Real Analysis

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1. Foundations

Definition 1.1 **Analysis Concepts**

1. **Metric:** An abstract notion of distance in a space (not necessarily $\mathbb{R}^n$).
2. **Topology:** An abstract notion of convergence (even in spaces with no underlying notion of distance).

2. Russell's Paradox

Let

$$S = \{T : T \text{ is a set and } T \notin T\}. \tag{1}$$

Is $S \in S$?

3. Constructing Sets

1. **Unions:** If $S = \{T_i\}_{i \in I}$, then

   $$\bigcup_{i \in I} T_i = \{x : \exists i \in I \text{ such that } x \in T_i\} \tag{2}$$

   is a set.

2. **Subsets with Conditions:** If S is a set and $\varphi(x)$ is a condition on elements, then

   $$\{x \in S : \varphi(x)\} \tag{3}$$

   is a set.

3. **Power Set:** If S is a set, then

   $$\mathscr{P}(S) = \{T : T \subseteq S\} \tag{4}$$

   is a set.

4. Cartesian Product

If A and B are sets, then

$$A \times B = \{(a, b) : a \in A, b \in B\}. \tag{5}$$

More generally, if $\{S_i\}_{i \in I}$ is a collection of sets, we can form the product

$$\prod_{i \in I} S_i. \tag{6}$$

An element is a tuple $(s_i)_{i \in I}$ such that $s_i \in S_i$. Formally,

$$\prod_{i \in I} S_i = \left\{ f : I \to \bigcup_{i \in I} S_i : f(i) \in S_i \text{ for all } i \in I \right\}. \tag{7}$$

5. Axiom of Choice (AC)

> **Proposition 5.1** — **Axiom of Choice**
>
> A Cartesian product of non-empty sets is non-empty.

6. Functions

A function $f : A \to B$ assigns each element of A exactly one element of B. Formally,

$$f \subseteq A \times B \text{ is a function} \iff \forall x \in A, \exists! y \in B \text{ such that } (x, y) \in f. \quad (8)$$

6.1. Types of Functions

1. **Injective:** $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$.
2. **Surjective:** $\forall y \in B, \exists x \in A$ such that $f(x) = y$.
3. **Bijective:** f is both injective and surjective.

> **Definition 6.1** — **Cardinality Equivalence**
>
> Two sets A and B have the same cardinality if there exists a bijection $f : A \to B$. We write $A \sim B$.

> **Theorem 6.2** — **Cantor's Theorem**
>
> For any set S, the power set $\mathscr{P}(S)$ has strictly greater cardinality than S: $S \neg \sim \mathscr{P}(S)$.

7. Cardinality

7.1. Properties

1. $A \sim A$ (reflexive)
2. $A \sim B \implies B \sim A$ (symmetric)
3. $A \sim B$ and $B \sim C \implies A \sim C$ (transitive)

7.2. Notations

1. $A \leq B$: there exists an injective map $f : A \to B$
2. $A = B$: $A \sim B$
3. $A < B$: $A \leq B$ and $A \neg \sim B$

8. Schröder-Bernstein Theorem

Theorem 8.1 **Schröder-Bernstein Theorem**

If there are injective maps $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.

9. Finite and Infinite Sets

Definition 9.1 **Finite Sets**

A set S is finite if $\mid S \mid = \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. Otherwise it is infinite.

Definition 9.2 **Dedekind-Infinite Sets**

A set S is Dedekind-infinite if there exists a bijection from S to a proper subset of itself. Otherwise, it is Dedekind-finite.

10. Countability

Definition 10.1 **Countable Sets**

A set S is **countable** if $S \leq \mathbb{N}$. If countable and infinite, we say it is **countably infinite**. Otherwise, it is **uncountable**.

Theorem 10.2 **Countable Union of Countable Sets**

Let I be a countable set, and let $\{S_i\}_{i \in I}$ be a countable collection of countable sets. Then

$$\bigcup_{i \in I} S_i \tag{9}$$

is countable.