Master in Innovation and Research in Informatics

SMDE - Queueing Theory

Evaluation of G/G/1 Systems

Authors:
Asaf Badouh
Pau Rodríguez Esmerats

Professor:
Esteve Codina

January 12, 2018

Contents

0.1 Initial analysis

	Interarrival Times τ	Service times X
Assigned Distributions	Normal	Lognormal
Parameters	($\mu = 77$, $\sigma = 15$)	(μ, $\sigma = 1.4286$)

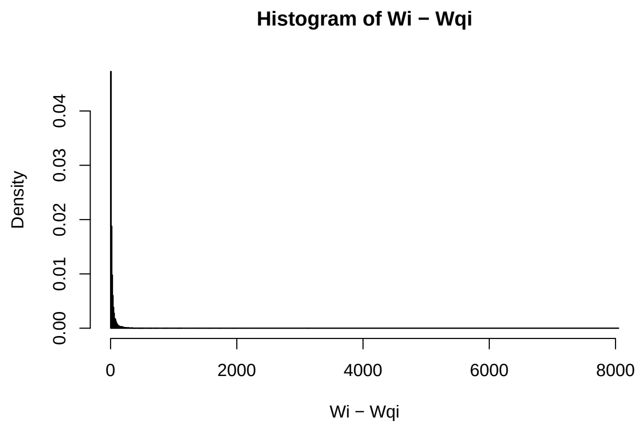
We execute a small simulation with 10.000 clients to verify that the simulation is running correctly. Our probability distribution for modelling the services times is a Lognormal with $\sigma = 1.4286$. To obtain the other parameter of the distribution, the μ, we choose the first traffic factor, $\rho = 0.4$, which allows us to compute the μ.

$$\rho = \frac{\lambda}{s\mu} = \frac{E[x]}{E[\tau]} = e^{\mu} \cdot \frac{e^{\frac{\sigma^2}{2}}}{E[x]} \Rightarrow \mu = ln\left(\frac{\rho}{e^{\frac{\sigma^2}{2}}} \cdot E[\tau]\right)$$

The resulting μ is 2.4070657.

In our simulation, the arrival times are defined by a Normal distributioni with parameters $\mu = 77, \sigma = 15$. We generate 10.000 clients int the simulation, and analyse the service times.

The histogram of the service times is shown below:



We can clearly observe that the service times follow a heavy-tailed distribution, as more than 75% of the histogram bins have few samples and they are all consecutively placed until the end of it.

The mean and sample variance of the services times are the following:

1. Mean of the service times: $W_{s_{avg}}$=30.4132959
2. Standard deviation of the service times: $W_{s_{std}}$=77.5293542
3. Coeficient of variation: $C_x = \frac{\sigma_x}{E[x]}$=2.5491928

The Theoretical values for a Lognormal with μ=2.4070657 and σ=1.4286 are the follwing:

1. Mean of the service times: $E[x]$=30.8
2. Standard deviation of the service times: $sqrtVar[x]$=79.7090561
3. Coeficient of variation: $C_x = \frac{\sigma_x}{E[x]}$=2.5879564

We observe that the sample statistics are close to the theoretical values. Our simulation can be considered to be correct.

0.2 Allen Cuneen's aproximation formula for W_q and L_q

For each loading factor ρ, we derive the required μ value for the Lognormal distribution:

- $s = 1$, $\lambda = \frac{1}{E[\tau]}$, $\mu = \frac{1}{E[x]}$
- $E[x] = m \cdot e^{\frac{\sigma^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}$
- $\rho = \frac{\lambda}{s\mu} = \frac{E[x]}{E[\tau]} = e^{\mu} \cdot \frac{e^{\frac{\sigma^2}{2}}}{E[x]} \Rightarrow \mu = ln\left(\frac{\rho}{e^{\frac{\sigma^2}{2}}} \cdot E[\tau]\right)$

We use the Allen Cuneen's approximation formula for L_q:

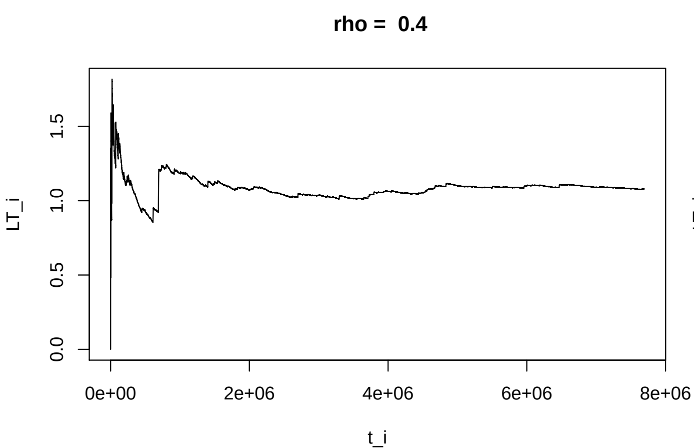
- $L_q \approx L_{q_{M/M/1}} \cdot \left(\frac{C_\tau^2 + C_x^2}{2}\right)$
- with: $C_x = \sqrt{\omega - 1} = \sqrt{e^{\sigma^2} - 1}$ and $C_\tau = \frac{\sigma_\tau}{E[\tau]}$
- and derive $W_q = \frac{L_q}{\lambda}$

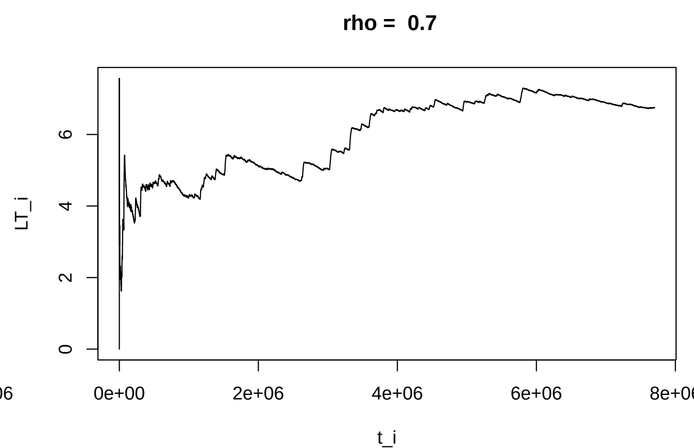
Using the Allen Cuneen's approximation formula, we can compute the W_q and L_q for each loading factor:

ρ	μ	W_q	L_q
0.4	2.4070657	69.1507968	0.8980623
0.7	2.9666815	423.5486306	5.5006316
0.85	3.1608375	1249.0362678	16.2212502
0.925	3.2453949	2958.3575269	38.4202276

0.3 Simulation

First, for each ρ, we're going to calculate what is the amount of clients needed to get in the steady state of the waiting system.



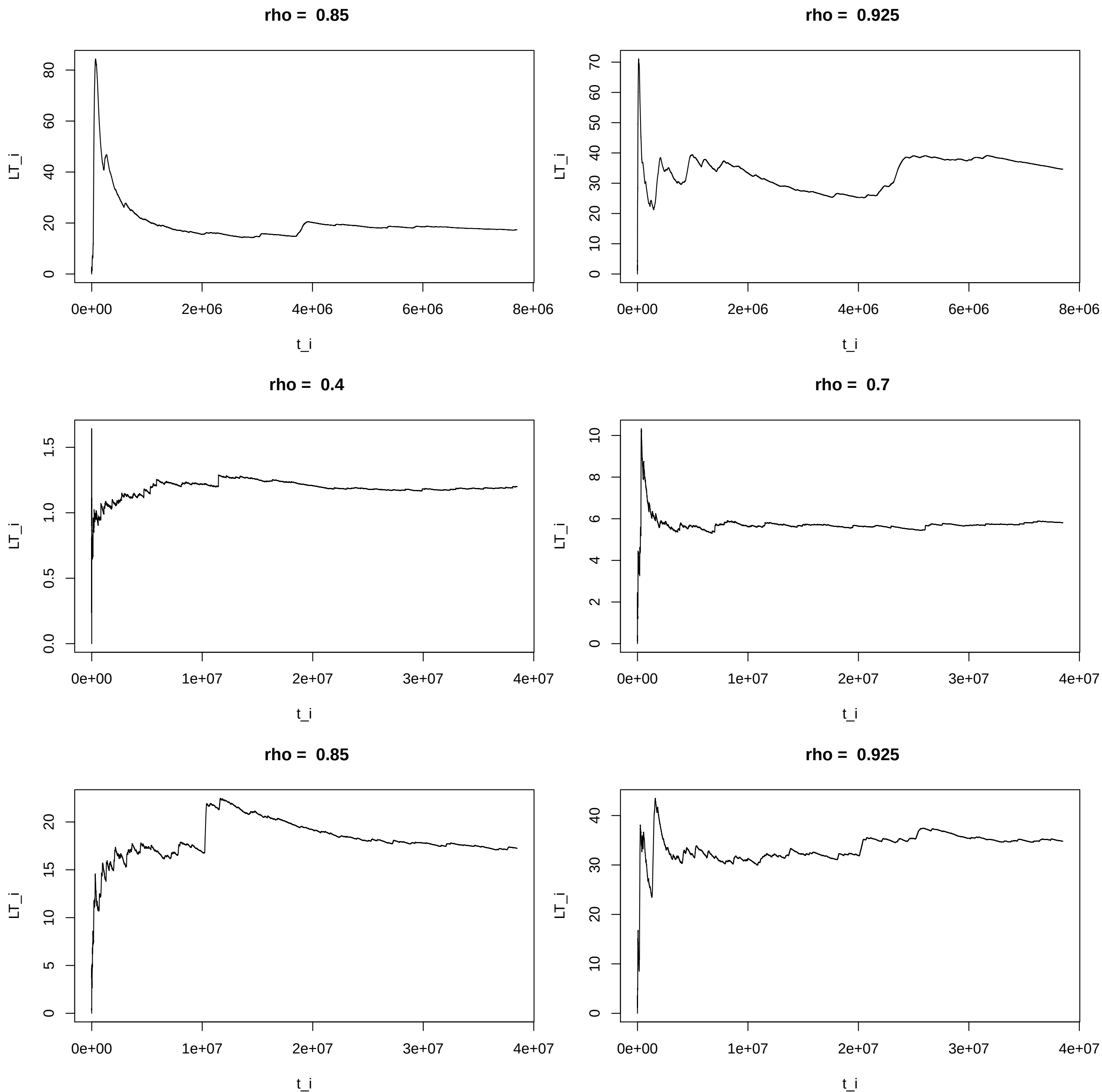


We observe that, apart from the simulation with loading factor 0.4, the other simulations have not attained the steady state.

If we repeat the simulations with a number of clients between 200000 and 500000, the steady state is attained with all loading factors. We have not tested more than 500000 clients.

0.3.1 Loading factor $\rho = 0.4$

We generate 10 simulations with 100000 clients, each with a different seed. For each simulation, we check if the steady state is attained without any abrupt increase or decrease in the value of the average occupancy. In case there's any abrupt increase or decrease, we change the seed. If for many seeds, this phenomena is still happening, we increase the number of clients.

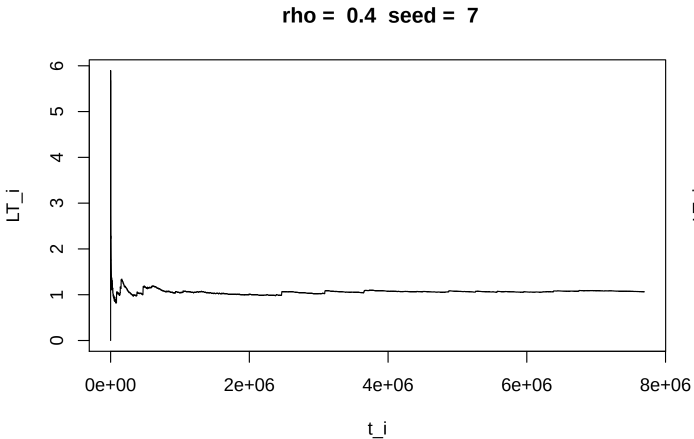


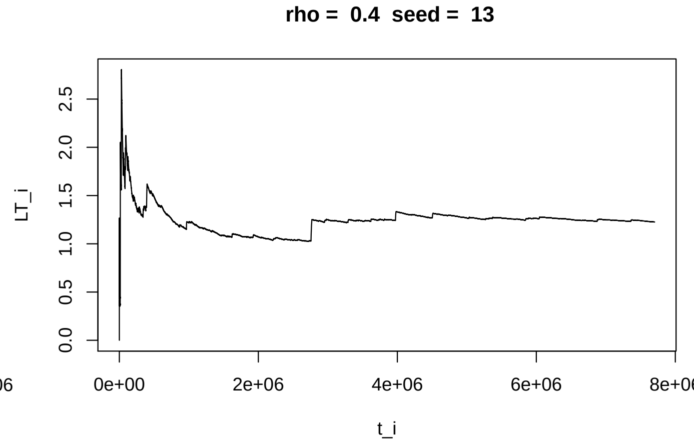
We observe that for the seeds 10101, 1078, 960 and 51, there's an abrupt change in the average occupancy. We change those seeds and redo the simulation.

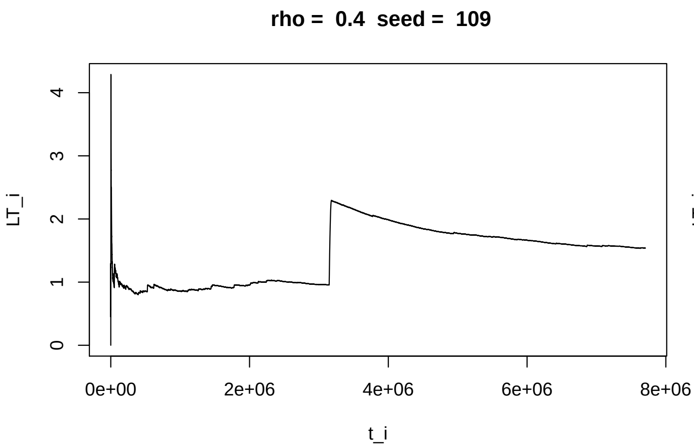
We changed a total of 4 out of 10 seeds.

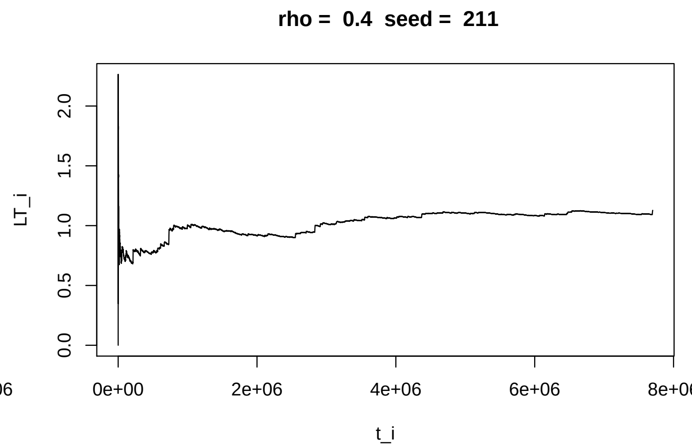
We now compute the confidence interval for L_q and W_q. We will use a t-Student distribution with critical value 1 - 0.95 and 9 degrees of freedom to compute a 95% confidence interval.

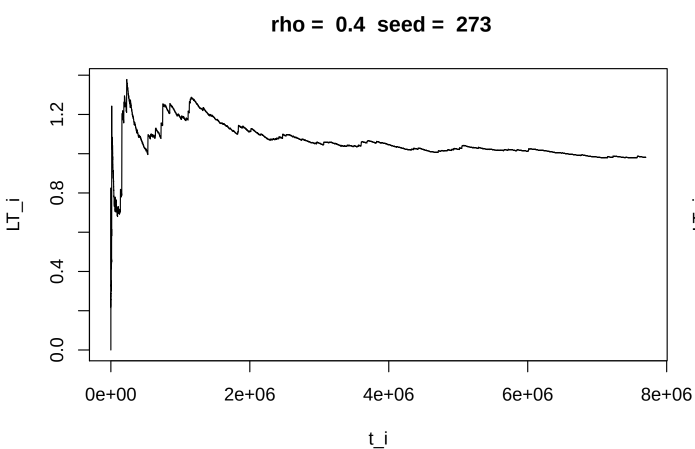
- $L_{q_1}, L_{q_2}, ..., L_{q_{10}}$, and $\overline{L}_q = \frac{1}{n}\sum_{i=1}^{n} L_{q_i}$
- $S^2_{L_q} = \frac{1}{n-1}\sum_{i=1}^{n}(L_{q_i} - \overline{L}_q)^2$

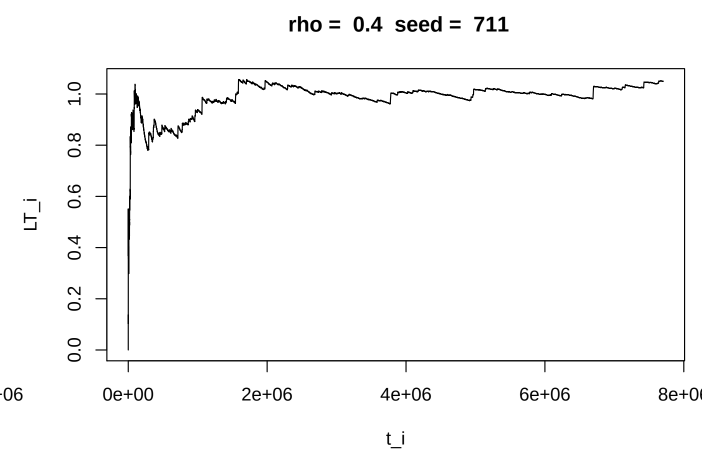










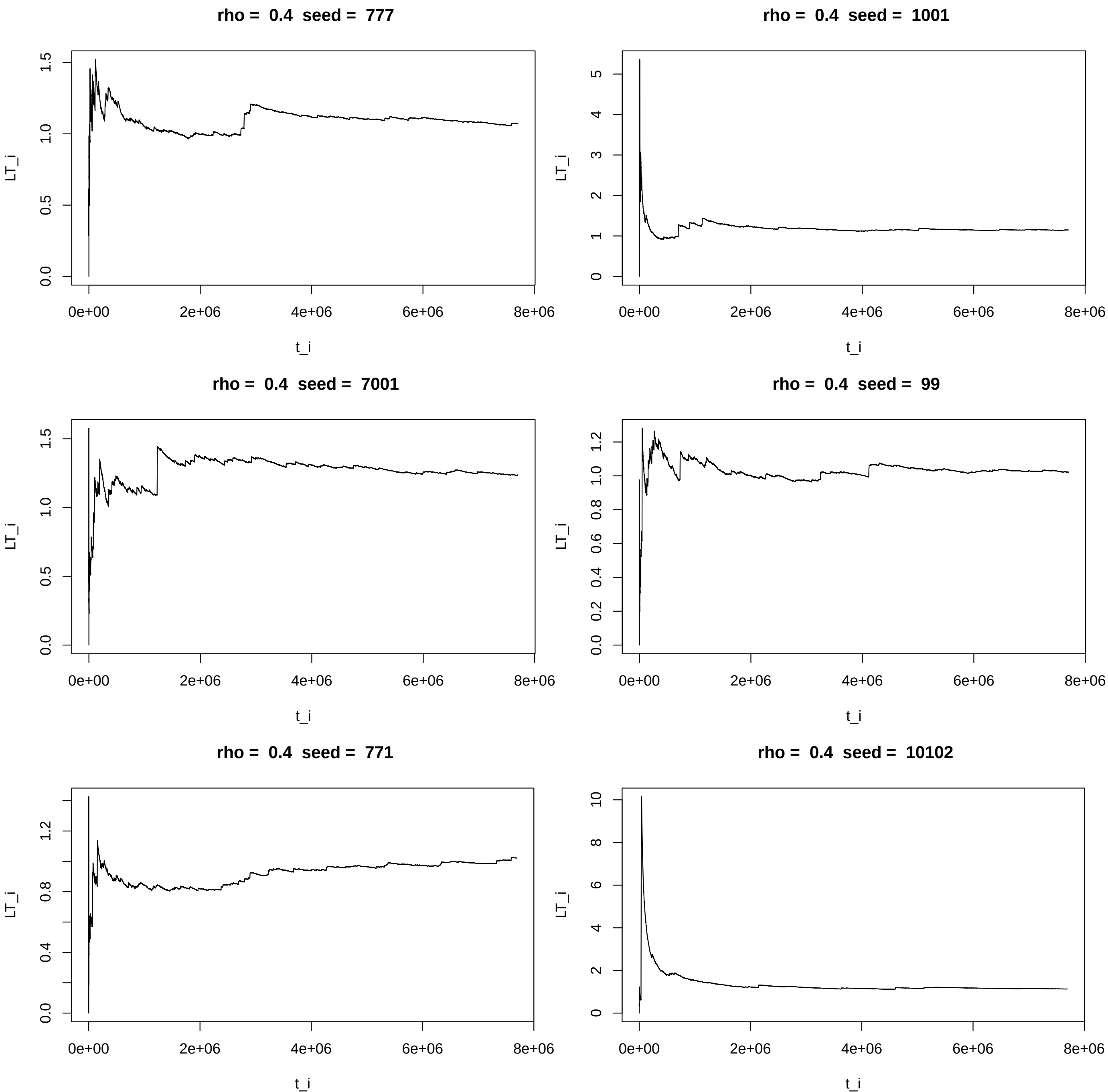


To get:

$$C.I.(L_q) = \overline{L}_q \pm t_{1-\alpha,n-1} \cdot \sqrt{\frac{S^2_{L_q}}{n}}$$

And then:

- $W_{q_1}, W_{q_2}, ..., W_{q_{10}}$, and $\overline{W}_q = \frac{1}{n}\sum_{i=1}^{n} W_{q_i}$
- $S^2_{W_q} = \frac{1}{n-1}\sum_{i=1}^{n}(W_{q_i} - \overline{W}_q)^2$



to get:

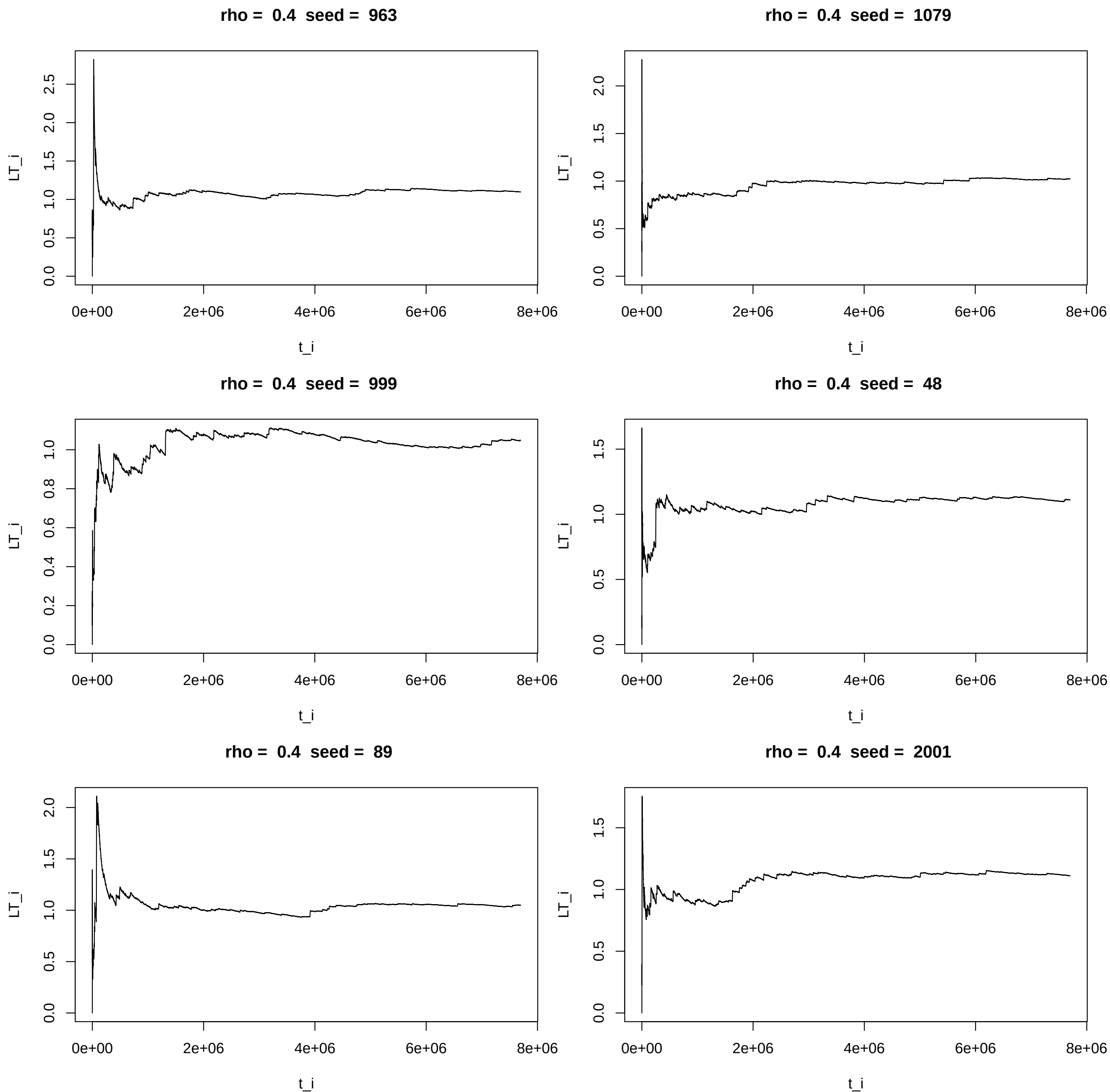
$$C.I.(W_q) = \overline{W}_q \pm t_{1-\alpha,n-1} \cdot \sqrt{\frac{S^2_{W_q}}{n}}$$

The computations produce the followings confidence intervals for the average queue length and waiting time:

ρ	-C.I W_q	+C.I W_q	-C.I L_q	+C.I L_q
0.4	0.6629266	0.7216626	51.0353731	55.5687961

0.3.2 Loading factor $\rho = 0.7$

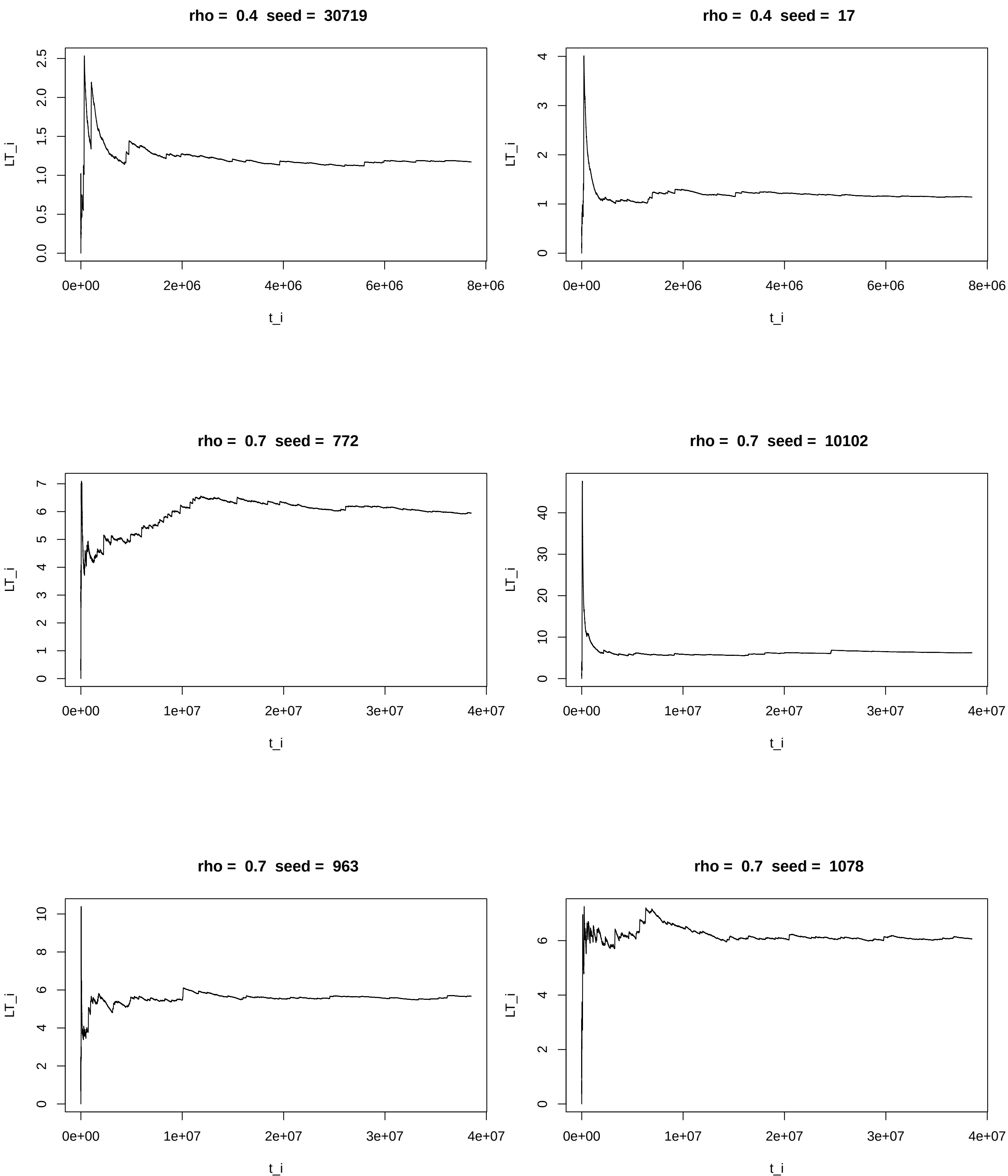
We follow the same procedure to generate 10 simulations with 100000 clients, as we did for the previous $\rho =$, correcting any invalid seeds for the ramdom number generator. The result is that we need to increase the number of clients to 500000 to be sure that the system has arribed to the steady state.

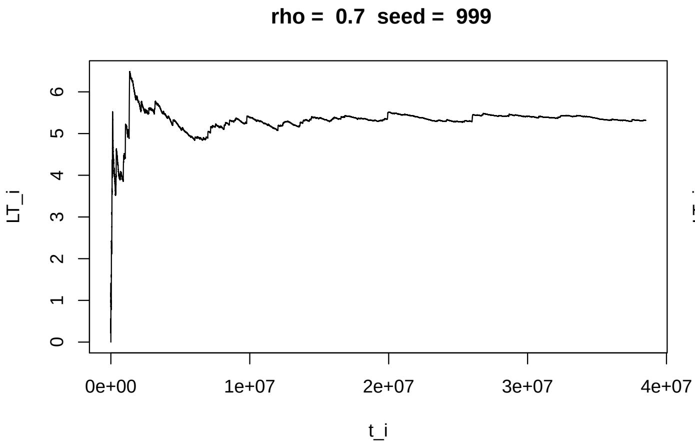


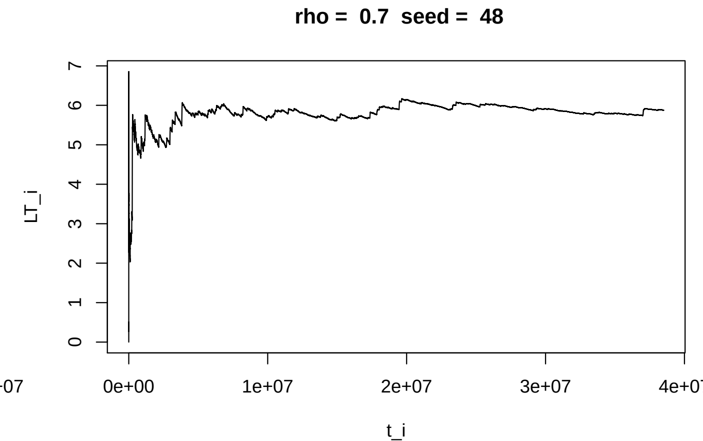
ρ	W_q	-C.I W_q	+C.I W_q	L_q	-C.I L_q	+C.I L_q
0.4	69.1507968	51.0353731	55.5687961	0.8980623	0.6629266	0.7216626
0.7	423.5486306	379.3389492	412.4915102	5.5006316	4.9266678	5.3560657
0.85	1249.0362678	1099.7052017	1232.9696842	16.2212502	14.2854646	16.016791
0.925	2958.3575269	2542.0094174	2889.0200971	38.4202276	33.0165379	37.5147854

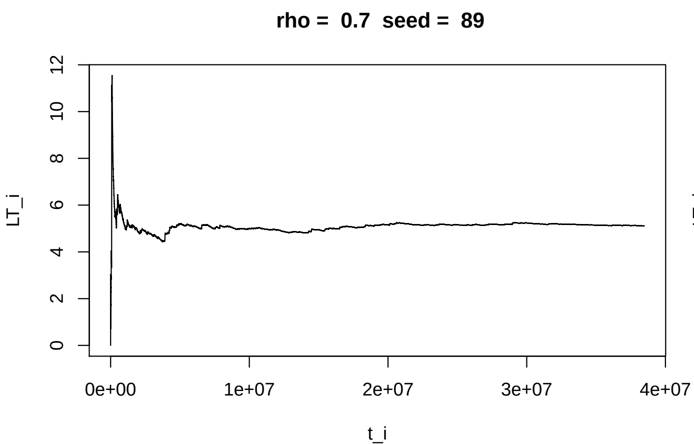
As we can see in the table above Allen-Cuneen's approximation values for all loading factors are outside of the confidence interval of our simulations, always above. Overall, the approximations of both, W_q and L_q, are not very far from the simulation's confidence interval. So the Allen-Cuneen's aproximation models a system where the occupancy is higher and the waiting times in the queue are also greater than what our simulation produces. How can we explain that?

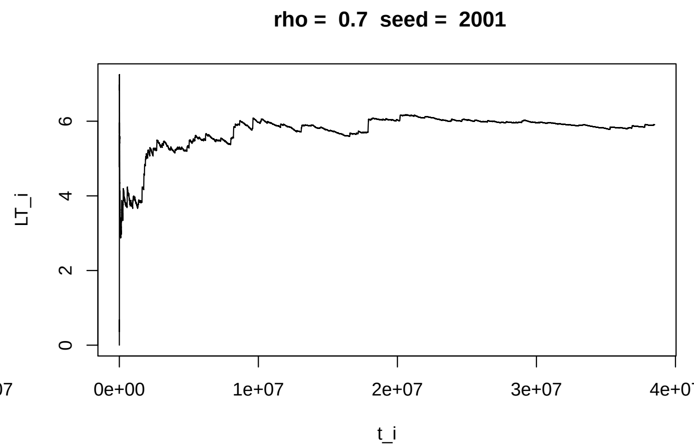
First of all, our random number generator has not been tested. As discussed in class, it is impossible to generate real randomness, so every rng must be tested for multiple desirable properties. In our particular case, even though we used the rng that ships with R, which is considered to ship with well tested generated

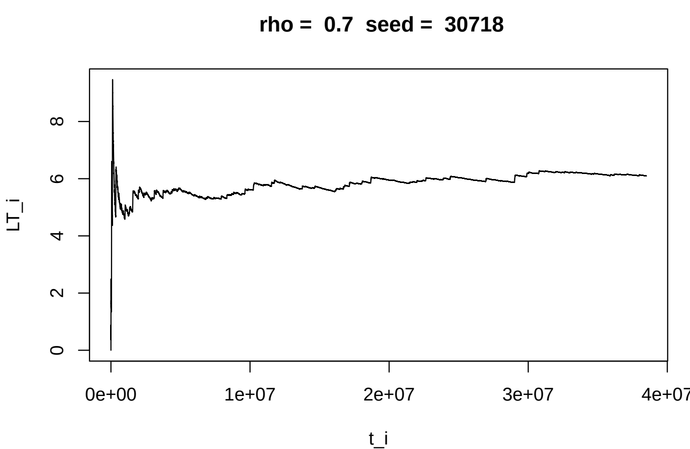

rho = 0.4 seed = 30719
rho = 0.4 seed = 17
rho = 0.7 seed = 772
rho = 0.7 seed = 10102
rho = 0.7 seed = 963
rho = 0.7 seed = 1078
LT_i
t_i
0e+00
2e+06
4e+06
6e+06
8e+06
1e+07
2e+07
3e+07
4e+07

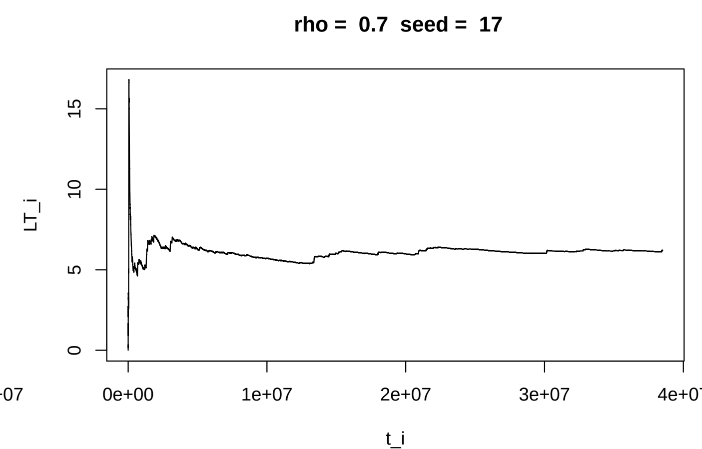

rho = 0.7 seed = 999
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

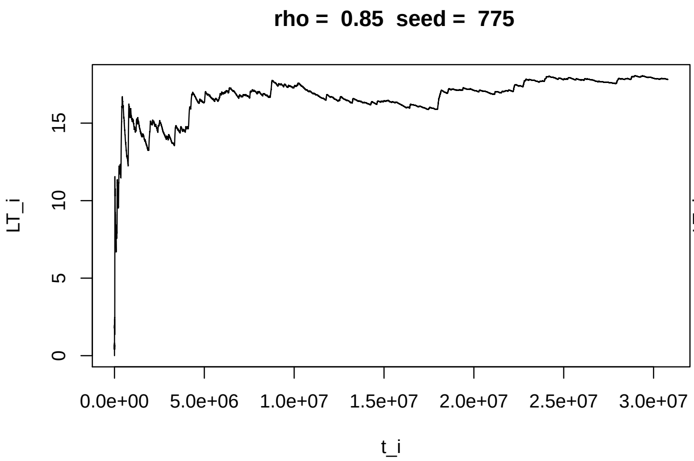

rho = 0.7 seed = 48
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

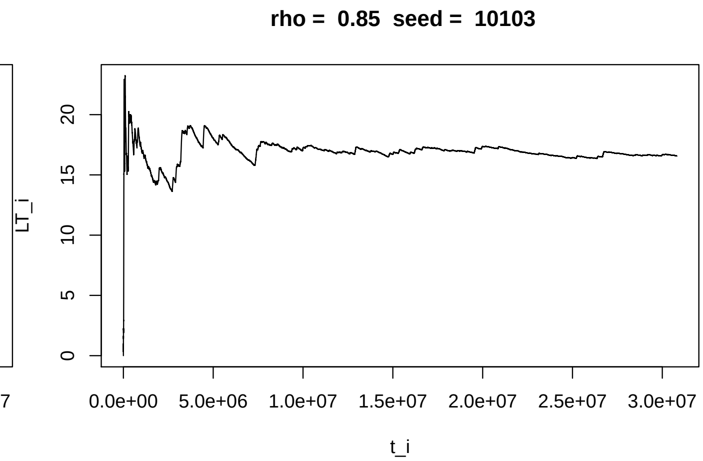

rho = 0.7 seed = 89
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

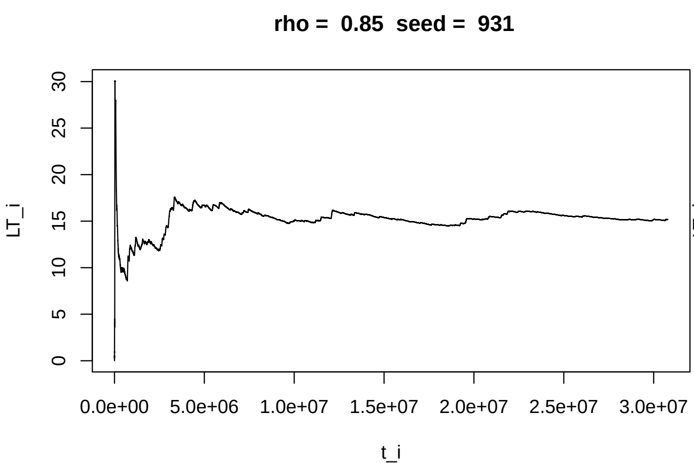

rho = 0.7 seed = 2001
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

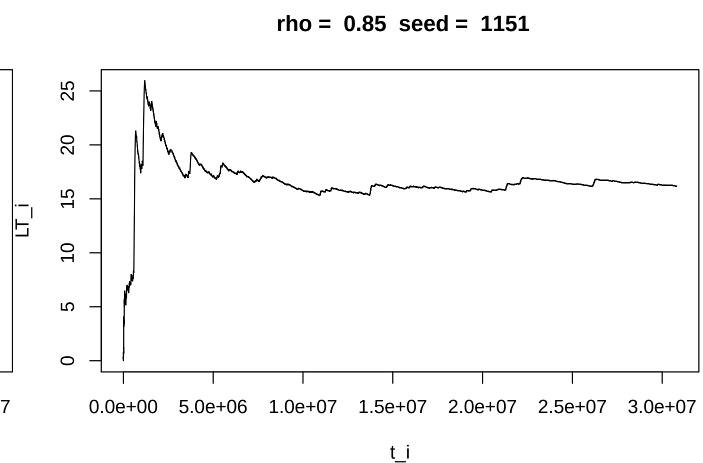

rho = 0.7 seed = 30718
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

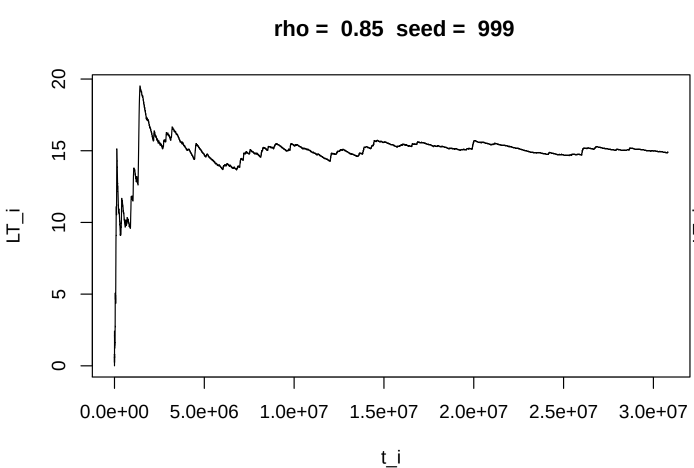

rho = 0.7 seed = 17
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07

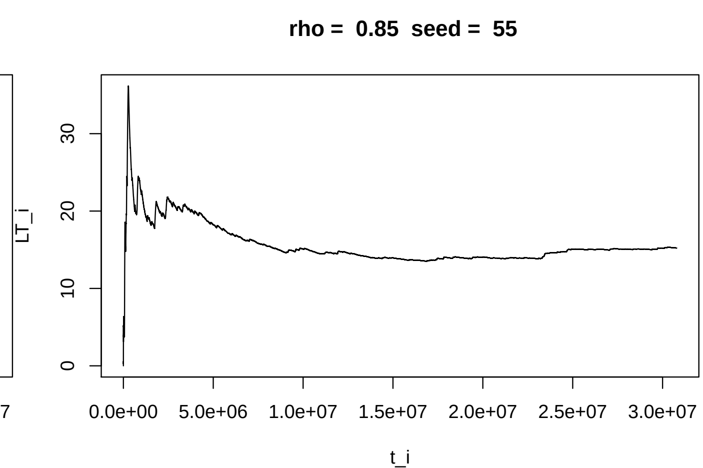

rho = 0.85 seed = 775
LT_i
t_i
0
5
10
15
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07


rho = 0.85 seed = 10103
LT_i
t_i
0
5
10
15
20
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07

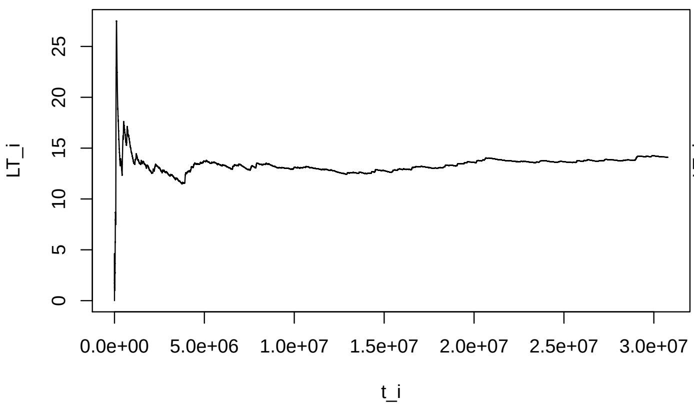

rho = 0.85 seed = 931
LT_i
t_i
0
5
10
15
20
25
30
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07


rho = 0.85 seed = 1151
LT_i
t_i
0
5
10
15
20
25
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07

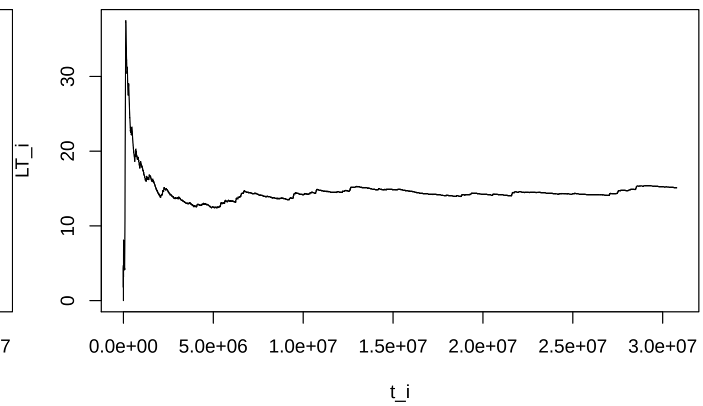

rho = 0.85 seed = 999
LT_i
t_i
0
5
10
15
20
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07


rho = 0.85 seed = 55
LT_i
t_i
0
10
20
30
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
3.0e+07

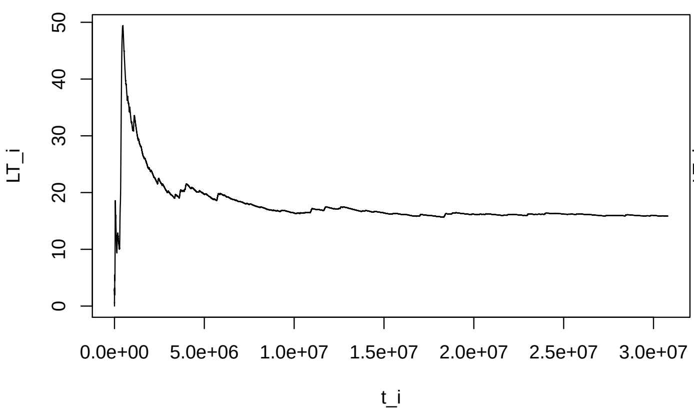
rho = 0.85 seed = 89



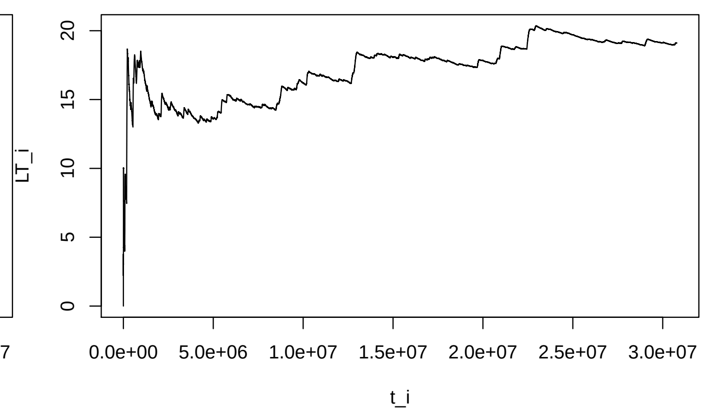
rho = 0.85 seed = 3001



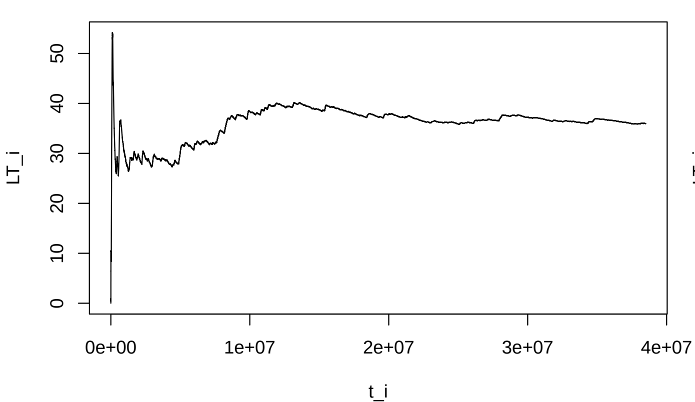
rho = 0.85 seed = 9718



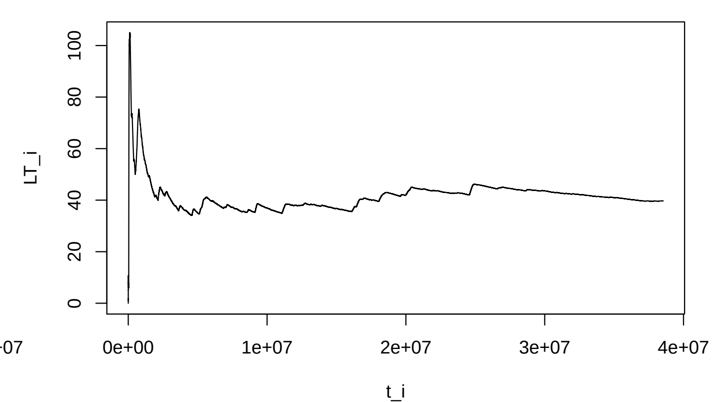
rho = 0.85 seed = 387



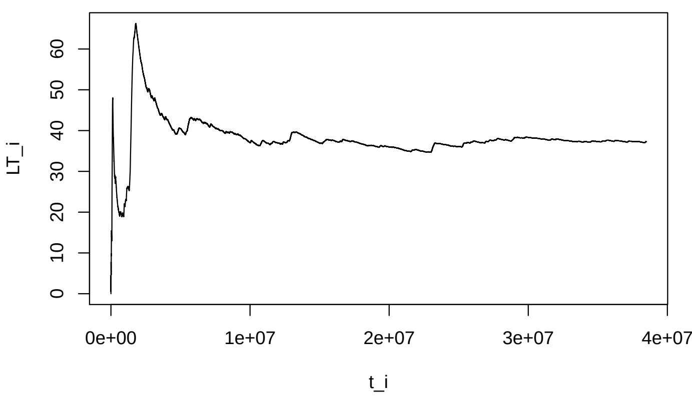
rho = 0.925 seed = 772



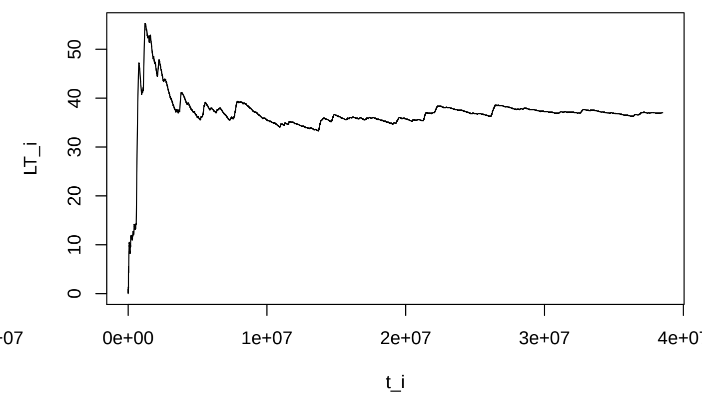
rho = 0.925 seed = 10102



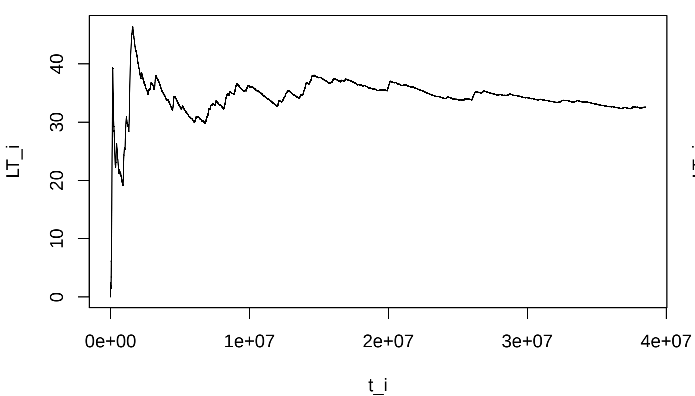
rho = 0.925 seed = 972



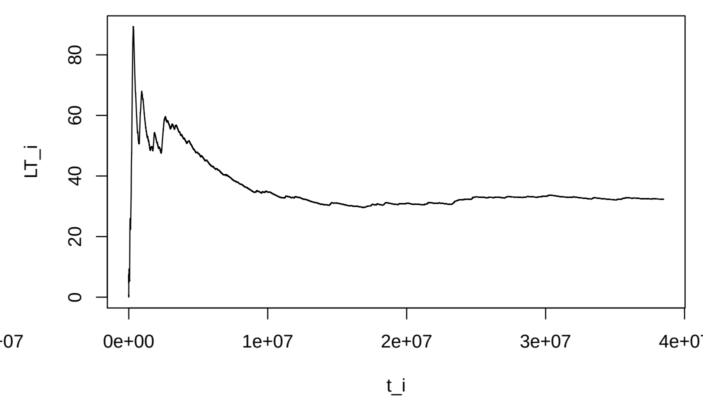
rho = 0.925 seed = 1151



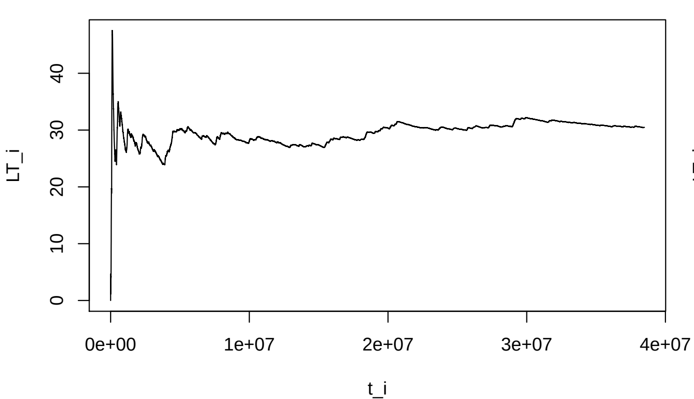
rho = 0.925 seed = 999



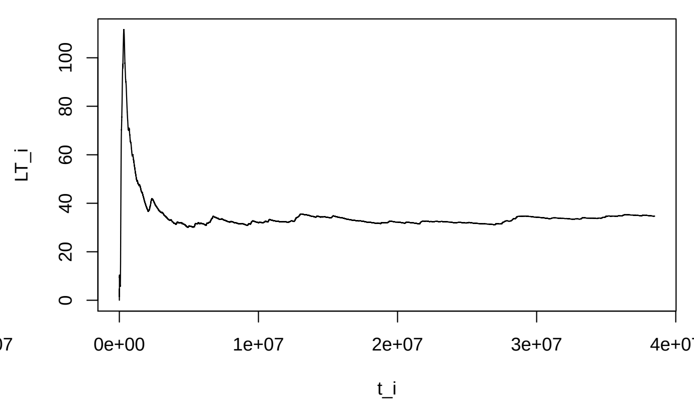
rho = 0.925 seed = 55

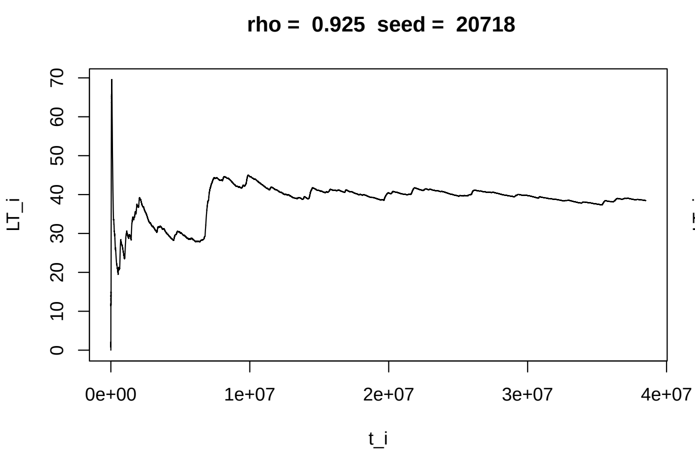


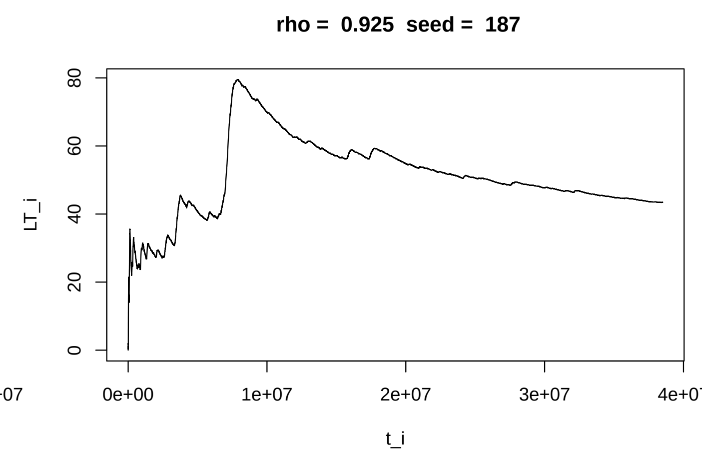
rho = 0.925 seed = 89



rho = 0.925 seed = 3001




rho = 0.925 seed = 20718
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07
0
10
20
30
40
50
60
70


rho = 0.925 seed = 187
LT_i
t_i
0e+00
1e+07
2e+07
3e+07
4e+07
0
20
40
60
80