EC 2: Baire classes and the Borel σ-algebra **Aidan Backus**

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The goal of this writeup is to prove that the Baire classes of functions $\mathbb{R} \to \mathbb{R}$ do not stabilize on any countable ordinal α. This result was proven by Henri Lebesgue by way of transfinite induction for $\alpha < \omega_1$ in the late 1890s (where ω_1 denotes the first uncountable ordinal), according to Thomas Hawkins's book "Lebesgue's Theory of Integration: Its Origins and Development".

This proof is less direct, relying on the structure of the Borel σ-algebra over $\mathbb{R}$. As one may expect from a student who is taking H104, several of the lemmata hinge on the properties of the Cantor set $\mathcal{C}$. Since most of these proofs dance dangerously close to what cannot be proven from ZFC, it's important to list out our assumptions.

We assume the axiom of choice so that we can well-order certain ordinals and trees, and assume that the the power set map $\kappa \mapsto 2^\kappa$ (where κ is a cardinal) is injective. On the other hand, we don't assume the continuum hypothesis, which would trivialize these results.

Definition 1. Let P be an uncountable Polish space and Σ_1 denote the topology of P. For each ordinal α, let Π_α denote the complements of sets in Σ_α, $\Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha$, and, if $\alpha > 1$, Σ_α denote the family of sets

$$A = \bigcup_{i<\gamma} A_i$$

where $A_i \in \Pi_\beta$ for $\beta < \alpha$ and $\gamma \leq \omega$. This collection of families is known as the *Borel hierarchy.*

Note that where we write Γ_α (for $\Gamma = \Sigma, \Pi, \Delta$), most texts in descriptive set theory will write Γ^0_α. In general ranks of the form Γ^β_α are used in recursion theory to describe how difficult it is to describe a set in a formal language (for example, there is a notion of a Π^β_α-indescribable cardinal). However, we will only ever need $\beta = 0$, so we omit it.

Lemma 2. *$\Sigma_\alpha \cup \Pi_\alpha \subseteq \Delta_{\alpha+1}$ for each ordinal α.*

Proof. We can write $X \in \Sigma_\alpha$ as a countable union of $X_i \in \Pi_\alpha$, by de Morgan's laws. Therefore $\Sigma_\alpha \subseteq \Sigma_{\alpha+1}$. Furthermore, $\Delta_{\alpha+1}$ is closed under complements and

$$\Sigma_\alpha \subseteq \{\bigcup A_i : A_i \in \bigcup \Sigma_\alpha\} = \Pi_{\alpha+1}$$

so that $\Sigma_\alpha \subseteq \Delta_{\alpha+1}$ and $\Pi_\alpha \subseteq \Delta_{\alpha+1}$. □

Definition 3. The smallest σ-algebra containing Σ_1 is known as the *Borel σ-algebra*, written $\mathcal{B}$.

Theorem 4. *A set X is contained in an element of the Borel hierarchy if and only if $X \in \mathcal{B}$.*

Proof. First notice that if $X \in \Delta_\alpha$, then $X^C \in \Delta_\alpha$. Furthermore, countable union maps Δ_α into Δ_α and likewise with countable intersection. Finally, $P \in \Delta_1$. Therefore $\bigcup \Delta_\alpha \subseteq \mathcal{B}$.

Conversely, if $X \in \mathcal{B}$, it must have been generated by the repeated applications of complementation and countable union to Σ_1, implying that $X \in \Delta_\alpha$ for some sufficiently large α. □

It follows pretty much immediately that we have the following chain of inclusion maps:

Definition 5. Suppose $\Sigma_\eta = \Sigma_{\eta+1}$ and for each $\beta < \eta$, $\Sigma_\beta \neq \Sigma_\eta$. Then we will call η the *stabilizer* of $\mathcal{B}$.

The key to victory will be to show that, in fact, $\eta = \omega_1$, and this proof will make up the bulk of this writeup. I used David Marker's notes on descriptive set theory as a reference in the computation of η.

As above, let $\mathcal{C}$ denote the Cantor set, and as a matter of convenience, if $U \subseteq \mathcal{C} \times \mathcal{C}$ and $x \in \mathcal{C}$, write $U_x = \{y \in \mathcal{C} : (x, y) \in U\}$, the projection of U onto $\mathcal{C}$ by x. Also recall that $\mathcal{C}^\alpha$ is homeomorphic to $\mathcal{C}$ for $\alpha \leq \omega$. In particular, $\mathcal{C} \times \mathcal{C}$ is Polish, and we have a canonical map $f : \mathcal{C} \to \mathcal{C}^\omega$, say $f(p) = (p_1, p_2, \dots)$. For convenience, let $\mathfrak{c} = \operatorname{card} \mathbb{R}$ denote the cardinality of the continuum.

Definition 6. Let $U \in \Sigma_\alpha(\mathcal{C} \times \mathcal{C})$. If, for each $X \in \Sigma_\alpha(\mathcal{C})$, we can find $x \in X$ such that $U_x = X$, we say that U is *α-universal* over $\mathcal{C}$.

In other words, an α-universal set U is one which completely parametrizes $\Sigma_\alpha(\mathcal{C})$ by way of projection.

Lemma 7. *If $\alpha < \omega_1$, there exists an α-universal set over $\mathcal{C}$.*

Proof. $\mathcal{C}$ is second-countable, so we have a family X_i $(i < \omega)$ of open sets which generates $\Sigma_1(\mathcal{C})$. Moreover, $\mathcal{C}$ can be viewed as the points p at the bottom of a tree of height ω, where one took the left branch at the nth row if p is in the left third of $\mathcal{C}$ at the nth row, and took the right branch otherwise. Let $p(n) = 0$ if we took the left branch, and $p(n) = 1$ otherwise.

Define
$$U_1 = \bigcup_{i<\omega} \{(x, y) \in \mathcal{C} \times \mathcal{C} : x(i) = 0 \text{ and } y \in X_i\}.$$
If $X \in \Sigma_1(\mathcal{C})$, then we can take $x \in \mathcal{C}$ such that $x(i) = 0$ if and only if $X_i \in X$. Then $U_x = X$, so U_1 is 1-universal.

Now suppose that we can find U_β, β-universal, for each $\beta < \alpha$. Then we can find a sequence of ordinals β_i such that $\beta_i \leq \beta_{i+1}$ and $\sup(\beta_i + 1) = \alpha$. Put
$$V_\alpha = \bigcup_{i<\omega} \{i\} \times U_{\beta_i}^C \in \Sigma_\alpha$$
and
$$U_\alpha = \{(x, y) \in \mathcal{C} \times \mathcal{C} : \exists i < \omega \ (i, x_i, y) \in V_\alpha\} \in \Sigma_\alpha$$
where x_i is the ith entry in the tuple $f(x)$ determined by the canonical map $f : \mathcal{C} \to \mathcal{C}^\omega$ denoted above.

If $Y \in \Sigma_\alpha$ then we can find $Y_i \in \Pi_{\beta_i}$ such that $Y = \bigcup Y_i$. Choosing a sequence $x_i \in \mathcal{C}$ such that $y \in Y$ if and only if $(x_i, y) \in U^C_{\beta_i}$ and $x = (x_1, x_2, \dots)$, then $(x, y) \in U_\alpha$ if and only if $y \in Y$. Thus U_α is α-universal.

Therefore for each $\alpha < \omega_1$, we have U_α which is α-universal. □

Lemma 8. $\eta(\mathcal{C}) \geq \omega_1$.

Proof. Fix $\alpha < \omega_1$ like usual, let U be α-universal over $\mathcal{C}$, and let $X = \{x \in \mathcal{C} : (x, x) \notin U\}$. $X \in \Pi_\alpha$, being the complement of a Σ_α projection of U. Thus $X \in \Sigma_{\alpha+1}$. Clearly X is nonempty, since $U \neq \mathcal{C} \times \mathcal{C}$.

Suppose for the sake of a contradiction that $X \in \Sigma_\alpha$, so that it could be possible for $\eta(\mathcal{C}) = \alpha$. Then if $x \in X$, $(x, x) \in U$, by universality of U. So $x \notin X$, and thus X is empty. □

Lemma 9. *If $Q \subseteq P$ are both Polish then for each α, $\Sigma_\alpha(Q) = \{A \cap Q : A \in \Sigma_\alpha(P)\}$.*

Proof. If $\alpha = 1$ this is nothing more than the inheritance principle. On the other hand, if this property holds for each $\beta < \alpha$, then for each $\beta < \alpha$, $\Pi_\beta(Q) = \{B \cap Q : B \in \Pi_\beta(P)\}$ by complementation. So $\Sigma_\alpha(Q)$ is the family of countable unions of $B_i \cap Q$s for $B_i \in \Pi_\beta(P)$ and $i < \omega$. Thus

$$\bigcup (B_i \cap Q) = (\bigcup B_i) \cap Q$$

and so $\Sigma_\alpha(Q) = \{A \cap Q : A \in \Sigma_\alpha(P)\}$. □

Lemma 10. $\eta \geq \omega_1$.

Proof. There exists an embedding of $\mathcal{C}$ in P, because it is Polish and uncountable. Furthermore $\mathcal{C}$ is itself Polish, so for each α, $\Sigma_\alpha(\mathcal{C}) = \{A \cap \mathcal{C} : A \in \Sigma_\alpha(P)\}$. If $\eta(P)$ was countable, then for each $\beta > \eta(P)$, $\Sigma_\beta(\mathcal{C}) = \{A \cap \mathcal{C} : A \in \Sigma_{\eta(P)}(P)\}$ so that $\eta(\mathcal{C}) = \eta(P)$, which is a contradiction. □

Lemma 11. $\operatorname{card} \mathcal{B} = \mathfrak{c}$ *and* $\eta \leq \omega_1$.

Proof. Each $X \in \mathcal{B}$ is the root of a tree T of height η, where each node has countably many children – each node represents the complement of the union of its children, as depicted below:

Since each node has countably many children, we might as well well-order them and assume that each node has in fact ω many children. Note that η must in fact be an ordinal or else we could inject the proper class of ordinals into $\mathcal{P}(P)$ for any Polish P. In particular, T has a final row.

For some ordinal α, this row consists of α many families of ω leaves L, where each family corresponds to a single parent node and each leaf corresponds to a set in Σ_1. This row determines each row above it, by an easy transfinite induction, and in particular determines X. How many different possible final rows can there be?

To answer this, finally invoke the hypothesis that P is Polish, and in particular second countable. Each $L \in \Sigma_1$ is determined by a union of countably many countable sets. Therefore there are at most $\mathfrak{c}$ many possible values of L, and thus $\alpha \leq \mathfrak{c}$. So $\operatorname{card} \mathcal{B} \leq \alpha\omega \leq \mathfrak{c}\omega = \mathfrak{c}$ (in the sense of cardinal multiplication). It follows that one could not possible have uncountably many rows in T since then $\operatorname{card} \mathcal{B} \geq \omega_1^\omega > \mathfrak{c}$. So $\eta \leq \omega_1$.

But since P is uncountable, it has at least $\mathfrak{c}$ open sets, as was proven on the homework, so $\operatorname{card} \mathcal{B} \geq \mathfrak{c}$. Therefore $\operatorname{card} \mathcal{B} = \mathfrak{c}$ as desired. $\square$

This completes the computation of η. In summary:

Theorem 12. $\eta = \omega_1$.

As an aside, this implies that $\mathcal{B}$ is unsuitable for measure theory on $\mathbb{R}$. Clearly the Lebesgue σ-algebra $\mathcal{L}$ must contain $\mathcal{B}$, but this containment must be strict.

Theorem 13. *There exists a Lebesgue measurable set which is not contained in $\mathcal{B}(\mathbb{R})$.*

Proof. Suppose not; i.e. suppose $\mathcal{B} = \mathcal{L}$ and note that, with respect to the Lebesgue measure, $\operatorname{card} \mathcal{C} = \mathfrak{c}$, where $\mathcal{C}$ denotes the Cantor set. By Cantor's theorem, the power set $\mathcal{P}(\mathcal{C})$ must have a cardinality strictly greater than $\mathfrak{c}$. The transfinite pigeonhole principle implies then that there exist uncountably many $X \in \mathcal{P}(\mathcal{C})$ which are nonmeasurable. But $X^* \leq \mathcal{C}^* = 0$, so X has zero outer measure and is therefore measurable, a contradiction. $\square$

Finally, we can approach the original subject matter: the Baire classes over $\mathbb{R}$. First, we need to generalize the notion of a Baire class to any ordinal, rather than merely the finite ordinals.

Definition 14. Define $\mathfrak{B}_0 = C^0(P, P)$. Given an ordinal $\alpha > 0$, define $\mathfrak{B}_\alpha$ to be the set of pointwise limits of sequences of functions in $\mathfrak{B}_\beta$, for $\beta < \alpha$. If $f \in \mathfrak{B}_\alpha$ but $f \notin \mathfrak{B}_{\alpha+1}$, say that f is of *Baire class* α. The set of $\mathfrak{B}_\alpha$s is known as the *Baire hierarchy*. For convenience, write $\mathfrak{B} = \mathfrak{B}_{\omega_1}$.

Clearly we have

$$\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}_\omega \subseteq \cdots \subseteq \mathfrak{B}$$

but the following lemma will show that these inclusions are proper.

Lemma 15. *Let $X \subseteq P$. Then $\chi_X \in \mathfrak{B}_\alpha$ if and only if $X \in \Sigma_\alpha \cup \Pi_\alpha$.*

Proof. We shall prove

$$\psi(\alpha) = (X \in \Sigma_\alpha \cup \Pi_\alpha \implies \chi_X \in \mathfrak{B}_\alpha)$$

first. By taking converses and $1 - \chi_X$ we may assume without loss of generality that $X \in \Sigma_\alpha$.

If $\alpha = 1$, then we can find a continuous function f whose zero locus is the closed set X^C. By dilating f at the appropriate sets we can, given $\epsilon > 0$ and $x \in X$, make $d(f(x), 1)) < \epsilon$. So there is a sequence of continuous functions which converges to χ_X.

On the other hand, if we know that $\psi(\beta)$ is true for $\beta < \alpha$, then we can take a sequence of $X_i \in \Pi_{\beta_i}$ ($\beta_i < \alpha$) such that $\bigcup X_i = X \in \Sigma_\alpha$. Take the sequence of functions given by

$$f_n(x) = \max_{i=1}^{n} \chi_{X_i}(x)$$

which clearly converges to χ_X, so $\chi_X \in \mathfrak{B}_\alpha$. So by transfinite induction, $\psi(\alpha)$ is true for all α.

Now to prove

$$\phi(\alpha) = (\chi_X \in \mathfrak{B}_\alpha \implies X \in \Sigma_\alpha \cup \Pi_\alpha).$$

In the case $\alpha = 1$, we can approximately χ_X arbitrarily well by mollifiers f, which have a zero locus outside either X or X^C (without loss of generality, justified as above, we can take the zero locus to be X^C). In particular given $x \in X$ and $\epsilon > 0$ we can find f whose zero locus is X^C such that $|f(x) - 1| < \epsilon$. With this, X must be open since it is the preimage of $P \setminus \{0\}$. So $X \in \Sigma_1$, satisfying $\phi(1)$.

Otherwise, suppose we know $\phi(\beta)$ for $\beta < \alpha$. Then given a sequence of χ_{X_i}s in $\mathfrak{B}_\beta$, we can take their pointwise limit and recover $\chi_X \in \mathfrak{B}_\alpha$. Any such pointwise limit must be determined by countably many unions and intersections of Σ_β sets, i.e. X_is. Therefore $X \in \Sigma_\alpha$. $\square$

From the above lemma, we immediately have the following result, which was the original goal of this writeup.

Theorem 16. *$\mathfrak{B}_\alpha \neq \mathfrak{B}_{\alpha+1}$ for each $\alpha < \omega_1$.*

Therefore, we can extend the notion of Baire classes to any countable ordinal.

As a consequence, we notice the stark limitations of the Riemann and Borel integrals.

Definition 17. Let P be a Polish space, $\Omega \in \mathcal{B}(P)$, and say that a function $f : \Omega \to P$ is *Borel integrable* if its undergraph $\mathcal{U}f$ is measurable with respect to the product measure μ on $P \times P$ generated by $\mathcal{B}(P)$. If f is Borel integrable, define its *Borel integral* over Ω by

$$\int_\Omega f = \mu\mathcal{U}f.$$

Note that the Borel integral is just the Lebesgue integral where the σ-algebra is restricted to $\mathcal{B}$.

Theorem 18. *There exists a function $\mathbb{R} \to \mathbb{R}$ which is Riemann integrable but not Borel integrable and a function which is Borel integrable but not Riemann integrable. However, both are Lebesgue measurable.*

Proof. Recall that there exists a zero set $S \subset \mathcal{C}$ which is not in $\mathcal{B}$. So $\chi_S \notin \mathfrak{B}$, and therefore is not Borel integrable. However, the discontinuity of χ_S is contained in $\mathcal{C}$, a zero set, so by the Riemann-Lebesgue theorem, χ_S is Riemann integrable. Furthermore, $S \in \mathcal{L}$, so χ_S is clearly Lebesgue measurable.

It is well-known that $\chi_{\mathbb{Q}}$ is not Riemann integrable but is at least Lebesgue integrable. Showing that $\mathbb{Q} \in \mathcal{B}$ will prove that $\chi_{\mathbb{Q}}$ is Borel integrable. Clearly any singleton $A \in \Pi_1$. Moreover, there exists a well-ordering of $\mathbb{Q}$, so there is a sequence A_i of singletons such that

$$\mathbb{Q} = \bigcup_{i<\omega} A_i$$

so $\mathbb{Q} \in \Sigma_2 \subset \mathcal{B}$. $\square$