. Let V be connected and bounded and H be the space of positive harmonic functions on V. Put

$$\tau(z, w) = \sup_{h \in H} \left| \frac{h(w)}{h(z)} \right|.$$

The function $d = \log \circ \tau : V \times V \to [0, \infty)$ is called the *Harnack metric* of V.

It's not clear that this function is even well-defined (why is the sup always finite?), let alone that it's a metric – and the most difficult question of all, that it's a metric which is actually worthy of study.

Lemma 7.30. *The Harnack metric is well-defined and a metric.*

Proof. Let C be as in 7.27. Then there exist $h \in H$ and $\delta > 0$ such that

$$\tau(z, w) < \frac{h(z)}{h(w)} + \delta \leq C + \delta < \infty.$$

Thus d is well-defined.

Moreover, we have $\tau(z, w) \geq 1$ with $\tau(z, w) = 1$ iff $z = w$, so the metric is positive-definite. It is symmetric because 7.27 is.

Finally, observe that since log is an isomorphism between the additive and multiplicative groups in $\mathbf{R}$, we must prove $\tau(a, c) \leq \tau(a, b)\tau(b, c)$ and then the triangle inequality follows. Indeed, if $h \in H$ then

$$h(a) \leq \tau(a, b)h(b) \leq \tau(a, b)\tau(b, c)h(c)$$

so

$$\frac{h(a)}{h(c)} \leq \tau(a, b)\tau(b, c)$$

and since this bound holds for each h,

$$\tau(a, c) \leq \tau(a, b)\tau(b, c).$$

$\square$

The value of the Harnack metric is that from its point of view, the boundary is infinitely far away from any point in the ball. In this sense, we can identify the ball with the plane; the "boundary" of $\mathbf{C}$ is the point at infinity ∞, which is infinitely far away from any point in $\mathbf{C}$.

Lemma 7.31. *If $d : V \times V \to [0, \infty)$ is the Harnack metric, then for $\zeta \in \partial V$ and $w \in V$,*

$$\lim_{z \to \zeta} d(z, w) = \infty.$$

Proof. Suppose that V is contained in a ball of radius R. Then $\log|\zeta - w| \leq \log R$. Now let

$$h(z) = \log R - \log|\zeta - w|.$$

Then $h \geq 0$ and is harmonic, being a linear combination of harmonic functions. But then

$$\lim_{z \to \zeta} d(z, w) \geq \lim_{z \to \zeta} \log\left(\frac{R}{|z - w|}\right) = \infty.$$

$\square$

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