1.
$$A = \begin{bmatrix} 1 & 2a \\ 2 & 3 \end{bmatrix}, \quad \mathbb{Z}_5$$

a) $\det(A) \neq 0$

$3 - 4a \neq 0$

$a \neq \frac{3}{4} = 3 \cdot 4^{-1}$

$= 3 \cdot 4 = 2$

$\Downarrow$

$a \neq 2$ $\quad a \in \{0, 1, 3, 4\}$

$4 \cdot 4^{-1} = 1$

$4 \cdot 0 = 0$
$4 \cdot 1 = 4$
$4 \cdot 2 = 3$
$4 \cdot 3 = 2$
$4 \cdot 4 = 1 \Rightarrow 4^{-1} = 4$

b) $AX = B$ $\quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$X = A^{-1} B$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}^T$$

$-2 = 3$
$-1 = 4$

$\det A = 3 - 2 = 1$

$$= 1 \cdot \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 22 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

② a) Prostor nad kojim je definiran skalarni umnožak.
Skalarni produkt nad X je:

$$(\cdot|\cdot) : X \times X \to \mathbb{R} \text{ (ili } \mathbb{C})$$

Za s. p. vrijedi:

a) $(x|x) \geq 0$ i $(x|x) = 0 \Longleftrightarrow x = 0$

b) $(\lambda x|y) = \lambda(x|y)$

c) $(x|y) = (y|x)$ (za $\mathbb{C}$: $(x|y) = \overline{(y|x)}$)

d) $(x+y|z) = (x|z) + (y|z)$

b) $T: |(x|y)| \leq \|x\| \cdot \|y\|$

$x, y \in X$

$t \in \mathbb{R}$ $\quad f : \mathbb{R} \to \mathbb{R}$

$$f(t) = (x+yt|x+yt) \geq 0$$

$$= \|x\|^2 + 2(x|y)t + \|y\|t^2 \geq 0$$

$$\Downarrow$$

$$D \leq 0$$

$$b^2 - 4ac = 4(x|y)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$(x|y)^2 \leq \|x\|^2 \cdot \|y\|^2 \quad /\sqrt{\ }$$

$$(x|y) \leq \|x\| \cdot \|y\|$$

c) $L(e_1, e_2, e_3)^\perp$ $\quad e_1 = (1, -1, 2, 0)$ $\quad \mathbb{R}^4$

$e_2 = (0, 1, -3, 1)$

$e_3 = (2, 1, -5, 3)$

$\dim L(e_1, e_2, e_3)^\perp = ?$

$x = (a_1, a_2, a_3, a_4)$

$(x|e_1) = 0 \qquad a_1 - a_2 + 2a_3 + 0 \cdot a_4 = 0$

$(x|e_2) = 0 \qquad 0 + a_2 - 3a_3 + a_4 = 0$

$(x|e_3) = 0 \qquad 2a_1 + a_2 - 5a_3 + 3a_4 = 0$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & -9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a_1 = a_3 - a_4$

$a_2 = 3a_3 - a_4$

$$x = (a_3 - a_4, 3a_3 - a_4, a_3, a_4) = a_3 \underbrace{\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{e} + a_4 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{f}$$

e i f su baza jer : 1) razapinju $L(e_1, e_2, e_3)^\perp$

2) linearno su nezavisni

$\Downarrow$

$$\dim L(e_1, e_2, e_3)^\perp = 2$$

② ③ $Y = \{ p(t) \in P_3 \mid \underbrace{p''(0) + \int_0^1 p(\tau)\,d\tau = 0}_{I},\ \underbrace{\int_{-1}^{1} p(\tau)\,d\tau = 0}_{II} \}$

$p(x) = a + bx + cx^2 + dx^3$

$p''(0) = (b + 2cx + 3dx^2)'\big|_{x=0} = (6dx + 2c)\big|_{x=0} = 2c$

$\int_0^1 (a + bx + cx^2 + dx^3)\,dx = ax\Big|_0^1 + \frac{1}{2}bx^2\Big|_0^1 + \frac{1}{3}cx^3\Big|_0^1 + \frac{1}{4}dx^4\Big|_0^1 =$

$= a + \frac{1}{2}b + \frac{1}{3}c + \frac{1}{4}d$

$\int_{-1}^{1} \dots = ax\Big|_{-1}^{1} + \frac{1}{2}bx^2\Big|_{-1}^{1} + \frac{1}{3}cx^3\Big|_{-1}^{1} + \frac{1}{4}dx^4\Big|_{-1}^{1} = 2a + 0 + \frac{2}{3}c + 0$

$= 2a + \frac{2}{3}c$

$I \to a + \frac{1}{2}b + \frac{7}{3}c + \frac{1}{4}d = 0$

$II \to 2a + \frac{2}{3}c = 0$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{7}{3} & \frac{1}{4} \\ 2 & 0 & \frac{2}{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{7}{3} & \frac{1}{4} \\ 0 & -1 & -4 & -\frac{1}{2} \end{bmatrix}$$

$0 \quad -\frac{1}{2} \quad -2 \quad -\frac{1}{4}$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 4 & \frac{1}{2} \end{bmatrix}$$

$a = -\frac{1}{3}c$

$b = -4c - \frac{1}{2}d$

$(-\frac{1}{3}c, -4c - \frac{1}{2}d, c, d)$

1) T: $p_1 + p_2 \in Y$

$(-\frac{1}{3}c_1, -4c_1 - \frac{1}{2}d_1, c_1, d_1) + (-\frac{1}{3}c_2, -4c_2 - \frac{1}{2}d_2, c_2, d_2) =$

$= (-\frac{1}{3}(c_1 + c_2), -4(c_1 + c_2) - \frac{1}{2}(d_1 + d_2), c_1 + c_2, d_1 + d_2)$

$= (-\frac{1}{3}C, -4C - \frac{1}{2}D, C, D) \in Y$

2) T: $\alpha p \in Y$

$$\left(-\frac{1}{3}\alpha c, -4\alpha c - \frac{1}{2}\alpha d, \alpha c, \alpha d\right) = \begin{bmatrix} \alpha c = C \\ \alpha d = D \end{bmatrix} =$$

$$= \left(-\frac{1}{3}C, -4C - \frac{1}{2}D, C, D\right)$$

iz 1) i 2) $\Rightarrow$ Y je podprostor od P_3

baza

$$c\underbrace{\left(-\frac{1}{3}, -4, 1, 0\right)}_{\vec{e}_1} + d\underbrace{\left(0, -\frac{1}{2}, 0, 1\right)}_{\vec{e}_2}$$

ovo je baza jer: 1) razapinje Y

2) $\vec{e}_1$ i $\vec{e}_2$ linearno su nezavisni

$\Downarrow$

$\dim Y = 2$

④

$M = \left\{ x \in \mathbb{R}^4 \mid x_1 + x_3 - x_4 = 0, \ x_1 + x_2 + 2x_3 - 2x_4 = 0 \right\}$

$N = L\left((1,0,0,0), \ (1,-1,0,-1) \right)$

$M \cap N$

M

$\underline{m = (x_1, x_2, x_3, x_4)}$

$$x_1 + 0 + x_3 - x_4 = 0$$
$$x_1 + x_2 + 2x_3 - 2x_4 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$x_1 = -x_3 + x_4$$
$$x_2 = -x_3 + x_4$$

$$m = (-x_3 + x_4, -x_3 + x_4, x_3, x_4)$$
$$= x_3 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{m}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{m}_2}$$

$\vec{m}_1$ i $\vec{m}_2$ čine bazu $\overbrace{\text{ }}^{\text{na } M}$ jer:

1) razapinju M
2) linearno su nezavisni

$\Downarrow$

$\dim M = 2$

$\underline{M \cap N}$

$$\lambda_1 m_1 + \lambda_2 m_2 = \lambda_3 n_1 + \lambda_4 n_2$$

$$\lambda_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \lambda_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \lambda_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

$$-\lambda_1 + \lambda_2 + \lambda_4 = 0$$

$$\lambda_1 = 0$$

$$0 + \lambda_2 + 0 + \lambda_4 = 0$$

$$\begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -\lambda_4 \\ \lambda_3 &= -2\lambda_4 \end{aligned}$$

$f \in \{M \cap N\}$ $\quad \vec{f} = \lambda_1 \vec{m}_1 + \lambda_2 \vec{m}_2 = 0 - \lambda_4 \vec{m}_2$

$$= \lambda_4 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{f}_1}$$

$\vec{f}_1$ je baza na $M \cap N$ jer:

1) razapinje $M \cap N$

2) vektor je linearno nezavisan

5. $g(x) = x$

$h(x) = \frac{3}{2}x^2 - \frac{1}{2}$ $\qquad f(x) = \begin{cases} x, & x \in \langle -1, 0 \rangle \\ 2x & x \in [0, 1 \rangle \end{cases}$

$L^2(-1, 1)$

$T: (g|h) = 0 \qquad \int_{-1}^{1} g(x)h(x)\,dx = \int_{-1}^{1} \left(\frac{3}{2}x^3 - \frac{1}{2}x\right)dx = \frac{3}{8}x^4\Big|_{-1}^{1} - \frac{1}{4}x^2\Big|_{-1}^{1}$

$$= \frac{3}{8} - \frac{3}{8} - \left(\frac{1}{4} - \frac{1}{4}\right) = 0$$

$$P_f = \frac{(f|g)}{\|g\|^2}g + \frac{(f|h)}{\|h\|^2}h$$

$$(f|g) = \int_{-1}^{0} x^2\,dx + \int_{0}^{1} 2x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{0} + \frac{2}{3}x^3\Big|_{0}^{1} = 0 - \left(\frac{-1}{3}\right) + \frac{2}{3} - 0$$

$$= 1$$

$$(f|h) = \int_{-1}^{0}\left(\frac{3}{2}x^3 - \frac{1}{2}x\right)dx + 2\int_{0}^{1}\left(\frac{3}{2}x^3 - \frac{1}{2}x\right)dx$$

$$= \left(\frac{3}{8}x^4 - \frac{1}{4}x^2\right)\Big|_{-1}^{0} + 2\left(\frac{3}{8}x^4 - \frac{1}{4}x^2\right)\Big|_{0}^{1}$$

$$= \left(0 - 0 - \left[\frac{3}{8} - \frac{1}{4}\right]\right) + 2\left(\frac{3}{8} - \frac{1}{4} - 0 - 0\right)$$

$$= -\frac{1}{8} + \frac{2}{8} = \frac{1}{8}$$

$$\|g\|^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$$\|h\|^2 = \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx = \int_{-1}^{1}\left(\frac{9}{4}x^4 - \frac{3}{2}x^2 + \frac{1}{4}\right)dx$$

$$= \frac{9}{20}x^5\Big|_{-1}^{1} - \frac{1}{2}x^3\Big|_{-1}^{1} + \frac{1}{4}x\Big|_{-1}^{1}$$

$$= \frac{9}{20}(1+1) - \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 2 = \frac{9}{10} - 1 + \frac{1}{2} = -\frac{1}{10} + \frac{1}{2} = \frac{4}{10} = \frac{2}{5}$$

$$\Rightarrow P_f = \frac{3}{2}x + \underbrace{\frac{1}{8}\cdot\frac{5}{2}}_{\frac{5}{16}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{3}{2}x + \frac{15}{32}x^2 - \frac{5}{32}$$

(6.) $\|A\| = |a| + 2|b| + |d| \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

a)

1) POZITIVNOST

T: $\|A\| \geq 0$ i $\|A\| = 0 \iff A = 0$

$|a| + 2|b| + |d| \geq 0$ ✓ $\qquad \|A\|$ je 0 akko $A = 0$ ✓

2) HOMOGENOST

T: $\|\lambda A\| = \lambda \|A\|$

$$\|\lambda A\| = \left\|\begin{matrix} \lambda a & \lambda b \\ 0 & \lambda c \end{matrix}\right\| = |\lambda a| + 2|\lambda b| + |\lambda d| = \lambda(|a| + 2|b| + |d|)$$

$$= \lambda \|A\|$$

✓

3) T: $\|x+y\| \leq \|x\| + \|y\|$

$$|a_1+a_2| + 2|b_1+b_2| + |d_1+d_2| \leq |a_1| + |a_2| + 2|b_1| + 2|b_2| + |d_1| + |d_2|$$

budući da vrijedi $|m+n| \leq |m| + |n|$,
nejednakost vrijedi

b) Zbog pozitivnosti!

za matrice oblika $a, b, d = 0$ i $c \neq 0$
dobije se da je norma matrice 0,
što krši pozitivnost.

17. a) jezgra: skup svih vektora $a \in X$ za koje vrijedi

$$\mathcal{A}(a) = 0$$

1) zatvorenost na +

$a, b \in \ker(X)$

T: $\mathcal{A}(a) + \mathcal{A}(b) = 0 + 0 = 0$

2) zatvorenost na množenje sa skalarom

$a \in \ker(X)$

$\mathcal{A}(\alpha a) = \Big/$ vrijedi homogenost za lin. op. $\Big/ = \alpha \mathcal{A}(a)$

$= \alpha \cdot 0 = 0$

b) slika: $\{ y \in Y \mid y = \mathcal{A}(x) \text{ za neki } x \in X \}$

1) zatvorenost na +

$a, b \in \mathrm{Im}(X)$

$a + b = \mathcal{A}(a') + \mathcal{A}(b') = \mathcal{A}(a' + b')$ ✓

2) zatvorenost na mn. za α

$a \in \mathrm{Im}(X)$

$\alpha a = \alpha \mathcal{A}(a') = \mathcal{A}(\alpha a')$ ✓

b) $\underline{A : \mathbb{R}^4 \to \mathbb{R}^3}$

$$a' = a - b + 2c + 0$$
$$b' = a - 3b + 7c - 3d$$
$$c' = 2a - c + 3d$$

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & -3 & 7 & -3 \\ 2 & 0 & -1 & 3 \end{bmatrix}$$

$\mathcal{A}(x) = 0$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 1 & -3 & 7 & -3 & 0 \\ 2 & 0 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 0 & -2 & 5 & -3 & 0 \\ 0 & 2 & -3 & 3 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$a' = 3$

$b' = 9$

$c' = 3$

$(3, 9, 3)$ je baza jer:

1) razapinje jezgru

2) je linearno nezavisan