Convex and Non-Convex Optimisation

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1. Mathematical Background

Definition 1.1 **Mathspeak**

1. Axiom: A foundational statement accepted without proof. All other results are built ontop.
2. Proposition: A proved statement that is less central than a theorem, but still of interest.
3. Lemma: A helper'' proposition proved to assist in establishing a more important result.
4. Corollary: A statement following from a theorem or proposition, requiring little to no extra proof.
5. Definition: A precise specification of an object, concept or notation.
6. Theorem: A non-trivial mathematical statement proved on the basis of axioms, definitions and earlier results.
7. Remark: An explanatory or clarifying note that is not part of the formal logical chain but gives insight / context.
8. Claim / Conjecture: A statement asserted that requires a proof.

Definition 1.2 **Vector Norm**

A vector norm on $\mathbb{R}^n$ is a function $\|\cdot\|$ from $\mathbb{R}^n$ to $\mathbb{R}$ such that:

a) $\|\mathbf{x}\| \geq 0, \forall\, \mathbf{x} \in \mathbb{R}^n$ and $\|\mathbf{x}\| = 0 \Longleftrightarrow \mathbf{x} = \mathbf{0}$
b) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|\ \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (Triangle Inequality)
c) $\|\alpha\,\mathbf{x}\| = |\alpha|\ \|\mathbf{x}\|\ \forall \alpha \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n$

Definition 1.3 **Continuous Derivatives**

The notation

$$f \in C^k(\mathbb{R}^n) \tag{1}$$

means that the function $f : \mathbb{R}^n \to \mathbb{R}$ possesses continuous derivatives up to order k on $\mathbb{R}^n$.

Example

a) $f \in C^1(\mathbb{R}^n)$ implies each $\frac{\partial f}{\partial x_i}$ exists, and $\nabla f(x)$ is continuous on $\mathbb{R}^n$
b) $f \in C^2(\mathbb{R}^n)$ implies each $\frac{\partial f^2}{\partial x_i y_i}$ exists, and $\nabla^2 f(x)$ forms a continuous Hessian matrix.

Theorem 1.4 **Cauchy Shwarz-Inequality**

$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \tag{2}$$

Definition 1.5 **Closed and Bounded Sets**

A set $\Omega \subset \mathbb{R}^n$ is *closed* if it contains all the limits of convergent sequences of points in Ω.

A set $\Omega \subset \mathbb{R}^n$ is *bounded* if $\exists K \in \mathbb{R}^+$ for which $\Omega \subset B[0, K]$, where $B[0, K] = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq K\}$ is the ball with centre 0.

Definition 1.6 **Standard Vector Function Forms**

If $f_0 \in \mathbb{R}, \mathbf{g} \in \mathbb{R}^n, G \in R^{\{n \times n\}}$:
a) Linear: $f(\mathbf{x}) = \mathbf{g}^T \mathbf{x}$
b) Affine: $f(\mathbf{x}) = \mathbf{g}^T \mathbf{x} + f_0$
c) Quadratic: $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T G \mathbf{x} + \mathbf{g}^T \mathbf{x} + f_0$

Definition 1.7 **Symmetric**

Let $A \in \mathbb{R}^{\{n \times n\}}$ be a symmetric matrix. Then:

a) A has n real eigenvalues.
   a) There exists an orthogonal matrix Q ($Q^\top Q = I$) such that $A = QDQ^\top$ where $D = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ and $Q = [v_1 ... v_n]$ with v_i an eigenvector of A corresponding to eigenvalue λ_i.
   b) $\det(A) = \prod\{i = 1\}^n \lambda_i$ and $\operatorname{tr}(A) = \sum_{\{i=1\}}^n \lambda_i = \sum_{\{i=1\}}^n A_{\{ii\}}$.
   c) A is positive definite $\Longleftrightarrow \lambda_i > 0$ for all $i = 1, ..., n$.
   d) A is positive semi-definite $\Longleftrightarrow \lambda_i \geq 0$ for all $i = 1, ..., n$.
   e) A is indefinite $\Longleftrightarrow$ there exist i, j with $\lambda_i > 0$ and $\lambda_j < 0$.

Definition 1.8 **Leading Principal Minors / Sylvester's Criterion**

A symmetric matrix A is **positive definite** if and only if all leading principal minors of A are positive. The ith principal minor δ_i of A is the determinant of the leading $i \times i$ submatrix of A.

If $\delta_i, i = 1, 2, ..., n$ has the sign of $(-1)^i, i = 1, 2, ..., n$, that is, the values of δ_i are alternatively negative and positive, then A is **negative definite**.

Note that *PSD* only applies if you check **all** principal minors, of which there are $2^n - 1$, as opposed to checking n submatrices here.

2. Convexity

Definition 2.1 **Convex**

A set $\Omega \subseteq \mathbb{R}^n$ is convex $\iff \theta ve(x) + (1-\theta)\,\mathbf{y} \in \Omega$ for all $\theta \in [0,1]$ and for all $\mathbf{x}, \mathbf{y} \in \Omega$. **Note:** there is no such thing as a *concave* set.

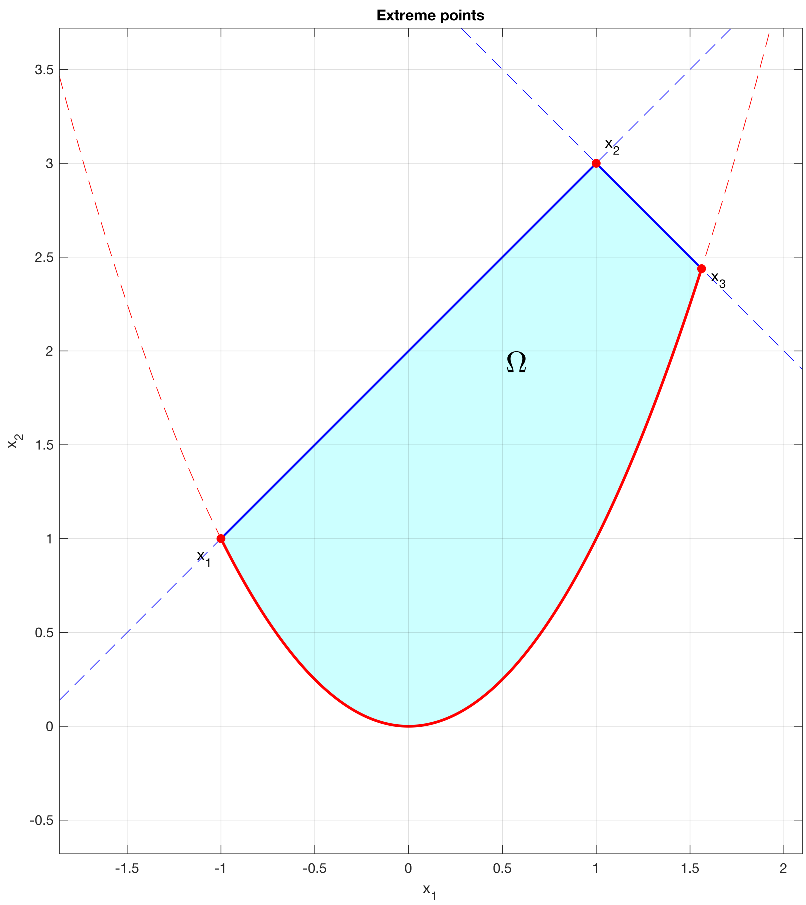
Proposition 2.2 **Intersection of Convex Sets**

Let $\Omega_1, \ldots, \Omega_n \subseteq \mathbb{R}^n$ be convex, then their intersections $\Omega = \Omega_1 \cap \ldots \cap \Omega_n$ is convex.

Definition 2.3 **Extreme Points**

Let $\Omega \subseteq \mathbb{R}^n$ be a convex set $\bar{\mathbf{x}} \in \Omega$ is an extreme point of $\Omega \iff \mathbf{x}, \mathbf{y} \in \Omega, \theta \in [0,1]$ and $\bar{\mathbf{x}} = \theta\,\mathbf{x} + (1-\theta)y \implies \mathbf{x}, \mathbf{y} \in \mathbb{R}$, or $x = y$.

In other words, a point is in an extreme point if it cannot be on a line segment in Ω.



Definition 2.4 **Convex Combination**

The convex combination of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)} \in \mathbb{R}^m$ is

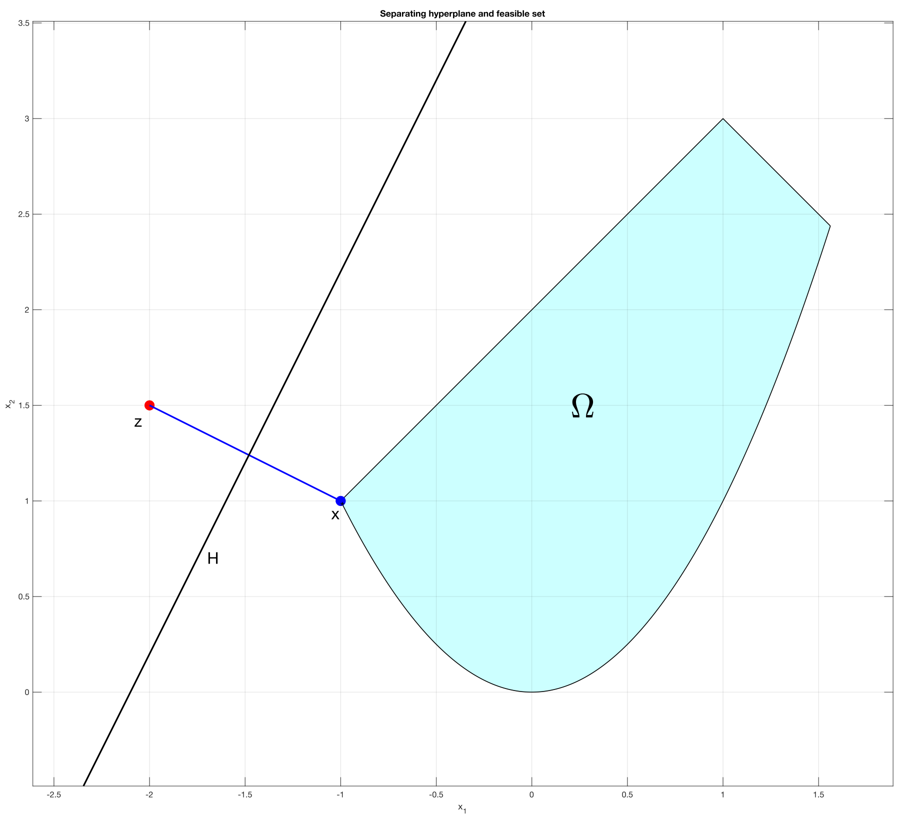
$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \, \mathbf{x}^{(i)}, \text{ where } \sum_{i=0}^{m} \alpha_i = 1 \text{ and } \alpha_i \geq 0, i = 1, \dots, m \tag{3}$$

Definition 2.5 **Convex Hull**

The convex hull $\mathrm{conv}(\Omega)$ of a set Ω is the set of all convex combinations of points in Ω.

Theorem 2.6 **Separating Hyperplane**

Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty closed convex set and let $z \notin \Omega$. There exists a hyperplane $H = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{a}^T \mathbf{u} = \beta\}$ such that $\mathbf{a}^T \mathbf{z} < \beta$ and $\mathbf{a}^T \mathbf{x} \geq \beta$ for all $x \in \Omega$.



Definition 2.7 **Convex / Concave Functions**

A function $f : \Omega \to \mathbb{R}$ (with Ω convex) is

- **convex** if $f(\theta\,\mathbf{x} + (1-\theta)\,\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$;
- **strictly convex** if strict inequality holds whenever $\mathbf{x} \neq \mathbf{y}$;
- **concave** if $-f$ is convex.

3. Unconstrained Optimisation

Definition 3.1 **Standard Form**

$$\underset{\mathbf{x}\in\Omega}{\text{minimise}}\, f(\mathbf{x}) \tag{4}$$

Remark: $\max f(\mathbf{x}) = -\min\{-f(\mathbf{x})\}$

Definition 3.2 **Hessian**

$f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. The Hessian $\nabla^2 f : \mathbb{R}^n \to \mathbb{R}^{\{n\times n\}}$ of f at x is

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix} \tag{5}$$

Theorem 3.3 **First order necessary conditions**

If x^* is a local minimizer and $f \in C^1(\mathbb{R}^n)$ then $\nabla f(x^*) = 0$.

Definition 3.4 **(Unconstrained) Stationary point**

x^* is an unconstrained stationary point $\iff \nabla f(x^*) = 0$

Example

local min, local max, saddle point.

Definition 3.5 **Saddle point**

A stationary point $\mathbf{x}^* \in \mathbb{R}^n$ is a saddle point of f if for any $\delta > 0$ there exist $\mathbf{x}, \mathbf{y}$ with $\|\mathbf{x} - \mathbf{x}^*\| < \delta$, $\|\mathbf{y} - \mathbf{x}^*\| < \delta$ such that:

$$f(\mathbf{x}) < f(\mathbf{x}^*) \quad \text{and} \quad f(\mathbf{y}) > f(\mathbf{x}^*) \tag{6}$$

Proposition 3.6 **Second order necessary conditions**

If $f in C^2(\mathbb{R}^n)$ then
a) Local minimiser $\implies \nabla f(\mathbf{x}^*) = 0$ *and* $\nabla^2 f(\mathbf{x}^*)$ positive semi-definite.
b) Local maximiser $\implies \nabla f(\mathbf{x}^*) = 0$ *and* $\nabla^2 f(\mathbf{x}^*)$ negative semi-definite.

Corollary 3.7 **Local maximiser**

$\bar{\mathbf{x}}$ is a local maximiser $\implies \nabla f(\bar{\mathbf{x}}) = \mathbf{0}$ and $\nabla^2 f(\bar{\mathbf{x}})$ negative semi-definite.

Theorem 3.8 **Second order sufficient conditions**

If $\nabla f(\mathbf{x}^*) = 0$ then
a) $\nabla^2 f(\mathbf{x}^*)$ positive definite $\implies \mathbf{x}^*$ is a *strict* local minimiser.
b) $\nabla^2 f(\mathbf{x}^*)$ negative definite $\implies \mathbf{x}^*$ is a *strict* local maximiser.
c) $\nabla^2 f(\mathbf{x}^*)$ indefinite $\implies \mathbf{x}^*$ is a saddle point.

a) $\nabla^2 f(\mathbf{x}^*)$ positive semi-definite $\implies \mathbf{x}^*$ is *either* a local minimiser or a saddle point!
b) $\nabla^2 f(\mathbf{x}^*)$ negative semi-definite $\implies \mathbf{x}^*$ is *either* a local maximiser or a saddle point! Be careful with these.

Corollary 3.9 **Global Optimums**

[From the sufficiency of stationarity as above, and under the convexity / concavity of $f \in C^2(\mathbb{R}^n)$:
a) f convex $\implies$ any stationary point is a global minimiser.
b) f *strictly* convex $\implies$ stationary point is the *unique* global minimiser.
c) f concave $\implies$ any stationary point is a global maximiser.
d) f *strictly* concave $\implies$ stationary point is the *unique* global maximiser.

]

4. Equality Constraints

Definition 4.1 **Standard Form**

$$\begin{aligned} \underset{\mathbf{x} \in \Omega}{\text{minimise}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{c}_i(\mathbf{x}) = 0 \end{aligned} \tag{7}$$

Definition 4.2 **Lagrangian**

Definition 4.3 **Regular Point**

Definition 4.4 **Matrix of Constraint Gradients**

$$A(\mathbf{x}) = [\nabla \mathbf{c}_i(\mathbf{x}) \;\; \dots \;\; \mathbf{c}_m(\mathbf{x})] \tag{8}$$

Definition 4.5 **Jacobian**

$$\begin{aligned} J(\mathbf{x}) &= A(\mathbf{x})^T \\ &= \begin{bmatrix} \nabla \mathbf{c}_i (\mathbf{x})^T \\ \vdots \\ \mathbf{c}_m (\mathbf{x})^T \end{bmatrix} \end{aligned} \tag{9}$$

Proposition 4.6 **First order necessary optimality conditions**

Corollary 4.7 **Constrained Stationary Point**

Proposition 4.8 **Second order sufficient conditions**

5. Inequality Constraints

Definition 5.1 **Standard Form**

$$\begin{aligned} \underset{\mathbf{x} \in \Omega}{\text{minimise}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{c}_i(\mathbf{x}) = 0, \quad i = 1, ..., m_E \\ & \mathbf{c}_i(\mathbf{x}) \leq 0, \quad i = m_E + 1, ..., m \end{aligned} \tag{10}$$

Definition 5.2 **Convex Problem**

The problem (NLP) is a standard form convex optimisation problem if the objective function f is convex on the feasible set, $\mathbf{c}_i$ is affine for each $i \in$ E, and $\mathbf{c}_i$ is convex for each $i \in \mathcal{I}$.

Definition 5.3 **Active Set**

The set of active constraints at a feasible point $\mathbf{x}$ is $\mathcal{A}(\mathbf{x}) = \{i \in 1, ..., m : \mathbf{c}_i(\mathbf{x}) = 0\}$ **Note** that this concept only applies to inequality constraints.

Definition 5.4 **Regular Point**

Proposition 5.5 **Constrained Stationary Point**

Theorem 5.6 **Karush Kuhn Tucker (KKT) necessary optimality conditions**

kkt generalises lagrange multipliers

Theorem 5.7 **Second-order sufficient conditions for strict local minimum**

Theorem 5.8 **KKT sufficient conditions for global minimum**

Theorem 5.9 **Wolfe Dual Problem**

strong duality, weak duality?

Note **Reduced Hessian**

The reduced Hessian W_Z^* is the projection of the Lagrandian's Hessian onto the tangent space of the constraints at the point x^*

6. General Constrained Optimisation

why does the reduced hessian exist for both? is there any difference when solving?

7. Numerical Methods (unconstrained)

Definition 7.1 **Rates of convergence of iterative methods**

Algorithm 7.2 **Line Search Algorithms**

Algorithm 7.3 **Steepest Descent Method**

Algorithm 7.4 **Newton's Method**

Algorithm 7.5 **Conjugate Gradient Method**

8. Penalty Methods

Definition 8.1 **Penalty function**

Remark

a) $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function $\Longrightarrow \max\left\{\mathbf{c}(\mathbf{x}), 0\right\}^2$ is a convex function
b) $\frac{\partial}{\partial x_i}[\max\{\mathbf{c}(\mathbf{x}), 0\}]^2 = 2\max\{\mathbf{c}(\mathbf{x}), 0\}\frac{\partial}{\partial x_i}$

Theorem 8.2 **Convergence Theorem**

9. Optimal Control Theory

Definition 9.1 **Standard Form**

Definition 9.2 **Hamiltonian**

break this up so it is understandable

Definition 9.3 **Co-state Equations**

Theorem 9.4 **Pontryagin Maximum Principle**

partially free target

non-autonmous problem