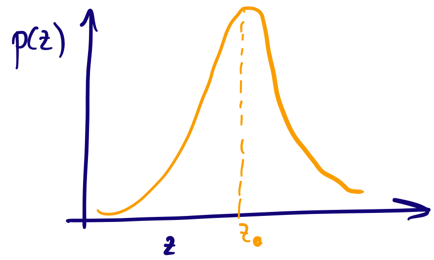
The unperturbed / fiducial dN/dz is assumed to be a Gaussian

$$p_0(z) = A\, e^{-(z-z_0)^2/2\sigma^2}$$



We allow for the central redshift and width of this distribution to shift as a function of sky position $\hat{n}$.

$$p'(z)(\hat{n}) = A\, e^{-[z - z_0'(\hat{n})]^2/2[\sigma'(\hat{n})]^2}$$

where $z_0'(\hat{n}) = z_0 + z_{shift}(\hat{n})$

and

$$\sigma'(\hat{n}) = \sigma + \sigma_{shift}(\hat{n})$$

We therefore can write

$$\Delta p(r\hat{n}) = p'(r\hat{n}) - p_0(r\hat{n})$$

$$= A\, e^{-[z - z_0'(\hat{n})]^2/2[\sigma'(\hat{n})]^2} - A\, e^{-(z-z_0)^2/2\sigma^2}$$

To make contact with the calculation for the power spectrum bias, we want $\Delta\rho_{\ell m}(k)$. How do we go from $\Delta\rho(r\hat{n})$ to $\Delta\rho_{\ell m}(k)$?

$$\Delta\rho_{\ell m}(k) = \int dr\, d\hat{n}\, r^2 \sqrt{\frac{2}{\pi}}\, k\, j_\ell(kr)\, Y^*_{\ell m}(\hat{n})\, \Delta\rho(r\hat{n})$$

we can see that the radial integral is related to the Hankel transform because

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x),$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\, k \int dr\, r^2 j_\ell(kr)\, g(r) = \int dr\, \frac{r^2 k}{kr} J_{\ell+\frac{1}{2}}(kr)\, g(r) = \int dr\, r\, J_{\ell+\frac{1}{2}}(kr)\, g(r)$$

where J_ν is the Bessel function of the 1st kind of order ν (for $\nu \geq -\frac{1}{2}$). Hence,

$$\Delta\rho_{\ell m}(k) = \int d\hat{n}\, Y^*_{\ell m}(\hat{n}) \int dr\, r\, J_{\ell+\frac{1}{2}}(kr)\, \Delta\rho(r\hat{n}).$$

$$= \int dr\, r\, J_{\ell+\frac{1}{2}}(kr) \int d\hat{n}\, Y^*_{\ell m}(\hat{n}) \left[A\, e^{-[z(r) - z_0'(\hat{n})]^2 / 2[\sigma'(\hat{n})]^2} - A\, e^{-(z(r) - z_0)^2 / 2\sigma^2} \right]$$

We can regard this as taking the $(\ell+\frac{1}{2})^{th}$-order Hankel transform of every m mode after having taken the spherical harmonic transform of the map (at each pixel, $z_0'(\hat{n})$ and $\sigma'(\hat{n})$ are constants).

Overall, to get $C_\ell^{\Delta\phi}(k\,;z) = \frac{1}{2\ell+1} \sum_m \Delta\phi_{\ell m}(k, z(r))\, \Delta\phi^*_{\ell m}(k, z(r))$

: 1. Evaluate $\theta(r,\hat{n}) \equiv \left[A\, e^{-[z(r) - z_0'(\hat{n})]^2 / 2[\sigma'(\hat{n})]^2} - A\, e^{-(z(r)-z_0)^2/2\sigma^2} \right]$ at each pixel $\hat{n}$ and at every value of r where FFTLog wants you to sample. If FFTLog wants N_{FFTLog} points, this will be an (N_{FFTLog}, N_{pix}) array.

2. Use Healpix to take SHT of each r slice. This gives you an $(N_{FFTLog}, \sim\frac{N_{pix}}{2})$. Let's call it $\Theta_{\ell m}(r_i)$. Once you've precomputed this, you'll use it $\forall$ the Hankel xform.
Since $\theta(r,\hat{n})$ is real, we can work w/ complex SH coefficients and eliminate half the storage needed

3. To get $\Delta\rho_{\ell m}(k_i)$, use FFTLog to get the $(\ell+\frac{1}{2})^{th}$ order Hankel transform of $\Theta_{\ell m}(r_i)$. This will give you an array of $\Delta\rho_{\ell m}(k_i)$ at all the N_{FFTLog} values of k. This basically entails doing a Hankel transform for every m: i.e., $\frac{1}{2}N_{pix}$ of them.

4. You then get $C_\ell^{\Delta\phi}(k_i)$ by summing the squares of the $\Delta\rho_{\ell m}(k_i)$ for all the m's at this ℓ.