

# Theory of Differential Equations

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# 1. Definitions

order = the power the differential is raised to. linear = the dependent variable and it's derivatives are all not non-linear.

$$\underbrace{\frac{d^2y}{dt}}_{\text{linear}} \quad \underbrace{\cos(x)\frac{dy}{dx}}_{\text{linear}} \quad \underbrace{\frac{dy}{dt}\frac{d^3y}{dt^3}}_{\text{non-linear}} \quad \underbrace{y' = e^y}_{\text{non-linear}} \quad \underbrace{y\frac{dy}{dx}}_{\text{non-linear}} \quad (1)$$

autonomous = independent variable does not appear in the equation

non-autonomous = independent variable *does* appear in the equation

ansatz = our initial guess for the form of a solution, i.e.  $y_p = A \cos(t) + B \sin(t)$

indicial equation = a quadratic equation that pops out during the application of the Frobenius method

analytic = a function is analytic at a point if it can be expressed as a convergent power series in a neighborhood of that point

ordinary point = when  $p(x)$  and  $q(x)$  are analytic at that point

regular singular point = if  $P(x) = (x - x_0)p(x)$  and  $Q(x) = (x - x_0)^2q(x)$  are both analytic at  $x_0$ .

irregular singular point = not regular.

mean convergence = a sequence of functions  $f_n$  converges in mean to  $f$  on  $[a, b]$  if

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0$$

pointwise convergence = a sequence of functions  $f_n$  converges pointwise to  $f$  on  $[a, b]$  if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \text{ for every } x \in [a, b]$$

uniform convergence = a sequence of functions  $f_n$  converges uniformly to  $f$  on  $[a, b]$  if

$$\lim_{n \rightarrow \infty} \sup_{x \in [a, b]} |f_n(x) - f(x)| = 0$$

equilibrium point = a point where the derivative of the dependent variable with respect to the independent variable is zero stability

- stable node = trajectories approach the equilibrium point from all directions and eigenvalues are real and negative
- unstable bicritical node ("star") = trajectories move away from the equilibrium point in all directions and eigenvalues are real and positive
- stable centre = trajectories orbit around the equilibrium point with eigenvalues that are purely imaginary
- unstable saddle point = trajectories approach the equilibrium point in one direction and move away in another, with eigenvalues having opposite signs
- unstable focus = trajectories spiral away from the equilibrium point with eigenvalues having positive real parts and non-zero imaginary parts

## 2. Solving Methods

### 2.1. First Order

#### 2.1.1. standard form

$$\frac{dy}{dx} = f(x, y) \quad (2)$$

#### 2.1.2. separable

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x) dx \quad (3)$$

#### 2.1.3. reduction to separable

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad (4)$$

with substitution:  $y(x) = xv(x)$

#### 2.1.4. linear standard form

$$\frac{dy}{dx} + p(x)y = q(x) \quad (5)$$

##### 2.1.4.1. integrating factor

note, the coefficient of  $y'(x)$  must be 1.

$$\varphi(x) = \exp\left(\int p(x) dx\right) \quad (6)$$

multiplying the Linear Standard Form 5 with  $\varphi(x)$  yields:

$$\frac{d}{dx}(\varphi y) = \varphi(x)q(x) \implies y = \varphi^{-1} \int \varphi q(x) dx \quad (7)$$

#### 2.1.5. exact

A first-order ODE is exact if it can be written in the form:

$$M(x, y) dx + N(x, y) dy = 0 \quad (8)$$

where  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ . The solution is then given by:  $F(x, y) = C$  where  $F(x, y)$  satisfies  $\frac{\partial F}{\partial x} = M(x, y)$  and  $\frac{\partial F}{\partial y} = N(x, y)$

## 2.2. Second Order

### 2.2.1. standard form

$$y'' + p(x)y' + q(x)y = r(x) \quad (9)$$

### 2.2.2. reducible to first order

$$\frac{d^2y}{dx^2} + f\left(y, \frac{dy}{dx}\right) = 0 \quad (10)$$

is reducible to the first-order ODE

$$p \frac{dp}{dy} + f(y, p) = 0 \quad (11)$$

with substitution  $p = \frac{dy}{dx}$

### 2.2.3. constant coefficients

when  $p(x)$  and  $q(x)$  are constants:

$$y'' + a_1y' + a_0y = 0 \quad (12)$$

#### 2.2.3.1. homogenous

solve the characteristic equation:

$$\lambda^2 + a_1\lambda + a_0 = 0 \quad (13)$$

cases:

- $\lambda_1, \lambda_2$  are real and distinct
- $\lambda_1, \lambda_2$  are real and coincide (same)
- $\lambda_1, \lambda_2$  are complex conjugates

in each case, the solution of  $y(x)$  becomes:

- $y(x) = C \exp(\lambda_1 x) + D \exp(\lambda_2 x)$
- $y(x) = C \exp(\lambda_1 x) + Dx \exp(\lambda_1 x)$
- $y(x) = C \exp(\alpha x) \cos(\beta x) + D \exp(\alpha x) \sin(\beta x) = \exp(\alpha x)(A \cos(\beta x) + B \sin(\beta x))$  by DeMoivre's Theorem

#### 2.2.3.2. inhomogenous $\rightarrow$ method of undetermined coefficients

$$y(x) = y_{h(x)} + y_{p(x)} \quad (14)$$

guesses for  $y_{p(x)}$ :

- for  $r(x) = P_{n(x)}$  (polynomial of degree  $n$ ), try  $y_{p(x)} = Q_{n(x)}$
- for  $r(x) = e^{\alpha x}$ , try  $y_{p(x)} = Ae^{\alpha x}$
- for  $r(x) = \sin(\beta x)$  or  $r(x) = \cos(\beta x)$ , try  $y_{p(x)} = A \sin(\beta x) + B \cos(\beta x)$
- for products of the above forms, try products of the corresponding forms
- if  $y_{p(x)}$  is already a solution of the homogeneous equation, multiply by  $x$  or  $x^k$  until linearly independent

### 2.2.4. variation of parameters

This method works for any 2nd order inhomogenous ODE if the complementary solution is known.

Theorem: The general solution of the 2nd order inhomogenous ODE:

$$y'' + b_1(x)y' + b_0(x)y = f(x) \quad (15)$$

is given by  $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

where  $y_1$  and  $y_2$  are linearly independent solutions of the homogenous ODE such that the Wronskian  $W(x) \neq 0$  and

$$u_1(x) = - \int \frac{y_2(x)f(x)}{W(x)} dx \quad (16)$$

and

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(x)} dx \quad (17)$$

### 2.2.5. power series method

note, that we embark on this approach because the second order standard form 2.2.1 is not solveable in general with *elementary functions*!

pick ansatz of the form

$$y = \sum_{n=0}^{\infty} a_n z^n \quad (18)$$

and take derivatives as required. for example:

$$\frac{dy}{dz} = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad \frac{d^2 y}{dz^2} = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \quad (19)$$

and substitute them into the ODE. Then solve by rearranging indices as necessary to obtain a recurrence relation. Apply the initial conditions and then guess the closed-form solution of the recurrence relation. Change back to the original variables if required.

If  $x_0$  is an ordinary point Section 1 of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad (20)$$

then the general solution in a neighbourhood  $|x - x_0| < R$  may be represented as a power series.

### 2.2.6. method of frobenius

Theorem: If  $x_0 = 0$  is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad (21)$$

then there exists at least one series solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}, c_0 \neq 0 \quad (22)$$

for some constant  $r$  (index).

#### 2.2.6.1. general indicial equation

$$r(r-1) + p_0 r + q_0 = 0 \quad (23)$$

## 2.3. n order

admits  $n$  linearly independent solutions.

### 2.3.1. power series expansion (not sure if it works for n order)

For an  $n^{\text{th}}$  order linear ODE with variable coefficients:

$$a_{n(x)}y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x) \quad (24)$$

We assume a solution of the form:

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad (25)$$

Taking derivatives and substituting yields a recurrence relation for coefficients  $c_k$ , typically allowing us to determine  $c_n$  in terms of  $c_0, c_1, \dots, c_{n-1}$ .

### 2.3.2. reduction of order

any  $n^{\text{th}}$  order ODE can be formulated as a system of  $n$  first order ODE's.

For  $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ , set  $y_i = y^{(i-1)}$  for  $i = 1, 2, \dots, n$  to obtain:

$$y_{i'} = y_{i+1} \quad (26)$$

for  $i = 1, 2, \dots, n - 1$

$$y_{n'} = f(x, y_1, y_2, \dots, y_n) \quad (27)$$

## 2.4. partial differential equations

### 2.4.1. standard form (linear, homogenous, 2nd order pde)

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + \frac{D(\partial u)}{\partial x} + \frac{E(\partial u)}{\partial y} + Fu = 0 \quad (28)$$

parabolic equation:  $B^2 - 4AC = 0$  Heat Equation 4.11

hyperbolic equation:  $B^2 - 4AC > 0$  Wave Equation 4.12

elliptic equation:  $B^2 - 4AC < 0$  Laplace Equation 4.13

### 2.4.2. separation of variables

$$U(x, y) = X(x)Y(y) \quad (29)$$

then  $U_x = YX'$  and  $U_y = Y'X$  rewrite the PDE with these substitutions, then divide through by  $XY$ . Integrate and solve.

### 2.4.3. change of variables

When a PDE is difficult to solve directly, changing variables can transform it into a simpler form.

For a second-order PDE, the transformation  $u = u(\xi, \eta)$  where  $\xi = \xi(x, y)$  and  $\eta = \eta(x, y)$  requires computing:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \quad (30)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} \quad (31)$$

And similarly for second-order derivatives. The canonical transformations are:

- For hyperbolic:  $\xi = x + y, \eta = x - y$  (characteristic coordinates)
- For parabolic:  $\xi = x, \eta = y - f(x)$  (transformation along characteristics)
- For elliptic:  $\xi = x + iy, \eta = x - iy$  (complex characteristics)

## 3. systems / dynamical systems

- $\lambda_2 < \lambda_1 < 0 \implies$  stable node
- $0 < \lambda_1 < \lambda_2 \implies$  unstable node
- $\lambda_1 = \lambda_2, \lambda_1 > 0 \implies$  unstable star
- $\lambda_1 = \lambda_2, \lambda_1 < 0 \implies$  stable star
- $\lambda_1 < 0 < \lambda_2 \implies$  unstable saddle node
- $\Re(\lambda_1) = 0 \implies$  centre, stable
- $\Re(\lambda_1) < 0 \implies$  stable focus
- $\Re(\lambda_1) > 0 \implies$  unstable focus

real canonical form For a linear system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , the real canonical form depends on the eigenvalues of  $\mathbf{A}$ :

- Real distinct eigenvalues  $\lambda_1 \neq \lambda_2$ :

$$\mathbf{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (32)$$

- Real repeated eigenvalues  $\lambda_1 = \lambda_2$  with linearly independent eigenvectors:



$$\mathbf{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \quad (33)$$

- Real repeated eigenvalues  $\lambda_1 = \lambda_2$  with one linearly independent eigenvector:

$$\mathbf{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad (34)$$

- Complex conjugate eigenvalues  $\lambda = \alpha \pm i\beta$ :

$$\mathbf{A}_{\text{canonical}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad (35)$$

## 4. functions

### 4.1. wronskian

$$W(f_1, f_2, \dots, f_n)(x) = \det \left( \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix} \right) \quad (36)$$

note that if a set of functions is linearly dependent, then its Wronskian will equal 0.

### 4.2. power series, Taylor series and maclaurin series expansions

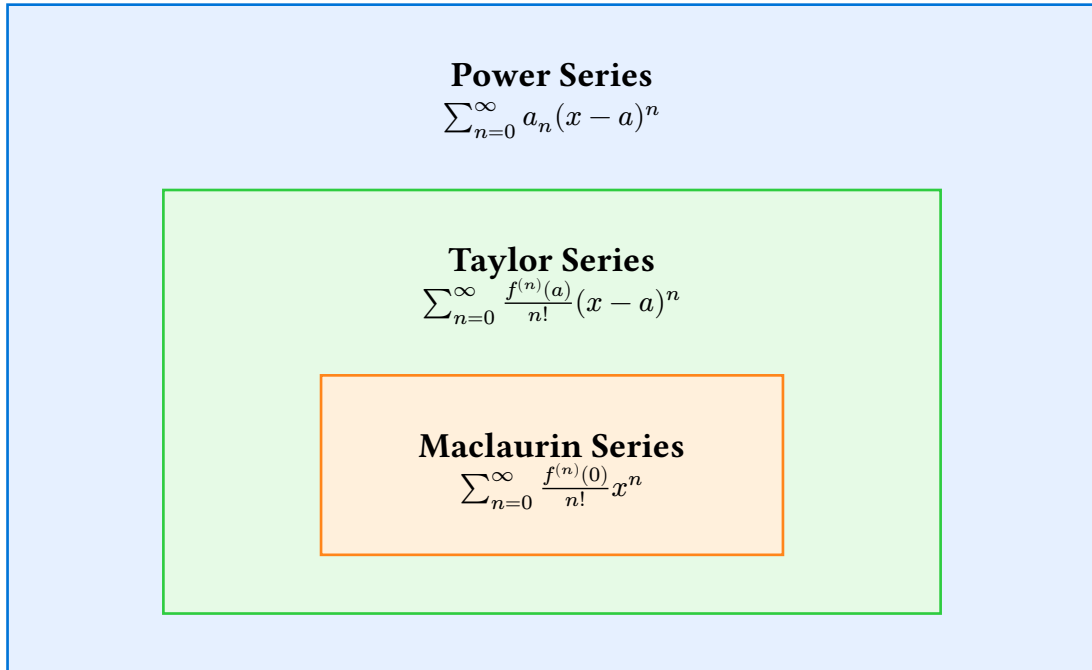


Figure 1: Relationship between power series, Taylor series, and Maclaurin series, showing proper subset relationships

### 4.3. orthogonality

A set of functions  $\{\varphi_n\}_{n=1,2,3,\dots}$  is said to be orthogonal on the interval  $[a, b]$  with respect to the inner product defined by

$$(f, g)_w = \int_a^b w(x) f(x) g(x) dx \quad (37)$$

with weight function  $w(x) > 0$ , if  $(\varphi_n, \varphi_m)_w = 0$  for  $m \neq n$ .

#### 4.4. orthonormality

a set  $\{\varphi_n\}_{n=1,2,3,\dots}$  is *orthonormal* when in addition to being Section 4.3,  $(\varphi_n, \varphi_n) = 1$ , for  $n = 1, 2, 3, \dots$

#### 4.5. cauchy-euler

$x^2 y'' + a_1 x y' + a_0 y = 0$  you can solve this by either letting  $x = e^t$  or using the ansatz  $y = x^\lambda$  the characteristic equation is  $\lambda^2 + (a_1 - 1)\lambda + a_0 = 0$  if you are blessed with the inhomogenous case of above, just use method of undetermined coefficients Section 2.2.3.2.

#### 4.6. legendre

legendre's (differential) equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \quad (38)$$

legendre's polynomials

#### 4.7. bessel

bessel's differential equation

$$y'' x^2 + x y' + (x^2 - \nu^2)y = 0 \quad (39)$$

bessel function of the first kind of order  $\alpha$ :

$$J_{\alpha(x)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)} \Gamma(m+\alpha+1) \left(\frac{x}{2}\right)^{2m+\alpha} \quad (40)$$

implies

$$\frac{d}{dx} x [x^\alpha J_{\alpha(x)}] = x^\alpha J_{\alpha-1}(x) \quad (41)$$

implies

$$\int_0^r x^n J_{n-1}(x) dx = r^n J_n(r) \quad (42)$$

for  $n = 1, 2, 3, \dots$

thus the de admits solutions case 1:  $2\nu \notin \mathbb{Z}$

$$y(x) = A J_{\nu(x)} + B J_{-\nu}(x) \quad (43)$$

$J_{\nu(x)}, J_{-\nu}(x)$  linearly independent case 2:  $2\nu \in \mathbb{Z}$

$$y(x) = A J_{\nu(x)} + B Y_{\nu}(x) \quad (44)$$

case 3:  $\nu \in \mathbb{Z}$   $J_{\nu(x)}, J_{-\nu}(x)$  linearly independent

$$y(x) = A J_{\nu(x)} + B Y_{\nu}(x) \quad (45)$$

#### 4.8. laguerre's equation

$$xy'' + (1 - x)y' + ny = 0 \quad (46)$$

#### 4.9. hermite's equation

$$y'' - 2xy' + 2ny = 0 \quad (47)$$

#### 4.10. sturm-liouville form

$$(py')' + (q + \lambda r)y = 0 \quad (48)$$

note that Section 4.7, Section 4.8, Section 4.9 and Section 4.6 equations can all be written in this form. furthermore, **any** 2nd order linear homogenous ODE  $y'' + a_1(x)y' + [a_2(x) + \lambda a_3(x)]y = 0$  may be written in this form.

#### 4.11. heat equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (49)$$

#### 4.12. wave equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (50)$$

#### 4.13. laplace's equation (pde )

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (51)$$

#### 4.14. fourier series

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)) \quad (52)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx, n = 0, 1, 2, \dots \quad (53)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx, n = 1, 2, \dots \quad (54)$$

#### 4.15. parseval's identity

$$\frac{\|f\|^2}{L} = \frac{1}{L} \int_{-L}^L f^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (55)$$