## A recurrent relationship for evaluating G/G/1 queues

- $x_i$ = service time for client $i$. (INPUT)

- $\theta_i$ = exit time instant from W.S. . for client $i$

- $t_i^S$ = arrival time instant to the service system.

- $t_i$ = entrance time instant to the W.S.; it can be obtained through the sequence of interarrival times $\tau_i = t_{i+1} - t_i$:  $t_i = \sum_{\ell=1}^{i-1} \tau_\ell$. (It can be considered an INPUT)

- $w_i = \theta_i - t_i$ = sojourn time of client $i$ in W.S.

- $w_{q,i} = t_i^S - t_i$ sojourn time of client $i$ in queue.

- $\mathcal{L}_i = w_i$: contribution of $i$-th client to the occupancy.

The following recurrent relationships allow for the generation of arrival time instants in service system $t_i^S$ and the exit times from the W.S. $\theta_i$ for i=1,2,3,4, $n$ clients:

Inicialitzation: $\mathcal{L} = 0; W = 0 \; \mathcal{L}_q = 0; W_q = 0; \theta_0 = -\infty, \; t_1 = 0$

For  $i = 1, 2, 3, ...n$ :

1. $t_i^S = max\{\theta_{i-1}, t_i\}$

2. Generate a service time $x_i$ from the specified distribution.

3. $\theta_i = t_i^S + x_i$

4. If $i < n$ then: Generate $\tau_i$ from the specified distribution;  $t_{i+1} = t_i + \tau_i$.

5. Printing statistics:

   a) $\mathcal{L}_i = w_i = \theta_i - t_i; \mathcal{L} = \mathcal{L} + \mathcal{L}_i; \mathcal{L}_{T_i} = \dfrac{\mathcal{L}}{t_i - t_1} \; ; W = W + w_i$

   b) $\mathcal{L}_{q,i} = w_{q,i} = t_i^S - t_i; \mathcal{L}_q = \mathcal{L}_q + \mathcal{L}_{q,i}; W_q = W_q + w_{q,i}$

After client $n$ report:

$$W = W/n \,; \; W_q = W_q/n \,; \; \mathcal{L} = \dfrac{\mathcal{L}}{t_n - t_1} \; ; \; \mathcal{L}_q = \dfrac{\mathcal{L}_q}{t_n - t_1}$$

In order to see if the previous values are stable, plot $\mathcal{L}_{T_i}$ versus $t_i$. It may be also useful to collect the generated samples $w_i$ and $w_{q,i}$ for statistical analysis (plot of histograms, basic statistics, ...)

**EVALUATION OF G/G/1 SYSTEMS**

The objective of this work is twofold:

- To compare results obtained by simulation for a given G/G/1 system with the Allen-Cuneen's approximation formula. The student will be assigned a specified arrival process and a family of service time distributions (i.e.: service times following a Weibull or lognormal distribution with parameters to be specified by the student accordingly to the required loading factors)

- Analyze the performance of a single server queueing system by simulation when service times correspond to a long-tailed distribution of probability.

1. **Evaluation of the effects of service times following a long-tailed distribution**

    The Weibull or lognormal distribution will be used in this exercise as example of a long tailed distribution for the service time $x$. For the case of the Weibull, its probability distribution function, mean and variance are given by:

    $$F_X(x) = 1 - \exp(-(x/b)^a),$$

    $$E[x] = b\Gamma(\frac{a+1}{a}), \ Var[x] = b^2\left(\Gamma(\frac{a+2}{a}) - \Gamma^2(\frac{a+1}{a})\right)$$

    ( where $\Gamma(\cdot)$ is the Euler function with $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1/2) = \sqrt{\pi}$ )

    whereas for the lognormal distribution its probability distribution function, mean and variance are given by:

    $$F_X(x) = F_z\left(\frac{\log x - \mu}{\sigma}\right),$$

    $$\mu = \log m \qquad m = \exp(\mu) \quad E[x] = m\omega^{\frac{1}{2}}$$
    $$\sigma = (\log\omega)^{\frac{1}{2}} \qquad \omega = \exp(\sigma^2) \quad Var[x] = m^2\omega(\omega-1)$$

    Analysis of the service time . Perform initially a small simulation using just 10.000 clients with the only purpose of capturing a sample of values for the service times and analyze the sample. Report basic statistics (mean, variance, coefficient of variation, ...) and a histogram for the sample. Compare it with the theoretical values for the service time distribution.

    The arrival process will be specified to the students by an assignment list. The number of clients will be 100.000

    For the sequence of traffic factors $\rho$ =0.4, 0,7, 0.85, 0.925, select for each of them a value of the parameter $b$ so that the corresponding loading factor $\rho$ is achieved. Evaluate average waiting times in queue ($W_q$) using Allen Cuneen's approximation formula for each of the previous loading factors.

    Write a small program for evaluating a G/G/1 system implementing the previous recurrence relationships in order to evaluate the resulting waiting system in which arrivals are given to the student by the assignment list and service times are given by a Weibull (or lognormal) distribution so that $a$ (or $\sigma$ in case of the lognormal ) is given in the assignment list and $b$ (or $\mu$ in case of the lognormal ) varies accordingly to the loading factor $\rho$ to be evaluated. The student will be assigned the value for $a$ (in all cases $a < 1$) or $\sigma$. Write the corresponding functions for generating random numbers for the specified arrival process and for a Weibull or lognormal distribution (for service times).

    Using different initial seeds carry out 10 simulations and obtain a confidence interval for the mean waiting time $W_q$ and queue length $L_q$. Plot the evolution of the average occupancy $\mathcal{L}_{T_i}$ versus

$t_i$ in the previous simulations to see if stable values have been reached using 100.000 clients. Compare the waiting times at queues obtained by simulation with those obtained using Allen-Cuneen's Approximation formula.

Make any appropriate comments you consider in order at the view of the previous results.

Deliver:

a) a report of no more than 20 pages answering/commenting all the previous aspects and

b) the computer program developed in 2. (sources, executables and input data files as well as instructions for its use.)

No exercise will be positively evaluated without the delivery of a running program.