Theory of Differential Equations

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1. Definitions

order = the power the differential is raised to.

linear = the dependent variable and it's derivatives are all not non-linear.

$$\underbrace{\frac{\mathrm{d}^2 y}{\mathrm{d}t}}_{\text{linear}} \quad \underbrace{\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{linear}} \quad \underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}}_{\text{non-linear}} \quad \underbrace{y' = e^y}_{\text{non-linear}} \quad \underbrace{y\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{non-linear}} \tag{1}$$

autonomous = independent variable does not appear in the equation

non-autonomous = independent variable *does* appear in the equation

ansatz = our initial guess for the form of a solution, i.e. $y_p = A\cos(t) + B\sin(t)$

indicial equation = a quadratic equation that pops out during the application of the Frobenius method

analytic = a function is analytic at a point if it can be expressed as a convergent power series in a neighborhood of that point

ordinary point = when $p(x)$ and $q(x)$ are analytic at that point

regular singular point = if $P(x) = (x - x_0)p(x)$ and $Q(x) = (x - x_0)^2 q(x)$ are both analytic at x_0.

irregular singular point = not regular.

mean convergence = a sequence of functions f_n converges in mean to f on $[a, b]$ if $\lim_{n\to\infty} \int_a^b |f_{n(x)} - f(x)|^2 \,\mathrm{d}x = 0$

pointwise convergence = a sequence of functions f_n converges pointwise to f on $[a, b]$ if $\lim_{n\to\infty} f_{n(x)} = f(x)$ for every $x \in [a, b]$

uniform convergence = a sequence of functions f_n converges uniformly to f on $[a, b]$ if $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_{n(x)} - f(x)| = 0$

Equilibrium Points and Stability

equilibrium point = a point where the derivative of the dependent variable with respect to the independent variable is zero

stable node = trajectories approach the equilibrium point from all directions and eigenvalues are real and negative

unstable bicritical node ("star") = trajectories move away from the equilibrium point in all directions and eigenvalues are real and positive

stable centre = trajectories orbit around the equilibrium point with eigenvalues that are purely imaginary

unstable saddle point = trajectories approach the equilibrium point in one direction and move away in another, with eigenvalues having opposite signs

unstable focus = trajectories spiral away from the equilibrium point with eigenvalues having positive real parts and non-zero imaginary parts

2. Solving Methods

2.1. First Order

2.1.1. standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2}$$

2.1.2. separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \Longrightarrow \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x \tag{3}$$

2.1.3. reduction to separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \tag{4}$$

with substitution: $y(x) = xv(x)$

2.1.4. linear standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x) \tag{5}$$

2.1.4.1. integrating factor

note, the coefficient of $y'(x)$ must be 1.

$$\varphi(x) = \exp\left(\int p(x)\,\mathrm{d}x\right) \tag{6}$$

multiplying the Linear Standard Form 5 with $\varphi(x)$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varphi y) = \varphi(x)q(x) \Longrightarrow y = \varphi^{-1} \int \varphi q(x)\,\mathrm{d}x \tag{7}$$

2.1.5. exact

A first-order ODE is exact if it can be written in the form:

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{8}$$

where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The solution is then given by: $F(x, y) = C$ where $F(x, y)$ satisfies $\frac{\partial F}{\partial x} = M(x, y)$ and $\frac{\partial F}{\partial y} = N(x, y)$

2.2. Second Order

2.2.1. standard form

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

2.2.2. reducible to first order

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{10}$$

is reducible to the first-order ODE

$$p\frac{\mathrm{d}p}{\mathrm{d}y} + f(y, p) = 0 \tag{11}$$

with substitution $p = \frac{\mathrm{d}y}{\mathrm{d}x}$

2.2.3. constant coefficients

when $p(x)$ and $q(x)$ are constants:

$$y'' + a_1 y' + a_0 y = 0 \tag{12}$$

2.2.3.1. homogenous

solve the characteristic equation:

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{13}$$

cases:

- λ_1, λ_2 are real and distinct
- λ_1, λ_2 are real and coincide (same)
- λ_1, λ_2 are complex conjugates

in each case, the solution of $y(x)$ becomes:

- $y(x) = C\exp(\lambda_1 x) + D\exp(\lambda_2 x)$
- $y(x) = C\exp(\lambda_1 x) + Dx\exp(\lambda_1 x)$
- $y(x) = C\exp(\alpha x)\cos(\beta x) + D\exp(\alpha x)\sin(\beta x) = \exp(\alpha x)(A\cos(\beta x) + B\sin(\beta x))$ by DeMoivre's Theorem

2.2.3.2. inhomogenous → method of undetermined coefficients

$$y(x) = y_{h(x)} + y_{p(x)} \tag{14}$$

guesses for $y_{p(x)}$:

- for $r(x) = P_{n(x)}$ (polynomial of degree n), try $y_{p(x)} = Q_{n(x)}$
- for $r(x) = e^{\alpha x}$, try $y_{p(x)} = Ae^{\alpha x}$
- for $r(x) = \sin(\beta x)$ or $r(x) = \cos(\beta x)$, try $y_{p(x)} = A\sin(\beta x) + B\cos(\beta x)$
- for products of the above forms, try products of the corresponding forms
- if $y_{p(x)}$ is already a solution of the homogeneous equation, multiply by x or x^k until linearly independent

2.2.4. variation of parameters

This method works for any 2nd order inhomogenous ODE if the complementary solution is known.

Theorem: The general solution of the 2nd order inhomogenous ODE:

$$y'' + b_1(x)y' + b_0(x)y = f(x) \tag{15}$$

is given by $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

where y_1 and y_2 are linearly independent solutions of the homogenous ODE such that the Wronskian $W(x) \neq 0$ and

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,\mathrm{d}x \tag{16}$$

and

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,\mathrm{d}x \tag{17}$$

2.2.5. power series method

note, that we embark on this approach because the second order standard form 2.2.1 is not solveable in general with *elementary functions*!

pick ansatz of the form

$$y = \sum_{n=0}^{\infty} a_n z^n \tag{18}$$

and take derivatives as required. for example:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \sum_{n=1}^{\infty} n a_n z^{n-1} \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \tag{19}$$

and substitute them into the ODE. Then solve by rearranging indices as necessary to obtain a recurrence relation. Apply the initial conditions and then guess the closed-form solution of the recurrence relation. Change back to the original variables if required.

If x_0 is an ordinary point Section 1 of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{20}$$

then the general solution in a neighbourhood $\mid x - x_0 \mid < R$ may be represented as a power series.

2.2.6. method of frobenius

Theorem: If $x_0 = 0$ is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{21}$$

then there exists at least one series solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}, c_0 \neq 0 \tag{22}$$

for some constant r (index).

2.2.6.1. general indicial equation

$$r(r-1) + p_0 r + q_0 = 0 \tag{23}$$

2.3. n order

admits n linearly independent solutions.

2.3.1. power series expansion (not sure if it works for n order)

For an n^{th} order linear ODE with variable coefficients:

$$a_{n(x)} y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = f(x) \tag{24}$$

We assume a solution of the form:

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \tag{25}$$

Taking derivatives and substituting yields a recurrence relation for coefficients c_k, typically allowing us to determine c_n in terms of $c_0, c_1, \ldots, c_{n-1}$.

2.3.2. reduction of order

any n^{th} order ODE can be formulated as a system of n first order ODE's.

For $y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$, set $y_i = y^{(i-1)}$ for $i = 1, 2, \ldots, n$ to obtain:

$$y_{i'} = y_{i+1} \tag{26}$$

for $i = 1, 2, \ldots, n - 1$

$$y_{n'} = f(x, y_1, y_2, \ldots, y_n) \tag{27}$$

2.4. partial differential equations

2.4.1. standard form (linear, homogenous, 2nd order pde)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \frac{D(\partial u)}{\partial x} + \frac{E(\partial u)}{\partial y} + Fu = 0 \qquad (28)$$

parabolic equation: $B^2 - 4AC = 0$ (Heat Equation 4.11)

hyperbolic equation: $B^2 - 4AC > 0$ (Wave Equation 4.12)

elliptic equation: $B^2 - 4AC < 0$ (Laplace Equation 4.13)

2.4.2. separation of variables

$$U(x,y) = X(x)Y(y) \qquad (29)$$

then $U_x = YX'$ and $U_y = Y'X$ rewrite the PDE with these substitutions, then divide through by XY. Integrate and solve.

2.4.3. change of variables

When a PDE is difficult to solve directly, changing variables can transform it into a simpler form.

For a second-order PDE, the transformation $u = u(\xi, \eta)$ where $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ requires computing:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad (30)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad (31)$$

And similarly for second-order derivatives. The canonical transformations are:

- For hyperbolic: $\xi = x + y, \eta = x - y$ (characteristic coordinates)
- For parabolic: $\xi = x, \eta = y - f(x)$ (transformation along characteristics)
- For elliptic: $\xi = x + iy, \eta = x - iy$ (complex characteristics)

3. systems / dynamical systems

- $\lambda_2 < \lambda_1 < 0 \Longrightarrow \text{stable node}$
- $0 < \lambda_1 < \lambda_2 \Longrightarrow \text{unstable node}$
- $\lambda_1 = \lambda_2, \lambda_1 > 0 \Longrightarrow \text{unstable star}$
- $\lambda_1 = \lambda_2, \lambda_1 < 0 \Longrightarrow \text{stable star}$
- $\lambda_1 < 0 < \lambda_2 \Longrightarrow \text{unstable saddle node}$
- $\Re(\lambda_1) = 0 \Longrightarrow \text{centre, stable}$
- $\Re(\lambda_1) < 0 \Longrightarrow \text{stable focus}$
- $\Re(\lambda_1) > 0 \Longrightarrow \text{unstable focus}$

real canonical form For a linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, the real canonical form depends on the eigenvalues of $\boldsymbol{A}$:

- Real distinct eigenvalues $\lambda_1 \neq \lambda_2$:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{32}$$

- Real repeated eigenvalues $\lambda_1 = \lambda_2$ with linearly independent eigenvectors:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \tag{33}$$

- Real repeated eigenvalues $\lambda_1 = \lambda_2$ with one linearly independent eigenvector:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \tag{34}$$

- Complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{35}$$

4. functions

4.1. wronskian

$$\mathrm{W}(f_1, f_2, \ldots, f_n)(x) = \det\left(\begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_{n(x)} \\ f_{1'}(x) & f_{2'}(x) & \ldots & f_{n'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}\right) \tag{36}$$

note that if a set of functions is linearly dependent, then its Wronskian will equal 0.

4.2. power series, taylor series and maclaurin series expansions

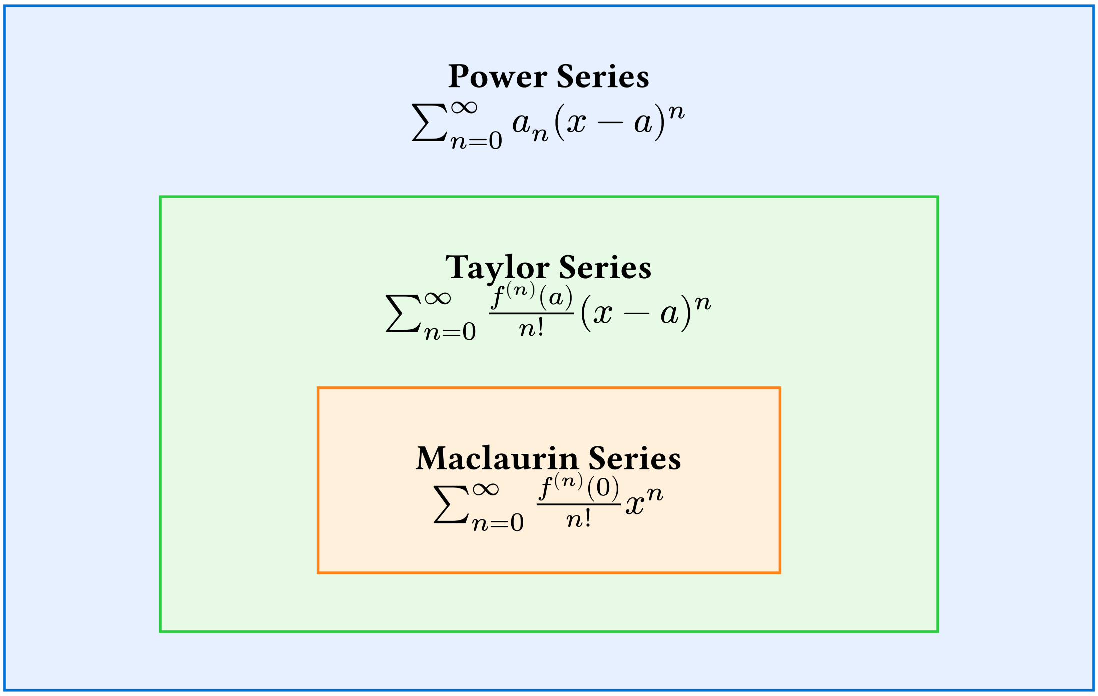

Figure 1: Relationship between power series, Taylor series, and Maclaurin series, showing proper subset relationships

4.3. orthogonality

A set of functions $\{\varphi_n\}_{n=1,2,3,\ldots}$ is said to be orthogonal on the interval $[a, b]$ with respect to the inner product defined by

$$(f, g)_w = \int_a^b w(x) f(x) g(x) \, \mathrm{d}x \tag{37}$$

with weight function $w(x) > 0$, if $(\varphi_n, \varphi_m)_w = 0$ for $m \neq n$.

4.4. orthonormality

a set $\{\varphi_n\}_{n=1,2,3,\ldots}$ is *orthonormal* when in addition to being Section 4.3, $(\varphi_n, \varphi_n) = 1$, for $n = 1, 2, 3, \ldots$.

4.5. cauchy-euler

$x^2 y'' + a_1 x y' + a_0 y = 0$ you can solve this by either letting $x = e^t$ or using the ansatz $y = x^\lambda$ the characteristic equation is $\lambda^2 + (a_1 - 1)\lambda + a_0 = 0$ if you are blessed with the inhomogenous case of above, just use method of undetermined coefficients Section 2.2.3.2.

4.6. legendre

legendre's (differential) equation

$$(1 - x^2) y'' - 2 x y' + n(n+1) y = 0 \tag{38}$$

legendre's polynomials

4.7. bessel

bessel's differential equation

$$y''x^2 + xy' + (x^2 - \nu^2)y = 0 \tag{39}$$

bessel function of the first kind of order α:

$$J_{\alpha(x)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)} \Gamma(m+\alpha+1) \left(\frac{x}{2}\right)^{2m+\alpha} \tag{40}$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}} x \left[x^\alpha J_{\alpha(x)}\right] = x^\alpha J_{\alpha-1}(x) \tag{41}$$

implies

$$\int_0^r x^n J_{n-1}(x) \,\mathrm{d}x = r^n J_{n(r)} \tag{42}$$

for $n = 1, 2, 3, ...$

thus the de admits solutions case 1: $2\nu \notin \mathbb{Z}$

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \tag{43}$$

$J_\nu(x)$, $J_{-\nu}(x)$ linearly independent case 2: $2\nu \in \mathbb{Z}$

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \tag{44}$$

case 3: $\nu \in \mathbb{Z}$ $J_\nu(x)$, $J_{-\nu}(x)$ linearly independent

$$y(x) = AJ_\nu(x) + BY_\nu(x) \tag{45}$$

4.8. laguerre's equation

$$xy'' + (1-x)y' + ny = 0 \tag{46}$$

4.9. hermite's equation

$$y'' - 2xy' + 2ny = 0 \tag{47}$$

4.10. sturm-liouville form

$$(py')' + (q + \lambda r)y = 0 \tag{48}$$

note that Bessel 4.7, Laguerre 4.8, Hermite 4.9 and Legendre 4.6 equations can all be written in this form. furthermore, **any** 2nd order linear homogenous ODE $y'' + a_1(x)y' + [a_2(x) + \lambda a_3(x)]y = 0$ may be written in this form.

4.11. heat equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{49}$$

4.12. wave equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{50}$$

4.13. laplace's equation (pde)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{51}$$

4.14. fourier series

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \tag{52}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\cos(nx)\,\mathrm{d}x, n = 0, 1, 2, ... \tag{53}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\sin(nx)\,\mathrm{d}x, n = 1, 2, ... \tag{54}$$

4.15. parseval's identity

$$\frac{\| f \|^2}{L} = \frac{1}{L}\int_{-L}^{L} f^2\,\mathrm{d}x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{55}$$