FIR Filters

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1 Mean Delay

The concept of mean delay falls out of the concept of group delay. Group delay is the negative derivative of the phase response of a filter. For symmetric FIR filters, the phase response tends to be linear. As a result, the mean delay becomes equal to the group delay. The mean delay signifies the number of samples by which the amplitude envelopes of an incoming signal are delayed by the action of some kind of intermediate system.

1.1 Moving Average Filter

$$y[n] = \frac{1}{N}(x[n] + x[n-1] + .. + x[n-N+1])$$

Taking the z-transform of this difference equation,

$$Y(z) = \frac{1}{N}(X(z) + z^{-1}X(z) + .. + z^{N-1}X(z))$$
$$Y(z) = \frac{1}{N}X(z)(1 + z^{-1} + ... + z^{N-1})$$
$$H(z) = \frac{1}{N}(1 + z^{-1} + ... + z^{N-1})$$

The Frequency Response Function (FRF) can be obtained along the unit circle by setting $z = e^{j\hat{\omega}}$

$$H(e^{j\hat{\omega}}) = \frac{1}{N}(1 + e^{-j\hat{\omega}} + e^{-2j\hat{\omega}} + ... + e^{-(N-1)j\hat{\omega}})$$

We know how to sum a geometric series,

$$\sum_{k=1}^{N} \alpha^k = \frac{1-\alpha^N}{1-\alpha}$$

This implies the following reduction for our FRF

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \frac{1}{N}(\frac{1-e^{-j\hat{\omega}N}}{1-e^{-j\hat{\omega}}}) \\ &= \frac{sin(\hat{\omega}N/2)}{Nsin(\hat{\omega}/2)}e^{-j\hat{\omega}(N-1)/2} \end{aligned}$$

Now, the phase response can be read of directly from this FRF,

$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}(N-1)/2$$

Now, the group delay is defined as the negative derivative of the phase response of the filter. Since the phase response of the filter is linear, its group delay should be constant (mod the wrap around at pi and -pi). As a result, the mean delay will be equal to the group delay

$$MeanDelay = \frac{N-1}{2} \tag{1}$$

Another way to arrive at this expression of mean delay is to make a center of mass argument. The mean delay for such an FIR filter can be given by the center of mass of the impulse response of that FIR filter.

Consider an expression for the center of mass of an impulse response $h[n]$

$$c_h = \frac{\sum_{n=1}^{N} nh[n]}{\sum_{n=1}^{N} h[n]}$$

Using this idea of center of mass, we can arrive at the following expression for the mean delay:

$$MeanDelay = \frac{N-1}{2}$$

1.2 Harmonically Tapped Filter

The difference equation for the harmonically tapped filter is given as follows:

$$y[n] = \frac{2}{M(M+1)}(Mx[n] + (M-1)x[n-1] + ... + x[M-N+1])$$

As for the moving average filter, I will use the idea of the center of mass of the impulse response to find the mean delay of the harmonically tapped filter.

To find the impulse response of the harmonically tapped filter, we set the input $x[n] = \delta[n]$. The response of the filter then is called the impulse response $h[n]$.

Using the center of mass formula described above,

$$\begin{aligned} c_h &= \frac{\sum_{n=0}^{M-1} nh[n]}{\sum_{n=0}^{M-1} h[n]} \\ &= \frac{(M-1) + 2(M-2) + 3(M-3) + ... + (M-1)}{M + M - 1 + .. + 1} \end{aligned}$$

Using Mathematica, I found that the numerator simplifies as follows:

$$\sum_{n=0}^{M-1} nh[n] = \frac{M(M-1)(M+1)}{6}$$

The denominator is simply a sum of an arithmetic series which simplifies as follows:

$$\sum_{n=0}^{M-1} n = \frac{M(M+1}{2}$$

Getting back to the center of mass calculation,

$$\begin{aligned} c_h &= \frac{2M(M+1)(M-1)}{6M(M+1)} \\ &= \frac{M-1}{3} \end{aligned}$$

For a 5-point harmonically tapped filter, this simplifies to

$$\begin{aligned} c_h &= \frac{5-1}{3} \\ &= \frac{4}{3} \end{aligned}$$

Since the mean delay is given by the center of mass of the impulse response,

$$MeanDelay = \frac{M-1}{3} \tag{2}$$

2 SNR Boost

Consider an input signal $x[n]$ which has some white gaussian noise given by $z[n]$. Now this signal is fed into an FIR filter with impulse response $h[n]$ and produces an output given by $y[n]$. In this section, I will derive an expression for the ratio of output noise to input noise as a result of the FIR filter.

Without any loss of generality, let us assume that the input signal $x[n]$ is DC with all variance appearing as noise. Let us also assume that the white gaussian noise is zero mean.

$$\begin{aligned} \sigma_y^2 &= E[(Y-\mu_y)(Y-\mu_y)^T] \\ &= E[Hxx^TH^T + Hzz^TH^T] \\ &= H(E[xx^T] + E[zz^T])H^T \\ &= H(\sigma_x^2 + \sigma_z^2)H^T \\ &= HH^T(\sigma_x^2 + \sigma_z^2) \end{aligned}$$

As mentioned before, we can think of a DC $x[n]$ and think about what the FIR filter does to its noise. In that case, the term $\sigma_x^2 = 0$ and we are left with:

$$\sigma_y^2 = HH^T\sigma_z^2$$

Since H is the matrix form of the 1-D impulse response, and since the impulse response can be represented as a row vector, the above equation can be reduced to the following:

$$\sigma_y^2 = \sum_{i=1}^{N-1} h_i^2\sigma_z^2$$
$$\frac{\sigma_y^2}{\sigma_x^2} = \sum_{i=1}^{N-1} h_i^2$$

Thus, we have proved that the ratio of the variances of the output to the input noises can be given by the sum of the squares of the filter coefficients.

The SNR Boost then, which is given by the ratio of standard deviations, can be given as follows

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\sum_{i=0}^{N-1} h_i^2} \tag{3}$$

2.1 Moving Average Filter

Using equation 3, we can find the SNR Boost for an N-point moving average filter.

$$\begin{aligned}\frac{\sigma_y^2}{\sigma_x^2} &= \sum_{i=1}^{N-1} h_i^2 \\ &= N\frac{1}{N^2} \\ &= \frac{1}{N}\end{aligned}$$

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\frac{1}{N}} \tag{4}$$

2.2 Harmonically Tapped Filter

Again using equation 3, we can find the SNR boost for our M point harmonically tapped filter

$$\begin{aligned}\frac{\sigma_y^2}{\sigma_x^2} &= \sum_{i=1}^{N-1} h_i^2 \\ &= \frac{1}{(M(M+1)/2)^2}(M^2 + (M-1)^2 + ... + 1^2) \\ &= \frac{M(M+1)(2M+1)/6}{M^2(M+1)^2/4} \\ &= \frac{2}{3}\frac{2M+1}{M(M+1)}\end{aligned}$$

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\frac{4M+2}{3M(M+1)}} \tag{5}$$

3 Questions

3.1 SNR boost of 5

To get an SNR boost of 5, we want the ratio of input to output noise to be 5.

For the moving average filter, this happens when

$$\sqrt{N} = 5$$

$$N = 5^2 = 25 \tag{6}$$

For the harmonically tapped filter, this happens when $\frac{\sigma_y}{\sigma_x}$ is 1/5. This happens when

$$M = 33 \tag{7}$$

3.2 Mean Delay at depth Required for SNR Boost of 5

For the moving average filter, the mean delay is 12 samples and for the harmonically tapped filter, the mean delay is about 10 or 11 samples. Thus the harmonically tapped filter has a lower mean delay than the moving average filter.

3.3 Comparison

The trade off between SNR boost and Mean delay is evident from the discussion above. The moving average filter because it gives the same SNR boost as the harmonically tapped filter in 8 fewer samples. For that given SNR boost, the harmonically tapped filter gives a mean delay of about 1 sample less than the moving average filter. If we cared about SNR boost and mean delay in equal measure, then I would be inclined to choose the moving average filter over the harmonically tapped filter.