SMDE_assignment03

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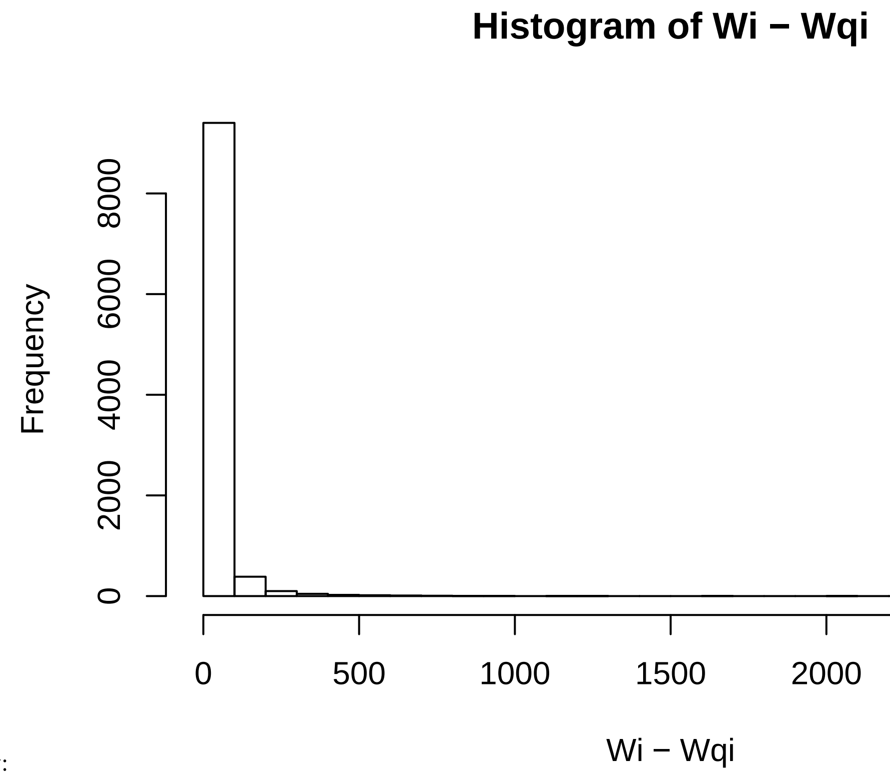
Initial analysis

We execute a small simulation with 10.000 clients to verify that the simulation is running correctly. Our probability distribution for modelling the services times is a Lognormal with $\sigma = 1.4286$. To obtain the other parameter of the distribution, the μ, we choose the first traffic factor, $\rho = 0.4$, which allows us to compute the μ.

$$\rho = \frac{\lambda}{s\mu} = \frac{E[x]}{E[\tau]} = e^{\mu} \cdot \frac{e^{\frac{\sigma^2}{2}}}{E[x]} \Rightarrow \mu = ln\left(\frac{\rho}{e^{\frac{\sigma^2}{2}}} \cdot E[\tau]\right)$$

The resulting μ is 2.4070657.

In our simulation, the arrival times are defined by a Normal distributioni with parameters $\mu = 77, \sigma = 15$. We generate 10.000 clients int the simulation, and analyse the service times.



The histogram of the service times is shown below:

The mean and sample variance of the services times are the following:

1. Mean of the service times: $W_{s_{avg}}$=30.5550545
2. Standard deviation of the service times: $W_{s_{std}}$=76.406743
3. Coeficient of variation: $C_x = \frac{\sigma_x}{E[x]}$=2.5006253

The Theoretical values for a Lognormal with μ=2.4070657 and σ=1.4286 are the follwing:

1. Mean of the service times: $E[x]$=30.8
2. Standard deviation of the service times: $sqrtVar[x]$=79.7090561
3. Coeficient of variation: $C_x = \frac{\sigma_x}{E[x]}$=2.5879564

We observe that the sample statistics are close to the theoretical values. Our simulation can be considered to be correct.

Allen Cuneen's aproximation formula for W_q and L_q

For each loading factor ρ, we derive the required μ value for the Lognormal distribution:

$$s = 1$$

$$\lambda = \frac{1}{E[\tau]}$$

$$\mu = \frac{1}{E[x]}$$

$$E[x] = m \cdot e^{\frac{\sigma^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}$$

$$\rho = \frac{\lambda}{s\mu} = \frac{E[x]}{E[\tau]} = e^{\mu} \cdot \frac{e^{\frac{\sigma^2}{2}}}{E[x]} \Rightarrow \mu = ln\left(\frac{\rho}{e^{\frac{\sigma^2}{2}}} \cdot E[\tau]\right)$$

We use the Allen Cuneen's approximation formula for L_q:

$$L_q \approx L_{q_{M/M/1}} \cdot \left(\frac{C_\tau^2 + C_x^2}{2}\right)$$

With:

$$C_x = \sqrt{\omega - 1} = \sqrt{e^{\sigma^2} - 1} C_\tau = \frac{\sigma_\tau}{E[\tau]}$$

And derive W_q:

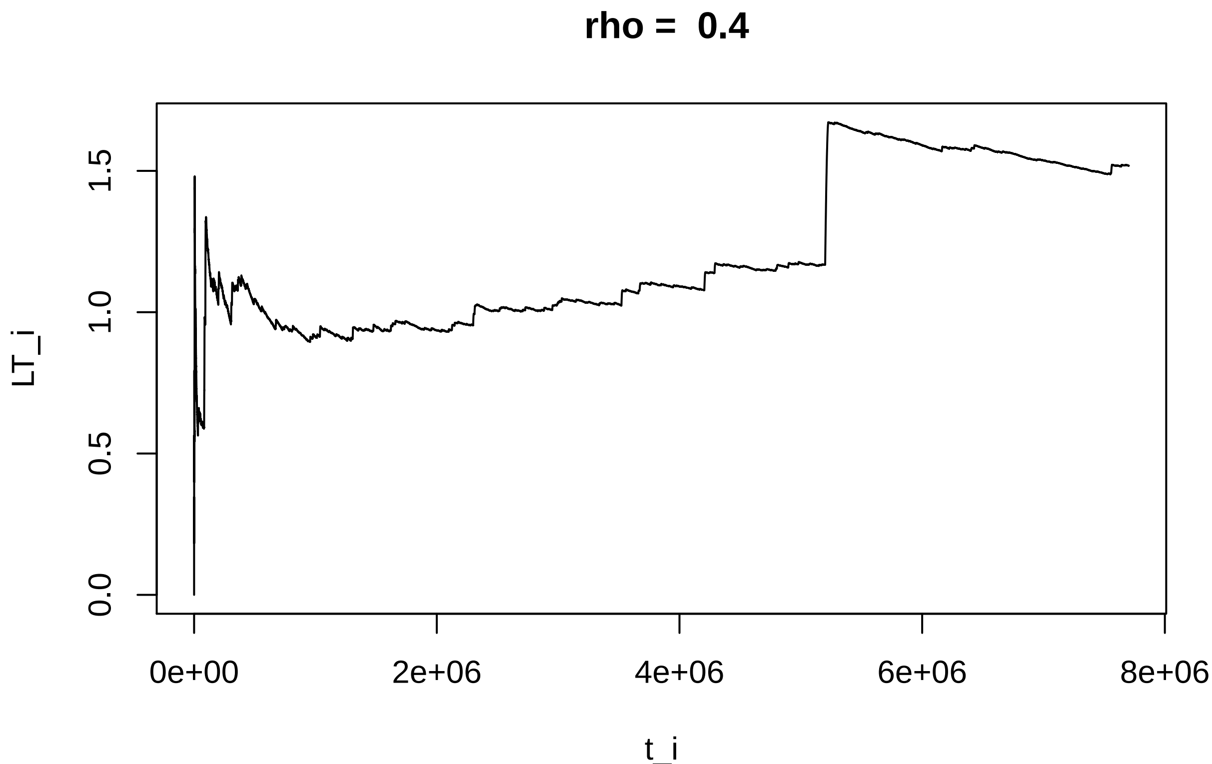
$$W_q = \frac{L_q}{\lambda}$$

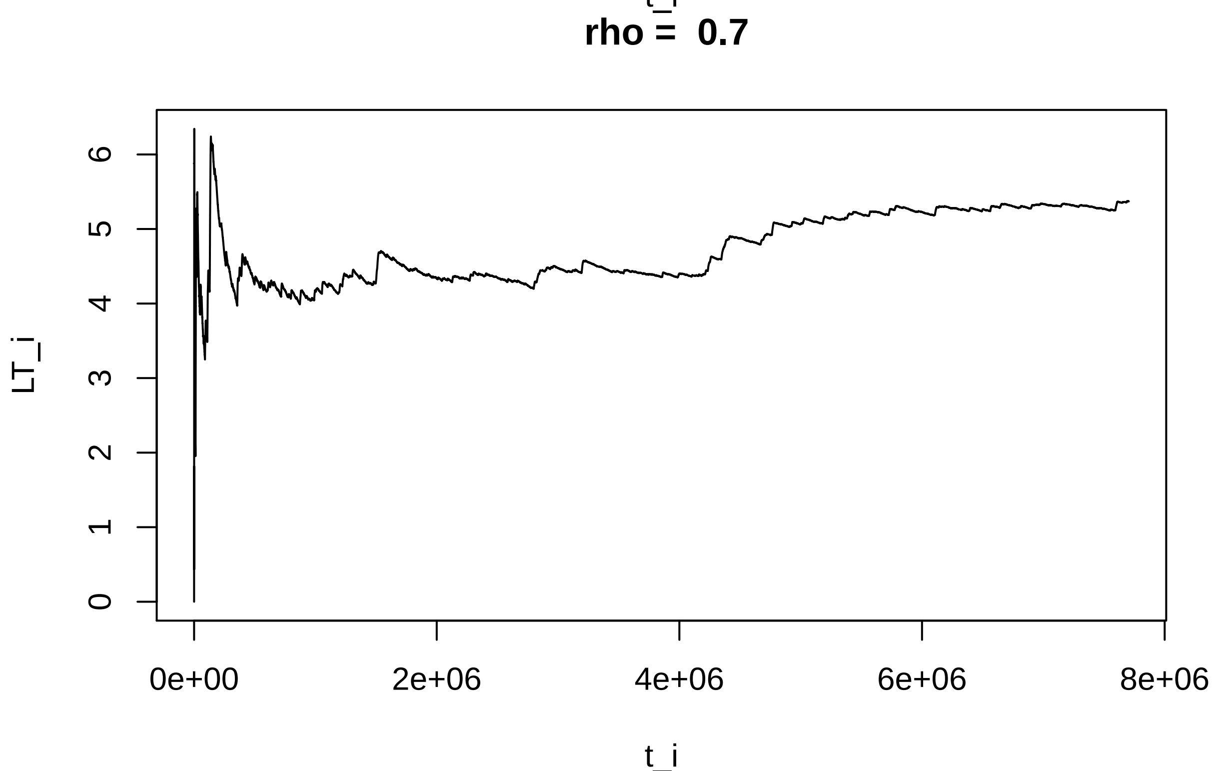
Using the Allen Cuneen's approximation formula, we can compute the W_q and L_q for each loading factor:

ρ	μ	W_q	L_q
0.4	2.4070657	69.1507968	0.8980623
0.7	2.9666815	423.5486306	5.5006316
0.85	3.1608375	1249.0362678	16.2212502
0.925	3.2453949	2958.3575269	38.4202276

Simulation

First, for each ρ, we're going to calculate what is the amount of clients needed to get in the steady state of the waiting system.


rho = 0.4
LT_i
0.0
0.5
1.0
1.5
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.7
LT_i
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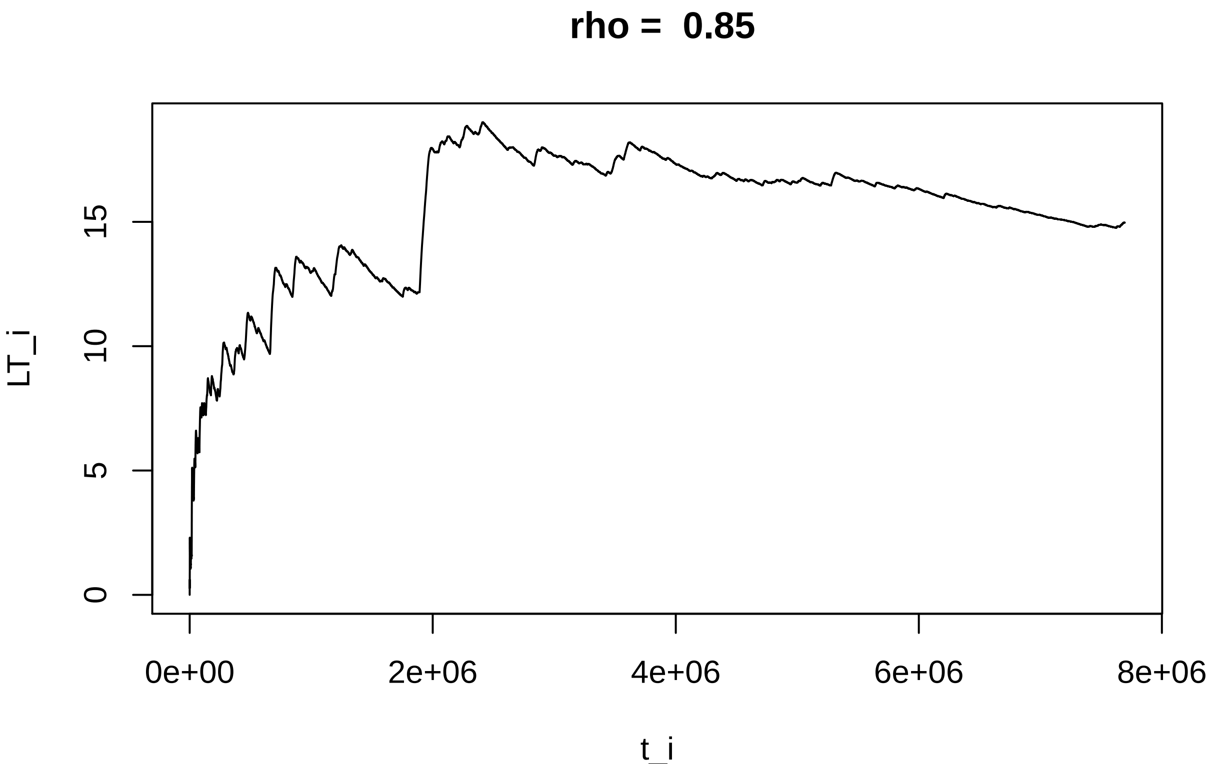
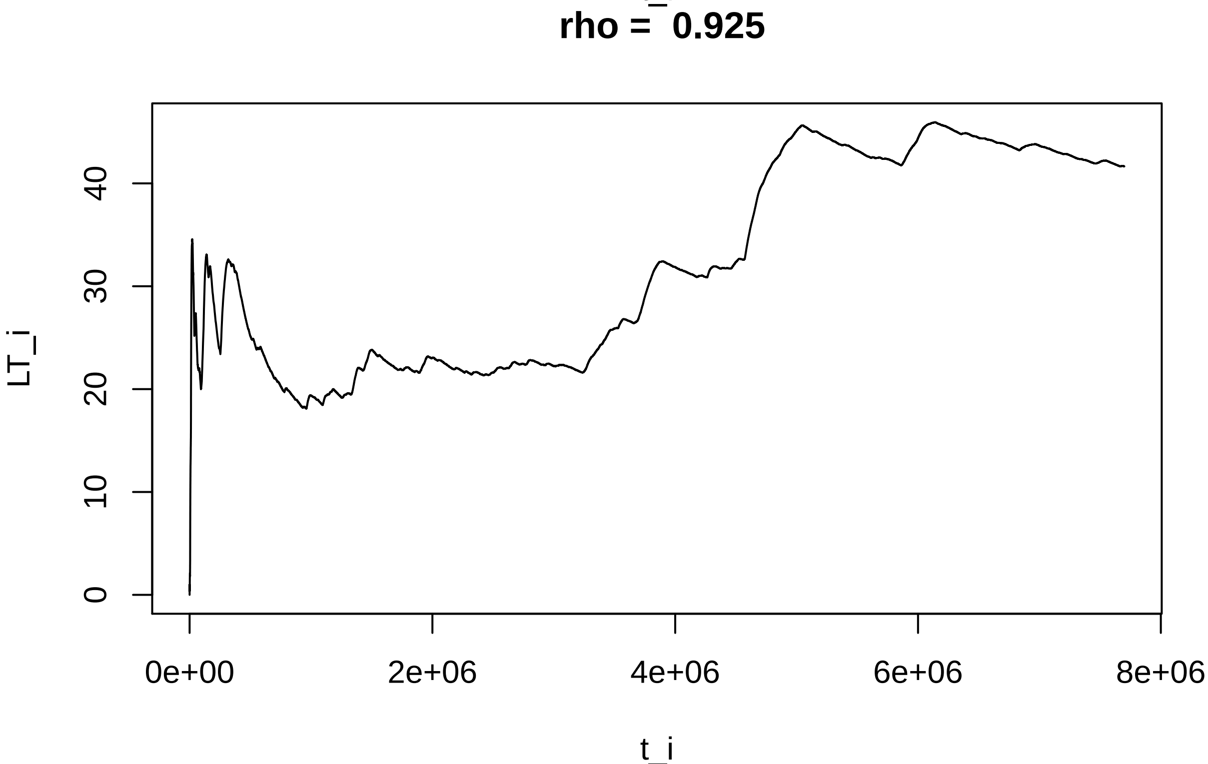
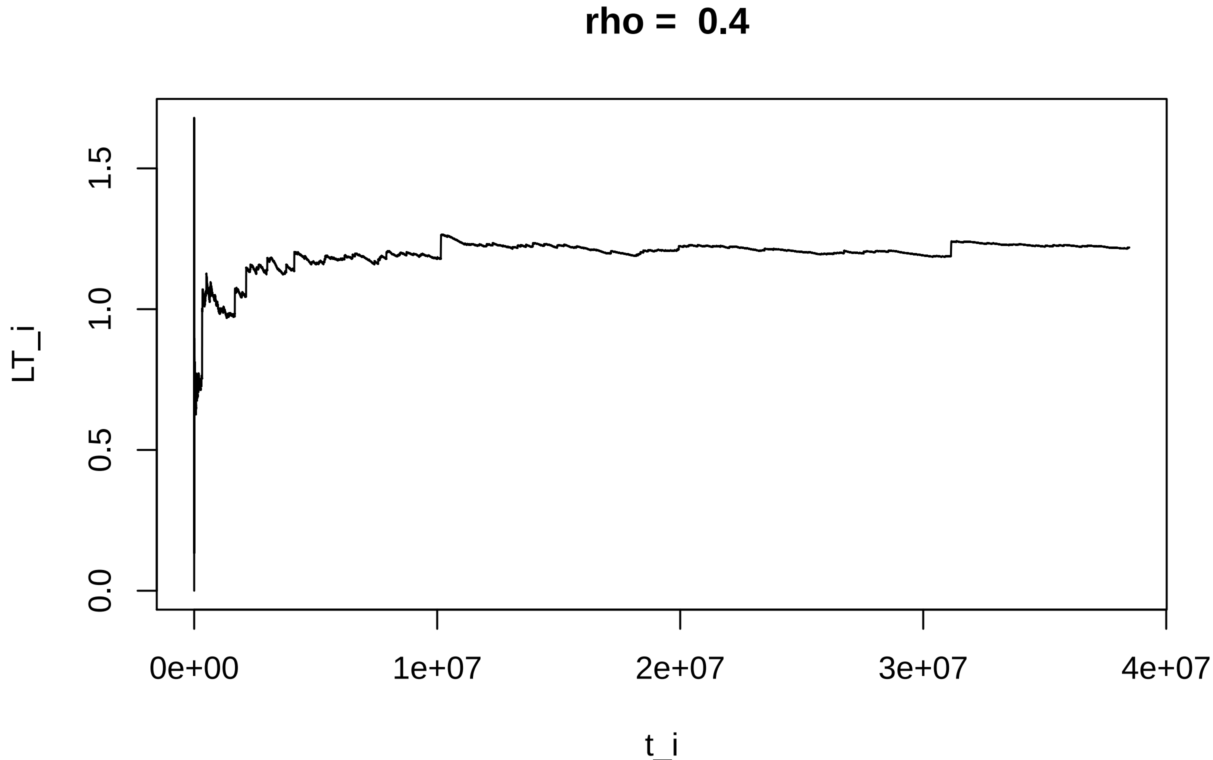
1
2
3
4
5
6
0e+00
2e+06
4e+06
6e+06
8e+06
t_i





We observe that, apart from the simulation with loading factor 0.4, the other simulations have not attained the steady state.

If we repeat the simulations with a number of clients between 200000 and 500000, the steady state is attained with all loading factors. We have not tested more than 500000 clients.


rho = 0.4
LT_i
0.0
0.5
1.0
1.5
0e+00
1e+07
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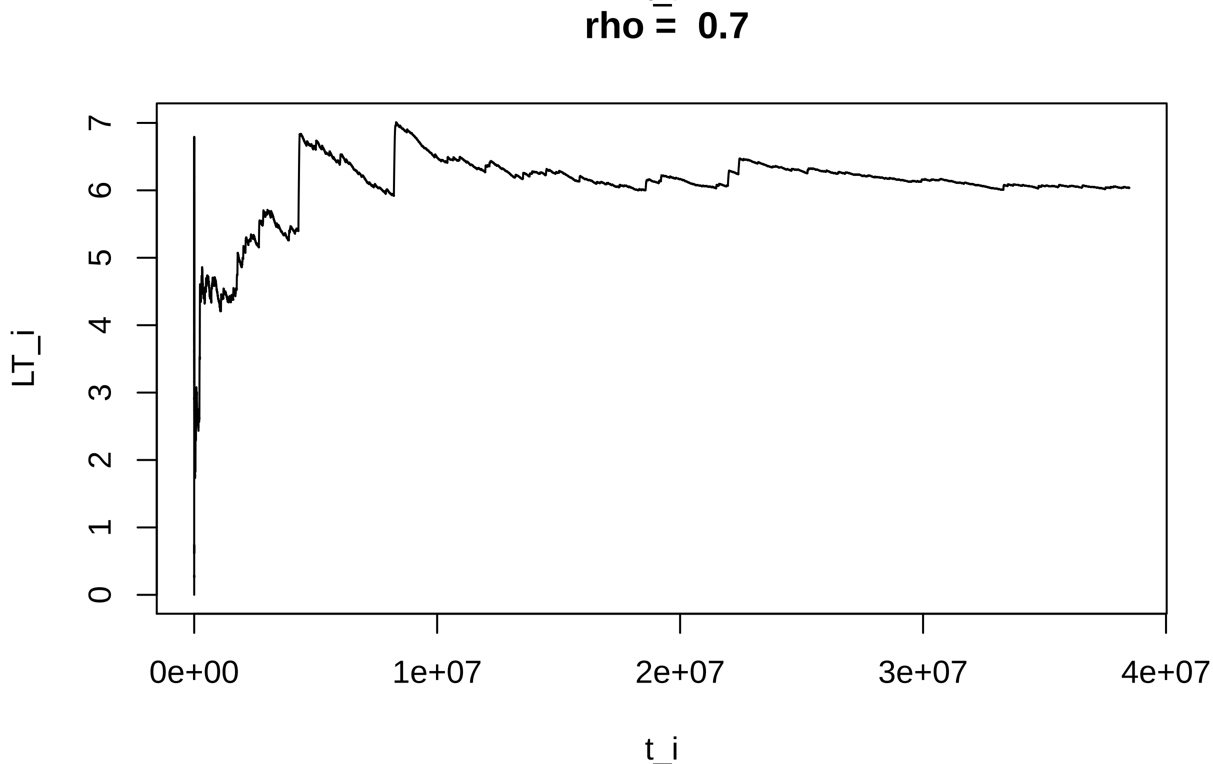
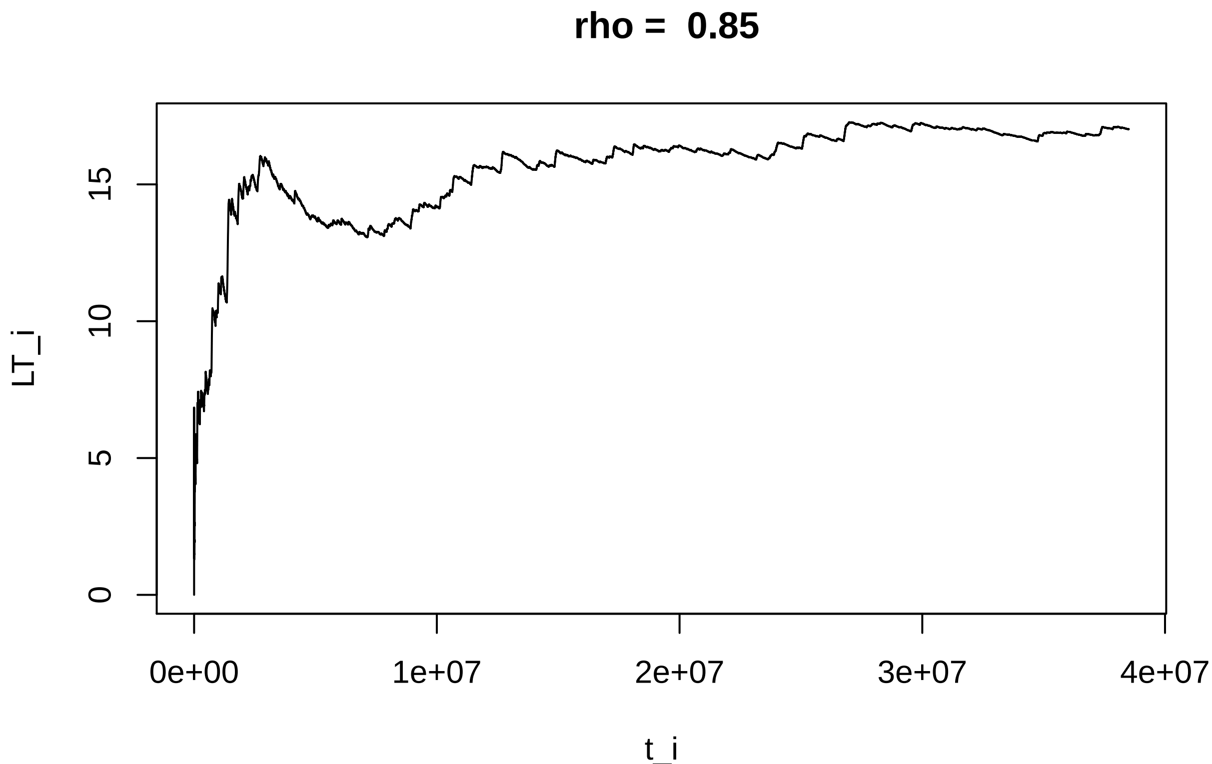
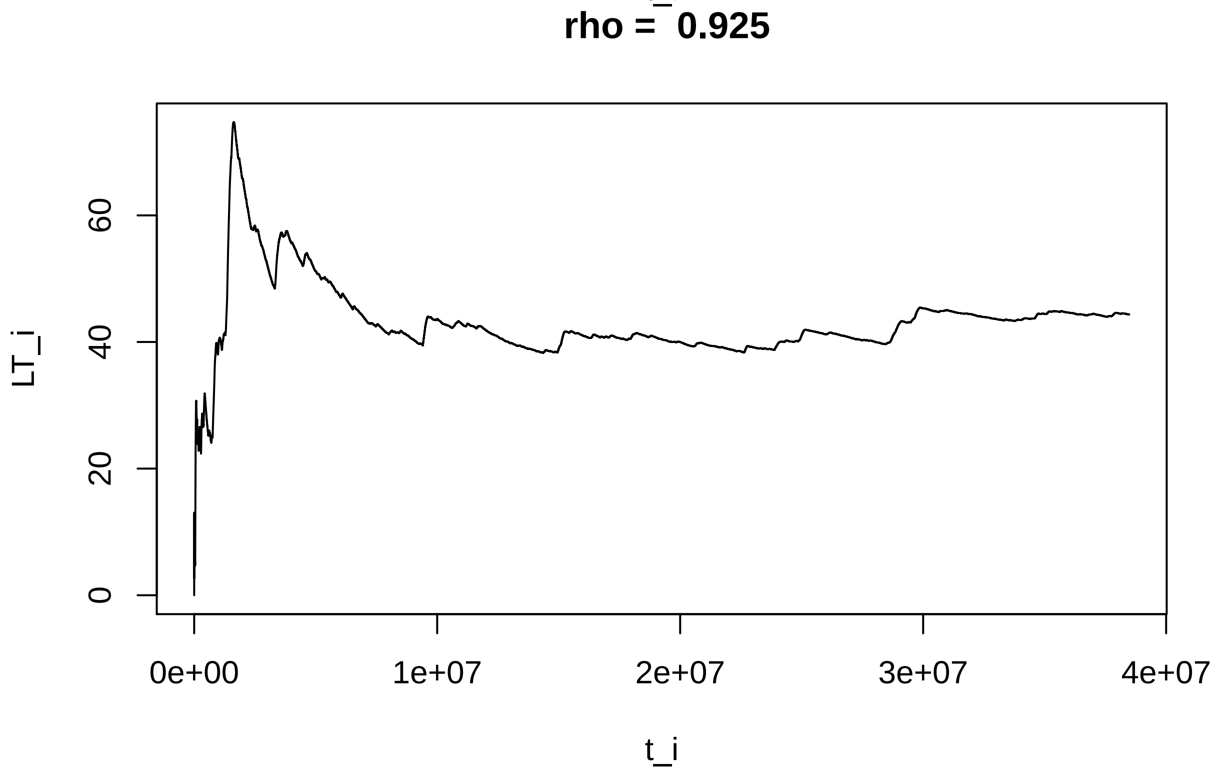
3e+07
4e+07
t_i


rho = 0.7
LT_i
0
1
2
3
4
5
6
7
0e+00
1e+07
2e+07
3e+07
4e+07
t_i

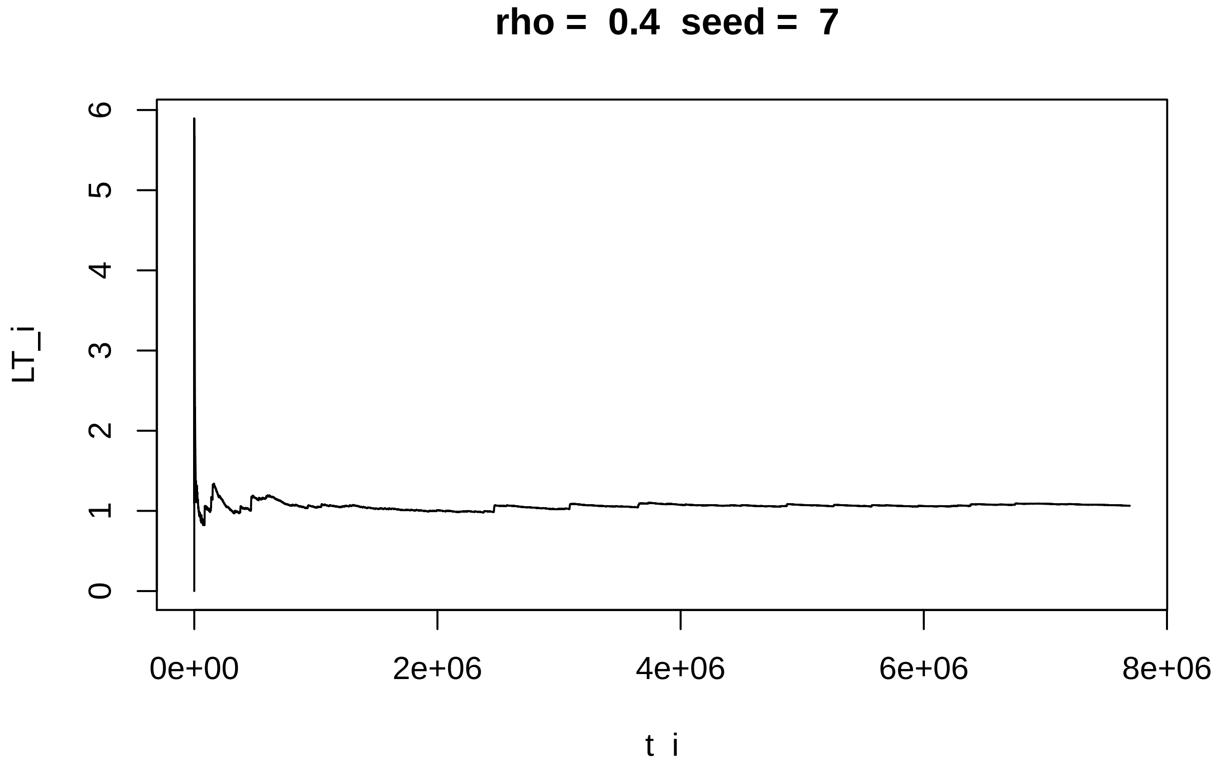


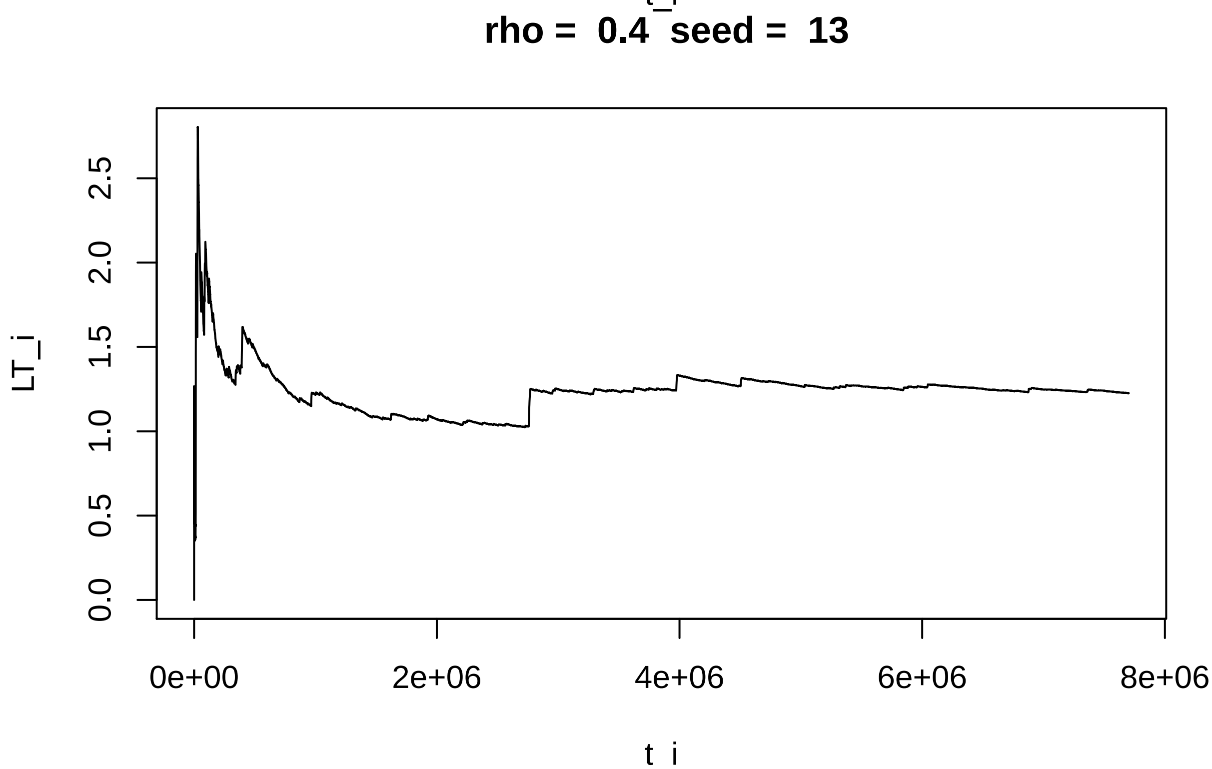


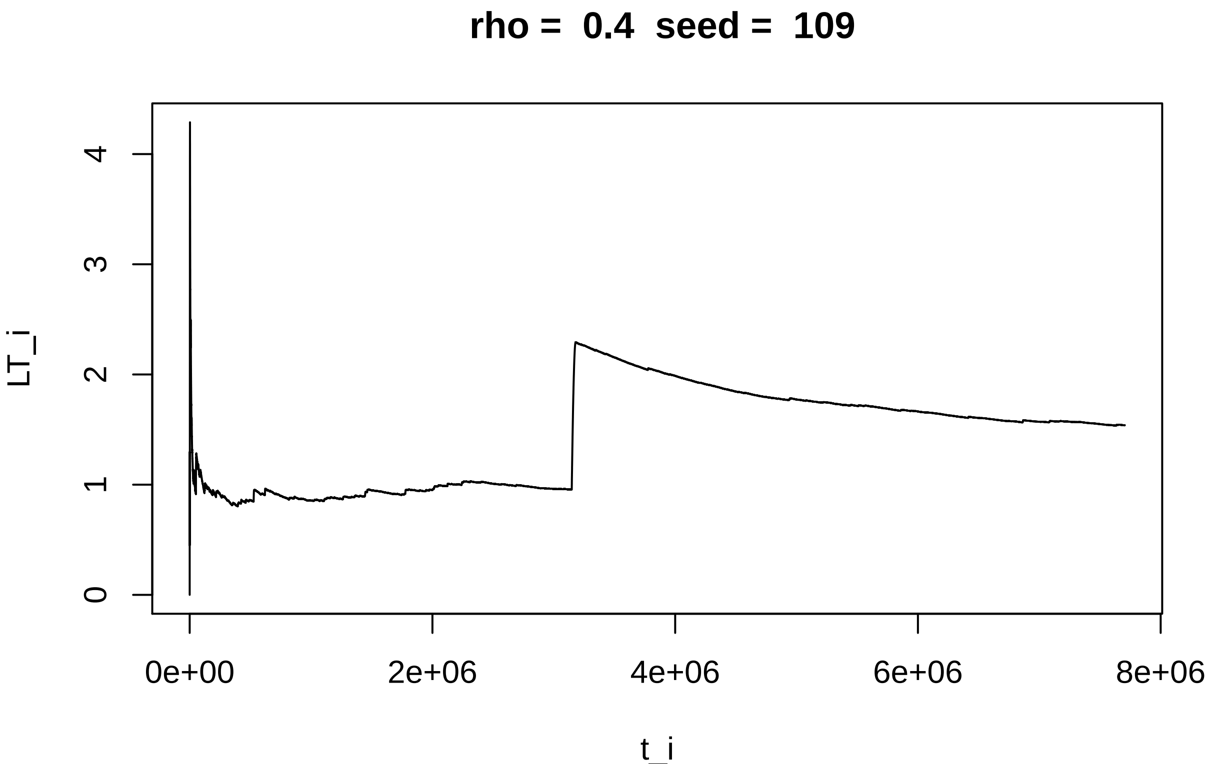
$\rho = \mathbf{0.4}$

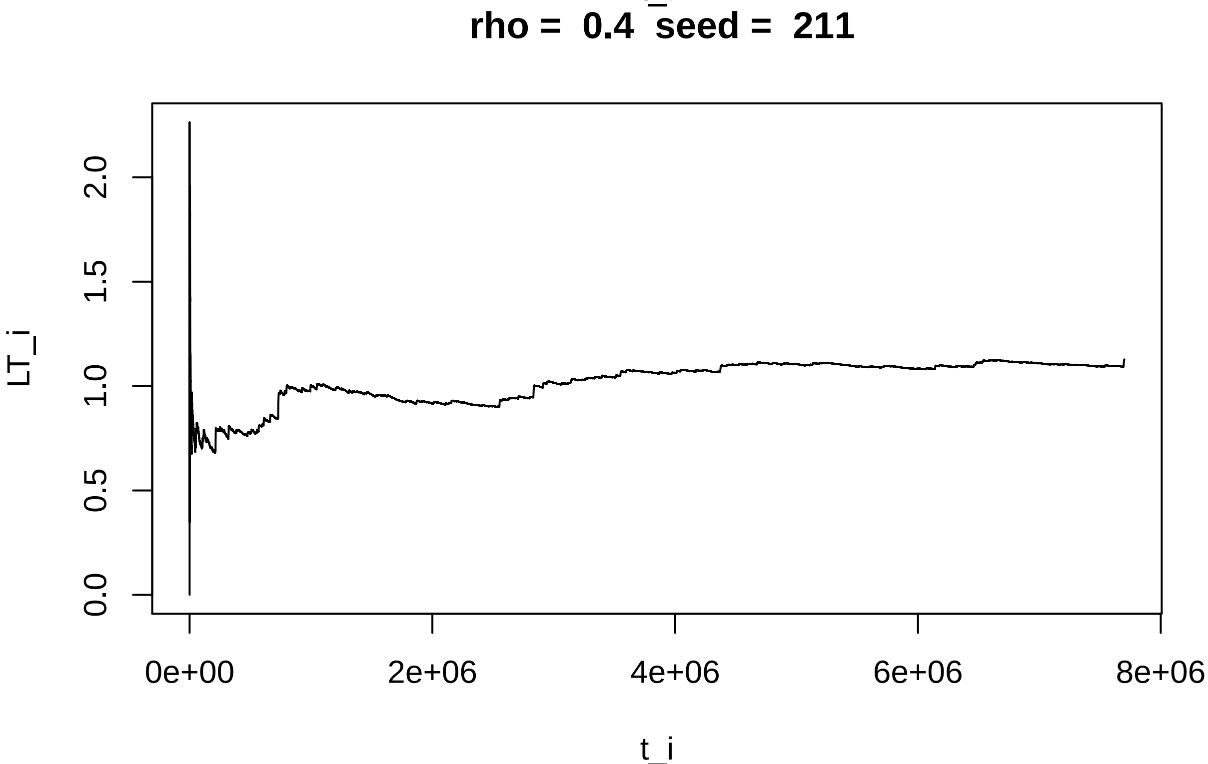
We generate 10 simulations with 100000 clients, each with a different seed. For each simulation, we check if the steady state is attained without any abrupt increase or decrease in the value of the average occupancy.

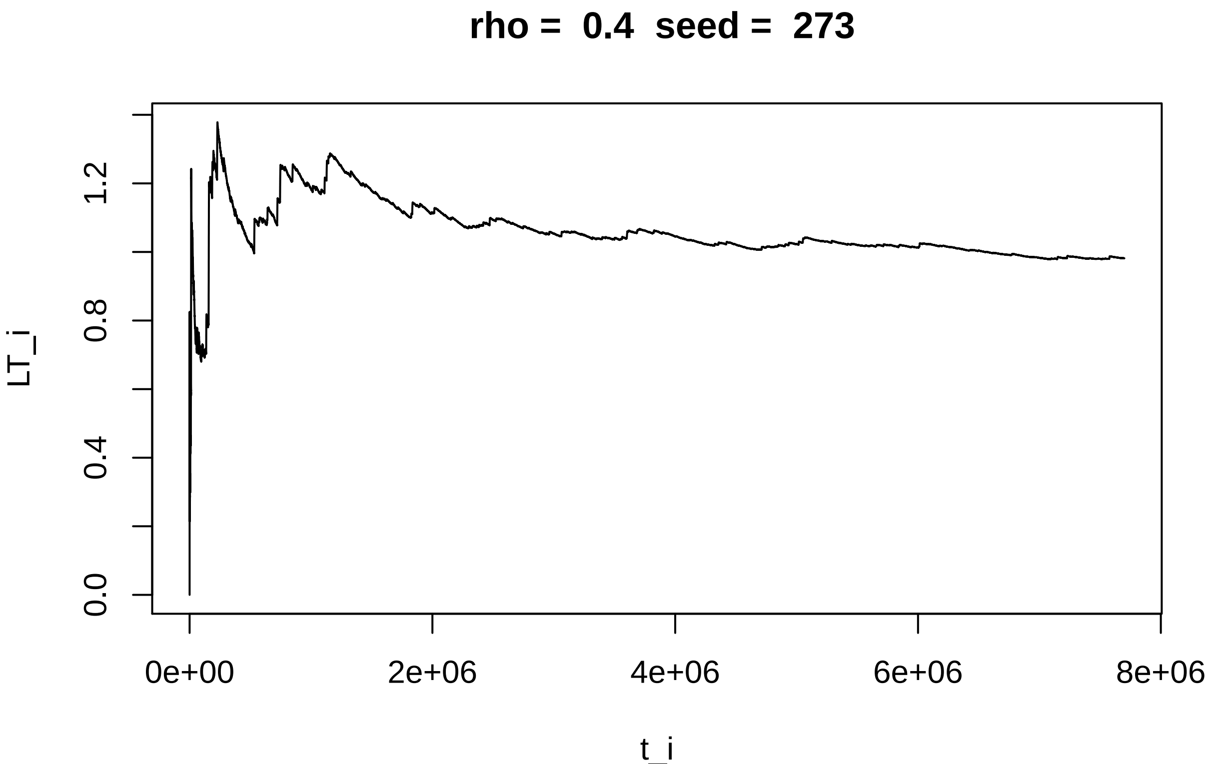
In case there's any abrupt increase or decrease, we change the seed. If for many seeds, this phenomena is still happening, we increase the number of clients.

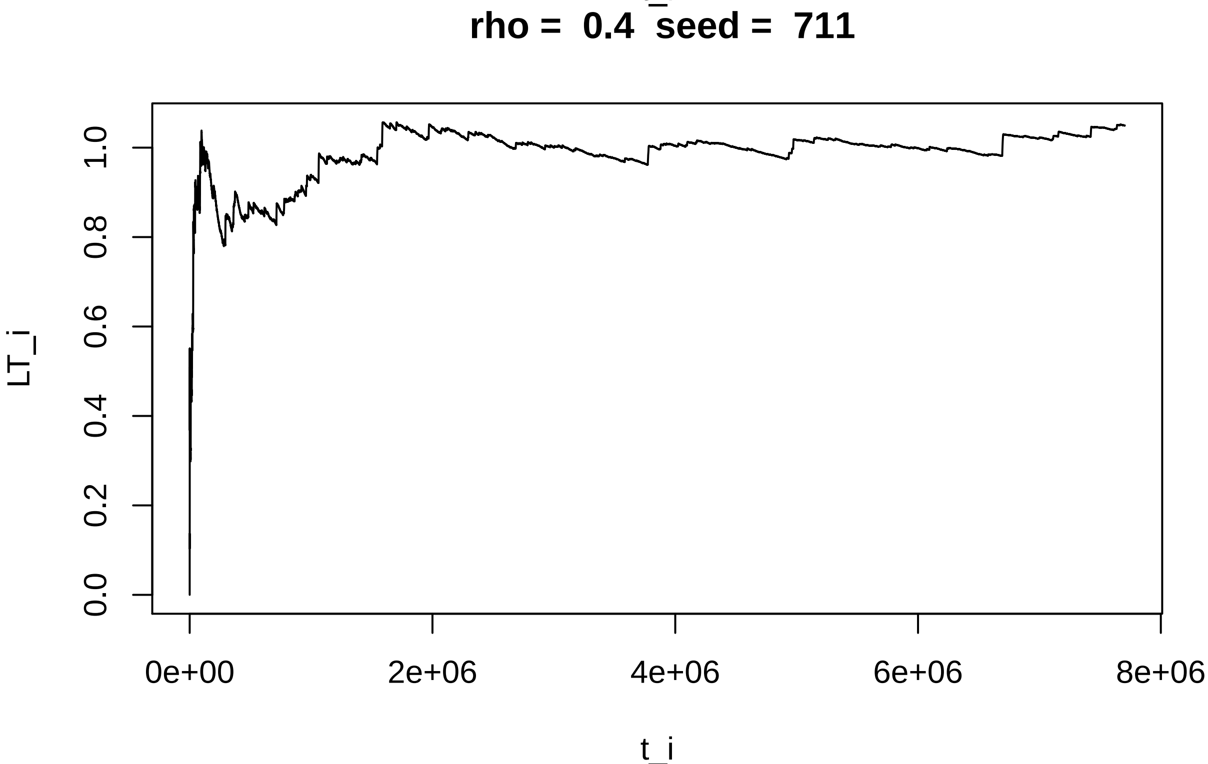





rho = 0.4 seed = 109
LT_i
0
1
2
3
4
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 211
LT_i
0.0
0.5
1.0
1.5
2.0
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 273
LT_i
0.0
0.4
0.8
1.2
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 711
LT_i
0.0
0.2
0.4
0.6
0.8
1.0
0e+00
2e+06
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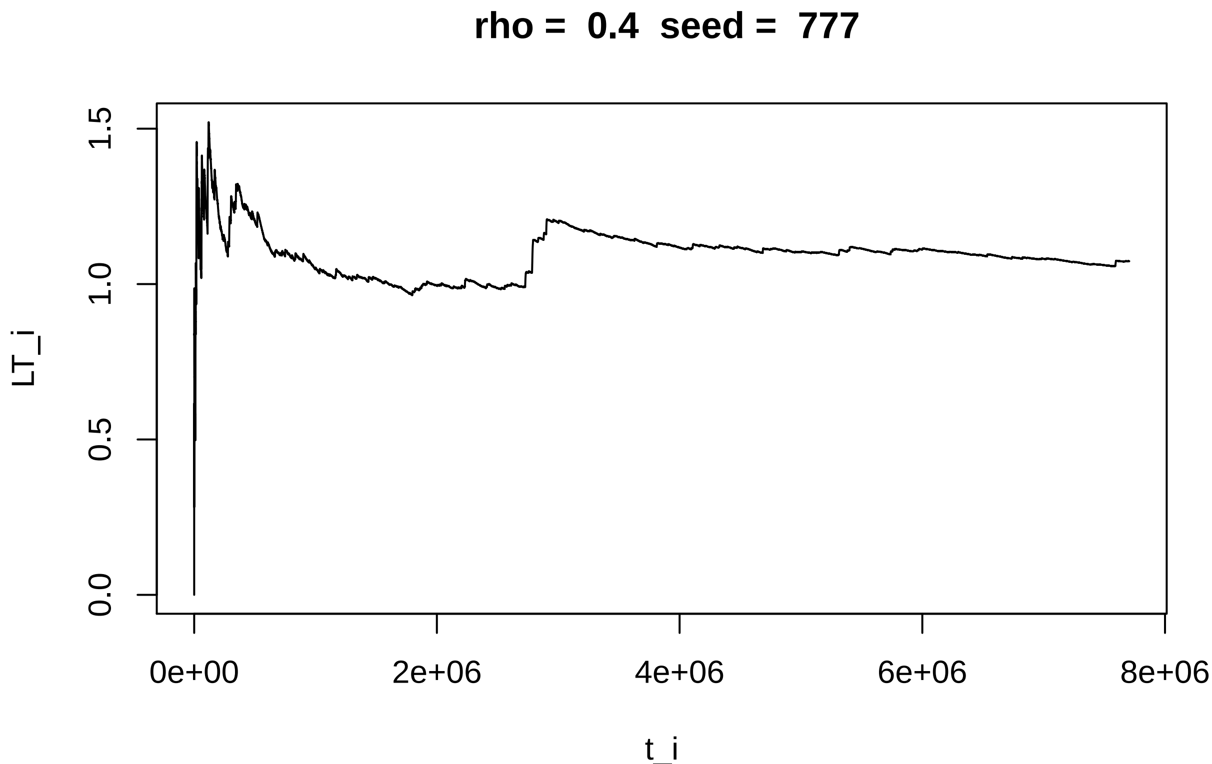
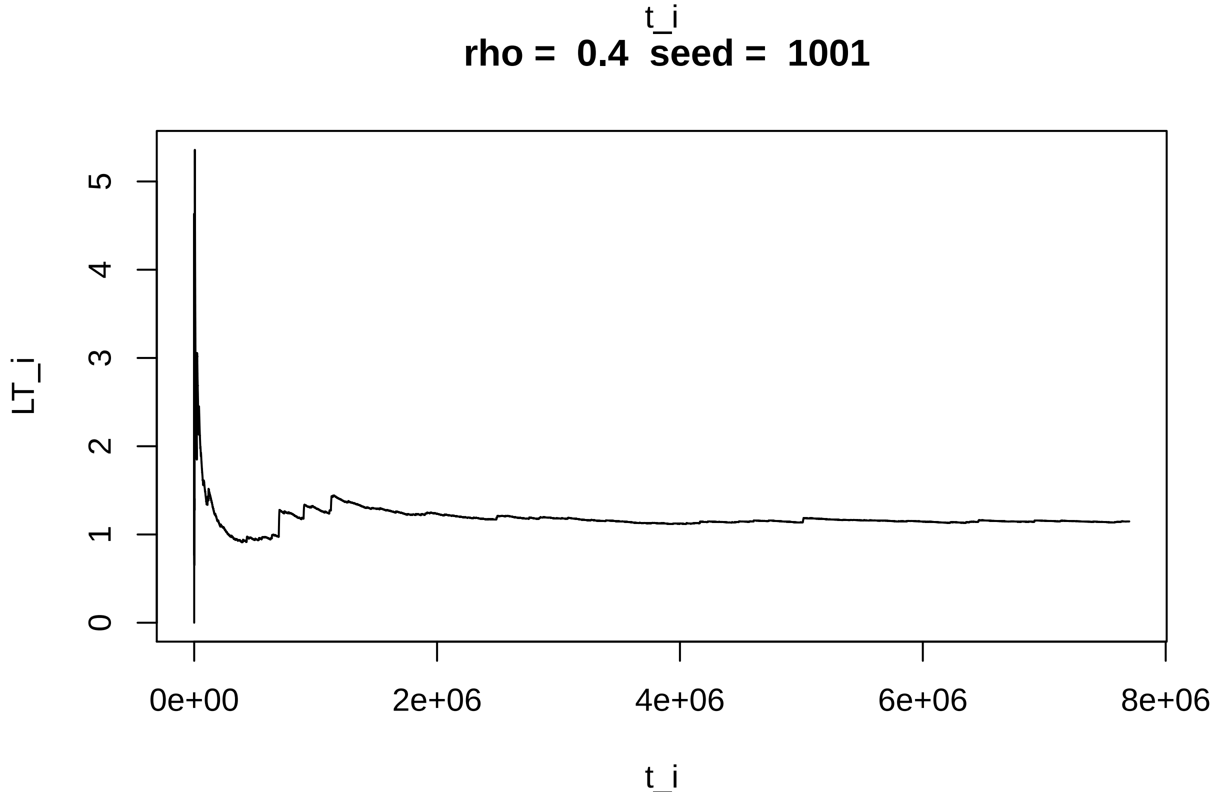
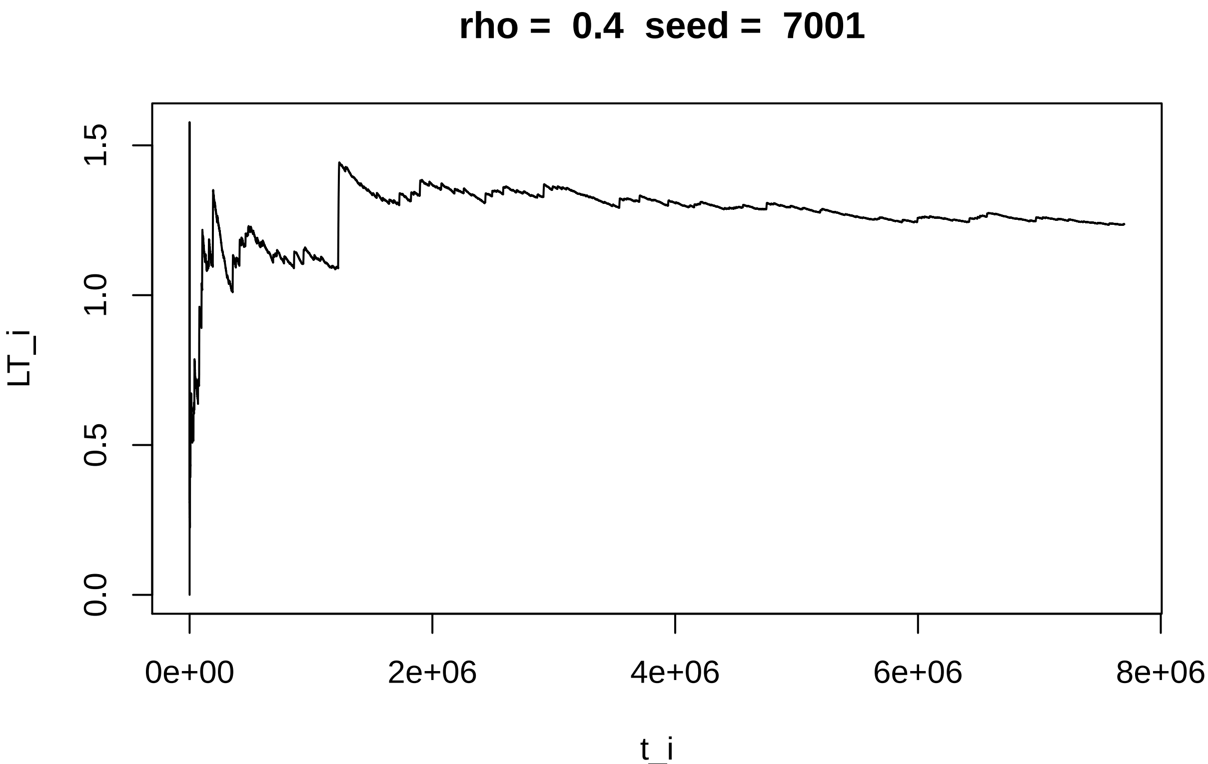
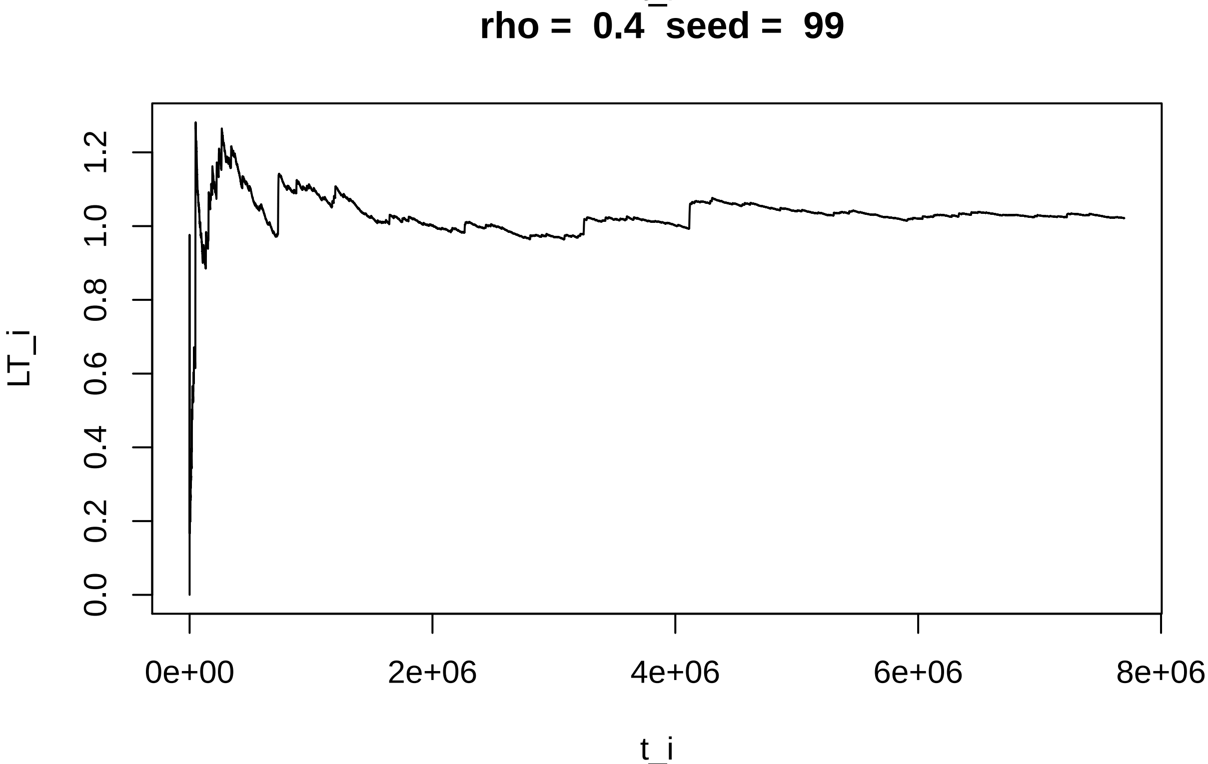
6e+06
8e+06
t_i

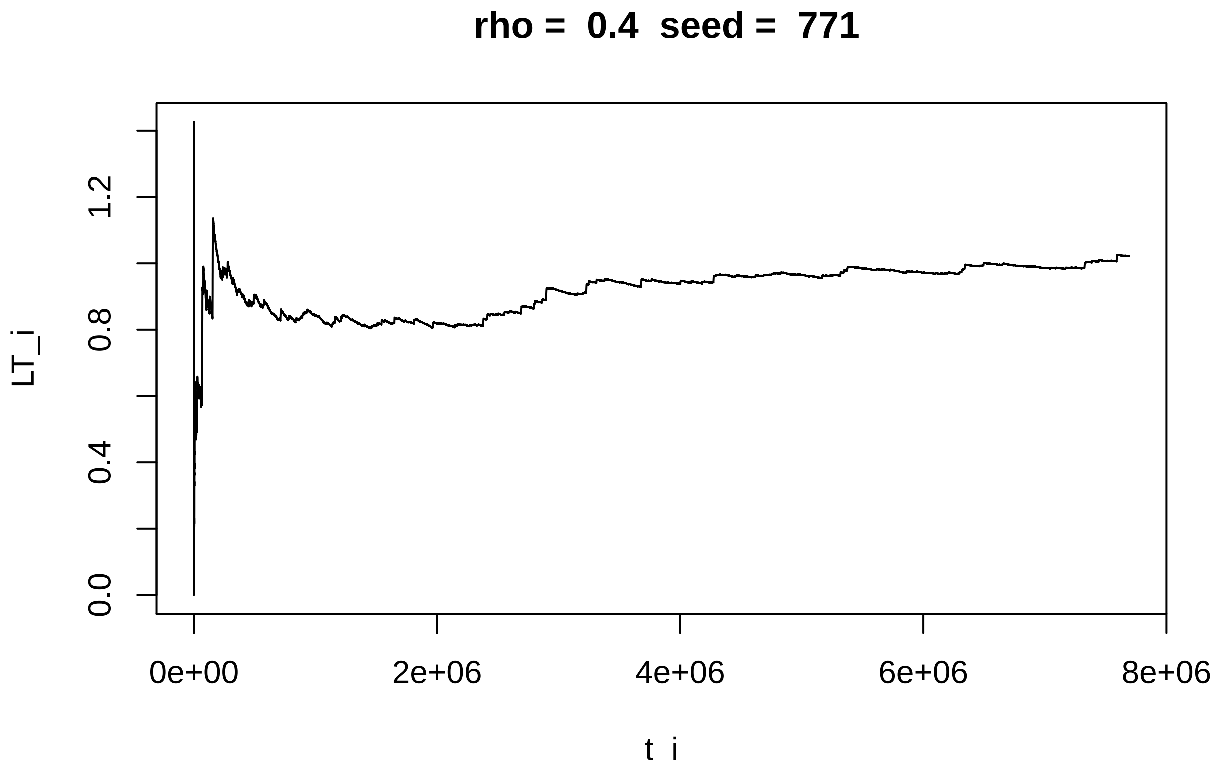

rho = 0.4 seed = 777
LT_i
t_i
0.0
0.5
1.0
1.5
0e+00
2e+06
4e+06
6e+06
8e+06

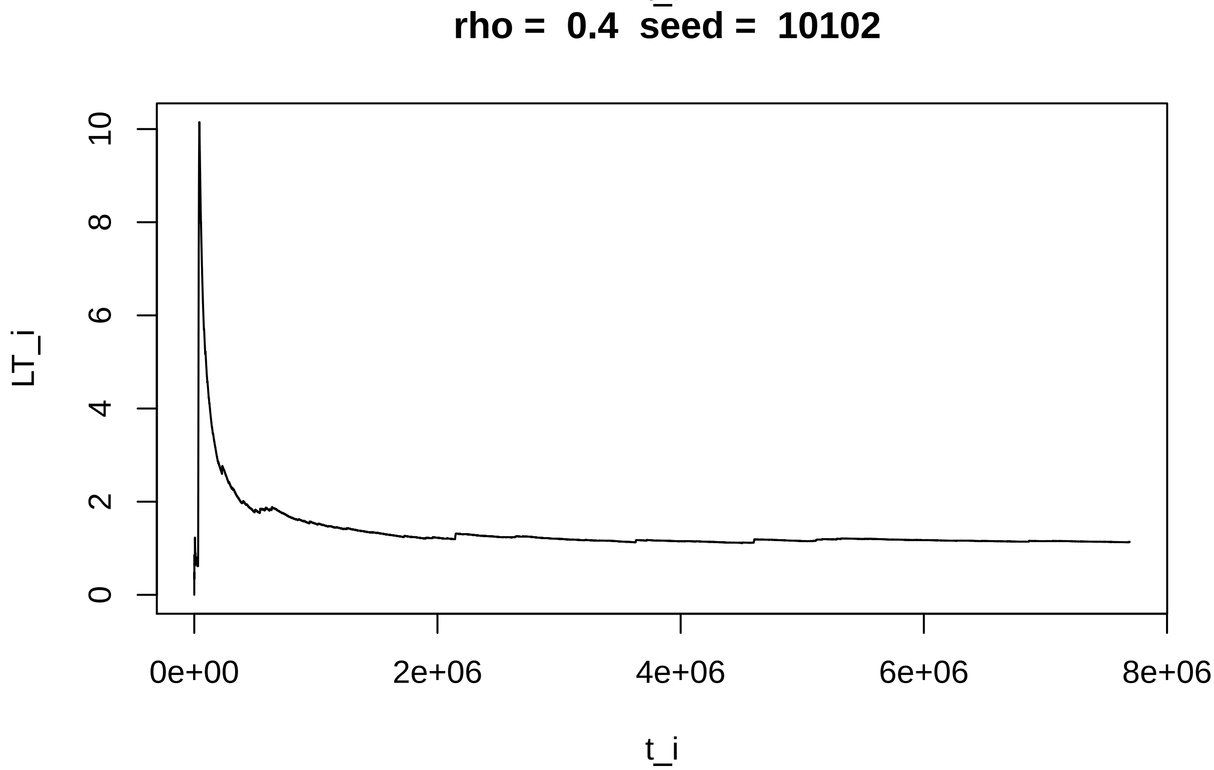

rho = 0.4 seed = 1001
LT_i
t_i
0
1
2
3
4
5
0e+00
2e+06
4e+06
6e+06
8e+06

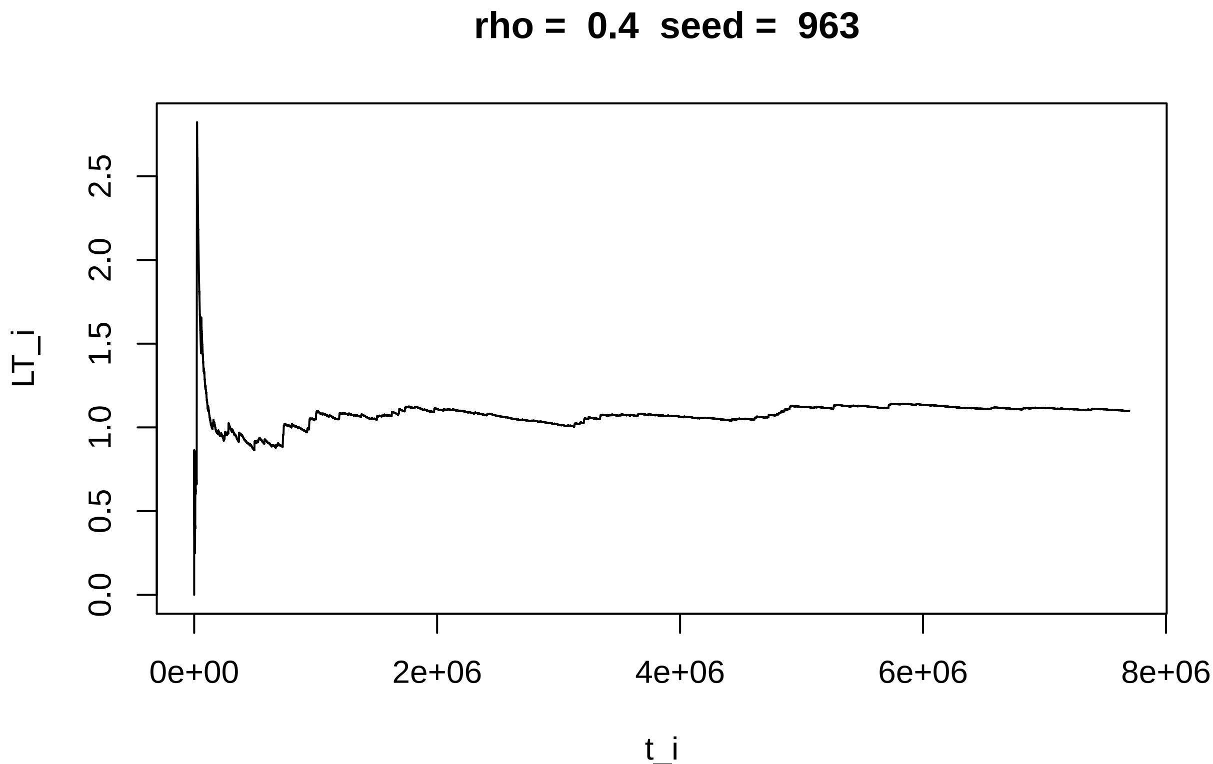


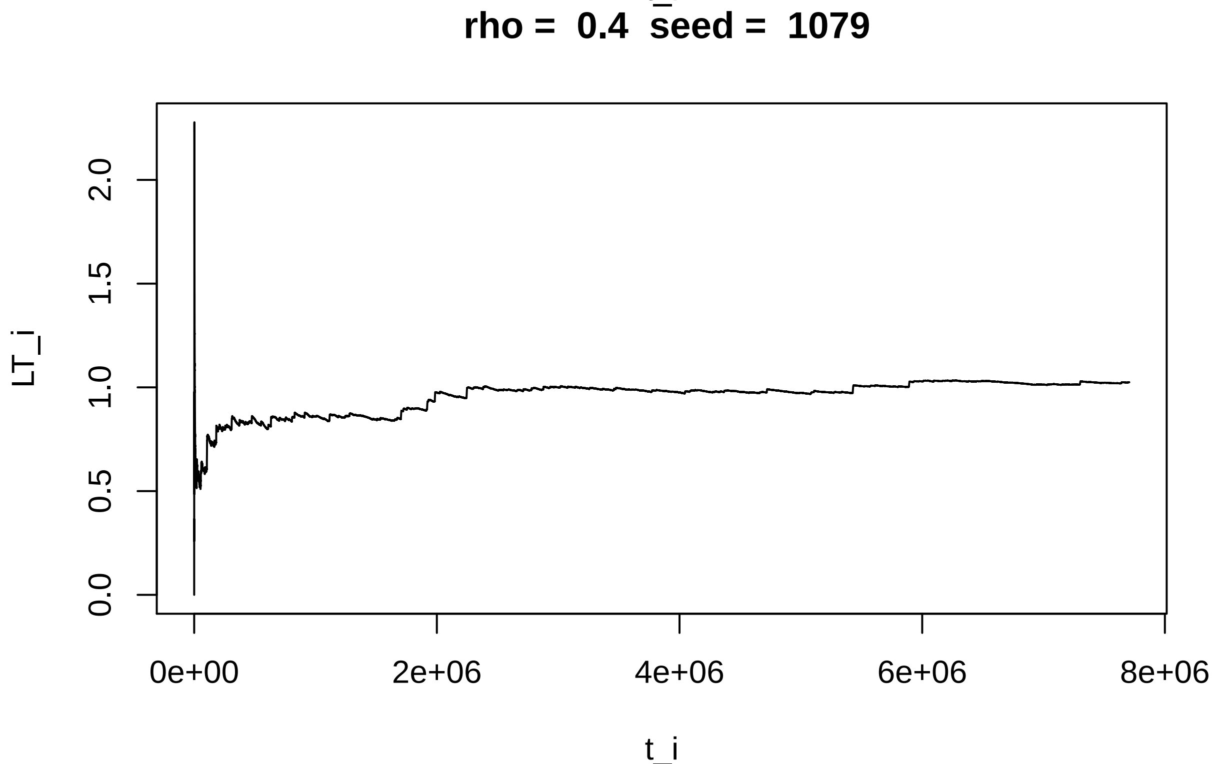


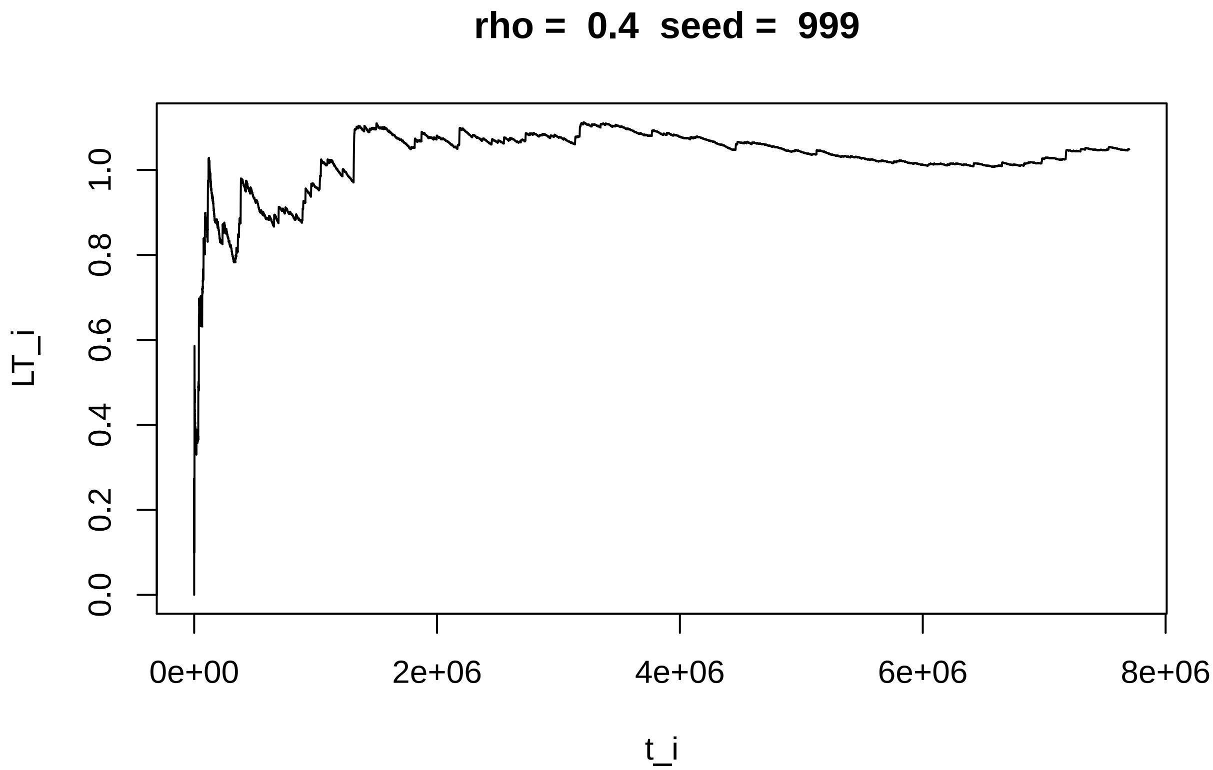
We observe that for the seeds 10101, 1078, 960 and 51, there's an abrupt change in the average occupancy. We change those seeds and redo the simulation.

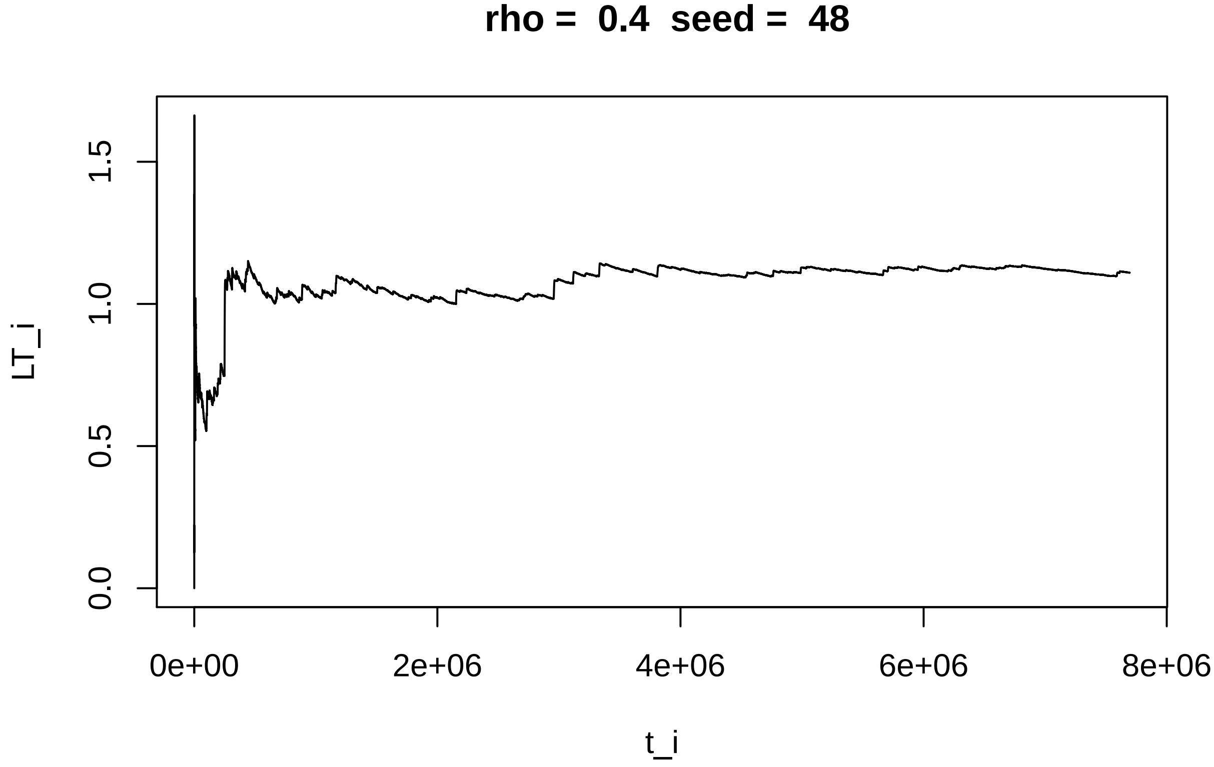

rho = 0.4 seed = 771
LT_i
0.0
0.4
0.8
1.2
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 10102
LT_i
0
2
4
6
8
10
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 963
LT_i
0.0
0.5
1.0
1.5
2.0
2.5
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 1079
LT_i
0.0
0.5
1.0
1.5
2.0
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 999
LT_i
0.0
0.2
0.4
0.6
0.8
1.0
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 48
LT_i
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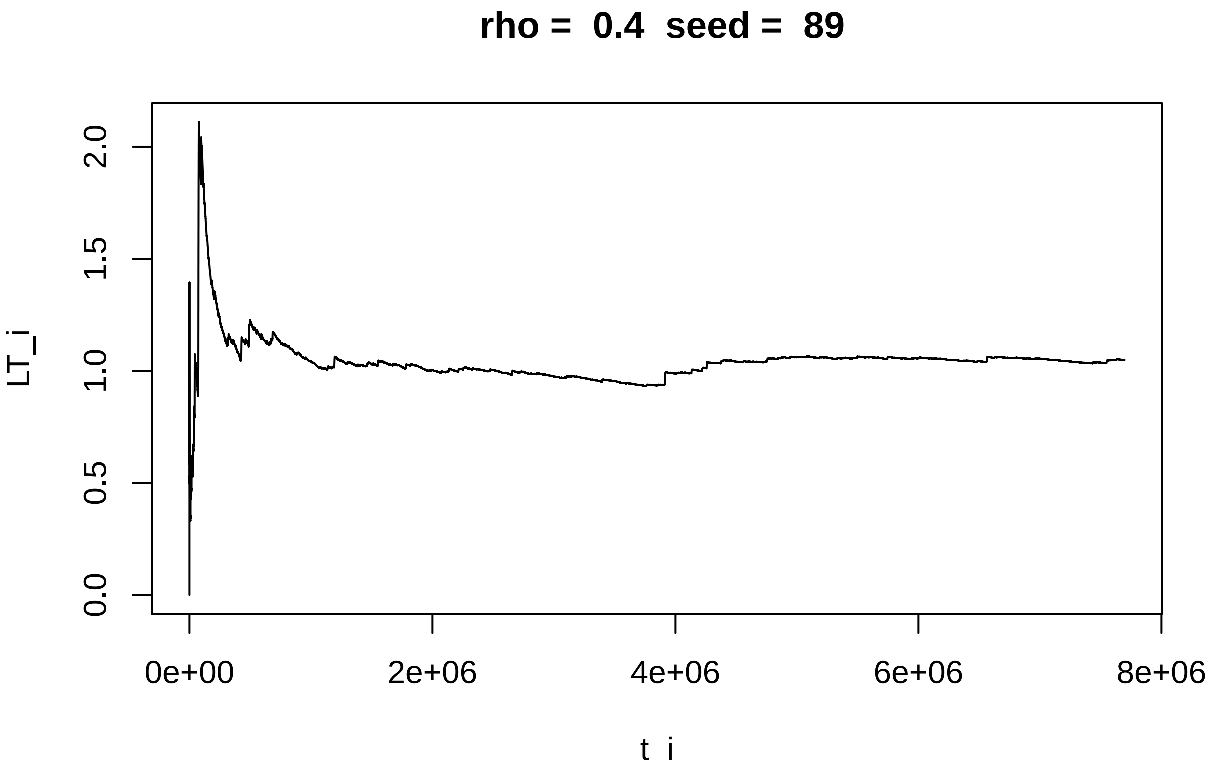
0.5
1.0
1.5
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.4 seed = 89
LT_i
t_i
0.0
0.5
1.0
1.5
2.0
0e+00
2e+06
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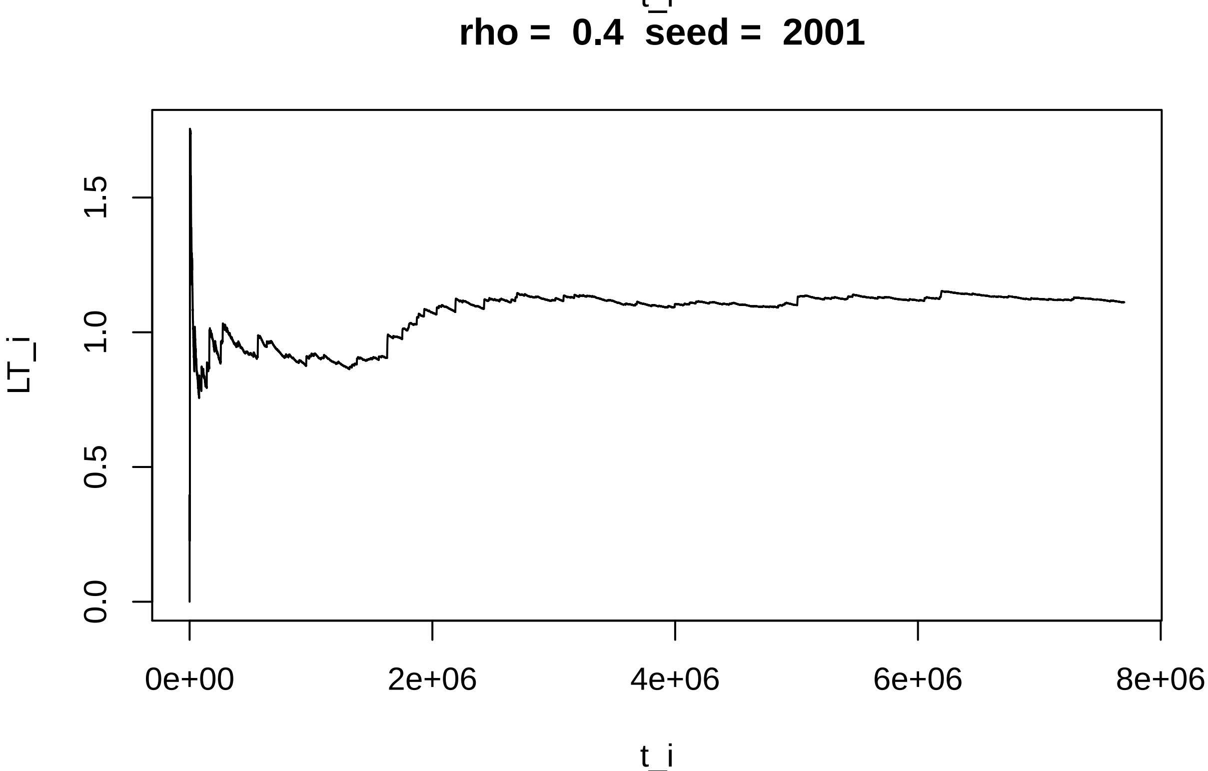
6e+06
8e+06


rho = 0.4 seed = 2001
LT_i
t_i
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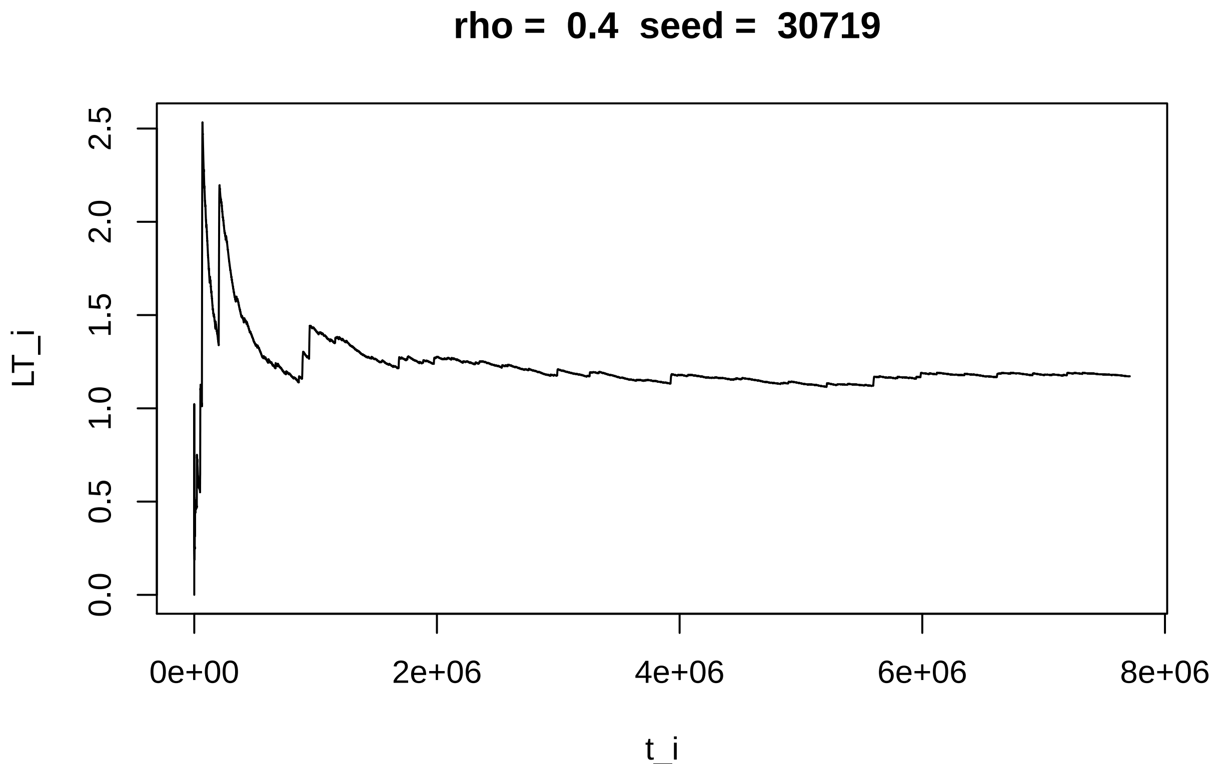
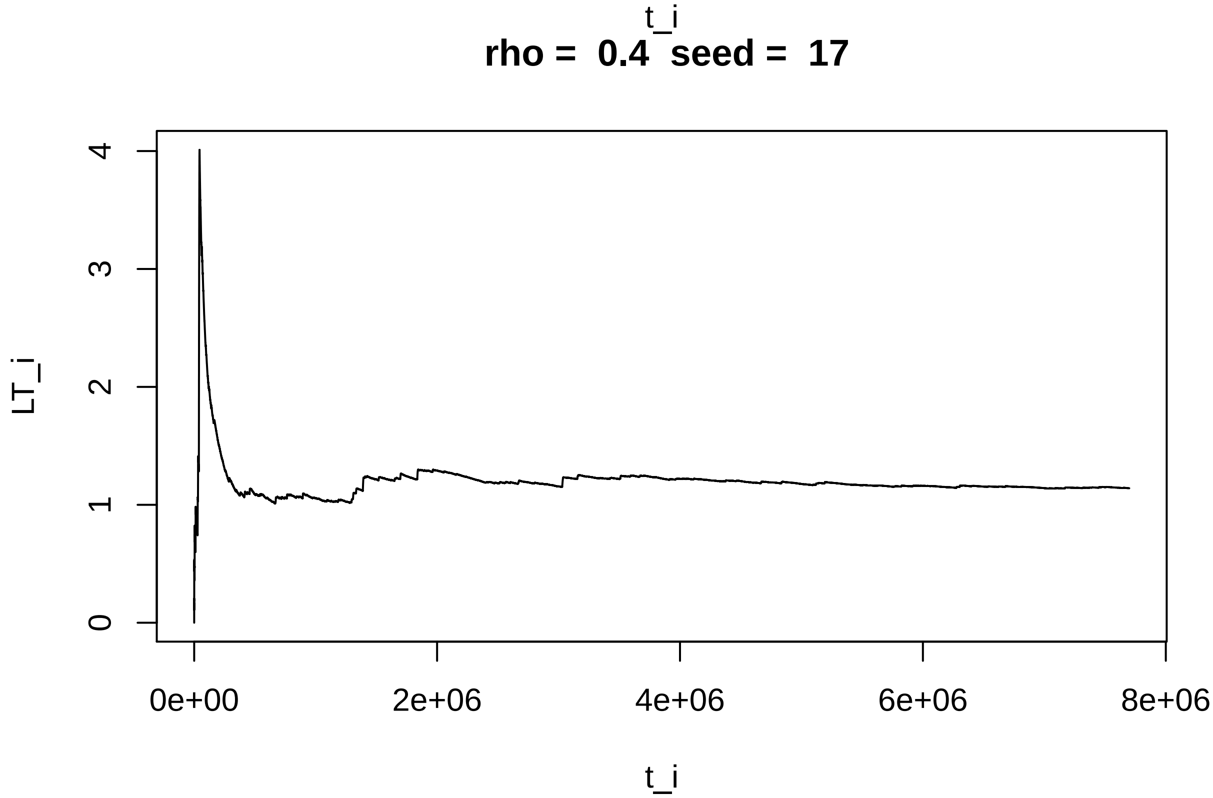
0.5
1.0
1.5
0e+00
2e+06
4e+06
6e+06
8e+06





We changed a total of 4 out of 10 seeds.

We now compute the confidence interval for L_q and W_q. We will use a t-Student distribution with critical value 1 - 0.95 and 9 degrees of freedom to compute a 95% confidence interval.

$$L_{q_1}, L_{q_2}, ..., L_{q_{10}}$$

$$\overline{L}_q = \frac{1}{n}\sum_{i=1}^{n} L_{q_i}$$

$$S^2_{L_q} = \frac{1}{n-1}\sum_{i=1}^{n}(L_{q_i} - \overline{L}_q)^2$$

$$C.I.(L_q) = \overline{L}_q \pm t_{1-\alpha,n-1} \cdot \sqrt{\frac{S^2_{L_q}}{n}}$$

$$W_{q_1}, W_{q_2}, ..., W_{q_{10}}$$

$$\overline{W}_q = \frac{1}{n}\sum_{i=1}^{n} W_{q_i}$$

$$S^2_{W_q} = \frac{1}{n-1}\sum_{i=1}^{n}(W_{q_i} - \overline{W}_q)^2$$

$$C.I.(W_q) = \overline{W}_q \pm t_{1-\alpha,n-1} \cdot \sqrt{\frac{S^2_{W_q}}{n}}$$

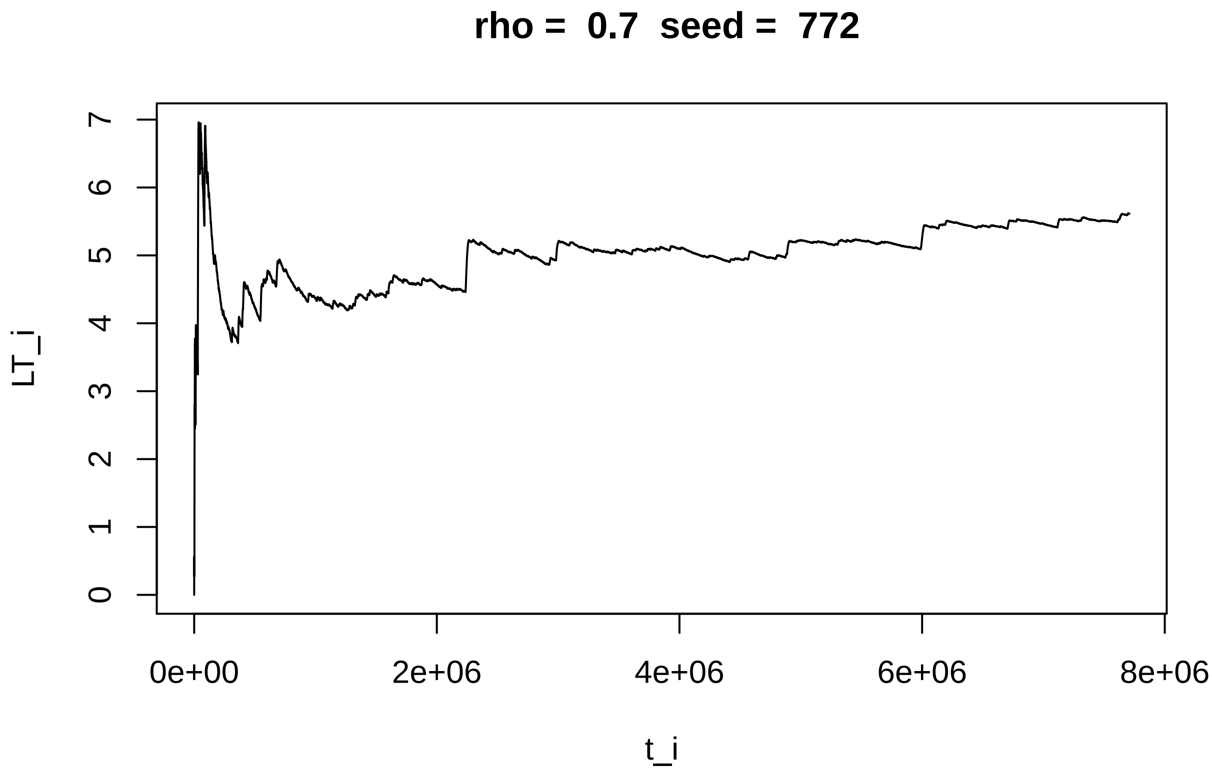
The computations produce the followings confidence intervals for the average queue length and waiting time:

```
##    NA        NA        NA       NA      NA
## 1 0.4 0.6629266 0.7216626 51.03537 55.5688
```

ρ	-C.I W_q	+C.I W_q	-C.I L_q	+C.I L_q
0.4	0.6629266	0.7216626	51.0353731	55.5687961

ρ = 0.7

We generate 10 simulations with 100000 clients, each with a different seed. We check if the steady state is attained. We also check for irregularities in the simulation results, in which case we change the random number generator seed and repeat the simulation.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3
4
5
6
7
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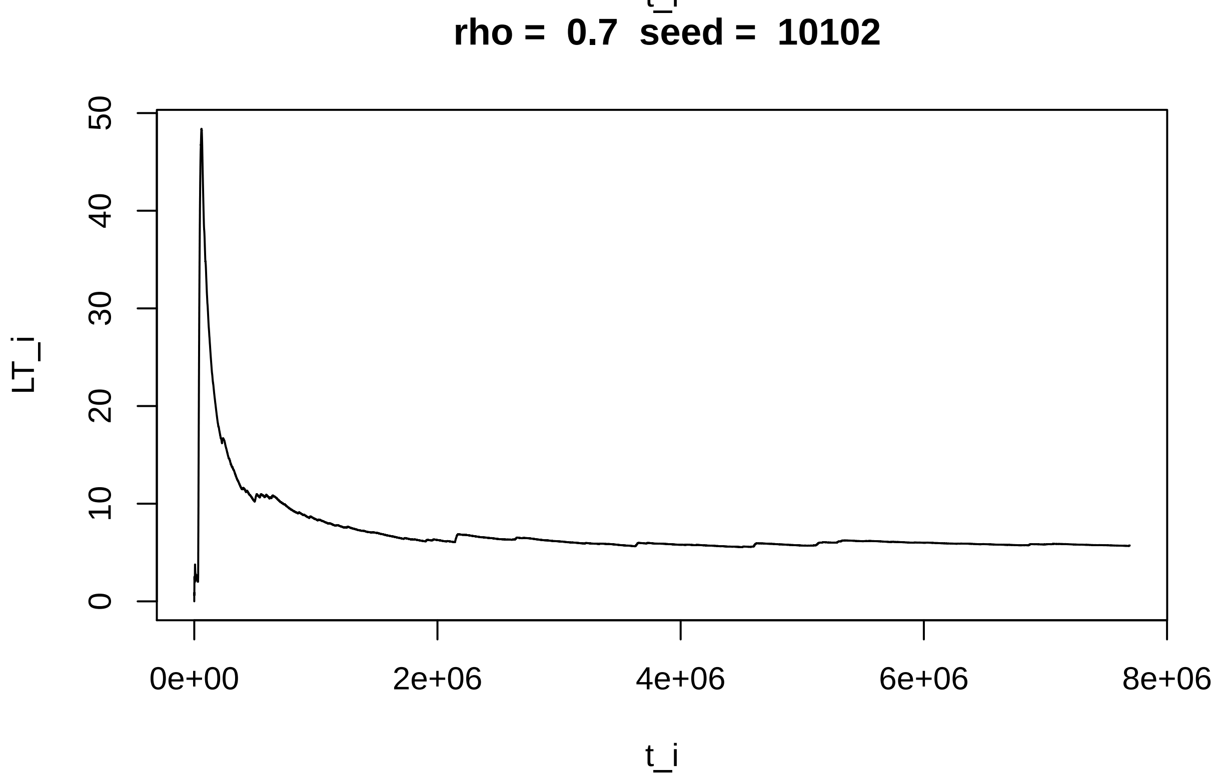
4e+06
6e+06
8e+06
t_i


rho = 0.7 seed = 10102
LT_i
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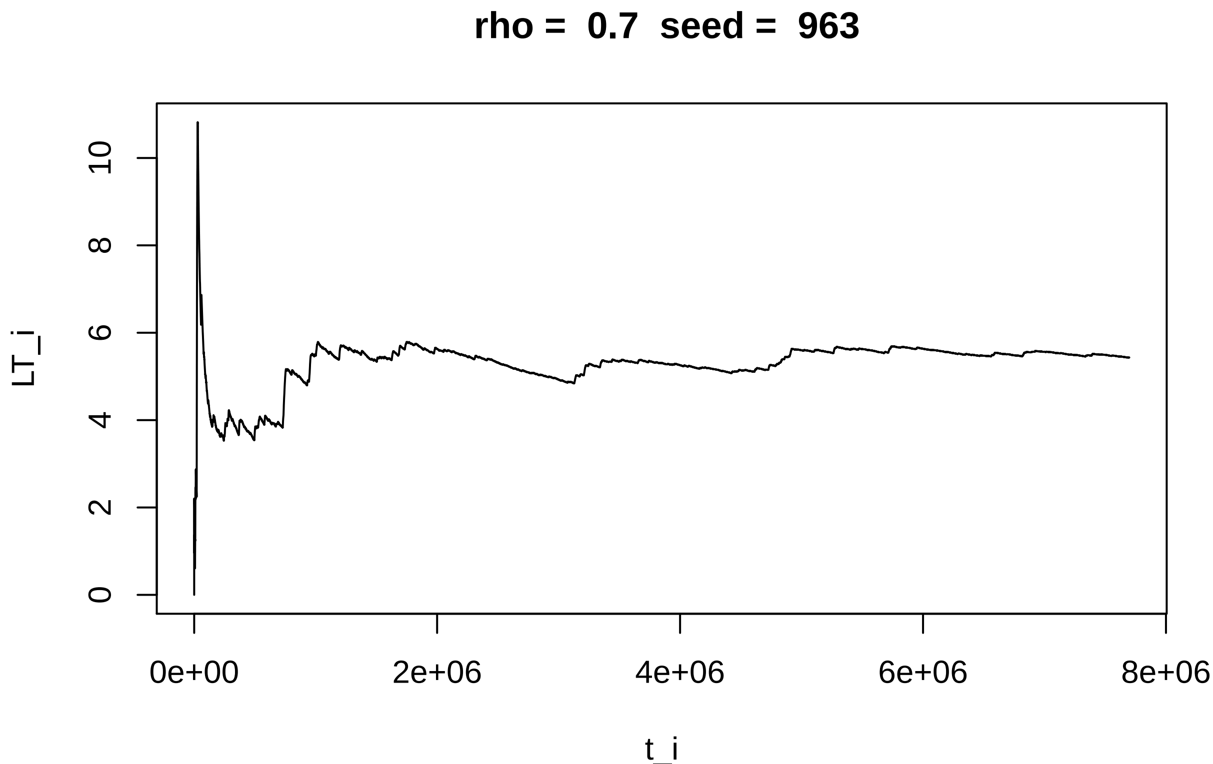
10
20
30
40
50
0e+00
2e+06
4e+06
6e+06
8e+06
t_i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LT_i
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2
4
6
8
10
0e+00
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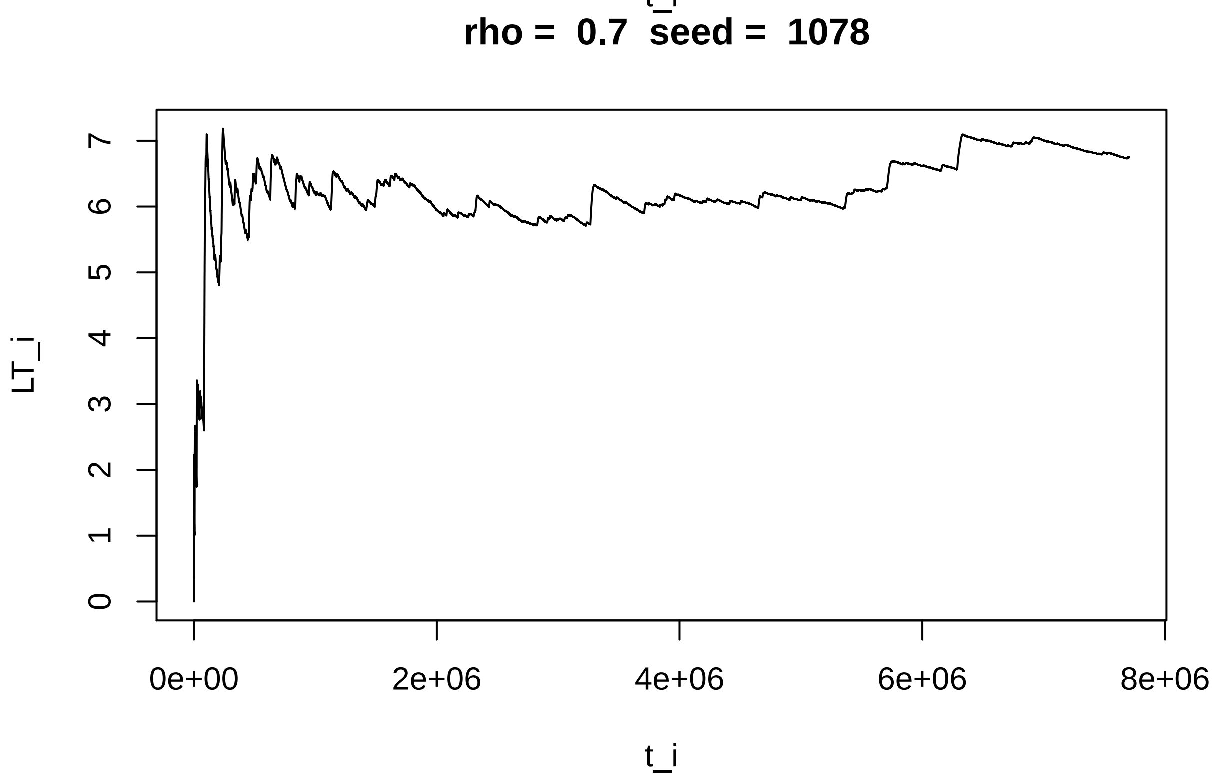
4e+06
6e+06
8e+06
t_i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LT_i
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5
6
7
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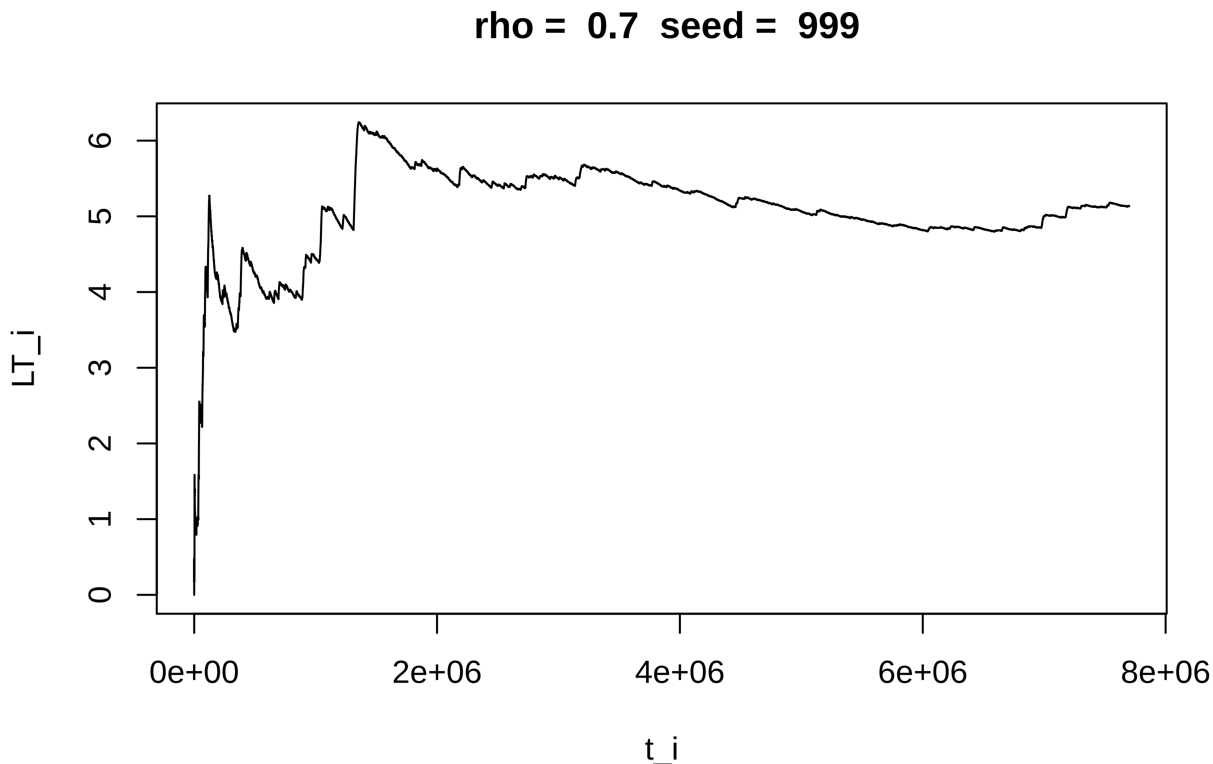
4e+06
6e+06
8e+06
t_i


rho = 0.7 seed = 999
LT_i
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3
4
5
6
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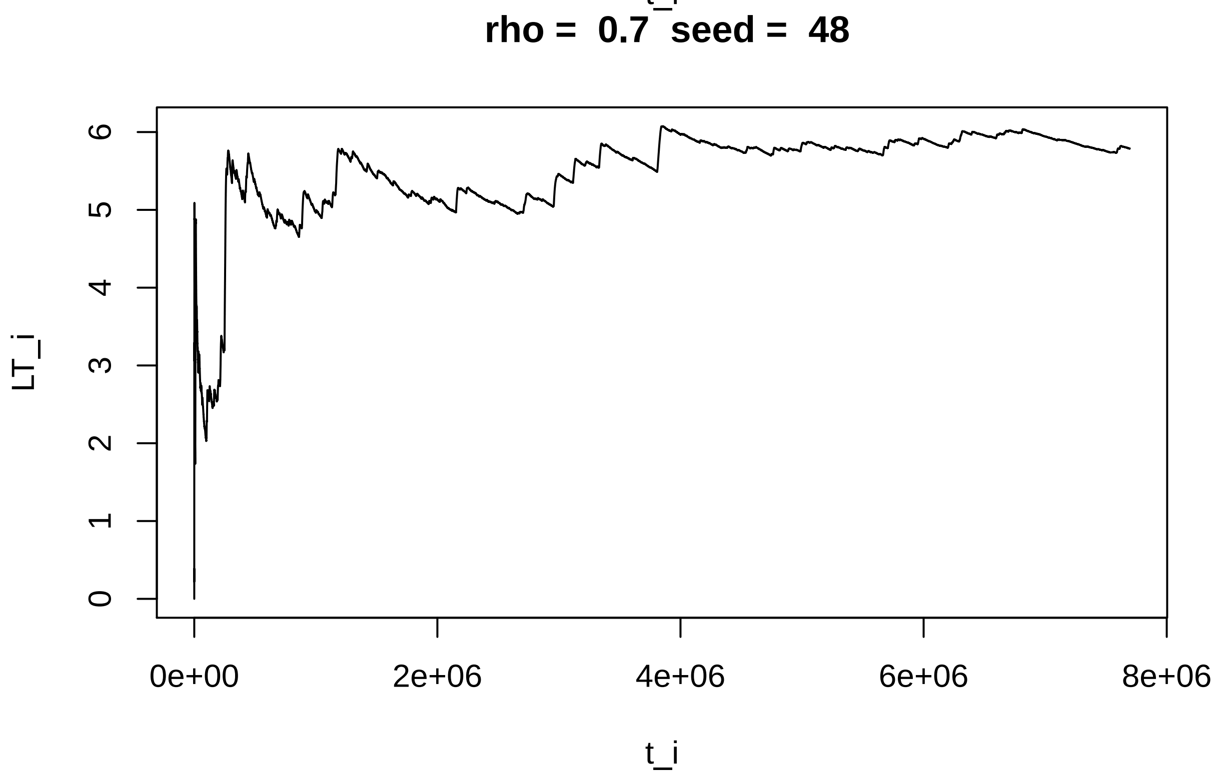
4e+06
6e+06
8e+06
t_i


rho = 0.7 seed = 48
LT_i
0
1
2
3
4
5
6
0e+00
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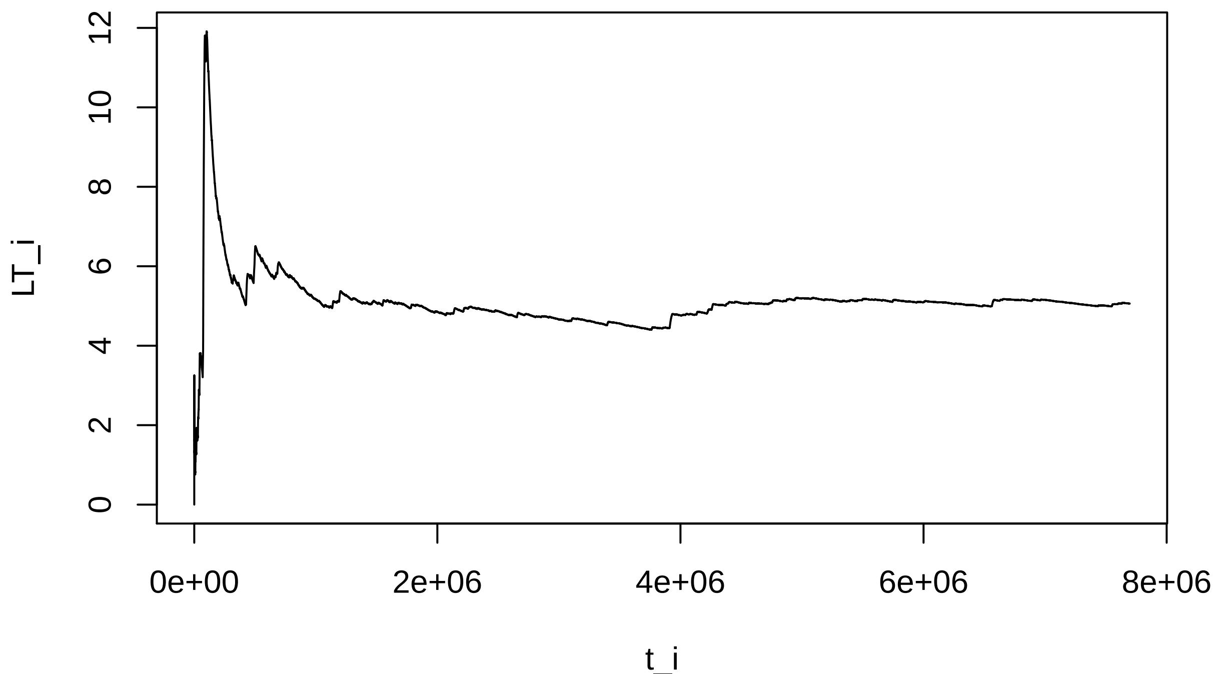
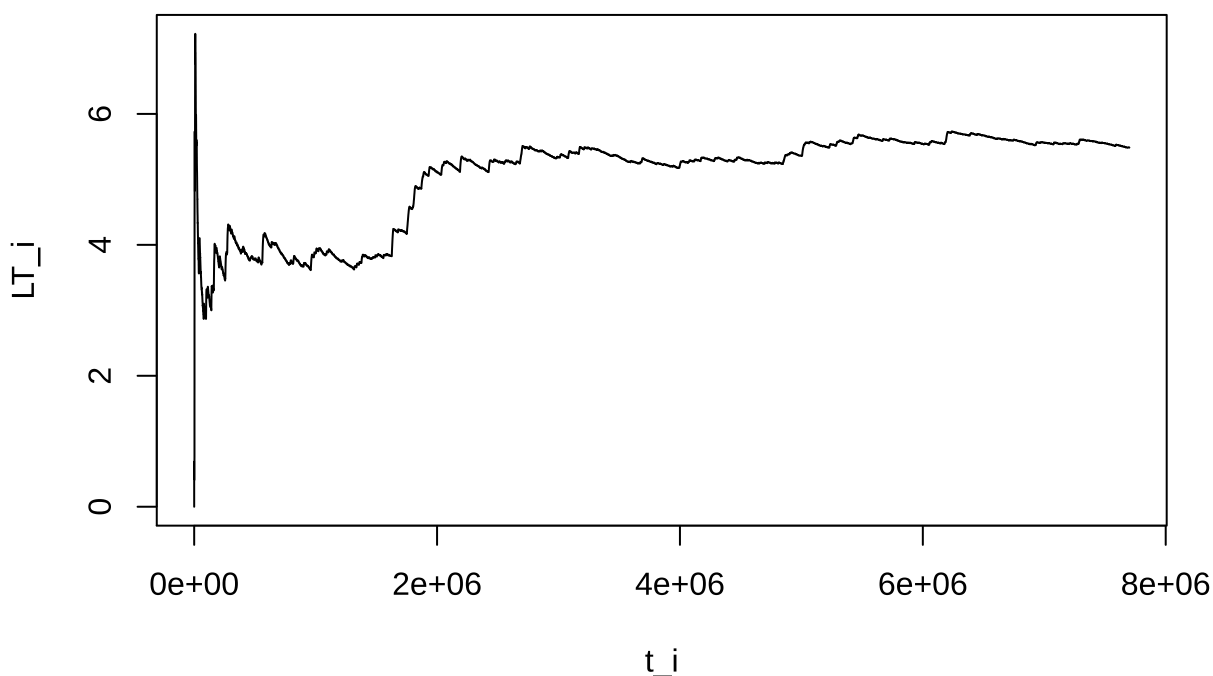
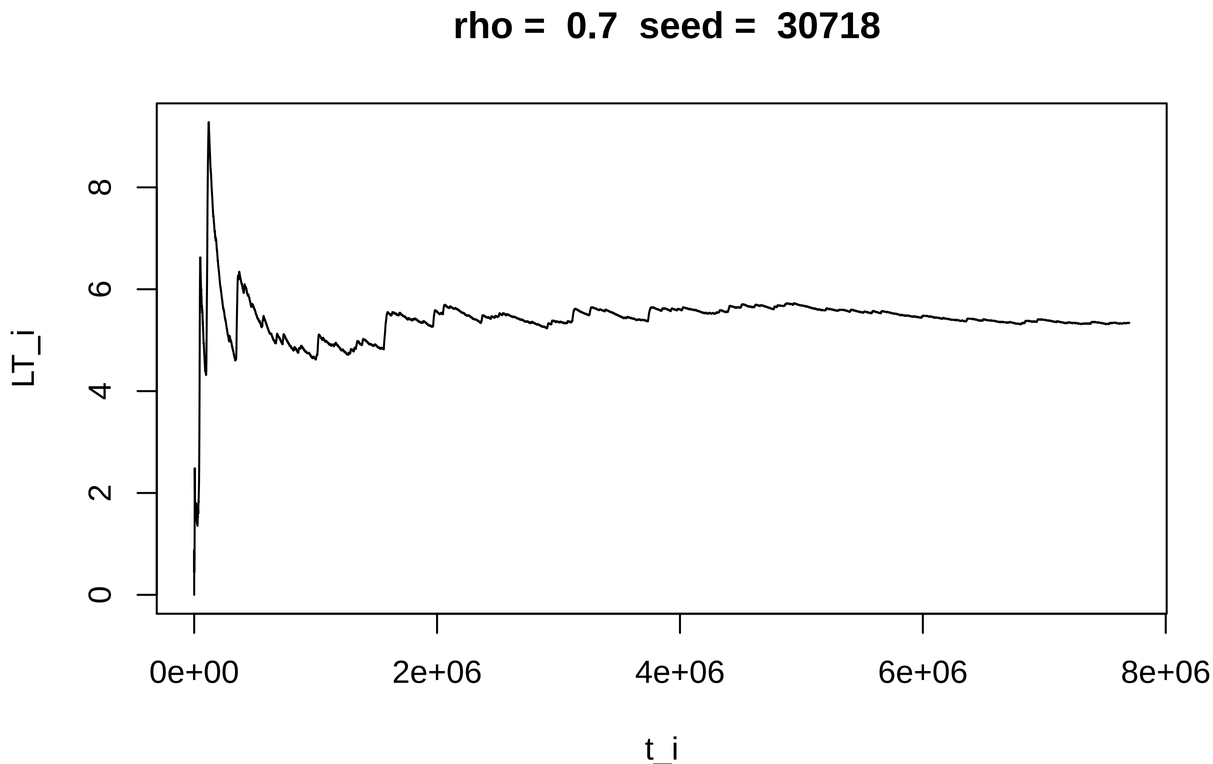
4e+06
6e+06
8e+06
t_i

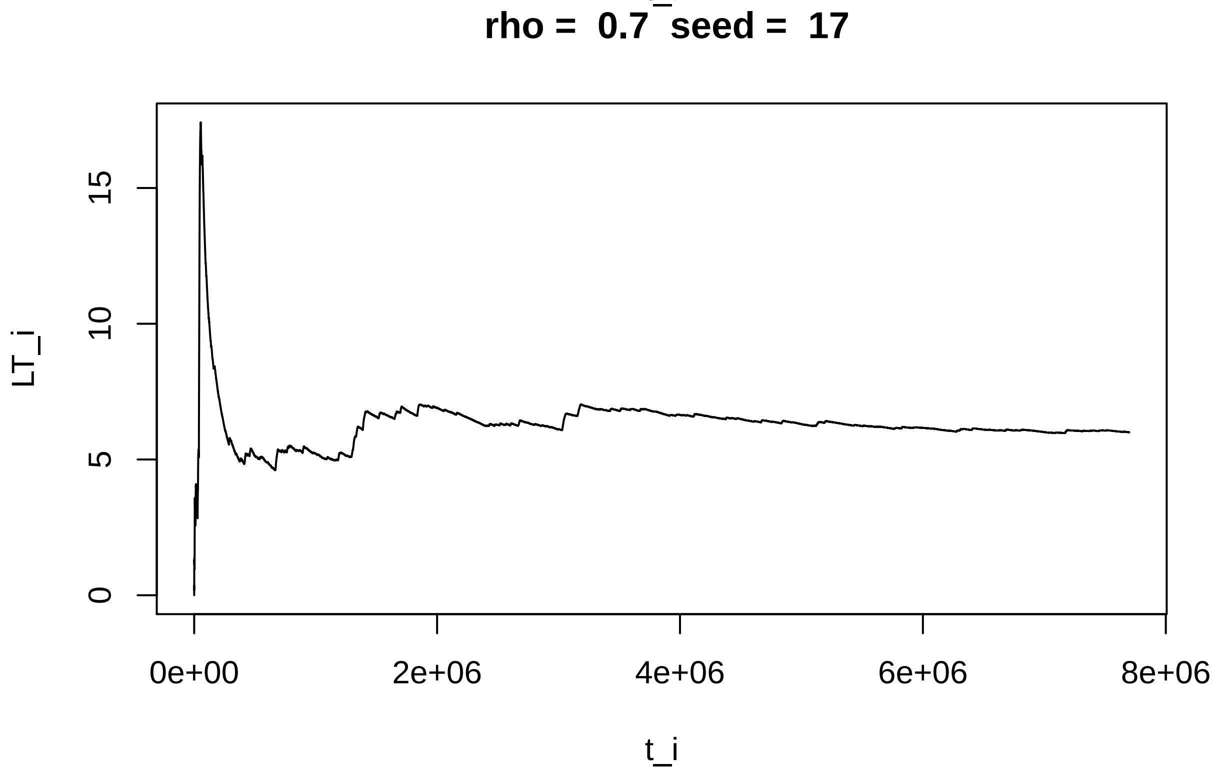
rho = 0.7 seed = 89



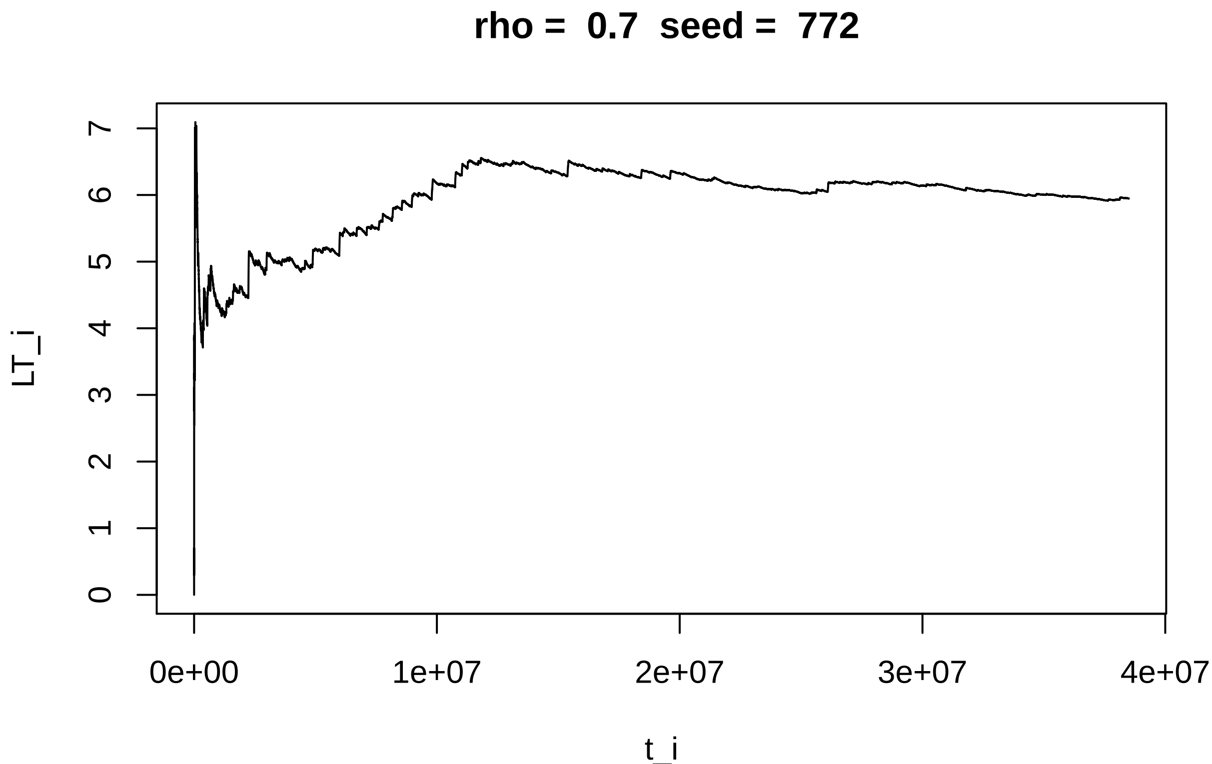
rho = 0.7 seed = 2001

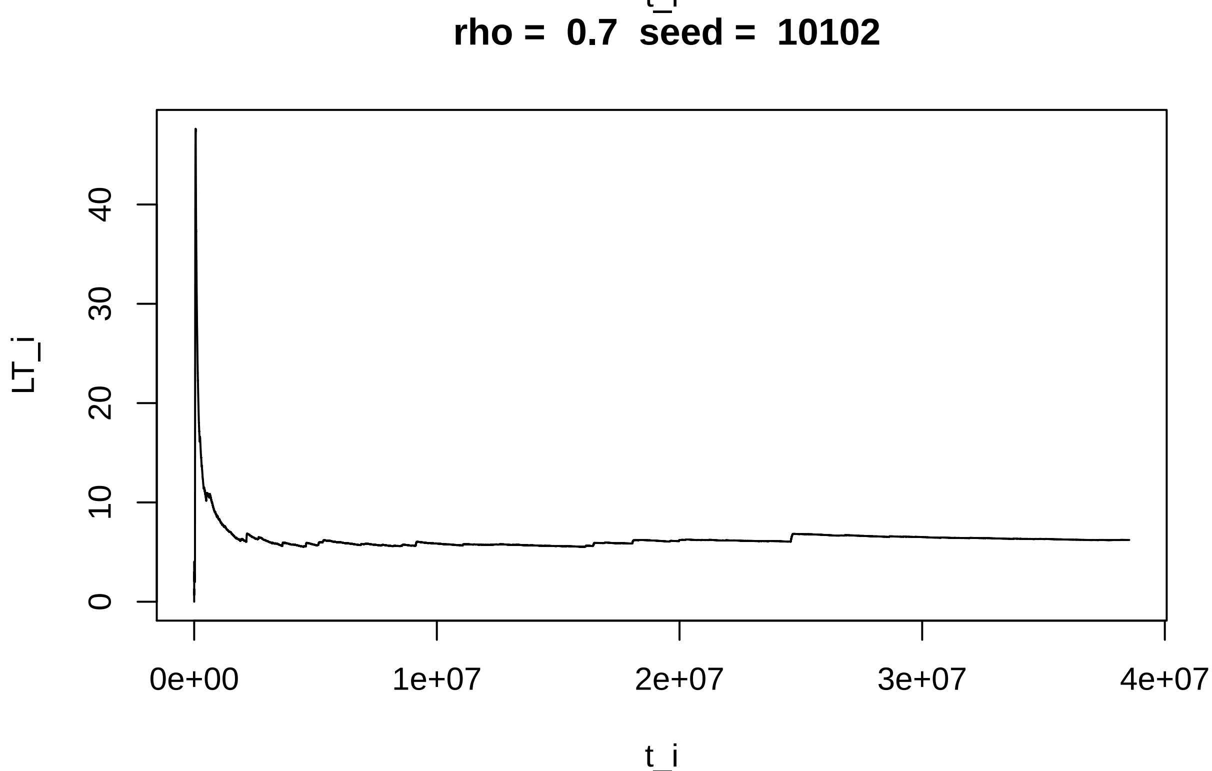


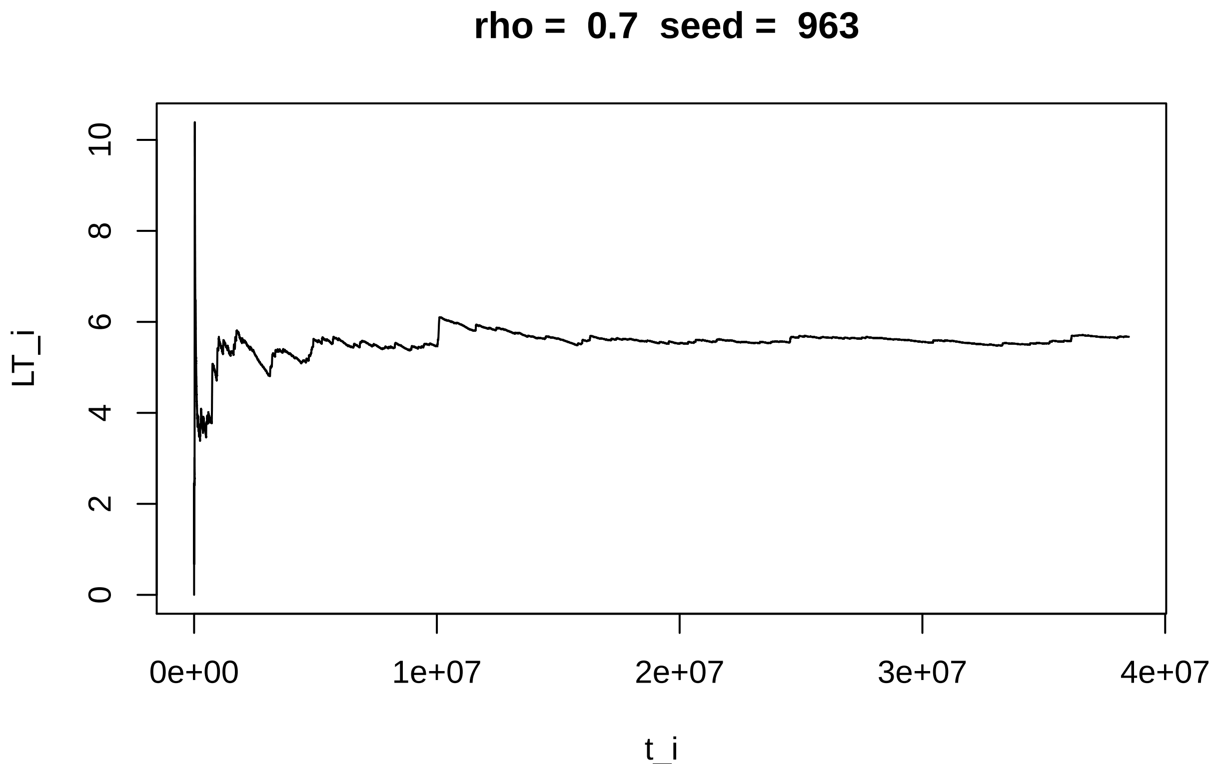


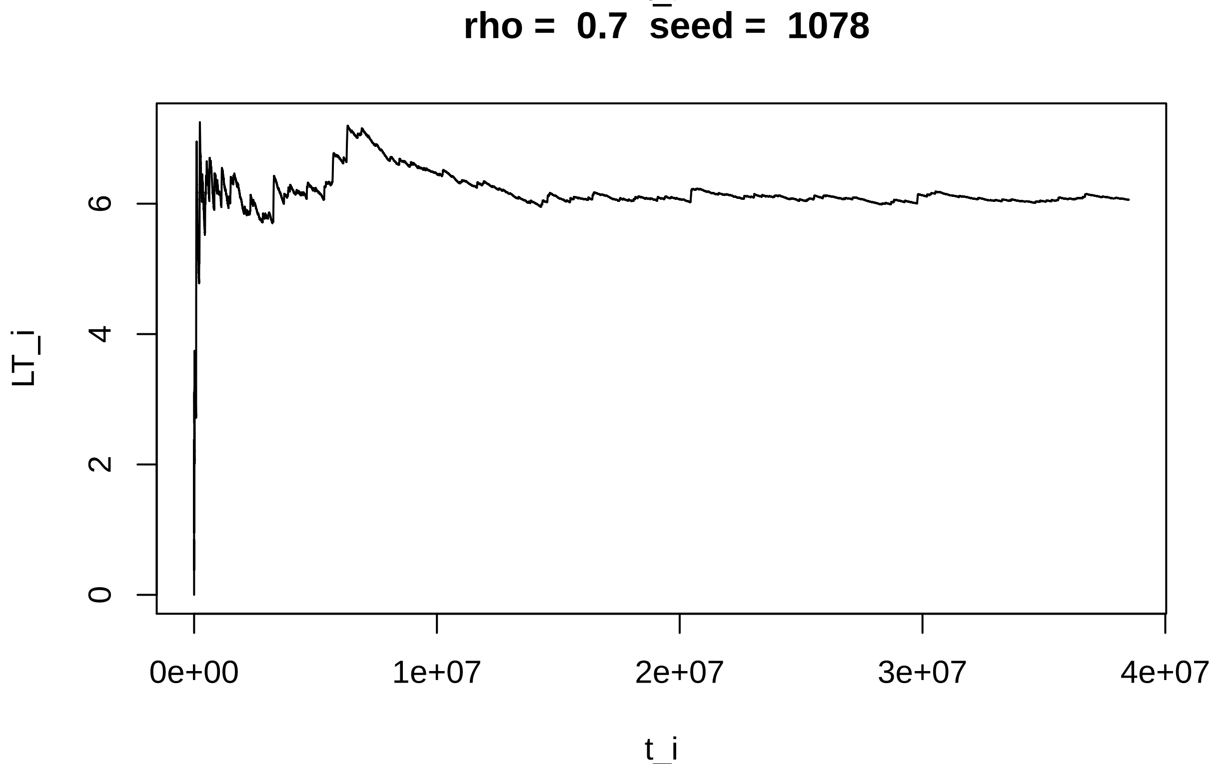


We observe that 5 out of 10 simulations are not in a steady state at the end of the simulation. We increase the number of clients to 500000 and repeat the simulations.

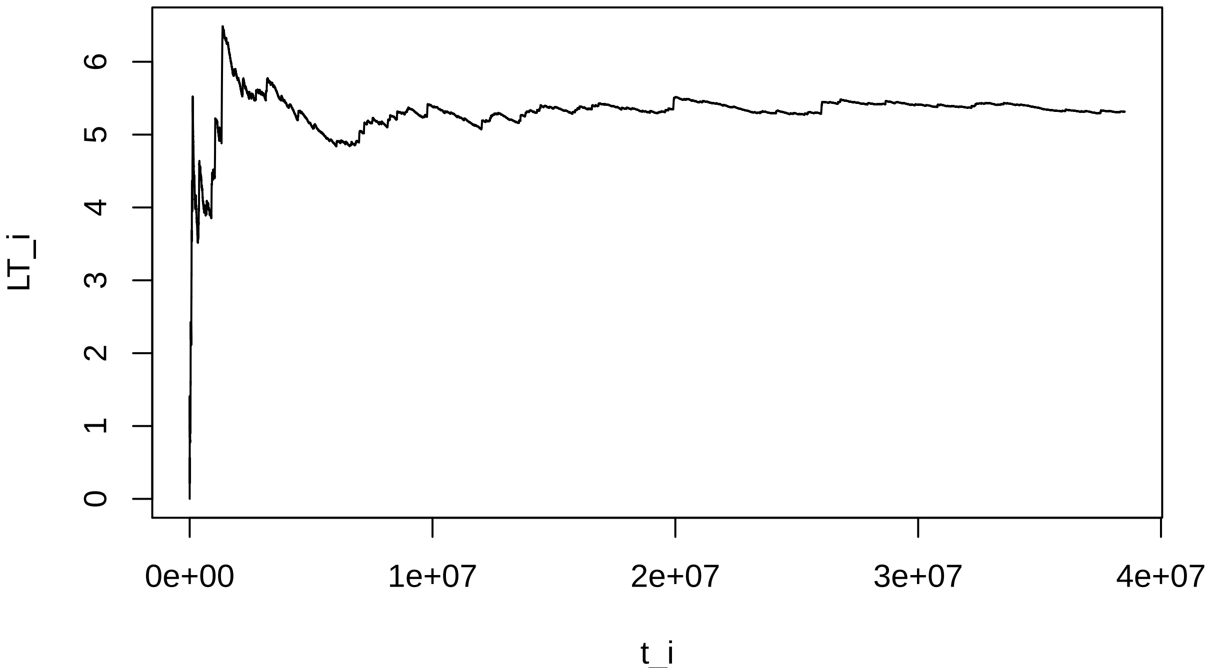

rho = 0.7 seed = 772
LT_i
0
1
2
3
4
5
6
7
0e+00
1e+07
2e+07
3e+07
4e+07
t_i


rho = 0.7 seed = 10102
LT_i
0
10
20
30
40
0e+00
1e+07
2e+07
3e+07
4e+07
t_i

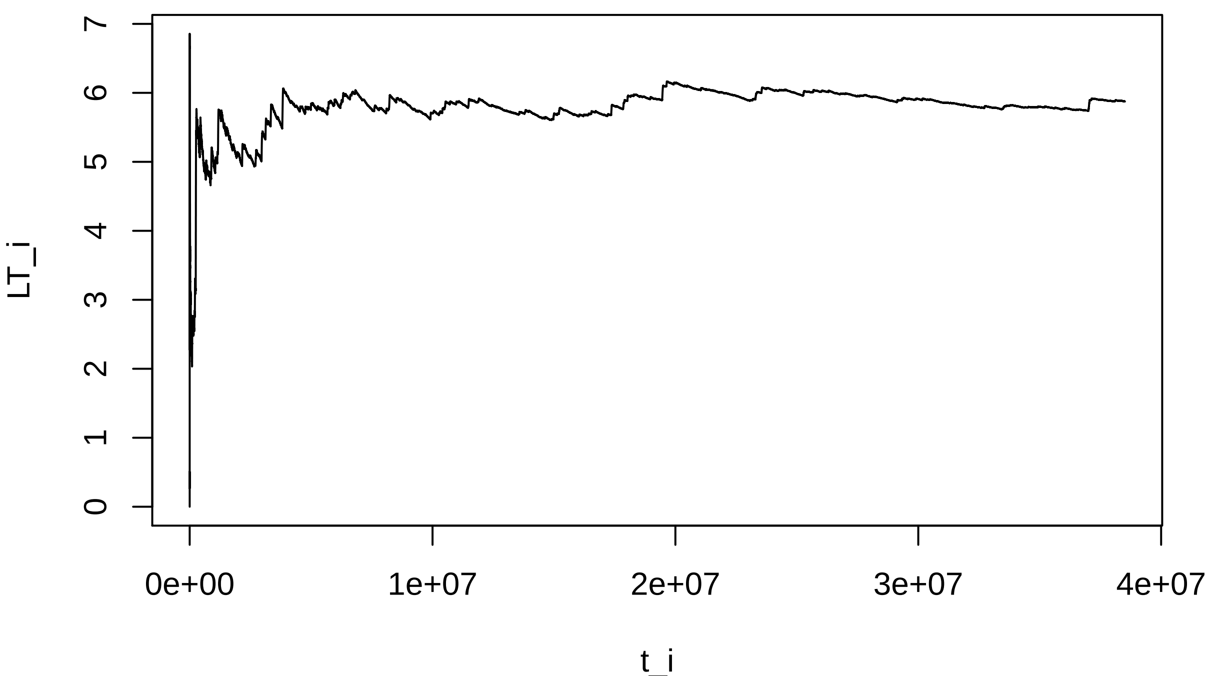

rho = 0.7 seed = 963
LT_i
t_i
0
2
4
6
8
10
0e+00
1e+07
2e+07
3e+07
4e+07

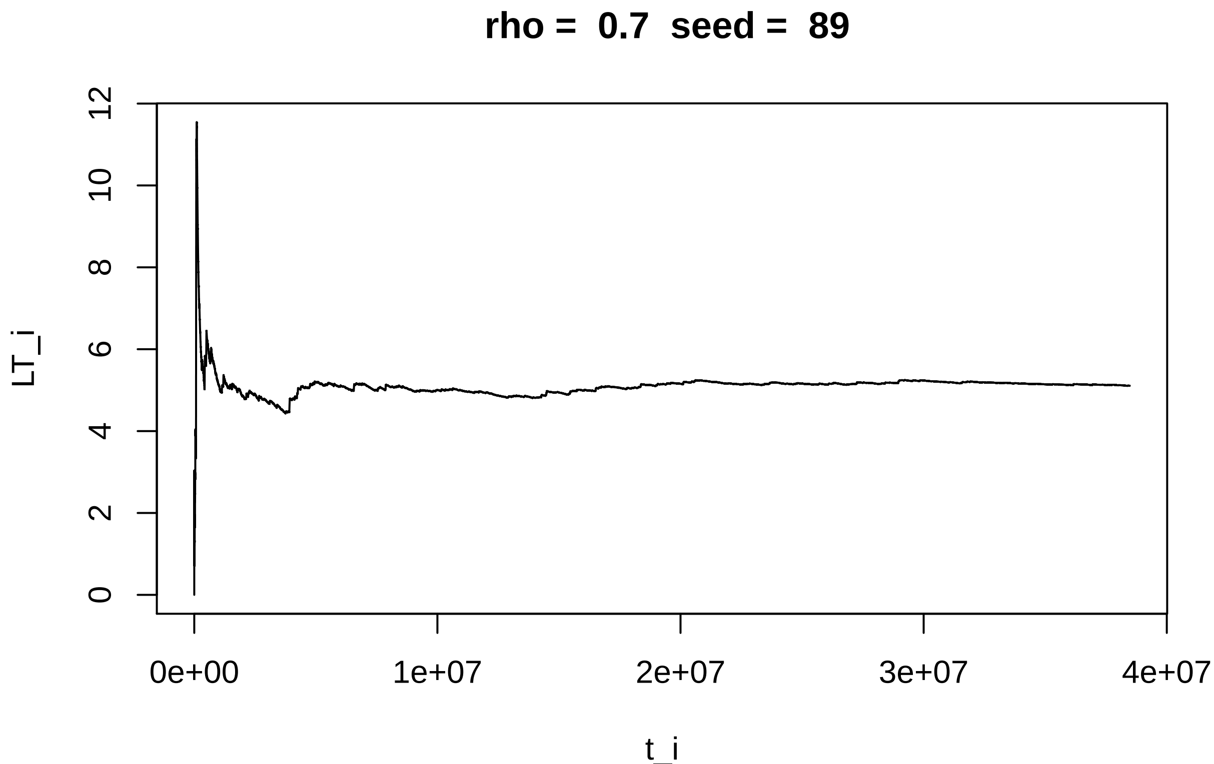

rho = 0.7 seed = 1078
LT_i
t_i
0
2
4
6
0e+00
1e+07
2e+07
3e+07
4e+07

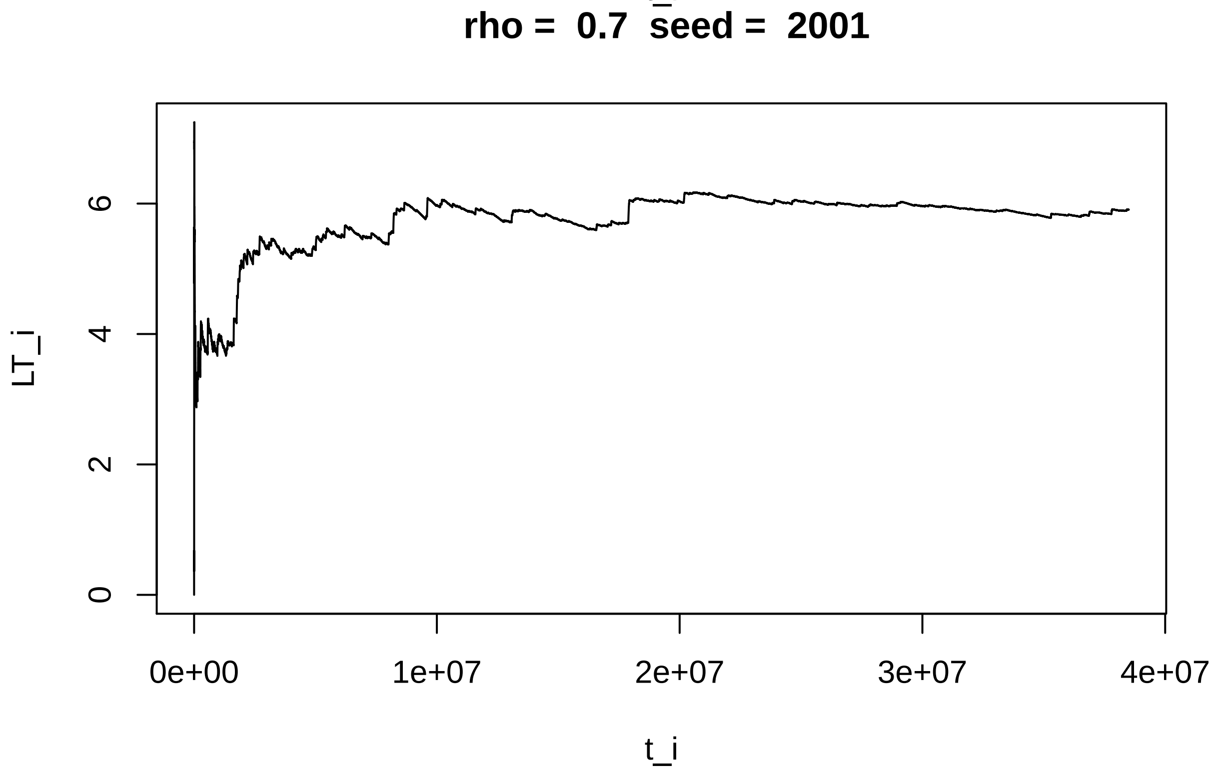
rho = 0.7 seed = 999

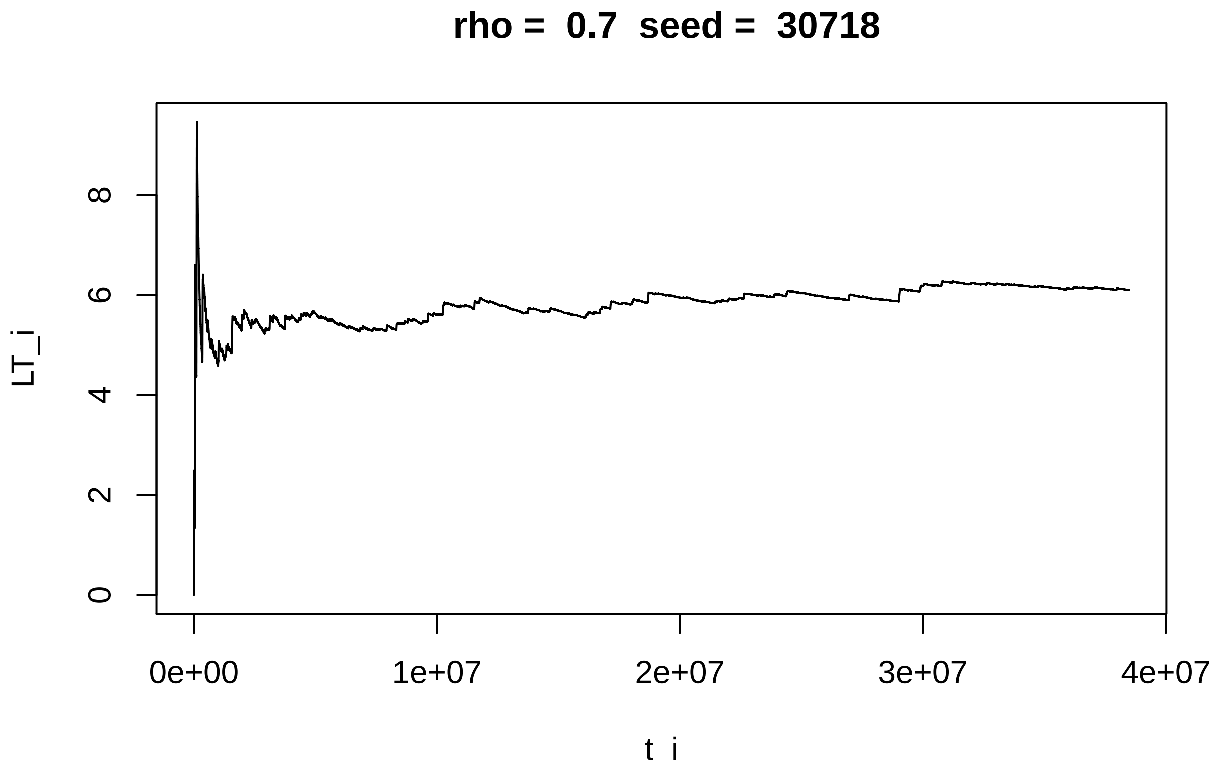


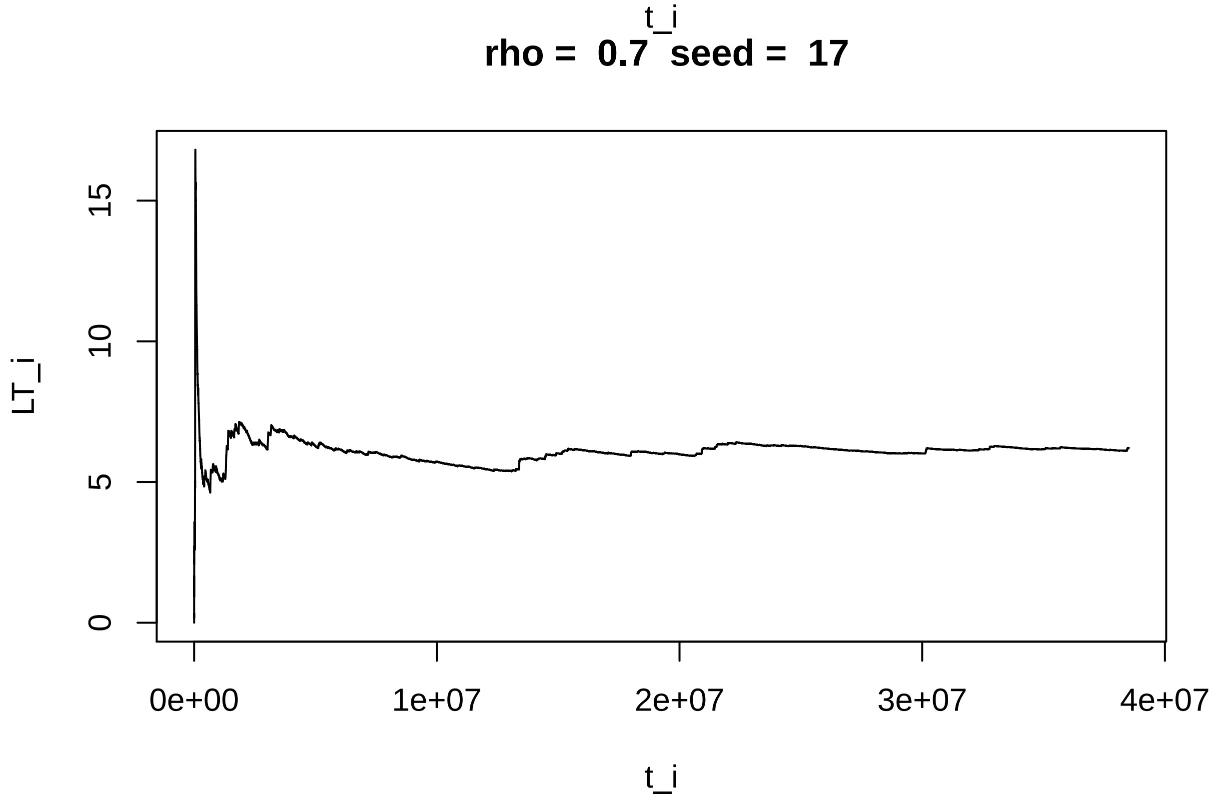
rho = 0.7 seed = 48




rho = 0.7 seed = 89
LT_i
t_i
0
2
4
6
8
10
12
0e+00
1e+07
2e+07
3e+07
4e+07


rho = 0.7 seed = 2001
LT_i
t_i
0
2
4
6
0e+00
1e+07
2e+07
3e+07
4e+07



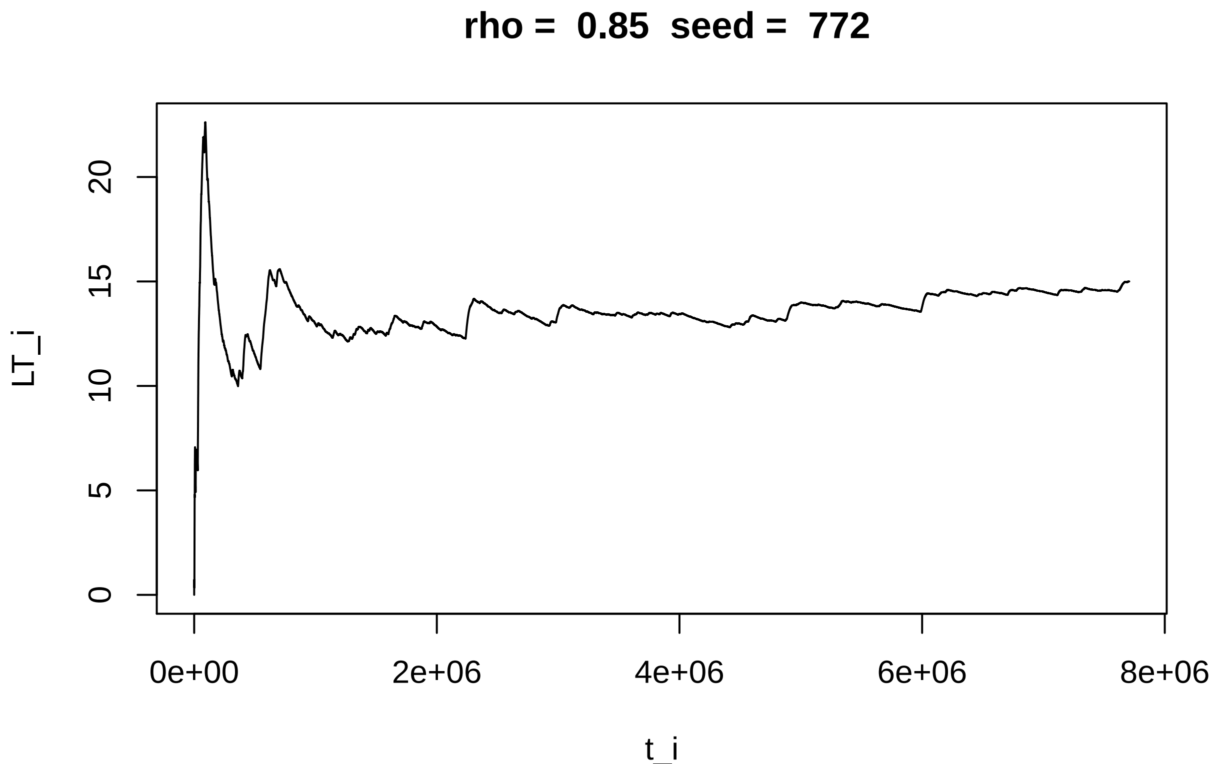


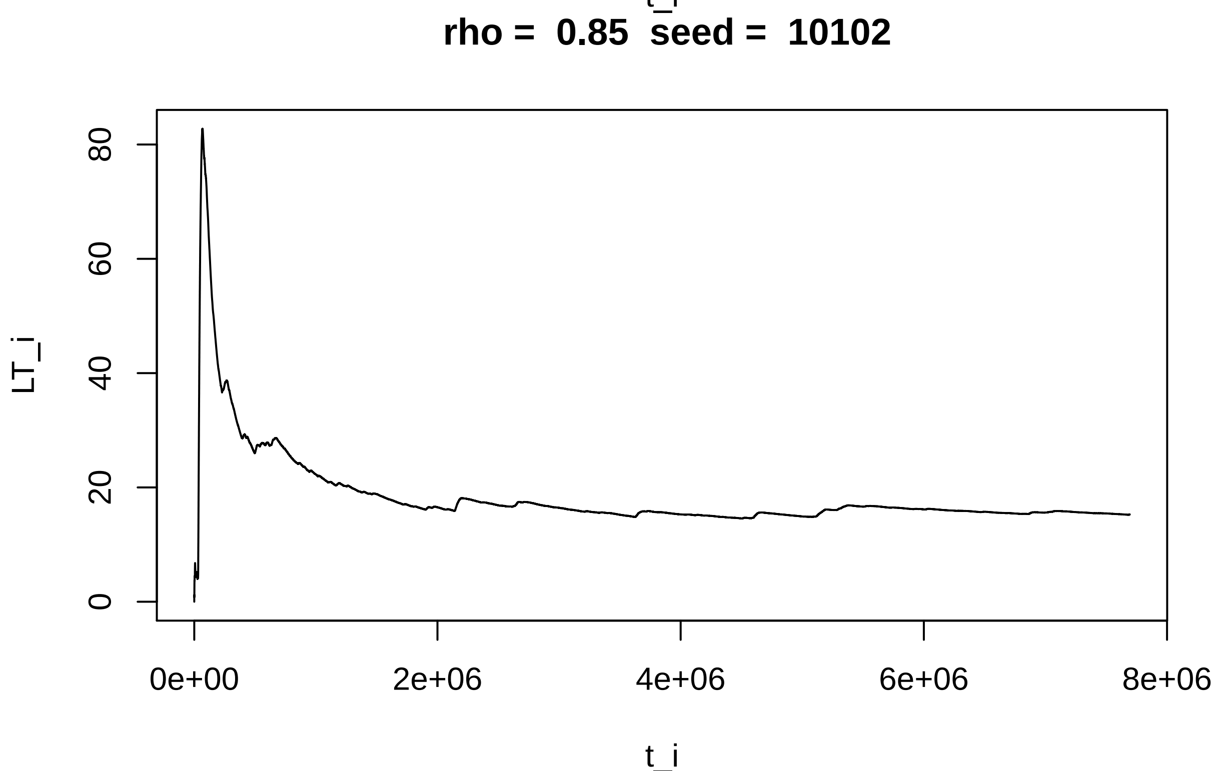
We now compute the confidence interval for L_q and W_q using the same procedure we detailed for $\rho = 0.7$. We will use a t-Student distribution with critical value 1 - 0.95 and 9 degrees of freedom to compute a 95% confidence interval. The computations produce the followings confidence intervals for the average queue length and waiting time:

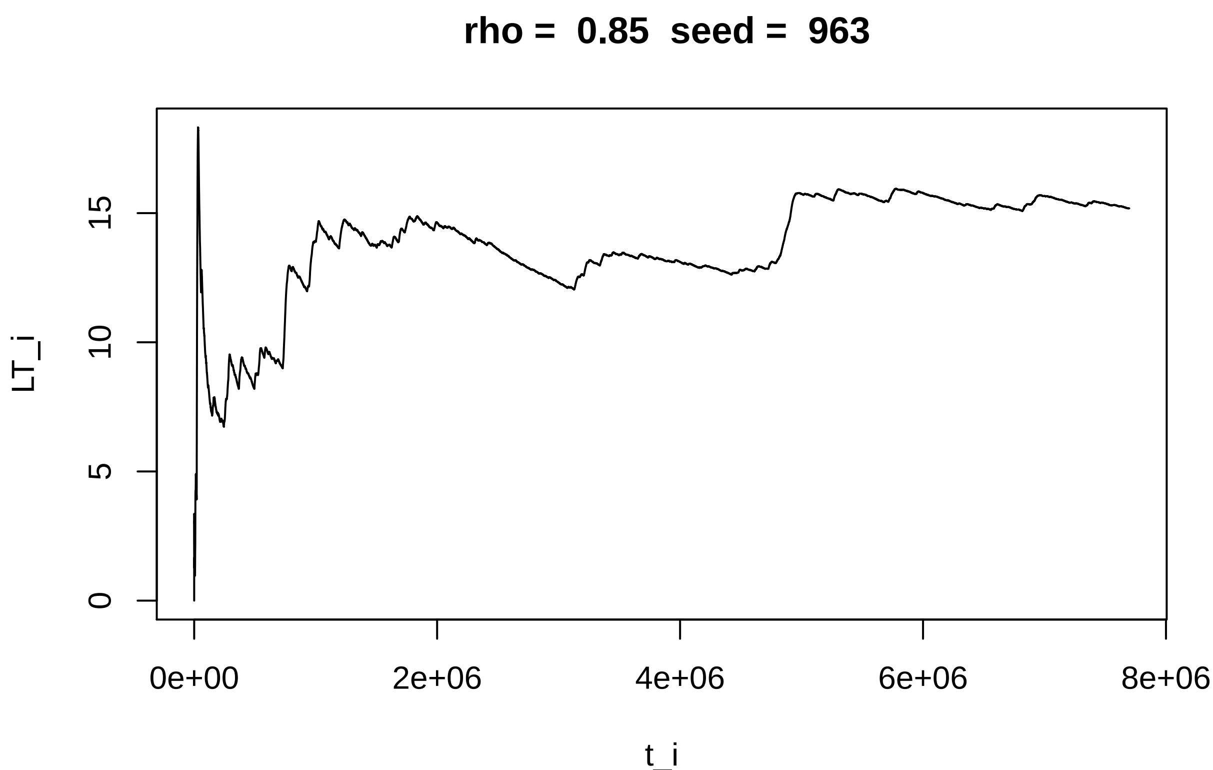
ρ	-C.I W_q	+C.I W_q	-C.I L_q	+C.I L_q
0.7	4.9266678	5.3560657	379.3389492	412.4915102

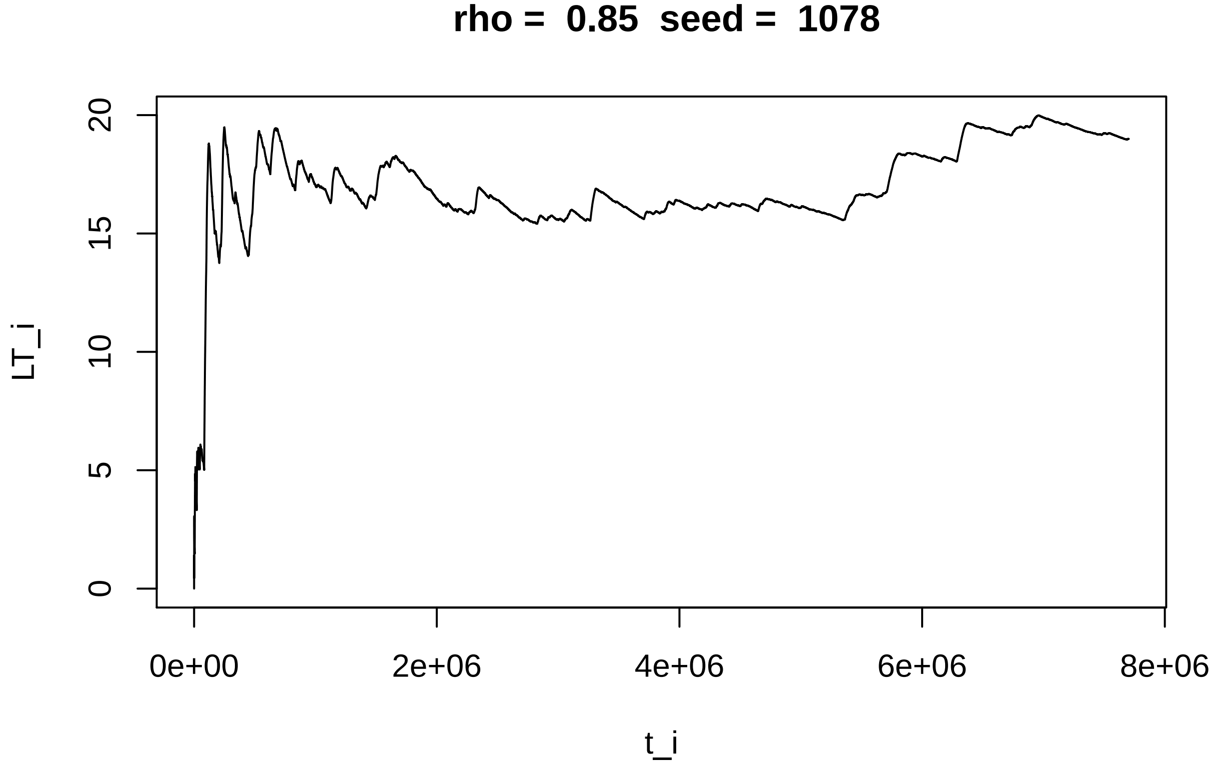
$\rho =$ **0.85**

We generate 10 simulations with 100000 clients, each with a different seed. We check if the steady state is attained. We also check for irregularities in the simulation results, in which case we change the random number generator seed and repeat the simulation.

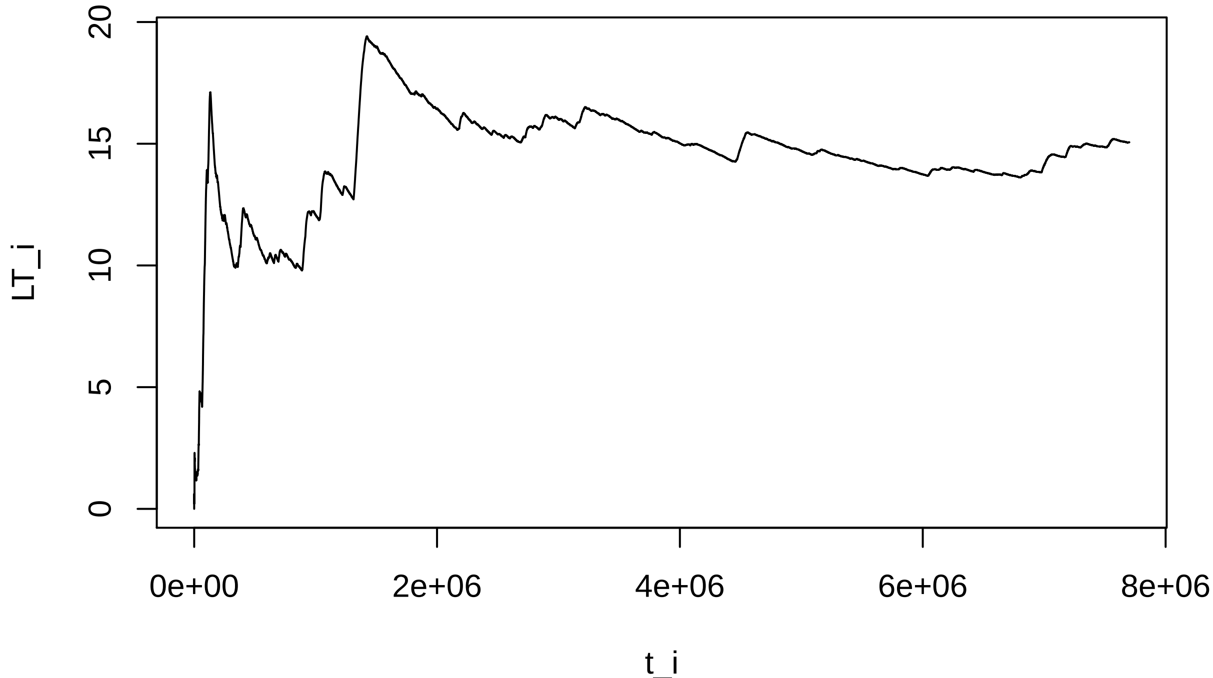

rho = 0.85 seed = 772
LT_i
t_i
0
5
10
15
20
0e+00
2e+06
4e+06
6e+06
8e+06


rho = 0.85 seed = 10102
LT_i
t_i
0
20
40
60
80
0e+00
2e+06
4e+06
6e+06
8e+06

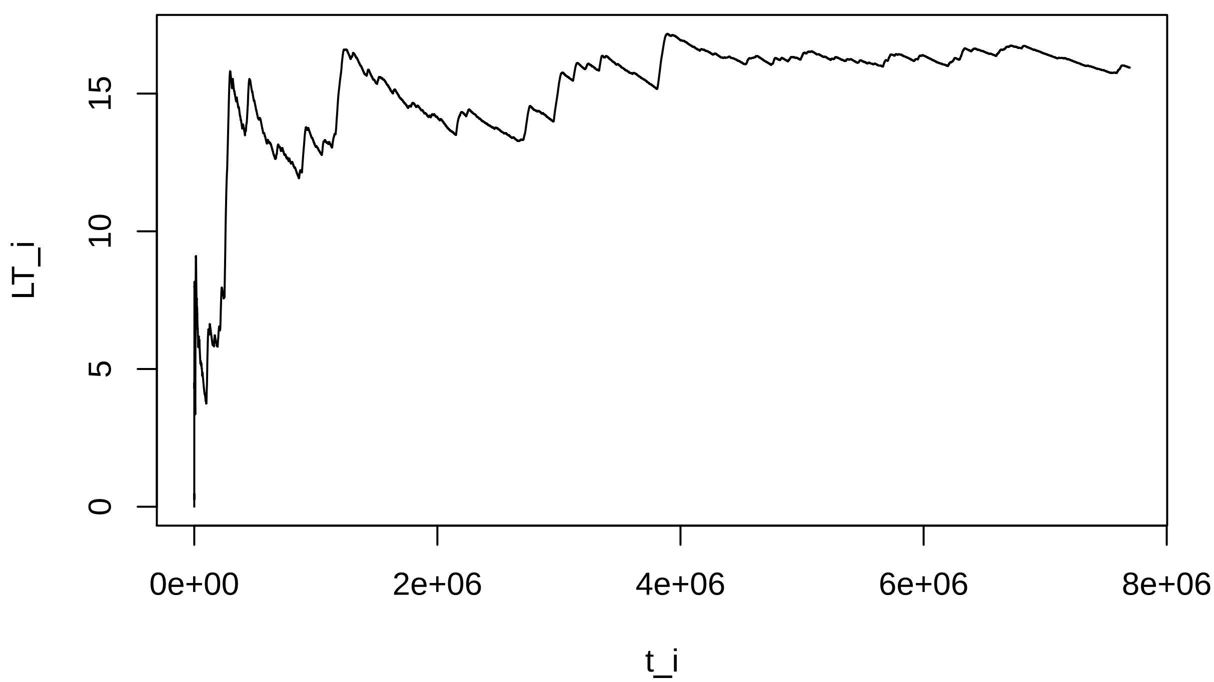

rho = 0.85 seed = 963
LT_i
0
5
10
15
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

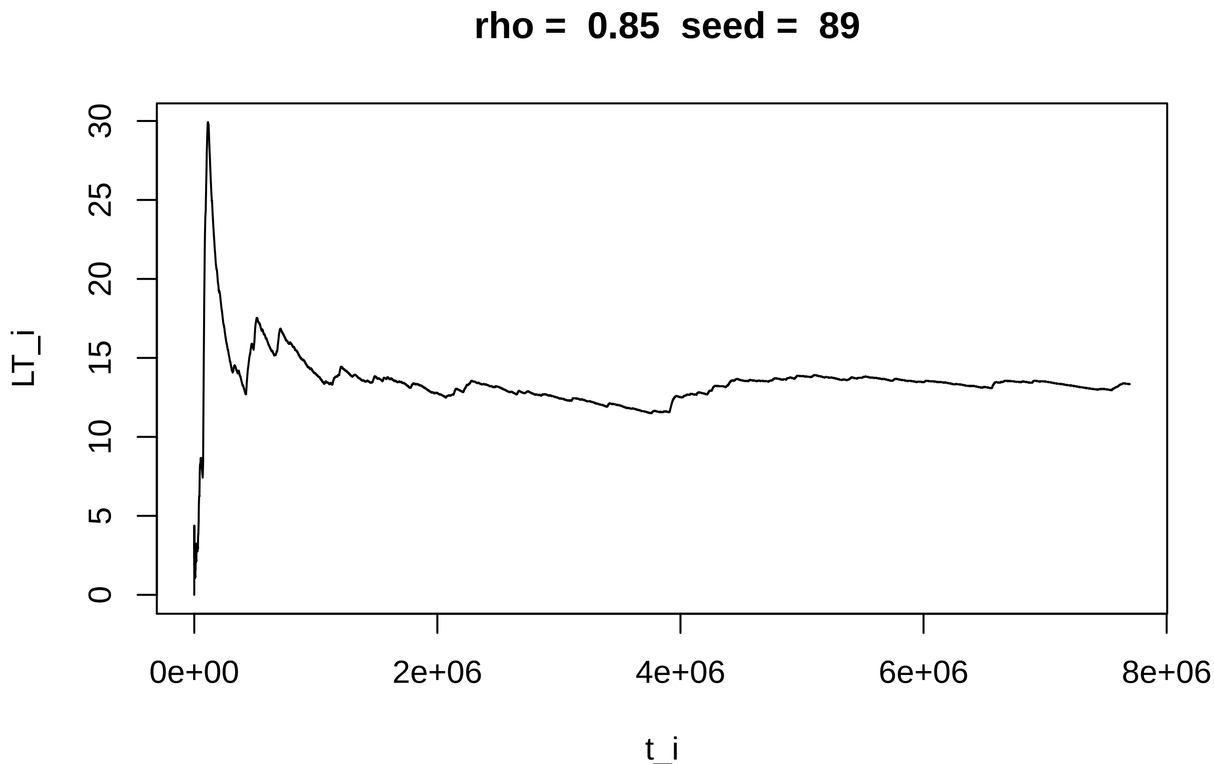

rho = 0.85 seed = 1078
LT_i
0
5
10
15
20
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

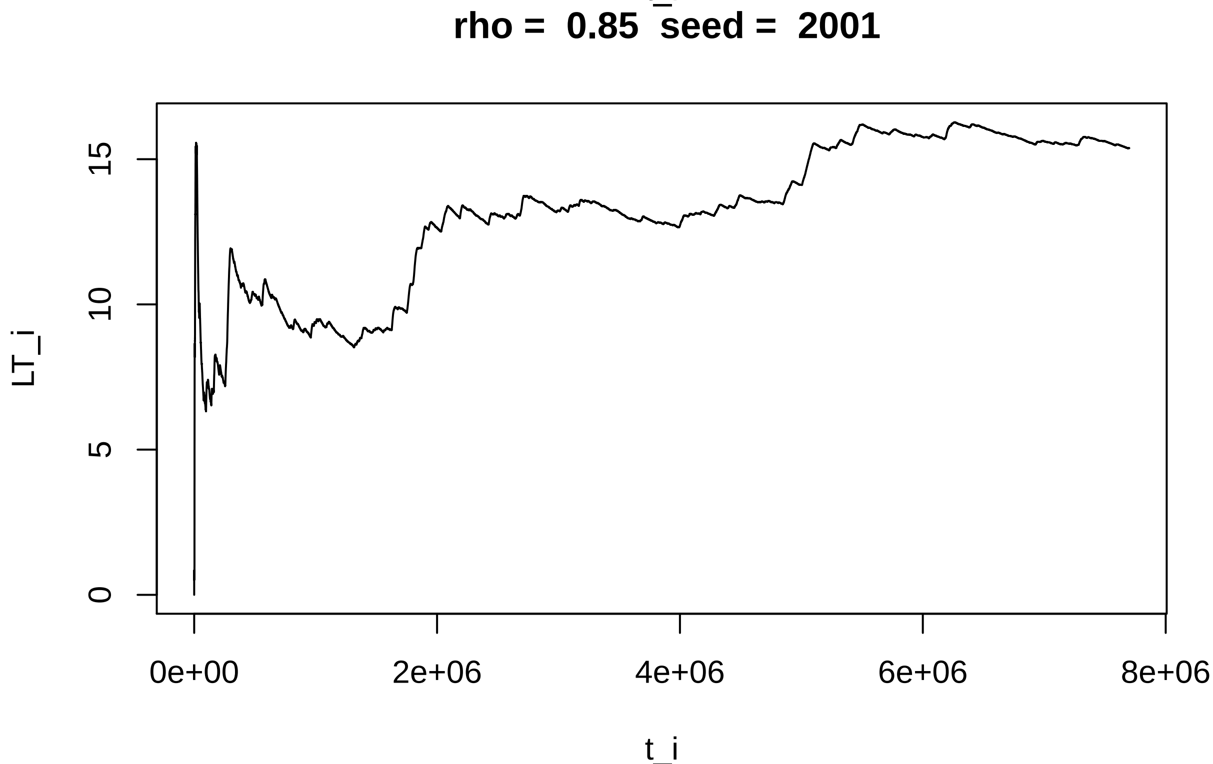
rho = 0.85 seed = 999

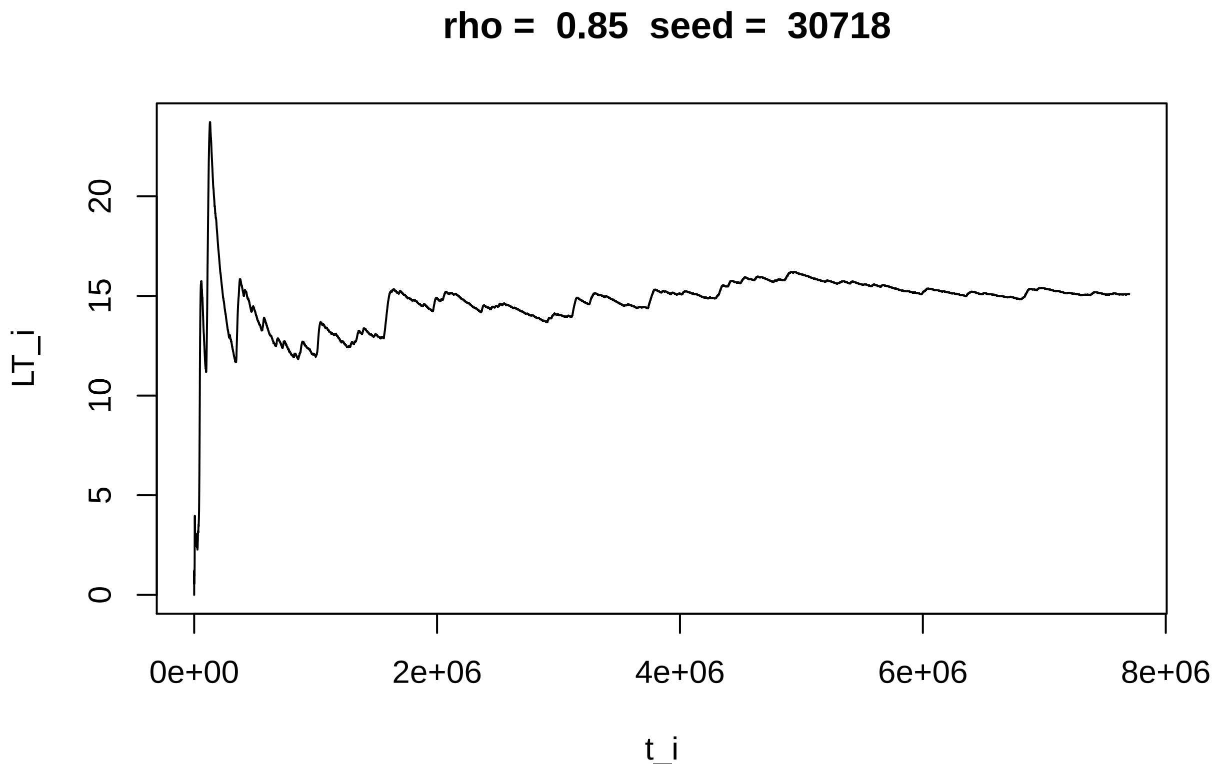


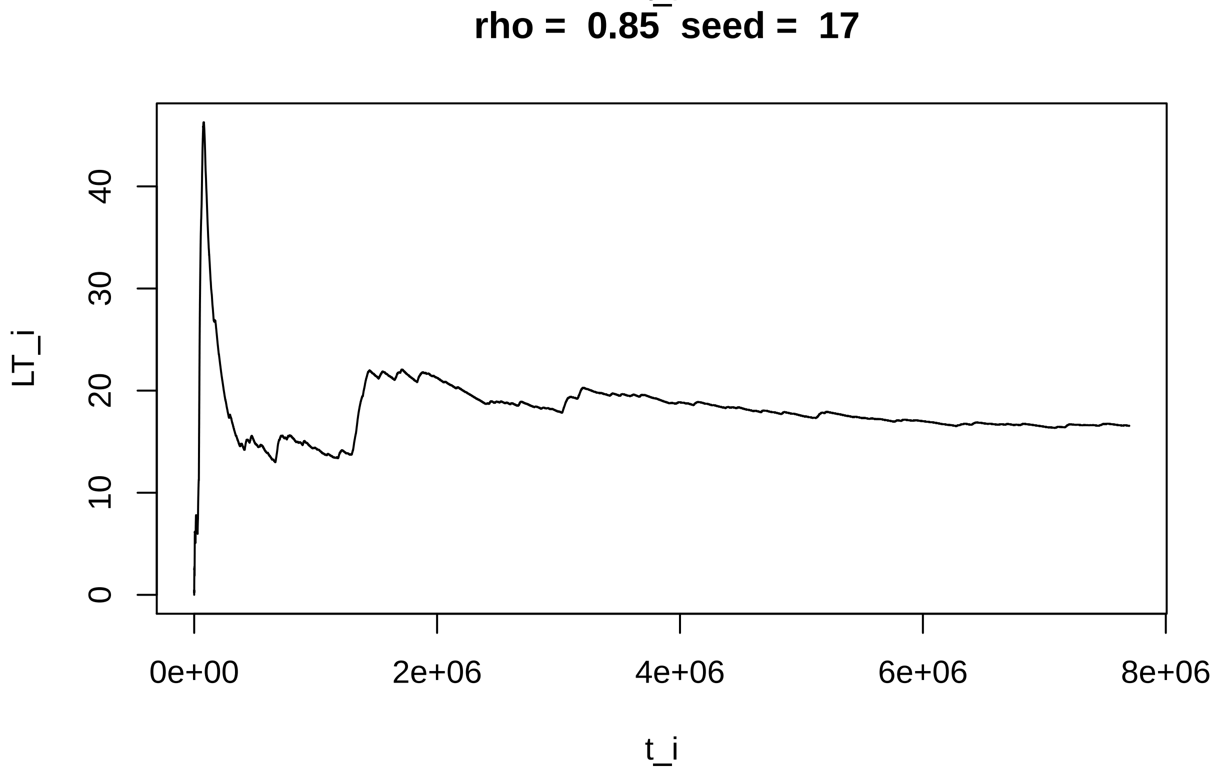
rho = 0.85 seed = 48



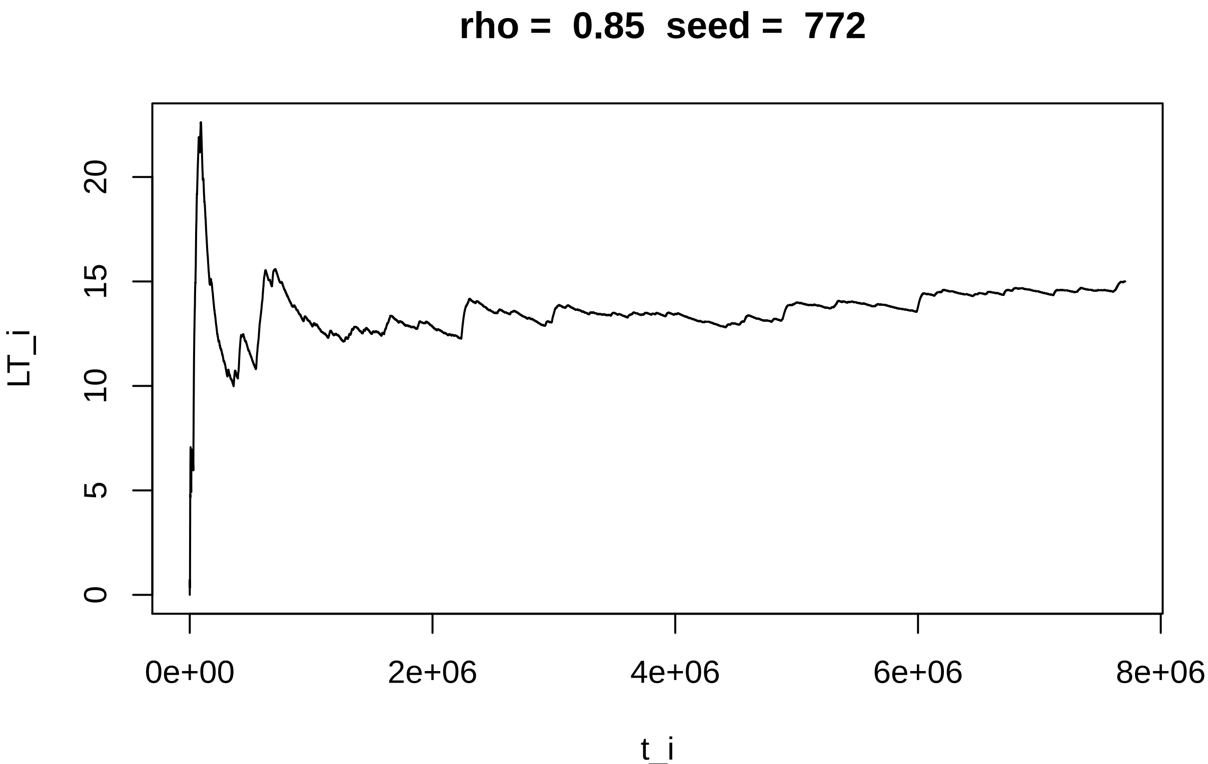

rho = 0.85 seed = 89
LT_i
t_i
0
5
10
15
20
25
30
0e+00
2e+06
4e+06
6e+06
8e+06

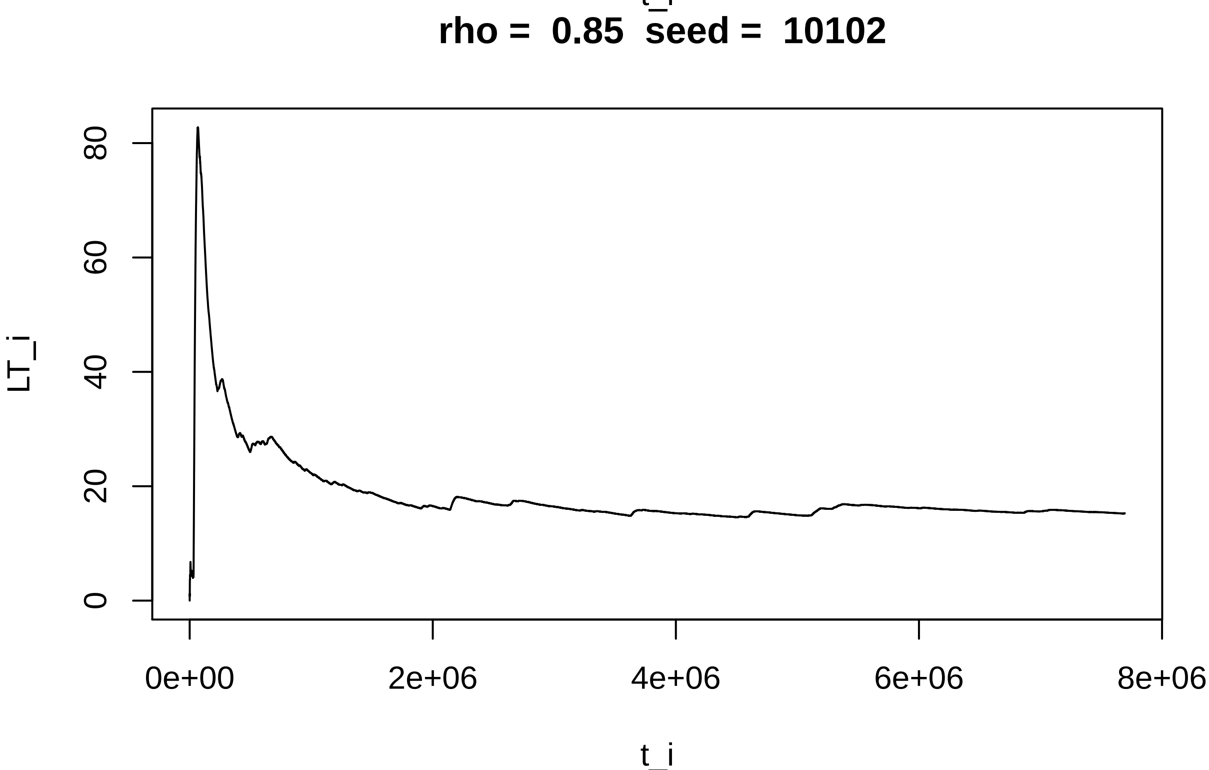

rho = 0.85 seed = 2001
LT_i
t_i
0
5
10
15
0e+00
2e+06
4e+06
6e+06
8e+06

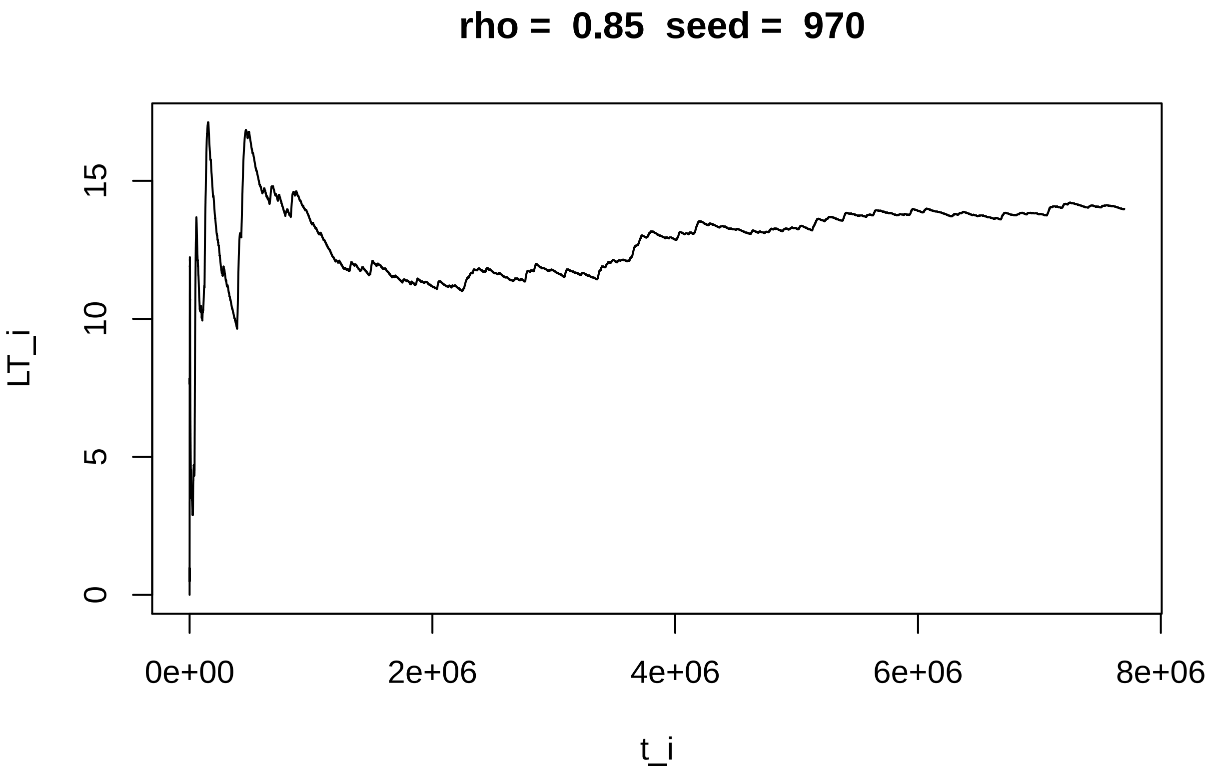


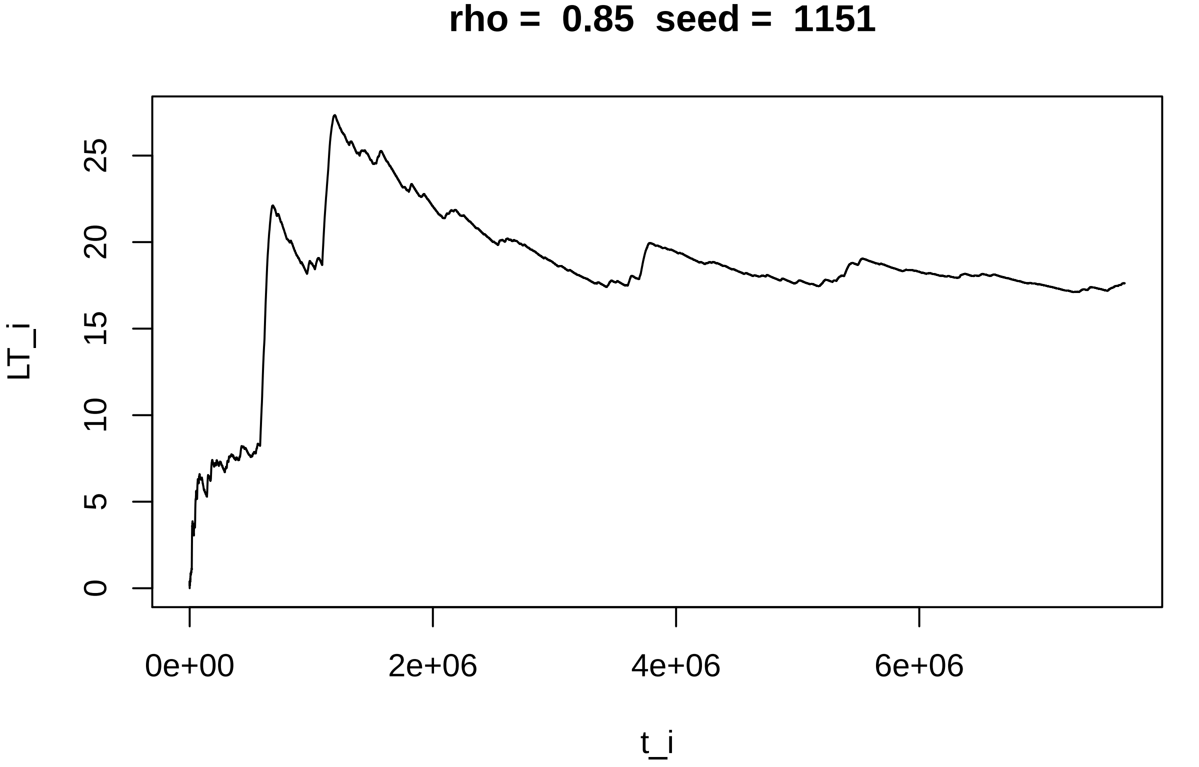


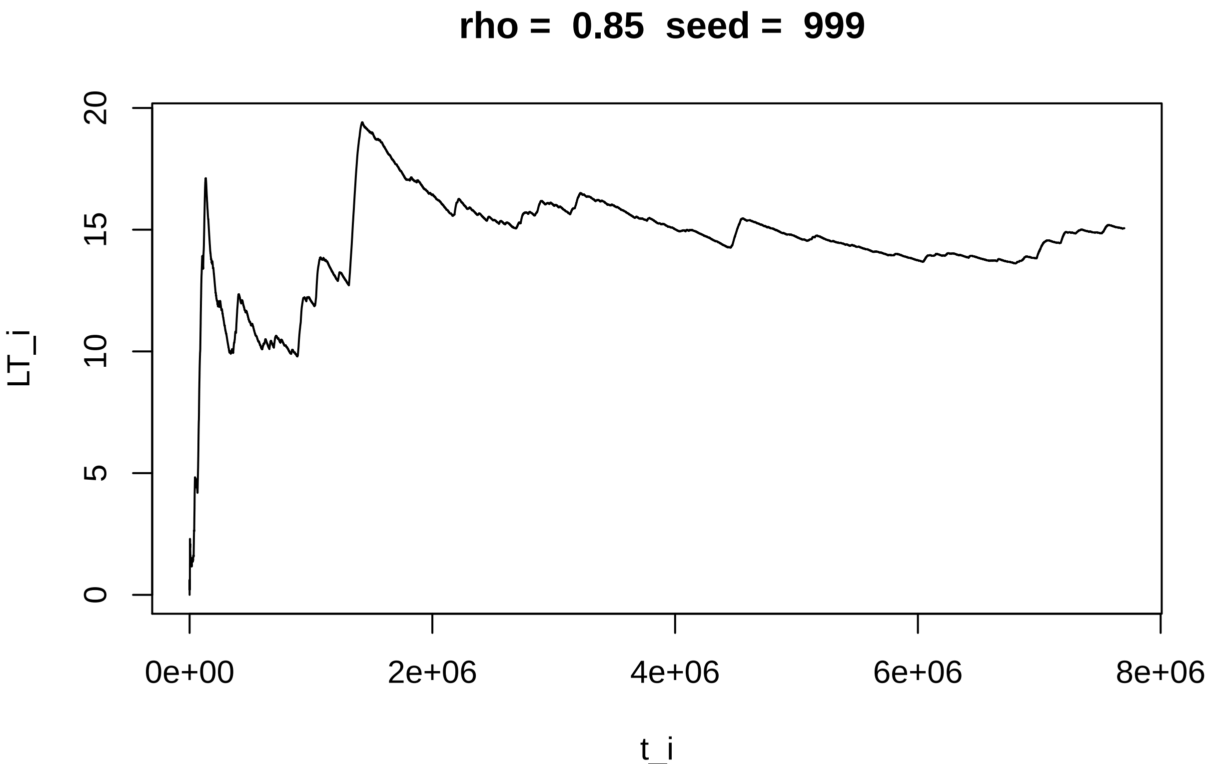
We observe that seeds 963, 1078, 48 and 2001 produce irregularities in the simulation, therefore we have to change them and repeat the simulation.

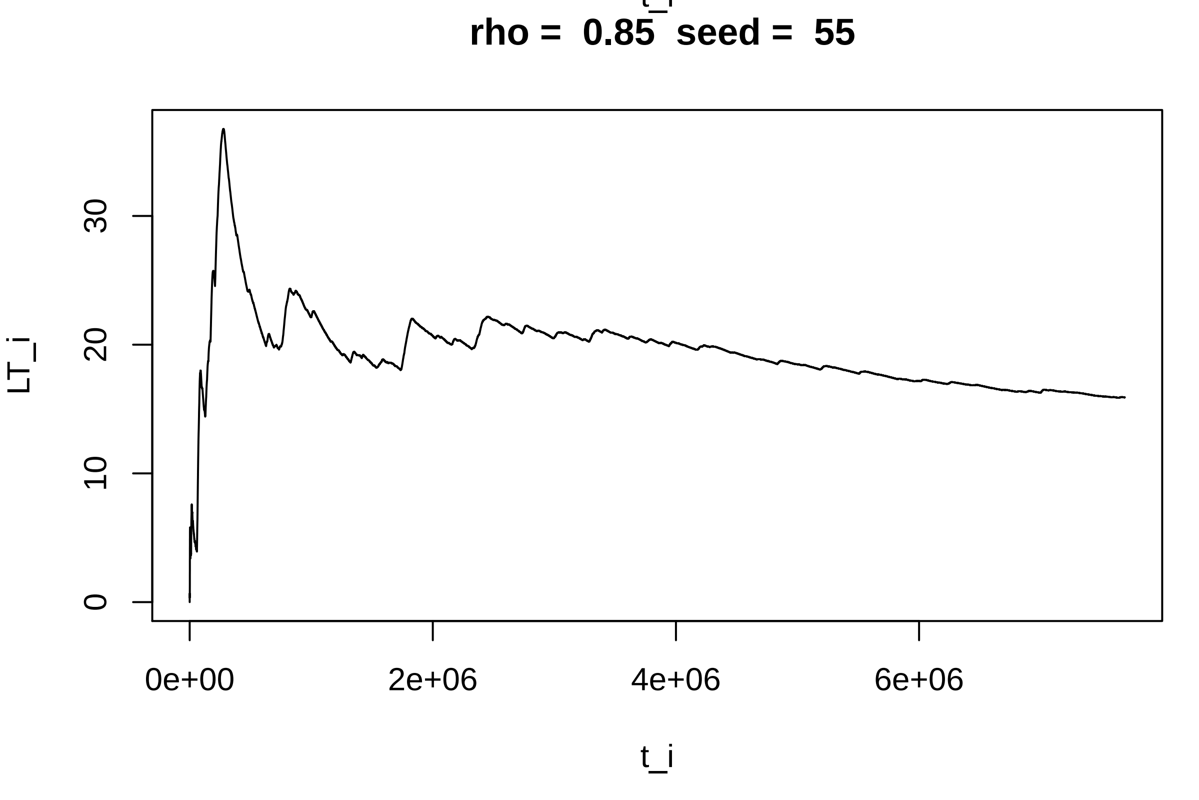

rho = 0.85 seed = 772
LT_i
0
5
10
15
20
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

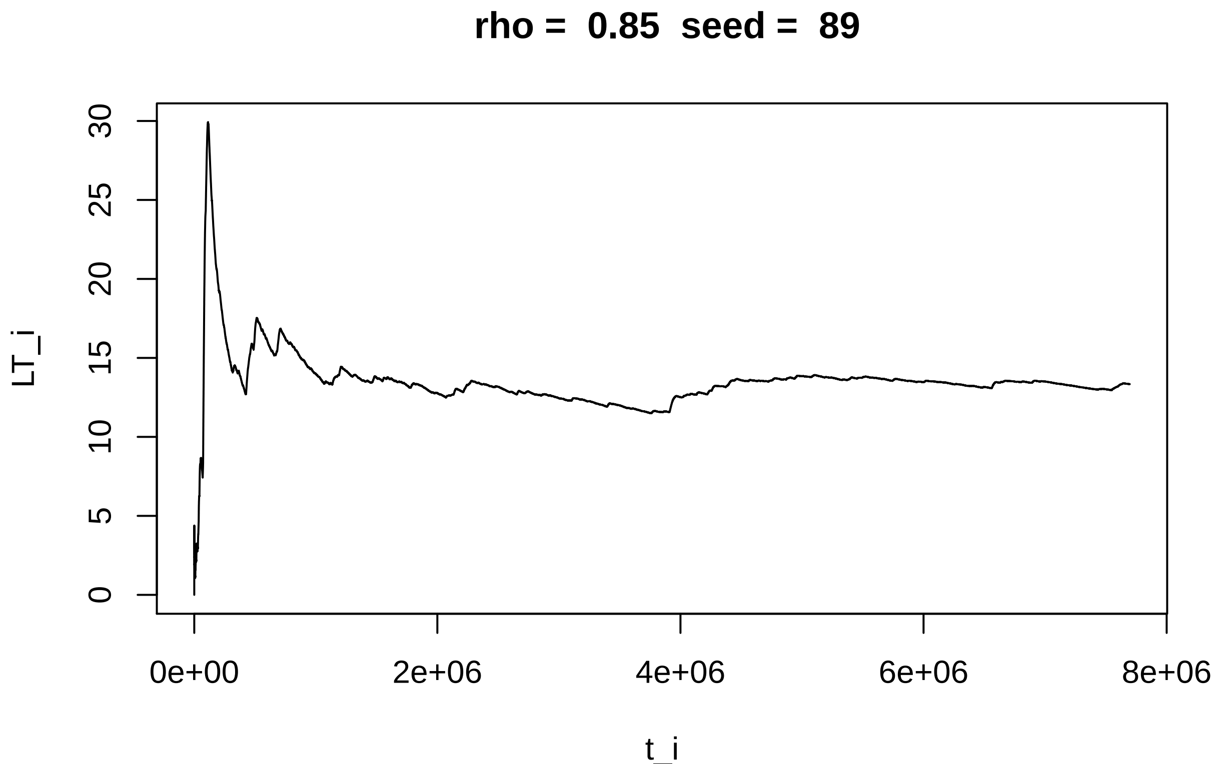

rho = 0.85 seed = 10102
LT_i
0
20
40
60
80
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

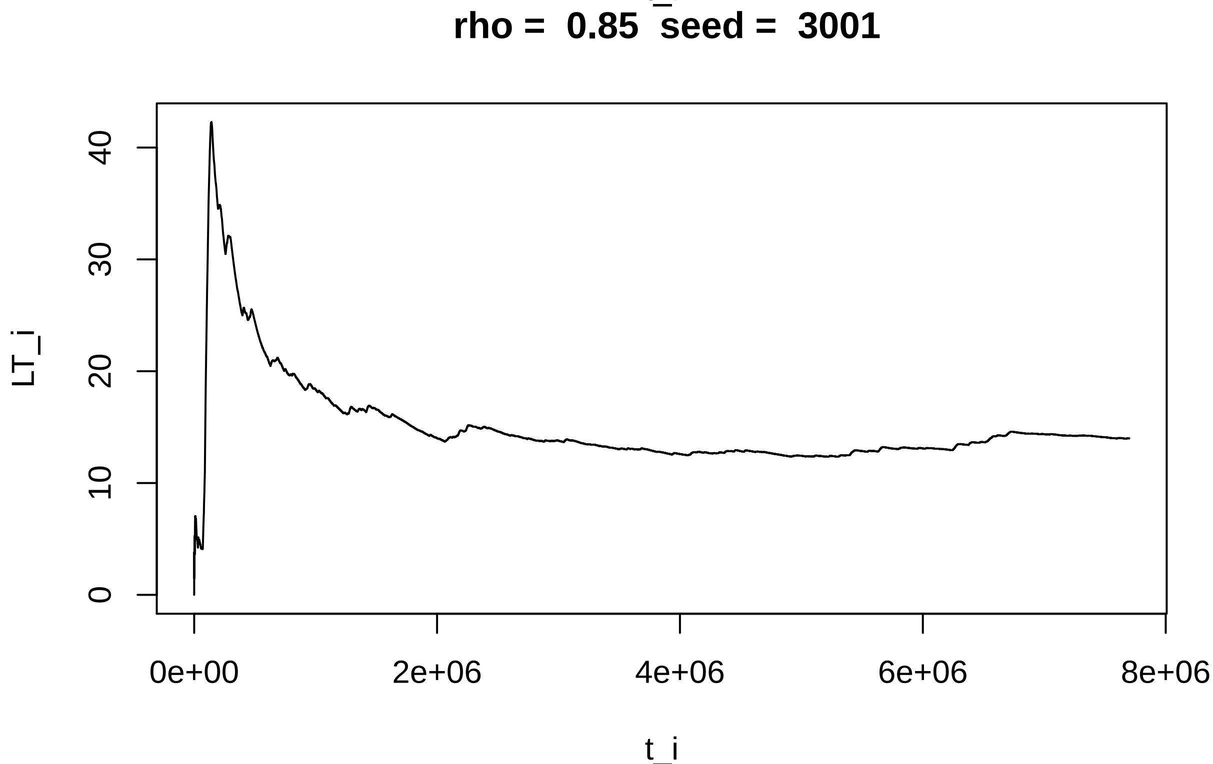

rho = 0.85 seed = 970
LT_i
0
5
10
15
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

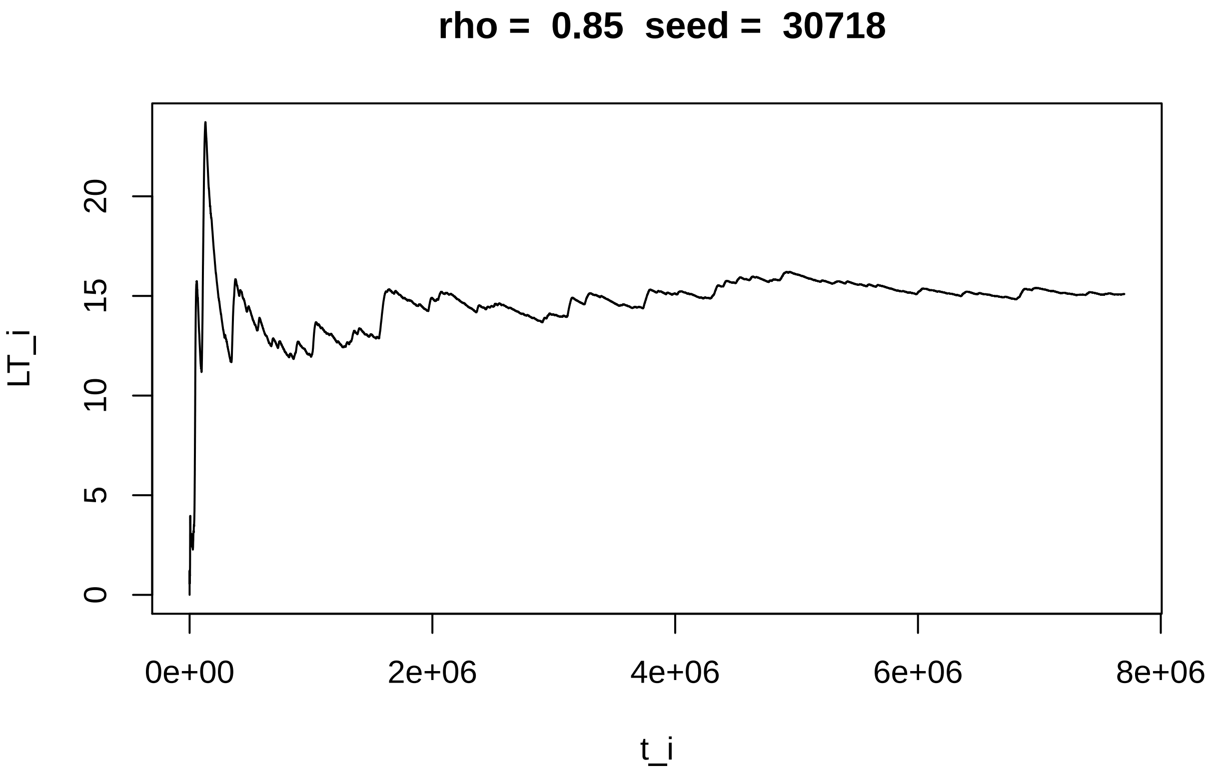

rho = 0.85 seed = 1151
LT_i
0
5
10
15
20
25
0e+00
2e+06
4e+06
6e+06
t_i

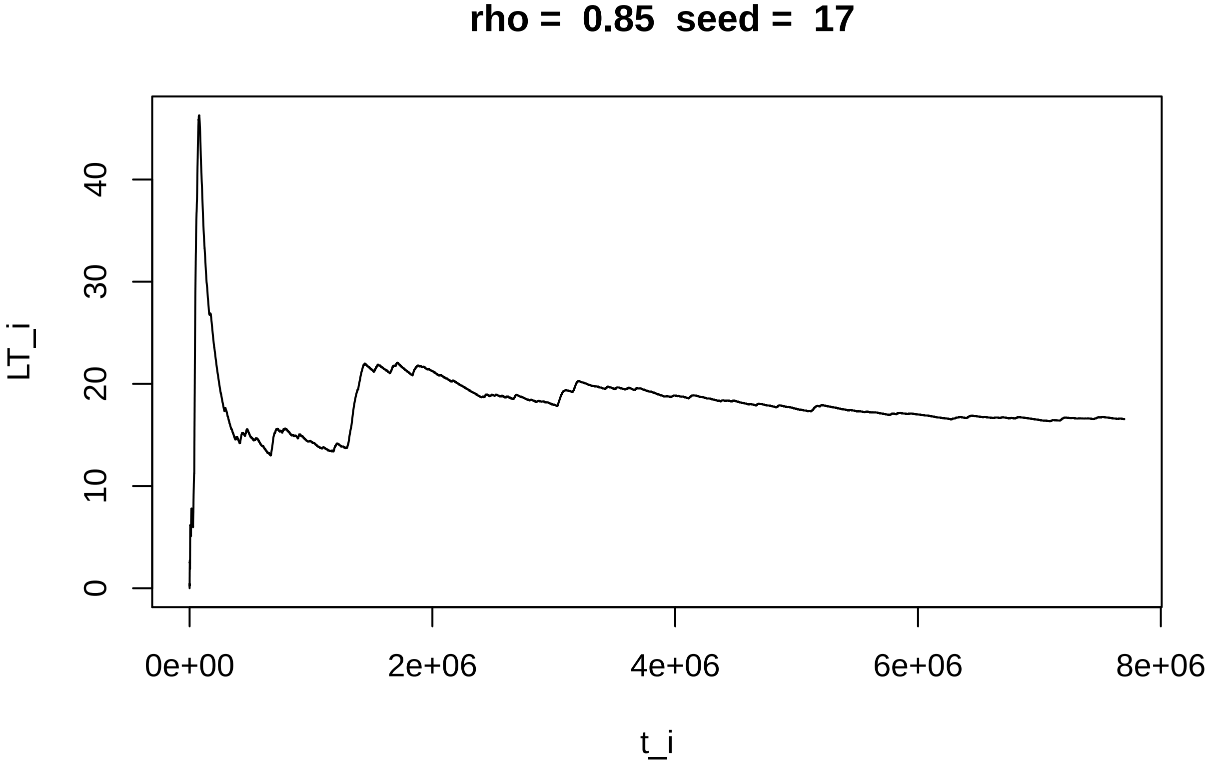

rho = 0.85 seed = 999
LT_i
0
5
10
15
20
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.85 seed = 55
LT_i
0
10
20
30
0e+00
2e+06
4e+06
6e+06
t_i


rho = 0.85 seed = 89
LT_i
0
5
10
15
20
25
30
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.85 seed = 3001
LT_i
0
10
20
30
40
0e+00
2e+06
4e+06
6e+06
8e+06
t_i



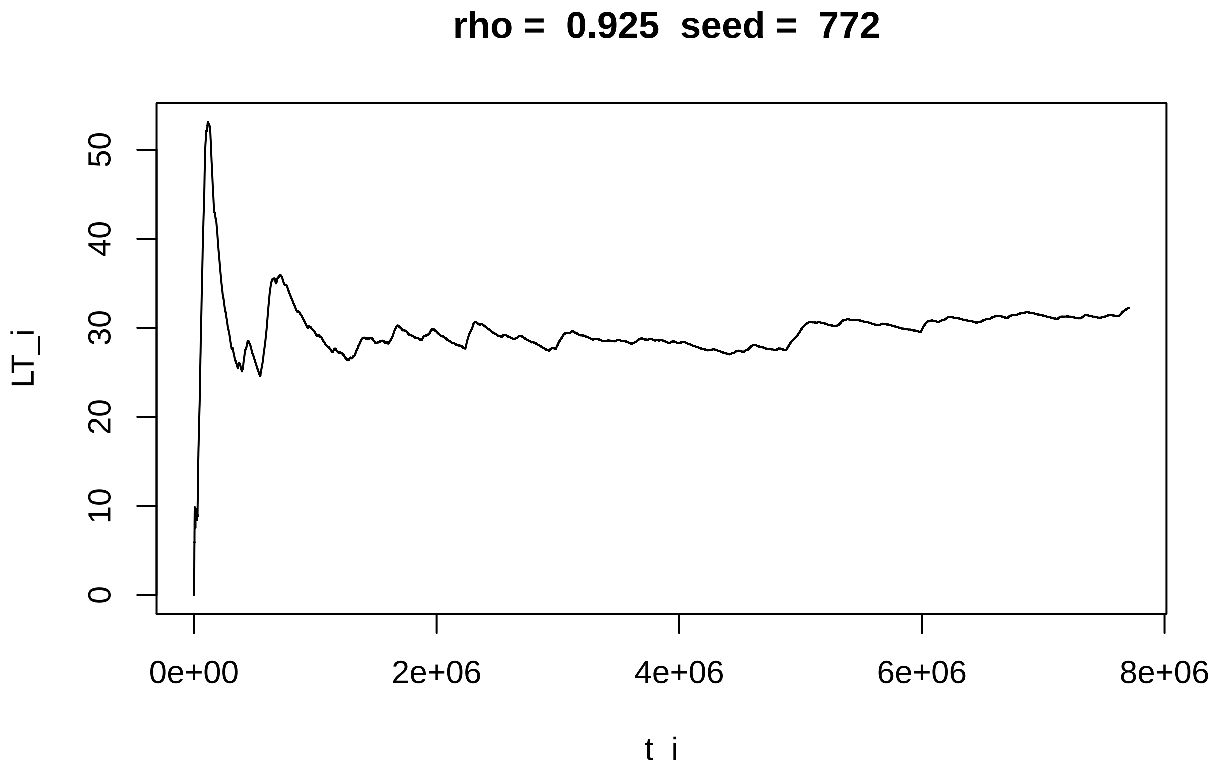


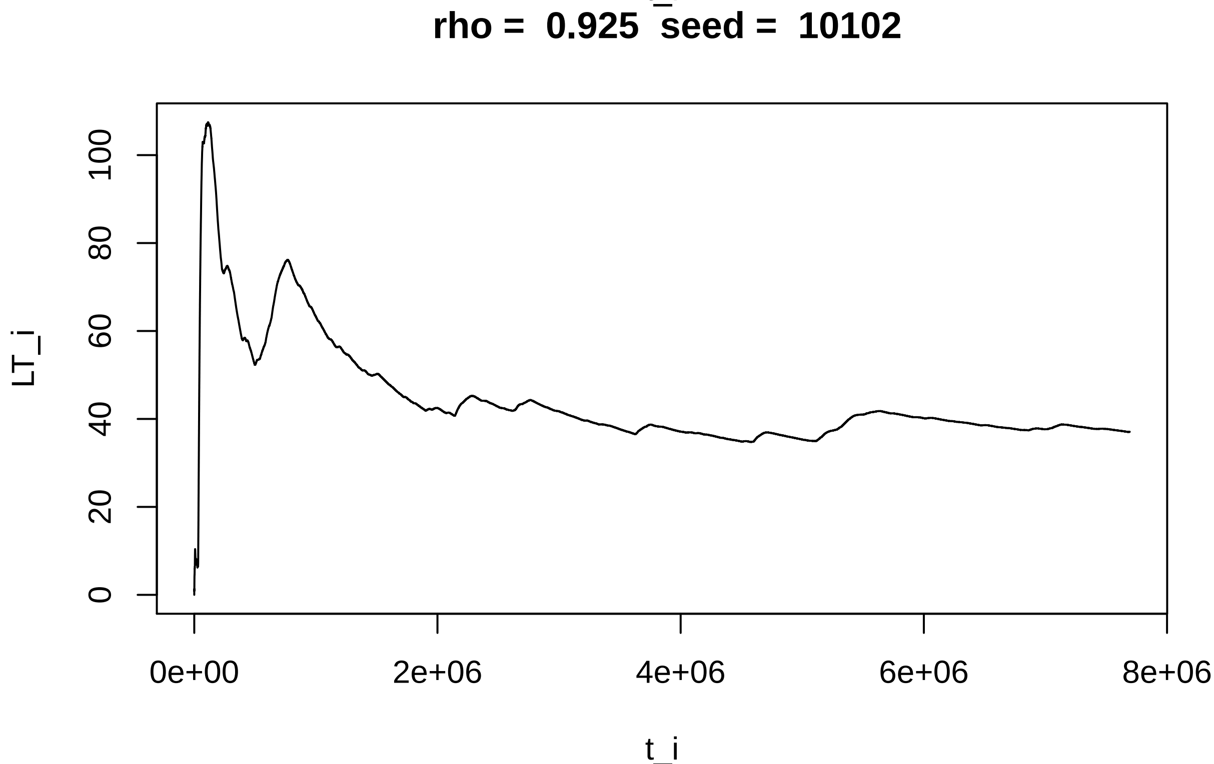
We now compute the confidence interval for L_q and W_q using the same procedure we detailed for $\rho = 0.85$. We will use a t-Student distribution with critical value 1 - 0.95 and 9 degrees of freedom to compute a 95% confidence interval. The computations produce the followings confidence intervals for the average queue length and waiting time:

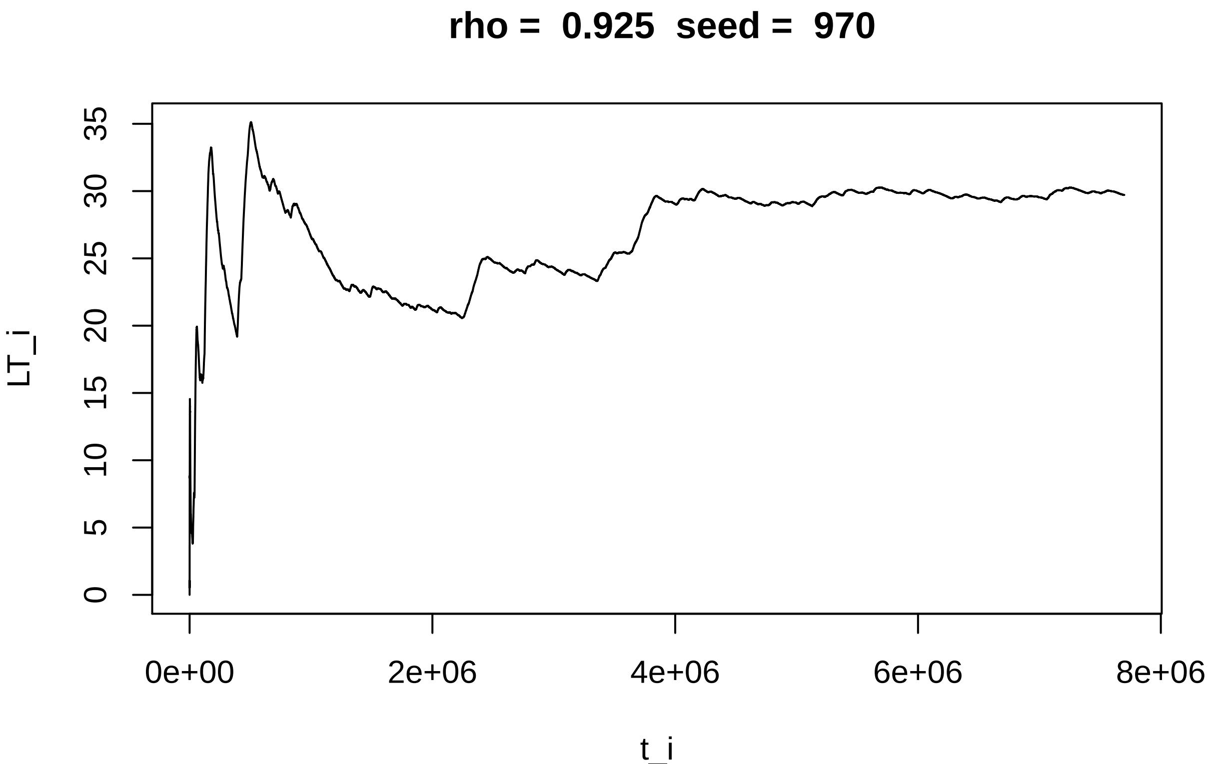
ρ	-C.I W_q	+C.I W_q	-C.I L_q	+C.I L_q
0.85	13.5934229	15.0731043	1046.5794927	1160.0665074

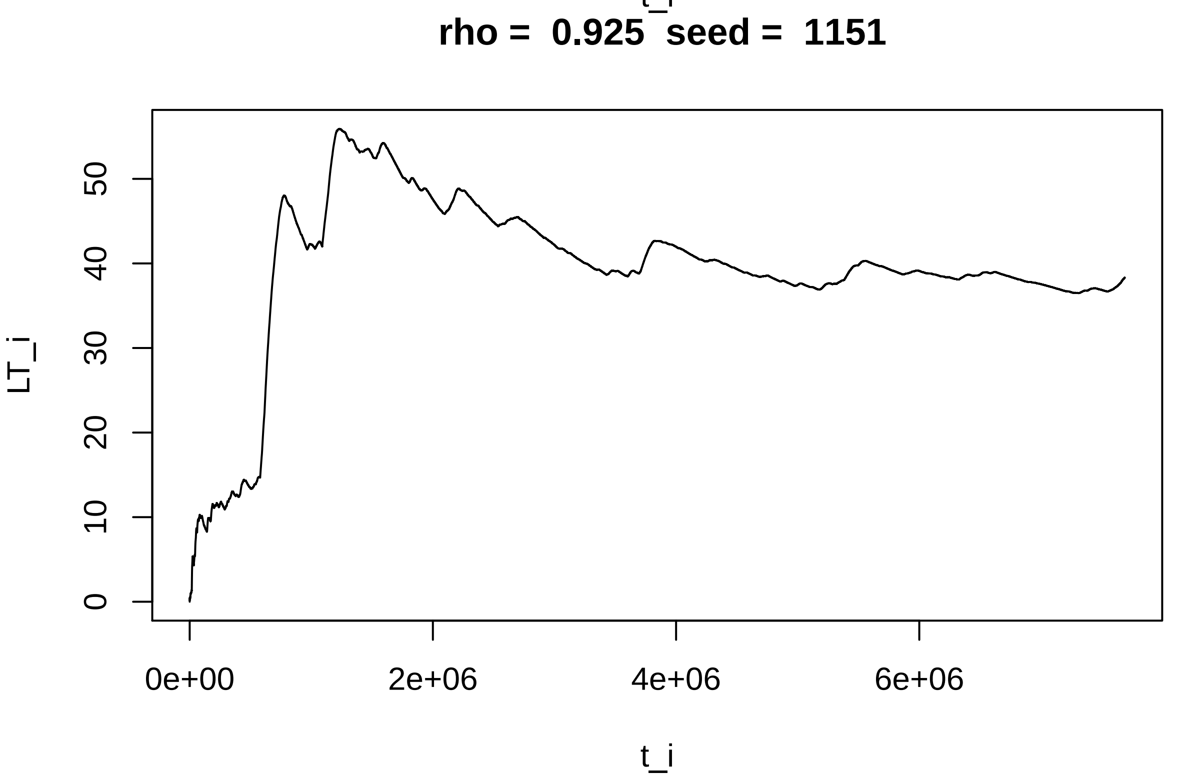
ρ = 0.925

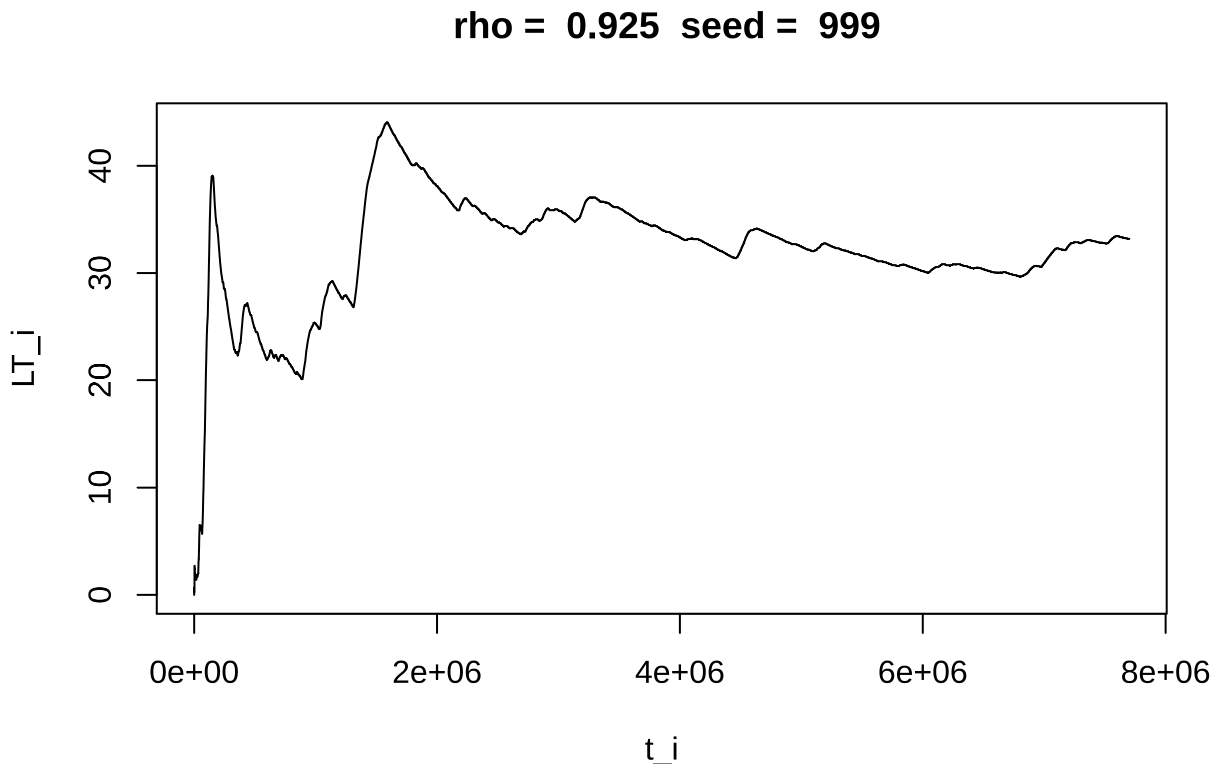
We generate 10 simulations with 100000 clients, each with a different seed. We check if the steady state is attained. We also check for irregularities in the simulation results, in which case we change the random number generator seed and repeat the simulation.

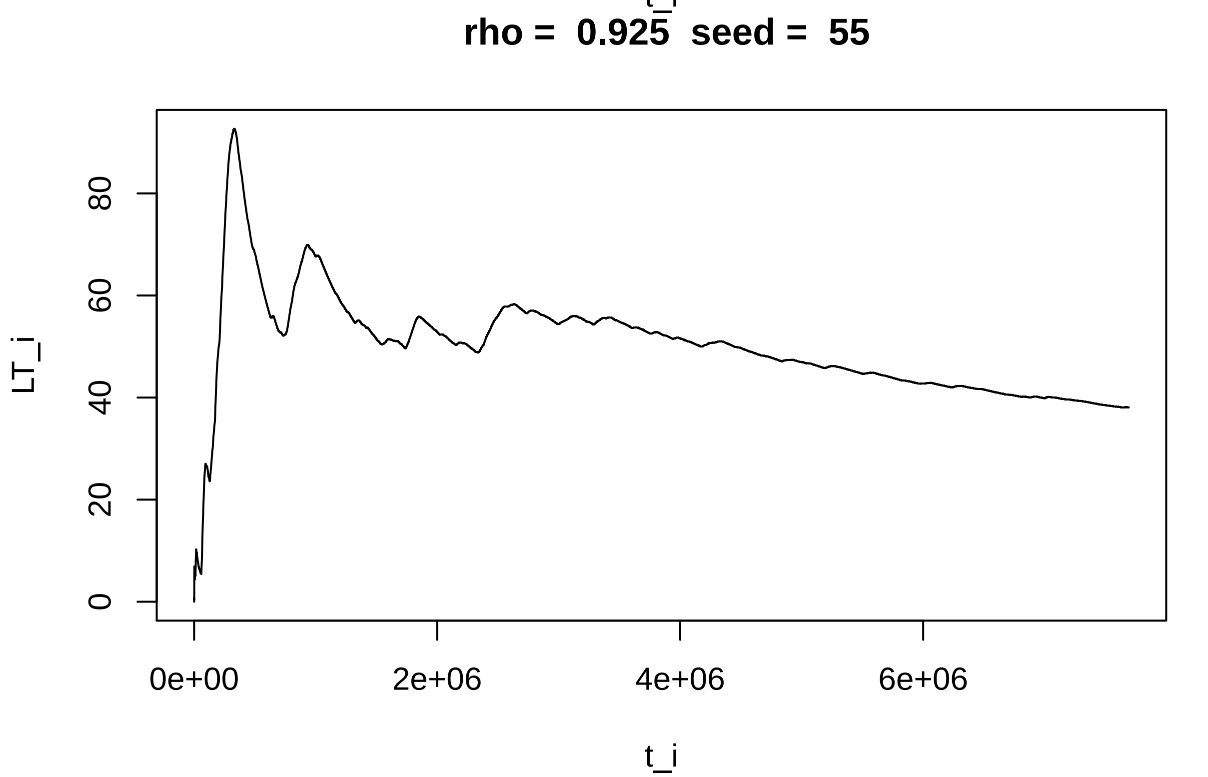

rho = 0.925 seed = 772
LT_i
t_i
0
10
20
30
40
50
0e+00
2e+06
4e+06
6e+06
8e+06

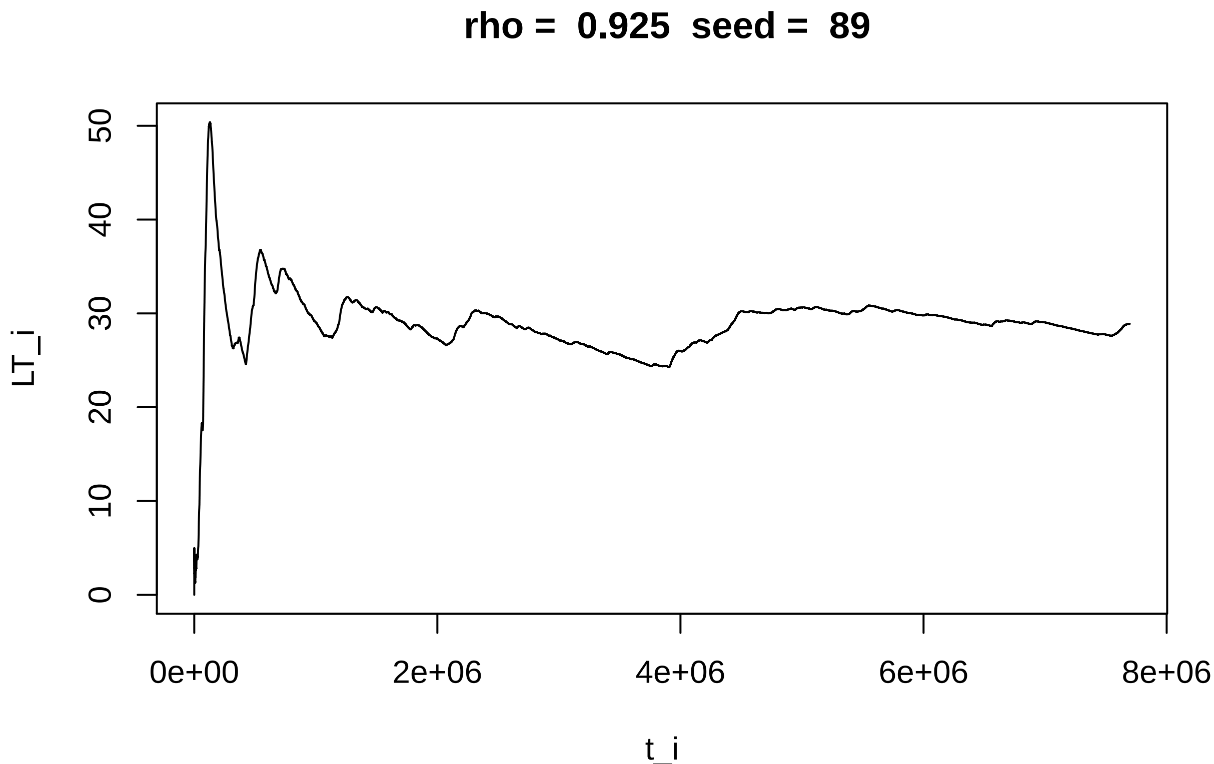

rho = 0.925 seed = 10102
LT_i
t_i
0
20
40
60
80
100
0e+00
2e+06
4e+06
6e+06
8e+06

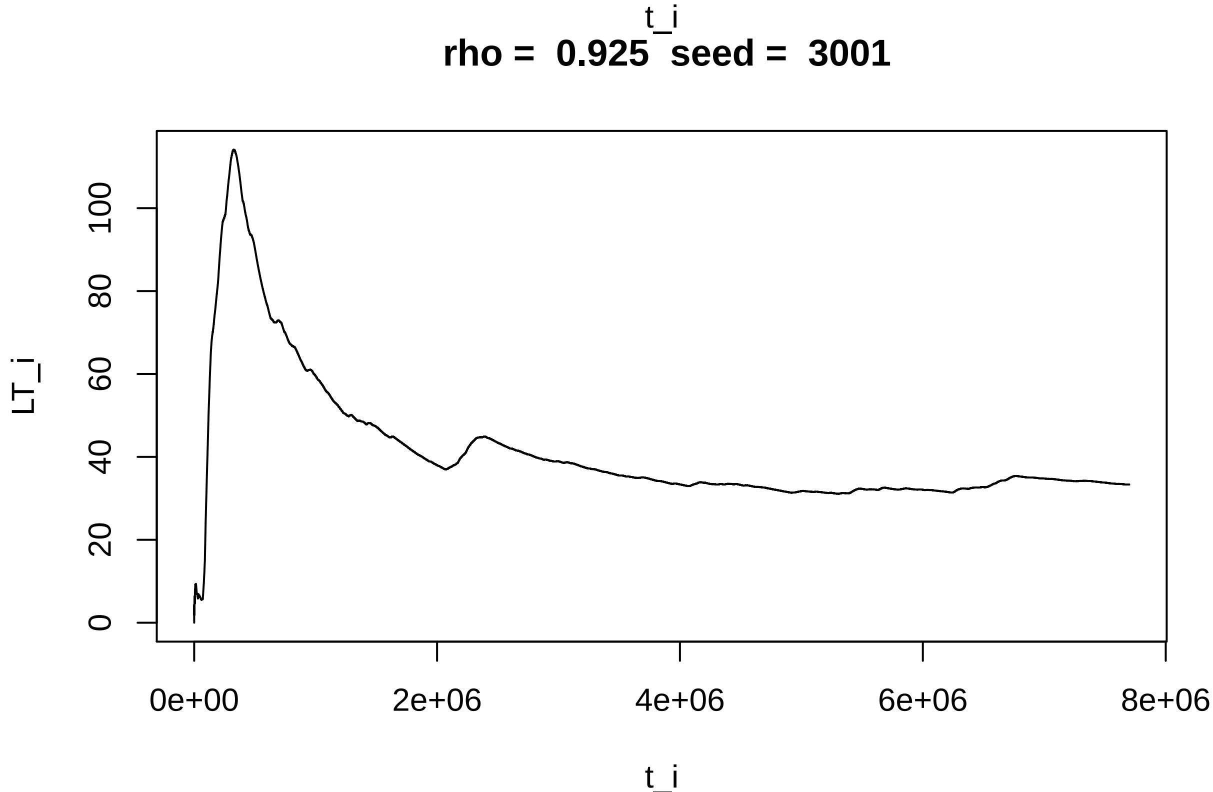

rho = 0.925 seed = 970
LT_i
0
5
10
15
20
25
30
35
0e+00
2e+06
4e+06
6e+06
8e+06
t_i

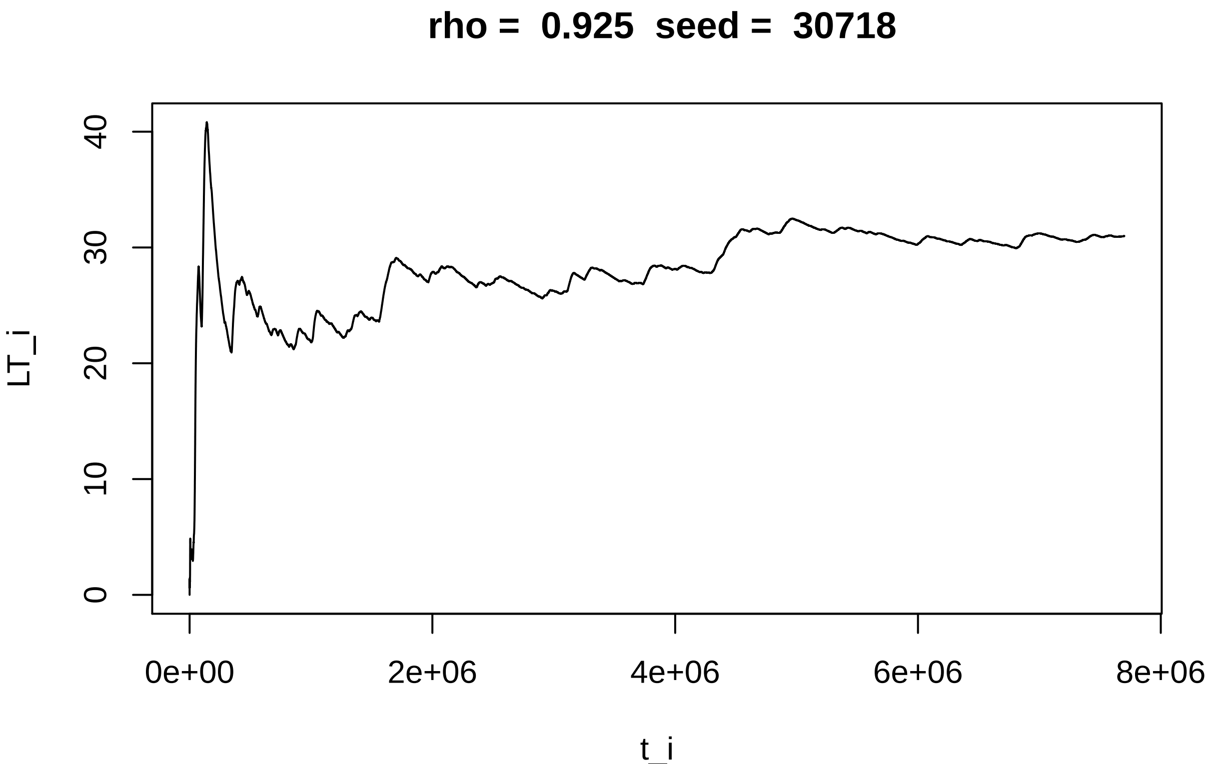

rho = 0.925 seed = 1151
LT_i
0
10
20
30
40
50
0e+00
2e+06
4e+06
6e+06
t_i

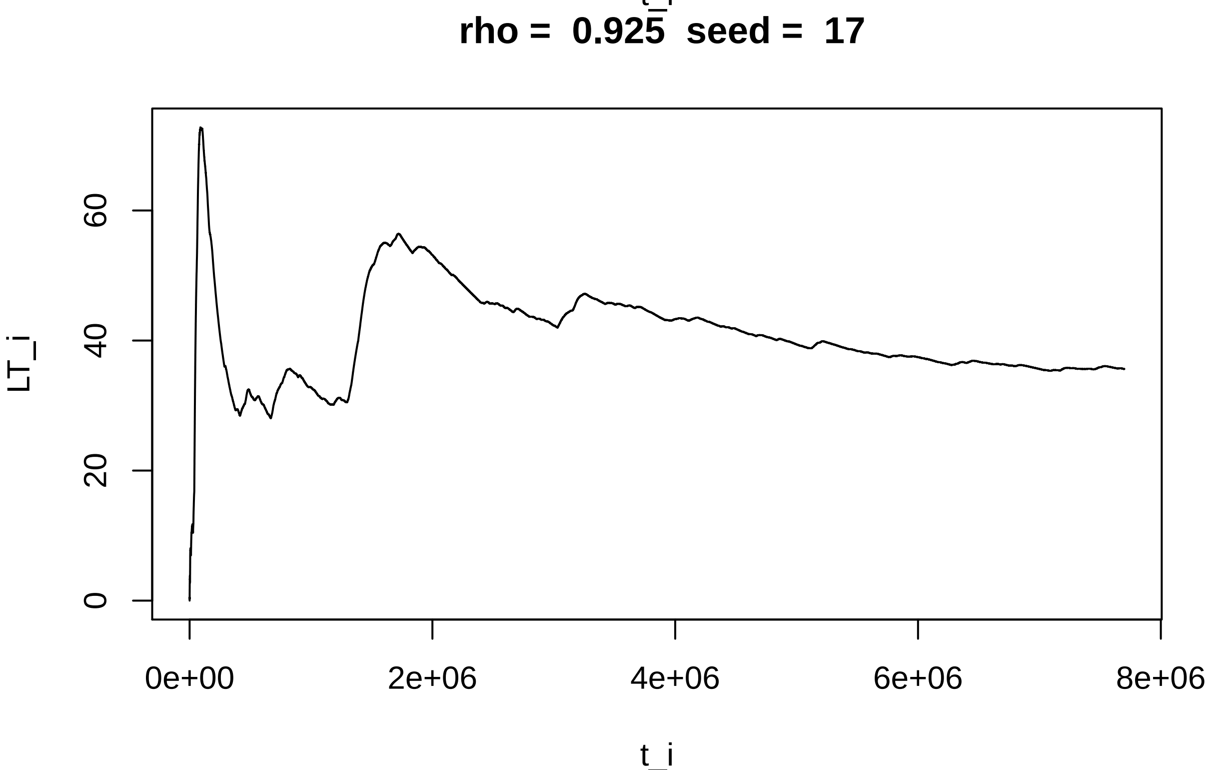

rho = 0.925 seed = 999
LT_i
0
10
20
30
40
0e+00
2e+06
4e+06
6e+06
8e+06
t_i


rho = 0.925 seed = 55
LT_i
0
20
40
60
80
0e+00
2e+06
4e+06
6e+06
t_i


rho = 0.925 seed = 89
LT_i
t_i
0
10
20
30
40
50
0e+00
2e+06
4e+06
6e+06
8e+06


rho = 0.925 seed = 3001
LT_i
t_i
0
20
40
60
80
100
0e+00
2e+06
4e+06
6e+06
8e+06





We observe that . . . PENDING <—- can we remove this?

We now compute the confidence interval for L_q and W_q using the same procedure we detailed for $\rho = 0.925$. We will use a t-Student distribution with critical value 1 - 0.95 and 9 degrees of freedom to compute a 95% confidence interval. The computations produce the followings confidence intervals for the average queue length and waiting time:

ρ	-C.I W_q	+C.I W_q	-C.I L_q	+C.I L_q
0.925	30.8481543	34.8003273	2375.1731581	2678.1414924

Comparison of Allen Cuneen's approximation and the simulation

ρ	W_q	-C.I W_q	+C.I W_q	L_q	-C.I L_q	+C.I L_q
0.4	69.1507968	51.0353731	55.5687961	0.8980623	0.6629266	0.7216626
0.7	423.5486306	379.3389492	412.4915102	5.5006316	4.9266678	5.3560657
0.85	1249.0362678	1046.5794927	1160.0665074	16.2212502	13.5934229	15.0731043
0.925	2958.3575269	2375.1731581	2678.1414924	38.4202276	30.8481543	34.8003273

As we can see in the table above Allen-Cuneen's approximation values for all loading factors are outside of the confidence interval of our simulations, always above. Overall, the approximations of both, W_q and L_q, are not very far from the simulation's confidence interval.

So the Allen-Cuneen's aproximation models a system where the occupancy is higher and the waiting times in the queue are also greater than what our simulation produces. How can we explain that?

First of all, our random number generator has not been tested. As discussed in class, it is impossible to generate real randomness, so every rng must be tested for multiple desirable properties. In our particular case, even though we used the rng that ships with R, which is considered to ship with well tested generated random number streams, we noticed that while repeatedly running the simulation we needed to change the seed from time to time in order to avoid irregularities or abrupt changes. We can conclude that we need to test the rng for appropriate period, given that we need around 200.000 samples in each simulation.

Secondly, the Allen Cuneen's approximation uses the exact values for exponential models and a correction factor for our model. Causes to the difference between aproximation and simulation:

1. Our model is not so close to a heavy tail distribution, so the correction factor is not accurate in the Allen Cuneen's formula.
2. We cannot use the Allen Cuneen's approximation formula with our model. We need to use the Kôllerstrôm formula, as our system is in "heavy traffic" conditions (it is a heavy tail distribution and that means heavy traffic?)
3. Something related to Ctau and Cx given that arrivals is a Normal distr. and service times is lognormal?

ρ	W_q	-C.I W_q	+C.I W_q	L_q	-C.I L_q	+C.I L_q
0.4	0.8336202	51.0353731	55.5687961	2.8977492	0.6629266	0.7216626
0.7	1.6672404	379.3389492	412.4915102	17.6182641	4.9266678	5.3560657
0.85	3.3344807	1046.5794927	1160.0665074	51.8958797	13.5934229	15.0731043
0.925	6.6689615	2375.1731581	2678.1414924	122.8694038	30.8481543	34.8003273

We can observe that Köllerström formula does not apply to this situation. We conclude that we are not in heavy traffic conditions.