Theory of Differential Equations

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1. Definitions

order = the power the differential is raised to.

linear = the dependent variable and it's derivatives are all not non-linear.

$$\underbrace{\frac{\mathrm{d}^2 y}{\mathrm{d}t}}_{\text{linear}} \quad \underbrace{\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{linear}} \quad \underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}}_{\text{non-linear}} \quad \underbrace{y' = e^y}_{\text{non-linear}} \quad \underbrace{y\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{non-linear}} \tag{1}$$

autonomous = independent variable does not appear in the equation

non-autonomous = independent variable *does* appear in the equation

ansatz = our initial guess for the form of a solution, i.e. $y_p = A\cos(t) + B\sin(t)$

indicial equation = a quadratic equation that pops out during the application of the Frobenius method

analytic = a function is analytic at a point if it can be expressed as a convergent power series in a neighborhood of that point

ordinary point = when $p(x)$ and $q(x)$ are analytic at that point

regular singular point = if $P(x) = (x - x_0)p(x)$ and $Q(x) = (x - x_0)^2 q(x)$ are both analytic at x_0.

irregular singular point = not regular.

mean convergence = a sequence of functions f_n converges in mean to f on $[a, b]$ if $\lim_{n\to\infty} \int_a^b |f_{n(x)} - f(x)|^2 \,\mathrm{d}x = 0$

pointwise convergence = a sequence of functions f_n converges pointwise to f on $[a, b]$ if $\lim_{n\to\infty} f_{n(x)} = f(x)$ for every $x \in [a, b]$

uniform convergence = a sequence of functions f_n converges uniformly to f on $[a, b]$ if $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_{n(x)} - f(x)| = 0$

Equilibrium Points and Stability

equilibrium point = a point where the derivative of the dependent variable with respect to the independent variable is zero

stable node = trajectories approach the equilibrium point from all directions and eigenvalues are real and negative

unstable bicritical node ("star") = trajectories move away from the equilibrium point in all directions and eigenvalues are real and positive

stable centre = trajectories orbit around the equilibrium point with eigenvalues that are purely imaginary

unstable saddle point = trajectories approach the equilibrium point in one direction and move away in another, with eigenvalues having opposite signs

unstable focus = trajectories spiral away from the equilibrium point with eigenvalues having positive real parts and non-zero imaginary parts

2. Solving Methods

2.1. First Order

2.1.1. standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2}$$

2.1.2. separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \Longrightarrow \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x \tag{3}$$

2.1.3. reduction to separable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \tag{4}$$

with substitution: $y(x) = xv(x)$

2.1.4. linear standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x) \tag{5}$$

2.1.4.1. integrating factor

note, the coefficient of $y'(x)$ must be 1.

$$\varphi(x) = \exp\left(\int p(x)\,\mathrm{d}x\right) \tag{6}$$

multiplying the Linear Standard Form 5 with $\varphi(x)$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varphi y) = \varphi(x)q(x) \Longrightarrow y = \varphi^{-1} \int \varphi q(x)\,\mathrm{d}x \tag{7}$$

2.1.5. exact

A first-order ODE is exact if it can be written in the form:

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{8}$$

where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The solution is then given by: $F(x, y) = C$ where $F(x, y)$ satisfies $\frac{\partial F}{\partial x} = M(x, y)$ and $\frac{\partial F}{\partial y} = N(x, y)$

2.2. Second Order

2.2.1. standard form

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

2.2.2. reducible to first order

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{10}$$

is reducible to the first-order ODE

$$p\frac{\mathrm{d}p}{\mathrm{d}y} + f(y, p) = 0 \tag{11}$$

with substitution $p = \frac{\mathrm{d}y}{\mathrm{d}x}$

2.2.3. constant coefficients

when $p(x)$ and $q(x)$ are constants:

$$y'' + a_1 y' + a_0 y = 0 \tag{12}$$

2.2.3.1. homogenous

solve the characteristic equation:

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{13}$$

cases:

- λ_1, λ_2 are real and distinct
- λ_1, λ_2 are real and coincide (same)
- λ_1, λ_2 are complex conjugates

in each case, the solution of $y(x)$ becomes:

- $y(x) = C\exp(\lambda_1 x) + D\exp(\lambda_2 x)$
- $y(x) = C\exp(\lambda_1 x) + Dx\exp(\lambda_1 x)$
- $y(x) = C\exp(\alpha x)\cos(\beta x) + D\exp(\alpha x)\sin(\beta x) = \exp(\alpha x)(A\cos(\beta x) + B\sin(\beta x))\text{ by DeMoivre's Theorem}$

2.2.3.2. inhomogenous $\rightarrow$ method of undetermined coefficients

$$y(x) = y_{h(x)} + y_{p(x)} \tag{14}$$

guesses for $y_{p(x)}$:

- for $r(x) = P_{n(x)}$ (polynomial of degree n), try $y_{p(x)} = Q_{n(x)}$
- for $r(x) = e^{\alpha x}$, try $y_{p(x)} = Ae^{\alpha x}$
- for $r(x) = \sin(\beta x)$ or $r(x) = \cos(\beta x)$, try $y_{p(x)} = A\sin(\beta x) + B\cos(\beta x)$
- for products of the above forms, try products of the corresponding forms
- if $y_{p(x)}$ is already a solution of the homogeneous equation, multiply by x or x^k until linearly independent

2.2.4. variation of parameters

This method works for any 2nd order inhomogenous ODE if the complementary solution is known.

Theorem. The general solution of the 2nd order inhomogenous ODE:

$$y'' + b_1(x)y' + b_0(x)y = f(x) \tag{15}$$

is given by $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

where y_1 and y_2 are linearly independent solutions of the homogenous ODE such that the Wronskian $W(x) \neq 0$ and

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,\mathrm{d}x \tag{16}$$

and

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,\mathrm{d}x \tag{17}$$

2.2.5. power series method

note, that we embark on this approach because the second order standard form 2.2.1 is not solveable in general with *elementary functions*!

pick ansatz of the form

$$y = \sum_{n=0}^{\infty} a_n z^n \tag{18}$$

and take derivatives as required. for example:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \sum_{n=1}^{\infty} n a_n z^{n-1} \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \tag{19}$$

and substitute them into the ODE. Then solve by rearranging indices as necessary to obtain a recurrence relation. Apply the initial conditions and then guess the closed-form solution of the recurrence relation. Change back to the original variables if required.

If x_0 is an ordinary point Section 1 of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{20}$$

then the general solution in a neighbourhood $|x - x_0| < R$ may be represented as a power series.

2.2.6. method of frobenius

Theorem. If $x_0 = 0$ is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad (21)$$

then there exists at least one series solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}, c_0 \neq 0 \quad (22)$$

for some constant r (index).

2.2.6.1. general indicial equation

$$r(r-1) + p_0 r + q_0 = 0 \quad (23)$$

2.3. n order

admits n linearly independent solutions.

2.3.1. power series expansion (not sure if it works for n order)

For an n^{th} order linear ODE with variable coefficients:

$$a_{n(x)} y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = f(x) \quad (24)$$

We assume a solution of the form:

$$y(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad (25)$$

Taking derivatives and substituting yields a recurrence relation for coefficients c_k, typically allowing us to determine c_n in terms of $c_0, c_1, \ldots, c_{n-1}$.

2.3.2. reduction of order

any n^{th} order ODE can be formulated as a system of n first order ODE's.

For $y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$, set $y_i = y^{(i-1)}$ for $i = 1, 2, \ldots, n$ to obtain:

$$y_{i'} = y_{i+1} \quad (26)$$

for $i = 1, 2, \ldots, n-1$

$$y_{n'} = f(x, y_1, y_2, \ldots, y_n) \quad (27)$$

2.4. partial differential equations

2.4.1. standard form (linear, homogenous, 2nd order pde)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \frac{D(\partial u)}{\partial x} + \frac{E(\partial u)}{\partial y} + Fu = 0 \tag{28}$$

parabolic equation: $B^2 - 4AC = 0$ (Heat Equation 4.11)

hyperbolic equation: $B^2 - 4AC > 0$ (Wave Equation 4.12)

elliptic equation: $B^2 - 4AC < 0$ (Laplace Equation 4.13)

2.4.2. separation of variables

$$U(x, y) = X(x)Y(y) \tag{29}$$

then $U_x = YX'$ and $U_y = Y'X$ rewrite the PDE with these substitutions, then divide through by XY. Integrate and solve.

2.4.3. change of variables

When a PDE is difficult to solve directly, changing variables can transform it into a simpler form.

For a second-order PDE, the transformation $u = u(\xi, \eta)$ where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ requires computing:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{30}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{31}$$

And similarly for second-order derivatives. The canonical transformations are:
- For hyperbolic: $\xi = x + y, \eta = x - y$ (characteristic coordinates)
- For parabolic: $\xi = x, \eta = y - f(x)$ (transformation along characteristics)
- For elliptic: $\xi = x + iy, \eta = x - iy$ (complex characteristics)

3. systems / dynamical systems

- $\lambda_2 < \lambda_1 < 0 \Longrightarrow \text{stable node}$
- $0 < \lambda_1 < \lambda_2 \Longrightarrow \text{unstable node}$
- $\lambda_1 = \lambda_2, \lambda_1 > 0 \Longrightarrow \text{unstable star}$
- $\lambda_1 = \lambda_2, \lambda_1 < 0 \Longrightarrow \text{stable star}$
- $\lambda_1 < 0 < \lambda_2 \Longrightarrow \text{unstable saddle node}$
- $\Re(\lambda_1) = 0 \Longrightarrow \text{centre, stable}$
- $\Re(\lambda_1) < 0 \Longrightarrow \text{stable focus}$
- $\Re(\lambda_1) > 0 \Longrightarrow \text{unstable focus}$

real canonical form For a linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, the real canonical form depends on the eigenvalues of $\boldsymbol{A}$:

- Real distinct eigenvalues $\lambda_1 \neq \lambda_2$:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{32}$$

- Real repeated eigenvalues $\lambda_1 = \lambda_2$ with linearly independent eigenvectors:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \tag{33}$$

- Real repeated eigenvalues $\lambda_1 = \lambda_2$ with one linearly independent eigenvector:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \tag{34}$$

- Complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$:

$$\boldsymbol{A}_{\text{canonical}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{35}$$

4. functions

4.1. wronskian

$$\mathrm{W}(f_1, f_2, ..., f_n)(x) = \det\left(\begin{pmatrix} f_1(x) & f_2(x) & ... & f_{n(x)} \\ f_1^{\prime(x)} & f_2^{\prime(x)} & ... & f_n^{\prime(x)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & ... & f_n^{(n-1)}(x) \end{pmatrix}\right) \tag{36}$$

note that if a set of functions is linearly dependent, then its Wronskian will equal 0.

4.2. power series, taylor series and maclaurin series expansions

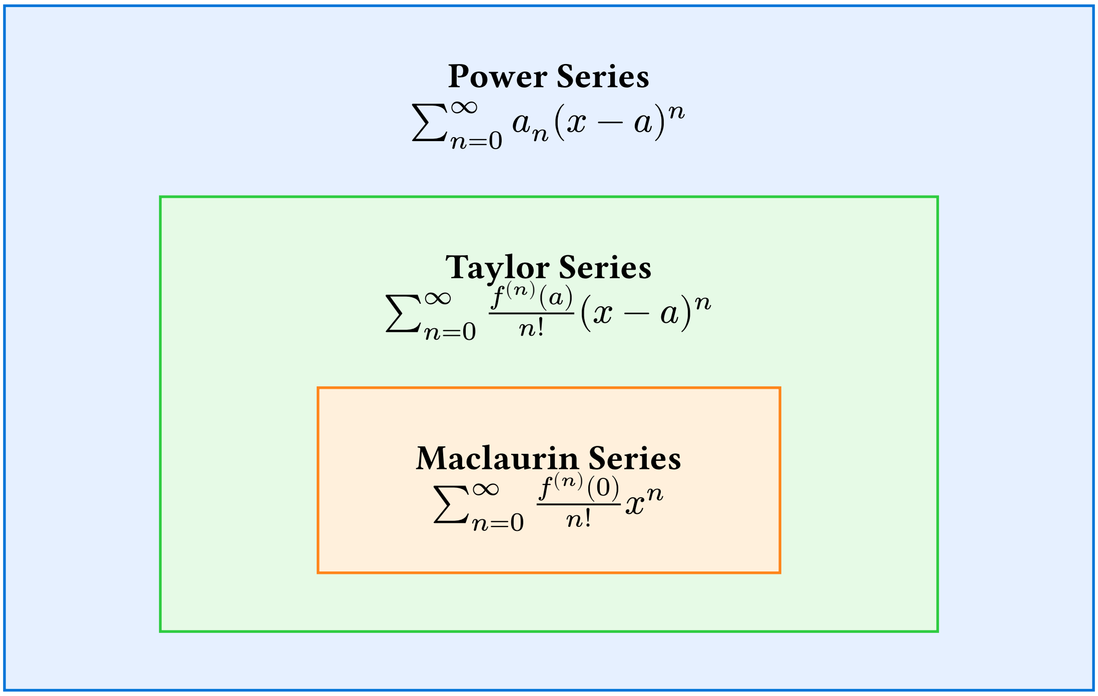

Figure 1: Relationship between power series, Taylor series, and Maclaurin series, showing proper subset relationships

4.3. orthogonality

A set of functions $\{\varphi_n\}_{n=1,2,3,\ldots}$ is said to be orthogonal on the interval $[a, b]$ with respect to the inner product defined by

$$(f, g)_w = \int_a^b w(x) f(x) g(x) \, \mathrm{d}x \tag{37}$$

with weight function $w(x) > 0$, if $(\varphi_n, \varphi_m)_w = 0$ for $m \neq n$.

4.4. orthonormality

a set $\{\varphi_n\}_{n=1,2,3,\ldots}$ is *orthonormal* when in addition to being Section 4.3, $(\varphi_n, \varphi_n) = 1$, for $n = 1, 2, 3, \ldots$.

4.5. cauchy-euler

$x^2 y'' + a_1 x y' + a_0 y = 0$ you can solve this by either letting $x = e^t$ or using the ansatz $y = x^\lambda$ the characteristic equation is $\lambda^2 + (a_1 - 1)\lambda + a_0 = 0$ if you are blessed with the inhomogenous case of above, just use method of undetermined coefficients Section 2.2.3.2.

4.6. legendre

legendre's (differential) equation

$$(1 - x^2) y'' - 2 x y' + n(n + 1) y = 0 \tag{38}$$

legendre's polynomials

4.7. bessel

bessel's differential equation

$$y''x^2 + xy' + (x^2 - \nu^2)y = 0 \tag{39}$$

bessel function of the first kind of order α:

$$J_{\alpha(x)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)} \Gamma(m+\alpha+1) \left(\frac{x}{2}\right)^{2m+\alpha} \tag{40}$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}} x \left[x^\alpha J_{\alpha(x)}\right] = x^\alpha J_{\alpha-1}(x) \tag{41}$$

implies

$$\int_0^r x^n J_{n-1}(x) \,\mathrm{d}x = r^n J_{n(r)} \tag{42}$$

for $n = 1, 2, 3, ...$

thus the de admits solutions case 1: $2\nu \notin \mathbb{Z}$

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \tag{43}$$

$J_\nu(x)$, $J_{-\nu}(x)$ linearly independent case 2: $2\nu \in \mathbb{Z}$

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \tag{44}$$

case 3: $\nu \in \mathbb{Z}$ $J_\nu(x)$, $J_{-\nu}(x)$ linearly independent

$$y(x) = AJ_\nu(x) + BY_\nu(x) \tag{45}$$

4.8. laguerre's equation

$$xy'' + (1-x)y' + ny = 0 \tag{46}$$

4.9. hermite's equation

$$y'' - 2xy' + 2ny = 0 \tag{47}$$

4.10. sturm-liouville form

$$(py')' + (q + \lambda r)y = 0 \tag{48}$$

note that Bessel 4.7, Laguerre 4.8, Hermite 4.9 and Legendre 4.6 equations can all be written in this form. furthermore, **any** 2nd order linear homogenous ODE $y'' + a_1(x)y' + [a_2(x) + \lambda a_3(x)]y = 0$ may be written in this form.

4.11. heat equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{49}$$

4.12. wave equation (pde)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{50}$$

4.13. laplace's equation (pde)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{51}$$

4.14. fourier series

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)) \tag{52}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\cos(nx)\,\mathrm{d}x, n = 0, 1, 2, \dots \tag{53}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\sin(nx)\,\mathrm{d}x, n = 1, 2, \dots \tag{54}$$

4.15. parseval's identity

$$\frac{\parallel f \parallel^2}{L} = \frac{1}{L}\int_{-L}^{L} f^2\,\mathrm{d}x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{55}$$