Theory of Differential Equations

Aayush Bajaj

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1. Definitions

order = the power the differential is raised to. linear = the dependent variable and it's derivatives are all not non-linear.

$$\underbrace{\frac{\mathrm{d}^2 y}{\mathrm{d}t}}_{\text{linear}} \quad \underbrace{\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{linear}} \quad \underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}}_{\text{non-linear}} \quad \underbrace{y' = e^y}_{\text{non-linear}} \quad \underbrace{y\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{non-linear}} \tag{1}$$

autonomous = independent variable does not appear in the equation non-autonomous = independent variable *does* appear in the equation

ansatz = our initial guess for the form of a solution, i.e. $y_p = A\cos(t) + B\sin(t)$ indicial equation = a quadratic equation that pops out during the application of the Frobenius method

analytic = a function is analytic at a point if it can be expressed as a convergent power series in a neighborhood of that point ordinary point = when $p(x)$ and $q(x)$ are analytic at that point regular singular point = if $P(x) = (x - x_0)p(x)$ and $Q(x) = (x - x_0)^2 q(x)$ are both analytic at x_0. irregular singular point = not regular.

mean convergence = pointwise convergence = uniform convergence =

equilibrium point = stability
- stable node =
- unstable bicritical node ("star") =
- stable centre =
- unstable saddle point =
- unstable focus =

2. Solving Methods

2.1. First Order

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2.1.1. standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2}$$

2.1.2. separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \Longrightarrow \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x \tag{3}$$

2.1.3. reduction to separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \tag{4}$$

substitution: $y(x) = xv(x)$

2.1.4. linear standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x) \tag{5}$$

2.1.4.1. integrating factor

note, the coefficient of $y'(x)$ must be 1.

$\varphi(x) = \exp(\int p(x)\,\mathrm{d}x)$

multiplying the Linear Standard Form 5 with $\varphi(x)$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varphi y) = \varphi(x)q(x) \Longrightarrow y = \varphi^{-1} \int \varphi q(x)\,\mathrm{d}x \tag{6}$$

2.1.5. exact

2.2. Second Order

2.2.1. standard form

$$y'' + p(x)y' + q(x)y = r(x) \tag{7}$$

2.2.2. reducible to first order

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{8}$$

is reducible to the first-order ODE

$$p\frac{\mathrm{d}p}{\mathrm{d}y} + f(y, p) = 0 \tag{9}$$

with substitution $p = \frac{\mathrm{d}y}{\mathrm{d}x}$

2.2.3. constant coefficients

when $p(x)$ and $q(x)$ are constants:

$$y'' + a_1 y' + a_0 y = 0 \tag{10}$$

2.2.3.1. homogenous

solve the characteristic equation:

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{11}$$

cases:
- λ_1, λ_2 are real and distinct
- λ_1, λ_2 are real and coincide (same)
- λ_1, λ_2 are complex conjugates

in each case, the solution of $y(x)$ becomes:
- $y(x) = C\exp(\lambda_1 x) + D\exp(\lambda_2 x)$
- $y(x) = C\exp(\lambda_1 x) + Dx\exp(\lambda_1 x)$
- $y(x) = C\exp(\alpha x)\cos(\beta x) + D\exp(\alpha x)\sin(\beta x) = \exp(\alpha x)(A\cos(\beta x) + B\sin(\beta x))$ by DeMoivre's Theorem

2.2.3.2. inhomogenous -> method of undetermined coefficients

$$y(x) = y_{h(x)} + y_{p(x)} \tag{12}$$

guesses for $y_{p(x)}$:

2.2.4. variation of parameters

This method works for any 2nd order inhomogenous ODE if the complementary solution is known.

Theorem: The general solution of the 2nd order inhomogenous ODE:

$$y'' + b_1(x)y' + b_0(x)y = f(x) \tag{13}$$

is given by $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

where y_1 and y_2 are linearly independent solutions of the homogenous ODE such that the Wronskian $W(x) \neq 0$ and

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,\mathrm{d}x \tag{14}$$

and

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,\mathrm{d}x \tag{15}$$

2.2.5. power series method

note, that we embark on this approach because the second order standard form 2.2.1 is not solveable in general with *elementary functions*!

pick ansatz of the form

$$y = \sum_{n=0}^{\infty} a_n z^n \tag{16}$$

and take derivatives as required. for example:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \sum_{n=1}^{\infty} n a_n z^{n-1} \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \tag{17}$$

and substitute them into the ODE. Then solve by rearranging indices as necessary to obtain a recurrence relation. Apply the initial conditions and then guess the closed-form solution of the recurrence relation. Change back to the original variables if required.

If x_0 is an ordinary point Section 1 of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{18}$$

then the general solution in a neighbourhood $\mid x - x_0 \mid < R$ may be represented as a power series.

2.2.6. method of frobenius

Theorem: If $x_0 = 0$ is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{19}$$

then there exists at least one series solution of the form $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$ for some constant r (index).

2.2.6.1. general indicial equation

$$r(r-1) + p_0 r + q_0 = 0 \tag{20}$$

2.3. n order

admits n linearly independent solutions.

2.3.1. power series expansion (not sure if it works for n order)

2.3.2. reduction of order

any n^{th} order ODE can be formulated as a system of n first order ODE's.

2.4. partial differential equations

2.4.1. standard form (linear, homogenous, 2nd order pde)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \frac{D(\partial u)}{\partial x} + \frac{E(\partial u)}{\partial y} + Fu = 0 \tag{21}$$

parabolic equation: $B^2 - 4AC = 0$ Heat Equation 4.11 hyperbolic equation: $B^2 - 4AC > 0$ Wave Equation 4.12 elliptic equation: $B^2 - 4AC < 0$ Laplace Equation 4.13

2.5. separation of variables

$$U(x, y) = X(x)Y(y) \tag{22}$$

then $U_x = YX'$ and $U_y = Y'X$ rewrite the PDE with these substitutions, then divide through by XY. Integrate and solve.

2.6. change of variables

3. systems / dynamical systems

- $\lambda_2 < \lambda_1 < 0 \Longrightarrow$ stable node
- $0 < \lambda_1 < \lambda_2 \Longrightarrow$ unstable node
- $\lambda_1 = \lambda_2, \lambda_1 > 0 \Longrightarrow$ unstable star
- $\lambda_1 = \lambda_2, \lambda_1 < 0 \Longrightarrow$ stable star
- $\lambda_1 < 0 < \lambda_2 \Longrightarrow$ unstable saddle node
- $\Re(\lambda_1) = 0 \Longrightarrow$ centre, stable
- $\Re(\lambda_1) < 0 \Longrightarrow$ stable focus
- $\Re(\lambda_1) > 0 \Longrightarrow$ unstable focus

real canonical form

4. functions

4.1. wronskian

$$\mathrm{W}(f_1, f_2, ..., f_n)(x) = \det\left(\begin{pmatrix} f_1(x) & f_2(x) & ... & f_{n(x)} \\ f_{1'}(x) & f_{2'}(x) & ... & f_{n'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & ... & f_n^{(n-1)}(x) \end{pmatrix}\right) \tag{23}$$

note that if a set of functions is linearly dependent, then its Wronskian will equal 0.

4.2. power series, taylor series and maclaurin series expansions

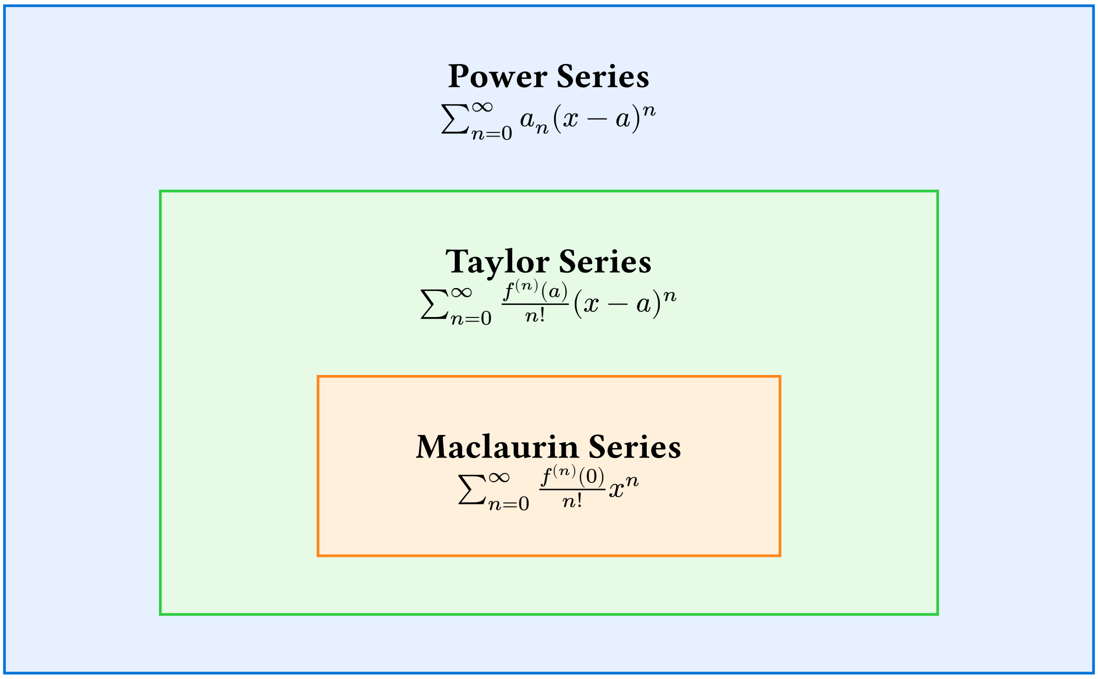

Figure 1: Relationship between power series, Taylor series, and Maclaurin series, showing proper subset relationships

4.3. orthogonality

A set of functions $\{\varphi_n\}_{n=1,2,3,\ldots}$ is said to be orthogonal on the interval $[a, b]$ with respect to the inner product defined by

$$(f, g)_w = \int_a^b w(x) f(x) g(x) \,\mathrm{d}x \tag{24}$$

with weight function $w(x) > 0$, if $(\varphi_n, \varphi_m)_w = 0$ for $m \neq n$.

4.4. orthonormality

a set $\{\varphi_n\}_{n=1,2,3,\ldots}$ is *orthonormal* when in addition to being Section 4.3, $(\varphi_n, \varphi_n) = 1$, for $n = 1, 2, 3, \ldots$.

4.5. cauchy-euler

$x^2 y'' + a_1 x y' + a_0 y = 0$ you can solve this by either letting $x = e^t$ or using the ansatz $y = x^\lambda$ the characteristic equation is $\lambda^2 + (a_1 - 1)\lambda + a_0 = 0$ if you are blessed with the inhomogenous case of above, just use method of undetermined coefficients Section 2.2.3.2.

4.6. legendre

legendre's (differential) equation

$$(1 - x^2) y'' - 2 x y' + n(n+1) y = 0 \tag{25}$$

legendre's polynomials

4.7. bessel

bessel's differential equation

$$y'' x^2 + x y' + (x^2 - \nu^2) y = 0 \tag{26}$$

bessel function of the first kind of order α:

$$J_{\alpha(x)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)} \Gamma(m+\alpha+1) \left(\frac{x}{2}\right)^{2m+\alpha} \tag{27}$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}} x \left[x^\alpha J_{\alpha(x)}\right] = x^\alpha J_{\alpha-1}(x) \tag{28}$$

implies

$$\int_0^r x^n J_{n-1}(x) \,\mathrm{d}x = r^n J_{n(r)} \tag{29}$$

for $n = 1, 2, 3, \ldots$

thus the de admits solutions case 1: $2\nu \notin \mathbb{Z}$

$$y(x) = A J_{\nu(x)} + B J_{-\nu}(x) \tag{30}$$

$J_{\nu(x)}$,$J_{-\nu}(x)$ linearly independent case 2: $2\nu \in \mathbb{Z}$

$$y(x) = A J_{\nu(x)} + B J_{-\nu}(x) \tag{31}$$

case 3: $\nu \in \mathbb{Z}$ $J_{\nu(x)}$,$J_{-\nu}(x)$ linearly independent

$$y(x) = A J_{\nu(x)} + B Y_\nu(x) \tag{32}$$

4.8. laguerre's equation

$$x y'' + (1-x) y' + n y = 0 \tag{33}$$

4.9. hermite's equation

$$y'' - 2 x y' + 2 n y = 0 \tag{34}$$

4.10. sturm-liouville form

$$(p y')' + (q + \lambda r) y = 0 \tag{35}$$

note that Section 4.7, Section 4.8, Section 4.9 and Section 4.6 equations can all be written in this form. furthermore, **any** 2nd order linear homogenous ODE $y'' + a_1(x) y' + [a_2(x) + \lambda a_3(x)] y = 0$ may be written in this form.

4.11. heat equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{36}$$

4.12. wave equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \tag{37}$$

4.13. laplace's equation (pde)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{38}$$

4.14. fourier series

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \tag{39}$$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) \,\mathrm{d}x, n = 0, 1, 2, ...$ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) \,\mathrm{d}x, n = 1, 2, ...$

parseval's identity

$$\frac{\| f \|^2}{L} = \frac{1}{L} \int_{-L}^{L} f^2 \,\mathrm{d}x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{40}$$