Goodstein Sequences and Unprovability in Peano Arithmetic

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Abstract

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Goodstein's theorem is a true finitary statement about the natural numbers which is nevertheless unprovable in the theory of Peano Arithmetic. Throughout this thesis, we prove this claim, following closely the methods of Buchholz and Wainer [1987]. This process leads us through various important results in the study of proof theory.

We begin by establishing the ordinal numbers and ordinal arithmetic, from which we prove Goodstein's theorem. In this process, we introduce the ordinal ε_0, around which many of our results center. Moreover, we establish a hierarchy of fast growing functions up to ε_0, following Ketonen and Solovay [1981]. Using these functions, we determine a formula for Goodstein's function following Caicedo [2007], and prove bounding results for elementary functions.

Moreover, we establish the machinery required to define PA, namely first order logic and a Tait-style proof system. In defining PA, we embed in our language the Csillag-Kalmar elementary functions, which allows for a neat definition of the provably computable functions of PA.

As was done in Gentzen [1964] to prove the consistency of Peano Arithmetic, we embed PA in an infinitary system PA_∞. This allows us to perform cut-elimination, essentially reducing the variation of how a proof is derived at the cost of a longer proof. This allows us to obtain a bound for the provably computable functions, which we can show does not hold for Goodstein's function by the formula derived previously. This can be used to show that Goodstein's theorem is unprovable in PA.

Contents

1 Introduction

Gödel's incompleteness theorems are widely known in the mathematical community; however, their impact on everyday mathematics is not immediately clear. We shed some light on this topic by exploring Goodstein's theorem, a "true" statement about a sequence of natural numbers, which nevertheless is unprovable in Peano Arithmetic. In doing so, we establish some foundational techniques of proof theory and ordinal analysis.

Our story begins with Hilbert's program (a review of which is given in Zach [2023]), a quest to provide a foundational basis for all of mathematics by specifying a formal system. The goal was to create an axiomatic system able to prove its own consistency by finitary means. These efforts were shown to be futile by Gödel [1931] when he proved his incompleteness theorems. Gödel's first incompleteness theorem states that every consistent axiomatic[1] system strong enough to formalise arithmetic must be incomplete, which means that there are statements the system cannot prove or disprove. In particular, Gödel's second incompleteness theorem states that such systems cannot prove their own consistency.

This result typically leaves people with more questions than answers. What are the consequences of these theorems to everyday mathematics? Which statements are unprovable? To answer these, one might turn to examining the *Gödel sentence*, the unprovable statement constructed in Gödel's first incompleteness

[1]Here we implicitly require that the axiomatisation is "effective", meaning we have some algorithm to find our provable statements.

theorem. However, the Gödel sentence is a self-referential statement which roughly states "this statement is unprovable". Moreover, to express this statement in the language of arithmetic, an encoding of theorems as numbers was used. The novelty of this approach led many to believe that Gödel's theorems had little consequence, as unprovable statements were not seen as relevant outside of the study of formal systems. However, in Peano Arithmetic (PA), the standard model of finitary arithmetic, there are unprovable statements that impinge upon the conduct of various fields of mathematics, such as Number Theory and Combinatorics.

Gödel's theorems launched the search for a true statement, used in everyday mathematics which is unprovable in Peano Arithmetic, as well as an associated consistency proof of PA. Gentzen [1964] devised a proof for the consistency of PA using the theory of Primitive Recursive Arithmetic with the addition of transfinite induction up to ε_0 applied to primitive recursive predicates. Gentzen's proof is the most widely accepted proof of the consistency of PA because he used mainly finitary arguments and Primitive Recursive Arithmetic is simpler than PA hence its consistency is much less controversial.

An infinitary aspect is introduced to this proof by the inclusion of transfinite induction up to the ordinal ε_0. This ordinal was first introduced by Cantor [1895] as the least fixed point of the operation $\alpha \mapsto \omega^\alpha$. Transfinite induction up to ε_0 may be stated in the language of PA, making use of an encoding of the ordinals below ε_0 as numbers, but it is not provable in PA. This fact can be seen by combining Gentzen's proof with Gödel's first incompleteness theorem.

In an attempt to find a true unprovable statement about natural numbers in PA, Goodstein [1944] invented *Goodstein sequences*[2]. From these sequences, he produced the statement "every (extended) Goodstein sequence eventually reaches zero" and proved that this was equivalent to transfinite induction up to ε_0. At this time, the link between unprovability and transfinite induction up to ε_0 was unclear, so the connection with unprovability was not made. However, using a different method, Kirby and Paris [1982] confirmed Goodstein's

[2]We note that Goodstein worked with more general sequences than those we consider in this work, which we now refer to as "extended Goodstein sequences".

suspicions by proving that the statement "every Goodstein sequence eventually reaches zero", now commonly known as *Goodstein's theorem*, is indeed unprovable in PA. This was done using model-theoretic methods, and the combinatorial machinery developed by Ketonen and Solovay [1981].

The significance of this result arises from the remarkably simple definition of a Goodstein sequence. A Goodstein sequence starting at m is obtained iteratively by increasing the base in which we write the previous term of the sequence, and subtracting 1. We say that a Goodstein sequence *terminates* when it reaches zero. From this sequence we define *Goodstein's function* $\mathcal{G}$, which given a number m returns the number of iterations it takes for a Goodstein sequence starting at m to terminate. Although these sequences grow in size very quickly, it is easy to see using infinite ordinal numbers that this sequence eventually reaches zero, hence our function $\mathcal{G}$ is total. This fact is referred to as *Goodstein's theorem*, and was proven by Goodstein himself.

Given the finitary nature of this statement, it is quite surprising that this infinite quality to the proof of Goodstein's theorem is strictly necessary. We analyse this claim by following a simpler proof given by Buchholz and Wainer [1987], which also builds on the machinery built in Ketonen and Solovay [1981]. The main idea of this proof revolves around the method of *cut-elimination*, first established by Gentzen [1964] to prove the consistency of Peano Arithmetic.

This method gives us a more general result, informally that all *provably computable (total)* functions in Peano Arithmetic must necessarily not grow too quickly — a property that does not hold for Goodstein's function $\mathcal{G}$, hence Peano Arithmetic cannot prove that $\mathcal{G}$ is total. This means that Goodstein's theorem is not provable in PA. We prove the fast growing nature of Goodstein's function by providing a formula for $\mathcal{G}$ from which this is evident, which was established by Caicedo [2007].

Having established the general idea behind the thesis, we delve more into the specifics required to prove these theorems. The theory of ordinals is fundamental to our approach, so in Chapter 2 we first establish results in *ordinal arithmetic*, introducing the ordinal ε_0 and culminating in a representation of ordinals in *Cantor normal form*. After having established the theory of ordinals, we state and prove Goodstein's theorem using these infinite ordinals.

Further to this, in Section 3.1 we properly define what "growing too quickly" means in a mathematical sense, following Ketonen and Solovay [1981]. We do this by introducing the notation of *domination*, where a function f *dominates* a function g if after some point, f is always larger than g. Then, we establish a hierarchy of functions $(F_\alpha)_{\alpha<\varepsilon_0}$ where F_α dominates F_β whenever alpha is smaller than beta. This will be key to our analysis, as it turns out that "growing too quickly" in this case means not dominated by any F_α for $\alpha < \varepsilon_0$. In Section 3.2 we further give the formula for Goodstein's function $\mathcal{G}$ explicitly in terms of these fast growing functions, from which we can prove that $\mathcal{G}$ dominates F_α for every $\alpha < \varepsilon_0$.

Moreover, we further establish useful classes of functions and results pertaining to these functions in Section 3.3. Most notably, we define the computable functions, which informally are those functions that you can calculate using a pen and paper, as well as an enumerable subset of the computable functions which we call the elementary functions. From Grzegorczyk [1964], we state a result that allows us to represent a computable function using elementary functions, and prove a domination result for elementary-in-F_α functions.

Furthermore, we start Chapter 4 by laying the foundations for our proof system of Peano Arithmetic, first establishing first order logic in Section 4.1, and then establishing a proof system in Section 4.2. For our proof system, we make use of Tait-style sequent calculus due to its convenient technical properties, which we compare and contrast to the more popular Hilbert-style proof systems.

We define our system for Peano Arithmetic in Section 4.3.1, and in doing so embed each elementary function in the language. While this does not make our system any more powerful, alongside with defining provably computable functions in terms of elementary functions, this trick allows us to treat equations involving elementary functions as atomic, eventually leading to our results.

What remains are a series of technical results, which allow us to bound our provably computable functions. We first show in Section 4.3.2 that it is possible to embed Peano Arithmetic in an infinitary version of arithmetic, referred to as PA_∞. Unlike in Peano Arithmetic, we may have an infinite number of premises in PA_∞. This added dimension allows us to perform 'Cut-elimination', essentially simplifying our proofs, which we explain in Section 4.3.3. From this, we proceed in Section 4.3.4 to obtain bounding results for theorems of Peano Arithmetic, eventually culminating in the proof that every provably computable function in Peano Arithmetic is necessarily dominated by some F_α, for $\alpha < \varepsilon_0$. Using previous results, we can deduce that Goodstein's function $\mathcal{G}$ is not provably computable, hence Goodstein's theorem, which states that $\mathcal{G}$ is total, is not provable in Peano Arithmetic.

Our main aim with this text is to provide a clear, self-contained description of the steps required to show Goodstein's theorem is unprovable in Peano Arithmetic. This is intended to demystify the proof techniques involved, as well as include a full account of all theorems used. To do this, we have taken from multiple sources which make use of slightly different variations of fast growing hierarchies, and so we had to modify certain proofs and statements so that only a single hierarchy is used. Moreover, we provide a proof for Caicedo's claim that the formula derived for Goodstein's function $\mathcal{G}$ can be used to obtain that $\mathcal{G}$ dominates every fast growing function.

Throughout this text we will typically use Greek symbols $\alpha, \beta, \xi, \delta$ to denote ordinals and φ, ψ to denote formulas and letters n, m to denote natural numbers. We refer to the set of natural numbers as ω when referring to ordinals, and as $\mathbb{N}$ more generally.

2

Ordinal Numbers

In this Chapter we introduce the ordinal numbers, which will be an integral tool in our analysis of Goodstein's function and Peano Arithmetic in general. In Section 2.1 we define these numbers and establish some important results, which we utilise in Section 2.2 to prove Goodstein's theorem. The results and definitions of this Chapter are taken from Jech [2007].

2.1 | An Introduction to Ordinals

The concept of an *order* is fundamental to mathematics. From a young age we are introduced to the *natural numbers* $1, 2, 3, 4, \ldots$ and so on. In doing so, an implicit order $1 < 2 < 3 < 4 < \ldots$ is created. *Ordinal numbers* are a natural extension of the natural numbers, allowing us to 'count' beyond finite quantities. In doing so, we establish a way to characterise the order of all *well-ordered* sets uniquely by an ordinal.

Due to their ability to characterise well-ordered sets, these numbers have a wide application in mathematics. In particular, ordinal numbers are widely used throughout proof theory in ordinal analysis. The scope of this thesis is to explore these proof-theoretic techniques, and hence we mainly state the necessary results, giving proofs of important results and to illustrate how proofs with ordinal numbers work.

We begin by setting the foundations for ordinal arithmetic by defining the ordinal numbers, as well as the addition, multiplication and exponentiation operations. This culminates in the Cantor normal form representations of ordinals, which becomes the basis of many of our proofs.

2.1.1 | Defining Ordinal Numbers

In this section, we define the ordinal numbers and provide some foundational results. This allows us to provide an overview of the ordinals below ε_0 and define the order type of a well-ordered set, as well as establish the technique of transfinite induction which we will use throughout this thesis.

Definition 2.1.1. A relation $\leq$ on a set X is a *(partial)-order relation* on X if it satisfies for all $x, y \in X$

(i) (Reflexivity): $x \leq x$.

(ii) (Anti-Symmetry): $x \leq y$ and $y \leq x$ implies $x = y$.

(iii) (Transitivity): $x \leq y$ and $y \leq z$ implies $x \leq z$.

Moreover, an order relation is a *linear-order relation* if it is connected, that is for every $x, y \in X$, either $x \leq y$ or $y \leq x$. In this case, we say that X is *linearly ordered* by $\leq$.

Definition 2.1.2. A binary relation $\leq$ on a set X is said to be *well-founded* if every non-empty subset $Y \subseteq X$ has a $\leq$-minimal element, that is there exists some $x \in Y$ such that for every $y \in Y$, $y \leq x$ does not hold.

Definition 2.1.3. A relation $\leq$ on a set X is said to be *well-ordered* if it is a well-founded linear-order.

Definition 2.1.4. A set α is said to be an *ordinal number* if it is transitive and well-ordered by $\underline{\in}$.

This is equivalent to α satisfying the following properties:

(A_α) (Transitive) If $\beta \in \alpha$ then $\beta \subseteq \alpha$.

(B_α) (Linearly ordered) If $\beta, \gamma \in \alpha$, then either $\beta = \gamma$, $\beta \in \gamma$ or $\gamma \in \beta$.

(C_α) (Well founded) If $\varnothing \neq B \subseteq \alpha$, then there exists a $\gamma \in B$ such that $\gamma \cap B = \varnothing$.

When referring to ordinals α, β we will denote $\alpha \in \beta$ as $\alpha < \beta$, and $\alpha \in \beta$ or $\alpha = \beta$ as $\alpha \leq \beta$.

Lemma 2.1.5. *If α and β are ordinal numbers, then either $\alpha = \beta$, $\alpha < \beta$ or $\beta < \alpha$.*

Lemma 2.1.6. *If α is an ordinal, the set $\alpha \cup \{\alpha\}$ is also an ordinal. This ordinal is referred to as the (ordinal immediate) successor of α and is denoted by $\alpha + 1$. Likewise, α is referred to as the (ordinal immediate) predecessor of $\alpha + 1$.*

Lemma 2.1.7. *If $\mathcal{F}$ is a family of ordinals, then $\bigcup \mathcal{F}$ is an ordinal. This ordinal is the supremum of $\mathcal{F}$ with respect to the linear order of ordinals and is denoted by $\sup \mathcal{F}$.*

Definition 2.1.8. Ordinal numbers can be separated into two categories: successor ordinals and limit ordinals. An ordinal number is a successor ordinal if it has an immediate predecessor. Otherwise it is a limit ordinal.

Example 2.1.9.

(i) The empty set $\varnothing$ is an ordinal, as properties ($A_\varnothing$)–($C_\varnothing$) are vacuously true.

(ii) The sets $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ are ordinals. This can be seen by checking the properties as in the definition, or by Lemma 2.1.6.

(iii) The set $A = \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ is not an ordinal, as $\{\varnothing\} \in A$ but $\varnothing \notin A$, hence A is not transitive under $\underset{=}{\in}$.

Definition 2.1.10 (Natural Numbers)**.** The empty set $\varnothing$, being the least ordinal is denoted by 0. We continue defining the natural numbers using the successor operation, letting $1 = 0 + 1 = \{\varnothing\}$, $2 = 1 + 1 = \{\varnothing, \{\varnothing\}\}$ and so on.

The least nonzero limit ordinal is denoted by ω, and consists of the set of all natural numbers.

Lemma 2.1.11. *There does not exist a strictly decreasing infinite sequence of ordinals $(\alpha_n)_{n \in \omega}$. In other words, any strictly decreasing sequence of ordinals must be finite.*

Proof. This property follows directly from the well-foundedness of ordinals. Let there be a decreasing sequence $(\alpha_n)_{n\in\omega}$ and let $B = \{\alpha_n : n \in \omega\}$. Clearly, $B \subseteq \alpha_0 + 1$, hence by (C_{α_0+1}) there exists some $n \in \omega$ such that $\alpha_n \cap B = \varnothing$. However $\alpha_{n+1} < \alpha_n$, which means by definition that $\alpha_{n+1} \in \alpha_n$, hence $\alpha_{n+1} \in \alpha_n \cap B$, contradicting (C_{α_0+1}). □

Theorem 2.1.12 (Transfinite Induction)**.** *Let C be a class of ordinals and assume that:*

(i) $0 \in C$.

(ii) *If $\alpha \in C$, then $\alpha + 1 \in C$.*

(iii) *If α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.*

Then C is the class of all ordinals.

This means that we can use induction to prove a property φ about ordinals holds, by showing that the class C of all ordinals such that φ holds obeys the above properties. Hence to prove that φ holds for all ordinals it suffices to show that:

(i) $\varphi(0)$ holds.

(ii) If $\varphi(\alpha)$ holds, then $\varphi(\alpha + 1)$ holds.

(iii) If α is a nonzero limit ordinal and $\varphi(\beta)$ holds for all $\beta < \alpha$, then $\varphi(\alpha)$ holds.

This framework allows us to construct a picture of the ordinals, making use of the successor and union operations defined above. To represent these ordinals, we make use of the addition, multiplication and exponentiation operators which are defined formally later in Definitions 2.1.17, 2.1.18 and 2.1.19.

We construct the natural numbers using the successor function as explained previously. The supremum of all the natural numbers gives us the ordinal ω.

$$\varnothing = 0,\ \{\varnothing\} = 1,\ \{\varnothing, \{\varnothing\}\} = 2,\ \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = 3,\ \dots\ \omega$$

We continue by applying the successor function to ω, obtaining all ordinal numbers in the form $\omega + n$, where $n < \omega$. The supremum of all such ordinals is $\omega + \omega$, which is equal to $\omega \cdot 2$.

$$\omega + 1,\ \omega + 2,\ \omega + 3,\ \omega + 4,\ \dots\ \omega + \omega = \omega \cdot 2$$

We can repeat this process to obtain all ordinals in the form $\omega \cdot n$ for any natural number n. Taking the supremum of all such ordinals, we obtain the ordinal $\omega \cdot \omega$ which is equal to ω^2.

$$\omega \cdot 2,\ \omega \cdot 3,\ \omega \cdot 4,\ \dots\ \omega \cdot \omega = \omega^2$$

This process can be repeated with ω^2, taking the successor and the supremum of all successors to obtain the ordinal ω^3.

$$\begin{gathered}
\omega^2,\ \omega^2 + 1,\ \omega^2 + 2,\ \omega^2 + 3,\ \dots\ \omega^2 + \omega \\
\omega^2 + \omega + 1,\ \omega^2 + \omega + 2,\ \omega^2 + \omega + 3,\ \dots\ \omega^2 + \omega + \omega = \omega^2 + \omega \cdot 2 \\
\omega^2 + \omega \cdot 2,\ \omega^2 + \omega \cdot 3,\ \omega^2 + \omega \cdot 4,\ \dots\ \omega^2 + \omega \cdot \omega = \omega^2 \cdot 2 \\
\omega^2 \cdot 2,\ \omega^2 \cdot 3,\ \omega^2 \cdot 4,\ \dots\ \omega^3
\end{gathered}$$

The same process can be used to obtain all ordinals in the form ω^n for any natural number n, the union of which gives us ω^ω.

$$\omega,\ \omega^2,\ \omega^3,\ \omega^4,\ \dots\ \omega^\omega$$

This can be extended even further to obtain ordinals ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$ and so on. This gives us all the ordinals in the form $\omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ with $n < \omega$ terms. The supremum of these ordinals is $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ with ω terms, and is denoted by ε_0.

$$\omega,\ \omega^\omega,\ \omega^{\omega^\omega},\ \omega^{\omega^{\omega^\omega}},\ \dots\ \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = \varepsilon_0$$

The ordinal ε_0 plays a special role in our analysis, being the first ordinal to satisfy the equation $x = \omega^x$. This important ordinal is commonly used in induction proofs as often induction up to ε_0 is enough. For example in the proof of the consistency of Peano Arithmetic given by Gentzen [1964], assuming induction up to ε_0 is enough. Moreover, we will see later that we only use ordinals less than ε_0 in our proof of Goodstein's Theorem.

A foundational fact about the ordinal numbers is that they can be uniquely used to represent the order of a well-ordered set. This fact will be useful to us in proving results about ordinal arithmetic, so we state it here.

Definition 2.1.13. Linearly ordered sets $(X, \leq)$ and $(Y, \leq)$ are *order isomorphic* if there exists a bijection $f : X \to Y$ such that such that for all $x_1, x_2 \in X$, $x_1 \leq x_2$ holds if and only if $f(x_1) \leq f(x_2)$.

Lemma 2.1.14. *Order isomorphism behaves as an equivalence relation on linearly ordered sets, that is*

(a) $(X, \leq)$ *is order isomorphic to* $(X, \leq)$.

(b) If $(X, \leq)$ *is order isomorphic to* $(Y, \leq)$, *then* $(Y, \leq)$ *is order isomorphic to* $(X, \leq)$.

(c) If $(X, \leq)$ *is order isomorphic to* $(Y, \leq)$ *and* $(Y, \leq)$ *is order isomorphic to* $(Z, \sqsubseteq)$, *then* $(X, \leq)$ *is order isomorphic to* $(Z, \sqsubseteq)$.

Theorem 2.1.15. *For every well-ordered set* $(X, \leq)$, *there exists a unique ordinal* α *such that* $(X, \leq)$ *is order isomorphic to* $(\alpha, \leq)$.

Definition 2.1.16. We define the *order type* $\mathrm{Otp}(X)$ of a well-ordered set $(X, \leq)$ as $\mathrm{Otp}(X) = \alpha$ where α is the unique ordinal such that X is order isomorphic to α. Note that by Theorem 2.1.15, we know that $\mathrm{Otp}(X)$ is well-defined.

2.1.2 | Ordinal Arithmetic

Throughout this section we state some basic results of operations on ordinals, namely addition, multiplication and exponentiation. Due to their similarities to the natural numbers, ordinal numbers behave similarly under arithmetic operations but have some subtle differences, such as commutativity not holding for addition and multiplication. We highlight these properties, offering examples when certain properties for the natural numbers do not hold for ordinals.

Definition 2.1.17 (Ordinal Addition)**.** For all ordinals α, β,

(i) $\beta + 0 = \beta$.

(ii) $\beta + (\alpha + 1) = (\beta + \alpha) + 1$.

(iii) For limit $\alpha \neq 0$, $\beta + \alpha = \sup\{\beta + \gamma : \gamma < \alpha\}$.

Definition 2.1.18 (Ordinal Multiplication)**.** For all ordinals α, β,

(i) $\beta \cdot 0 = 0$.

(ii) $\beta \cdot (\alpha + 1) = (\beta \cdot \alpha) + \beta$.

(iii) For limit $\alpha \neq 0$, $\beta \cdot \alpha = \sup\{\beta \cdot \gamma : \gamma < \alpha\}$.

Definition 2.1.19 (Ordinal Exponentiation)**.** For all ordinals α, β,

(i) $\beta^0 = 1$.

(ii) $\beta^{\alpha+1} = \beta^\alpha \cdot \beta$.

(iii) For limit $\alpha \neq 0$, $\beta^\alpha = \sup\{\beta^\gamma : \gamma < \alpha\}$.

Definition 2.1.20 (Lexicographic Ordering)**.** Let $(F_t)_{t\in T}$ be an indexed family of nonempty linearly ordered sets, where each F_t is linearly ordered by $\leq_t$, and let T be linearly ordered by $\leq$. The *lexicographical order* $\leq_{lex}$ on the Cartesian product $\prod_{t\in T} F_t$ is defined by the formula

$$f \leq_{lex} g \iff f = g \vee \exists s \in T\, (s = min\{t \in T : f(s) \neq g(s)\} \wedge f(s) < g(s))\,.$$

Definition 2.1.21. If $(A, <_A)$ and $(B, <_B)$ are two disjoint linearly ordered sets, then the *sum* of these sets $(A \cup B, <)$ is defined such that

$$\begin{aligned} c < d \iff\ & c, d \in A \wedge c <_A d \\ & \vee\ c, d \in B \wedge c <_B d \\ & \vee\ c \in A \wedge d \in B. \end{aligned}$$

Lemma 2.1.22. Otp$(A \cup B)$ *under the linear ordering in Definition 2.1.21 is equivalent to* Otp$((A \times \{0\}) \cup (B \times \{1\}))$ *under lexicographic ordering.*

Theorem 2.1.23 (Alternative Forms)**.** *If A and B are linearly ordered sets such that* $\mathrm{Otp}(A) = \alpha$*, and* $\mathrm{Otp}(B) = \beta$*, then under lexicographic ordering,*

(i) $\mathrm{Otp}((A \times \{0\}) \cup (B \times \{1\})) = \alpha + \beta$.

(ii) $\mathrm{Otp}(B \times A) = \alpha \cdot \beta$.

Lemma 2.1.24. *The ordinal functions $\alpha + \beta$, $\alpha \cdot \beta$ and α^β are continuous in the second variable,* i.e. *if γ is a limit ordinal and $\beta = \sup_{\nu<\gamma}\{\beta_\nu\}$, then*

(i) $\alpha + \beta = \sup_{\nu<\gamma}\{\alpha + \beta_\nu\}$.

(ii) $\alpha \cdot \beta = \sup_{\nu<\gamma}\{\alpha \cdot \beta_\nu\}$.

(iii) $\alpha^\beta = \sup_{\nu<\gamma}\{\alpha^{\beta_\nu}\}$.

Lemma 2.1.25 (Left Monotonicity)**.** *Let α, β, γ be ordinals. Then the following results hold:*

(i) $\beta < \gamma \iff \alpha + \beta < \alpha + \gamma$.

(ii) For $\alpha \neq 0$, $\beta < \gamma \iff \alpha \cdot \beta < \alpha \cdot \gamma$.

(iii) For $\alpha > 1$, $\beta < \gamma \iff \alpha^\beta < \alpha^\gamma$.

Proof. The proofs of the above results are routine and follow a similar pattern, so we only give a proof for case *(iii)* by assuming case *(ii)*.

We begin by proving the direction $\beta < \gamma \implies \alpha^\beta < \alpha^\gamma$. The case when γ is a limit ordinal can be proved directly, as by definition, $\alpha^\gamma = \bigcup\{\alpha^\xi : \xi < \gamma\}$, hence $\beta < \gamma$ implies that $\alpha^\beta \in \alpha^\gamma$, that is $\alpha^\beta < \alpha^\gamma$.

For the case when $\gamma = \delta + 1$, we proceed by induction on γ. If $\beta = \delta$, then using the left monotonicity of ordinal multiplication, $\alpha^\beta = \alpha^\delta < \alpha^\delta \cdot \alpha = \alpha^{\delta+1}$. Otherwise if $\beta < \delta$, then we have by the inductive hypothesis that $\alpha^\beta < \alpha^\delta$, and by the first case $\alpha^\delta < \alpha^{\delta+1}$, hence $\alpha^\beta < \alpha^{\delta+1}$.

To prove the other direction, let $\alpha^\beta < \alpha^\gamma$. Then assume for contradiction that $\beta \geq \gamma$. Either $\beta = \gamma$, in which case $\alpha^\beta = \alpha^\gamma$ or $\beta > \gamma$, which by the first direction implies that $\alpha^\beta > \alpha^\gamma$. Hence we conclude that $\beta < \gamma$. □

Corollary 2.1.26 (Left Cancellation)**.** *If α, ξ_1, ξ_2 are ordinals, then*

(i) $\alpha + \xi_1 \leq \alpha + \xi_2$ *implies that* $\xi_1 \leq \xi_2$.

(ii) $\alpha \neq 0$ *and* $\alpha \cdot \xi_1 \leq \alpha \cdot \xi_2$ *implies that* $\xi_1 \leq \xi_2$.

(iii) $\alpha > 1$ *and* $\alpha^{\xi_1} \leq \alpha^{\xi_2}$ *implies that* $\xi_1 \leq \xi_2$.

This can be easily extended to show equality of ξ_1 and ξ_2 in the case where the expressions are equal.

Proof. The proofs for the statements above are a direct consequence of Lemma 2.1.25. We only prove case *(i)* as the rest follow similarly. To prove *(i)* assume $\xi_1 > \xi_2$. By left monotonicity of addition, $\alpha + \xi_1 > \alpha + \xi_2$, contradicting the premise that $\alpha + \xi_1 \leq \alpha + \xi_2$. Hence $\xi_1 \leq \xi_2$. □

Example 2.1.27. Note that right monotonicity does not always hold, as demonstrated in the examples below.

(i) $1 < 2$, however $1 + \omega = \omega = 2 + \omega$, hence in this case right monotonicity with respect to addition does not hold.

(ii) $1 < 2$, however $1 \cdot \omega = \omega = 2 \cdot \omega$, hence in this case right monotonicity with respect to multiplication does not hold.

(iii) $2 < 3$, however $2^\omega = \omega = 3^\omega$, hence in this case right monotonicity with respect to exponentiation does not hold.

However, we can still state a weaker result for right monotonicity, making use of $\leq$.

Lemma 2.1.28 (Weak Right Monotonicity)**.** *Let α, β, γ be ordinals such that $\beta \leq \gamma$. Then the following results hold:*

(i) $\beta + \alpha \leq \gamma + \alpha$.

(ii) $\beta \cdot \alpha \leq \gamma \cdot \alpha$.

(iii) $\beta^\alpha \leq \gamma^\alpha$.

Lemma 2.1.29. *For any ordinals α, β:*

(i) $\beta \leq \alpha + \beta$.

(ii) *If* $\alpha > 0$, $\beta \leq \alpha \cdot \beta$.

Lemma 2.1.30. *If $\alpha \leq \gamma$, there exists a unique ordinal β such that $\alpha + \beta = \gamma$.*

Proof. Let $\alpha \leq \gamma$. Consider the set $S = \{\xi : \alpha < \xi \leq \gamma\}$ with the same order $\leq$, and let $\beta = \mathrm{Otp}(S)$. Then α and S are two disjoint linearly ordered sets, such that $\alpha \cup S = \gamma$. Clearly, $\mathrm{Otp}(\gamma) = \gamma$, hence by Theorem 2.1.23 and Lemma 2.1.22, we have that $\gamma = \mathrm{Otp}(\alpha \cup S) = \mathrm{Otp}(\alpha) + \mathrm{Otp}(S) = \alpha + \beta$.

The uniqueness of β can be deduced from Corollary 2.1.26, as if $\alpha + \beta_1 = \alpha + \beta_2$, then $\beta_1 = \beta_2$. □

Lemma 2.1.31. *If $0 < \alpha \leq \gamma$, there exists a greatest ordinal β such that $\alpha \cdot \beta \leq \gamma$.*

Proof. The set $\{\delta : \alpha \cdot \delta > \gamma\}$ is non-empty as by Lemma 2.1.29, $\alpha \cdot (\gamma + 1) \geq \gamma + 1 > \gamma$, and hence has a least element δ.

If δ is a limit ordinal, then by Lemma 2.1.24, $\alpha \cdot \delta = \alpha \cdot \lim_{\xi < \delta}(\xi) = \lim_{\xi < \delta}(\alpha \cdot \xi)$, hence $\lim_{\xi < \delta}(\alpha \cdot \xi) > \gamma$ which implies that there exists a $\xi < \delta$ such that $\alpha \cdot \xi > \gamma$, which contradicts our choice of δ.

Hence $\delta = \beta + 1$ for some ordinal β. Then β must be the greatest ordinal such that $\alpha \cdot \beta \leq \gamma$. □

Lemma 2.1.32. *If $1 < \alpha \leq \gamma$, there exists a greatest ordinal β such that $\alpha^\beta \leq \gamma$.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.1.31. □

Lemma 2.1.33 (Euclidean Division for Ordinals)**.** *If γ is an ordinal number and $\alpha \neq 0$, then there exists a unique β and a unique $\rho < \alpha$ such that $\alpha \cdot \beta + \rho = \gamma$.*

Moreover, if $\alpha \leq \gamma$, then $\beta \neq 0$ and $\rho < \gamma$.

Proof. If $\alpha > \gamma$, existence holds by setting $\beta = 0$ and $\rho = \gamma$. Otherwise, using Lemmas 2.1.31 and 2.1.30, let β be the largest ordinal such that $\alpha \cdot \beta \leq \gamma$, and let ρ be the unique ρ such that $\alpha \cdot \beta + \rho = \gamma$. It is clear that $\beta \neq 0$, as $\alpha \cdot 1 \leq \gamma$, meaning $\beta \geq 0$, and hence $\rho < \gamma$. Moreover, $\rho < \alpha$ as otherwise $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha \leq \gamma$, contradicting our choice of β.

To show uniqueness, let $\gamma = \alpha \cdot \beta_1 + \rho_1 = \alpha \cdot \beta_2 + \rho_2$, such that $\rho_1, \rho_2 < \alpha$. If $\beta_1 = \beta_2$, then by Lemma 2.1.30, $\rho_1 = \rho_2$, so we may assume that $\beta_1 < \beta_2$. Then, using the left monotonicty of multiplication we have that $\alpha \cdot \beta_1 + (\alpha + \rho_2) = \alpha \cdot (\beta_1 + 1) + \rho_2 \leq \alpha \cdot \beta_2 + \rho_2 = \alpha \cdot \beta_1 + \rho_1$. Then $\alpha + \rho_2 \leq \rho_1$, contradicting the fact that $\rho_1 < \alpha$. $\square$

Lemma 2.1.34 (Left Distributivity). *Let α, β, γ be ordinals. Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.*

2.1.3 | The Cantor Normal Form

These results allow us to represent all the ordinals in Cantor normal form. Similar to how each natural number can be represented in base 2, or base n in general, here we are representing an ordinal in base ω. This form allows us to work easily with ordinals under ε_0, as in this representation the exponents are necessarily smaller than the original ordinal. In this section, we put the previous results on ordinals together to prove the Cantor normal form theorem.

Lemma 2.1.35. *If $\alpha_0 > \alpha_1 > \cdots > \alpha_l$ and $k_1, k_2, \ldots, k_l \in \omega$, then $\omega^{\alpha_0} > \sum_{i=1}^{l} \omega^{\alpha_i} \cdot k_i$.*

Proof. By left monotonicity, we know that $\omega^{\alpha_n} < \omega^{\alpha_1}$ for any $n > 1$. Hence by weak right monotonicity, $\sum_{i=1}^{l} \omega^{\alpha_i} \cdot k_i \leq \sum_{i=1}^{l} \omega^{\alpha_1} \cdot k_i$, and by left distributivity this is equal to $\omega^{\alpha_1} \cdot \sum_{i=1}^{l} k_i$. Since the finite sum of finite ordinals is also finite, by the left monotonicity of exponentiation we have that $\omega^{\alpha_1} \cdot \sum_{i=1}^{l} k_i < \omega^{\alpha_1} \cdot \omega = \omega^{\alpha_1 + 1}$. Then since $\alpha_1 < \alpha_0$, $\alpha_1 + 1 \leq \alpha_0$, hence by left monotonicity $\omega^{\alpha_1+1} \leq \omega^{\alpha_0}$, so we can conclude that $\sum_{i=1}^{l} \omega^{\alpha_i} \cdot k_i < \omega^{\alpha_0}$.

$\square$

Theorem 2.1.36 (Cantor Normal Form). *Every ordinal $\alpha > 0$ can be expressed uniquely as the sum*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$$

where $\beta_1 > \beta_2 > \cdots > \beta_l$ and $k_1, k_2, \ldots, k_l$ are finite non-zero ordinals. We refer to this sum as the Cantor normal form *of α.*

Proof. We use induction to prove existence of the Cantor normal form. If $\alpha = 1$, then trivially $\alpha = \omega^0 \cdot 1$. Let $\alpha > 1$ be arbitrary, and assume that every $\xi < \alpha$ can be written in Cantor normal form.

By Lemma 2.1.32, there exists a greatest β such that $\omega^\beta \leq \alpha$ (if $\alpha < \omega$ then this holds for $\beta = 0$), and by Lemma 2.1.33 there exist a unique γ and ρ such that $\rho < \omega^\beta$ and $\alpha = \omega^\beta \cdot \gamma + \rho$. Moreover since $\omega^\beta \leq \alpha$, $\gamma \neq 0$ and $\rho < \alpha$. Note that γ is finite as if $\gamma \geq \omega$, $\alpha = \omega^\beta \cdot \gamma + \rho \geq \omega^\beta \cdot \gamma \geq \omega^\beta \cdot \omega = \omega^{\beta+1}$, contradicting the maximality of β. Hence we can let $\beta = \beta_1$ and $\gamma = k_1$.

If $\rho = 0$, then $\alpha = \omega^{\beta_1} \cdot k_1$ is in Cantor normal form. Otherwise, since $\rho < \alpha$, by the inductive hypothesis $\rho = \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$, with $\beta_2 > \cdots > \beta_l$ and $k_2, \ldots k_l$ finite and non-zero. Also, $\beta_1 > \beta_2$ as $\omega^{\beta_2} \leq \rho < \omega^{\beta_1}$. Hence we can write α as the sum $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$ where $\beta_1 > \beta_2 > \cdots > \beta_l$ and $k_1, k_2, \ldots k_l$ are finite non-zero ordinals, as required.

We now prove uniqueness by induction on α. The expansion $1 = \omega^0 \cdot 1$ is clearly unique, as changing β_1 will introduce a quantity greater than 1, and changing k_1 will lead to a different natural number.

Let $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l = \omega^{\gamma_1} \cdot t_1 + \omega^{\gamma_2} \cdot t_2 + \cdots + \omega^{\gamma_m} \cdot t_m$ be expansions of α in Cantor normal form. Assume that $\beta_1 < \gamma_1$. Then, we have that $\gamma_1 > \beta_1 > \beta_2 > \cdots > \beta_l$, so we can apply Lemma 2.1.35 to conclude that $\omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l < \omega^{\gamma_1}$ which is less than or equal to $\omega^{\gamma_1} \cdot t_1 + \cdots + \omega^{\gamma_m} \cdot t_m = \alpha$, a contradiction. Hence we conclude that $\beta_1 = \gamma_1$.

So we can write α as $\alpha = \gamma \cdot k_1 + \rho = \gamma \cdot t_1 + \sigma$, where $\gamma = \omega^{\beta_1} = \omega^{\gamma_1}$ and $\rho = \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_1} \cdot k_l$, $\sigma = \omega^{\gamma_2} \cdot t_2 + \cdots + \omega^{\gamma_m} \cdot t_m$. By Lemma 2.1.35, ρ and σ are less than γ, so by Lemma 2.1.33, $k_1 = t_1$ and $\rho = \sigma$. Hence by the inductive hypothesis, the Cantor normal form expansions of ρ and σ are unique, making the Cantor normal form expansion of α unique. □

2.2 | Goodstein's Theorem

Goodstein's Theorem is a finitary statement regarding a sequence of natural numbers, recursively defined by a simple operation of changing base and subtracting by 1. In this section, we rigorously define this function, by first defining the change of base operation. This allows us to state Goodstein's theorem: that every Goodstein sequence eventually reaches zero after a finite number of steps.

Given the previously established foundations for ordinals, we proceed to provide a proof for this finitary statement using our infinite ordinals. While this tool may seem overly powerful for our finite statement, in later sections we see why an infinite mechanism is necessarily required to prove this statement.

2.2.1 | Defining a Goodstein Sequence

To state Goodstein's theorem, we must first define the sequence in question. We do this by defining hereditary base notation, which allows us to define our sequence in an inductive way.

Definition 2.2.1. Given $b \geq 2$, a natural number n is said to be written in *base-b notation* if it is written as a sum of powers of b,

$$n = b^{m_1} \cdot k_1 + b^{m_2} \cdot k_2 + \dots b^{m_l} \cdot k_l$$

with $n \geq 0$, $m_1 > m_2 > \cdots > m_l$ and $0 < k_i < b$ for all $i = 1, 2, \dots, l$.

Note that the base-b representation of 0 is always 0, and corresponds to the case where $l = 0$.

Definition 2.2.2 (Hereditary Base Notation)**.** Given $b \geq 2$, a natural number n is said to be written in *hereditary base-b* if it is first written in base b, then all its exponents are written in base-b, then the exponents of the exponents are written in base-b and so on.

Lemma 2.2.3. *For every $b \geq 2$, every natural number n can be written uniquely in base-b notation. Moreover, this can be extended to show that every natural number can be written uniquely in hereditary base-b.*

Example 2.2.4. In ordinary base-b notation:

- $10 = 3^2 + 3^0$, for $b = 3$.
- $82 = 2^6 + 2^4 + 2^1$, for $b = 2$.

- $20000 = 5^6 + 5^5 + 5^4 \cdot 2$, for $b = 5$.

While 10 is already in hereditary base-3 notation, the rest would need to have their exponents rewritten in base-b notation. Hence in hereditary base-b notation,

- $10 = 3^2 + 3^0$, for $b = 3$.
- $82 = 2^{2^{2^1}+2^1} + 2^{2^{2^1}} + 2^1$, for $b = 2$.
- $20000 = 5^{5^1+1} + 5^{5^1} + 5^4 \cdot 2$, for $b = 5$.

Note how in the case of 82, we have to first write the exponents in hereditary base-2 notation resulting in the expression $2^{2^2+2^1} + 2^{2^2} + 2^1$, and further rewrite the exponents of the exponents by writing 2 as 2^1. This process might need to be repeated further in certain cases: for example writing 65536 in hereditary base-2 notation requires four steps, as $65536 = 2^{16} = 2^{2^4} = 2^{2^{2^2}} = 2^{2^{2^{2^1}}}$.

Definition 2.2.5. The Goodstein sequence starting at $n \geq 0$ is a sequence of natural numbers $g_1(n),\ g_2(n),\ g_3(n)\ldots$ defined as follows:

(Base Case) $g_1(n) = n$.

(Inductive Case) For $i > 1$ and $g_{i-1}(n) > 0$, obtain $g_i(n)$ by the following algorithm:

(a) Write $g_{i-1}(n)$ in hereditary base-(i);

(b) Increment the base by changing each occurrence of i with $i + 1$;

(c) Subtract one from the above number to obtain $g_i(n)$.

(Termination) If $g_k(n) = 0$ for some k, then the sequence terminates at k and $g_i(n) = 0$ for all $i > k$.

Definition 2.2.6. Let n, b be natural numbers with $b \geq 2$, and let β be an ordinal. We define the term $(n)_{b \to \beta}$ to be the number obtained by writing n in hereditary base-b notation and substituting each b with β.

This notation allows us to simplify the inductive case of the Goodstein sequence, as we can write $g_i(n)$ as $(g_{i-1}(n))_{i\to i+1} - 1$.

Note how we allow β to be an infinite ordinal, which means that $(n)_{b\to\beta}$ can also be an infinite ordinal. For example, letting $\beta = \omega$, $(6)_{2\to\omega} = (2^2 + 2^1)_{2\to\omega} = \omega^\omega + \omega$, which is an infinite ordinal.

Example 2.2.7. We first give a small example of a Goodstein sequence, starting at $n = 3$.

$$
\begin{aligned}
g_1(3) &= 3 \\
g_2(3) &= (3)_{2\to3} - 1 = (2^1 + 2^0)_{2\to3} - 1 = 3^1 + 3^0 - 1 = 3 \\
g_3(3) &= (3)_{3\to4} - 1 = (3^1)_{3\to4} - 1 = 4 - 1 = 3 \\
g_4(3) &= (3)_{4\to5} - 1 = (4^0 \cdot 3)_{4\to5} - 1 = 5^0 \cdot 3 - 1 = 2 \\
g_5(3) &= (2)_{5\to6} - 1 = (5^0 \cdot 2)_{5\to6} - 1 = 6^0 \cdot 2 - 1 = 1 \\
g_6(3) &= (1)_{6\to7} - 1 = (6^0)_{6\to7} - 1 = 7^0 - 1 = 0
\end{aligned}
$$

By definition, $g_k(3) = 0$ for every $k \geq 6$. In this case, the Goodstein sequence starting from 3 terminates quickly – however this cannot be said for larger integers. For $n \geq 4$ the terms grow rapidly, due to the nature of the change of base function. In fact, at $n = 4$ it takes more than 10^{10^8} steps for this sequence to terminate. To illustrate this, we further give some terms of the Goodstein sequence starting at $n = 8$, noting how the value of these terms grows faster at each iteration.

$$
\begin{aligned}
g_1(8) &= 8 \\
g_2(8) &= (8)_{2\to3} - 1 = (2^{2^1+1})_{2\to3} - 1 = 3^{3^1+1} - 1 = 80 \\
g_3(8) &= (80)_{3\to4} - 1 = (3^3 \cdot 2 + 3^2 \cdot 2 + 3^1 \cdot 2 + 2)_{3\to4} - 1 = 553 \\
g_4(8) &= (553)_{4\to5} - 1 = (4^4 \cdot 2 + 4^2 \cdot 2 + 4^1 \cdot 2 + 1)_{4\to5} - 1 = 6310 \\
g_5(8) &= (6310)_{5\to6} - 1 = (5^5 \cdot 2 + 5^2 \cdot 2 + 5^1 \cdot 2)_{5\to6} - 1 = 93395 \\
g_6(8) &= (93395)_{6\to7} - 1 = (6^6 \cdot 2 + 6^2 \cdot 2 + 6^1 + 5)_{6\to7} - 1 = 1647196
\end{aligned}
$$

2.2.2 | Proving Goodstein's Theorem using Ordinals

Theorem 2.2.8 (Goodstein's Theorem)**.** *Every Goodstein sequence terminates. In other words, for every $n \geq 0$ there exists a natural number k such that $g_k(n) = 0$.*

To prove Goodstein's theorem we need the following lemma, which allows us to assign an ordinal number to each term $g_i(n)$ in every Goodstein sequence.

Lemma 2.2.9. *For any natural numbers s, b such that $s > 0$ and $b \geq 2$,*

$$(s-1)_{b\to\omega} < (s)_{b\to\omega}.$$

Proof. We prove this theorem by induction on s. For $s = 1$, $(0)_{b\to\omega} = 0$ and $(1)_{b\to\omega} = 1$, so the inequality trivially holds. Let $s = b^{m_1} \cdot k_1 + b^{m_2} \cdot k_2 + \cdots + b^{m_l} \cdot k_l$ be in base-b notation. Define β_i to be $(m_i)_{b\to\omega}$ for $1 \leq i \leq l$. Then the ordinal $(s)_{b\to\omega}$ can be written as

$$(s)_{b\to\omega} = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l.$$

We further split this analysis into two cases, considering separately the case when $(s)_{b\to\omega}$ is a successor or a limit ordinal. In the successor case, we can prove our statement directly without using the inductive hypothesis. If $(s)_{b\to\omega}$ is a successor ordinal, then $\beta_l = 0$, since the least term of the summation has to be a natural number. Then, $(s-1)_{b\to\omega} = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_{l-1}} \cdot k_{l-1} + (k_l - 1)$ is the predecessor of $(s)_{b\to\omega}$, hence $(s-1)_{b\to\omega} < (s)_{b\to\omega}$ as desired.

Consider now the case when $(s)_{b\to\omega}$ is a limit ordinal. In this case, $\beta_l > 0$. Using the fact that

$$b^{m_l} - 1 = b^{m_l - 1} \cdot (b-1) + b^{m_l - 2} \cdot (b-1) + \cdots + b - 1$$

we get that $(s-1)_{b\to\omega}$ is equal to the expression

$$\omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot (k_l - 1) + \omega^{(m_l-1)_{b\to\omega}} \cdot (b-1) + \omega^{(m_l-2)_{b\to\omega}} \cdot (b-1) + \cdots + b - 1.$$

Hence to prove our inequality, by left monotonicity it suffices to show that $\omega^{(m_l)_{b\to\omega}} > \omega^{(m_l-1)_{b\to\omega}} \cdot (b-1) + \omega^{(m_l-2)_{b\to\omega}} \cdot (b-1) + \cdots + b - 1$.

By construction $s \geq 2^{m_l}$, hence $s > m_l$. This means that by the induction hypothesis $(m_l)_{b\to\omega} > (m_l - 1)_{b\to\omega} > (m_l - 2)_{b\to\omega} > \cdots > (1)_{b\to\omega} > (0)_{b\to\omega}$. Hence we can apply Lemma 2.1.35 to conclude that

$$\omega^{(m_l)_{b\to\omega}} > \omega^{(m_l-1)_{b\to\omega}} \cdot (b-1) + \omega^{(m_l-2)_{b\to\omega}} \cdot (b-1) + \cdots + b - 1$$

as required.

□

Equipped with this lemma, we can now prove Goodstein's theorem by making use of the well-founded property of ordinals.

Proof. Let $g_1(n),\ g_2(n), \ldots$ be the Goodstein sequence starting from $n > 0$ and assume that $g_k(n) \neq 0$ for every k. Define a sequence of ordinals $\alpha_1,\ \alpha_2, \ldots$ by letting $\alpha_i = (g_i(n))_{i+1\to\omega}$. Using Lemma 2.2.9 we show that

$$(g_i(n))_{i+1\to\omega} = ((g_{i-1}(n))_{i\to i+1} - 1)_{i+1\to\omega} < ((g_{i-1}(n))_{i\to i+1})_{i+1\to\omega} = (g_{i-1}(n))_{i\to\omega}$$

Hence $\alpha_i < \alpha_{i-1}$, meaning α_i is an infinite sequence of decreasing ordinals. As shown in Lemma 2.1.11, by the well founded property of ordinals there does not exist an infinite sequence of decreasing ordinals, contradicting our assumption. This implies that there exists some k such that $g_k(n) = 0$ hence our Goodstein sequence must terminate.

□

Remark 2.2.10. We note that in this proof we only use ordinals less than ε_0, as for any natural numbers m, b where $b \geq 2$ we have that $(m)_{b\to\omega} < \varepsilon_0$.

Definition 2.2.11. We define the *Goodstein function* $\mathcal{G}(n)$ to be the function $\mathcal{G} : \omega \to \omega$ which maps n to k where k is the number at which the Goodstein sequence starting at n terminates. In other words, k is the least number such that $g_k(n) = 0$.

We know that $\mathcal{G}$ is indeed a function, as by Goodstein's theorem for all $n \in \omega$, $n > 0$, there exists some $k \in \omega$ such that $g_k(n) = 0$. Moreover, $g_1(0) = 0$ hence $\mathcal{G}(0) = 1$, so $\mathcal{G}(n)$ is defined for all $n \in \omega$.

Example 2.2.12. By Example 2.2.7, we know that $g_6(3)$ is the first value at which the Goodstein sequence starting at 3 becomes 0, hence in this case $\mathcal{G}(3) = 6$.

3

Fast Growing Hierarchy

This Chapter pertains results involving the fast growing hierarchy $(F_\alpha)_{\alpha\in\varepsilon_0}$. We introduce this hierarchy and important results in Section 3.1, and use this hierarchy to derive a formula for Goodstein's function in Section 3.2. We further introduce important classes of functions in Section 3.3, culminating in a domination result of the elementary-in-F_α functions.

3.1 | Domination Results for a Fast Growing Hierarchy

In this section we introduce the fast growing hierarchy $(F_\alpha)_{\alpha\in\varepsilon_0}$, which is a trivial variant of the Löb-Wainer fast growing hierarchy, introduced by Ketonen and Solovay [1981]. We are mainly concerned with *domination results* for this hierarchy. Informally, a function *dominates* another if after some point, its value is always greater than the other function's. We will show that for any $\alpha > \beta$, F_α eventually dominates F_β, following the definitions and proofs of Ketonen and Solovay [1981].

Definition 3.1.1. A function $f : \mathbb{N}^m \to \mathbb{N}$ is said to be *dominated* by $g : \mathbb{N} \to \mathbb{N}$ if there exists some $N \in \mathbb{N}$ such that whenever $\max(x_1, \ldots, x_m) > N$,

$$f(x_1, \ldots, x_m) < g(\max(x_1, \ldots, x_m))$$

Whenever this happens, we say that g *dominates* f.

If $f : \mathbb{N} \to \mathbb{N}$, that is $m = 1$, we can simplify our expression further. In this case, f is dominated by g if there exists some $N \in \mathbb{N}$ such that for all $x > N$,

$$f(x) < g(x)$$

We proceed in defining a fundamental sequence for every limit ordinal $\alpha \leq \epsilon_0$, $(d(\alpha, n))_{n \in \omega}$, where $\lim_{n \to \omega} d(\alpha, n) = \alpha$. This sequence will be used to define our F_α's for limit α. To define these sequences, we first obtain some lemmas which will allow us to write α in a different form.

Lemma 3.1.2. *Any ordinal α satisfying $0 < \alpha < \varepsilon_0$ can be written in a unique way as $\alpha = \omega^\beta(\gamma + 1)$ where $\beta < \alpha$.*

Proof. Using Theorem 2.1.36, α has a normal form representation, $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$. Moreover, using left distributivity of ordinals we can write α as $\omega^\beta(\delta + 1)$ where $\beta = \beta_l$ and $\delta = \omega^{\beta_1 - \beta_l} \cdot k_1 + \omega^{\beta_2 - \beta_l} \cdot k_2 + \cdots + (k_l - 1)$. The uniqueness of this representation follows from the uniqueness of the normal form.

For contradiction, we assume that $\beta \geq \alpha$. We show using induction that for all $n \in \omega$, $\alpha \geq \omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ where ω is exponentiated n times. The base case is trivial. Let $\xi = \omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ exponentiated k times, and assume that $\alpha \geq \xi$. Now by left monotonicity of exponentiation,

$$\alpha = \omega^\beta(\gamma + 1) \geq \omega^\alpha(\gamma + 1) \geq \omega^\xi(\gamma + 1) \geq \omega^\xi$$

where ω^ξ is ω exponentiated $k + 1$ times. Hence by induction $\alpha \geq \varepsilon_0$, which contradicts our hypothesis.

$\square$

Definition 3.1.3 (Fundamental Sequence)**.** For every limit ordinal α with $0 < \alpha \leq \varepsilon_0$ and $k \in \omega$ we define $d(\alpha, k)$ an increasing cofinal sequence in α inductively as below:

(i) If $\alpha = \omega^{\beta+1}(\gamma + 1)$, then $d(\alpha, k) = \omega^{\beta+1}\gamma + \omega^\beta k$.

(ii) If $\alpha = \omega^\beta(\gamma + 1)$ for limit β, then $d(\alpha, k) = \omega^\beta\gamma + \omega^{d(\beta,k)}$.

(iii) If $\alpha = \varepsilon_0$, then

$$d(\varepsilon_0, k) = \begin{cases} \omega, & \text{if } k = 0 \\ \omega^{d(\varepsilon_0,\, k-1)}, & \text{otherwise.} \end{cases}$$

We further expand this definition for successor or zero α inductively by

(iv) $d(0,k) = 0$.

(v) $d(\gamma+1,k) = \gamma$.

Using the construction in 3.1.2, we present the equivalent representation of $d(\alpha,k)$ in normal form, for limit $0 < \alpha < \varepsilon_0$.

Lemma 3.1.4. *Let $\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l$ be a limit ordinal written in the normal form. Then,*

(i) $d(\alpha,k) = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\gamma+1} \cdot (k_l - 1) + \omega^{\gamma} \cdot k$ for successor $\beta_l = \gamma + 1$.

(ii) $d(\alpha,k) = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot (k_l - 1) + \omega^{d(\beta_l,k)}$ for limit β_l.

Definition 3.1.5 (Fast Growing Hierarchy)**.** We define $F_\alpha : \omega \to \omega$ for all $\alpha < \varepsilon_0$ inductively as

(i) $F_0(n) = n+1$.

(ii) $F_{\alpha+1}(n) = F_\alpha^{n+1}(n) = F_\alpha(F_\alpha(\dots(n)\dots))$ with F_α applied $n+1$ times.

(iii) $F_\alpha(n) = F_{d(\alpha,n)}(n)$ for limit $\alpha \neq 0$.

Definition 3.1.6. Let $\beta \leq \alpha \leq \varepsilon_0$. We define the term $\alpha \underset{n}{\to} \beta$ if there exists a sequence of ordinals $\alpha_1, \alpha_2, \dots, \alpha_r$ where $\alpha_1 = \alpha$, $\alpha_r = \beta$ and $\alpha_{i+1} = d(\alpha_i, n)$, for $0 \leq i < r$.

The following proposition is evident from Definition 3.1.6.

Proposition 3.1.7. *Let $\alpha, \beta, \gamma < \varepsilon_0$ be ordinals. Then,*

(a) If $\alpha \underset{n}{\to} \beta$, $\alpha \underset{n}{\to} \gamma$ and $\beta > \gamma$, then $\beta \underset{n}{\to} \gamma$.

(b) If $\alpha \underset{n}{\to} \beta$ and $\beta \underset{n}{\to} \gamma$, then $\alpha \underset{n}{\to} \gamma$.

Lemma 3.1.8. *Let $\alpha, \beta < \varepsilon_0$ be ordinals, with normal form representations $\alpha = \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_l} \cdot k_l$ and $\beta = \omega^{\delta_1} \cdot m_1 + \cdots + \omega^{\delta_s} \cdot m_s$. If $\gamma_l \geq \delta_1$, then*

$$d(\alpha + \beta, n) = \alpha + d(\beta, n)$$

hence if $\beta \underset{n}{\rightarrow} \xi$, then $\alpha + \beta \underset{n}{\rightarrow} \alpha + \xi$.

Proof. This proof follows from the fact that the definition of $d(\alpha, n)$ changes the term with omega to the smallest power, as can be seen in Lemma 3.1.4 for the limit case, and trivially in the successor case. Hence if $\gamma_l \geq \delta_1$, the normal form of $\alpha + \beta$ is the concatenation of the two representations, meaning we only change β in $d(\alpha + \beta, n)$, hence $d(\alpha + \beta, n) = \alpha + d(\beta, n)$.

The last claim can be shown using induction. $\beta \underset{n}{\rightarrow} \xi$ means that there exists a sequence $\beta_1, \beta_2, \ldots, \beta_r$ with $\beta_1 = \beta$, $\beta_r = \xi$ and $\beta_{i+1} = d(\beta_i, n)$. Notice how for each β_i, the largest power that appears for β_i in the normal form representation must at most δ_1, hence we can use our result above. Hence $d(\alpha + \beta_i, n) = \alpha + d(\beta_i, n)$, and so our result holds. □

Notice how in the above proof, it suffices to show that after one step of $\underset{n}{\rightarrow}$ our result holds. From now on, we will make use of the fact that to prove that something holds for $\alpha \underset{n}{\rightarrow} \beta$, it typically suffices to show it is true for the case where $\beta = d(\alpha, n)$, due to the transitivity of $\underset{n}{\rightarrow}$.

Lemma 3.1.9. *Let $0 < \alpha \leq \varepsilon_0$ and $n \geq 0$. Then $\alpha \underset{n}{\rightarrow} 0$.*

Proof. The proof is straightforward by strong induction on α. Let $\alpha > 0$. If $d(\alpha, n) = 0$ then we are done, otherwise by the induction hypothesis $d(\alpha, n) \underset{n}{\rightarrow} 0$ so our result holds. □

Corollary 3.1.10. *Let $k < l < \omega$, $\alpha < \varepsilon_0$ and $n \geq 0$. Then, $\omega^\alpha \cdot l \underset{n}{\rightarrow} \omega^\alpha \cdot k$.*

Proof. By Lemma 3.1.9, we have that $\omega^\alpha \cdot (l - k) \underset{n}{\rightarrow} 0$. Furthermore, by Lemma 3.1.8, we have that $\omega^\alpha \cdot l = \omega^\alpha \cdot k + \omega^\alpha \cdot (l - k) \underset{n}{\rightarrow} \omega^\alpha \cdot k$. □

Lemma 3.1.11. *Let $n \geq 1$ and $\delta < \varepsilon_0$. Then $\omega^{\delta+1} \underset{n}{\rightarrow} \omega^\delta$.*

Proof. We have that $\omega^{\delta+1} \underset{n}{\rightarrow} d(\omega^{\delta+1}, n) = \omega^\delta \cdot n$, and by Corollary 3.1.10, $\omega^\delta \cdot n \underset{n}{\rightarrow} \omega^\delta$, so our result holds by transitivity of $\underset{n}{\rightarrow}$. □

Lemma 3.1.12. *Let $\alpha < \varepsilon_0$ and $n \geq 1$. If $\alpha \underset{n}{\rightarrow} \beta$ then $\omega^\alpha \underset{n}{\rightarrow} \omega^\beta$.*

Proof. It is enough to consider the case where $\beta = d(\alpha, n)$. If α is a limit, then $d(\omega^\alpha, n) = \omega^{d(\alpha,n)} = \omega^\beta$ as required. If α is a successor, then $\alpha = \beta + 1$ and so the proof follows by Lemma 3.1.11 □

Theorem 3.1.13. *Let $\lambda \leq \varepsilon_0$ be a limit ordinal. If $i < j < \omega$ then $d(\lambda, j) \underset{1}{\rightarrow} d(\lambda, i)$.*

Proof. We make use of the previous lemmas, and prove this by induction on λ.

We first consider the case where $\lambda = \omega^\beta(\gamma + 1)$. If $\gamma > 0$, we can deduce using the induction hypothesis that $d(\omega^\beta, j) \underset{1}{\rightarrow} d(\omega^\beta, i)$. Hence by Lemma 3.1.8,

$$d(\lambda, j) = \omega^\beta \cdot \gamma + d\left(\omega^\beta, j\right) \underset{1}{\rightarrow} \omega^\beta \cdot \gamma + d\left(\omega^\beta, i\right) = d(\lambda, i)$$

as required.

If $\gamma = 0$, we further consider separately the cases where β is a successor and a limit. The case where β is a successor is immediate from Lemma 3.1.10 and the definition of $d(\omega^\beta, n)$. If β is a limit ordinal, we have by the inductive hypothesis that $d(\beta, j) \underset{1}{\rightarrow} d(\beta, i)$. Hence by Lemma 3.1.12, $d(\omega^\beta, j) = \omega^{d(\beta,j)} \underset{1}{\rightarrow} \omega^{d(\beta,i)} = d(\omega^\beta, i)$ as required.

If $\lambda = \varepsilon_0$, we prove this by induction on j. For $j = 1$, $d(\varepsilon_0, 1) = \omega^{d(\varepsilon_0,0)} = \omega^\omega$. If $i < j$, then $i = 0$ necessarily, and hence $d(\varepsilon_0, 0) = \omega$. We have that $d(\omega^\omega, 1) = \omega$, hence $d(\varepsilon_0, 1) \underset{1}{\rightarrow} d(\varepsilon_0, 0)$.

For $j = k + 1$, by our inductive hypothesis $d(\varepsilon_0, k) \underset{1}{\rightarrow} d(\varepsilon_0, i)$ for any $0 \leq i < k$. By Lemma 3.1.12, $d(\varepsilon_0, k+1) = \omega^{d(\varepsilon_0,k)} \underset{1}{\rightarrow} \omega^{d(\varepsilon_0,i)} = d(\varepsilon_0, i+1)$. Hence we can use the transitivity of $\underset{1}{\rightarrow}$ to conclude that $d(\varepsilon_0, k+1) \underset{1}{\rightarrow} d(\varepsilon_0, i)$ for all $0 \leq i < k+1$. □

Corollary 3.1.14. *Let $\beta < \alpha \leq \varepsilon_0$, and $n > i$. Then if $\alpha \underset{i}{\rightarrow} \beta$, then $\alpha \underset{n}{\rightarrow} \beta$.*

Proof. The proof is by induction on α. It suffices to show that this holds for α limit and $\beta = d(\alpha, i)$. By Theorem 3.1.13, we have that $d(\alpha, n) \underset{1}{\rightarrow} d(\alpha, i)$, hence by our inductive hypothesis, $d(\alpha, n) \underset{n}{\rightarrow} d(\alpha, i)$, and hence because $\alpha \underset{n}{\rightarrow} d(\alpha, n)$, our result follows by transitivity of $\underset{n}{\rightarrow}$. □

Proposition 3.1.15. *Let $\alpha \leq \varepsilon_0$. Then,* *(a)* $F_\alpha(n) > n$.

(b) *If* $n > m$ *then* $F_\alpha(n) > F_\alpha(m)$.

(c) *If* $\alpha = \beta + 1$, $F_\alpha(n) \geq F_\beta(n)$ *and moreover if* $n \geq 1$, *the inequality is strict.*

(d) *If* $\alpha \underset{n}{\rightarrow} \beta$, *then* $F_\alpha(n) \geq F_\beta(n)$.

Proof. This proof is done by induction on α. For $\alpha = 0$, the claims are evident. Consider first the case when $\alpha = \beta + 1$.

$F_\alpha(n) = F_\beta^{n+1}(n) \geq F_\beta(n) > n$ by the inductive hypothesis, proving *(a)*. If $n \geq 1$ $F_\beta^{n+1}(n) = F_\beta(F_\beta^n(n)) > F_\beta(n)$, concluding the proof for *(c)*.

If $\alpha \underset{n}{\rightarrow} \beta$ in one step, $F_\alpha(n) \geq F_\beta(n)$ by *(c)*. If $\alpha \underset{n}{\rightarrow} \beta$ in more than one step, we use Proposition 3.1.7 and the inductive hypothesis to conclude *(d)*.

For *(b)*, if $n > m$, then $F_\alpha(n) = F_\beta^{n+1}(n) > F_\beta^{n+1}(m) \geq F_\beta^{m+1}(m) = F_\alpha(m)$ by the inductive hypothesis and by *(a)*.

Now, suppose α is a limit ordinal. Then, $F_\alpha(n) = F_{d(\alpha,n)}(n) > n$, proving *(a)*, and if $\alpha \underset{n}{\rightarrow} \beta$ in one step we have $\beta = d(\alpha, n)$ so $F_\alpha(n) = F_\beta(n)$ proving *(d)*. By Theorem 3.1.13, $d(\alpha, n) \underset{i}{\rightarrow} d(\alpha, m)$ for all $m < n$, hence by *(d)* and the inductive hypothesis, $F_{d(\alpha,n)}(n) \geq F_{d(\alpha,m)}(n) > F_{d(\alpha,m)}(m) = F_\alpha(m)$, proving *(b)*. □

Lemma 3.1.16. *Let* $\alpha, \beta \leq \varepsilon_0$. *Then,*

(a) *If* $\alpha > \beta$, *then for some* $1 \leq n < \omega$, $\alpha \underset{n}{\rightarrow} \beta$.

(b) *Let* $n \geq 1$ *and* $\alpha \underset{n}{\rightarrow} \beta$. *If* $\alpha > \beta + 1$, *then* $\alpha \xrightarrow[n+1]{} \beta + 1$.

Proof. We prove *(a)* by induction on α. Consider first the case when $\alpha = \gamma + 1$. If $\gamma = \beta$, then $\gamma + 1 \underset{n}{\rightarrow} \beta$ holds for all n. Otherwise $\gamma > \beta$ and so by the inductive hypothesis $\gamma \underset{n}{\rightarrow} \beta$ for some $n \geq 1$ and hence $\alpha \underset{n}{\rightarrow} \beta$.

For α limit, since $\alpha > \beta$ there exists an m such that $d(\alpha, m) > \beta$. Hence by our inductive hypothesis there exists an $i \geq 1$ such that $d(\alpha, m) \underset{i}{\rightarrow} \beta$. By Corollary 3.1.14, letting $n = \max\{m, i\}$ we have that $\alpha \underset{n}{\rightarrow} d(\alpha, m)$ and $d(\alpha, m) \underset{n}{\rightarrow} \beta$, concluding our argument.

To prove *(b)*, we also make use of induction on α. Since $\alpha \underset{n}{\rightarrow} \beta$, we have that $d(\alpha, n) \geq \beta$. If $d(\alpha, n) > \beta + 1$, then the statement follows by the inductive hypothesis. If $d(\alpha, n) = \beta + 1$, then by Corollary 3.1.14 the result follows. Hence we can assume that $d(\alpha, n) = \beta$. We can also assume that $\alpha = \omega^\delta$, as by Lemma 3.1.8 the other cases follow.

We consider first the case where $\delta = \eta + 1$. Then $\beta = \omega^\eta \cdot n$ and $\alpha \xrightarrow[n+1]{} d(\alpha, n+1) = \omega^\eta \cdot (n+1) = \omega^\eta \cdot n + \omega^\eta$, so it is enough to prove that $\omega^\eta \xrightarrow[n+1]{} 1$ by Lemma 3.1.8. By the inductive hypothesis, it is sufficient to show that $\omega^\eta \xrightarrow[n]{} 0$, which follows by Lemma 3.1.9.

We now consider the case when δ is a limit ordinal $< \varepsilon_0$. We have that $\alpha \xrightarrow[n+1]{} \omega^{d(\delta, n+1)}$, and we are required to prove that $\omega^{d(\delta,n+1)} \xrightarrow[n+1]{} \omega^{d(\delta,n)} + 1$. By our inductive hypothesis we have that $\delta \xrightarrow[n+1]{} d(\delta, n) + 1$, so $d(\delta, n+1) \xrightarrow[n+1]{} d(\delta, n) + 1$. Hence by Lemma 3.1.12, $\omega^{d(\delta,n+1)} \xrightarrow[n+1]{} \omega^{d(\delta,n)+1}$. Moreover by Lemma 3.1.11, $\omega^{d(\delta,n)+1} \xrightarrow[n]{} \omega^{d(\delta,n)}$, so by our inductive hypothesis, $\omega^{d(\delta,n)+1} \xrightarrow[n+1]{} \omega^{d(\delta,n)} + 1$, as required.

For $\alpha = \varepsilon_0$, we make use of Theorem 3.1.13 and Corollary 3.1.14 to conclude that $d(\varepsilon_0, n+1) \xrightarrow[n]{} d(\varepsilon_0, n)$. Hence by our inductive hypothesis, $d(\varepsilon_0, n+1) \xrightarrow[n+1]{} d(\varepsilon_0, n) + 1$, as desired. $\square$

Proposition 3.1.17. *Let $\alpha, \beta < \varepsilon_0$. Then if $\alpha > \beta$, F_α dominates F_β. Moreover if $\alpha \xrightarrow[n]{} \beta$ and $n \geq 1$, then $F_\alpha(m) > F_\beta(m)$ for all $m > n$.*

Proof. By Lemma 3.1.16 *(a)*, for any $\alpha > \beta$ we always have some $n \geq 1$ such that $\alpha \xrightarrow[n]{} \beta$, so it suffices to show F_α dominates F_β whenever $\alpha \xrightarrow[n]{} \beta$.

Let $m > n$, to show that $F_\alpha(m) > F_\beta(m)$. If $\alpha = \beta + 1$, we have by Proposition 3.1.15 *(c)* that $F_\alpha(m) > F_\beta(m)$. Otherwise, since $m - 1 \geq n$, we know by Corollary 3.1.14 that $\alpha \xrightarrow[m-1]{} \beta$ and hence by Lemma 3.1.16 *(b)* $\alpha \xrightarrow[m]{} \beta + 1$. By Proposition 3.1.15 *(d)* we have that $F_\alpha(m) \geq F_{\beta+1}(m)$, and furthermore by *(c)* we have that $F_{\beta+1}(m) > F_\beta(m)$, hence $F_\alpha(m) > F_\beta(m)$. $\square$

We further extend this domination result by showing that for any $\beta < \varepsilon_0$ there exists some fixed n depending only on β such that for every $\alpha > \beta$, $F_\alpha(m) > F_\beta(m)$ for all $m > n$. To do this, we first define the *norm* of an ordinal $\alpha < \varepsilon_0$.

Definition 3.1.18. We define the *norm* of an ordinal $\alpha < \varepsilon_0$ inductively by setting $\|0\| = 0$, and if $\omega^{\alpha_1} \cdot k_1 + \ldots \omega^{\alpha_l} \cdot k_l$ is the Cantor normal form of $\alpha > 0$, then $\|\alpha\| = \sum_{i=1}^{l} k_i(\|\alpha_i\| + 1)$.

Proposition 3.1.19. *Let $\beta < \varepsilon_0$ and let $n = \|\beta\|$. Then $\alpha \underset{n}{\to} \beta$ for any ordinal $\alpha < \varepsilon_0$ such that $\alpha > \beta$. Hence for every $x > \|\beta\|$, $F_\alpha(x) > F_\beta(x)$.*

Proof. We prove the first statement, noting that the second statement follows immediately from the first by Proposition 3.1.17. We prove this by induction on β, noting that the statement holds for $\beta = 0$ by Lemma 3.1.9. Hence we fix $\beta > 0$ and assume that the statement holds for all $\beta' < \beta$. We let $n = \|\beta\|$.

We proceed further by induction on α, assuming that for all $\beta < \alpha' < \alpha$ that $\alpha' \underset{n}{\to} \beta$. Clearly, $\beta + 1 \underset{n}{\to} \beta$. Note that we can write our ordinals as $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_l}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$ by simply expanding the normal form.

We first consider the case when $l \geq k$ and $\alpha_i = \beta_i$ for all $1 \leq i \leq k$. In this case, $l > k$ and $\alpha = \beta + \delta$ where $\delta = \omega^{\alpha_{k+1}} + \cdots + \omega^{\alpha_l}$ and $\delta > 0$ as otherwise $\alpha = \beta$. By Lemma 3.1.9, $\delta \underset{n}{\to} 0$ and moreover by Lemma 3.1.8, $\beta + \delta \underset{n}{\to} \beta + 0 = \beta$, so we are done.

Considering the case when $l < k$, there must be a minimal $r \leq l$ such that $\alpha_r \neq \beta_r$. Otherwise, if $\alpha_i = \beta_i$ for every $1 \leq i \leq l$ we get that $\beta > \alpha$, which contradicts our hypothesis. Letting $\alpha^* = \omega^{\alpha_r} + \cdots + \omega^{\alpha_l}$ and $\beta^* = \omega^{\beta_r} + \cdots + \omega^{\beta_k}$, we have that $\alpha^* > \beta^*$. If $r > 1$, then $\beta^* < \beta$, so by the outermost inductive hypothesis, $\alpha^* \underset{n}{\to} \beta^*$, and hence by Lemma 3.1.8, $\alpha \underset{n}{\to} \beta$.

We only have left to consider the case when $r = 1$, that is $\alpha = \alpha^*$ and $\beta = \beta^*$. If $k > 1$, then by Lemmas 3.1.8 and 3.1.9, $\alpha \underset{n}{\to} \omega^{\alpha_1}$ and $\beta < \omega^{\alpha_1} < \alpha$, hence by the innermost inductive hypothesis $\omega^{\alpha_1} \underset{n}{\to} \beta$, from which $\alpha \underset{n}{\to} \beta$ follows. We are left to prove the statement for $k = 1$, that is $\alpha = \omega^{\alpha_1}$. We further split this into two cases, considering separately when $\alpha_1 > \beta_1 + 1$ and $\alpha_1 = \beta_1 + 1$.

In the first case, we know that $\beta_1 < \beta$, hence letting $m = \|\beta_1\|$ we can use the outermost inductive hypothesis to deduce that $\alpha_1 \underset{m}{\to} \beta_1$. By Lemma 3.1.16 *(b)*, $\alpha_1 \underset{m+1}{\longrightarrow} \beta_1 + 1$ and noting that $m < n$, we apply Lemma 3.1.14 to deduce that $\alpha_1 \underset{n}{\to} \beta_1 + 1$. Then by Lemma 3.1.12, $\omega^{\alpha_1} \underset{n}{\to} \omega^{\beta_1+1}$ and since $\alpha > \omega^{\beta_1+1} > \beta$, by the innermost inductive hypothesis, $\omega^{\beta_1+1} \underset{n}{\to} \beta$ as required.

Finally, consider the case when $\alpha_1 = \beta_1 + 1$. We first note that $n \geq k$ by definition of $\|\beta\|$, hence $\omega^{\beta_1} \cdot n \geq \beta$. Now, $\alpha = \omega^{\beta_1+1} \underset{n}{\to} \omega^{\beta_1} \cdot n$. If $\omega^{\beta_1} \cdot n = \beta$ we are done, otherwise $\omega^{\beta_1} \cdot n > \beta$, hence by the innermost inductive hypothesis, $\omega^{\beta_1} \cdot n \underset{n}{\to} \beta$, so the result follows.

□

3.2 | A Formula for Goodstein Sequence Termination

In this section, we seek to find a formula to define Goodstein's function $\mathcal{G}(n)$, which gives the number of iterations it takes for a Goodstein sequence starting at n to terminate. We do this by adapting the work of Caicedo [2007], originally established for the original Löb-Wainer hierarchy, to work for our variant of the hierarchy. We further prove the claim made in this paper that $\mathcal{G}(n)$ dominates every F_α.

Definition 3.2.1. Consider a sequence starting with a hereditary base-b representation of n, and continuing in the usual way as a Goodstein sequence as in Definition 2.2.5. We define $B_b(n)$ to be the base at which this sequence becomes zero, starting from base b.

Example 3.2.2. We calculate $B_2(3)$ by defining the sequence (a_n) as in Definition 3.2.1.

$a_1 = 3 = 2^1 + 2^0$ with base 2.
$a_2 = (a_1)_{2\to3} - 1 = 3^1 + 3^0 - 1 = 3^1$ with base 3.
$a_3 = (a_2)_{3\to4} - 1 = 4^1 - 1 = 4^0 \cdot 3$ with base 4.
$a_4 = (a_3)_{4\to5} - 1 = 5^0 \cdot 3 - 1 = 5^0 \cdot 2$ with base 5.
$a_5 = (a_4)_{5\to6} - 1 = 6^0 \cdot 2 - 1 = 6^0 \cdot 1$ with base 6.
$a_6 = (a_5)_{6\to7} - 1 = 7^0 \cdot 1 - 1 = 0$ with base 7.

Hence we have found that $B_2(3) = 7$. Note how in this case our a_n corresponds exactly with $g_n(3)$, as we are starting in base 2. However this new notation also allows us to start from different bases. For example, we can calculate that $B_5(7) = 15$ by the following sequence:

$a_1 = 7 = 5^1 + 5^0 \cdot 2$ with base 5.
$a_2 = (a_1)_{5\to 6} - 1 = 6^1 + 6^0 \cdot 2 - 1 = 6^1 + 6^0 \cdot 1$ with base 6.
$a_3 = (a_2)_{6\to 7} - 1 = 7^1 + 7^0 \cdot 1 - 1 = 7^1$ with base 7.
$a_4 = (a_3)_{7\to 8} - 1 = 8^1 - 1 = 7$ with base 8.
$\vdots$
$a_{10} = (a_9)_{13\to 14} = 2 - 1 = 1$ with base 14.
$a_{11} = (a_{10})_{14\to 15} = 1 - 1 = 0$ with base 15.

Remark 3.2.3. It follows from the definition above that for every natural number $n > 0$, $B_2(n) = \mathcal{G}(n) + 1$, as the base at which a Goodstein sequence terminates is always $\mathcal{G}(n) + 1$.

Definition 3.2.4. Let $\alpha < \varepsilon_0$ be represented in normal form as $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$. We define the exponential polynomial $p_\alpha(x)$ inductively as

$$p_\alpha(x) = x^{p_{\beta_1}(x)} k_1 + x^{p_{\beta_2}(x)} k_2 + \cdots + x^{p_{\beta_l}(x)} k_l$$

where $p_n(x) = n$ for all $n \in \omega$.

Definition 3.2.5. We define $N(\alpha)$ for an ordinal α inductively as follows:

(i) $N(n) = n$ for all $n < \omega$.

(ii) $N(\alpha) = \max\{N(\beta_1), \ldots, N(\beta_l), k_1, \ldots, k_l\}$ where $\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l$ is the normal form representation of α.

Given the above definitions, the following results are immediate.

Lemma 3.2.6. *For $b, m \in \omega$, we have that:*

(i) $N\left((m)_{b\to\omega}\right) < b$.

(ii) $p_{(m)_{b\to\omega}}(b) = m$.

(iii) *If $b > N(\alpha)$, then $(p_\alpha(b))_{b\to\omega} = \alpha$.*

Definition 3.2.7. Let α, β be ordinals such that $\beta \leq \alpha < \varepsilon_0$, and let $b \in \omega$. We define the relation $\alpha \underset{b}{\rightharpoonup} \beta$ whenever there exists a sequence $\alpha_1 \geq \cdots \geq \alpha_k$ where $\alpha_1 = \alpha$, $\alpha_k = \beta$, α_i is limit and $\alpha_{i+1} = d(\alpha_i, b)$ for $i < k$.

Note that the above is a slight modification to the relation $\alpha \underset{b}{\to} \beta$ as introduced in Definition 3.1.6, excluding the ability to go down to the predecessor of an ordinal. This change is made so that if $\alpha \underset{b}{\rightharpoonup} \beta$, then $F_\alpha = F_\beta$ by the definition of F. Note that this is not true in the original relation, as in fact for every α, $\alpha \underset{n}{\to} 0$.

Lemma 3.2.8. *If $\alpha \underset{b}{\rightharpoonup} \beta$ then $F_\alpha(b) = F_\beta(b)$ and $p_\alpha(b) = p_\beta(b)$.*

Proof. To prove these results, it is enough to show that they hold after one step, that is if $\beta = d(\alpha, b)$, then $F_\alpha(b) = F_\beta(b)$ and $p_\alpha(b) = p_\beta(b)$.

The first equality is immediate, as $\alpha \underset{b}{\rightharpoonup} \beta$ means that $\beta = d(\alpha, b)$ and $F_\alpha(b) = F_{d(\alpha,b)}(b) = F_\beta(b)$ by definition.

The remaining equation requires some more work to prove. Let $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_l} \cdot k_l$ be in normal form. We make use of Lemma 3.1.4, to determine the form of $\beta = d(\alpha, b)$, noting that β_l is non-zero since α is necessarily a limit ordinal.

For successor $\beta_l = \gamma + 1$, we have that

$$d(\alpha, b) = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\gamma+1} \cdot (k_l - 1) + \omega^\gamma \cdot b$$

The polynomial $p_\beta(b)$ can be written as

$$p_\beta(b) = b^{p_{\beta_1}(b)} \cdot k_1 + \cdots + b^{p_{\gamma+1}(b)} \cdot (k_l - 1) + b^{p_\gamma(b)} \cdot b$$

It is clear to see that $p_\gamma(b) + 1 = p_{\gamma+1}(b)$ and hence since $b^{p_\gamma(b)} \cdot b = b^{p_\gamma(b)+1}$ we get that $p_\alpha(b) = p_\beta(b)$ as desired.

For limit β_l we make use of induction. We have that

$$d(\alpha, b) = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot (k_l - 1) + \omega^{d(\beta_l, b)}$$

Since $\beta_l < \alpha$, we apply the inductive hypothesis to conclude that $p_{d(\beta_l,b)}(b) = p_{\beta_l}(b)$, hence it follows immediately that $p_\alpha(b) = p_\beta(b)$. □

Lemma 3.2.9. *For every ordinal α such that $0 < \alpha < \varepsilon_0$ and $b > N(\alpha)$, we have that*

$$\alpha \underset{b}{\rightharpoonup} (p_\alpha(b) - 1)_{b \to \omega} + 1.$$

Proof. We prove this by transfinite induction on α, noting that the base case of $\alpha = 1$ is straightforward, as $p_1(b) - 1 = 0$. For $\alpha = \beta + 1$, we have that $p_\alpha(b) - 1 = p_\beta(b) + 1 - 1 = p_\beta(b)$. Moreover, since $b > N(\alpha)$, then $b > N(\beta)$ so $(p_\beta(b))_{b\to\omega} = \beta$, and so we get $(p_\alpha(b) - 1)_{b\to\omega} + 1 = \beta + 1 = \alpha$.

For the limit case, we first show that it suffices to prove the required property for $\alpha = \omega^\beta$. Let $\gamma = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l$, $b > N(\gamma)$ and assume that the property holds for any ω^β. Then,

$$\begin{aligned}
(p_\gamma(b) - 1)_{b\to\omega} + 1 &= \left(b^{p_{\beta_1}(b)} \cdot k_1 + \cdots + b^{p_{\beta_l}(b)} \cdot k_l - 1\right)_{b\to\omega} + 1 \\
&= \left(b^{p_{\beta_1}(b)} \cdot k_1 + \cdots + b^{p_{\beta_l}(b)} \cdot (k_l - 1) + b^{p_{\beta_l}(b)} - 1\right)_{b\to\omega} + 1 \\
&= \omega^{\left(p_{\beta_1}(b)\right)_{b\to\omega}} \cdot k_1 + \cdots + \omega^{\left(p_{\beta_l}(b)\right)_{b\to\omega}} \cdot (k_l - 1) + \left(b^{p_{\beta_l}(b)} - 1\right)_{b\to\omega} + 1 \\
&= \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot (k_l - 1) + \left(b^{p_{\beta_l}(b)} - 1\right)_{b\to\omega} + 1
\end{aligned}$$

noting that the last equation holds by Lemma 3.2.6 *(iii)* as $b > N(\gamma)$ implies that $b > N(\beta_i)$ for $1 \le i \le l$. Hence since by our assumption $\omega^{\beta_l} \underset{b}{\rightharpoondown} (b^{p_{\beta_l}(b)} - 1)_{b\to\omega} + 1$, from Lemma 3.1.8 we can conclude that

$$\begin{aligned}
\gamma &\underset{b}{\rightharpoondown} \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot (k_l - 1) + \left(b^{p_{\beta_l}(b)} - 1\right)_{b\to\omega} + 1 \\
&= (p_\gamma(b) - 1)_{b\to\omega} + 1
\end{aligned}$$

as required.

Therefore, we only have left to prove that $\omega^\beta \underset{b}{\rightharpoondown} (p_{\omega^\beta}(b) - 1)_{b\to\omega} + 1$. We do this by induction on β. If $\beta = 0$, then this reduces to the case when $\alpha = 1$, which is trivial. Consider $\beta > 0$ and suppose that for every $\gamma < \beta$, $\omega^\gamma \underset{b}{\rightharpoondown} (p_{\omega^\gamma}(b) - 1)_{b\to\omega} + 1$. Then, β has a normal form $\beta = \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_l} \cdot k_l$ with $\gamma_l < \beta$ (as otherwise $\beta \ge \omega^\beta$). By the inductive hypothesis and the reasoning above, we get that $\beta \underset{b}{\rightharpoondown} (p_\beta(b) - 1)_{b\to\omega} + 1$.

Letting $\gamma = (p_\beta(b) - 1)_{b\to\omega}$, we have that $\gamma < \beta$ as $\beta \underset{b}{\rightharpoondown} \gamma + 1$, and moreover $N(\gamma) < b$ by Lemma 3.2.6 *(i)*. Hence we can apply the inductive hypothesis to get that $\omega^\gamma \underset{b}{\rightharpoondown} (p_{\omega^\gamma}(b) - 1)_{b\to\omega} + 1 = (b^{p_\gamma(b)} - 1)_{b\to\omega} + 1$. Putting everything together, we can conclude that

$$\omega^\beta \underset{b}{\rightharpoondown} \omega^{\gamma+1} \underset{b}{\rightharpoondown} \omega^\gamma \cdot b = \omega^\gamma \cdot (b-1) + \omega^\gamma \underset{b}{\rightharpoondown} \omega^\gamma \cdot (b-1) + \left(b^{p_\gamma(b)} - 1\right)_{b\to\omega} + 1.$$

Now, $p_{\omega^\beta}(b) - 1 = b^{p_\beta(b)} - 1 = b^{p_\beta(b)-1} \cdot (b-1) + b^{p_\beta(b)-1} - 1$ and hence

$$(p_{\omega^\beta}(b) - 1)_{b\to\omega} + 1 = \omega^\gamma \cdot (b-1) + (b^{p_\gamma(b)} - 1)_{b\to\omega} + 1$$

as required.

□

We caution that Lemma 3.2.9 does not necessarily hold for $b = N(\alpha)$. For example, taking $\alpha = \omega \cdot 2 + 1$, $(p_\alpha(2) - 1)_{2\to\omega} = (2^2)_{2\to\omega} = \omega^\omega$ but $\alpha \underset{2}{\rightharpoondown} \alpha$ only, as it is a successor.

Lemma 3.2.10. *For all $\alpha < \varepsilon_0$ and all $b > N(\alpha)$ such that $b \geq 2$,*

$$B_b\left(b^{p_\alpha(b)} - 1\right) = F_\alpha(b-1).$$

Proof. This proof is done by induction on α. For the successor case, we assume the result is true for all $\xi \leq \alpha$ and prove the result for $\alpha + 1$, letting $b > N(\alpha + 1)$. We have that $p_{\alpha+1}(b) = p_\alpha(b) + 1$, hence $b^{p_{\alpha+1}(b)} - 1 = b^{p_\alpha(b)}(b-1) + b^{p_\alpha(b)} - 1$. By the inductive hypothesis we know that $B_b(b^{p_\alpha(b)} - 1) = F_\alpha(b-1)$, hence starting from base b from term $B_b(b^{p_{\alpha+1}(b)} - 1)$ and continuing the usual way until we get to base $F_\alpha(b-1)$ gives us

$$B_b\left(b^{p_\alpha(b)}(b-1) + b^{p_\alpha(b)} - 1\right) = B_{F_\alpha(b-1)}\left((F_\alpha(b-1))^{p_\alpha(F_\alpha(b-1))}(b-1)\right).$$

To simplify our working, we let $c = F_\alpha(b-1)$. As $b - 1 > 0$, we can increment the base to get

$$B_c\left(c^{p_\alpha(c)}(b-1)\right) = B_{c+1}\left((c+1)^{p_\alpha(c+1)}(b-1) - 1\right)$$

and can further write this expression as

$$B_{c+1}\left((c+1)^{p_\alpha(c+1)}(b-2) + (c+1)^{p_\alpha(c+1)} - 1\right).$$

Since $c + 1 = F_\alpha(b-1) + 1 > b > N(\alpha + 1) \geq N(\alpha)$, we can apply the inductive hypothesis again to conclude that

$$B_{c+1}\left((c+1)^{p_\alpha(c+1)} - 1\right) = F_\alpha(c).$$

Now $F_\alpha(c) = F_\alpha^2(b-1)$ hence we can increment the base as before to deduce that

$$B_b\left(b^{p_{\alpha+1}(b)}-1\right) = B_{F_\alpha^2(b-1)}\left(\left(F_\alpha^2(b-1)\right)^{p_\alpha\left(F_\alpha^2(b-1)\right)}(b-2)\right)$$

If $b-2>0$, we can iterate this process further, noting that $F_\alpha^k(b-k)+1>b$ always holds for $k>0$, obtaining

$$B_b\left(b^{p_{\alpha+1}(b)}-1\right) = B_{F_\alpha^k(b-1)}\left(\left(F_\alpha^k(b-1)\right)^{p_\alpha\left(F_\alpha^k(b-1)\right)}(b-k)\right)$$

for $b=k$. Hence the right hand term evaluates to zero, which means that the base at which we first reach zero is

$$B_b\left(b^{p_{\alpha+1}(b)}-1\right) = F_\alpha^b(b-1) = F_{\alpha+1}(b-1).$$

For limit α, we again write $b^{p_\alpha(b)}-1$ as $b^{p_\alpha(b)-1}(b-1)+b^{p_\alpha(b)-1}-1$. By case *(ii)* of Lemma 3.2.6, we know that $p_{(p_\alpha(b)-1)_{b\to\omega}}(b) = p_\alpha(b)-1$, hence letting $\gamma = (p_\alpha(b)-1)_{b\to\omega}$, noting that $b>N(\gamma)$ necessarily holds, we can apply the inductive hypothesis to deduce that

$$B_b\left(b^{p_\alpha(b)-1}-1\right) = B_b\left(b^{p_\gamma(b)}-1\right) = F_\gamma(b-1).$$

Hence from this point onwards we can continue the argument in the same manner, applying the above to deduce that

$$B_b\left(b^{p_\alpha(b)}-1\right) = B_{F_\gamma(b-1)}\left(\left(F_\gamma(b-1)\right)^{p_\gamma(F_\gamma(b-1))}(b-1)\right)$$

and continuing this process for $b-1>0$ until we reach

$$B_b\left(b^{p_\alpha(b)}-1\right) = B_{F_\gamma^k(b-1)}\left(\left(F_\gamma^k(b-1)\right)^{p_\gamma\left(F_\gamma^k(b-1)\right)}(b-k)\right)$$

for $k=b$, which implies that

$$B_b\left(b^{p_\alpha(b)}-1\right) = F_\gamma^b(b-1) = F_{\gamma+1}(b-1).$$

We have by Lemma 3.2.9 that $\alpha \underset{b}{\rightharpoondown} (p_\alpha(b)-1)_{b\to\omega}+1 = \gamma+1$, hence applying Lemma 3.2.8, we get that $F_{\gamma+1}(b-1) = F_\alpha(b-1)$, as required. □

Corollary 3.2.11. *For any natural numbers b, m such that $b \geq 2$,*

$$B_b\left(b^m - 1\right) = F_{(m)_{b\to\omega}}(b-1).$$

Proof. By Lemma 3.2.6, we know that $m = p_{(m)_{b\to\omega}}(b)$ and that $b > N((m)_{b\to\omega})$, hence we can apply Lemma 3.2.10 to conclude that $B_b(b^m - 1) = F_{(m)_{b\to\omega}}(b-1)$. □

Theorem 3.2.12. *Suppose n is a positive natural number and let n be written in base-2 notation as*

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}$$

where $m_1 > m_2 > \cdots > m_k$. Let $\alpha_i = (m_i)_{2\to\omega}$. Then

$$\mathcal{G}(n) = F_{\alpha_1}\left(F_{\alpha_2}\left(\ldots\left(F_{\alpha_k}(2)\right)\ldots\right)\right) - 1$$

Proof. The proof of this theorem follows from Corollary 3.2.11 by induction on n. The base case is straightforward, as $1 = 2^0$, and $F_0(2) - 1 = 2$, and clearly $\mathcal{G}(1) = 2$. Let the base-2 representation of n be $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}$, and let $t = 2^{m_2} + \cdots + 2^{m_k}$. The $\mathcal{G}(t)$th term of the Goodstein sequence starting at n can be written as

$$g_{\mathcal{G}(t)}(n) = B_2(t)^{(m_1)_{2\to B_2(t)}}$$

since for the first $\mathcal{G}(t)$ iterations, we only increase the base of the largest term, while decrementing t. The next term of our sequence is

$$g_{\mathcal{G}(t)+1}(n) = (B_2(t)+1)^{(m_1)_{2\to B_2(t)+1}} - 1$$

which allows us to use Corollary 3.2.11 with $b = B_2(t) + 1$ and $m = (m_1)_{2\to B_2(t)+1}$ to obtain

$$B_{B_2(t)+1}\left(g_{\mathcal{G}(t)+1}(n)\right) = F_{(m)_{b\to\omega}}(b-1) = F_{(m_1)_{2\to\omega}}\left(B_2(t)\right)$$

noting that $(m)_{b\to\omega} = (m_1)_{2\to\omega}$ because changing base from 2 to b, then changing base from b to ω is equivalent to immediately changing base from 2 to ω.

It is clear to see that $B_2(n) = B_{B_2(t)+1}(g_{\mathcal{G}(t)+1}(n))$, as at the $(\mathcal{G}(t)+1)$th term of the sequence starting from n, we have reached base $B_2(t)+1$. Moreover, by Remark 3.2.3, we have that $\mathcal{G}(n) = B_2(n) + 1$, hence

$$\mathcal{G}(n) = F_{(m_1)_{2\to\omega}}\left(B_2(t)\right) - 1.$$

Note that if $t = 0$, $B_2(t) = 2$, and hence our result follows immediately. Otherwise, letting $\alpha_i = (m_i)_{2\to\omega}$ for $1 \le i \le k$, using the inductive hypothesis we obtain

$$B_2(t) = \mathcal{G}(t) + 1 = F_{\alpha_2}\left(\dots\left(F_{\alpha_k}(2)\right)\dots\right)$$

hence combining the two equations together we get that

$$\mathcal{G}(n) = F_{\alpha_1}\left(F_{\alpha_2}\left(\dots\left(F_{\alpha_k}(2)\right)\dots\right)\right) - 1$$

as desired. □

Example 3.2.13. This allows us to calculate $\mathcal{G}(n)$ for every n. We first confirm that this formula works for a known value of $\mathcal{G}$, taking $n = 3$.

In this case $3 = 2^1 + 2^0$, hence we have $\alpha_1 = (1)_{2\to\omega} = 1$ and $\alpha_2 = (0)_{2\to\omega} = 0$. Plugging these values in, we get that

$$\mathcal{G}(3) = F_1\left(F_0(2)\right) - 1 = F_1(3) - 1 = F_0^4(3) - 1 = 7 - 1 = 6$$

which matches the observation in Example 2.2.12.

Moreover we can actually calculate $\mathcal{G}(8)$ now. We have that $8 = 2^3$, and $(3)_{2\to\omega} = (2^1 + 2^0)_{2\to\omega} = \omega + 1$. Putting these values into our equation we get

$$\mathcal{G}(8) = F_{\omega+1}(2) - 1 = F_\omega^3(2) - 1.$$

The value of F_ω grows incredibly fast. This gives some justification as to why our sequences seemingly blow up at $n = 4$, as this is when 2 first appears in the exponent, meaning ω appears in the index of the F's.

To illustrate an even larger example, taking $n = 22$ we have that $22 = 2^4 + 2^2 + 2^1$ and hence $\alpha_1 = \omega^\omega$, $\alpha_2 = \omega$ and $\alpha_3 = 1$, so we get the equation

$$\mathcal{G}(22) = F_{\omega^\omega}\left(F_\omega(F_1(2))\right) - 1.$$

This formula is especially useful as it allows us to prove that for every α, the Goodstein function $\mathcal{G}(n)$ dominates F_α. To do this, we first prove some straightforward lemmas about the growth of F_α.

Lemma 3.2.14. *For any natural number $n > 0$, $F_1^n(m) = 2^n \cdot m + 2^n - 1$. Moreover, $F_2(m) = 2^{m+1} \cdot m + 2^{m+1} - 1$.*

Proof. We prove the first identity by induction on n. For $n = 1$, $F_1(m) = F_0^{m+1}(m) = 2m + 1 = 2^1(m) + 2^1 - 1$. Let $n = k + 1$ and suppose that the hypothesis holds for $n = k$. Then,

$$\begin{aligned} F_1^{k+1}(m) &= F_1\left(F_1^k(m)\right) \\ &= F_1\left(2^k(m) + 2^k - 1\right) \\ &= F_0^{2^k(m)+2^k}\left(2^k(m) + 2^k - 1\right) \\ &= 2^k(m) + 2^k - 1 + 2^k(m) + 2^k = 2^{k+1}(m) + 2^{k+1} - 1. \end{aligned}$$

Moreover, the last equation holds as $F_2(m) = F_1^{m+1}(m) = 2^{m+1}(m) + 2^{m+1} - 1$ by above. □

Lemma 3.2.15. *For any integer $m \geq 2$, $F_{(m)_{2\to\omega}}(2) > 2^{m+2}$ and moreover if $m > 2$ $F_{(m)_{2\to\omega}}(2) > F_{(m-1)_{2\to\omega}}(2^{m+2})$.*

Proof. For $m = 2$, by Lemma 3.2.14, $F_2(2) = 2^4 + 2^3 - 1 > 2^4$. Let $m > 2$, and assume $F_{(k)_{2\to\omega}}(2) > 2^{k+2}$ holds for all $k < m$. Since $p_{(m)_{2\to\omega}}(2) = m$ and by Lemma 3.2.9, we have that $(m)_{2\to\omega} \underset{2}{\rightharpoondown} (m-1)_{2\to\omega} + 1$. Moreover, one can show using straightforward induction and Lemma 3.2.9 that for every $k \geq 2$, $(k)_{2\to\omega} \underset{2}{\rightarrow} 2$. Hence our result follows by Lemma 3.2.14 and Proposition 3.1.15 as follows:

$$\begin{aligned} F_{(m)_{2\to\omega}}(2) &= F_{(m-1)_{2\to\omega}+1}(2) \\ &= F^3_{(m-1)_{2\to\omega}}(2) \\ &> F^2_{(m-1)_{2\to\omega}}\left(2^{m+1}\right) \\ &\geq F_{(m-1)_{2\to\omega}}\left(F_2\left(2^{m+1}\right)\right) \\ &> F_{(m-1)_{2\to\omega}}\left(2^{2^{m+1}+1} \cdot 2^{m+1} + 2^{2^{m+1}+1} - 1\right) \\ &> F_{(m-1)_{2\to\omega}}\left(2^{m+2}\right) > 2^{m+2} \end{aligned}$$

noting that the last line gives us both results we require. □

Theorem 3.2.16. *For every ordinal $\alpha < \varepsilon_0$, $\mathcal{G}$ dominates F_α.*

Proof. Let α be fixed, and consider the ordinal $\gamma = (p_{\alpha+1}(2))_{2\to\omega}$. Clearly, $\gamma > \alpha$ and $N(\gamma) = 1$, hence letting $t = p_\gamma(2)$, $(t)_{2\to\omega} = \gamma > \alpha$. Let $N = \max\{\|\alpha\|, 2^{t+1}\}$, and consider any $x > N$.

Suppose $x = 2^{m_1} + \cdots + 2^{m_k}$, with $m_1 > \cdots > m_k$. Then by Lemma 3.2.15,

$$\mathcal{G}(x) \geq F_{(m_1)_{2\to\omega}}(2) - 1 \geq F_{(m_1-1)_{2\to\omega}}(2^{m_1+2}) > F_{(m_1-1)_{2\to\omega}}(x)$$

and since $x > 2^{t+1}$, $m_1 \geq t+1$ and hence $(m_1 - 1)_{2\to\omega} > \alpha$, so by Proposition 3.1.19, $\mathcal{G}(x) > F_\alpha(x)$ as required.

□

3.3 | Results on Computable Functions

In this section we flesh out some important classes of functions, namely the computable, primitive recursive and elementary functions. We highlight some important theorems without proof in order to give a better picture of the nature of these functions, as well as facilitate the analysis of provably computable functions in the proceeding chapter by proving a domination result for elementary-in-F_α functions.

We first define some functions and notation from which we obtain our computable functions, then define the computable and primitive recursive functions following lecture notes from van den Dries [2016]. Note that to simplify notation, we let $\mathbf{u}$ denote a vector in $\mathbb{N}^n$, where n varies depending on the context.

Definition 3.3.1. For a relation $R \in \mathbb{N}^n$, we define the characteristic function of R, $\chi_R : \mathbb{N}^n \to \mathbb{N}$ as

$$\chi_R(a) = \begin{cases} 1, & \text{if } a \in R \\ 0, & \text{otherwise} \end{cases}$$

Example 3.3.2. Considering the binary relation $\leq$ on natural numbers, $\chi_\leq(m,n) = 1$ if $m \leq n$, and 0 otherwise.

Definition 3.3.3. The *projection* (*coordinate*) function $\pi_i^n : \mathbb{N}^n \to \mathbb{N}$ is defined as $\pi_i^n(x_1, \ldots, x_n) = x_i$.

Definition 3.3.4. Given a function $f : \mathbb{N}^n \to \mathbb{N}$, we define the *minimisation* of f whenever $f(\mathbf{u}, x) = 0$ holds for some x as

$$\mu x[f(\mathbf{u}, x) = 0] = \text{the least } x \text{ such that } f(\mathbf{u}, x) = 0.$$

We further define the *bounded minimisation* of f,

$$\mu x_{\leq y}[f(\mathbf{u},x)=0]=\begin{cases}\text{the least } x\leq y \text{ such that } f(\mathbf{u},x)=0, & \text{if such an } x \text{ exists}\\ 0, & \text{otherwise.}\end{cases}$$

Definition 3.3.5. The class of *computable* (*recursive*) functions is the smallest class of functions which have domain $\mathbb{N}^n$ for some $n \in \mathbb{N}$, containing as initial functions addition, multiplication, projection, and the characteristic function of $\leq$, which is closed under:

(i) *Substitution:*
If $g : \mathbb{N}^m \to \mathbb{N}$ is computable and $h_1, \dots, h_m : \mathbb{N}^n \to \mathbb{N}$ are computable, then so is $f : \mathbb{N}^n \to \mathbb{N}$ given by

$$f(\mathbf{u}) = g(h_1(\mathbf{u}), \dots, h_m(\mathbf{u})).$$

(ii) *Minimisation:*
If $g : \mathbb{N}^{n+1} \to \mathbb{N}$ is computable, and for all $\mathbf{u} \in \mathbb{N}^n$ there exists $x \in \mathbb{N}$ such that $g(\mathbf{u}, x) = 0$, then the function $f : \mathbb{N}^n \to \mathbb{N}$ given by

$$f(\mathbf{u}) = \mu x[g(\mathbf{u}, x) = 0]$$

is computable.

Informally, these functions are meant to represent functions which can be calculated by an algorithm in a finite number of steps, for example with a pen and paper. In fact, this class of functions coincides with those which are computable by a Turing machine. The accuracy of this representation is asserted by the Church-Turing Thesis, and is widely accepted.

Computable functions are extensively studied as they essentially represent any problem solvable by a computer, and moreover the study of these functions is not trivial, mainly due to closure under minimisation. In particular, the condition that for all $\mathbf{u} \in \mathbb{N}^n$ there exists $x \in \mathbb{N}$ such that $g(\mathbf{u}, x) = 0$ is not constructive, and in fact there is no fully constructive way to generate the computable functions. We consider a subset of computable functions, namely the *primitive recursive* functions, which can indeed be generated constructively.

Definition 3.3.6. The class of *primitive recursive* functions PR is the smallest class of functions which have domain $\mathbb{N}^n$ for some $n \in \mathbb{N}$, containing as initial functions the nullary function $\mathbb{N}^0 \to \mathbb{N}$ with value 0, the successor function $S(x) = x + 1$ and the projection functions, which is closed under:

(i) *Substitution:*
If $g : \mathbb{N}^m \to \mathbb{N}$ is primitive recursive and $h_1, \ldots, h_m : \mathbb{N}^n \to \mathbb{N}$ are primitive recursive, then so is $f : \mathbb{N}^n \to \mathbb{N}$ given by

$$f(\mathbf{u}) = g(h_1(\mathbf{u}), \ldots, h_m(\mathbf{u})).$$

(ii) *Primitive Recursion:*
If $h_1 : \mathbb{N}^n \to \mathbb{N}$ and $h_2 : \mathbb{N}^{n+2} \to \mathbb{N}$ are computable, then the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ given by

$$\begin{aligned} f(0, \mathbf{u}) &= h_1(\mathbf{u}) \\ f(y+1, \mathbf{u}) &= h_2(y, \mathbf{u}, f(y, \mathbf{u})) \end{aligned}$$

is computable.

Every primitive recursive function is indeed computable, however not every computable function is primitive recursive. The most well known example of this is the Ackermann function, which grows faster than any primitive recursive function and hence cannot be primitive recursive. A similar argument can be used to show that Goodstein's function $\mathcal{G}(x)$ is also not primitive recursive.

This connection between fast growing functions and primitive recursive functions is made clear by the characterisation of PR using the Grzegorczyk hierarchy $\mathcal{E}^n$. To define this hierarchy, we make use of a hierarchy of functions on the naturals $(f_n)_{n\in\omega}$. We follow the characterisation and definitions as in Grzegorczyk [1964]

Definition 3.3.7. We define the Grzegorczyk fast growing functions $(f_n)_{n\in\mathbb{N}}$ for $n = 0, 1, 2$ as

$$\begin{aligned} f_0(m, x) &= x + 1 \\ f_1(m, x) &= m + x \\ f_2(m, x) &= (m+1) \cdot (x+1). \end{aligned}$$

Moreover for $n \geq 3$ we define f_n recursively by

$$f_{n+1}(0,x) = f_n(x+1,x+1)$$
$$f_{n+1}(m+1,x) = f_{n+1}(m, f_{n+1}(m,x)).$$

We highlight the similarities here to the hierarchy introduced in Section 3.1. For $n \geq 2$, $x > 0$, $f_{n+1}(x,y)$ can be seen as applying f_{n+1} x times to the argument y, and so, giving some liberty to notation, $f_{n+1}(y) = f_n^{y+1}(y+1)$. The main difference between the two is that f_2 introduces multiplication, which means that $(f_n)_{n\in\omega}$ contains some larger functions than $(F_n)_{n\in\omega}$, which is why we use this different hierarchy. However, it is easy to see that $(F_\alpha)_{\alpha\in\varepsilon_0}$ grows larger than $(f_n)_{n\in\omega}$, that is each f_n is dominated by some F_α.

Definition 3.3.8 (Grzegorczyk Hierarchy)**.** For $n \in \omega$, we define $\mathcal{E}^n$ to be the smallest class of functions containing as initial functions the successor function $S(x) = x+1$, the projection functions $\pi_1^2(x,y) = x$ and $\pi_2^2(x,y) = y$, and f_n which is closed under:

(i) *Substitution:*
If $g : \mathbb{N}^m \to \mathbb{N}$ is in $\mathcal{E}^n$ and $h_1, \dots, h_m : \mathbb{N}^n \to \mathbb{N}$ are in $\mathcal{E}^n$, then so is $f : \mathbb{N}^n \to \mathbb{N}$ given by

$$f(\mathbf{u}) = g(h_1(\mathbf{u}), \dots, h_m(\mathbf{u})).$$

(ii) *Limited Recursion:*
If $h_1, h_2, h_3 \in \mathcal{E}^n$ where $h_1, h_3 : \mathbb{N}^n \to \mathbb{N}$ and $h_2 : \mathbb{N}^{n+2} \to \mathbb{N}$, then the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ given by

$$f(0,\mathbf{u}) = h_1(\mathbf{u})$$
$$f(y+1,\mathbf{u}) = h_2(y,\mathbf{u},f(y,\mathbf{u}))$$
$$f(y,\mathbf{u}) \leq h_3(\mathbf{u})$$

is in $\mathcal{E}^n$.

In particular, the class $\mathcal{E}^3$ is the class of Csillag-Kalmar elementary functions. This class can be defined in many different ways, for example by considering closure under substitution, bounded sums and bounded products of a set of base functions, however these are all equivalent.

Definition 3.3.9 (Csillag-Kalmar elementary functions)**.** We define the class of *elementary functions* $\mathcal{E}$ to be $\mathcal{E}^3$.

Lemma 3.3.10 (Characterisation of elementary functions)**.** *Let $\mathcal{E}'$ be the smallest class including as initial functions the successor function, addition, multiplication, exponentiation which is closed under substitution and limited recursion. Then, $\mathcal{E}' = \mathcal{E}$, that is $\mathcal{E}'$ is the class of elementary functions.*

We further state without proof some results regarding the Grzegorczyk hierarchy which we will make use of in later chapters, culminating in the characterisation of primitive recursive functions.

Lemma 3.3.11. *For every n, $\mathcal{E}^n$ is closed under the operation of bounded minimisation, $\mu x_{\leq y}$. That is, if $g(\mathbf{u}, x) : \mathbb{N}^{n+1} \to \mathbb{N}$ is contained in $\mathcal{E}^n$, then the function $f(\mathbf{u}, y) : \mathbb{N}^{n+1} \to \mathbb{N}$ defined as*

$$f(\mathbf{u}, y) = \mu x_{\leq y}[g(\mathbf{u}, x) = 0]$$

is an element of $\mathcal{E}^n$.

Theorem 3.3.12. *For every n, $\mathcal{E}^n \subset \mathcal{E}^{n+1}$ Moreover, the function $f_{n+1}(x, x) \in \mathcal{E}^{n+1}$ dominates every $g \in \mathcal{E}^n$, hence*

$$\mathcal{E}^n \subsetneq \mathcal{E}^{n+1}.$$

Theorem 3.3.13. *The class PR is the union of the classes $\mathcal{E}^n$, that is*

$$PR = \bigcup_{n \in \omega} \mathcal{E}^n.$$

Combining Theorems 3.3.12 and 3.3.13, we get that each primitive recursive function is dominated by f_n for some n.

Remark 3.3.14. Goodstein's function $\mathcal{G}$ is a computable function that is not primitive recursive. The fact that $\mathcal{G}$ is not primitive recursive can be seen from the fact that $\mathcal{G}$ dominates each F_α (Theorem 3.2.16), and because every f_n is dominated by some F_α.

To see that Goodstein's function is computable, we note that the Goodstein sequence $g_k(n)$ can be represented as a function $g : \mathbb{N}^2 \to \mathbb{N}$ [1], where $g(k, n) =$

[1] This is not exact, as $g_0(n)$ is not defined, but this can easily be solved by setting $g_0(n) = n + 1$, and $g_1(n) = g_0(n) - 1$.

$g_k(n)$. It can easily be seen that g is a primitive recursive (hence computable) function, since each term is defined from the previous term and the index. Moreover, we can write $\mathcal{G}(x)$ as

$$\mathcal{G}(x) = \mu k[g(k,x) = 0]$$

hence $\mathcal{G}(x)$ is a computable function.

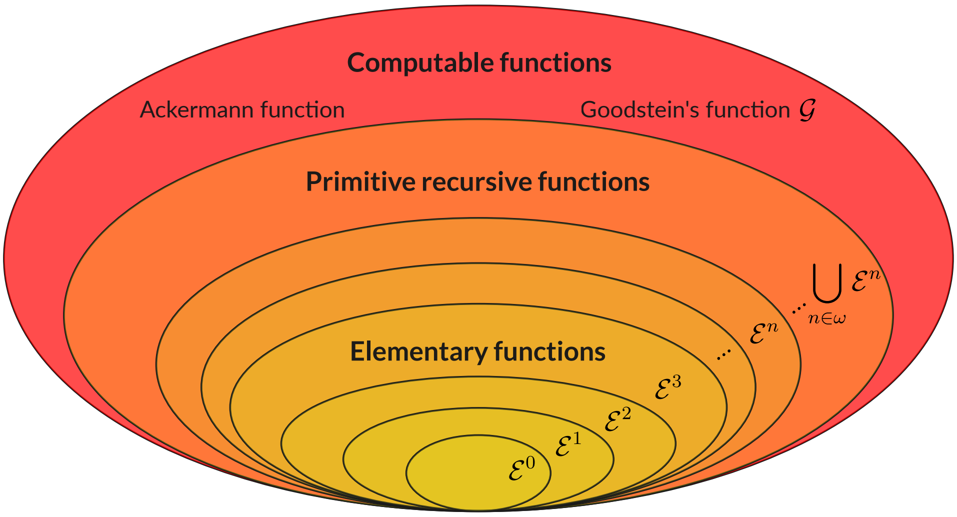

Figure 3.1: Diagram of computable and primitive recursive functions.

We also present an alternate characterisation of the Grzegorczyk Hierarchy $(\mathcal{E}^n)_{n\in\omega}$ for $n \geq 3$ as given by Cobham [1965], which will allow us to easily define functions in $\mathcal{E}^n$. In this characterisation, we make reference to Turing machines. A Turing machine is a model of computation which manipulates symbols on a tape according to a set of rules. We forgo an in-depth explanation of these machines as this is out of scope for the thesis, however a full explanation can be found in Papadimitriou [1995]; Sipser [2012].

Theorem 3.3.15. *A function $f : \mathbb{N}^m \to \mathbb{N}$ belongs to $\mathcal{E}^n$ for $n \geq 3$ if and only if there exists a Turing Machine which computes $f(\mathbf{u})$ in $s_f(\mathbf{u})$ steps, where $s_f(\mathbf{u}) \in \mathcal{E}^n$.*

While not every computable function is primitive recursive, every such function can be represented using primitive recursive functions. In fact Grzegorczyk

[1964] proved a stronger theorem: in particular every computable function can be represented using $\mathcal{E}^0$ functions and the μx operator.

Theorem 3.3.16 (Kleene Normal Form)**.** *Every computable function can be expressed in the form*[2]

$$f(\mathbf{u}) = V(\mu x[T(\mathbf{u}, x) = 0])$$

where $V, T \in \mathcal{E}^0$.

We conclude this section with a result bounding functions generated by adding F_α to the initial functions of the elementary functions. This result will ultimately allow us to bound our provably computable functions in Chapter 4. We adapt this proof from a similar result from Löb and Wainer [1970].

Definition 3.3.17. We define the class of *elementary-in-f* functions $E(f)$ to be the smallest class of functions containing f and the initial functions of $\mathcal{E}$, closed under substitution and limited recursion.

To prove the desired result, we must first establish some basic lemmas about our fast growing hierarchy.

Lemma 3.3.18. *Let* $\alpha < \varepsilon_0$, $\alpha > 0$ *and* $n \geq 1$. *Then* $\alpha \underset{n}{\to} 1$, *hence* $F_\alpha(n) \geq F_1(n)$.

Proof. Every α can be expressed as $\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l$ and by Lemmas 3.1.9 and 3.1.8, we get that $\alpha \underset{n}{\to} \omega^{\beta_1}$. If $\beta_1 = 0$ we are done, otherwise applying Lemma 3.1.9 again we obtain that $\beta_1 \underset{0}{\to} 0$ and hence by Lemma 3.1.12 $\omega^{\beta_1} \underset{n}{\to} 1$. From this, we get that $F_\alpha(n) \geq F_1(n)$ by Proposition 3.1.15 *(d)*. ☐

Lemma 3.3.19. *Let* $\alpha < \varepsilon_0$ *and* $\alpha \geq \omega$. *Then for any* $n \geq 1$, $\alpha \underset{n}{\to} \omega$. *Hence* $F_\alpha(n) \geq F_\omega(n)$.

Proof. Suppose that α can be written as $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_l}$ in Cantor Normal Form, with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. If $l \geq 2$, by Lemma 3.1.9 $\omega^{\beta_2} + \cdots + \omega^{\beta_l} \underset{n}{\to} 0$, and hence by Lemma 3.1.8, $\alpha \underset{n}{\to} \omega^{\beta_1}$.

[2] In the original paper, a stronger statement is proved with the operator $\iota x[T(\mathbf{u}, x) = 0]$ that returns the *unique* x such that $T(\mathbf{u}, x) = 0$.

We know that $\beta_1 > 0$, as otherwise $\alpha < \omega$, and hence we can apply Lemma 3.3.18 to get that $\beta_1 \underset{n}{\rightarrow} 1$. Hence by Lemma 3.1.12, $\omega^{\beta_1} \underset{n}{\rightarrow} \omega^1 = \omega$. As before, $F_\alpha(n) \geq F_\omega(n)$ follows from Proposition 3.1.15 *(d)*. □

Lemma 3.3.20. *For any natural numbers $t, n > 0$ and ordinal α, $F^t_{\alpha+1}(n) \geq F^{t(n+1)}_\alpha(n)$. Consequentially, for any $k < \omega$, $F^t_k(n) \geq F_0^{t(n+1)^k}(n) = n + t(n+1)^k$.*

Proof. We prove the first statement by induction on t, noting that the second statement follows by a simple induction on k from the first result. For $t = 1$, $F_{\alpha+1}(n) = F^{n+1}_\alpha(n)$ by definition, so we are done.

Suppose our result holds for t. Then, by the inductive hypothesis, and noting that $F^{t(n+1)}_\alpha(n) + 1 > n + 1$,

$$\begin{aligned}
F^{t+1}_{\alpha+1}(n) &= F_{\alpha+1}\left(F^t_{\alpha+1}(n)\right) \\
&> F_{\alpha+1}\left(F^{t(n+1)}_\alpha(n)\right) \\
&= F_\alpha^{F^{t(n+1)}_\alpha(n)+1}\left(F^{t(n+1)}_\alpha(n)\right) \\
&> F^{n+1}_\alpha\left(F^{t(n+1)}_\alpha(n)\right) = F^{(t+1)(n+1)}_\alpha(n).
\end{aligned}$$

□

Theorem 3.3.21. *Let $\alpha < \varepsilon_0$ be a non-zero ordinal. Then for $\beta = \max\{\alpha, \omega\}$, for every $f \in E(F_\alpha)$, there exists a number p such that for all inputs $x_1, \ldots, x_n$,*

$$f(x_1, \ldots, x_n) < F^p_\beta(\max(x_1, \ldots, x_n)).$$

Consequentially, every $f \in E(F_\alpha)$ is dominated by $F_{\beta+1}$.

Proof. We prove the first part of this theorem by induction on the generation of elementary-in-F_α functions, using the characterisation in Lemma 3.3.10. We first prove the property for initial functions $x+1$, $x+y$, $x \cdot y$ and F_α, then show that if the property holds for some functions, then it holds for the function defined by substitution or limited recursion using these functions.

To prove the property holds for initial functions, we make heavy use of Proposition 3.1.15 *(a)*, noting that this inequality implies that $F_\beta(n) \geq n+1$. For the successor function, our property holds with $p = 2$ as $F^2_\beta(x) \geq F_\beta(x) + 1 > x + 1$.

For addition, the case when $x = y = 0$ is trivial as $F_\beta(\max(x,y)) > 0$ is always true. Otherwise, we know that β and $k = \max(x,y)$ are not zero, so by Lemma 3.3.18, $F_\beta(k) \geq F_1(k)$. Hence our result follows as $F_1(k) = F_0^{k+1}(k) > F_0^y(x) = x + y$.

For multiplication, let $k = \max(x,y)$. For $k < 2$, $F_\beta(k) > x \cdot y$ trivially, so we may assume $k \geq 2$. Then by Lemma 3.3.19,

$$F_\beta(k) \geq F_\omega(k) = F_k(k) \geq F_2(k)$$

and by Lemma 3.2.14, $F_2(k) = 2^{k+1} \cdot k + 2^{k+1} - 1 > k \cdot k \geq x \cdot y$.

If we consider exponentiation, letting $k = \max(x,y) > 0$ again we have by Lemma 3.3.19 that $F_\beta(k) \geq F_\omega(k) = F_k(k)$. By Lemma 3.3.20, $F_k(k) \geq k + (k+1)^k > k^k \geq x^y$ as desired.

If $f = F_\alpha$, we note that if $\alpha < \omega$ then as a consequence of Proposition 3.1.17 there exists some p such that $F_\alpha(p) \leq F_\alpha(x+p) < F_\omega(x+p)$. Moreover, $F_\omega(x+p) \leq F_\omega(F_\omega^p(x))$, hence $F_\omega^{p+1}(x) > F_\alpha(x)$ for any x. If $\alpha \geq \omega$, then $\beta = \alpha$ and the result follows with $p = 2$, as $F_\alpha(F_\alpha(x)) > F_\alpha(x)$.

If f is defined from h_1, h_2, h_3 by limited recursion, and $h_3(\mathbf{u}) < F_\beta^p(\max(\mathbf{u}))$, then $f(y, \mathbf{u}) < h_3(\mathbf{u}) < F_\beta^p(\max(y, \mathbf{u}))$ hence the same p suffices for f.

Now, suppose f is defined from $g, h_1, \ldots h_m$ by substitution and there exist $q, p_1, \ldots p_m$ such that $g(\mathbf{x}) < F_\beta^q(\max(\mathbf{x}))$ and $p_i(\mathbf{x}) < F_\beta^{p_i}(\max(\mathbf{x}))$ for $1 \leq i \leq m$. Letting $p = \max(p_1, \ldots, p_m)$ we get

$$\begin{aligned} f(\mathbf{x}) &< F_\beta^q(\max(h_1(\mathbf{x}), \ldots h_m(\mathbf{x}))) \\ &< F_\beta^q\left(\max\left(F_\beta^{p_1}(\max(\mathbf{x})), \ldots, F_\beta^{p_m}(\max(\mathbf{x}))\right)\right) \\ &= F_\beta^q\left(F_\beta^p(\max(\mathbf{x}))\right) \\ &= F_\beta^{p+q}(\max(\mathbf{x})) \end{aligned}$$

so $p + q$ is the appropriate upper bound.

The final part of the theorem follows immediately from the first. We know that for every $f \in E(F_\alpha)$, there exists a p such that $f(\mathbf{x}) < F_\beta^p(\max(\mathbf{x}))$. Hence, for all inputs $\mathbf{x}$ such that $\max(\mathbf{x}) > p$,

$$f(\mathbf{x}) < F_\beta^p(\max(\mathbf{x})) < F_\beta^{\max(\mathbf{x})+1}(\max(\mathbf{x})) = F_{\beta+1}(\max(\mathbf{x})).$$

□

4

Unprovability in Peano Arithmetic

In this Chapter we lay out the common technique of Cut-elimination, first pioneered by Gentzen [1964], which is used throughout proof theory for unprovability results. In doing so, we prove a general result about provably computable functions in Peano Arithmetic, which we apply to Goodstein's function to show it is not provable.

4.1 | First Order Logic

We must first define the framework in which we can define our system of Peano Arithmetic. We build up from the basics, defining the structure required to build a first order logic. We start by defining the alphabet, from which we can build our language. From this, we further define terms and formulas and make our terminology clear. We follow definitions as in Halbeisen [2011] and O'Leary [2016].

Definition 4.1.1. A *first order alphabet* A is made up of the following symbols:

(a) **Variables**, typically denoted by $x, y, z, \ldots$ which are placeholders for objects of the domain under consideration.

(b) **Logical connectives**, which are $\neg$, $\vee$ and $\wedge$.

(c) **Logical quantifiers**, which are $\forall$ and $\exists$.

(d) **Equality symbols**, "=" and "≠" which stand for the particular binary equality relation and its negation.

(e) **Constant symbols**, which stand for fixed individual objects in the domain.

(f) **Function symbols**, which stand for fixed functions taking objects as arguments and returning objects as values. Each such symbol is associated with a positive natural number, referred to as the *arity* of the function. A symbol with arity n is sometimes referred to as n-ary.

(g) **Relation symbols**, which stand for fixed relations between objects in the domain. Again, we associate an arity to each relation symbol.

Symbols (a)-(d) are common throughout all first order logics, and are hence referred to as *logical symbols*. Symbols (e)-(g) vary depending on the mathematical objects we wish to talk about and are likewise referred to as *non-logical symbols*. In addition to these symbols, auxiliary symbols such as brackets are used to make expressions more readable.

Definition 4.1.2. Given a first order alphabet, we construct *terms* as follows:

(T1) Each variable is a term.

(T2) Each constant symbol is a term.

(T3) If $t_1, \dots, t_n$ are terms and f is an n-ary function symbol, then $f t_1 \dots t_n$ is a term.

Note that certain binary functions such as $+$ and $\cdot$ are more commonly written in infix notation, that is as $x + y$ rather than $+xy$. The two representations can be interchanged without any consequences, hence we shall adopt the infix notation for readability purposes.

Definition 4.1.3. Given a first order alphabet, we construct *formulas* as follows:

(F1) If t_1 and t_2 are terms, then $t_1 = t_2$ and $t_1 \neq t_2$ are formulas.

(F2) If $t_1, \dots, t_n$ are terms and R is an n-ary relation symbol, then $R t_1 \dots t_n$ is a formula.

(F3) If φ is a formula, then $\neg\varphi$ is a formula.

(F4) If φ, ψ are formulas, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are formulas.

(F5) If φ is a formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are formulas.

Formulas of the form (F1) and (F2) are referred to as *atomic formulas*, as they are the building blocks of all other formulas.

Note that while the implication symbol $\rightarrow$ is not defined, $\varphi \rightarrow \psi$ is taken as an abbreviation, denoting $\neg\varphi \vee \psi$. This keeps our language minimal, while still allowing us to reason about implications.

We further define the length of a formula, based on the number of logical connectives and quantifiers in the formula, excluding negation. This exclusion is justified further on, as we take two formulas to be equal if they have the same negation normal form, as defined later in Definition 4.2.1.

Definition 4.1.4. The *length* of a formula is defined inductively as follows:

- $\text{len}(\varphi) = 0$ if φ is an atomic formula.
- $\text{len}(\neg\varphi) = \text{len}(\varphi)$.
- $\text{len}(\varphi \vee \psi) = \text{len}(\varphi \wedge \psi) = \max\{\text{len}(\varphi), \text{len}(\psi)\} + 1$.
- $\text{len}(\exists x\varphi) = \text{len}(\forall x\varphi) = \text{len}(\varphi) + 1$.

Definition 4.1.5 (Free and Bound Variable Occurrences)**.** Given a formula φ and a variable x, we say that an occurrence of x in φ is *free* according to the following rules:

- A variable occurrence in an atomic formula is free.
- A variable occurrence in $\varphi \wedge \psi$, $\varphi \vee \psi$ is free if and only if the corresponding occurrence in φ or ψ is free.
- A variable occurrence in $\neg\varphi$, $\forall y\varphi$, $\exists y\varphi$ where $y \neq x$ is free if and only if the corresponding occurrence in φ is free.
- Any occurrence of x in $\forall x\varphi$ and $\exists x\varphi$ is not free.

If an occurrence of a variable is not free, it is *bound*.

A variable x is said to be *free* (respectively *bound*) in φ if there exists a free (respectively bound) occurrence of x in φ. If x is free in φ, we may write the formula as $\varphi(x)$.

Definition 4.1.6. A formula φ is a *sentence* if φ does not contain any free variables.

Definition 4.1.7. If $\varphi(x)$ is a formula, then a *substitution* $\varphi[x/t]$ is the formula obtained by replacing all free instances of x by t. When there is no ambiguity, we may write $\varphi[x/t]$ as $\varphi(t)$. A substitution $\varphi[x/t]$ is said to be *admissible* if no free occurrence of x in φ is in the range of a quantifier that binds any variable contained in t.

We will work up to *alpha equivalence*, meaning we are allowed to rename bound variables, as this preserves the meaning of the original formula. We will only consider admissible substitutions, as otherwise a substitution may change the meaning of the original formula. This does not restrict the expressivity of our language, as since we are working up to alpha equivalence, we can always rename bound variables of a given formula to make a substitution admissible.

4.2 | Tait-Style Sequent Calculus

This section pertains to our proof system. We make use of a sequent calculus, as is commonly done in a proof theoretic analysis. We define our system and the terminology we will be using throughout this chapter in Section 4.2.1. Moreover, in Section 4.2.2 we justify our use of the sequent calculus by comparing it to a Hilbert-style system, and showing that our system is at least as powerful by proving some common axioms of Hilbert-style systems in our system. In doing so, we derive some useful results about our proof system which we utilise in later sections.

4.2.1 | Defining a Tait-Style Sequent Calculus

From the foundations laid out in Section 4.1, we can now devise the proof system in which we will prove our unprovability results. We make use of Tait-style sequent calculus as in Buchholz and Wainer [1987], as this formalisation facilitates our analysis of provably computable functions.

In this formalisation, we explicitly define the negation of a formula $\neg\varphi$ as follows.

Definition 4.2.1 (negation normal form)**.** The negation $\neg\varphi$ of a formula φ is defined inductively by the following rules:

- $\neg(f(t_1,\ldots,t_n) = t_{n+1})$ denotes $f(t_1,\ldots,t_n) \neq t_{n+1}$.
- $\neg(\varphi \wedge \psi)$ denotes $\neg\varphi \vee \neg\psi$.
- $\neg(\varphi \vee \psi)$ denotes $\neg\varphi \wedge \neg\psi$.
- $\neg\forall x\varphi$ denotes $\exists x\neg\varphi$.
- $\neg\exists x\varphi$ denotes $\forall x\neg\varphi$.

If φ cannot be reduced further by the above rules, φ is said to be in *negation normal form*.

If two formulas have the same negation normal form, they are taken to be equivalent. We define a *sequent* to be a finite set of formulas, $\{\varphi_1,\ldots,\varphi_k\}$. From this point onward, we let Γ denote an arbitrary sequent, and Γ,φ denote the union $\Gamma\cup\{\varphi\}$. The intended meaning of $\Gamma = \{\varphi_1,\ldots,\varphi_k\}$ is the disjunction $\varphi_1 \vee \cdots \vee \varphi_k$.

A proof system is made up of logical axioms and rules of inference, to which the axioms of a specific theory may be added. Below we specify the logical axioms and rules of inference of our proof system, barring any mention of theory-specific axioms.

Definition 4.2.2 (Tait-style Proof System)**.** We define our Tait-style proof system by the logical axioms and inference rules as follows:

1. **Logical Axioms**: $\Gamma, \neg\varphi, \varphi$ is an axiom for every formula φ.

2. **Inferences Rules**:

$$(\wedge) \quad \frac{\Gamma, \varphi_0 \qquad \Gamma, \varphi_1}{\Gamma, \varphi_0 \wedge \varphi_1} \qquad\qquad (\vee) \quad \frac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \vee \varphi_1} \quad i = 0 \text{ or } i = 1$$

$$(\forall) \quad \frac{\Gamma, \varphi(x)}{\Gamma, \forall x \varphi(x)} \quad x \text{ not free in } \Gamma \qquad (\exists) \quad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x \varphi(x)} \quad \varphi[x/t] \text{ admissible}$$

$$(\text{Cut}) \quad \frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}$$

Definition 4.2.3. A sequent Γ has a *derivation* if there exists a finite tree built according to the rules of inference and axioms of our Tait-style proof system, such that the leaves of the tree are the axioms and the root is the sequent. If there exists a derivation of Γ, then Γ referred to as a *theorem*. If Γ is the singleton $\{\varphi\}$, we simply say φ is a theorem.

It is helpful for us to define some terminology to discuss an application of a rule of inference. We will refer to the upper sequents as the *premises* of the rule, and the lower sequent as the *conclusion*. Moreover, the displayed formula in the conclusion is referred to as the *principal* formula, and the displayed formulas in the premise are referred to as *minor* formulas. The remaining formulas are referred to as *side* formulas.

To give a concrete example, if we consider the $\wedge$-rule, Γ, φ_0 and Γ, φ_1 are premises and $\Gamma, \varphi_0 \wedge \varphi_1$ is the conclusion. The principal formula is $\varphi_0 \wedge \varphi_1$ and φ_0, φ_1 are minor formulas. All occurrences of Γ appear as side formulas.

4.2.2 | Comparison to Hilbert-Style Proof Systems

The limited choice of axioms is purposeful as it makes proving things about theorems easier. This can be contrasted with the way a Hilbert-style proof system is typically formulated, with a larger set of axioms and limited inference rules. We will consider some of the axioms and rules most commonly used in Hilbert-style proof systems to make a connection between our system and these alternate systems.

We will only consider one rule of inference — the Modus Ponens rule, which is found in most Hilbert-style systems. This rule will be shown to be a consequence of our Cut-rule. Another common inference rule used to reason about predicates is the Generalization rule, however we will not consider this as it is equivalent to the $(\forall)$-rule. We will also show how three axioms of propositional logic often present in these alternate systems are theorems in the Tait-style system.

Definition 4.2.4 (Modus Ponens)**.** From φ and $\varphi \to \psi$, infer ψ. Equivalently, written in sequent calculus:

$$\text{(MP)} \quad \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Definition 4.2.5 (Hilbert-Style Axioms)**.** For any formulas φ, ψ, θ

(H1) $\varphi \to (\psi \to \varphi)$

(H2) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

(H3) $\varphi \to (\psi \to \theta) \to ((\varphi \to \psi) \to (\varphi \to \theta))$

To prove that the above hold in our Tait-style proof system, we first prove some basic properties that hold about our system. We note that these properties hold regardless of the axioms added for a specific theory, so in particular they hold in Peano Arithmetic as well.

Lemma 4.2.6 (Weakening)**.** *If* Γ *is a theorem and* $\Gamma \subseteq \Gamma'$ *then* Γ' *is a theorem.*

Proof. Since an arbitrary Γ is present in each axiom, we can simply take this to be Γ' in the proof of Γ. Since our inference rules are applied to formulas in Γ, the resulting proof is a proof for Γ', Γ, which is equal to Γ'.

The only possible complication involves the $(\forall)$-rule, as this restricts x to not be free in Γ', however since we are working up to alpha-equivalence we can apply the rule using a fresh variable y to get $\forall y.\varphi(y)$, then rename the bound variable to x. □

Since weakening of premises is allowed, we can take advantage of this and use the alternative forms

$$(\wedge)\ \frac{\Gamma_0, \varphi_0 \qquad \Gamma_1, \varphi_1}{\Gamma_0, \Gamma_1, \varphi_0 \wedge \varphi_1} \quad \text{and} \quad (\text{Cut})\ \frac{\Gamma_0, \varphi_0 \qquad \Gamma_1, \neg\varphi_0}{\Gamma_0, \Gamma_1}$$

of the conjunction and cut rules. This is done to maintain readability of large sequent proofs, and make it clearer which premises are being used in the derivation.

Note how in Definition 4.2.3 we generate theorems inductively, as a theorem is generated by rules of inference from axioms. This allows us to prove properties of theorems inductively. We first prove the property holds for every axiom. Then, we prove that if the property holds for the premises of an inference rule, it must hold for the conclusion. We illustrate an example of such a proof below.

Lemma 4.2.7. *If $\Gamma, \varphi_0 \vee \varphi_1$ is a theorem, then $\Gamma, \varphi_0, \varphi_1$ is a theorem.*

Proof. We prove this result by induction on the generation of theorems. We first consider the case when $\Gamma, \varphi_0 \vee \varphi_1$ is an axiom. This only occurs if Γ contains an axiom, hence $\Gamma, \varphi_0, \varphi_1$ holds trivially.

If $\Gamma, \varphi_0 \vee \varphi_1$ was obtained using the Cut-rule, written in sequent calculus as

$$\frac{\Gamma, \varphi_0 \vee \varphi_1, \theta \qquad \Gamma, \varphi_0 \vee \varphi_1, \neg\theta}{\Gamma, \varphi_0 \vee \varphi_1}\ (\text{Cut})$$

then the inductive hypothesis holds for the premises $\Gamma, \varphi_0 \vee \varphi_1, \theta$ and $\Gamma, \varphi_0 \vee \varphi_1, \neg\theta$, hence $\Gamma, \varphi_0, \varphi_1, \theta$ and $\Gamma, \varphi_0, \varphi_1, \neg\theta$ are theorems. Using this, we can use the Cut-rule again to obtain the required result.

$$\frac{\Gamma, \varphi_0, \varphi_1, \theta \qquad \Gamma, \varphi_0, \varphi_1, \neg\theta}{\Gamma, \varphi_0, \varphi_1}\ (\text{Cut})$$

If $\Gamma, \varphi_0 \vee \varphi_1$ was obtained by the $(\wedge)$-rule, then the resulting conjunction must be contained in Γ, hence $\Gamma = \Gamma', \theta_0 \wedge \theta_1$, and was obtained by

$$\frac{\Gamma', \theta_0, \varphi_0 \vee \varphi_1 \qquad \Gamma', \theta_0, \varphi_1 \vee \varphi_1}{\Gamma', \theta_0 \wedge \theta_1, \varphi_0 \vee \varphi_1}\ (\wedge)$$

The premise $\Gamma', \theta_i, \varphi_0 \vee \varphi_1$ for $i \in \{0, 1\}$ is a theorem, hence we can apply the inductive hypothesis to deduce that $\Gamma', \theta_i, \varphi_0, \varphi_1$ is a theorem. Hence we can use the $(\wedge)$-rule again to obtain

$$\frac{\Gamma', \theta_0, \varphi_0, \varphi_1 \qquad \Gamma', \theta_0, \varphi_1, \varphi_1}{\Gamma', \theta_0 \wedge \theta_1, \varphi_0, \varphi_1}\ (\wedge)$$

where $\Gamma', \theta_0 \wedge \theta_1, \varphi_0, \varphi_1 = \Gamma, \varphi_0, \varphi_1$ is a theorem, as required.

The case where $\Gamma, \varphi_0 \vee \varphi_1$ is derived by the $(\forall)$ and $(\exists)$ rules is similar to the above, as since $\varphi_0 \vee \varphi_1$ does not match the principal formula in these rules, the inference rule is applied to some formula in Γ, meaning we can apply the inductive hypothesis to the premises and reapply the inference rule to obtain our desired result. Similarly if our formula is derived by the $(\vee)$-rule applied to some formula in Γ, we can obtain our result using the same process.

The final case is when $\Gamma, \varphi_0 \vee \varphi_1$ is derived by the $(\vee)$-rule to obtain $\varphi_0 \vee \varphi_1$, that is

$$\frac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \vee \varphi_i}\ (\vee)$$

In this case, we have that Γ, φ_i is a theorem, so by Weakening, $\Gamma, \varphi_0, \varphi_1$ is also a theorem as required.

$\square$

These results allow us to prove that Modus Ponens indeed holds for theorems.

Lemma 4.2.8 (Modus Ponens)**.** *If φ and $\varphi \to \psi$ are theorems, then ψ is a theorem.*

Proof. Note that $\varphi \to \psi$ is simply an abbreviation of $\neg\varphi \vee \psi$, and hence by Lemma 4.2.7, $\neg\varphi, \psi$ is a theorem. By Weakening, since φ is a theorem, so is φ, ψ hence we can simply apply the Cut-rule to obtain ψ.

$$\frac{\psi, \varphi \qquad \psi, \neg\varphi}{\psi}\ (\mathrm{Cut})$$

$\square$

To prove the Hilbert-style axioms, we make use of some well known facts about logical connectives in order to make the proof of the second axiom more concise.

Proposition 4.2.9 (Commutativity)**.** *For any formulas φ, ψ*

(i) $\varphi \vee \psi$ *is a theorem if and only if* $\psi \vee \varphi$ *is a theorem.*

(ii) $\varphi \wedge \psi$ *is a theorem if and only if* $\psi \wedge \varphi$ *is a theorem.*

Proof. The proofs for these properties are similar, so we will only present the proof for the first direction of *(i)*. That is, we will prove that if $\varphi \vee \psi$ is a theorem then $\psi \vee \varphi$ is a theorem. By Lemma 4.2.8, it is enough to show that $\varphi \vee \psi \rightarrow \psi \vee \varphi$ is a theorem.

This expression is equivalent to $\neg(\varphi \vee \psi) \vee (\psi \vee \varphi)$ which can be further written as $(\neg\varphi \wedge \neg\psi) \vee (\psi \vee \varphi)$ in negation normal form. We can derive this from axioms as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi, \neg\varphi \qquad \psi, \neg\psi}{\varphi, \psi, \neg\varphi \wedge \neg\psi}\,(\wedge)}{\varphi \vee \psi, \psi, \neg\varphi \wedge \neg\psi}\,(\vee)}{\varphi \vee \psi, \neg\varphi \wedge \neg\psi}\,(\vee)}{(\varphi \vee \psi) \vee (\neg\varphi \wedge \neg\psi), \neg\varphi \wedge \neg\psi}\,(\vee)}{(\varphi \vee \psi) \vee (\neg\varphi \wedge \neg\psi)}\,(\vee)$$

□

Note here how we take φ, φ to be equivalent to φ, since this notation represents the set $\{\varphi\} \cup \{\varphi\}$ which is simply $\{\varphi\}$. This allows us to reduce various premises into one conclusion using the $(\vee)$-rule. In the above proof, we show each step of the proof for demonstration purposes, but from now onward we will group steps together so that proofs do not become too cumbersome.

Proposition 4.2.10 (Associativity)**.** *For any formulas φ, ψ, θ*

(i) $(\varphi \vee \psi) \vee \theta$ *is a theorem if and only if* $\varphi \vee (\psi \vee \theta)$ *is a theorem.*

(ii) $(\varphi \wedge \psi) \wedge \theta$ *is a theorem if and only if* $\varphi \wedge (\psi \wedge \theta)$ *is a theorem.*

Proof. As before, we will only prove one direction in one case, as the remaining cases can be proven similarly. We prove the first direction of *(i)*, that is if $(\varphi \vee \psi) \vee \theta$ is a theorem then $\varphi \vee (\psi \vee \theta)$ is a theorem. By Lemma 4.2.8, it suffices to show that $(\varphi \vee \psi) \vee \theta \rightarrow \varphi \vee (\psi \vee \theta)$ holds, which is equivalent to proving $((\neg\varphi \wedge \neg\psi) \wedge \neg\theta) \vee (\varphi \vee (\psi \vee \theta))$. We illustrate a proof below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi,\varphi \qquad \neg\psi,\psi}{\varphi,\psi,\neg\varphi\wedge\neg\psi}(\wedge) \qquad \neg\theta,\theta}{\varphi,\psi,\theta,(\neg\varphi\wedge\neg\psi)\wedge\neg\theta}(\wedge)}{\varphi\vee(\psi\vee\theta),(\neg\varphi\wedge\neg\psi)\wedge\neg\theta}(\vee)\text{ applied 5 times}}{((\neg\varphi\wedge\neg\psi)\wedge\neg\theta)\vee(\varphi\vee(\psi\vee\theta))}(\vee)\text{ applied twice}$$

□

Equipped with these results, we can now prove that the Hilbert-style axioms are indeed theorems.

Proposition 4.2.11 (Hilbert-Style Axioms)**.** *For any formulas φ,ψ,θ, the following formulas are theorems.*

(H1) $\varphi \to (\psi \to \varphi)$.

(H2) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

(H3) $\varphi \to (\psi \to \theta) \to (\varphi \to \psi \to (\varphi \to \theta))$.

Proof. To prove *(H1)*, we prove the equivalent formula $\neg\varphi \vee (\neg\psi \vee \varphi)$. This is done trivially, by applying the $(\vee)$-rule consecutively from the axiom $\neg\psi,\varphi,\neg\varphi$ in the order required.

The proof for *(H2)* is similarly straightforward, and is done by proving the equivalent formula $(\neg\varphi \wedge \psi) \vee (\neg\psi \vee \varphi)$. The proof is illustrated below.

$$\dfrac{\dfrac{\dfrac{\varphi,\neg\varphi \qquad \psi,\neg\psi}{\varphi,\neg\psi,\neg\varphi\wedge\psi}(\wedge)}{\neg\psi\vee\psi,\neg\varphi\wedge\psi}(\vee)\text{ applied twice}}{(\neg\varphi\wedge\psi)\vee(\neg\psi\vee\psi)}(\vee)\text{ applied twice}$$

To prove *(H3)*, we prove the equivalent formula

$$\neg(\neg\varphi \vee (\neg\psi \vee \theta)) \vee (\neg(\neg\varphi \vee \psi) \vee (\neg\varphi \vee \theta))$$

which is not reduced completely to negation normal form in order to shorten the length of the proof. We have that $\neg(\neg\varphi \vee (\neg\psi \vee \theta)),(\neg\varphi \vee (\neg\psi \vee \theta))$ is an axiom, hence a theorem, so by Lemma 4.2.7 $\neg(\neg\varphi \vee (\neg\psi \vee \theta)),\neg\varphi,\neg\psi,\theta$ is a theorem. Using the same reasoning, we have that $\neg(\neg\varphi \vee \psi),\neg\varphi,\psi$ is a theorem. Hence we can apply the Cut-rule on ψ and the $(\vee)$ rule to these theorems to obtain our result, as illustrated below by the truncated tree of the proof starting from the two theorems just proven.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\neg\varphi\vee(\neg\psi\vee\theta)),\neg\varphi,\neg\psi,\theta \qquad \neg(\neg\varphi\vee\psi),\neg\varphi,\psi}{\neg(\neg\varphi\vee(\neg\psi\vee\theta)),\neg\varphi,\theta,\neg(\neg\varphi\vee\psi)}\,(\mathrm{Cut})}{\neg(\neg\varphi\vee(\neg\psi\vee\theta)),\neg\varphi\vee\theta,\neg(\neg\varphi\vee\psi)}\,(\vee)\text{ twice}}{\neg(\neg\varphi\vee(\neg\psi\vee\theta)),\neg(\neg\varphi\vee\psi)\vee(\neg\varphi\vee\theta)}\,(\vee)\text{ twice}}{\neg(\neg\varphi\vee(\neg\psi\vee\theta))\vee(\neg(\neg\varphi\vee\psi)\vee(\neg\varphi\vee\theta))}\,(\vee)\text{ twice}$$

□

4.3 | Peano Arithmetic

Now that we have defined all the machinery required, we can start working in Peano Arithmetic. We first define the system of Peano Arithmetic in Section 4.3.1, by defining the language and adding the axioms specific to PA. In Section 4.3.2, we further define an infinitary system PA_∞, which allows for an infinite number of premises, and show that we can embed PA in this system.

While derivations in PA_∞ are no longer finite trees which makes proofs more unwieldy, this infinitary system obeys some desirable properties, as we show in Section 4.3.3. Namely, any theorem of PA_∞ can be derived using only atomic formulas as the minor formulas in any instance of the Cut-rule. This process is referred to as Cut-elimination.

From this result, we continue in Section 4.3.4 to obtain bounding results for statements which are provable in PA_∞. This culminates in the result stating that each provably computable function in PA must necessarily be dominated by some F_α where $\alpha < \varepsilon_0$. This is exactly the result we need, as putting this together with results from Chapter 3, we obtain the fact that Goodstein's function cannot be provably computable in PA.

4.3.1 | Defining Peano Arithmetic

We now formally define our system of Peano Arithmetic as a first order logic in a Tait-style proof system. We begin first by defining the alphabet of Peano

Arithmetic, hence the terms and formulas, by specifying its non-logical symbols.

Definition 4.3.1. The first order alphabet of Peano Arithmetic contains the constant symbol 0, the relation symbol $\leq$, and a symbol for each elementary function $f \in \mathcal{E}$.

Some functions in $\mathcal{E}$ which we will specifically refer to include:

- The unary successor function S, denoting $S(x) = x + 1$.
- The binary addition and multiplication functions $+$, $\cdot$ respectively, which we typically write in infix notation.
- The binary exponential function $\exp(x, y)$ denoted as x^y.

We will refer to a term in the form $SS \dots S0$ with the successor function S applied $n \geq 0$ times as the numeral n. A direct analogy can be done here as to how we defined the natural numbers in Chapter 2, and indeed PA was defined as an axiomatic system for the natural numbers.

Note that our inclusion of all the elementary functions as function symbols is a deliberate choice. In a minimal axiomatisation, only the successor function is defined, as from this we can prove all statements about elementary functions, however this approach requires using quantifiers to represent certain equations involving elementary functions. Defining these functions in the language explicitly means formulas in the form $f(\mathbf{x}) = y$ and $f(\mathbf{x}) \neq y$, where $f \in \mathcal{E}$ can be atomic formulas, hence treated as the basic building blocks of the language. This treatment is necessary for certain proceeding arguments to work.

To give meaning to the alphabet defined for PA, we define axioms which are specific to number theory. Together with the axioms and inference rules of our proof system, these allow us to reason about whether formulas are provable in our system. Note that to characterise the elementary functions we make use of the alternate characterisation in Lemma 3.3.10.

Definition 4.3.2. The axioms of Peano Arithmetic are of three kinds

1. **Logical Axioms:** $\Gamma, \neg\varphi, \varphi$ for every formula φ.

2. **Elementary Axioms:** For any substitutions for x, y, z, the following hold

$(=)$	$\Gamma, x = x$	$\Gamma, x \neq y, y = x$	$\Gamma, x \neq y, y \neq z, x = z$
$(\leq)$	$\Gamma, 0 \leq x$	$\Gamma, x \neq y, x \leq y$	$\Gamma, x > y, x \leq S(y)$
	$\Gamma, x \leq y, y \leq x$	$\Gamma, x > y, y > z, x \leq z$	$\Gamma, x > y, y > x, x = y$
	$\Gamma, x > S(y), x \leq y, x = S(y)$		
(S)	$\Gamma, Sx \neq 0$	$\Gamma, Sx \neq Sy, x = y$	$\Gamma, x \neq y, Sx = Sy$
$(+)$	$\Gamma, x + 0 = x$		
	$\Gamma, x + y \neq z, x + Sy = Sz$		
	$\Gamma, x \neq x', y \neq y', z \neq z', x + y \neq z, x' + y' = z'$		
	$\Gamma, x + y \neq z, x + y \neq z', z = z'$		
$(\cdot)$	$\Gamma, x \cdot 0 = 0$		
	$\Gamma, x \cdot S(y) = (x \cdot y) + x$		
(exp)	$\Gamma, x^0 = S(0)$		
	$\Gamma, x^{S(y)} = x \cdot x^y$		
(LR)	If f is defined by limited recursion from h_1, h_2, h_3		
	$\Gamma, f(0, \mathbf{u}) = h_1(\mathbf{u})$		
	$\Gamma, f(y+1, \mathbf{u}) = h_2(y, \mathbf{u}, f(y, \mathbf{u}))$		
	$\Gamma, f(y+1, \mathbf{u}) \leq h_3(\mathbf{u})$		

3. **Induction Axioms:** $\Gamma, \neg\varphi(0), \exists x(\varphi(x) \wedge \neg\varphi(Sx)), \forall x\varphi(x)$ for every formula φ.

The logical rules of inference for Peano Arithmetic are of five kinds:

$(\wedge)$ $\dfrac{\Gamma, \varphi_0 \qquad \Gamma, \varphi_1}{\Gamma, \varphi_0 \wedge \varphi_1}$ $\qquad$ $(\vee)$ $\dfrac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \vee \varphi_1}$ $i = 0$ or $i = 1$

$(\forall)$ $\dfrac{\Gamma, \varphi(x)}{\Gamma, \forall x\varphi(x)}$ x not free in Γ $\qquad$ $(\exists)$ $\dfrac{\Gamma, \varphi(t)}{\Gamma, \exists x\varphi(x)}$ $\varphi[x/t]$ admissible

(Cut) $\dfrac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}$

Definition 4.3.3. A sequent Γ has a *derivation in PA* if there exists a finite tree built according to the rules of inference and axioms of Peano Arithmetic, such that the leaves of the tree are the axioms and the root is the sequent. If there exists a derivation of Γ, then Γ referred to as a *theorem of PA*. If Γ is the singleton $\{\varphi\}$, we simply say φ is a theorem of PA.

Note that since the rules of inference and axioms of the Tait-style proof system as defined in Definition 4.2.2 are contained in the rules of inference and axioms of PA, a theorem is by definition a theorem of PA. The converse does not hold, as the elementary axioms allow us to prove more statements.

Example 4.3.4. We illustrate some of the Peano Arithmetic specific rules in action by proving that $\forall x(0+x=x)$ is a theorem of PA. This derivation makes use of the induction axiom as well as some facts about addition, and the alternate form of the Cut-rule.

$$\dfrac{\dfrac{\dfrac{0+x\neq x, 0+Sx=Sx}{0+x\neq x\vee 0+Sx=Sx}\ (\vee)}{\forall x(\neg\varphi(x)\vee\varphi(Sx))}\ (\forall) \qquad \dfrac{0+0=0 \qquad 0+0\neq 0, \exists x(\varphi(x)\wedge\neg\varphi(Sx)), \forall x\varphi(x)}{\exists x(\varphi(x)\wedge\neg\varphi(Sx)), \forall x(0+x=x)}\ (\text{Cut})}{\forall x(0+x=x)}\ (\text{Cut})$$

Definition 4.3.5. A function f is said to be *provably computable (total)* in PA if there are two elementary functions V and T such that

(i) $f(x_1,\ldots,x_k)=V(\mu y[T(x_1,\ldots,x_k,y)=0])$.

(ii) $\forall x_1\ldots\forall x_k\exists y(T(x_1,\ldots,x_k,y)=0)$ is a theorem PA.

Note that by Theorem 3.3.16, condition (i) holds for any computable function. Condition (ii) is the real restriction, asserting that the fact that there always exists a y such that $T(x_1,\ldots,x_k,y)=0$ is a verifiable statement in PA. This definition corresponds to being *provably total* in PA, as if there always exists some y such that $T(x_1,\ldots,x_k,y)=0$, then $V(\mu y[T(x_1,\ldots,x_k,y)=0])$ must be defined for every $x_1,\ldots,x_k$, meaning that f is total.

4.3.2 | Embedding PA in Infinitary System of Arithmetic

In this section, we further define an infinitary system of arithmetic, which we will refer to as PA_∞, in which we will embed PA. This system replaces the induction axiom and the $(\forall)$-rule by the (ω)-rule,

$$\frac{\Gamma, \varphi(0) \qquad \Gamma, \varphi(1) \qquad \Gamma, \varphi(2) \qquad \Gamma, \varphi(3) \qquad \ldots}{\Gamma, \forall x \varphi(x)}$$

in order to be able to perform Cut-elimination. This proof theoretic method, coined by Gentzen [1964], allows us to remove all but the most trivial instances of the Cut-rule. This is done to be able to bound the length of our proofs, which allows us to reach our provability results, following closely the methods in Buchholz and Wainer [1987].

Due to the $(\forall)$-rule being eliminated from PA_∞, we are able to eliminate free variables in the proof of a sentence, as we can now replace the free variable with infinitely many premises. Hence in PA_∞ we can safely restrict the terms considered to numerals.

We also add the unary relation $x \in N$ to the language of PA_∞. The intended interpretation of this relation is that x is a natural number, but further on we will restrict this N to be a bounded set. This will allow us to later consider formulas that are true where N is taken to be some bounded set, thereby essentially bounding the variables and numerals that appear in these formulas.

Definition 4.3.6. We define the relation $\vdash^{\alpha} \Gamma$, meaning Γ *is derivable in* PA_∞ with the ordinal bound $\alpha < \varepsilon_0$ inductively using the following axioms:

$\vdash^{\alpha} \Gamma, \varphi$ whenever φ is a true atomic sentence of PA.

$\vdash^{\alpha} \Gamma, 0 \in N$.

$\vdash^{\alpha} \Gamma, n \in N, n \notin N$ for any n numeral.

and the following inference rules:

(N) If $\vdash^{\alpha} \Gamma, n \in N$ then $\vdash^{\alpha+1} \Gamma, S(n) \in N$.

$(\vee)$ If $\vdash^{\alpha} \Gamma, \varphi_i$ $(i = 0$ or $i = 1)$ then $\vdash^{\alpha+1} \Gamma, \varphi_0 \vee \varphi_1$.

($\wedge$) If $\vdash^{\alpha} \Gamma, \varphi_i$ ($i = 0$ and $i = 1$) then $\vdash^{\alpha+1} \Gamma, \varphi_0 \wedge \varphi_1$.

($\exists$) If $\vdash^{\alpha} \Gamma, \varphi(n)$ (for some n) then $\vdash^{\alpha+1} \Gamma, \exists x \varphi(x)$.

(ω) If $\vdash^{\alpha} \Gamma, \varphi(n)$ (for every n) then $\vdash^{\alpha+1} \Gamma, \forall x \varphi(x)$.

(Cut) If $\vdash^{\alpha} \Gamma, \varphi$ and $\vdash^{\alpha} \Gamma, \neg\varphi$ then $\vdash^{\alpha+1} \Gamma$.

(Accumulation) If $\vdash^{\alpha} \Gamma$ and $\beta \xrightarrow{k} \alpha$ where $k = \max\{3\} \cup \{3n+1 : n \notin N \text{ is in } \Gamma\}$ then $\vdash^{\beta} \Gamma$.

Although we define these rules using the $\vdash^{\alpha}$ notation, we may write things down using the same proof tree notation as Section 4.3.1 when it is clear we are working in PA_∞. The only difference is we may now have an infinite number of premises, as in the (ω)-rule. Consequentially, a proof in PA_∞ is a well-founded infinitely branching tree.

In order to embed PA in our infinitary system, we first define φ^N. This modified formula ensures that our formula is true for numerals such that $n \in N$ holds.

Definition 4.3.7. For each formula φ of PA, let φ^N be the result of replacing any instance of $\exists x(\psi)$ with $\exists x(x \in N \wedge \psi^N)$ and any instance of $\forall x(\psi)$ with $\forall x(x \notin N \vee \psi^N)$. For a sequent $\Gamma = \{\varphi_1, \ldots, \varphi_k\}$ we define Γ^N to be $\{\varphi_1^N, \ldots, \varphi_k^N\}$.

In order to prove the Embedding Lemma more easily, we first prove the Weakening property for our infinitary system as done before for PA.

Lemma 4.3.8 (Weakening). *If* $\Gamma \subseteq \Gamma'$ *and* $\vdash^{\alpha} \Gamma$, *then* $\vdash^{\alpha} \Gamma'$

Proof. This follows similarly to the proof of Weakening of PA, where we can simply add the extra formulas in $\Gamma' \backslash \Gamma$ in the axioms. The fact that the proof is still valid for the Accumulation rule follows from Corollary 3.1.14 as if $k' = \max\{3\} \cup \{3n+1 : n \notin N \text{ is in } \Gamma'\}$ and $k = \max\{3\} \cup \{3n+1 : n \notin N \text{ is in } \Gamma\}$ then $k' \geq k$, so if $\beta \xrightarrow{k} \alpha$ then $\beta \xrightarrow{k'} \alpha$. Hence we can apply the Accumulation rule in the same way as in the original derivation. The fact that the proof remains valid for the other rules is trivial from the definition. □

As before, this allows us to use the alternate versions for the ($\vee$) and (Cut) rules.

Lemma 4.3.9 (Embedding). *If Γ is a theorem of PA containing free variables $x_1, \ldots, x_r$ then there is an ordinal $\alpha = w\ell$ for some integer ℓ, such that for all numerals $n_1, \ldots, n_r$,*

$$\mid\!\frac{\alpha}{\quad} n_1 \notin N, \ldots, n_r \notin N, \Gamma^N(n_1, \ldots, n_r)$$

where $\Gamma^N(n_1, \ldots, n_r)$ is the substitution $\Gamma^N[x_1/n_1, \ldots, x_r/n_r]$.

Proof. We will prove this result by induction over the generation of theorems in PA.

If Γ is a logical axiom, then $\neg\theta, \theta$ is contained in Γ. Clearly, $(\neg\theta)^N = \neg(\theta^N)$, so we can take $\varphi = \theta^N$, meaning it suffices to prove $\mid\!\frac{\alpha}{\quad} \neg\varphi, \varphi$, noting that any $n \notin N$ can be added by Weakening. We note how φ is now a formula in the language of PA together with the relation $x \in N$. Hence, it is sufficient to prove that for every such formula φ, $\mid\!\frac{\alpha}{\quad} \neg\varphi, \varphi$ for $\alpha = \omega\ell$ where $\ell = \text{len}(\varphi)$. We prove this last statement by numerical induction on $\text{len}(\varphi)$.

If $\text{len}(\varphi) = 0$ then either φ is a true atomic sentence or φ is the formula $n \in N$. In both cases, $\neg\varphi, \varphi$ is an axiom hence $\mid\!\frac{0}{\quad} \neg\varphi, \varphi$.

Suppose $\varphi = \psi_0 \wedge \psi_1$ and $\ell = \text{len}(\varphi)$. Then, since $\text{len}(\psi_i) < \text{len}(\psi_0 \wedge \psi_1)$ by our inductive hypothesis, $\mid\!\frac{\omega\ell_i}{\quad} \neg\psi_i, \psi_i$ for $\ell_i = \text{len}(\psi_i) < \ell$. Hence $\mid\!\frac{\omega(\ell-1)}{\qquad} \neg\psi_i, \psi_i$, as if $\ell_i < \ell - 1$, we can use the Accumulation rule by Corollary 3.1.10. Hence in two steps we can obtain the desired formula:

$$\cfrac{\cfrac{\cfrac{\omega(\ell-1)\ \cfrac{\vdots}{\neg\psi_0, \psi_0} \qquad \omega(\ell-1)\ \cfrac{\vdots}{\neg\psi_1, \psi_1}}{\neg\psi_0, \neg\psi_1, \psi_0 \wedge \psi_1}\ (\wedge)\ \omega(\ell-1)+1}{\neg\psi_0 \vee \neg\psi_1, \neg\psi_1, \psi_0 \wedge \psi_1}\ (\vee)\ \omega(\ell-1)+2}{\neg\psi_0 \vee \neg\psi_1, \psi_0 \wedge \psi_1}\ (\vee)\ \omega(\ell-1)+3$$

Note how $\neg\psi_0 \vee \neg\psi_1$ is equal to $\neg(\psi_0 \wedge \psi_1)$, meaning the derived sequent is the one required. Since $\omega(\ell-1)+\omega \xrightarrow{3} \omega(\ell-1)+3$ and by definition of k, $k \geq 3$, by Corollary 3.1.14 we can apply the Accumulation rule to obtain $\mid\!\frac{\omega\ell}{\quad} \neg\psi_0 \vee \neg\psi_1, \psi_0 \wedge \psi_1$ as desired. The proof for Γ derived by the $(\vee)$-rule follows similarly.

Suppose $\varphi = \exists x \psi(x)$ and $\ell = \text{len}(\varphi)$. Then $\text{len}(\psi(n)) = \ell - 1$, so by our inductive hypothesis, for every numeral n, $\mid\!\frac{\omega(\ell-1)}{\qquad} \neg\psi(n), \psi(n)$. Hence by the $(\exists)$-rule we have that $\mid\!\frac{\omega(\ell-1)+1}{\qquad} \neg\psi(n), \exists x\psi(x)$ for every n. We can now apply the (ω)-rule to deduce $\mid\!\frac{\omega(\ell-1)+2}{\qquad} \forall x \neg\psi(x), \exists x\psi(x)$ and hence we can apply the Accumulation rule as before to deduce $\vdash^{\omega\ell} \forall x \neg\psi(x), \exists x\psi(x)$. The proof for Γ derived by the ($\forall$)-rule follows similarly.

If Γ is an elementary axiom, substituting each free variable in the principal formula results in an atomic sentence, hence our result follows trivially by the axioms of PA_∞.

If our sequent is the induction axiom $\Gamma, \neg\varphi(0), \exists x(\varphi(x) \wedge \neg\varphi(Sx)), \forall x\varphi(x)$ by Weakening and deleting any free variables occurring in Γ, it suffices to show $\vdash^{\omega k} \neg\varphi^N(0), \exists x(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx))$ for a natural number k. We do this by proving that for $\ell = \mathrm{len}(\varphi^N(x))$

$$\vdash^{\omega\ell+3n} \neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \varphi^N(n)$$

for every n. This gives us the required result, as by Weakening and Accumulation we have

$$\vdash^{\omega(\ell+1)} \neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), n \notin N, \varphi^N(n)$$

hence we can obtain our result by the following applications:

$$\begin{array}{rcl}
 & \vdots & \\
\omega(\ell+1) & \overline{\neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), n \notin N, \varphi^N(n)} & \\
\omega(\ell+1)+2 & \overline{\neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), n \notin N \vee \varphi^N(n)} & (\vee)\text{ twice} \\
\omega(\ell+1)+3 & \overline{\neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \forall x\left(x \notin N \vee \varphi^N(x)\right)} & (\omega) \\
\omega(\ell+2) & \overline{\neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \forall x\left(x \notin N \vee \varphi^N(x)\right)} & (\text{Accumulation})
\end{array}$$

Finally, we prove the required statement

$$\vdash^{\omega\ell+3n} \neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \varphi^N(n)$$

by induction on n. The case for $n = 0$ follows trivially as our sequent contains the formulas $\neg\varphi^N(0), \varphi^N(0)$ hence is true by the first case of this proof. We assume that our result holds for k, that is

$$\vdash^{\omega\ell+3k} \neg\varphi^N(0), \exists x\left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \varphi^N(k)$$

and prove it for $Sk = k+1$. Let $\psi = \exists x(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx))$. By the first case of this proof, we know that $\vdash^{\omega\ell} \neg\varphi^N(Sk), \varphi^N(Sk)$, where $\ell = \mathrm{len}(\varphi^N(x))$. By the accumulation rule, we further get $\vdash^{\omega\ell+3k} \neg\varphi^N(Sk), \varphi^N(Sk)$ hence, applying the alternate ($\wedge$)-rule with the inductive hypothesis we obtain

$$\vdash^{\omega\ell+3k+1} \neg\varphi^N(0), \psi, \varphi^N(k) \wedge \neg\varphi^N(Sk), \varphi^N(Sk).$$

We can also obtain $\vdash^{\omega\ell+3k+1} k \in N$ by applying the (N)-rule k times starting from $\vdash^{\omega\ell+3k+1-k} 0 \in N$. Applying the ($\wedge$) rule to our derived sequents gives us

$$\vdash^{\omega\ell+3k+2} \neg\varphi^N(0), \psi, k \in N \wedge \varphi^N(k) \wedge \neg\varphi^N(Sk), \varphi^N(Sk)$$

on which we can apply the ($\exists$)-rule to obtain

$$\vdash^{\omega\ell+3(k+1)} \neg\varphi^N(0), \psi, \exists x \left(x \in N \wedge \varphi^N(x) \wedge \neg\varphi^N(Sx)\right), \varphi^N(Sk).$$

The derived formula here is precisely ψ, hence we have obtained our desired result.

Suppose $\Gamma, \forall x\varphi(x)$ was derived from $\Gamma, \varphi(x)$ using the ($\forall$)-rule. Deleting mention of free variables occurring in Γ, we can assume that $\vdash^{\omega\ell} n \notin N, \Gamma^N, \varphi^N(n)$ hence applying the ($\vee$)-rule twice we have that $\vdash^{\omega\ell+2} \Gamma^N, n \notin N \vee \varphi^N(n)$. This holds for every n, hence we can apply the (ω)-rule and Accumulation rule to obtain our desired result $\vdash^{\omega(\ell+1)} \Gamma^N, \forall x(x \notin N \vee \varphi^N(x))$.

Suppose $\Gamma, \exists x\varphi(x)$ was derived in PA from $\Gamma, \varphi(t)$ using the ($\exists$)-rule. Then t is either a numeral $m = S^m(0)$, or $S^m(x)$ for some variable x, where S is applied m times. If $t = S^m(x)$ where x is not free in Γ then we can replace x with 0 in the proof of $\Gamma, \varphi(t)$ as $\Gamma[x/0] = \Gamma$, and since $\varphi(S^m(x))$ is proven for any x, it also holds for $x = 0$. Hence we only need to consider the case when $t = S^m(0)$ and $t = S^m(x)$ and x occurs free in Γ.

We delete any occurrence of other free variables in Γ, φ in order to be able to use the inductive hypothesis. In the first case, we have by the inductive hypothesis that $\vdash^{\omega\ell} \Gamma^N, \varphi^N(S^m(0))$, in particular by Weakening we have $\vdash^{\omega\ell} S^m(0) \notin N, \Gamma^N, \varphi^N(S^m(0))$. In the second case we have that $\vdash^{\omega\ell} n \notin N, \Gamma^N(n), \varphi^N(S^m(n))$ for every n. In both cases, we have that $\vdash^{\omega\ell} n \notin N, \Gamma^N(n), \varphi^N(S^m(n))$ for some n. By Corollary 3.1.10 we can apply Accumulation to obtain

$$\vdash^{\omega(\ell+m)} n \notin N, \Gamma^N(n), \varphi^N(S^m(n)).$$

We also have that $\left|\frac{\omega\ell}{}\right. n \notin N, n \in N$, from which we can obtain $\left|\frac{\omega(\ell+m)}{}\right. n \notin N, S^m(n) \in N$ by applying the (N)-rule and Accumulation repeatedly as below.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n \notin N, n \in N}{n \notin N, S(n) \in N}\ \omega\ell+1\ (N)}{n \notin N, S(n) \in N}\ \omega(\ell+1)\ (\text{Weakening})}{n \notin N, S^2(n) \in N}\ \omega(\ell+1)+1\ (N)}{\dfrac{n \notin N, S^2(n) \in N}{\vdots}}\ \omega(\ell+2)\ (\text{Weakening})}{n \notin N, S^m(n) \in N}\ \omega(\ell+m)
$$

Hence by the $(\wedge)$-rule $\left|\frac{\omega(\ell+m)+1}{}\right. n \notin N, \Gamma^N(n), S^m(n) \in N \wedge \varphi^N(S^m(n))$ and by the $(\exists)$-rule, $\left|\frac{\omega(\ell+m)+2}{}\right. n \notin N, \Gamma^N(n), \exists x(x \in N \wedge \varphi^N(x))$ hence by Accumulation our result holds with ordinal bound $\omega(\ell+m+1)$.

The rest of the cases, where Γ is derived by the $(\vee)$, $(\wedge)$ and Cut rules follow from the inductive hypothesis. In the Cut and $(\wedge)$ rules, we use Corollary 3.1.10 and the Accumulation rule to line up the two different premises. Then for all cases we simply reapply the rule and use Accumulation to get the desired ordinal bound.

□

4.3.3 | Cut-Elimination in PA_∞

With Peano Arithmetic embedded in our infinitary system, we can now begin building towards the main technique used to bound our proofs: Cut-elimination. To do this, we must first define the *cut-rank* of a proof, which bounds the length of formulas used in the Cut-rule.

Definition 4.3.10. A derivation of Γ in PA_∞ has *cut-rank* r if for every application of the Cut-rule with minor formulas φ and $\neg\varphi$, $\text{len}(\varphi) < r$. If Γ has a derivation with ordinal bound α and cut-rank $\leq r$, we denote this by $\left|\frac{\alpha}{r}\right. \Gamma$.

In this section we will prove various results about PA_∞, eventually culminating in the result that any Γ provable in PA_∞ has a 'cut free' derivation, that is a derivation with cut-rank 0.

Lemma 4.3.11 (Inversion)**.**

(i) *If* $\vdash_r^\alpha \Gamma, \varphi_0 \wedge \varphi_1$ *then* $\vdash_r^\alpha \Gamma, \varphi_0$ *and* $\vdash_r^\alpha \Gamma, \varphi_1$.

(ii) *If* $\vdash_r^\alpha \Gamma, \forall x \varphi(x)$ *then* $\vdash_r^\alpha \Gamma, \varphi(n)$ *for every* n.

Proof. The proof of this lemma is very similar to the proof of Lemma 4.2.7, by induction on the generation of derivations in PA_∞. The case where our formulas appear only as side formulas in the derivation follow exactly as before, where we use the inductive hypothesis on the premises used and can reapply the same rule with no issues. Hence we consider only the cases when our formulas appear as principal formulas.

For *(i)*, this means that $\vdash^\alpha \Gamma, \varphi_0 \wedge \varphi_1$ was derived using the $(\wedge)$ rule, that is $\vdash^\beta \Gamma, \varphi_i$ for every $i \in \{0,1\}$ and $\alpha = \beta + 1$. As $\alpha \xrightarrow{k} \beta$ for any $k \geq 3$, we can apply the Accumulation rule to conclude $\vdash^\alpha \Gamma, \varphi_i$ for $i \in \{0,1\}$.

In the case of *(ii)*, this means $\vdash^\alpha \Gamma, \forall x \varphi(x)$ was derived using the (ω)-rule, hence $\alpha = \beta + 1$ and $\vdash^\beta \Gamma, \varphi(n)$ holds for every n. Therefore we can use the Accumulation rule as before to conclude that $\vdash^\alpha \Gamma, \varphi(n)$ holds for every n. Note that in both of these cases, the cut-rank remains unchanged. □

Lemma 4.3.12 (Reduction Lemma)**.** *Suppose* $\vdash_r^\alpha \Gamma_0, \neg\varphi$ *where* $\omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_l} \cdot k_l$ *is the Cantor Normal Form of* α *and* φ *is of the form* $\exists x \psi(x)$ *or* $\psi_0 \vee \psi_1$ *and of length* $r+1$. *Then if* $\vdash_r^\beta \Gamma, \varphi$, *where* $\beta < \omega^{\beta_l + 1}$, *we have* $\vdash_r^{\alpha+\beta} \Gamma_0, \Gamma$.

Proof. We prove this by transfinite induction on β, considering the different cases by which $\vdash_r^\beta \Gamma, \varphi$ could be derived. The base case with $\beta = 0$, reduces to the case when our sequent is an axiom, as whenever we apply an inference rule, the ordinal bound increases.

The case where Γ, φ is an axiom is straightforward, as φ is in the form $\exists x \psi(x)$ or $\psi_0 \vee \psi_1$ hence does not pattern match the principal formulas of any axiom. This means that the principal formulas are contained entirely in Γ, making Γ an axiom, so our result follows trivially.

Consider the case when $\vdash_r^\beta \Gamma, \varphi$ was derived by the Accumulation rule, that is $\vdash_r^\delta \Gamma, \varphi$ for some δ such that $\beta \xrightarrow{k} \delta$ for $k = \max\{3\} \cup \{3n+1 : n \notin N \text{ is in } \Gamma \cup \{\varphi\}\}$.

Then we can apply our inductive hypothesis to conclude that $\vdash^{\alpha+\delta}_{r} \Gamma_0, \Gamma$. By Lemma 3.1.8, we have that $\alpha+\beta \xrightarrow{k} \alpha+\delta$. We also have that if $k' = \max\{3\} \cup \{3n+1 : n \notin N \text{ is in } \Gamma \cup \Gamma_0\}$, then $k' \geq k$. Hence we can apply Corollary 3.1.14 and the Accumulation rule to obtain our result.

Consider the case when $\vdash^{\beta}_{r} \Gamma, \varphi$ is derived using a rule other than Accumulation, where φ is not the principal formula. Suppose $\Gamma = \Gamma', \psi$ where ψ is the principal formula and our sequent is derived from premises $\vdash^{\delta}_{r} \Gamma', \psi_i, \varphi$ where $\beta = \delta + 1$. By the induction hypothesis, we have $\vdash^{\alpha+\delta}_{r} \Gamma_0, \Gamma', \psi_i$, and applying the respective rule we get that $\vdash^{\alpha+\beta}_{r} \Gamma_0, \Gamma', \psi$. Note that since $\vdash^{\beta}_{r} \Gamma, \varphi$ has cut-rank $\leq r$, if obtained by the Cut-rule ψ_i has length $\leq r$, hence the cut-rank is preserved.

The remaining cases are when φ is the principal formula, which means $\vdash^{\beta}_{r} \Gamma, \varphi$ was either derived by the $(\exists)$ or the $(\vee)$ rules. If derived by the $(\exists)$-rule we have that $\varphi = \exists x \psi(x)$ and $\vdash^{\delta}_{r} \Gamma, \psi(n)$ for a fixed n, $\beta = \delta + 1$. By Weakening, we have that $\vdash^{\delta}_{r} \Gamma, \psi(n), \varphi$. By our hypothesis, we also have that $\vdash^{\beta}_{r} \Gamma_0, \neg\varphi$, that is $\vdash^{\alpha}_{r} \Gamma, \forall x \neg\psi(x)$. By Lemma 4.3.11, this means that $\vdash^{\alpha}_{r} \Gamma_0, \neg\psi(n)$. We can align the ordinal bound here to $\alpha + \delta$ using Accumulation by Lemma 3.1.8. Hence we can apply the Cut-rule to the two sequents (noting that $\mathrm{len}(\psi) \leq \mathrm{len}(\varphi) - 1 = r$) to obtain our desired result.

$$
\dfrac{\alpha+\delta\ \dfrac{\vdots}{\Gamma, \psi(n)} \qquad \alpha+\delta\ \dfrac{\vdots}{\Gamma_0, \neg\psi(n)}}{\Gamma_0, \Gamma}\ \alpha+\beta \ \text{(Cut)}
$$

The case where $\vdash^{\beta}_{r} \Gamma, \varphi$ was derived using the $(\vee)$-rule follows similarly. Let $\varphi = \psi_0 \vee \psi_1$. We have that $\vdash^{\delta}_{r} \Gamma, \psi_i, \varphi$ for a fixed $i \in \{0, 1\}$, so we can conclude by the inductive hypothesis that $\vdash^{\alpha+\delta}_{r} \Gamma_0, \Gamma, \psi_i$. We also have that $\vdash^{\alpha}_{r} \Gamma_0, \neg\psi_0 \wedge \neg\psi_1$, hence by Lemma 4.3.11 $\vdash^{\alpha}_{r} \Gamma_0, \neg\psi_i$ holds. We again match up the ordinal bound to $\alpha+\delta$, and apply the Cut-rule for the matching i to obtain $\vdash^{\alpha+\beta}_{r} \Gamma_0, \Gamma$, noting that cut-rank remains $\leq r$ due to the length of ψ_i being bounded by r. $\square$

Theorem 4.3.13 (Cut-Elimination). *If $\vdash^{\alpha}_{r+1} \Gamma$, then $\vdash^{\omega^\alpha}_{r} \Gamma$. Therefore $\vdash^{\alpha^*}_{0} \Gamma$ where $\alpha^* = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}$ with ω exponentiated $r+1$ times.*

Proof. We prove this by induction on the derivation of $\vdash^{\alpha}_{r+1} \Gamma$. If derived by any rule other than the cut rule, the hypothesis follows immediately from the inductive hypothesis. We consider the case where the sequent is derived by the ($\vee$)-rule, noting that the rest of the cases are analogous. Let $\vdash^{\alpha}_{r+1} \Gamma, \varphi_0 \vee \varphi_1$ be derived by the ($\vee$)-rule from $\vdash^{\beta}_{r+1} \Gamma, \varphi_i$ for $i \in \{0,1\}$ and $\alpha = \beta + 1$. Then, by the inductive hypothesis we have that $\vdash^{\omega^\beta}_{r} \Gamma, \varphi_i$ hence applying the ($\vee$)-rule again we get $\vdash^{\omega^\beta+1}_{r} \Gamma, \varphi_0 \vee \varphi_1$.

Now, for $n \geq 2$, $\omega^\alpha = \omega^\beta \cdot \omega \underset{n}{\rightarrow} \omega^\beta \cdot n \underset{n}{\rightarrow} \omega^\beta \cdot 2$, by Corollary 3.1.10. By Lemma 3.1.9 and Lemma 3.1.12 we can conclude that $\omega^\beta \underset{n}{\rightarrow} \omega^0 = 1$, and hence $\omega \cdot 2 = \omega^\beta + \omega^\beta \underset{n}{\rightarrow} \omega^\beta + 1$. As we have proved $\omega^\alpha \underset{n}{\rightarrow} \omega^\beta + 1$ for all $n \geq 2$, we can use the Accumulation rule to obtain $\vdash^{\omega^\alpha}_{r} \Gamma, \varphi_0 \vee \varphi_1$ as desired.

Consider the remaining case, where $\vdash^{\alpha}_{r+1} \Gamma$ is derived by the Cut-rule from $\vdash^{\beta}_{r+1} \Gamma, \varphi$ and $\vdash^{\beta}_{r+1} \Gamma, \neg\varphi$, where $\alpha = \beta + 1$. Note that $\text{len}(\varphi) \leq r + 1$ because of the cut-rank. If $\text{len}(\varphi) < r + 1$ then we can continue as in the first case, as by the inductive hypothesis $\vdash^{\omega^\beta}_{r} \Gamma, \varphi$ and $\vdash^{\omega^\beta}_{r} \Gamma, \neg\varphi$ hence applying the Cut-rule we get $\vdash^{\omega^\beta+1}_{r} \Gamma$, noting the cut-rank remains at most r, and hence by Accumulation, $\vdash^{\omega^\alpha+1}_{r} \Gamma$.

Otherwise, $\text{len}(\varphi) = r + 1 > 1$, and so φ is in the form $\varphi_0 \vee \varphi_1$, $\exists x \varphi_0$, $\varphi_0 \wedge \varphi_1$ or $\forall x \varphi_0$. In the last two cases, by negation normal form $\neg\varphi$ is in the form $\varphi_0 \vee \varphi_1$ or $\exists x \varphi_0$. Hence we can apply Lemma 4.3.12 to $\vdash^{\omega^\beta}_{r} \Gamma, \varphi$ and $\vdash^{\omega^\beta}_{r} \Gamma, \neg\varphi$ noting all conditions hold due to above, and $\omega^\beta < \omega^{\beta+1}$. From this we obtain $\vdash^{\omega^\beta \cdot 2}_{r} \Gamma$, from which we can deduce $\vdash^{\omega^\alpha}_{r} \Gamma$ by the Accumulation rule, noting that $\omega^\alpha \underset{n}{\rightarrow} \omega^\beta \cdot 2$ for $n \geq 2$ by the same reasoning as before. □

4.3.4 | Bounding Results in Peano Arithmetic

Definition 4.3.14. We define a *positive* $\Sigma_1(N)$ *formula* to be any formula built up from atomic formulas, excluding $x \notin N$ and using only $\wedge$, $\vee$ and $\exists$ as logical connectives.

Definition 4.3.15. We say that a set $\Gamma = \{\varphi_1, \ldots, \varphi_m\}$ of positive $\Sigma_1(N)$ sentences is *true in* k if for some $1 \leq i \leq n$ we have that φ_i is true when N is interpreted as the finite set $\{n : 3n + 1 < k\}$.

Note that if φ is true in k then clearly φ is true in k' for any $k' \geq k$.

Lemma 4.3.16 (Bounding Lemma). *If Γ contains only positive $\Sigma_1(N)$ sentences and $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma$ with cut-rank 0, then Γ is true in $F_\alpha(k)$ where $k = \max\{3\} \cup \{3n_1 + 1, \dots, 3n_m + 1\}$.*

Proof. We prove this by induction on the generation of $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma$. If $n_1 \notin N, \dots, n_m \notin N, \Gamma$ holds as it is an axiom, then either Γ contains a true atomic sentence, $0 \in N$ or $n_i \in N$ for $1 \leq i \leq m$. Hence Γ is true in $F_\alpha(k)$, noting that in the latter case, $F_\alpha(k) \geq k > 3n_i + 1$, so it follows that $n_i \in \{n : 3n + 1 < F_\alpha(k)\}$.

Consider the case when $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma, S(n) \in N$ was derived using the (N)-rule from $\vdash^{\beta} n_1 \notin N, \dots, n_m \notin N, \Gamma, n \in N$ where $\alpha = \beta + 1$. Then by the inductive hypothesis, either Γ is true in $F_\beta(k)$ hence true in $F_\alpha(k)$ or $n \in N$ is true in $F_\beta(k)$. In the latter case, we note how $F_{\beta+1}(k) = F_\beta^{k+1}(k) \geq F_\beta(k) + k$, where last inequality follows directly from induction on Proposition 3.1.15 *(a)*. Hence $3(n+1) + 1 = 3n + 1 + 3 < F_\beta(k) + 3 \leq F_\beta(k) + k \leq F_\alpha(k)$, so $S(n) \in N$ holds in $F_\alpha(k)$.

Consider $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma, \varphi_0 \vee \varphi_1$ derived by the $(\vee)$-rule from $\vdash^{\beta} n_1 \notin N, \dots, n_m \notin N, \Gamma, \varphi_i$ for $i \in \{0, 1\}$, $\alpha = \beta + 1$. By the inductive hypothesis, we either get that Γ is true in $F_\beta(k)$ or φ_i is true in $F_\beta(k)$, hence $\varphi_0 \vee \varphi_1$ is true in $F_\beta(k)$, so our result follows. The case when $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma, \varphi_0 \wedge \varphi_1$ is derived by the $(\wedge)$-rule follows similarly from the inductive hypothesis, noting that for Γ, φ_0 and Γ, φ_1 to be true in $F_\beta(k)$, either Γ is true in $F_\beta(k)$ or both φ_0 and φ_1 are true in $F_\beta(k)$.

For the $(\exists)$-rule, consider $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma, \exists x \varphi$ derived from $\vdash^{\beta+1} n_1 \notin N, \dots, n_m \notin N, \Gamma, \varphi(n)$ where $\alpha = \beta + 1$ and n is some numeral. Note that since $\exists \varphi(n)$ is a positive $\Sigma_1(N)$ formula, $\varphi(n)$ is also a positive $\Sigma_1(N)$ formula. This means we can apply the inductive hypothesis to deduce that $\Gamma, \varphi(n)$ is true in $F_\beta(k)$ hence true in $F_\alpha(k)$. Hence $\Gamma, \exists x \varphi$ is true in $F_\alpha(k)$, as $\varphi(n)$ is true in $F_\alpha(k)$ implies that $\exists x \varphi$ is true in $F_\alpha(k)$.

If our sequent $\vdash^{\alpha} n_1 \notin N, \dots, n_m \notin N, \Gamma$ is obtained by the Accumulation rule, from $\vdash^{\beta} n_1 \notin N, \dots, n_m \notin N, \Gamma$ it is enough to note that $\alpha \underset{k}{\to} \beta$ for $k = \max\{3\} \cup \{3n_1 + 1, \dots, 3n_m + 1\}$. By the inductive hypothesis, Γ is true in $F_\beta(k)$ and by Proposition 3.1.15 *(d)*, we get that $F_\alpha(k) \geq F_\beta(k)$, hence Γ is true in $F_\alpha(k)$.

Note that the (ω)-rule could not have been applied to obtain a $\Sigma_1(N)$ formula, so it remains only to prove our statement holds after applying the Cut-rule. Let $\left|\frac{\alpha}{}\right. n_1 \notin N, \ldots, n_m \notin N, \Gamma$ be derived from $\left|\frac{\beta}{}\right. n_1 \notin N, \ldots, n_m \notin N, \Gamma, \varphi$ and $\left|\frac{\beta}{}\right. n_1 \notin N, \ldots, n_m \notin N, \Gamma, \neg\varphi$ where $\alpha = \beta + 1$. Note that since this is derived with cut-rank 0, $\mathrm{len}(\varphi) = 0$, which means φ is either an atomic sentence, or in the form $n \in N$ for a numeral n. In the first case φ is a $\Sigma_1(N)$ formula, so we can apply the inductive hypothesis to deduce that Γ, φ and $\Gamma, \neg\varphi$ are both true in $F_\beta(k)$. However it cannot be true that both φ and $\neg\varphi$ are true in $F_\beta(k)$, hence Γ is true in $F_\beta(k)$ hence true in $F_\alpha(k)$.

Otherwise if $\varphi = n \in N$, we get that $\left|\frac{\beta}{}\right. n_1 \notin N, \ldots, n_m \notin N, \Gamma, n \in N$ and $\left|\frac{\beta}{}\right. n_1 \notin N, \ldots, n_m \notin N, \Gamma, n \notin N$ both hold. Applying the inductive hypothesis, we get that $\Gamma, n \in N$ is true in $F_\beta(k)$ and Γ is true in $F_\beta(\max(k, 3n+1))$. From the first equation, we know either Γ is true in $F_\beta(k)$, which means our result holds, or $n \in N$ is true in $F_\beta(k)$. In the second case $3n+1 < F_\beta(k)$, so Γ is true in $F_\beta(\max(k, 3n+1)) \leq F_\beta(\max(k, F_\beta(k))) = F_\beta(F_\beta(k)) = F_\beta^2(k) < F_\beta^{k+1}(k) = F_\alpha(k)$. $\square$

Theorem 4.3.17. *Let f be provably computable in PA. Then, f is elementary in some F_α for some $\alpha < \varepsilon_0$ and hence there exists an ordinal $\beta < \varepsilon_0$ such that F_β dominates f.*

Proof. Suppose f is provably computable in PA, that is there exist elementary functions V, T such that

$$f(x_1, \ldots, x_m) = V(\mu y[T(x_1, \ldots, x_m, y) = 0])$$

and $\forall x_1 \ldots \forall x_m \exists y (T(x_1, \ldots, x_m, y) = 0)$ is a theorem of PA. We note that $T(x_1, \ldots, x_m, y) = 0$ is an atomic formula, as $T(x_1, \ldots, x_m, y)$ is a term, since T is a function defined in the language of PA. By the Embedding Lemma 4.3.9 (noting our formula has no free variables) and the Cut-Elimination Theorem 4.3.13 there exists some $\alpha = \omega^{\omega^{\cdot^{\cdot^{\omega \ell}}}} < \varepsilon_0$ such that

$$\left|\frac{0}{\alpha}\right. \forall x_1 (x_1 \notin N \vee \cdots \vee \forall x_m (x_m \notin N \vee \exists y (y \in N \wedge T(x_1, \ldots, x_m, y) = 0)) \ldots)$$

We note that Lemma 4.2.7 holds for PA_∞, and leaves the cut-rank unchanged. Moreover, by repeated application of the Inversion Lemma 4.3.11 *(ii)*, and Lemma

4.2.7, we get that for every numeral $n_1, \ldots, n_m$,

$$\left|\frac{0}{\alpha}\right. n_1 \notin N, \ldots, n_m \notin N, \exists y(y \in N \wedge T(n_1, \ldots, n_m, y) = 0).$$

By the Bounding Lemma 4.3.16, we have that $\exists y(y \in N \wedge T(n_1, \ldots, n_m, y) = 0)$ is true in $F_\alpha(k)$ where $k = \max(3, 3n_1 + 1, \ldots, 3n_m + 1)$. This means that for every $n_1, \ldots, n_m$, there exists a y, where $y < 3y + 1 < F_\alpha(k)$ such that $T(n_1, \ldots, n_m, y) = 0$.

By Lemma 3.3.11, the elementary functions are closed under bounded minimisation, hence

$$\mu y_{\leq b}[T(n_1, \ldots, n_m, y) = 0]$$

is an elementary function of $n_1, \ldots, n_m$ and b. Letting $b = F_\alpha(\max(3, 3n_1 + 1, \ldots, 3n_m + 1))$, we obtain an elementary-in-F_α definition of f, noting that $\mu y[T(x_1, \ldots, x_m, y) = 0]$ coincides with $\mu y_{\leq b}[T(x_1, \ldots, x_m, y) = 0]$ since there always exists a $y < b$ such that this equality holds. Concretely, we have that

$$f(x_1, \ldots, x_m) = V(\mu y_{\leq b}[T(x_1, \ldots, x_m, y) = 0]).$$

Putting everything together, by Theorem 3.3.21, every elementary-in-F_α function is dominated by F_β where $\beta = \max(\alpha, \omega) + 1$, hence f is necessarily dominated by F_β. □

Theorem 4.3.18. *The Goodstein function $\mathcal{G}$ is not provably computable.*

Proof. This follows immediately from the results we have already proved. By Theorem 3.2.16, Goodstein's function $\mathcal{G}$ dominates F_α for every $\alpha \in \varepsilon_0$, hence $\mathcal{G}$ cannot be dominated by some F_α. Hence, by the contrapositive of Theorem 4.3.17, $\mathcal{G}$ is not provably computable. □

Corollary 4.3.19. *The statement "every Goodstein sequence terminates" is unprovable in Peano Arithmetic.*

Proof. We first note that since $\mathcal{G}$ is not provably computable, at least one of the conditions of Definition 4.3.5 must fail. By Remark 3.3.14, we know that Goodstein's function is computable, hence by Theorem 3.3.16, there exists some elementary V, T such that $\mathcal{G}(x) = V(\mu y[T(x, y) = 0])$. This means the second condition must necessarily fail, that is,

$$\forall x \exists y(T(x, y) = 0)$$

is not a theorem of PA.

We further show that we can choose T and V so that $\forall x \exists y(T(x,y) = 0)$ expresses exactly the statement "every Goodstein sequence terminates". Recall that we can express the kth term of our Goodstein sequence as the computable function $g(k,n) = g_k(n)$. Because of this, we can define the function $T'(n,k,\ell)$ such that $T'(n,k,\ell) = 0$ if and only if within ℓ steps it is possible using a Turing Machine to check that the Goodstein sequence starting at n terminates at index k. This function is necessarily an elementary function by Theorem 3.3.15, as $s_{T'}(n,k,\ell) = \ell$ which is clearly an elementary function.

To get our elementary function T' in the required form, we make use of *pairing functions*. These allow us to encode the two numbers k,ℓ into one number y, without losing any information. The existence of these functions is well known; some examples of elementary pairing functions can be found in Grzegorczyk [1964]. We let U,V be elementary pairing functions, and define

$$T(n,y) = T'(n,V(y),U(y)).$$

Then, $T(n,y) = 0$ if and only if $V(y)$ is the number at which the Goodstein sequence starting at n terminates, and this can be verified in $U(y)$ steps.

From this definition, we have that $\mathcal{G}(n) = V(y)$ whenever $T(n,y) = 0$, hence it follows that we can write $\mathcal{G}$ as

$$\mathcal{G}(x) = V(\mu y[T(x,y) = 0]).$$

As this satisfies the first condition of Definition 4.3.5, we must necessarily have that the second condition fails, that is

$$\forall x \exists y(T(x,y) = 0)$$

is not a theorem of PA. This is exactly the statement we set out to prove, as $\exists y T(x,y) = 0$ holds if and only if the Goodstein sequence starting at x terminates. $\square$

In Buchholz and Wainer [1987], the authors also prove the converse of Theorem 4.3.17, which we state below without proof.

Theorem 4.3.20. *Every function which is dominated by some F_α for $\alpha < \varepsilon_0$ is provably computable in PA.*

This means that the provably computable functions in PA are fully characterised as those which are dominated by an element of our fast growing hierarchy $(F_\alpha)_{\alpha \in \varepsilon_0}$.

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