Colouring 2-crossing-critical graphs

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Introduction

- This presentation follows a joint work in progress, with Professor Drago Bokal, Professor John Baptist Gauci and Marietta Galea.
- Preliminary definitions and results are taken from [1].

Definition

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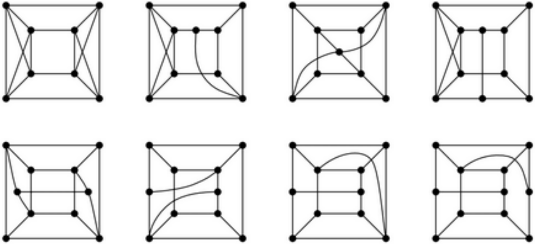
The crossing number $\mathrm{cr}(G)$ of a graph G is the smallest number of crossings over all of its plane drawings. Furthermore, for some $k \in \mathbb{N}$, G is k-crossing-critical if $\mathrm{cr}(G) \geq k$ but every proper subgraph $H \subseteq G$ has $\mathrm{cr}(H) < k$.



Tiles

Definition

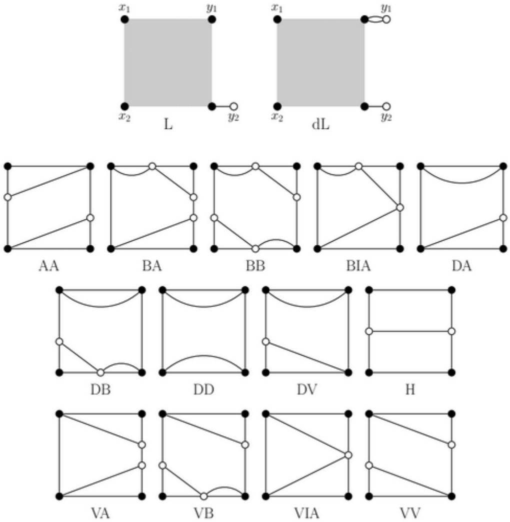
A tile is a tripe $T = (G, x, y)$, consisting of a graph G and two non-empty sequences $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ of distinct vertices of G, with no vertex appearing in both x and y. The sequence x (sequence y) is T's left wall (right wall, respectively).

Definition

Tiled graphs are joins of cyclic sequences of tiles. Defined inductively as below.

- A tile $T = (G, x, y)$ is compatible with a tile $T' = (G', x', y')$ if $|y| = |x'|$. Their join $T \otimes T' := (H, x, y')$ where H is obtained by the disjoint union of G and G' by identifying y_i with x'_i for each $i \in \{1, \ldots, |y|\}$.
- A sequence $\mathcal{T} = (T_1, \ldots, T_m)$ of tiles is compatible if T_i is compatible with T_{i+1} for each $i = 1, \ldots, m-1$. The join $\otimes\mathcal{T}$ is $T_1 \otimes T_2 \otimes \cdots \otimes T_m$.
- The cyclization of T with $|x| = |y|$ is the graph obtained from G by identifying x_i with y_i for each $i = 1, \ldots, |x|$.

Full characterisation of large 2-crossing critical graphs



Colouring results

Chromatic number was studied before [2], yielding the following results:

Theorem

Every large 2-crossing-critical graph is 4-colourable.

Theorem

A large 2-crossing critical graph is 2-colourable if and only if is the join of $T_0 \otimes T_1 \otimes \cdots \otimes T_{2m}$ where T_i is an elementary tile and

- *Tile T_0 contains an H-picture*
- *Each $sig(T_i)sig(T_{(i+1) \mod (2m+1)})$ starts with DDdLDD, DDLH, HdLH or HLDD for $0 \leq i \leq 2m+1$*
- *the number of L frames in $\{T_{2i}\}_{0 \leq i \leq m}$ is odd.*

Propagation

Definition

Consider a proper vertex colouring c of G. The colours $c(x_1), c(x_2)$ are the input colours of T, and the colours $c(y_1), c(y_2)$ are the output colours of T. If there exists such a colouring, we will denote this as $(c(x_1), c(x_2)) \rightsquigarrow (c(y_1), c(y_2))$.

Definition (Relabelling of colours)

Two k-colourings c and c' of T are equivalent if $c(v) = c(w)$ whenever $c'(v) = c'(w)$ for each pair of wall vertices $v, w \in \{x_1, x_2, y_1, y_2\}$. We call the induced equivalence classes k-propagations.

If two tiles have the same propagation, they will act the same in terms of colouring and hence are interchangeable.

Propagation Matrices

Definition

The k-propagation matrix $P_k(T)$ of a tile T is the matrix P_k such that given an ordering of input colours $(c_1, c_2, \ldots, c_t)$ and output colours $(o_1, o_2, \ldots, o_s)$, such that

$$P_k(T)_{i,j} = \begin{cases} 1, & \text{if } c_i \rightsquigarrow o_j \\ 0, & \text{otherwise} \end{cases}$$

Note that if the size of the left wall and the right wall is the same, we use take the same ordering for both input colours and output colours.

Lemma

The k-propagation matrix of a tile $T = T_1 \otimes T_2$ is simply $P_k(T_1)P_k(T_2)$, normalised such that every non-zero element is 1.

What does a propagation matrix tell you?

Lemma

The T is k-colourable if and only if $P_k(T)$ has at least one non-zero entry.

Lemma

The cyclization of T is k-colourable if and only if $P_k(T)$ has a 1 on the diagonal.

Finding 3-propagation matrices

Proposed algorithm to identify 4-chromatic graphs

1. Since we only had 42 graphs, we used a simple algorithm to find all propagations.
2. We then find the closure of all possible propagations by iteratively multiplying known propagation matrices together to create new ones.
3. During the above process, for every two matrices store their result.
4. Identify which matrices F do not have a loop. These correspond to graphs which are not 3-colourable.
5. Identify all possible 'predecessors' X of F (inclusive of F).
6. Find all ordered pairs (P, P') such that P' is a matrix corresponding to an elementary tile, and $PP' \in X$.

Automata

Constructing our desired automata:

We construct an automata $M_{\text{4chrom}} = (\Sigma, Q, \delta, q_0, F)$ with

- Alphabet $\Sigma = X' \subseteq X$, where each element of X' is an elementary tile;
- States $Q = X \bigcup \{I\}$, all 'predecessors' of F;
- Start state I (the identity matrix);
- Final states F, being all propagation matrices without a loop;
- Whenever $PP' \in Q$ for $P \in Q$, $P' \in \Sigma$, $\delta(P, P') = PP'$.

This automata will accept all sequences of tiles which result in a 4-chromatic graph and reject the rest!

Results

Results

Let $n \in \mathbb{N} \cup \{0\}$ and $V_i \in \{AIVL, VIAL\}$ for all $i \in \mathbb{N} \cup \{0\}$. Then, any large 4-chromatic 2-crossing-critical graph can be written in one of the following forms:

- $T \prod_{i=0}^{2n} V_i$ where $T \in \{ABL,\ VVL,\ DVL,\ BAL,\ DDdL,\ AIVdL,\ VIAdL,\ VDL,\ AIVL,\ VIAL\}$
- $T_1\, T_2 \prod_{i=0}^{2n+1} V_i$ where $(T_1,\ T_2) \in \{(AVL,\ VAL),\ (DVL,\ VDL),\ (VIAdL,\ DDL),\ (AIVdL,\ DDL),\ (BIAL,\ AIBL)\}$
- $(T_1)(\prod_{i=0}^{2n} V_i)(T_2)(\prod_{i=0}^{2n+1} V_i)$ where $(T_1,\ T_2) \in \{\ (VIAdL,\ DDL),\ (AIVdL,\ DDL),\ (AVL,\ VAL),\ (DVL,\ VDL)\}$

References

[1] D. Bokal et al. "Characterizing 2-crossing-critical graphs". In: *Advances in Applied Mathematics* 74 (2016), pp. 23–208. ISSN: 0196-8858. DOI: https://doi.org/10.1016/j.aam.2015.10.003. URL: https://www.sciencedirect.com/science/article/pii/S0196885815001165.

[2] Drago Bokal et al. *Properties of Large 2-Crossing-Critical Graphs*. 2021. arXiv: 2112.04854 [cs.DM].

Thanks!