

Appendix C

Calculus and Topology in \mathbb{R}^d

C.1. Recall that \mathbb{R}^d is the set of all d -tuples of real numbers; it is a vector space over \mathbb{R} . (The d in \mathbb{R}^d is for dimension; we write \mathbb{R}^d , rather than \mathbb{R}^n , in order to have n available for use as a subscript.) Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be elements of \mathbb{R}^d . The *norm* of x is defined by

$$\|x\| = \left(\sum_{i=1}^d x_i^2 \right)^{1/2}$$

and the *distance* between x and y is defined to be $\|x - y\|$.

C.2. If $x \in \mathbb{R}^d$ and if r is a positive number, then the *open ball* $B(x, r)$ with center x and radius r is defined by

$$B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}.$$

A subset U of \mathbb{R}^d is *open* if for each x in U there is a positive number r such that $B(x, r) \subseteq U$. A subset of \mathbb{R}^d is *closed* if its complement is open. A point x in \mathbb{R}^d is a *limit point* of the subset A of \mathbb{R}^d if for each positive r the open ball $B(x, r)$ contains infinitely many points of A (this is equivalent to requiring that for each positive r the ball $B(x, r)$ contain at least one point of A distinct from x). It is easy to check that a subset of \mathbb{R}^d is closed if and only if it contains all of its limit points.

If A is a subset of \mathbb{R}^d , then the *closure* of A is the set \bar{A} (or A^-) that consists of the points in A , together with the limit points of A ; \bar{A} is closed and is, in fact, the smallest closed subset of \mathbb{R}^d that includes A .

C.3. A subset A of \mathbb{R}^d is *bounded* if there is a real number M such that $\|x\| \leq M$ holds for each x in A .

C.4. (Proposition) *Let U be an open subset of \mathbb{R} . Then there is a countable collection \mathcal{U} of disjoint open intervals such that $U = \cup \mathcal{U}$.*

Proof. Let \mathcal{U} consist of those open subintervals I of U that are maximal, in the sense that the only open interval J that satisfies $I \subseteq J \subseteq U$ is I itself. Of course $\cup \mathcal{U} \subseteq U$. One can verify the reverse inclusion by noting that if $x \in U$, then the union of those open intervals that contain x and are included in U is an open interval that contains x and belongs to \mathcal{U} . It is easy to check (do so) that the intervals in \mathcal{U} are disjoint from one another. If for each I in \mathcal{U} we choose a rational number x_I that belongs to I , then (since the sets in \mathcal{U} are disjoint from one another) the map $I \mapsto x_I$ is an injection; thus \mathcal{U} has the same cardinality as some subset of \mathbb{Q} , and so is countable (see item A.6 in Appendix A). \square

C.5. A sequence $\{x_n\}$ of elements of \mathbb{R}^d converges to the element x of \mathbb{R}^d if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$; x is then called the *limit* of the sequence $\{x_n\}$ (note that here x and x_1, x_2, \dots are elements of \mathbb{R}^d ; in particular, x_1, x_2, \dots are *not* the components of x). A sequence in \mathbb{R}^d is *convergent* if it converges to some element of \mathbb{R}^d .

C.6. Let A be a subset of \mathbb{R}^d , and let x_0 belong to A . A function $f: A \rightarrow \mathbb{R}$ is *continuous at x_0* if for each positive number ε there is a positive number δ such that $|f(x) - f(x_0)| < \varepsilon$ holds whenever x belongs to A and satisfies $\|x - x_0\| < \delta$; f is *continuous* if it is continuous at each point in A . The function $f: A \rightarrow \mathbb{R}$ is *uniformly continuous* if for each positive number ε there is a positive number δ such that $|f(x) - f(x')| < \varepsilon$ holds whenever x and x' belong to A and satisfy $\|x - x'\| < \delta$. A function $f: A \rightarrow \mathbb{R}$ is *continuous on* (or *uniformly continuous on*) the subset A_0 of A if the restriction of f to A_0 is continuous (or uniformly continuous).

C.7. Let A be a subset of \mathbb{R}^d , let f be a real- or complex-valued function on A , and let a be a limit point of A . Then $f(x)$ has *limit L* as x approaches a , written $\lim_{x \rightarrow a} f(x) = L$, if for every positive ε there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ holds whenever x is a member of A that satisfies $0 < \|x - a\| < \delta$.

One can check that $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{n \rightarrow \infty} f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A , all different from a , such that $\lim_{n \rightarrow \infty} x_n = a$. (Let us consider the more difficult half of that assertion, namely that if $\lim_{n \rightarrow \infty} f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A , all different from a , such that $\lim_{n \rightarrow \infty} x_n = a$, then $\lim_{x \rightarrow a} f(x) = L$. We prove this by proving its contrapositive. So assume that $\lim_{x \rightarrow a} f(x) = L$ is not true. Then there exists a positive ε such that for every positive δ there is a value x in A such that $0 < \|x - a\| < \delta$ and $|f(x) - L| \geq \varepsilon$. If for each n we let $\delta = 1/n$ and choose an element x_n of A such that $0 < \|x_n - a\| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$, we will have a sequence $\{x_n\}$ of elements of A , all different from a , that satisfy $\lim_{n \rightarrow \infty} x_n = a$ but not $\lim_{n \rightarrow \infty} f(x_n) = L$.)

C.8. Let A be a subset of \mathbb{R}^d . An *open cover* of A is a collection \mathcal{S} of open subsets of \mathbb{R}^d such that $A \subseteq \cup \mathcal{S}$. A *subcover* of the open cover \mathcal{S} is a subfamily of \mathcal{S} that is itself an open cover of A .

Proofs of the following results can be found in almost any text on advanced calculus or basic analysis (see, for example, Bartle [4], Hoffman [60], Rudin [104], or Thomson et al. [117]).

C.9. (Theorem) Let A be a closed bounded subset of \mathbb{R}^d . Then every open cover of A has a finite subcover.

Theorem C.9 is often called the *Heine–Borel theorem*.

C.10. (Theorem) Let A be a closed bounded subset of \mathbb{R}^d . Then every sequence of elements of A has a subsequence that converges to an element of A .

C.11. It is easy to check that the converses of Theorems C.9 and C.10 hold: if A satisfies the conclusion of Theorem C.9 or of Theorem C.10, then A is closed and bounded. The subsets of \mathbb{R}^d that satisfy the conclusion of Theorem C.9 (hence the closed bounded subsets of \mathbb{R}^d) are often called *compact*. See also Appendix D.

C.12. (Theorem) Let C be a nonempty closed bounded subset of \mathbb{R}^d , and let $f: C \rightarrow \mathbb{R}$ be continuous. Then

- (a) f is uniformly continuous on C , and
- (b) f is bounded on C . Moreover, there are elements x_0 and x_1 of C such that $f(x_0) \leq f(x) \leq f(x_1)$ holds at each x in C .

C.13. (The Intermediate Value Theorem) Let A be a subset of \mathbb{R} , and let $f: A \rightarrow \mathbb{R}$ be continuous. If the interval $[x_0, x_1]$ is included in A , then for each real number y between $f(x_0)$ and $f(x_1)$ there is an element x of $[x_0, x_1]$ such that $y = f(x)$.

C.14. (The Mean Value Theorem) Let a and b be real numbers such that $a < b$. If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on the closed interval $[a, b]$ and differentiable at each point in the open interval (a, b) , then there is a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.