CS281A - Problem Set 2

Andrea Bajcsy

September 15, 2016

Problem 2.1.

(a) We can formulate the polynomial regression problem as a form of linear prediction by solving the general linear model equation $X\alpha = y$ where:

$$X = \begin{bmatrix} 1 & t_1 & t_1^2 & ... & t_1^D \\ 1 & t_2 & t_2^2 & ... & t_2^D \\ 1 & t_3 & t_3^2 & ... & t_3^D \\ ... & & & & \\ 1 & t_n & t_n^2 & ... & t_n^D \end{bmatrix} \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ ... \\ \alpha_D \end{bmatrix} y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ ... \\ y_n \end{bmatrix}$$

(b) Figure 1 shows a plot of the mean-squared error R(D) vs. Degree $D \in 1, 2, ...n-1$ when using the data in y.dat and t.dat. See back for code that performs least-squares fit of a polynomial of degree D.

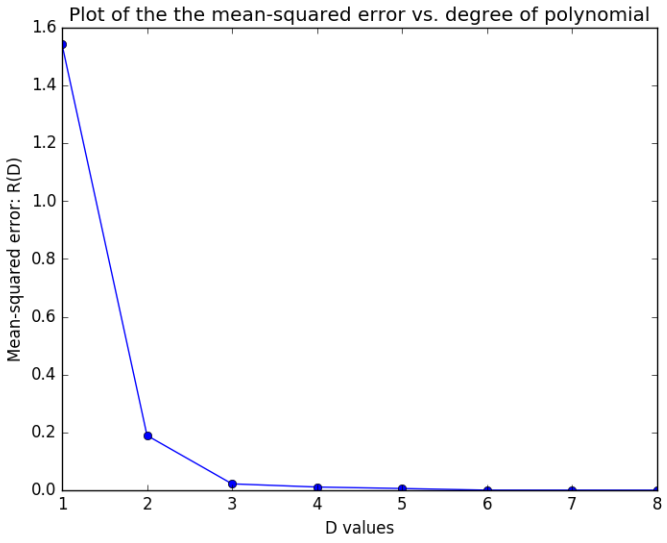

Figure 1: D vs. R(D)

(c) How does the MSE behave as a function of D and why? With the degree n-1 fit, we get no mean-squared error since the function fits to every data point. What happens if you try to fit a polynomial of degree n? Why? To fit a polynomial of degree n, we will be solving $X\alpha = y$, where

$X^{n\ x\ n}$.

$$\begin{bmatrix} 1 & t_1 & t_1^2 & ... & t_1^n \\ 1 & t_2 & t_2^2 & ... & t_2^n \\ 1 & t_3 & t_3^2 & ... & t_3^n \\ ... & & & & \\ 1 & t_n & t_n^2 & ... & t_n^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ ... \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ ... \\ y_n \end{bmatrix}$$

Using ordinary least-squares, we solve for $\alpha = (X^T X)^{-1} X^T y$.

(d) Figure 2 shows a plot of the degree $D \in 1, 2, ...n-1$ versus the mean-squared error R(D) and $\tilde{R}$ when using the data in y.dat, yfresh.dat, and t.dat. Why do you think that this plot is qualitatively different from the plot in part (b)? What does this tell you how the fitted degree D should be chosen?

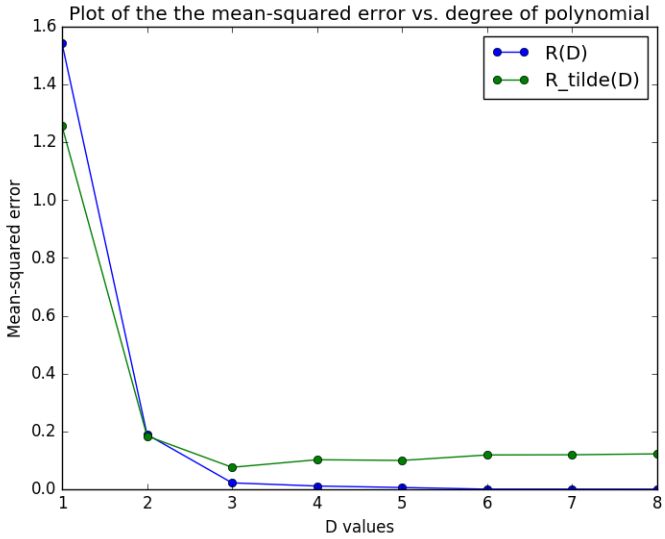

Figure 2: D vs. R(D) and $\tilde{R}$

(e) Figure 3 shows a plot of the degree $D \in 2, ...9$ versus the mean-squared error $\tilde{R}$ and F(D) when using the data in y.dat, yfresh.dat, and t.dat. How are the minimizing arguments of the two functions related? Why is this an interesting observation?

Problem 2.2.

(a) Prove that A is a convex function. –check that final claim about pos.def always holds!

Proof. By definition,

$$A(\eta) = log(\int_\gamma h(y) e^{\eta y}\, dy) \quad , \quad p_\eta(y) = h(y) e^{\eta y - A(\eta)}$$

To prove convexity, we want to take the second derivative. Let:

$$B(\eta) = \int_\gamma h(y) e^{\eta y}\, dy$$

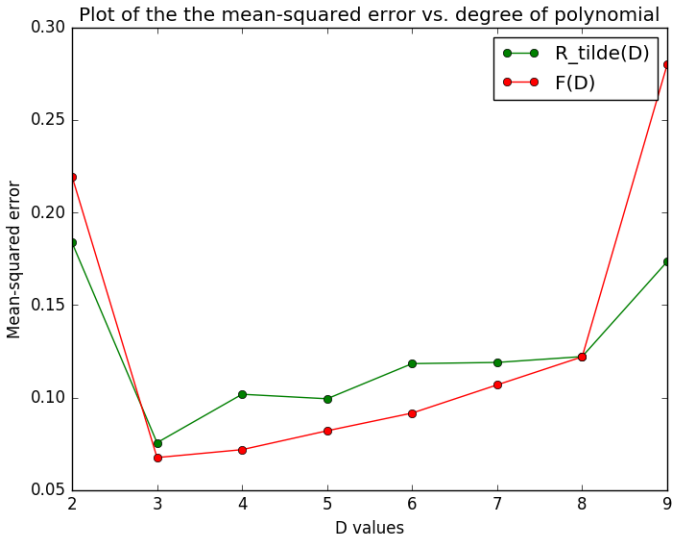

Figure 3: D vs. $\tilde{R}$ and F(D)

Then the first derivative we get:

$$\frac{\partial A(\eta)}{\partial \eta} = \left(\frac{1}{B(\eta)}\right)\left(\frac{\partial B(\eta)}{\partial \eta}\right) = \frac{\int_\gamma h(y)e^{\eta y} y\, dy}{\int_\gamma h(y)e^{\eta y}\, dy} = \frac{\int_\gamma h(y)e^{\eta y - A(\eta)} y\, dy}{\int_\gamma h(y)e^{\eta y - A(\eta)}\, dy} = E_{p_\eta}[y]$$

Taking the second derivative we have:

$$\frac{\partial}{\partial \eta}\frac{B'(\eta)}{B(\eta)} = \frac{\partial}{\partial \eta}\left(B'(\eta)\frac{1}{B(\eta)}\right) = \frac{B''(\eta)}{B(\eta)} - \frac{(B'(\eta))^2}{B(\eta)^2}$$

$$= \frac{\int_\gamma h(y)e^{\eta y} y^2\, dy}{\int_\gamma h(y)e^{\eta y}\, dy} \; - \; (E_{p_\eta}[y])^2 = \frac{\int_\gamma h(y)e^{\eta y - A(\eta)} y\, dy}{\int_\gamma h(y)e^{\eta y - A(\eta)}\, dy} \; - \; (E_{p_\eta}[y])^2$$

$$= E_{p_\eta}[y^2] \; - \; (E_{p_\eta}[y])^2 = Var_{p_\eta}[y] \; \succeq \; 0$$

Since Var_{p_η} is positive definite, we have shown that $A(\eta)$ is convex. □

(b) Express KL divergance in terms of $A(\eta)$ and $A'(\eta)$.

$$D(p_\eta || p_\eta) = E_\eta\left(log(\frac{h(y)e^{\eta y - A(\eta)}}{h(y)e^{\tilde{\eta} y - A(\tilde{\eta})}})\right)$$

$$= \int_y log\left(e^{(\eta-\tilde{n})y - (A(\eta)-A(\tilde{\eta}))} p_\eta(y)\right) dy$$

$$= \int_y \left((\eta - \tilde{n})y - (A(\eta) - A(\tilde{\eta}))\right) h(y)e^{\eta y - A(\eta)}\, dy$$

$$= (\eta - \tilde{n}) \int_y h(y)e^{\eta y - A(\eta)} y\, dy - (A(\eta) - A(\tilde{\eta})) \int_y h(y)e^{\eta y - A(\eta)}\, dy$$

$$= (\eta - \tilde{n})A' - A(\eta) + A(\tilde{\eta})$$

Since $A = \int_y h(y)e^{\eta y - A(\eta)}y$ and $\int_y p_\eta(y)dy = 1$ by definition.

(i) Bernoulli random variable:

$$p_\eta(y) = \eta^y(1-\eta)^{1-y}, y \in 0,1, n \in (0,1)$$

$$= e^{ylog(\eta)+(1-y)log(1-\eta)} = e^{ylog(\frac{\eta}{1-\eta})-log(1+e^{\frac{\eta}{1-\eta}})}$$

Thus, we have $A(\eta) = log(1+e^\eta)$ and $A^*(t) = sup_{\eta \in R}\ \{\eta t - log(1+e^\eta\}$. We now take the gradient of A^* with respect to η, set this to 0 in order to solve the optimization problem, and then solve for η in terms of t.

$$\nabla_\eta A^*(t) = t - \frac{e^\eta}{1+e^\eta}$$

$$0 = t - \frac{e^\eta}{1+e^\eta} \implies t = \frac{e^\eta}{1+e^\eta}$$

$$\frac{1}{t} = \frac{1+e^\eta}{e^\eta} = \frac{1}{e^\eta} + 1 \implies \frac{1}{t} - 1 = \frac{1}{e^\eta}$$

$$e^\eta = \frac{1}{\frac{1}{t}-1} \implies \eta = log(\frac{1}{\frac{1}{t}-1})$$

$$\eta = -log(\frac{1}{t} - 1))$$

Substituting this back into our equation, we get:

$$A^*(t) = -tlog(\frac{1}{t}-1) - log(1+e^{-log(\frac{1}{t}-1)}) = -tlog(\frac{1}{t}-1) + log(1-t)$$

$$= tlog(t) - tlog(1-t) + log(1-t) = tlog(t) + (1-t)log(1-t)$$

(ii) Gaussian random variable:

$$p_\eta(y) = \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}e^{yn - \frac{\eta^2}{2}}$$

Thus, we have $A(\eta) = \frac{\eta^2}{2}$ and $A^*(t) = sup_{\eta \in R}\ \left\{\eta t - \frac{\eta^2}{2}\right\}$.

$$\nabla_\eta A^*(t) = t - n$$

$$0 = t - n \implies t = n$$

Substituting this back into our equation, we get:

$$A^*(t) = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

(iii) Poisson random variable:

$$p_\eta(y) = \frac{1}{y!}e^{y\eta - e^\eta}$$

Thus, we have $A(\eta) = e^\eta$ and $A^*(t) = sup_{\eta \in R}\ \{\eta t - e^\eta\}$.

$$\nabla_\eta A^*(t) = t - e^\eta$$

$$0 = t - e^\eta \implies log(t) = n$$

Substituting this back into our equation, we get:

$$A^*(t) = tlog(t)e^{log(t)} = tlog(t) - t = t(log(t) - 1)$$

(d) Prove that conjugate dual is always a convex function

Proof. □

Problem 2.3.

(a) By definition, we have likelihood of η as:

$$l(\eta; y_1, ...y_n) = log(p(y_1, ..., y_n|\eta)) = log(h(y_1, ...y_n) + \eta^T(\sum_{i=1}^{n} y_i - nA(\eta))$$

We differentiate and solve for $\hat{\eta}$ to get the MLE (assuming the inverse function exists under suitable regularity conditions):

$$\frac{\partial}{\partial\eta} l(\eta; y_1, ...y_n) = \sum_{i=1}^{n} y_i - n\frac{\partial}{\partial\eta} A(\eta)$$

$$\frac{\partial}{\partial\eta} A(\eta) = \frac{\sum_{i=1}^{n} y_i}{n} \implies \hat{\eta} = (A'^{-1})(\frac{\sum_{i=1}^{n} y_i}{n})$$

(b) Closed-form estimates for MLE in Poisson, Bernoulli, Gaussian models:

Poisson

$$\frac{\partial}{\partial\eta} A(\eta) = e^\eta = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\hat{\eta} = log(\frac{\sum_{i=1}^{n} y_i}{n})$$

Bernoulli (where $\frac{\sum_{i=1}^{n} y_i}{n} \neq 1$)

$$\frac{\partial}{\partial\eta} A(\eta) = \frac{e^\eta}{1 + e^\eta} = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\hat{\eta} = log(\frac{\frac{\sum_{i=1}^{n} y_i}{n}}{1 - \frac{\sum_{i=1}^{n} y_i}{n}})$$

Gaussian

$$\frac{\partial}{\partial\eta} A(\eta) = \eta = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\hat{\eta} = \frac{\sum_{i=1}^{n} y_i}{n}$$

(c) (d)

Problem 2.4.

(a) Based on the definition of MLE in GLM as well as stochastic gradient, we can redefine the function $L(\theta)$, the gradient $\triangle^t L(\theta)$, and the $\tilde{\theta}^{t+1}$ update step as:

$$L(\theta) = -y_I x_I^T \theta^t + A(x_I^T \theta^t)$$
$$\triangle^t L(\theta) = -y_I x_I^T + x_I^T A'(x_I^T \theta^t)$$
$$\tilde{\theta}^{t+1} = \hat{\theta}^t - \gamma^t \triangle^t L(I)$$

(b) Explicit updates for Possion and Logistic cases:
Poisson

$$A(t) = e^t \implies \tilde{\theta}^{t+1} = \hat{\theta}^t - \gamma^t x_I^T e^{x_I^T \theta^t}$$

Logistic

$$A(t) = log(1 + e^t) \implies \tilde{\theta}^{t+1} = \hat{\theta}^t - \gamma^t x_I^T (\frac{e^{x_I^T \theta^t}}{1 + e^{x_I^T \theta^t}})$$

(c) (d) (e) (f)