NONSINGULAR CURVES

AIDAN BACKUS

Abstract. We show that every curve is birational to a nonsingular projective curve. This follows Hartshorne, Chapter 1.

1. Introduction

In complex analysis last semester, we proved a weaker form of the following theorem:

Theorem 1.1 (classification of nonsingular projective curves)**.** *Let X be a smooth projective curve over k. There is an invariant $g \in \mathbf{N}$ of X, called the* genus *of X, such that:*

1. *$g = \dim \Omega^1(X)$ where Ω^1 is the sheaf of 1-forms with coefficients in $\mathscr{O}_X$.*
2. *$g = \dim H^1(X, \mathscr{O}_X)$ (in the sense of sheaf cohomology).*
3. *If $k = \mathbf{C}$, then $g = \dim H^1(X^{an}, \mathbf{C})/2$ (in the sense of singular cohomology).*

Moreover:

1. *If $g = 0$, then X is isomorphic to $\mathbf{P}^1$.*
2. *If $g = 1$, then X is an elliptic curve. Moreover, the moduli space of elliptic curves is k.*
3. *There is an explicit threefold which is the moduli space of curves such that $g = 2$.*
4. *And so on...*

See also Chapter IV of Hartshorne.

The goal of this talk is to show:

Theorem 1.2. *In every birational class $[X]$ of curves there exists a unique smooth projective curve X. In fact, every smooth curve Y birational to X embeds in X.*

Assuming this is true for a moment, the classification of smooth projective curves is actually a classification of all curves modulo birationality.

Corollary 1.3 (resolution of singularities)**.** *For every curve X there exists a smooth curve X' and a birational map $X' \to X$.*

Date: October 2021.

2. Abstract curves

The definition of a scheme was based on the idea that we can study varieties up to isomorphism by understanding the regular functions on them. To study varieties up to birationality we need a larger class of functions (so that the notion of isomorphism is weaker), namely the rational functions.

Definition 2.1. An (abstract) *function field* K (of dimension 1) is a finitely generated field extension $k \to K$ with transcendence degree 1.

Let K be the function field of a smooth curve Y. Recall from the main lecture: for every $y \in Y$ there is a natural discrete valuation

$$v(y) : K \to \mathbf{Z}$$

where $v(y)(f)$ is the order of vanishing of f at y and with discrete valuation ring $\mathscr{O}_{Y,y}$. We always view this ring $\mathscr{O}_{Y,y}$ as a subring of the function field K.

Definition 2.2. If K is a function field of dimension 1, then the *projective curve of DVRs* C_K is, as a set, the set of all discrete valuation subrings of K.

Lemma 2.3. *Let X be a smooth curve. Then the map $x \mapsto \mathscr{O}_{X,x}$ is an injective map $X \to C_{K(X)}$.*

Proof. If $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$ as subrings of K, then we can choose an affine open set $U = \operatorname{Spec} A$ containing $\{x, y\}$ and then find maximal ideals $\mathfrak{m}, \mathfrak{n}$ with $\mathscr{O}_{X,x} = A_\mathfrak{m}$ and $\mathscr{O}_{X,y} = A_\mathfrak{n}$, thus $\mathfrak{m} \subseteq \mathfrak{n}$. By maximality of $\mathfrak{m}$ we obtain $\mathfrak{m} = \mathfrak{n}$ and hence $x = y$. □

Owing to the above lemma, it is natural to identify $C = C_{K(X)}$ with the "projective completion" of X. To make this rigorous, we need to turn C into a scheme; first we introduce a Zariski topology for C. The closed sets are the sets

$$Z_f = \{x \in C : f \notin x\}.$$

Thinking of x as a point, the assumption is that f does not restrict to an element of the stalk $\mathscr{O}_{C,x}$; that is, f has a pole at x. Thus our closed sets will be the sets of poles of rational maps. But the Zariski topology on a curve is nothing more than the cofinite topology, so we need to check that the closed sets Z_f are finite:

Lemma 2.4. *Let K be a function field. Then:*

1. *For every $f \in K$, Z_f is a finite set.*
2. *For every discrete valuation ring R in K there is a smooth affine curve X such that R is isomorphic to a local ring of X.*

Proof. See Hartshorne, Lemma I.6.5; make sure to review Dedekind domains beforehand. □

Now we introduce the structure sheaf of $C = C_K$. For every open set $U \subseteq C$, write $\mathscr{O}_C(U) = \bigcap_{R\in U} R$. The stalks of $\mathscr{O}_C$ are local rings of affine curves, so C is locally ringed.

Definition 2.5. By a *curve of DVRs* we mean an open subset of C_K for some function field K of dimension 1. The *category of curves of DVRs* is the relevant full subcategory of the category of locally ringed spaces.

At this point we have three (four over $\mathbf{C}$) different notions of smooth curve:

(1) Smooth quasiprojective varieties with Krull dimension 1.
(2) Curves of DVRs.
(3) Smooth, integral separated schemes of finite type and Krull dimension 1.
(4) Cofinite subsets of closed Riemann surfaces (if we're over $\mathbf{C}$).

We have $1 \leftrightarrow 3$ by Hartshorne, Chapter 2. $1 \leftrightarrow 4$ was argued in complex analysis last semester. It remains to show $1 \leftrightarrow 2$.

Proposition 2.6. *Every smooth quasiprojective curve is isomorphic to a curve of DVRs according to the embedding that sends a point to its local ring.*

Proof. Let K be the function field of the smooth quasiprojective curve Y and let U be the set of local rings of Y. We already showed that we have an isomorphism

$$\varphi : Y \to U$$

in the category of sets and we just need to show that U is a curve of DVRs and upgrade φ to a morphism of locally ringed spaces.

First we show that U is open in $C = C_K$. Since C has the cofinite topology, a superset of any open subset of C is open, so we just need to show that U contains a nonempty open subset of C. Thus we can replace Y by an open subset of Y. But Y is quasiprojective and therefore admits an open cover by affine curves, so we may assume that Y is affine, say $Y = \operatorname{Spec} A$.

Since $Y = \operatorname{Spec} A$, K is the field of fractions of A, there exists f such that

$$A = \frac{k[x_1, \dots, x_n]}{f(x_1, \dots, x_n)},$$

and U is the space of all discrete valuation rings of K containing A. So for every $R \in U$, $A \subseteq R$ iff $x_1, \dots, x_n \in R$, and hence

$$U = \bigcap_{i=1}^{n} \{R \in C : x_i \in R\} = \bigcap_{i=1}^{n} U_i.$$

Now the complement of U_i is finite by the previous lemma, so U_i is open and hence U is open.

Since U and Y both have the cofinite topology and φ is a bijection, it follows that φ is a homeomorphism. For any $V \subseteq Y$ open, $\mathscr{O}_Y(V) = \bigcap_{y \in Y} \mathscr{O}_{Y,y}$ which is exactly the definition of $\mathscr{O}_C(\varphi(V))$, so φ is an isomorphism of locally ringed spaces. $\square$

Corollary 2.7. *Every curve of DVRs is a scheme.*

Proof. Every point $x \in C_K$ is the local ring of a point y in a smooth affine curve Y. The above construction shows that an open neighborhood of y in Y is isomorphic to an open neighborhood of x. $\square$

Of course, Hartshorne Chapter I just states this result by saying that a curve of DVRs can be covered by affine varieties.

Lemma 2.8. *Let X be an abstract smooth curve, $x \in X$, Y a projective variety, and $\varphi : X \setminus \{x\} \to Y$ a morphism. Then φ extends uniquely to $\overline{\varphi} : X \to Y$.*

Proof. By assumption $Y \subseteq \mathbf{P}^n$ is closed, so by continuity if φ extends to a map $X \to \mathbf{P}^n$ then it extends to a map $X \to Y$. So we may assume that $Y = \mathbf{P}^n$.

Let $z_0, \dots, z_n$ be coordinates on $\mathbf{P}^n$ and let

$$U = \{[z] \in \mathbf{P}^n : z_0, z_1, \dots, z_n \neq 0\}.$$

If $\varphi(X \setminus \{x\}) \cap U$ is empty, then since X is irreducible, there exists a hyperplane $\{x_i = 0\}$ which contains $\varphi(X \setminus \{x\})$, so we may assume that φ actually maps into $\mathbf{P}^{n-1}$. However if $n = 0$ then $\mathbf{P}^{n-1}$ is empty, so after decreasing n finitely many times we may assume that φ meets U.

Let

$$f_{ij} = \varphi^*(z_i/z_j),$$

which is regular on an open subset of X (since z_i/z_j is regular on U) and hence $f_{ij} \in K$. Let R be the local ring at x and v its discrete valuation. Let $r_i = v(f_{i0})$, so that

$$v(f_{ij}) = r_i - r_j.$$

Let k be such that r_k is minimal; then

$$v(f_{ik}) = r_i - r_k \geq r_i - r_i = 0$$

and hence $f_{ik} \in R$; that is, f_{ik} is regular in a neighborhood of x. So we can extend φ to $\overline{\varphi}$ by setting

$$\overline{\varphi}(x) = [f_{0k}(x), \dots, f_{nk}(x)].$$

Then $\overline{\varphi}$ is the unique continuous extension of X.

To see that $\overline{\varphi}$ is a morphism of schemes, it suffices to check this on a small open set $V \ni \overline{\varphi}(x)$. First, if

$$V_0 = \operatorname{Spec} k[z_0/z_k, \dots, z_n/z_k],$$

then $\overline{\varphi}(x) \in V_0$ since $f_{kk}(x) = 1$. Also we defined $\overline{\varphi}$ so that $\overline{\varphi}^*(z_i/z_k) = f_{ik}$ which is regular in a neighborhood of x. Any function on V is a localization of a function on V_0 and hence pulls back to a function which is regular near x. $\square$

Note that the above proof is essentially the same proof that projective morphisms are proper. You can actually prove the above result using the valuative criterion of properness (see the appendix).

Lemma 2.9. *The curve of DVRs C_K is isomorphic to a smooth projective curve.*

Proof. Since $C = C_K$ has the cofinite topology, it is quasicompact. So C is a quasicompact scheme and hence can be covered by finitely many affine curves V_i, each of which has finite complement. Let

$$\varphi_i : V_i \to Y_i$$

be the projective completion of V_i[1]. Since V_i has finite complement, the previous lemma allows us to extend φ_i to a morphism $\overline{\varphi}_i : C \to Y_i$. Thus we obtain a product morphism

$$\varphi : C \to \prod_{i=1}^{n} Y_i.$$

Let Y be the closure in $\prod_i Y_i$ of the image of C, so Y is a projective variety and $\varphi : C \to Y$ is dominant. Therefore Y is a curve.

We claim that φ is an isomorphism. Let $x \in C$, thus there is i such that $x \in V_i$. Let $\Pi_i : Y \to Y_i$ be the projection map, so that pulling back along Π_i and then along φ, we obtain morphisms of local rings

$$\mathscr{O}_{Y_i,\varphi_i(x)} \to \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{C,x}.$$

But Π_i and φ are dominant so these morphisms of local rings are monomorphisms. Their composition is an isomorphism by definition of Y_i, so they must all be isomorphisms. In particular,

$$\varphi_x{}^* : \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{C,x}$$

is an isomorphism.

Let $y \in Y$. Then the localization of the integral closure of $\mathscr{O}_{Y,y}$ at a maximal ideal is a discrete valuation ring R of K. So $\mathscr{O}_{Y,y}$ is contained in a discrete valuation ring $R \in C$ such that the intersection of the maximal ideal of R with $\mathscr{O}_{Y,y}$ is the maximal ideal of $\mathscr{O}_{Y,y}$[2]. By definition, $\mathscr{O}_{Y,\varphi(R)} = R$. So by injectivity of the map that sends

[1] that is, embed V_i in $\mathbf{A}^{n_i} \subseteq \mathbf{P}^{n_i}$ and take the closure of V_i in $\mathbf{P}^{n_i}$

[2] In the main lecture, I believe we said that having a structure sheaf whose stalk consisted of discrete valuation rings was equivalent to smoothness. In particular, passing from $\mathscr{O}_{Y,y}$ to the discrete valuation ring R amounts to taking a resolution of singularities of Y at y.

points to their local rings, $y = \varphi(R)$. Therefore φ is surjective, and is the map that sends a point to its local ring, hence is injective.

Since φ is a homeomorphism and is stalkwise an isomorphism, φ is an isomorphism of schemes. □

Theorem 2.10. *The following categories are equivalent:*

(1) *Smooth projective curves with dominant morphisms.*
(2) *Quasiprojective curves with dominant rational maps.*
(3) *The opposite category of abstract function fields of dimension* 1 *with algebra homomorphisms.*

Proof. We have a forgetful functor from smooth projective curves to quasiprojective curves, and the contravariant functor that sends quasiprojective varieties to their function fields restricts to a contravariant functor on quasiprojective curves. The latter is an equivalence of categories.

We have just shown that for every function field K, the curve C of discrete valuation rings of K is a smooth projective curve. Now let $F : K_2 \to K_1$ be an algebra homomorphism. Since quasiprojective curves are equivalent to the opposite of abstract function fields, F induces a uniquely defined dominant rational map $C_1 \to C_2$. Let $\varphi : U \to C_2$ be a restriction of the rational map to a dominant morphism and let

$$\overline{\varphi} : C_1 \to C_2$$

by the extension of φ to C_1. Then $\overline{\varphi}$ is dominant. The fact that this extension is unique means that this assignment is functorial; that is, $K \mapsto C$, $F \mapsto \overline{\varphi}$ defines a contravariant functor from the opposite of abstract function fields to smooth projective curves. □

3. Rational and elliptic curves

In these examples we assume that k does not have characteristic 2 or 3 to save myself a headache.

Let Y be a smooth rational curve, for example

$$Y = \{(x, y) \in \mathbf{A}^2 : xy = 1\}.$$

According to our main theorem, Y embeds in $\mathbf{P}^1$ and $K(Y) = k(x)$ whenever x is an affine parameter on $\mathbf{P}^1$ (so $K(Y)$ is a pure transcendental extension of k). Let

$$\iota : Y \to \mathbf{P}^1$$

be the emebdding.

If ι is surjective then Y is isomorphic to $\mathbf{P}^1$. In that case, symmetries of Y correspond by our theorem to symmetries of $k(x)$. In complex analysis we showed that the group of symmetries of $k(x)$ is $PGL(k^2)$, namely the matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

acts on $k(x)$: identify points of $k(x)$ with rational functions $\mathbf{P}^1 \to \mathbf{P}^1$, and then identify those with homogeneous maps $k^2 \to k^2$, on which $PGL(k^2)$ acts by pullback. Thus the symmetry group of Y is $PGL(k^2)$.

If Y is not isomorphic to $\mathbf{P}^1$ then Y embeds in a proper open subset of $\mathbf{P}^1$. But $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$, so Y is a closed subset of an open subset U of $\mathbf{A}^1$. Since Y is clearly not finite it follows that $Y = U$.

Now suppose that instead

$$X = \{(x, y) \in \mathbf{A}^2 : y^2 = x^3 - x\}.$$

Writing $f(x, y) = y^2 - x^3 + x$,

$$df = (1 - 3x^2)\ dx + 2y\ dy$$

which is nonzero on X, so X is smooth; by definition it is affine. On the other hand, one can show that y is irreducible in $A = k[x, y]/(f(x, y))$. But $X \cap \{y = 0\}$ consists of the three points $\{-1, 0, 1\}$ and hence is reducible, so y is not prime and hence A cannot be a UFD.

Since Y is an open subset of $\mathbf{A}^1$, for every open $U \subseteq Y$, $\mathscr{O}_Y(U)$ is a localization of the UFD $k[x]$ and hence is itself a UFD. But Y was an arbitrary smooth rational curve, so X is not a rational curve. In addition, $K(Y) = k[x, y]/(xy - 1)$ is an extension field of $K(X)$, so we get a dominant rational map $X \to Y$. In fact, X is an elliptic curve, but we haven't proven that.

Irrationality has many consequences, including:

(1) There exists a dominant rational function on X.
(2) $K(X) = k[x, y]/(f(x, y))$ is not a pure transcendental extension of k.
(3) When combined with the Hodge theorem: $H^1(X^{an}, \mathbf{C})$ is nontrivial.

Appendix A. Alternative proofs

Lemma A.1. *Let X be an abstract smooth curve, $x \in X$, Y a projective variety, and $\varphi : X \setminus \{x\} \to Y$ a morphism. Then φ extends uniquely to $\overline{\varphi} : X \to Y$.*

Proof. Let ξ be the generic point of X; then the residue field of ξ is the function field K of X, and we can restrict φ to a map $\operatorname{Spec} K \to Y$ which factors through the inclusion $\operatorname{Spec} K \to X \setminus \{x\}$. By the valuative criterion applied to the natural map

$X \to \operatorname{Spec} k$, and the fact that $R = \mathscr{O}_{X,\{x\}}$ is a discrete valuation ring, $\operatorname{Spec} K \to Y$ extends uniquely from $\operatorname{Spec} K$ to $\mathscr{O}_{X,\{x\}}$, thus we have

$$\psi : \operatorname{Spec} R \to \operatorname{Spec} A$$

whenever $\operatorname{Spec} A$ is an affine open neighborhood of $\psi(x)$ (and hence also of $\psi(\xi)$). Hence $\psi^\sharp$ maps $A \to R$. If $\operatorname{Spec} B$ is an affine open neighborhood of x then we actually get a map $\psi^\sharp : A \to B_x$. But A is a finitely generated k-algebra, so $\psi^\sharp$ only sends finitely many generators of A to units in B_x and hence $\psi^\sharp$ actually corestricts to a map $A \to B_f$ for some $f \in B$, thus

$$\psi : \operatorname{Spec} B_f \to \operatorname{Spec} A.$$

If we take the germs of ψ, φ in $\mathscr{O}_{X,y} \subset K$ with $y \in \operatorname{Spec} B_f \setminus \{x\}$, then those germs restrict to the same map $\operatorname{Spec} K \to Y$ and so by the valuative criterion, $\psi = \varphi$ in $\mathscr{O}_{X,y}$. But y was arbitrary so $\psi = \varphi$ in

$$\operatorname{Spec} B_f \setminus \{x\} = \operatorname{Spec} B_f \cap (X \setminus \{x\}).$$

But $\{\operatorname{Spec} B_f, X \setminus \{x\}\}$ is an open cover of X, so the morphisms glue uniquely to a morphism $\overline{\varphi} : X \to Y$. $\square$