```
In[•]:= (*Complete reset of Mathematica and xAct*)Quit[]

In[1]:= (*Load xAct packages*)<< xAct`xTensor`
<< xAct`xCoba`

(*Define the manifold and metric*)
DefManifold[M4, 4, {μ, ν, ρ, σ, λ, κ, α, β, γ, δ}]
DefMetric[-1, metric[-μ, -ν], CD, {";", "∇"}]

(*Define abstract indices properly*)
a = α
b = β
c = γ
d = δ

(*Define the Weyl tensor in terms of Riemann tensor*)
WeylDefinition = WeylCD[-a, -b, -c, -d] → RiemannCD[-a, -b, -c, -d] -
   (1 / 2) * (metric[-a, -c] × RicciCD[-b, -d] - metric[-a, -d] × RicciCD[-b, -c] -
      metric[-b, -c] × RicciCD[-a, -d] + metric[-b, -d] × RicciCD[-a, -c]) + (1 / 6) *
    RicciScalarCD[] * (metric[-a, -c] × metric[-b, -d] - metric[-a, -d] × metric[-b, -c])

(*Define the covariant derivative of the Weyl tensor*)
WeylDerivative = CD[-d][WeylCD[-a, -b, -c, d]]

(*Substitute the Weyl tensor definition*)
WeylDerivativeExpanded = WeylDerivative /. WeylDefinition

(*Apply the contracted Bianchi identity*)
BianchiiRule1 = CD[-ν][RicciCD[-μ, ν]] → (1 / 2) * CD[-μ][RicciScalarCD[]]

(*Apply symmetries and simplify*)
WeylDerivativeSimplified = WeylDerivativeExpanded //. BianchiiRule1

(*Simplify further using built-in xAct rules*)
WeylDerivativeSimplified = Simplify[WeylDerivativeSimplified]

(*Use ToCanonical to properly simplify the Weyl tensor expression*)
WeylDerivativeCanonical = ToCanonical[WeylDerivativeSimplified]

Print["Weyl tensor definition:"]
Print[WeylDefinition]
Print[]

Print["Weyl tensor covariant derivative:"]
Print[WeylDerivative]
Print[]

Print["After substituting Weyl definition and applying Bianchi identities:"]
Print[WeylDerivativeCanonical]
Print[]
```

```
(*The final simplified form*)
FinalResult =
 CD[-d][WeylCD[c, a, b, d]] → CD[-b][RicciCD[c, a]] - CD[-a][RicciCD[c, b]] + (1/6) *
   (metric[c, b] × CD[-a][RicciScalarCD[]] - metric[c, a] × CD[-b][RicciScalarCD[]])

Print["Final simplified result:"]
Print[]
Print["In xAct notation:"]
Print[FinalResult]

Print[]
Print["Verification - Key properties used:"]
Print["1. Weyl tensor is traceless: C^a_{bac} = 0"]
Print["2. Contracted Bianchi identity: R^{ab}_{;b} = (1/2)R^{;a}"]
Print["3. Antisymmetry: [a;b] means antisymmetrization over indices a and b"]

(*Use xAct's built-in Weyl tensor simplification*)
Print[]
Print["Using xAct's built-in Weyl tensor simplification:"]
WeylBuiltIn = ToCanonical[CD[-d][WeylCD[-a, -b, -c, d]]]
Print[WeylBuiltIn]

(*Additional simplification using xAct rules*)
Print[]
Print["Final canonical form:"]
FinalCanonical = ToCanonical[CD[-δ][WeylCD[-α, -β, -γ, δ]]]
Print[FinalCanonical]
```

```
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}

CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.

Connecting to external MinGW executable...

Connection established.

------------------------------------------------------------

Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}

CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.

------------------------------------------------------------

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}

CopyRight (C) 2005-2021, David Yllanes
  and Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------
```

```
These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
** DefManifold: Defining manifold M4.
** DefVBundle: Defining vbundle TangentM4.
** DefTensor: Defining symmetric metric tensor metric[-μ, -ν].
** DefTensor: Defining antisymmetric tensor epsilonmetric[-α, -β, -γ, -δ].
** DefTensor: Defining tetrametric Tetrametric[-α, -β, -γ, -δ].
** DefTensor: Defining tetrametric Tetrametric†[-α, -β, -γ, -δ].
** DefCovD: Defining covariant derivative CD[-μ].
** DefTensor: Defining vanishing torsion tensor TorsionCD[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[α, -β, -γ].
** DefTensor: Defining Riemann tensor RiemannCD[-α, -β, -γ, -δ].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-α, -β].
** DefTensor: Defining Weyl tensor WeylCD[-α, -β, -γ, -δ].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-α, -β].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detmetric[]. Determinant.
```

Out[5]= α

Out[6]= β

Out[7]= γ

Out[8]= δ

Out[9]= $$W[\nabla]_{\alpha\beta\gamma\delta} \to \frac{1}{2}\left(-\mathrm{metric}_{\beta\delta}\, R[\nabla]_{\alpha\gamma} + \mathrm{metric}_{\beta\gamma}\, R[\nabla]_{\alpha\delta} + \mathrm{metric}_{\alpha\delta}\, R[\nabla]_{\beta\gamma} - \mathrm{metric}_{\alpha\gamma}\, R[\nabla]_{\beta\delta}\right) + \frac{1}{6}\left(-\mathrm{metric}_{\alpha\delta}\,\mathrm{metric}_{\beta\gamma} + \mathrm{metric}_{\alpha\gamma}\,\mathrm{metric}_{\beta\delta}\right) R[\nabla] + R[\nabla]_{\alpha\beta\gamma\delta}$$

Out[10]=
$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Out[11]=
$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Out[12]=
$$\nabla_\nu R[\nabla]_{\mu}{}^{\nu} \to \frac{\nabla_\mu R[\nabla]}{2}$$

Out[13]=

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Out[14]=

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Out[15]=

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Weyl tensor definition:

$$W[\nabla]_{\alpha\beta\gamma\delta} \to \frac{1}{2}\left(-\text{metric}_{\beta\delta}\ R[\nabla]_{\alpha\gamma} + \text{metric}_{\beta\gamma}\ R[\nabla]_{\alpha\delta} + \text{metric}_{\alpha\delta}\ R[\nabla]_{\beta\gamma} - \text{metric}_{\alpha\gamma}\ R[\nabla]_{\beta\delta}\right) + \frac{1}{6}\left(-\text{metric}_{\alpha\delta}\ \text{metric}_{\beta\gamma} + \text{metric}_{\alpha\gamma}\ \text{metric}_{\beta\delta}\right) R[\nabla] + R[\nabla]_{\alpha\beta\gamma\delta}$$

Weyl tensor covariant derivative:

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

After substituting Weyl definition and applying Bianchi identities:

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Out[25]=

$$\nabla_\delta W[\nabla]^{\gamma\alpha\beta\delta} \to -\nabla_\alpha R[\nabla]^{\gamma\beta} + \nabla_\beta R[\nabla]^{\gamma\alpha} + \frac{1}{6}\left(\text{metric}^{\gamma\beta}\ \nabla_\alpha R[\nabla] - \text{metric}^{\gamma\alpha}\ \nabla_\beta R[\nabla]\right)$$

Final simplified result:

In xAct notation:

$$\nabla_\delta W[\nabla]^{\gamma\alpha\beta\delta} \to -\nabla_\alpha R[\nabla]^{\gamma\beta} + \nabla_\beta R[\nabla]^{\gamma\alpha} + \frac{1}{6}\left(\text{metric}^{\gamma\beta}\ \nabla_\alpha R[\nabla] - \text{metric}^{\gamma\alpha}\ \nabla_\beta R[\nabla]\right)$$

Verification - Key properties used:

1. Weyl tensor is traceless: C^a_{bac} = 0

2. Contracted Bianchi identity: R^{ab}_{;b} = (1/2)R^{;a}

3. Antisymmetry: [a;b] means antisymmetrization over indices a and b

Using xAct's built-in Weyl tensor simplification:

Out[37]=

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

Final canonical form:

Out[41]=

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$

$$\nabla_\delta W[\nabla]_{\alpha\beta\gamma}{}^{\delta}$$