CV192, HW # 4

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1 Theoretical Questions

1.1

We will show the that convolution is associative:

$$(a*b)*c_{(i,j)} = \sum_{k,l}(a*b)\,(i-k,j-l)\,c\,(k,l) = \sum_{k,l}\sum_{k',l'} a\,(i-k-k',j-l-l')\,b\,(k',l')\,c\,(k,l)$$

We define $k^* = k + k'$ and $l^* = l + l'$ and get:

$$\sum_{k^*,l^*}\sum_{k,l} a\,(i-k^*,j-l^*)\,b\,(k^*-k,l^*-l)\,c\,(k,l) = \sum_{k^*,l^*} a\,(i-k^*,j-l^*)\,(b*c)(k^*,l^*) = a*(b*c)_{(i,j)}$$

1.2

We will show that $h*\delta = \delta*h = h$.

$$h*\delta = \sum_{k,l} h(i-k,j-l)\delta(k,l) = h(i-0,j-0)\cdot 1 = h(i,j)$$

$$\delta*h = \sum_{k,l} \delta(i-k,j-l)h(k,l) = h(k,l) \overset{*}{=} h(i,j)$$

* For i=k and j=l.

1.3

At first we will define $h_{flipped}$ for the convenience of performing convolution:

$$\text{h}_{flipped} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix}$$

The values of H are determined by h, we define the relative position as follows :

$$H_{i,j} = \begin{cases} h_{1,1}, & j = i-1-N \wedge imodN - 1 \geq 0 \\ h_{1,2}, & j = i-N \\ h_{1,3}, & j = i+1-N \wedge imodN + 1 < N \\ h_{2,1}, & j = i-1 \wedge imodN - 1 \geq 0 \\ h_{2,2}, & j = i \\ h_{2,3}, & j = i+1 \wedge imodN + 1 < N \\ h_{3,1}, & j = i-1+N \wedge imodN - 1 \geq 0 \\ h_{3,2}, & j = i+N \\ h_{3,3}, & j = i+1+N \wedge imodN + 1 < N \\ 0, & otherwise \end{cases}$$

1.4

i) In order to recover x, we need H to be invertible (not singular), so we get:

$$y = Hx \Longrightarrow x = H^{-1}y$$

ii) According to the given data we know that:

h=h$_{flipped} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, the matrix H we constructed in the previous exercise will be the Identity matrix $I = H \in M_{MN \times MN}$ so we get that H is invertible as wanted.
The recover of x will be:

$$x = H^{-1}y \Rightarrow x = Iy \Rightarrow x = y$$

1.5

The effect of convolving with:

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is moving each pixel 2 pixels to the right and 1 pixel down, we can inspect this by the 1 location in the filter (2 positions to the right and 1 position down from the center) or from the convolved image:

1.6

i) Let Y be a square orthogonal matrix, we can say that:

$$det(Y) = \pm 1.$$

Proof:

$$\begin{aligned}
&YY^T = I \\
&det(I) = 1 \\
&det(YY^T) = det(I) = 1 \\
&det(YY^T) = det(Y)det(Y^T) = 1 \Rightarrow det(Y) = det(Y^T) \\
&det(Y) = \pm 1
\end{aligned}$$

ii) In order to prove that Y^T is orthogonal we need to show that $Y^T(Y^T)^T = I$:

$$YY^T = I \Rightarrow Y^T = Y^{-1}$$

$$Y^T(Y^T)^T = Y^TY = Y^{-1}Y = I$$

iii) Y, Y^T are square orthogonal matrices, $y_i^T y_i$ is the i,i entry in Y^TY, therefore:

$$Y^TY = I \Rightarrow y_i^T y_i = 1$$

iv)

$$\|\boldsymbol{y}_i\|_{\ell_2} = \sqrt{\sum_{j=1}^{n} |y_{j,i}|^2} = \sqrt{y_i^T y_i} = \sqrt{1} = 1$$

v) $y_i^T y_j$ is equal to the (i,j) entry in $Y^TY = I$, therefore for $i \neq j, \quad y_i^T y_j = 0$. Meaning that the vectors y_i^T and y_j are orthogonal to each other and the angle between them is 90.

vi) Let G be the group for $n \times n$ orthogonal matrices, we will prove the 3 qualities of such group:
1) $I \in G$

$$II^T = I \Rightarrow I \in G$$

2) $A \in G, B \in G \Rightarrow AB \in G$

$$A, B \in G \Rightarrow (AB)(AB)^T = ABB^T A^T = AIA^T = AA^T = I \Rightarrow AB \in G$$

3) $A \in G$, A^{-1} exists and $A^{-1} \in G$

$$A \in G \Rightarrow AA^T = I \Rightarrow A^{-1} = A^T$$

A^T is orthogonal so:

$$A^T \in G \Rightarrow A^T = A^{-1} \in G$$

1.7

Let K be an $n \times n$ separable filter. We will show that K is not invertible:

$$det(K) = det(USV^T) = det(U)det(S)det(V^T) = det(U) \cdot 0 \cdot det(V^T) = 0$$

A matrix with deteminate equal to zero is not invertiable.

1.8

Bilateral filtering is not a linear operation.
A linear operation is one that holds the following constraint:

$$h(f + \lambda g) = h(f) + \lambda h(g)$$

If we look at the definition of the bilateral filter:

$$g(x) = \frac{1}{c} \sum_{x_i} f^r(|I(x) - I(x_i)|) f^d(|x - x_i|) I(x_i)$$

we can see that f^r, f^d may be defined as Gaussian, hence the functions will contain a multiplication of exponent function with squared error difference, which is obviously not linear. If we take the definition of linear operator over filters we get that the following must hold:

$$(c_1 I_1 + c_2 I_2) * h = c_1(i_1 * h) + c_2(I_2 * h)$$

While on the left side of the equation that exponent power will be:

$$(c_1 I_1(x) + c_2 I_2(x) - c_1 I_1(x_i) + c_2 I_2(x_i))^2$$

And on the right side we will have:

$$(c_1 I_1(x) - c_1 I_1(x_i)^2 + (c_2 I_2(x) - c_2 I_2(x_i))^2$$

That you can easily see is not equal.

1.9

We will calculate the Laplacian of an isotropic Gaussian:

$$G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} exp(-\frac{x^2+y^2}{2\sigma^2})$$

At, first we will calculate the first partial derivatives:

$$\frac{\partial G(x,y,\sigma)}{\partial x} = \frac{1}{2\pi\sigma^2} \cdot (-\frac{x}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2})$$

$$\frac{\partial G(x,y,\sigma)}{\partial y} = \frac{1}{2\pi\sigma^2} \cdot (-\frac{y}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2})$$

The second derivatives:

$$\begin{aligned}\frac{\partial^2 G(x,y,\sigma)}{\partial x} &= \frac{1}{2\pi\sigma^2} \cdot (-\frac{}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2}) + \frac{1}{2\pi\sigma^2} \cdot (-\frac{x}{\sigma^2}) \cdot (exp(-\frac{x^2+y^2}{2\sigma^2}) \cdot (-\frac{x}{\sigma^2})) \\ &= \frac{1}{2\pi\sigma^2} \cdot (\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2})\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 G(x,y,\sigma)}{\partial y} &= \frac{1}{2\pi\sigma^2} \cdot (-\frac{}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2}) + \frac{1}{2\pi\sigma^2} \cdot (-\frac{y}{\sigma^2}) \cdot (exp(-\frac{x^2+y^2}{2\sigma^2}) \cdot (-\frac{y}{\sigma^2})) \\ &= \frac{1}{2\pi\sigma^2} \cdot (\frac{y^2}{\sigma^4} - \frac{1}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2})\end{aligned}$$

Finally we get:

$$\begin{aligned}\nabla^2 G(x,y,\sigma) &= \frac{\partial^2 G(x,y,\sigma)}{\partial x} + \frac{\partial^2 G(x,y,\sigma)}{\partial y} \\ &= \frac{1}{2\pi\sigma^2} \cdot (\frac{x^2+y^2}{\sigma^4} - \frac{2}{\sigma^2}) \cdot exp(-\frac{x^2+y^2}{2\sigma^2}) = (\frac{x^2+y^2}{\sigma^4} - \frac{2}{\sigma^2}) \cdot G(x,y,\sigma)\end{aligned}$$

As requested.

1.10

Given that:

$$x' = xcos\theta - ysin\theta,$$

$$y' = xsin\theta + ycos\theta$$

We will show that:

$$\nabla^2 I(x,y) = \nabla^2 I(x'(x,y), y'(x.y))$$

Or equivalently:

$$\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = \frac{\partial^2 I}{\partial x'^2} + \frac{\partial^2 I}{\partial y'^2}$$

Proof:

$$\begin{aligned}
\frac{\partial^2 I}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial I}{\partial x} \\
&= \frac{\partial}{\partial x}(\frac{\partial I}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial I}{\partial y'}\frac{\partial y'}{\partial x}) \\
&= \frac{\partial}{\partial x}(\frac{\partial I}{\partial x'}cos\theta + \frac{\partial I}{\partial y'}sin\theta) \\
&= \frac{\partial I}{\partial x'}\frac{\partial}{\partial x}cos\theta + \frac{\partial I}{\partial y'}\frac{\partial}{\partial x}sin\theta \\
&= \frac{\partial I}{\partial x'}cos\theta(\frac{\partial I}{\partial x'}cos\theta + \frac{\partial I}{\partial y'}sin\theta) + \frac{\partial I}{\partial y'}sin\theta(\frac{\partial I}{\partial x'}cos\theta + \frac{\partial I}{\partial y'}sin\theta) \\
&= \frac{\partial^2 I}{\partial x'^2}cos^2\theta + 2(\frac{\partial^2 I}{\partial x'\partial y'}sin\theta cos\theta) + \frac{\partial^2 I}{\partial y'^2}sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 I}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial I}{\partial y} \\
&= \frac{\partial}{\partial y}(\frac{\partial I}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial I}{\partial y'}\frac{\partial y'}{\partial y}) \\
&= \frac{\partial}{\partial y}(\frac{\partial I}{\partial x'}(-sin\theta) + \frac{\partial I}{\partial y'}cos\theta) \\
&= \frac{\partial I}{\partial x'}\frac{\partial}{\partial y}(-sin\theta) + \frac{\partial I}{\partial y'}\frac{\partial}{\partial y}cos\theta \\
&= \frac{\partial I}{\partial x'}(-sin\theta)(\frac{\partial I}{\partial x'}(-sin\theta) + \frac{\partial I}{\partial y'}cos\theta) + \frac{\partial I}{\partial y'}cos\theta(\frac{\partial I}{\partial x'}(-sin\theta) + \frac{\partial I}{\partial y'}cos\theta) \\
&= \frac{\partial^2 I}{\partial y'^2}sin^2\theta - 2(\frac{\partial^2 I}{\partial x'\partial y'}sin\theta cos\theta) + \frac{\partial^2 I}{\partial y'^2}cos^2\theta
\end{aligned}$$

Combining our solutions we get:

$$\begin{aligned}
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} &= \frac{\partial^2 I}{\partial x'^2}cos^2\theta + 2(\frac{\partial^2 I}{\partial x'\partial y'}sin\theta cos\theta) + \frac{\partial^2 I}{\partial y'^2}sin^2\theta + \frac{\partial^2 I}{\partial y'^2}sin^2\theta - 2(\frac{\partial^2 I}{\partial x'\partial y'}sin\theta cos\theta) + \frac{\partial^2 I}{\partial y'^2}cos^2\theta \\
&= (cos^2\theta + sin^2\theta)(\frac{\partial^2 I}{\partial x'^2} + \frac{\partial^2 I}{\partial y'^2}) = \frac{\partial^2 I}{\partial x'^2} + \frac{\partial^2 I}{\partial y'^2}
\end{aligned}$$

1.11

We will show that:

$$\mathrm{GL(n)} = \left\{\boldsymbol{Q} | \boldsymbol{Q} \in \mathbb{R}^{n\times n}, \det \boldsymbol{Q} \neq 0\right\}$$

is a matrix group, In order to do so we will prove the required conditions:
G1:

$$det(I_{n\times n}) = 1 \neq 0, I \in \mathbb{R}^{n\times n} \Rightarrow, I \in \mathrm{GL(n)}$$

G2: Let A , $B \in$ GL(n)

$$\Rightarrow det(A) \neq 0 \neq det(B) \Rightarrow det(AB) = det(A)det(B) \neq 0$$

In addition $AB \in \mathbb{R}^{n\times n} \Rightarrow AB \in$ GL(n)

G3: Let $A \in$ GL(n)

$$det(A) \neq 0 \Rightarrow A^{-1} exist$$

Since $A^{-1} \in \mathbb{R}^{n\times n}$ and $det(A^{-1}) \neq 0, A^{-1} \in$ GL(n).

1.12

We will show that $\mathrm{GL}_+(\mathrm{n}) = \{\boldsymbol{Q}|\boldsymbol{Q} \in \mathbb{R}^{n\times n}, \det \boldsymbol{Q} > 0\}$ is a matrix group.
G1:

$$I_{n\times n} \in \mathbb{R}^{n\times n}, det(I_{n\times n}) = 1 \Rightarrow I_{n\times n} \in \mathrm{GL}_+(\mathrm{n})$$

G2: Let A,B$\in \mathrm{GL}_+(\mathrm{n})$

$$\Rightarrow A, B \in \mathbb{R}^{n\times n} \Rightarrow AB \in \mathbb{R}^{n\times n}$$

and

$$det(A) > 0, det(B) > 0 \Rightarrow det(AB) = det(A)det(B) > 0 \Rightarrow AB \in \mathrm{GL}_+(\mathrm{n})$$

G3: Let A$\in \mathrm{GL}_+(\mathrm{n}), det(A) > 0$ so A^{-1} exist.

$$A^{-1} \in \mathbb{R}^{n\times n}, \det\left(A^{-1}\right) = \frac{1}{\det(A)} > 0 \Rightarrow A^{-1} \in \mathrm{GL}_+$$

1.13

We will show that $G = \{\boldsymbol{Q}|\boldsymbol{Q} \in \mathbb{R}^{n\times n}, \det \boldsymbol{Q} < 0\}$ is not a matrix group.

Let $A, B \in G$:

$$det(A) < 0, det(B) < 0 \Rightarrow det(AB) = det(A)det(B) > 0 \Rightarrow AB \notin G$$

Since the G2 condition doesn't hold, G is not a matrix group.

1.14

Let $A, B \in R_{n\times n}$ symmetric matrices. We will show that usually $AB \neq (AB)^T$

$$(AB)^T = B^T A^T = BA$$

Hence, AB has to be symmetric as well, which is usually not true.

1.15

$$US(n) = \{Q|Q = SI_{n\times n}, S \in \mathbb{R}_{>0}$$

We will prove that US is a matrix group:
G1: We choose $S = 1 \in \mathbb{R}_{>0}$, therefore we get:

$$Q = SI_{n\times n} = I_{n\times n}, det(Q) = det(I_{n\times n}) = 1 \Rightarrow I_{n\times n} \in US(n)$$

G2: Let $A, B \in US(n)$ Hence the definition of A, B will be:

$$A = S_1 I_{n\times n}, S1 \in \mathbb{R}_{>0}$$

$$B = S_2 I_{n\times n}, S2 \in \mathbb{R}_{>0}$$

Therefore the definition of AB will be:

$$AB = S_1 I_{n\times n} S_2 I_{n\times n} = S_1 S_2 I_{n\times n} I_{n\times n} = S_1 S_2 I_{n\times n}$$

Where $S_1 S_2 \in \mathbb{R}_{>0} \Rightarrow AB \in US(n)$

G3: Let $A \in US(n)$, therefore:

$$A = SI_{n\times n}, S \in \mathbb{R}_{>0}$$

It is easy to see that $det(A) = det(SI) = S^n \neq 0$, hence A^{-1} exist.
Moreover:

$$A^{-1} = (SI_{n\times n})^{-1} = \frac{1}{S}(I_{n\times n})^{-1} = \frac{1}{S}I_{n\times n}$$

Since $S \in \mathbb{R}_{>0} \Rightarrow \frac{1}{S} \in \mathbb{R}_{>0} \Rightarrow A^{-1} \in US(n)$.