Geometry notes

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Chapter 1

Riemannian geometry

This chapter follows John Morgan's lectures on Ricci flow on YouTube.

Definition 1.1. Let M be a smooth manifold. A *Riemannian metric* on M is a smooth contravariant 2-tensor which is positive-definite. A *Riemannian manifold* is a smooth manifold with a choice of Riemannian metric. $\diamondsuit$

By definition, the Riemannian metric induces an inner product on each tangent space of M, which varies smoothly in M.

1.1 The covariant derivative

We want to take the derivative of a vector field. Now we can't quite do this because we need an isomorphism $T_xM \to T_{x+h}M$ between the tangent spaces at x and $x+h$. The covariant derivative induced by a Riemannian metric assigns isomorphisms $T_xM \to T_yM$ for all $x, y \in M$.

Given $E \to M$ a tensor bundle over M, let $C^\infty(M, E)$ be the space of smooth sections of E.

Definition 1.2. Let $E \to M$ be a tensor bundle. A *connection* on M is a family of derivations $\nabla_X : C^\infty(M, E) \to C^\infty(M, E)$, $X \in C^\infty(M, TM)$, such that $X \mapsto \nabla_X$ is linear over the ring $C^\infty(M)$. $\diamondsuit$

In other words, given a vector field X, sections Y, Z, and a function f,

$$\nabla_{fX} Y = f \nabla_X Y$$

and

$$\nabla_X(fY + Z) = f\nabla_X Y + X(f)Y + \nabla_X Z.$$

The interpretation is that we are differentiating Y along the integral curves of X. Now multiplying X by f just changes the velocity at which we are moving along the integral curves of X, which justifies the $C^\infty(M)$-linearity. As for the derivation property, it essentially follows because we want this thing to be a derivative.

Definition 1.3. A *metric connection* ∇ is a connection such that

$$\nabla_X(g(Y_1, Y_2)) = g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2).$$

$\diamondsuit$

The intuition here is that $\nabla_X(g(Y_1, Y_2))$ is a cubic form in (g, Y_1, Y_2), but a metric connection annihilates g in some sense. Indeed, we could think of ∇_X as acting on tensors instead of vectors (or more generally, sections of a fiber bundle $E \to X$ – here we view g as a section of the bundle of contravariant 2-tensors on M) in which case we would actually get $\nabla_X g = 0$.

Definition 1.4. A connection on $C^\infty(M, TM)$ is *torsion-free* if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

$\diamondsuit$

Note carefully that the notion of a torsion-free connection only makes sense for the tangent bundle; general tensor bundles may not have a Lie bracket action by X. This property is very mysterious geometrically, but is motivated by the following theorem.

Theorem 1.5 (Levi-Civita)**.** There is a unique torsion-free metric connection on (M, g).

Definition 1.6. The *Levi-Civita connection* or the *covariant derivative* is the unique torsion-free metric connection. $\diamondsuit$

Let us now write the Levi-Civita connection in coordinates. Let $(x^1, \dots, x^n)$ be local coordinates. Then we have a basis $\partial_1, \dots, \partial_n$ for the tangent space,

Definition 1.7. The *Christoffel symbol* Γ^k_{ij} of the metric g satisfies, in coordinates for which $(\partial_1, \dots, \partial_n)$ is a basis of the tangent space,

$$\nabla_{\partial_i}(\partial_j) = \Gamma^k_{ij}\partial_k.$$

$\diamondsuit$

Since the Levi-Civita connection is torsion-free we have

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Indeed, this follows because $[\partial_i, \partial_j] = 0$. Since the Levi-Civita connection is a metric connection,

$$0 = \nabla_\ell g_{ik} = g_{ik,\ell} - g_{mk}\Gamma^m_{i\ell} - g_{im}\Gamma^m_{k\ell}$$

which implies

$$2\Gamma^i_{k\ell} = g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}).$$

Notice that Γ is not a mixed 3-tensor, in spite of the notation, but rather is just a vector of smooth functions on M.

1.2 The geodesic equation

Let (M, g) be a Riemannian manifold with covariant derivative ∇. Now at every point along a curve γ we have the velocity

$$\dot{\gamma}(t) \in T_{\gamma(t)}M.$$

Then $\dot{\gamma}$ extends to a vector field in a neighborhood of the image of γ so $\nabla_{\dot{\gamma}}\dot{\gamma}$ makes sense.

Definition 1.8. A *geodesic* is a curve γ such that

$$\nabla_{\dot{\gamma}}(\dot{\gamma}) = 0.$$

That equation is called the *geodesic equation*. $\diamondsuit$

In local coordinates $(x^1, \ldots, x^n)$, one has $\dot{\gamma} = (\dot{x}^1, \ldots, \dot{x}^n)$ and the geodesic equation reads

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0.$$

Thus geodesics exist and are unique, by the Picard theorem for ODE.

We can use this to define a map $U \to M$ where U is a small neighborhood of the origin of T_xM, by sending a tangent vector to the geodesic initiated at that tangent vector. For example, on a sphere, one maps a tangent vector onto its respective great circle.

Definition 1.9. The partially-defined map $T_xM \to M$ is called the *exponential map* or *Gauss map* at x, and is denoted by exp. $\diamondsuit$

The motivation for the notation exp is that if M is a linear group, then the exponential map at the identity is exactly the map $A \mapsto e^A$, where A is a matrix in the Lie algebra of M.

1.3 Curvature

The above theory was entirely linear, but now we consider the quadratic theory. The motivation is that the value of the metric at a point – its data to first order – has no content. Indeed, every positive-definite quadratic form is the standard inner product of some coordinate system on the tangent space. So the data of the metric at a single point, or even to first-order approximation near that point, is vacuous.

Curvature will measure the extent to which second derivatives (in the Levi-Civita connection) do not commute.

Definition 1.10. Let

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Then R is the *Riemann curvature tensor*. $\diamondsuit$

By definition of the Lie bracket,

$$\nabla_{[X,Y]}f = \nabla_X\nabla_Y f - \nabla_Y\nabla_X f.$$

This does not hold if one replaces f with a vector field, which is the interesting case here.

The curvature tensor R is $C^\infty(M)$-bilinear, as one can easily check by writing out $R(fX, Y)$. Thus R is in fact a tensor. Moreover, the expression $R(X, Y)Z$ is $C^\infty(M)$-linear in Z.

We now consider the expression $g(R(X, Y)Z, W)$. This expression is $C^\infty(M)$-quadrilinear – that is, it is a 4-tensor. Thus we may define

$$(R^\ell_{ijk})\partial_\ell = R(\partial_i, \partial_j)\partial_k$$

at least in local coordinates. This 4-tensor tells us everything there is to know about the curvature of g. As usual, we can drop indices, say

$$R_{ij\ell k} = g_{mk}R^m_{ijk}.$$

Notice the convention that the top index is the *third* index when it goes downstairs.

What does the Riemann curvature tensor actually measure?

Definition 1.11. Let γ be a curve in M, $E \to M$ a tensor bundle, and ∇ a connection on $C^\infty(M, E)$. A *parallel section* of E to γ is a section $X \in C^\infty(M, E)$ with $\nabla_{\cdot\gamma}X = 0$. ♢

Intuitively, a parallel section to γ is one which is constant along γ. Of course this doesn't make sense because the fibers of E are isomorphic but noncanonically so – this is why we have to choose a connection before we can talk about being "constant". Now if $(x, v) \in E$, then there is a unique parallel section X with $X_{\gamma(0)} = (x, v)$, by the Picard theorem.

Definition 1.12. Let γ be a curve in M, $E \to M$ a tensor bundle, and ∇ a connection on $C^\infty(M, E)$. Let $(x, v) \in E$. The *parallel transport* of (x, v) along γ is the unique parallel section X of E with $X_{\gamma(0)} = (x, v)$. ♢

In other words, the parallel transport gives an isomorphism from the fiber of $\gamma(0)$ to the fiber of $\gamma(1)$.

Now let X, Y be vector fields on M, ∇ the Levi-Civita connection, and suppose that τ_X and τ_Y are the parallel transport maps of ∇ along the integral curves of X, Y for unit time. Intuitively, for each x, we get an isomorphism $\tau_X : T_{\exp(0)}M \to T_{\exp(X_x)}M$ where exp is the exponential map at x. Suppose that $[X, Y] = 0$. Then one has a quadrilateral Q defined by $tY, sX, -tY, -sX$, and the parallel transport

$$\tau_Q = \tau_{sX}^{-1}\tau_{tY}^{-1}\tau_{sX}\tau_{tY}$$

is not necessarily the identity, even though Q is a loop. This may hold even if Q is contractible! Moreover, one has

$$\nabla_X Z = \frac{d}{dt}\tau_{tX}Z|_{t=0}.$$

Therefore

$$\frac{\partial^2}{\partial t \partial s}\tau_Q Z|_{t=s=0} = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z = R(X, Y)Z.$$

So $R(X, Y)Z$ is the difference between Z and its parallel transport along an infinitesimal quadrilateral with edges proportional to X, Y.

Chapter 2

Riemann surfaces

In this chapter we begin discussing algebraic geometry. Algebraic geometry is the study of projective varieties over $\mathbb{C}$; compact Riemann surfaces give the dimension 1 case.

Definition 2.1. Let X be a surface. If φ_1, φ_2 are charts on X valued in $\mathbb{C}$, we say that φ_1, φ_2 are *compatible charts* if the transition functions $\varphi_2 \circ \varphi_1^{-1}$ is holomorphic. $\diamondsuit$

Definition 2.2. Let X a surface. A *complex atlas* on X is an open cover of X by compatible charts. Two complex atlases A, B are *compatible atlases* if for every $\varphi \in A, \psi \in B$, φ, ψ are compatible charts. $\diamondsuit$

Definition 2.3. A *complex structure* is an equivalence class of complex atlases with respect to compatibility. A *Riemann surface* is a connected surface with a choice of complex structure. $\diamondsuit$

Let us give examples of Riemann surfaces. We first note that open subsets of Riemann surfaces are Riemann surfaces, simply by taking restrictions of charts. Trivially $\mathbb{C}$ is a Riemann surface, where the identity is a global chart. So every open set in $\mathbb{C}$ is a Riemann surface.

Definition 2.4. The *Riemann sphere*, or *projective line*, $\mathbb{P}^1$, is the Riemann surface defined by taking the topological one-point compactification $\mathbb{C} \to S^2$, and taking the charts $\varphi : \mathbb{C} \to \mathbb{C}$ to be the identity and $\psi : \mathbb{P}^1 \setminus \{0\}$ to be $\psi(z) = 1/z$. $\diamondsuit$

To see that $\mathbb{P}^1$ is in fact a Riemann surface, we note that $\psi \circ \varphi^{-1}(z) = 1/z$, and similarly for $\varphi^{-1} \circ \psi$.

Definition 2.5. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Let Γ be the free abelian discrete group generated by ω_1, ω_2, equipped with its action on $\mathbb{C}$ by translation. Let $T_\Gamma = \mathbb{C}/\Gamma$ be the space of orbits of Γ and $\pi : \mathbb{C} \to T_\Gamma$ be the projective map. We put a complex structure on T_Γ as follows. Let U be a subset of a fundamental domain of the action of the Γ, so $\pi|U$ is a bijection. Then declare that the charts are $(\pi|U)^{-1}$. We say that T_Γ is a *complex torus* or *elliptic curve*. $\diamondsuit$

2.1 Holomorphic maps

We introduce the morphisms of the category of Riemann surfaces.

Definition 2.6. A map $f : X \to Y$ between Riemann surfaces is a *holomorphic map* if for every pair of charts $\varphi : U \to X$, $\psi : V \to Y$ there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r] & V \arrow[u, "\psi"'] \end{tikzcd}$$

in which the bottom arrow commutes. $\diamondsuit$

Definition 2.7. A holomorphic bijection $f : X \to Y$ is a *biholomorphic map* if f^{-1} exists and is holomorphic. If a biholomorphic map $X \to Y$ exists, we say that X, Y are *biholomorphic surfaces*. If f is biholomorphic onto its image, we say that f is an *embedding*. $\diamondsuit$

The fundamental problem for Riemann surfaces, introduced by Riemann himself, is: if S is a topological surface, how many distinct (i.e. up to biholomorphic isomorphism) complex structures exist on S? Solving this problem in general requires one to use Teichmüller theory and moduli stacks of curves, and is still an active area of research. This is different than in the smooth category, where there is exactly one smooth structure one can put on a surface.

Example 2.8. Liouville's theorem shows that $\mathbb{C}$ has at least two Riemann surface structures on it: the disc and the plane. $\diamondsuit$

Definition 2.9. A *holomorphic function* on X is a holomorphic map $X \to \mathbb{C}$. The space of holomorphic functions on X is denoted $\mathscr{O}(X)$. $\diamondsuit$

Theorem 2.10 (removal of singularities)**.** If $f \in \mathscr{O}(X \setminus A)$, A is finite, and f is bounded near A, then $f \in \mathscr{O}(X)$. $\diamondsuit$

Theorem 2.11 (analytic continuation)**.** If $f, g : X \to Y$ are holomorphic, $A \subseteq X$ is not discrete, and $f|A = g|A$, then $f = g$. $\diamondsuit$

These theorems are both local, so by taking charts we can pass to the case that X is an open subset of $\mathbb{C}$, in which case they are trivial. They are especially powerful when X is compact, so that A is either discrete or infinite.

Definition 2.12. We say that $f \in \mathscr{O}(U)$ is a *meromorphic function* on X if $U \subseteq X$ is open, $X \setminus U$ is discrete, and

$$\lim_{z \to z_0} |f|(z_0) = \infty$$

whenever $z_0 \notin U$. Points in $X \setminus U$ are called *poles* of f. The space of meromorphic functions on X is denoted $\mathscr{M}(X)$. $\diamondsuit$

We note that $\mathscr{O}$ and $\mathscr{M}$ both have the structure of a sheaf of rings. Since, if (U, z) is a chart, $\mathscr{M}(U)$ is exactly the space of meromorphic functions in the usual sense on some open subsets of $\mathbb{C}$, we may expand $f \in \mathscr{M}(X)$ as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z_k$$

at least in the coordinate chart (U, z).

Theorem 2.13. Let $f \in \mathscr{M}(X)$ and define $\hat{f} \in \mathscr{O}(X \to \mathbb{P}^1)$ by $\hat{f} = f$ away from the poles of f, and $\hat{f}(p) = \infty$ for every pole p of f. Conversely, $\hat{f} \in \mathscr{O}(X \to \mathbb{P}^1)$ is either constant or restricts to a meromorphic function on X.

Proof. By the definition of a one-point compactification, $\hat{f}$ is continuous. In particular, $\hat{f}$ is locally bounded in charts, so by removal of continuities f extends to the holomorphic function $\hat{f}$.

Conversely, let $A = \hat{f}^{-1}(\infty)$. If $A = X$ then $\hat{f}$ is constant. Otherwise, A is discrete, and $X \setminus A$ is open, so $f = \hat{f}|(X \setminus A)$ is a meromorphic function on X. □

We now check that locally, holomorphic maps are monomials.

Theorem 2.14. Let $f : X \to Y$ be a nonconstant holomorphic map, $a \in X$, $b = f(a)$. Then there are coordinates near a, b in which $a = b = 0$ and

$$f(z) = z^k$$

where $k \in \mathbb{N}$, $k \geq 1$, and k does not depend on the choice of coordinates.

Proof. Let $\varphi : U \to \mathbb{C}$ be a chart with $\varphi(a) = 0$ and $\psi : V \to \mathbb{C}$ a chart with $\psi(b) = 0$. Let $F : \varphi(U) \to \psi(V)$ be the local representation of f. Then F has a zero at 0, say of order k, thus

$$F(z) = z^k G(z)$$

where $G(0) \neq 0$. Taking a biholomorphic isomorphism does not change the order of a zero, so k does not depend on the choice of coordinates. Shrinking U if necessary, we can assume that G is nonzero on all of $\varphi(U)$. We can then ramify the logarithm to find a H with $G = H^k$. In these coordinates we have

$$F(z) = (zH(z))^k.$$

Let $\alpha(z) = zH(z)$. Then, after shrinking U as necessary again, α is a holomorphic embedding $\varphi(U) \to \mathbb{C}$. Indeed, $\alpha'(0) = H(0)$ is nonzero, so the claim follows by the inverse function theorem. We can then replace φ with $\alpha \circ \varphi$, and in those coordinates we have $F(z) = z^k$, as desired. □

Corollary 2.15 (open mapping theorem)**.** Every holomorphic map is open.

Proof. This can be checked locally, and locally the map is z^k, which is open. □

Corollary 2.16. Every injective holomorphic map is an embedding.

Proof. By the open mapping theorem, we just need to check that f^{-1} is holomorphic. But $f(z) = z^k$ and f is injective, so $k = 1$, so $f^{-1}(w) = w$ is holomorphic. □

Corollary 2.17 (maximum principle)**.** Let $f : X \to \mathbb{C}$ be a holomorphic function. Then f is constant or $|f|$ attains a maximum.

Proof. If $r = \max |f|$ is attained, say $r = |f(a)|$, then $f(X) \subseteq \overline{B(0, r)}$. By the open mapping theorem, $f(X)$ is open if f is not constant, even though $f(X)$ meets $\partial B(0, r)$. □

Corollary 2.18 (Liouville's theorem)**.** If X is compact and $f : X \to \mathbb{C}$ is holomorphic then f is constant.

Proof. The maximum principle and extreme value theorem imply this. □

Proposition 2.19 (GAGA for $\mathbb{P}^1$)**.** If $f \in \mathscr{M}(\mathbb{P}^1)$ then f is a rational map.

Proof. This holds if f is constant; otherwise, the set A of poles of f is discrete and compact, so A is finite, say $A = \{a_1, \dots, a_n\}$. After replacing f with $1/f$ if necessary (which does not affect whether f is a rational map) we may assume that $A \subset \mathbb{C}$. Therefore we may let

$$h_\ell(z) = \sum_{j=-k_\ell}^{-1} c_k (z - a_\ell)^j$$

be the principal part of f at a_ℓ. Then $f - (h_1 + \cdots + h_n) \in \mathscr{O}(\mathbb{P}^1)$ is constant; since the h_ℓ are rational maps the claim then holds. □

This is the essence of the GAGA principle of Serre, which says that a projective variety over $\mathbb{C}$ is the same thing as a complex compact manifold (that has enough meromorphic functions). Thus much of complex analysis is equivalent to algebraic geometry.

2.2 Covering spaces of Riemann surfaces

Since we assume that Riemann surfaces are connected, and they are locally euclidean, they are always path-connected, and thus we may define their fundamental group, and consider the covering space of a Riemann surface.

Example 2.20. The map $p_k : \mathbb{C}^* \to \mathbb{C}^*$, $p_k(z) = z^k$ is a covering space. Indeed, p_k' is never zero, so by the inverse function theorem p_k is a local homeomorphism. If $p_k(b) = a$, then there are $U \ni a$, $V \ni b$ such that p_k is a homeomorphism $V \to U$. In particular,

$$p_k^{-1}(U) = \bigcup_{j=0}^{k-1} \omega^j V$$

where ω generates the group of kth roots of unity. If V is taken small enough, then the $\omega^j V$ are disjoint, hence the claim. ♢

Example 2.21. For similar reasons, $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering space. Indeed, $\exp'$ is never zero, and so for every $a \in \mathbb{C}^*$ we can find $U \ni a$ such that

$$\log U = \bigcup_{n \in \mathbb{Z}} V + 2\pi n i$$

with V sufficiently small. $\diamondsuit$

Example 2.22. The projection $\mathbb{C} \to T_\Gamma$, T_Γ an elliptic curve, is a covering space. $\diamondsuit$

The above examples realize $\mathbb{C}$ as the universal cover of $\mathbb{C}^*$ and T_Γ. On the other hand, $\mathbb{P}^1$ is simply connected and so cannot be covered by $\mathbb{C}$.

The homotopy lifting property tells us when we can take a logarithm of a function. Consider the universal cover

$$\exp : \mathbb{C} \to \mathbb{C}^*.$$

If $f : X \to \mathbb{C}^*$ is holomorphic then we want to lift f to a map $\hat{f} : X \to \mathbb{C}$ which commutes with the universal cover. This is exactly the question of finding a logarithm of f.

Lemma 2.23. Let X, Y, Z be connected manifolds. Let $p : Y \to X$ be a local homeomorphism, $f : Z \to X$ continuous, and $g_1, g_2 : Z \to Y$ lifts of f along p. If there is z_0 with $g_1(z_0) = g_2(z_0)$ then $g_1 = g_2$.

Proof. Let $T \subseteq Z$ be the equalizer of g_1, g_2. Then $T \ni z_0$ is nonempty and closed. Let $w \in T$. Then there is a neighborhood U of $g_1(w) = g_2(w)$ which is homeomorphic along p to a neighborhood of $f(w)$. Then $g_i = p^{-1} \circ f$ near w so T is open. Therefore $T = Z$. $\square$

Lemma 2.24. If $p : Y \to X$ is holomorphic and a local homeomorphism and $g : Z \to Y$ is a continuous lift of a holomorphic map $Z \to X$ along p, then g is holomorphic.

Proof. Locally, g is $p^{-1} \circ f$, which is holomorphic by the inverse function theorem. $\square$

In particular, if the logarithm of a function exists, then it is automatically holomorphic.

In general lifts may not exist for local homeomorphisms, but they will for covering spaces. This motivates why covering spaces are useful for complex analysis. We recall the path-lifting property: paths always lift along covering spaces, and the lifts are in bijection with the possible starting points of the path. Similarly, homotopies of paths always lift along covering spaces, determined by the starting points of each of the paths (as long as the starting points of the paths together form a path). If we impose that the homotopy lift has a single starting point, then it also has a single endpoint.

We recall that if $p : Y \to X$ is a covering space then the pushforward $p_* : \pi_1(Y) \to \pi_1(X)$ is injective. Indeed, if γ is a loop in Y annihilated by p_* then there is a homotopy H in X which witnesses this, and H lifts to a homotopy in Y which witnesses that γ is nullhomotopic.

Theorem 2.25. Let $p : Y \to X$ be a covering space, $f : Z \to X$ continuous. Then f lifts to a map $Z \to Y$ iff $f_* \pi_1(Z) \subseteq p_*(\pi_1(Y))$.

Proof. If a lift exists then the claim follows by functoriality. Conversely, let $z_0, w \in Z$, and let $\alpha : z_0 \to w$ be a path in Z. Then there is a lift $\tilde{\alpha} : y_0 \to y_1$ of $f_*\alpha$ in Y. Let $\tilde{f}(w) = y_1$. If we have another path $\alpha' : z_0 \to w$ then $\alpha' \circ \alpha^{-1}$ is a loop at z_0, so $f_*(\alpha' \circ \alpha^{-1}) \in p_*(\pi_1(Y))$ by assumption. Thus there is a $\gamma \in \pi_1(Y)$ with $p_*\gamma = f_*(\alpha' \circ \alpha^{-1})$. On the other hand, $p_*(\tilde{\alpha}' \circ \tilde{\alpha}^{-1}) = p_*(\gamma)$. Since p_* is injective this implies $\tilde{\alpha} \circ \tilde{\alpha}^{-1} = \gamma$. So $\tilde{f}(w)$ does not depend on the choice of α. □

Corollary 2.26. Suppose that Z is simply connected, $p : Y \to X$ a covering space, $f : Z \to X$ continuous. Then f lifts to a map $Z \to Y$.

Proof. In the previous theorem, $f_*\pi_1(Z) = 0$. □

Corollary 2.27. If U is a simply connected Riemann surface, $f : U \to \mathbb{C}^*$ holomorphic, then f has a holomorphic logarithm.

Proof. A logarithm of f is a lift of f along the covering space $\exp : \mathbb{C} \to \mathbb{C}^*$. □

We recall that every connected manifold has a universal cover; that it is unique by abstract nonsense; and that it is the unique simply connected covering space.

Definition 2.28. A *Galois covering space* is a covering space $p : Y \to X$ such that the deck group of p acts transitively on each fiber of p. ♢

Example 2.29. The covering space $p_k : \mathbb{C}^* \to \mathbb{C}^*$ is Galois. If $p_k(z_1) = p_k(z_2)$ then there is a root of unity ω such that $z_2 = \omega z_1$. But ω is a deck transformation since $(\omega z)^k = z^k$. ♢

Theorem 2.30 (Galois theory for covering spaces)**.** Let X be a connected manifold and $p : \tilde{X} \to X$ its universal cover. Then p is Galois and its deck group is $\pi_1(X)$. Moreover, if $q : Y \to X$ is any covering space, then there is a unique covering space $f : \tilde{X} \to Y$ with $f : \tilde{X} \to Y$, and f is the universal cover of Y. In addition, there is a subgroup G of $\pi_1(X)$ such that $Y = \tilde{X}/G$ with quotient map f, and G is a normal subgroup of $\pi_1(X)$ iff q is Galois. If q is Galois, then $\pi_1(Y) = \pi(X)/G$. ♢

We omit the proof since this is not a topology class.

Example 2.31. The universal cover of $\mathbb{C}^*$ is $\exp : \mathbb{C} \to \mathbb{C}^*$. Its deck group is isomorphic to $\mathbb{Z}$ since φ commutes with exp exactly if $\varphi(z) = 2\pi ik$ for some $k \in \mathbb{Z}$. This gives a proof that $\pi_1(\mathbb{C}^*) = \mathbb{Z}$. ♢

A motivation for covering spaces is *uniformization* – that is, the classification of Riemann surfaces up to homeomorphism. Now $\mathbb{P}^1$ is simply connected, so it is its own universal cover. If T_Γ is an elliptic curve, then Γ acts on $\mathbb{C}$ by translation and $T_\Gamma = \mathbb{C}/\Gamma$, so the universal cover of T_Γ is $\mathbb{C}$. If $\mathcal{H}$ denotes the hyperbolic plane and $\mathbb{D}$ the disc, then $\exp : \mathcal{H} \to \mathbb{D}^*$ is the universal cover. But $\mathbb{P}^1$, $\mathcal{H}$, and $\mathbb{C}$ are the only possible universal covers of Riemann surfaces, as we will see.

2.3 Branched coverings

In complex analysis it is convenient to generalize the notion of covering space somewhat.

Definition 2.32. A *discrete map* is a holomorphic map with discrete fibers. ♢

Definition 2.33. Let $p : Y \to X$ be an open discrete map. A *branch point* $y \in Y$ is a point such that for every $V \ni y$ open, $p|V$ is not a homeomorphism. If p has no branch points, we say that p is an *unbranched map*. ♢

Lemma 2.34. An open discrete map is unbranched iff it is a local homeomorphism.

Proof. Just expand out the definitions. □

Lemma 2.35. The set of branch points of a nonconstant holomorphic map is discrete.

Proof. One has $f'|B = 0$, so if B clusters then $f' = 0$ identically. □

Lemma 2.36. If X is a Riemann surface, Y a connected surface, $p : Y \to X$ a local homeomorphism, then there is a unique complex structure on Y for which p is holomorphic.

Proof. Let (U, φ) be a chart on X. Define charts on Y by $\{(V, \varphi \circ p)$ where $p|V$ is a homeomorphism $V \to U$. This clearly defines a complex structure on Y. Uniqueness follows because we can invert p locally to get a biholomorphic isomorphism. □

Definition 2.37. A *proper map* is a holomorphic map f such that the preimage of every compact set under f is compact. ♢

Between compact Riemann surfaces, every holomorphic map is proper.

Lemma 2.38. Let $f : Y \setminus f^{-1}(A) \to X \setminus A$ be a proper map, B the branch points of f, $A = f(B)$. Then f is a covering space.

Proof. Let $x \in X \setminus A$. By properness, the preimage of x is finite, say $f^{-1}(x) = \{y_1, \ldots, y_n\}$. Let $V_i \ni x_i$ be such that $f|V_i$ is an embedding and the V_i are disjoint, then $W = \bigcap_i f(V_i)$ is nonempty. Then $f^{-1}(W)$ is evenly covered. □

Definition 2.39. Let $f : Y \to X$ be a holomorphic map. If $f(z) = z^k$ near y, then we say that the *order* of y, $v(f, y) = k$. ♢

It follows that

$$\sum_{y \in f^{-1}(x)} v(f, y)$$

does not depend on x provided that f is proper.

Lemma 2.40. Let f be a proper map. Let x be a branch point of f, $f^{-1}(x) = \{y_1, \ldots, y_m\}$, with k_i the order of y_i. Then $\sum_i k_i$ is the number of sheets of f as a covering space (away from its branch points).

Proof. The degree of a covering space is constant, and in a punctured neighborhood of x, f has degree $\sum_i k_i$. □

Corollary 2.41. Let X be a compact Riemann surface. Let $f \in \mathscr{M}(X)$; then the number of zeroes of f equals the number of poles of f, counted with multiplicity.

Proof. Expanding out the definitions, f defines a proper map $X \to \mathbb{P}^1$.

$$\sum_{x \in f^{-1}(0)} v(f,x) = \sum_{x \in f^{-1}(\infty) v(f,x)}$$

as desired. □

Let us compute the moduli space of covering spaces of $\mathbb{D}^*$. Recall that p_k denotes the map $p_k(z) = z^k$. In what follows we use $\mathcal{H}$ to mean the left half-plane, which isomorphic to the upper half-plane but makes notation easier.

Theorem 2.42. Let X be a Riemann surface, $f : X \to \mathbb{D}^*$ a holomorphic covering space. If f has infinitely many sheets then f is isomorphic to the covering space $\exp : \mathcal{H} \to \mathbb{D}^*$. Otherwise, f is isomorphic to the covering space $p_k : \mathbb{D}^* \to \mathbb{D}^*$.

Proof. Since exp is the universal cover, there is a covering space $\psi : \mathcal{H} \to X$ which commutes with $\exp, f$. By Galois theory, the deck group of ψ is a subgroup G of $\pi_1(\mathbb{D}^*) = \mathbb{Z}$. So either G is the trivial group or $G = k\mathbb{Z}$ acts on $\mathcal{H}$ by translation by $2\pi ink$. If G is trivial then ψ is an isomorphism. Otherwise X is a quotient of $\mathcal{H}$ by the action of G, so X is a cylinder. Every cylinder is isomorphic to $\mathbb{D}^*$, identifying one end of the tube with 0 and the other end with $\partial\mathbb{D}$. □

Theorem 2.43. Let X be a Riemann surface, $f : X \to \mathbb{D}$ a proper nonconstant map such that f is unbranched over $\mathbb{D}^*$. Then there is a $k \geq 1$ and an isomorphism $\varphi : X \to \mathbb{D}$ such that $f = p_k \circ \varphi$.

Proof. Let $X^* = f^{-1}(\mathbb{D}^*)$. Then $f|X^*$ is a covering space of $\mathbb{D}^*$, with only finitely many sheets since f is proper. Thus there is an isomorphism $\varphi^* : X^* \to \mathbb{D}^*$ of covering spaces between p_k and f. We claim that $f^{-1}(0)$ is a point, so that φ^* extends to an isomorphism of Riemann surfaces $\varphi : X \to D$.

So suppose that $f^{-1}(0) = \{b_1, \ldots, b_n\}$, B a disk around 0, and $f^{-1}(B) = \bigcup_i V_i$ where $b_i \in V_i$ and the V_i are disjoint. But $f^{-1}(B)$ is isomorphic to $p_k^{-1}(B)$ which is $D(0, r^{1/k})^*$, which is connected. Therefore $n = 1$. □

2.4 Sheaves on Riemann surfaces

Definition 2.44. Let X be a topological space and C a category. A *presheaf* $\mathscr{F}$ on X valued in C is a map which assigns each open set U of X to some $\mathscr{F}(U) \in C$ equipped with *restrictions*

$$\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$$

whenever $V \subseteq U$ which compose, thus $\rho^V_W \circ \rho^U_V = \rho^U_W$ and $\rho^U_U = 1$. We write $f|V = \rho^U_V(f)$ whenever $f \in \mathscr{F}(U)$. ♢

Example 2.45. The continuous functions C, holomorphic functions $\mathscr{O}$, and meromorphic functions $\mathscr{M}$ form presheaves on any Riemann surface valued in algebras over $\mathbb{C}$. ♢

Definition 2.46. Let X be a topological space and C a category. A *sheaf* $\mathscr{F}$ on X valued in C is a presheaf such that for every $U \subseteq X$ open and open covers $\mathcal{U}$ of U:

1. If $f, g \in \mathscr{F}(U)$ such that for every $V \in \mathcal{U}$, $f|V = g|V$, then $f = g$.
2. If for every $V \in \mathcal{U}$ there is $f_V \in \mathscr{F}(V)$ such that $f_V|V \cap W = f_W|V \cap W$ for all $V, W \in \mathcal{U}$, then there is $f \in \mathscr{F}(U)$ such that for every V, $f|V = f_V$.

♢

Definition 2.47. Let X be a topological space. Let $x \in X$ and $\mathscr{F}$ a presheaf on X valued in a category with colimits. The *stalk* $\mathscr{F}_x$ is the colimit of $\mathscr{F}(U)$ as U shrinks down to x. A *germ* at x is an element of $\mathscr{F}_x$. ♢

Let $\mathscr{F}$ be a presheaf, valued in a concrete category. We will construct a topological space $|\mathscr{F}|$ which as an underlying set is the disjoint union of the stalks of $\mathscr{F}$, so we get a fiber bundle structure

$$p : |\mathscr{F}| \to X$$

which sends all the germs at x to x.

Theorem 2.48. There is a topology on the set $|\mathscr{F}|$ such that p is continuous and a local homeomorphism.

Proof. We define a base of sets

$$[U, f] = \{[z, f] : z \in U\}$$

whenever $U \subseteq X$ is open and $f \in \mathscr{F}(U)$. Here $[z, f]$ denotes the image of f in $\mathscr{F}_x$ when we take the colimit over the diagram of open neighborhoods of z, so $[U, f]$ is the set of all germs obtained by restricting f. Every germ is in the image of some f so the $[U, f]$ form a cover of $|\mathscr{F}|$.

Now if $\varphi \in [U, f] \cap [V, g]$ and φ is a germ at x then $\varphi = f_x = g_x$. Therefore there is a $W \subseteq U \cap V$ such that $f|W = g|W$, by definition of colimit. Let $h = f|W$; then $[W, h] \subseteq [U, f] \cap [V, g]$. Therefore the $[U, f]$ form a topological base.

If $W \subseteq X$ is open then

$$p^{-1}(W) = \bigcup_{f \in \mathscr{F}(W)} [W, f]$$

is open and hence p is continuous. Moreover, if $[U, f]$ is given then

$$p|[U, f] : [U, f] \to U$$

is a homeomorphism. So p is locally a homeomorphism. □

Definition 2.49. The *étalé space* of $\mathscr{F}$ is $|\mathscr{F}|$. ♢

Sheaves are not useful in the smooth category, except in the study of de Rham cohomology. This is because if there are too many partitions of unity then the étalé space will fail to be Hausdorff. (I think this has to do with the fact that sheaf cohomology becomes trivial in that case.)

If f is a section of a sheaf and x is a point, the germ of f at x is denoted f_x.

Definition 2.50. A sheaf $\mathscr{F}$ has the *identity property* if for every open connected set Y and $f, g \in \mathscr{F}(Y)$, if there is a $a \in Y$ such that $f_a = g_a$ then $f = g$. ♢

Example 2.51. The sheaf $\mathscr{O}$ has the identity property. Indeed if $f_x = g_x$ then there is an open neighborhood W of x such that $f|W = g|W$, so $f = g$. ♢

Thus sheaves are, in fact, useful in the study of complex analysis, even though they're not in real analysis.

Proposition 2.52. If $\mathscr{F}$ is a sheaf with the identity property on a manifold, then $|\mathscr{F}|$ is Hausdorff.

Proof. Suppose that $f_x \neq g_y$. If $x \neq y$ then there are open sets $U \ni x, V \ni y$ which are disjoint, so $[U, f]$ and $[V, g]$ separate f_x, g_y. Otherwise, there are $U, V \ni x$ such that $f \in \mathscr{F}(U)$, $g \in \mathscr{F}(V)$. So there is a $W \subseteq U \cap V$ which is connected and open, since the underlying space is a manifold. We claim that $[W, f], [W, g]$ separate f, g. If not, then there is a $z \in W$ such that $f_z = g_z$, so by the identity theorem $f = g$ and hence $f_x = g_y$, a contradiction. □

Corollary 2.53. Let X be a Riemann surface. Then $|\mathscr{O}_X|$ has the structure of a disjoint union of Riemann surfaces such that $p : |\mathscr{O}_X| \to X$ is holomorphic.

Proof. Since $|\mathscr{O}_X|$ is a Hausdorff space and p is a local homeomorphism, $|\mathscr{O}_X|$ is a surface and p defines a complex structure on $|\mathscr{O}_X|$. □

Technically we should show that each component of $|\mathscr{O}_X|$ is second countable. However this is not actually very important, because this follows from the fact that $|\mathscr{O}_X|$ has a complex structure, according to a theorem of Rado. We omit the proof.

Let us use sheaves to study analytic continuation. Here we use I to mean the unit interval.

Definition 2.54. Let X be a Riemann surface and u a path in X starting at $a \in X$. An *analytic continuation* of a germ f_a on X along u is a lift $\hat{u} : I \to |\mathscr{O}_X|$ with $\hat{u}(0) = f_a$. ♢

Note that $p : |\mathscr{O}_X| \to X$ is not a covering space. Indeed, $p : |\mathscr{O}_\mathbb{C}| \to \mathbb{C}$ is not a covering space because a germ of $1/z$ cannot be analytically continued along any path which ends at 0. This is the canonical example of a local homeomorphism which is not a covering space.

What does the definition really mean? Let $u : I \to X$ be a path and f_a a germ at a. Then if we can lift u to $\hat{u}$ which starts at $\hat{u}(0) = f_a$, then for each point $t \in [0, 1]$, there is some basic open set $[V_t, f_t]$ in $|\mathscr{O}_X|$ around $\hat{u}(t)$. By compactness of I there are finitely many such basic open sets $[V_i, f_i]$. These drop to open sets $V_i \subseteq X$ and finitely many holomorphic functions $f_i : V_i \to \mathbb{C}$. Then they glue together to a function $f : W \to \mathbb{C}$ where $W = \bigcup_i V_i$ is an open set that contains $u(I)$. So from a germ and a path which lifts along p we can find an honest-to-god holomorphic function defined closed to that path.

Lemma 2.55. Let $d : |\mathscr{O}| \to |\mathscr{O}|$ be the differentiation map, thus

$$df_z = (f')_z.$$

Then d is a covering space.

Proof. The map d is well-defined because every germ extends to a holomorphic function on an honest-to-god open set, and as long as that open set is small enough the choice of open set does not matter.

Now let $f_a \in [U, f]$ be a germ. If U is small enough then U is simply connected so there is a $F \in \mathscr{O}(U)$ with $F' = f$. So

$$d^{-1}([U, f]) = \bigcup_C [U, F + c]$$

and since each of the $[U, F + c]$ locally look like Ω, if U is small enough then d is a covering space. Here F is far from $F + c$, $c \neq 0$, because they never define the same germ. □

Definition 2.56. Let Ω be an open subset of $\mathbb{C}$, $f \in \mathscr{O}(\Omega)$, γ a path in Ω. A *primitive* of f along γ is a path of germs F_t, $t \in I$, at $\gamma(t)$, such that $F_t' = f_{\gamma(t)}$. ♢

We abused notation to write F_t for the germ at $\gamma(t)$. The point is that a primitive along γ is only defined in a very small neighborhood of γ, rather than all of Ω.

Theorem 2.57 (fundamental theorem)**.** Let X be a Riemann surface, γ a path in X, and $f \in \mathscr{O}(X)$. Then

$$\int_\gamma f(z)\ dz = F_1(\gamma(1)) - F_0(\gamma(0)). \tag{2.1}$$

Proof. If X is convex then this is obvious (in that it can be proven using the Cauchy-Goursat triangle trick or something idk). But, as in the construction of an analytic continuation, we can cover $\gamma(I)$ by finitely many convex sets and apply (2.1) in each convex set. Since any two primitives are equal up to a constant what we get is independent of the choice of primitive. Thus we get (2.1) in general. □

Theorem 2.58 (Cauchy)**.** Let X be a Riemann surface, $\Gamma : \gamma_0 \to \gamma_1$ a based homotopy of paths in X, and $f \in \mathscr{O}(X)$. Then

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Consider the based homotopy of germs $G(s,t) = f_{\Gamma(s,t)}$ where $\Gamma(s,t) = \gamma_s(t)$. Lift G to a based homotopy $\tilde{G} : I^2 \to |\mathscr{O}|$ along d; thus for every $s \in I$, $\tilde{G}(s, 0) = \tilde{G}(0, 0)$. Then by definition of d, $\tilde{G}$ is a homotopy of primitives of f, so

$$\int_{\gamma_0} f(z)\ dz = \tilde{G}(0,1)(\Gamma(0,1)) - \tilde{G}(0,0)(\Gamma(0,0))$$

by (2.1). Moreover if $t\{0, 1\}$ then

$$\tilde{G}(1,t) = \tilde{G}(0,t)$$

since $\tilde{G}$ is a based homotopy, so

$$\begin{aligned}\int_{\gamma_1} f(z)\, dz &= \tilde{G}(1,1)(\Gamma(1,1)) - \tilde{G}(1,0)(\Gamma(1,0))\\ &= \tilde{G}(0,1)(\Gamma(0,1)) - \tilde{G}(0,0)(\Gamma(0,0))\\ &= \int_{\gamma_0} f(z)\, dz\end{aligned}$$

which was desired. □

Theorem 2.59 (monodromy theorem)**.** Let $u : u_0 \to u_1$ is a based homotopy of paths, $u_0(0) = a$, $u_0(1) = b$. If $\varphi \in \mathscr{O}_a$ has an analytic continuation along every path u_s, then the analytic continuation along u_0 is equal to the analytic continuation along u_1.

Proof. Let $\hat{u}$ be the lift of u to $|\mathscr{O}|$. Then $\hat{u}$ is unique since homotopy lifts are unique if they exist. □

Corollary 2.60. Let X be a simply connected Riemann surface, $a \in X$, and $\varphi \in \mathscr{O}_a$. If φ has an analytic continuation along every path starting at a, then there is a global section $f \in \mathscr{O}(X)$ such that $f_a = \varphi$.

Proof. Given any x, choose a path $\gamma : a \to x$, and let φ_x be the germ at x obtained by the analytic continuation of φ along γ. By monodromy, since $\pi_1(X) = 0$, φ_x does not depend on γ. So we can define $f(x) = \varphi_x(x)$. Then f is holomorphic in a neighborhood of x, so f is globally holomorphic. □

2.5 Analytic continuation to global sections

Let us now consider how to globally continue a germ, even in the case that $\pi_1(X)$ is nonzero. There are two approaches to this: multivalued functions, or to define the Riemann surface of the germ. The latter was the historical motivation for defining Riemann surfaces in the first place.

Definition 2.61. Let $p : Y \to X$ be an unbranched map, $p(y) = x$. The *pullback* by p

$$p_x^* : \mathscr{O}_x \to \mathscr{O}_y$$

on the stalks of $\mathscr{O}$ defined by $p^* f_x = (f \circ p)_y$. Since p is unbranched, we can also define the *pushforward* $p_* = (p^*)^{-1}$. ◇

Definition 2.62. Let $a \in X$, $\varphi \in \mathscr{O}_a$. An *analytic continuation* of φ is an unbranched covering space

$$p : Y \to X$$

equipped with a global section $f \in \mathscr{O}(Y)$ such that if $p(b) = a$ then $p_*(f_b) = \varphi$. ◇

Example 2.63. Let $a \in X^*$, $g(z) = z^{1/n}$ defined in some open set $U \ni a$. Consider

$$Y^* = \{(z, w) \in (\mathbb{C}^*)^2 : w = z^n\}.$$

Define $p : Y^* \to \mathbb{C}^*$, $p(z, w) = w$. Then $g(p(z, w)) = g(z^n) = z$. So we set f to be the projection onto the first coordinate of Y^* onto $\mathbb{C}$. Therefore (p, f) is an analytic continuation of g.

Notice that Y^* is isomorphic to $\mathbb{C}^*$. We can extend Y^* to

$$Y = \{(z, w) \in \mathbb{C}^2 : w = z^n\}$$

which is homeomorphic but not isomorphic to $\mathbb{C}$. We can see this by noting that p is not conformal to 0. $\diamondsuit$

Example 2.64. Let $Y^* = \{(z, w) \in (\mathbb{C}^*)^2 : w = e^z\}$. Then Y^* defines the analytic continuation of log, and defines an infinite-sheeted covering space of $\mathbb{C}^*$. $\diamondsuit$

Definition 2.65. Let $a \in X$. Let $p : Y \to X$, $f : Y \to \mathbb{C}$ define an analytic continuation of $\varphi \in \mathscr{O}_a$. We say that (p, f) is the *maximal analytic continuation* of φ if for every analytic continuation $q : Z \to X$, $g : Z \to \mathbb{C}$, of φ, there is an $F : Z \to Y$ such that $g = f \circ F$ and $q = p \circ F$. $\diamondsuit$

By general abstract nonsense, maximal analytic continuations are unique up to unique isomorphism.

Theorem 2.66. For every $\varphi \in \mathscr{O}_a$, $a \in X$, there is a unique maximal analytic continuation of φ.

Proof. Let Y be the connected component of φ in the étalé space $|\mathscr{O}|$. Let $p : Y \to X$ be the map that sends a germ to its basepoint. Let $f(\eta) = \eta_{p(\eta)}$ whenever $\eta \in Y$. Then $f(\varphi) = \varphi(a)$. Maximality follows from the definition of analytic continuation along paths; any larger analytic continuation would extend off of Y into another connected component of $|\mathscr{O}|$. $\square$

Chapter 3

Line bundles on Riemann surfaces

This chapter follows lectures of Georgios Daskalopolus.

3.1 Vector bundles

Let K be a topological field. Recall that a vector bundle of rank n over K on a topological space X consists of a continuous map

$$p : E \to X$$

such that $p^{-1}(x) = E_x$ is equipped with a homeomorphism $h_x : E_x \to K^n$ such that for every sufficiently small open $U \subseteq X$, $p^{-1}(U)$ is identified with $U \times K^n$ by a homeomorphism

$$h : p^{-1}(U) \to U \times K^n$$

such that $h|E_x = h_x$. A morphism of vector bundles $E \to F$ will be a continuous map which commutes with the projections p_E, p_F.

Definition 3.1. A *trivial vector bundle* on X is a vector bundle which is isomorphic to $X \times K^n$. $\diamondsuit$

Example 3.2. The tangent bundle TS^3 is trivial since S^3 is a Lie group, so we can parallel transport tangent vectors all around S^3 using the multiplication action of S^3 on S^3. The hairy ball theorem says that TS^2 is nontrivial. $\diamondsuit$

Suppose X is a manifold and $K \in \{\mathbb{R}, \mathbb{C}\}$. By a linear chart on X we mean a chart U such that $p^{-1}(U)$ is identified with $U \times K^n$. If U, V are linear charts we can form the transition map $h_{UV} : U \cap V \to \mathrm{GL}(n, K)$ which identifies the K^n in $U \times K^n$ with the K^n in $V \times K^n$.

Let X be a manifold. If $(U_i)_i$ is an open cover of X by charts and we are given maps $h_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$ which satisfy the cocycle relation, then there exists a vector bundle E with transition maps h_{ij}. If F is also a vector bundle with transition maps g_{ij} and $f : E \to F$ is a morphism of vector bundles, then in local coordinates we get maps

$$f_i : U_i \times K^n \to U_i \times K^n$$

which preserves U_i, i.e. for every $v \in K^n$ there is a $w \in K^n$ with $f_i(x,v) = (x,w)$. Conversely, if we are given local morphisms f_i which satisfy the transition relation

$$f_i(h_{ij}(v_j)) = f_i(v_i) = g_{ij}(f_j(v_j))$$

on the overlaps $(U_i \times U_j) \times K^n$; that is, $f_i \circ h_{ij} = g_{ij} \circ f_j$, then we can find a global morphism $f : E \to F$.

Definition 3.3. A *section* of a vector bundle $p : E \to X$ is a map $s : X \to S$ such that $p(s(x)) = x$ for every $x \in X$. The sheaf of sections will be denoted $\Gamma(E)$ or $\Omega^0(E)$. ♢

Every vector bundle has local sections. Indeed, if U is a linear chart of E, we can set

$$e_i(x) = (x, e_i)$$

where the e_i are a coordinate basis of K^n.

Suppose that $h_i : E_i \to U_i \times K^n$ is a local trivialization and $s \in \Gamma(E, X)$. Then we get local sections $s_i \in \Gamma(E, U_i)$ which satisfy

$$s_i = h_{ij} \circ s_j$$

where $h_{ij} = h_i \circ h_j^{-1}$ are the transition maps. Conversely, if we are given local sections which commute with the transition maps, then we may define a global section.

An important philosophy in the theory of vector bundles is that every operation in the category of vector spaces also holds in the category of vector bundles.

Definition 3.4. Let E, F be vector bundles over X with transition maps h, g respectively. Define the *direct sum vector bundle* $E \oplus F$ to be the vector bundle with transition maps $h_{ij} \oplus g_{ij}$. Define the *tensor product vector bundle* $E \otimes F$ to be the vector bundle with transition maps $h_{ij} \otimes g_{ij}$. Define the *dual vector bundle* E^* to be the vector bundle with transition maps h_{ji}^*. ♢

One can also define the Hom-bundle $\mathrm{Hom}(E, F)$, and presumably any other bundle associated to a functor from the category of vector spaces to itself.

3.2 Holomorphic line bundles

We restrict to the setting of $K = \mathbb{C}$, X a Riemann surface, and requiring that all maps involved are holomorphic.

Definition 3.5. A *line bundle* is a vector bundle of rank 1. ♢

In the setting of holomorphic line bundles we may not expect global sections to exist.

Lemma 3.6. Let $L \to X$ be a line bundle. Let s be a holomorphic global section which is never zero. Then L is trivial, as we can define an isomorphism $h : L \to X \times \mathbb{C}$, by

$$h(x, \alpha s(x)) = (x, \alpha s(x))$$

whenever $\alpha \in \mathbb{C}$.

Proof. Indeed, $s(x)$ spans E_x, so the definition of h makes sense. □

Proposition 3.7. Let X be a Riemann surface. The tangent bundle TX and cotangent bundle T^*X are holomorphic line bundles over X.

Proof. Suppose we have a cover by open sets U_i and coordinate maps $z_i : U_i \to \mathbb{C}$. The transition maps for T^*X are given by $g_{ij} = dz_j/dz_i$. The chain rule says that the sections of T^*X are 1-forms, as they should be. The tangent bundle is as desired, by duality. □

Fix a linear chart U. On U we define the vector fields over $\mathbb{R}$ $\partial_x, \partial_y \in \Gamma(TX, U)_{\mathbb{R}}$ by setting $\partial_x(z) = (z, e_1)$, and $\partial_y = (z, e_2)$. We also define the vector field over $\mathbb{C}$, $\partial_z \in \Gamma(TX, U)_{\mathbb{C}}$ by setting $\partial_z(z) = (z, 1)$. This gives a isomorphism over $\mathbb{R}$ of the real tangent bundle and the holomorphic tangent bundle, say by $\partial_x \mapsto \partial_z$, $\partial_y \mapsto i\partial_z$. We also define the 1-form dz to be the dual to ∂_z, thus $dz(\partial_z) = 1$.

We will write Ω^1_h for the sheaf of holomorphic 1-forms.

Let X be a Riemann surface. Let TX denote, for now, the real tangent bundle, and let $TX^{\mathbb{C}} = TX \otimes \mathbb{C}$, the tensor product taken over $\mathbb{R}$. Then $TX^{\mathbb{C}}$ is a rank-4 vector bundle over $\mathbb{R}$, so it can be given the structure of a rank-2 vector bundle over $\mathbb{C}$. More generally, if E is a real bundle, then we can define a complex bundle $E^{\mathbb{C}}$ which has the same transition functions, since $\mathrm{GL}(2, \mathbb{R}) \subset \mathrm{GL}(2, \mathbb{C})$. If we identify $2\partial_z = \partial_x - i\partial_y$ then $\partial_z, \overline{\partial_z}$ span $TX^{\mathbb{C}}$. We then write

$$TX^{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X$$

where $T^{1,0X}$ is spanned by ∂_z, $T^{0,1}X$ is spanned by $\overline{\partial_z}$. Then $T^{1,0}X, T^{0,1}X$ are line bundles over $\mathbb{C}$, and we have an $\mathbb{R}$-isomorphism $T^{1,0}X \to T^{0,1}X$. We identify the holomorphic tangent space with $T^{1,0}X$.