Cosmological Methods in ABACUS

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1. Cosmological Evolution Equations

The purpose of this note is to document the differential equations for evolving the homogeneous cosmological parameters.

The cosmological symplectic leapfrog equations are written in proper time (Quinn et al. 1997), so it is convenient to pick proper time as the independent variable. This will allow us to compute a fine grid to support the microsteps.

We care about the scale factor $a(t)$ and the Kick and Drift operators. In units of comoving positions and canonical velocites, the kicks occur by the quantity

$$\Delta\eta_K = \int_{t_1}^{t_2} \frac{dt}{a} \tag{1}$$

and the drifts by

$$\Delta\eta_D = \int_{t_1}^{t_2} \frac{dt}{a^2}. \tag{2}$$

I choose the notation η because η_K is the conformal time.

It is convenient, however, to re-scale these functions so that the Einstein-de Sitter solution is constant. This will avoid a buildup of numerical error at early times and avoids the singularity in η_D as $a \to 0$. Defining

$$f_n(a) = a^n H(a) \int \frac{dt}{a^n} = a^n H(a) \int \frac{da}{a^{n+1} H(a)}, \tag{3}$$

we have $f_1 = aH\eta_K$ and $f_2 = a^2 H\eta_D$. In EdS, the solutions are a constant $2/(3-2n)$. Note that f_2 is negative: $\eta_D = -2a^{-1/2}$. This is ok: η_D is singular at $a = 0$, but we only care about differences in η_D, which are positive for increasing time.

For f_n, we have the differential equation

$$\frac{df_n}{dt} = H(a)\left[1 + nf_n + \frac{1}{2} f_n \frac{d\ln H^2}{d\ln a}\right]. \tag{4}$$

We will write an ODE for $a(t)$ below. As usual, $H(a)$ has a simple form

$$H(a) = H_0\sqrt{\Omega_m a^{-3} + \Omega_K a^{-2} + \hat{\Omega}_X} \tag{5}$$

where $\hat{\Omega}_X = \rho_X(a)/\rho_{cr,0}$ is the density of dark energy at epoch a scaled to the critical density today. We have an evolution equation

$$\frac{d\ln\hat{\Omega}_X}{d\ln a} = -3(1 + w) \tag{6}$$

for an equation of state $w(a)$. The derivative of the Hubble parameter has a simple form:

$$\frac{d\ln H^2}{d\ln a} = \frac{-3a^{-3}\Omega_m - 2a^{-2}\Omega_K - 3(1+w)\hat{\Omega}_X}{a^{-3}\Omega_m + a^{-2}\Omega_K + \hat{\Omega}_X} \tag{7}$$

For a general choice of $w(a)$, one has to do the differential equation to find $\hat{\Omega}_X$ as a function of a. For some choices of $w(a)$, we can do the integral to get an analytic form for $\hat{\Omega}_X$. In particular, the $w_0 - w_a$ parameterization $w = w_0 + w_a(1-a)$ has the solution

$$\hat{\Omega}_X = \Omega_X a^{-3(1+w_0+w_a)} \exp^{3w_a(a-1)}. \tag{8}$$

To integrate $a(t)$, we follow a similar plan to define a scaled function so that the EdS solution is constant. We define

$$\alpha = \frac{2}{3t}\frac{a^{3/2}}{H_0\sqrt{\Omega_m}}, \tag{9}$$

which evolves as

$$\frac{d\alpha}{dt} = H\alpha\left(\frac{3}{2} - \frac{1}{Ht}\right). \tag{10}$$

We have $\alpha = 1$ in EdS. Once solved, we have

$$a = \left(\frac{3\alpha t}{2}H_0\sqrt{\Omega_m}\right)^{2/3}. \tag{11}$$

For the growth function $D(a)$, we again define a scaled function $\gamma = D/a$ so that we have a constant in the EdS limit. However, because the linear growth function equation is second-order, we need to track $d\gamma/dt$ as well. This quantity can be connected to the familiar $f = d\ln D/d\ln a$ as

$$f = \frac{1}{H\gamma}\frac{d\gamma}{dt} + 1. \tag{12}$$

The equation for γ is

$$\frac{d^2\gamma}{dt^2} = -4H\frac{d\gamma}{dt} - H^2\gamma\left[3 + \frac{1}{2}\frac{d\ln H^2}{d\ln a} - \frac{3}{2}\frac{\Omega_m H_0^2 a^{-3}}{H^2}\right], \tag{13}$$

which can quickly be split into a coupled first-order ODE for γ and $d\gamma/dt$.

We choose $\gamma = 1$ for the EdS limit.

The solutions for the EdS limit provide adequate initial conditions, but it is worth noting that the initial time must be very early if the spatial curvature is non-zero. The code currently uses $z = 10^6$ as the initial time, so the corrections from Λ are negligible. But the corrections from Ω_K

would be 10^{-6}. Similarly, if dark energy evolves sufficiently rapidly in redshift, one might want to correct the initial conditions.

One either picks even earlier redshifts or includes a first-order correction. For example, I believe that the first-order corrections for f_n are

$$f_n = \frac{2}{3-2n}\left[1 + \frac{1}{5-2n}\frac{\Omega_K a}{\Omega_m} - \frac{3w}{3-6w-2n}\frac{a^3\hat{\Omega}_X}{\Omega_m}\right]. \tag{14}$$

Using $f_0 = Ht$, we can derive the correction for α as

$$\alpha = 1 + \frac{4}{5}\frac{\Omega_K a}{\Omega_m} + \frac{3-3w}{3-6w}\frac{a^3\hat{\Omega}_X}{\Omega_m}. \tag{15}$$

These have not been checked enough, nor included in the code.

The expressions arise from inserting $f_n = 2/(3-2n) + g_n$ into

$$\frac{df_n}{d\ln a} = 1 + nf_n + \frac{1}{2}f_n\frac{d\ln H^2}{d\ln a} \tag{16}$$

to get

$$\frac{dg_n}{d\ln a} = (n-\frac{3}{2})g_n + \left(\frac{g_n}{2} + \frac{1}{3-2n}\right)\frac{d\ln H^2}{d\ln a}. \tag{17}$$

Working to lowest order in $d\ln H^2/d\ln a$ allows us to drop the g_n term in the prefactor. We can then rearrange to get

$$\frac{dg_n a^{-n+3/2}}{d\ln a} = \frac{a^{-n+3/2}}{3-2n}\frac{d\ln H^2}{d\ln a}. \tag{18}$$

Expanding $\ln H^2$ to lowest order is $\Omega_K a/\Omega_m + \hat{\Omega}_X a^3/\Omega_m$, and we can then do the derivative and integral with respect to $\ln a$ to get the above expression for f_n.

2. Power Spectrum normalization

If we define the inverse FFT as

$$f(x) = \sum_k \exp(-ikx)F(k), \tag{19}$$

then we have

$$\left\langle f(0)^2\right\rangle = \sum_k \left\langle |F(k)|^2\right\rangle. \tag{20}$$

Meanwhile, we should have that the variance of the density field should be

$$\left\langle \delta^2\right\rangle = \int \frac{d^3k}{(2\pi)^3}P(k) = \sum_k \frac{V_k}{(2\pi)^3}P(k) = \sum \frac{1}{V}P(k) \tag{21}$$

where $V_k = (2\pi/L)^3$ and $V = L^3$ for a cube of side L.

Hence, the Fourier transform of the density field should be normalized so that the real and complex part each have a variance of $P/2V$.

The displacement fields are then $q_x = (ik_x/k^2)\delta_k$, etc.

In the code we compute the FFTs by layering δ, q_x, q_y, and q_z into two complex arrays, with $A = \delta + iq_x$ and $B = q_y + iq_z$. The Fourier space arrays are loaded so that the real and imaginary parts satisfy Hermitian and anti-Hermitian properties, respectively, under the $k \to -k$ symmetry.

To make life simpler, we force all elements with at least one component at the Nyquist frequency to be zero. This should not affect the simulation science, as modes just beyond Nyquist have already been ignored! But it avoids the bookkeeping headache of handling the aliasing constraints at the Nyquist frequency.