Appendices to Accompany
Partial Ambiguity and Fuzzy Decision-making in Asset Markets

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Appendices A and B

Appendices A and B formalize the notions of expectations and expected utility for a partially ambiguous payoff. In the body of the main article, we conjectured that under partial ambiguity, the representative agent forms an imprecise fuzzy forecast of a future payoff that may be expressed linguistically as "around $\widehat{Y}$" where $\widehat{Y}$ is the benchmark forecast inferred from concentrated information. In order to develop the notions of expectations and expected utility for the case a partially ambiguous payoff, in both appendices A and B we follow a two-step procedure. Step

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1 (**defuzzification**) resolves mathematically the multi-valued nature of fuzzy statement "around $\widehat{Y}$" while treating $\widehat{Y}$ as a known, non-random magnitude. Step 2 (**expectation**) develops the results obtained in step 1 to allow for random (risky) $\widehat{Y}$. Appendix C derives condition (11) in the body of the article and Appendix D outlines the calibration procedure and provides some further details on the data used for calibration.

Appendix A. Expected Value of a Partially Ambiguous Payoff. Derivation of Equations (1)-(3)[2]

Step 1: Defuszzyfication Assume at first that forthcoming income is purely ambiguous. She establishes a mental benchmark $\widehat{Y}_{t+1}$ about her forthcoming income but thinks of it in approximate, fuzzy terms that can be expressed linguistically as "around $\widehat{Y}_{t+1}$." As a result, she believes that her forthcoming income is going to be $\widetilde{Y}_{t+1} \in \{\text{around } \widehat{Y}_{t+1}\}$. Forecast $\{\text{around } \widehat{Y}_{t+1}\}$ is a fuzzy set. We assume that the degree to which any given level of income Y belongs to fuzzy set $\{\text{around } \widehat{Y}\}$ is represented by the following triangular membership function:[3]

$$\mu(Y) = \begin{cases} \dfrac{Y-(1-z)\widehat{Y}}{z\widehat{Y}} & \text{if } (1-z)\widehat{Y} \leq Y \leq \widehat{Y} \\ \dfrac{(1+z)\widehat{Y}-Y}{z\widehat{Y}} & \text{if } \widehat{Y} \leq Y \leq (1+z)\widehat{Y} \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

[2]The methodology that we describe in this and the following appendices is an extension of the defuzzyfication strategy introduced in Hauenschild and Stahlecker [2]. For an introductory exposure on the fuzzy-set theory and related literature refer to Nguyen and Walker [5].

[3]Membership function (A.1) is fairly general and commonly used in the related literature. We chose this particular functional form because of its mathematical simplicity. However, any related membership function would ultimately lead to a set of results that are similar to those obtained in this paper.

Variable $z \in [0,1]$ in (A.1) is the coefficient of relative ambiguity associated with income earned in period $t+1$. It is the parameter that controls the degree of ambiguity in the investor's forecasts and it is proportional to the degree of disagreement present in scattered information. It must be apparent from (A.1) that: a) $\mu(Y) \in (0,1)$, b) any level of income in interval $(\widehat{Y} \pm z\widehat{Y})$ has a positive membership in fuzzy set $\{\text{around } \widehat{Y}\}$, c) the further away Y is from $\widehat{Y}$, the lower is its membership in the fuzzy set.

It is easy to show that the upper and lower bounds of an $\alpha - cut$ of (A.1) at a particular level of $\alpha^* \in [0,1]$ are given by:[4]

$$\{\widehat{Y} \pm \widehat{Y} z(1-\alpha^*)\} \tag{A.2}$$

Given the upper and lower bounds of the α-cut above, for extremely optimistic and pessimistic individuals who try to forecast the next-period income, term "around $\widehat{Y}$" would mean the upper and lower bounds in (A.2) respectively:

$$\widetilde{Y}^o_{t+1} = (1 + z(1-\alpha^*))\,\widehat{Y}_{t+1} \tag{A.3}$$

$$\widetilde{Y}^p_{t+1} = (1 - z(1-\alpha^*))\,\widehat{Y}_{t+1} \tag{A.4}$$

For an individual whose sentiment falls somewhere in between extreme optimism and extreme pessimism, term "around $\widehat{Y}$" can be represented as as a convex combination

[4] An α-cut of a fuzzy set $\widetilde{Y}$ and its corresponding membership function μ is given by the following crisp set $P_\alpha = \{Y \in \mathbb{R}_+ | \mu(Y) \geq \alpha\}$ where $\alpha \in (0,1]$. To obtain the bounds in (A.2), just set the top two piecewise components of membership function (A.1) equal to α^* and solve each for Y.

of extremely optimistic and pessimistic views:

$$\widetilde{Y}_{t+1} = q\widetilde{Y}^{p}_{t+1} + (1-q)\widetilde{Y}^{o}_{t+1} \tag{A.5}$$
$$= \widehat{Y}_{t+1} + (1-2q)(1-\alpha^{*})z\widehat{Y}_{t+1}$$

Here $q \in [0,1]$ is the Arrow-Hurwicz optimism-pessimism index (Hurwicz [4], Hurwicz [3], Arrow and Hurwicz [1]) with $q = 0$ representing extremely optimistic (ambiguity loving) and $q = 1$ extremely pessimistic (ambiguity averse) views. Following Hauenschild and Stahlecker [2], we aggregate (A.5) over all αs :

$$\begin{aligned} \widetilde{Y}_{t+1} &= \int_0^1 (\widehat{Y}_{t+1} + (1-2q)(1-\alpha)z\widehat{Y}_{t+1})\ d\alpha \\ &= (1 + (1-2q)\frac{z}{2})\widehat{Y}_{t+1} \end{aligned}$$

Thus,

$$\widetilde{Y}_{t+1} = \delta\widehat{Y}_{t+1} \tag{A.6}$$

where

$$\delta = 1 + (1-2q)\frac{z}{2} \tag{A.7}$$

Expression (A.6) is the defuzzified version of the fuzzy projection about forthcoming income; it pins down what exactly fuzzy belief "around $\widehat{Y}_{t+1}$" means to an individual. It must be apparent from (A.6) and (A.7) that ambiguous income "around $\widehat{Y}_{t+1}$" is perceived to be less (more) than $\widehat{Y}_{t+1}$ whenever the decision maker is pessimistic (optimistic) about future income and, therefore, whenever $0.5 < q \leq 1$ ($0 \leq q < 0.5$). Also, as ambiguity diminishes, i.e. as $z \to 0$, $\delta \to 1$ and $\widetilde{Y} \to \widehat{Y}$.

We finally assume that perceived ambiguity associated with distant future is the same as ambiguity associated with the proceeding period. This assumption implies that (A.6) must hold for any future period $s \in \{1, \infty\}$:[5]

$$\widetilde{Y}_{t+s} = \delta \widehat{Y}_{t+s} \tag{A.8}$$

Step 2: Expectation In order to introduce risk into our hitherto risk-less framework, we simply conjecture that the benchmark income $\widehat{Y}_{t+s}$ itself is a random variable with a given c.d.f. $F(\widehat{Y}_{t+1})$. It follows from (A.8) that:

$$E_t[\widetilde{Y}_{t+s}] = \delta E_t[\widehat{Y}_{t+s}] \tag{A.9}$$

where E_t represents conventional current expectations. Thus, current expectations of a perceived future random payoff "around $\widehat{Y}_{t+s}$" is equal to δ times the expectation of random payoff $\widehat{Y}_{t+s}$.

Appendix B. Expected Utility of a Partially Ambiguous Payoff. Derivation of Equations (4)-(5)

It should not be surprising that because of risk- and ambiguity-averse preferences, expected utility of ambiguous and risky income must not be the same as the utility of expected benchmark income.

Step 1: Defuzzification Again, first assume that forthcoming income $\widehat{Y}_{t+1}$ is purely ambiguous but not risky. What is the utility of approximately known income $\widehat{Y}_{t+1}$? For an extremely optimistic individual, the utility of payoff "around $\widehat{Y}_{t+1}$" can be

[5] We make this simplifying assumption for demonstration purposes only. It is technically possible to let ambiguity expand over the forecast horizon. However, with expanding ambiguity we land on a very similar set of results as without, although with more complexity and greater number of parameters and stability conditions involved.

represented using (A.3) as:

$$u^o(\widetilde{Y}_{t+1}) = u((1 + (1 - \alpha^*)z)\widehat{Y}_{t+1}) \tag{B.1}$$

where $u(\ldots)$ is a strictly concave and bounded function. Similarly, for an extremely pessimistic individual is:

$$u^p(\widetilde{Y}_{t+1}) = u((1 - (1 - \alpha^*)z)\widehat{Y}_{t+1}) \tag{B.2}$$

Utility of an individual whose optimism-pessimism is somewhere in between these two extremes can be presented as a convex combination of the two extremes:

$$\begin{aligned} u(\widetilde{Y}_{t+1}) =& q u^p(\widetilde{Y}_{t+1}) + (1 - q)u^o(\widetilde{Y}_{t+1}) \\ =& q u((1 - (1 - \alpha^*)z)\widehat{Y}_{t+1}) + (1 - q)u((1 + (1 - \alpha^*)z)\widehat{Y}_{t+1}) \end{aligned}$$

We again aggregate the above over all possible α-cuts between 0 and 1:

$$\begin{aligned} u(\widetilde{Y}_{t+1}) = q \int_0^1 u((1 - (1 - \alpha)z)\widehat{Y}_{t+1})d\alpha \\ + (1 - q) \int_0^1 u((1 + (1 - \alpha)z)\widehat{Y}_{t+1})d\alpha \end{aligned} \tag{B.3}$$

We assume the utility function is of the following CRRA form:

$$u(Y) = \frac{Y^{1-\gamma}}{1 - \gamma} \tag{B.4}$$

where γ is the investor's coefficient of risk aversion. It follows that the utility of income "around $\widehat{Y}_{t+1}$" can be expressed as:

$$\begin{aligned} u(\widetilde{Y}_{t+1}) &= q\int_0^1 \frac{((1-z(1-\alpha))\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma}d\alpha + (1-q)\int_0^1 \frac{((1+z(1-\alpha))\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma}d\alpha \\ &= \frac{\widehat{Y}_{t+1}^{1-\gamma}}{1-\gamma}\left[q\int_0^1 (1-z(1-\alpha))^{1-\gamma}d\alpha + (1-q)\int_0^1 (1+z(1-\alpha))^{1-\gamma}d\alpha\right] \end{aligned} \tag{B.5}$$

We rewrite the above in a more concise form as:

$$u(\widetilde{Y}_{t+1}) = \theta u(\widehat{Y}_{t+1}) \tag{B.6}$$

where $u(\widehat{Y}_{t+1}) = \frac{\widehat{Y}_{t+1}^{1-\gamma}}{1-\gamma}$ and θ is the expression in the square brackets of (B.5) that can be simplified to:

$$\theta = \frac{(2q-1) - q(1-z)^{2-\gamma} + (1-q)(1+z)^{2-\gamma}}{(2-\gamma)z} \tag{B.7}$$

The difference between utility of a payoff "around $\widehat{Y}_{t+1}$" and utility of a payoff precisely $\widehat{Y}_{t+1}$ is captured by coefficient θ.

Figure B.1 demonstrates how θ varies with variations in γ, q, and z. As it must be apparent from Figure B.1, $\theta > 1$ for a sufficiently pessimistic and risk-averse individual. Specifically, when $z = 0.05$, $\theta > 1$ for all values of $\gamma > 2$ and $q > 0.45$. With more ambiguity, even lower values of γ and z result in $\theta > 1$, e. g. when $z = 0.2$, $\theta > 1$ for all values of $\gamma > 1.5$ and $q > 0.40$. Intuition here is better understood by considering marginal utilities instead: $u'(\widetilde{Y}_{t+1}) = \theta u'(\widehat{Y}_{t+1})$. For a risk-averse and ambiguity-averse individual who believes that her income is going to be "around $\widehat{Y}_{t+1}$," additional known income provides θ times more marginal utility than for someone who believes that her income is going to be exactly $\widehat{Y}_{t+1}$.

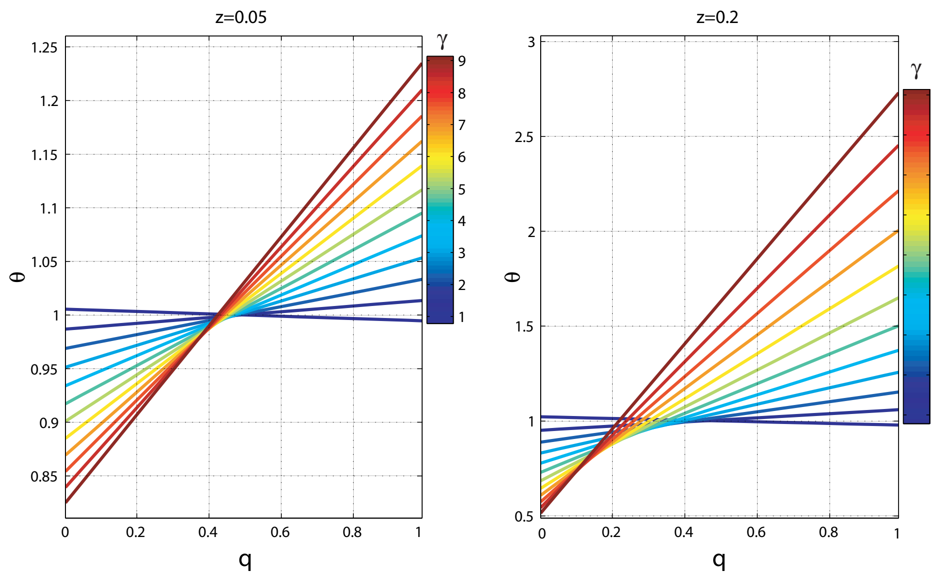

Figure B.1: Values of θ resulting from different values of risk aversion, γ, optimism-pessimism, q, and ambiguity, z.

Step 2: Expectation In order to incorporate risk into the utility analysis, we again allow the benchmark level of income $\widehat{Y}_{t+1}$ to be a random variable with some given c.d.f $F(\widehat{Y}_{t+1})$. It follows from (B.6) that for given θ

$$E_t[u(\widetilde{Y}_{t+1})] = \theta E_t[u(\widehat{Y}_{t+1})] \tag{B.8}$$

The above means that the expected utility of perceived income "around $\widehat{Y}_{t+1}$" is equal to θ times the expected utility of exactly $\widehat{Y}_{t+1}$.

If we again assume that perceived ambiguity associated with distant future is the same as ambiguity associated with the following period then (B.8) must hold for any future period $s \in \{1, \infty\}$:

$$E_t[u(\widetilde{Y}_{t+s})] = \theta E_t[u(\widehat{Y}_{t+s})] \tag{B.9}$$

Appendix C. Derivation of Equation (11)

Substitute equation (10) into (8):

$$1 = E_t \left[\frac{P^e_{t+1} + D_{t+1}}{P^e_t} \beta\theta \widehat{X}^{-\gamma}_{t+1} \right] \tag{C.1}$$

Since forthcoming dividends are partially ambiguous, the investor forms a fuzzy forecast of this variable. Thus,

$$1 = E_t \left[\frac{P^e_{t+1} + \widetilde{D}_{t+1}}{P^e_t} \beta\theta \widehat{X}^{-\gamma}_{t+1} \right] \tag{C.2}$$

Applying (A.8) to $\widetilde{D}_{t+1}$ and rearranging we get:

$$P^e_t = E_t \left[(P^e_{t+1} + \delta_{t+1} \widehat{D}_{t+1}) \beta\theta \widehat{X}^{-\gamma}_{t+1} \right] \tag{C.3}$$

In equilibrium, consumption growth must be equal to dividend growth, hence

$$\widehat{X}_{t+1} = \frac{\widehat{C}_{t+1}}{C_t} = \frac{\widehat{D}_{t+1}}{D_t} \tag{C.4}$$

Substitute the above into (C.3) and rearrange

$$P^e_t = E_t \left[\beta\theta \left(\frac{\widehat{D}_{t+1}}{D_t} \right)^{-\gamma} P^e_{t+1} + \delta\beta\theta \left(\frac{\widehat{D}_{t+1}}{D_t} \right)^{-\gamma} \widehat{D}_{t+1} \right] \tag{C.5}$$

Update the above one period forward

$$P^e_{t+1} = E_{t+1}\left[\beta\theta\left(\frac{\widehat{D}_{t+2}}{D_{t+1}}\right)^{-\gamma}P^e_{t+2} + \delta\beta\theta\left(\frac{\widehat{D}_{t+2}}{D_{t+1}}\right)^{-\gamma}\widehat{D}_{t+2}\right] \tag{C.6}$$

Substituting the above into the previous expression yields the following:

$$P^e_t = E_t\left[\beta^2\theta^2\left(\frac{\widehat{D}_{t+2}}{D_t}\right)^{-\gamma}P^e_{t+2} + \beta^2\theta^2\delta\left(\frac{\widehat{D}_{t+2}}{D_t}\right)^{-\gamma}\widehat{D}_{t+2} + \delta\beta\theta\left(\frac{\widehat{D}_{t+1}}{D_t}\right)^{-\gamma}\widehat{D}_{t+1}\right] \tag{C.7}$$

Continuous i-fold forward substitution leads to the following expression:

$$P^e_t = E_t\left[(\beta\theta)^i\left(\frac{\widehat{D}_{t+i}}{D_t}\right)^{-\gamma}P^e_{t+i} + \sum_{s=1}^{i}\delta(\beta\theta)^s\left(\frac{\widehat{D}_{t+s}}{D_t}\right)^{-\gamma}\widehat{D}_{t+s}\right] \tag{C.8}$$

When calibrating the model, we are able to retain only the solutions for which the following condition holds:[6]

$$\lim_{i\to\infty}(\beta\theta)^i\left(\frac{\widehat{D}_{t+i}}{D_t}\right)^{-\gamma} = 0 \tag{C.9}$$

Thus, with appropriate parameterization (C.8) can be written as:

$$P^e_t = E_t\left[\sum_{s=1}^{\infty}\delta(\beta\theta)^s\left(\frac{\widehat{D}_{t+s}}{D_t}\right)^{-\gamma}\widehat{D}_{t+s}\right] \tag{C.10}$$

[6]Note that the expression for cumulative dividend growth $\frac{\widehat{D}_{t+i}}{D_t}$ can be decomposed into: $\frac{\widehat{D}_{t+i}}{\widehat{D}_{t+i-1}} \times \frac{\widehat{D}_{t+i-1}}{\widehat{D}_{t+i-2}} \times \cdots \times \frac{\widehat{D}_{t+1}}{D_t}$. Thus even if $(\beta\theta) > 0$, the whole expression, may converge.

or

$$P_t^e = E_t\left[\sum_{s=1}^{\infty}\delta(\beta\theta)^s\left(\frac{\widehat{D}_{t+s}}{D_t}\right)^{1-\gamma}\right]D_t \tag{C.11}$$

Since δ is finite and if (C.9) holds, the sum on the right-hand-side expression above must also converge. If growth of the benchmark level of dividends is i.i.d., so that $\frac{\widehat{D}_{t+k}}{\widehat{D}_{t+k-1}} = \epsilon_{t+k}$, then the above can be rewritten as:[7]

$$P_t^e = E_t\left[\sum_{s=1}^{\infty}\delta(\beta\theta)^s\prod_{k=1}^{s}\epsilon_{t+k}^{1-\gamma}\right]D_t \tag{C.12}$$

Thus, $P_t^e = wD_t$, where $w = E_t\left[\sum_{s=1}^{\infty}\delta(\beta\theta)^s\prod_{k=1}^{s}\epsilon_{t+k}^{1-\gamma}\right]$. ■

Appendix D. Data and Calibration Procedure

Real return on equity is obtained by summing real capital gain and real dividend yield. We obtain the gross ex-post real risk-free rate by dividing the gross nominal interest rate by the gross forthcoming percent change in the CPI. Real equity premia are subsequently obtained by subtracting the real risk-free rate from the real return on equity.

Our calibration procedure consists of three logical steps. Steps 1 and 2 calibrate the model parameters to the average observed equity premium and the risk-free rate respectively. Step 3 examines each combination of calibrated parameters to ensure that they do not violate stability condition (C.9).

Step 1: To calibrate the model to the average equity premium, we substitute (2) into

[7]Here again we utilize the fact that cumulative dividend growth can be decomposed into the product of period-by-period growth: $\frac{\widehat{D}_{t+i}}{D_t} = \frac{\widehat{D}_{t+i}}{\widehat{D}_{t+i-1}} \times \frac{\widehat{D}_{t+i-1}}{\widehat{D}_{t+i-2}} \times \cdots \times \frac{\widehat{D}_{t+1}}{D_t}$.

(21) and solve the resulting expression for the Arrow-Hurwicz optimism-pessimism index q:

$$q = \frac{1}{2} - \frac{e^{\sigma^2\gamma - \eta} - 1}{z} \tag{D.1}$$

We use the above to obtain all possible values of q^* that for given values of the coefficient of risk aversion $\gamma^* \in (0, 10)$ and the coefficient of relative ambiguity $z^* \in (0, 1)$ calibrate the model to the average observed net risk premium $\overline{\eta}$ and the variance of the log of consumption growth, $\overline{\sigma}^2$ (in the last column of Table 1). Thus,

$$q^* = \frac{1}{2} - \frac{e^{\overline{\sigma}^2\gamma^* - \overline{\eta}} - 1}{z^*} \tag{D.2}$$

Values of γ^* and z^* are obtained by dividing the parameter space $(0, 10) \times (0, 1)$ into a grid of $3,000$ equally spaced points. Thus the calibration process consists of $3,000$ iterations, one for each of these points. We obtain values for $\overline{\eta}$ and $\overline{\sigma}^2$ from the descriptive statistics in the last column of Table 1 as follows:[8]

$$\overline{\eta} = \ln\left(\frac{1.0786}{1.0052}\right) = 0.0705 \tag{D.3}$$

$$\overline{\sigma}^2 = \ln\left(\frac{Var(X)}{E(X)^2} + 1\right) = \ln\left(\frac{0.0288^2}{1.0203^2} + 1\right) = 0.000797 \tag{D.4}$$

Thus, Step 1 consists of obtaining $3,000$ values of q^* given $\overline{\eta} = 0.0705$ and $\overline{\sigma}^2 = 0.000797$ and a grid of different values of $\gamma^* \in (0, 10)$ and $z^* \in (0, 1)$.

Step 2: Upon completion of Step 1, we still need to calibrate the coefficient of time preference β. The calibrated value of $\beta = \beta^*$ must be consistent not only with the

[8]For the first expression, recall that the model solution for equilibrium equity return (19) is presented in a ratio form expressed in terms of gross equity and risk-free rates. To obtain the formula for the variance of log-consumption growth below, we used the following property of the log-normal distribution $Var(X) = (e^{\sigma^2} - 1)E(X)^2 \Rightarrow \sigma^2 = \ln\left(\frac{Var(X)}{E(X)^2} + 1\right)$

average risk premium but also with the average risk-free rate. Solving the expression for the predicted equilibrium risk-free rate in (18) for β leads to the following:

$$\beta = e^{\gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - r^f - \ln(\theta)} \tag{D.5}$$

The values of $\theta = \theta^*$ that are consistent with the average observed risk premium are obtained by feeding each combination of $\{\gamma^*, z^*, q^*\}$ from Step 1 into the right-hand side of (B.7).

$$\theta^* = \frac{(2q^* - 1) - q^*(1 - z^*)^{2-\gamma^*} + (1 - q^*)(1 + z^*)^{2-\gamma^*}}{(2 - \gamma^*)z^*} \tag{D.6}$$

We obtain the sample counterparts of r^f and μ from the last column of Table 1 as follows:

$$\overline{r}^f = \ln(1.0052) = 0.0052 \tag{D.7}$$

$$\overline{\mu} = \ln(E(X)) - \frac{1}{2}\sigma^2 = \ln(1.0203) - \frac{1}{2}0.000797 = 0.0197 \tag{D.8}$$

where formula $\overline{\mu} = \ln(E(X)) - \frac{1}{2}\sigma^2$ follows from the log-normality assumption. On each iteration of the calibration process, we substitute each $\gamma = \gamma^*$ and corresponding $\theta = \theta^*$ as well as $\overline{r}^f = 0.0052$, $\overline{\mu} = 0.0197$, and $\overline{\sigma}^2 = 0.000797$ into the right-hand side of (D.5). This process leads to the calibrated value of β^*:

$$\beta^* = e^{\gamma^*\overline{\mu} - \frac{1}{2}(\gamma^*)^2\overline{\sigma}^2 - \overline{r}^f - \ln(\theta^*)} \tag{D.9}$$

By working through the first two steps, we are able to obtain combinations of $\{\gamma^*, q^*, \beta^*, z^*\}$ that successfully calibrate the solutions of the model to the descriptive statistics in the last column of Table 1.

Step 3: As it was discussed at the beginning of this appendix, we must test each combination $\{\gamma^*, q^*, \beta^*, z^*\}$ against stability condition (C.9). To that end, we utilize condition $\frac{\widehat{C}_{t+1}}{C_t} = \frac{\widehat{D}_{t+1}}{D_t}$ as well as the assumption of log-normality of consumption growth and rewrite (C.9) as follows:

$$\lim_{i\to\infty} (\beta^*\theta^*)^i (1.0197)^{-i\gamma^*} = 0 \tag{D.10}$$

where θ^* is given by (D.6) and 1.0197 is one plus the average rate of log-consumption growth obtained in (D.7).[9] Convergence of the above sequence is guaranteed if the partial derivative of $(\beta^*\theta^*)^i (1.0197)^{-i\gamma^*}$ with respect to i is negative and thus if the following condition holds:

$$-\gamma^* \ln 1.097 + \ln \beta^* + \ln \theta^* < 0 \tag{D.11}$$

Hence, we calculate value $-\gamma^* \ln 1.097 + \ln \beta^* + \ln \theta^*$ for each quad-tuple $\{\gamma^*, q^*, \beta^*, z^*\}$ and if the value is negative we retain the quad-tuple and discard it if the value is positive.

References

[1] Arrow, K., Hurwicz, L., 1972. An optimality criterion for decision-making under ignorance. In: *Uncertainty and Expectations in Economics: Essays in Honor of GLS Shackle, CF Carter and JL Ford*, Oxford: Basil Blackwell.

[9] Here again we used decomposition $\frac{\widehat{C}_t+i}{C_t} = \frac{\widehat{C}_{t+i}}{\widehat{C}_{t+i-1}} \times \frac{\widehat{C}_{t+i-1}}{\widehat{C}_{t+i-2}} \times \cdots \times \frac{\widehat{C}_{t+1}}{C_t}$. Keeping in mind that the mean of log consumption growth $\overline{\mu}$ is the same as geometric average of (non log) consumption growth, $\frac{\widehat{C}_t+i}{C_t}$ can be written as $(1+\overline{\mu})^i$.

[2] Hauenschild, N., Stahlecker, P., 2001. Precautionary saving and fuzzy information. Economics Letters 70, 107–114.

[3] Hurwicz, L., 1951. The generalized bayes-minimax principle: A criterion for decision making under uncertainty. Cowles Comission Discussion Papers 355.

[4] Hurwicz, L., 1951. Optimality criteria for decision making under ignorance. Cowles Comission Discussion Papers 370.

[5] Nguyen, H. T., Walker, E. A., 2005. A First Course in Fuzzy Logic. Chapman Hall/CRC, third ed.