$$\frac{n_g(\hat{n})}{\bar{n}_g} = 1 + \delta_g^{(2D)}(\hat{n}) = \int dr \left[\bar{\phi}(r) + \Delta\phi(r\hat{n})\right]\left[\frac{N_g^{(3D)}(r\hat{n})}{\bar{N}_g^{(3D)}}\right]$$

where $\bar{\phi}(r) = \frac{H(z)}{c}\left\langle \frac{dn}{dz}\Big|_r \right\rangle$. We can rewrite it as

$$\frac{n_g(\hat{n})}{\bar{n}_g} = \int dr \left[\bar{\phi}(r) + \Delta\phi(r\hat{n})\right]\left[\delta_g^{(3D)}(r\hat{n}) + 1\right]$$

$$= \int dr \left[\bar{\phi}(r) + \Delta\phi(r\hat{n})\right] \delta_g^{(3D)}(r\hat{n})$$

$$+ \int dr\, \bar{\phi}(r) + \int dr\, \Delta\phi(r\hat{n})$$

but, by convention, $\int dr\, \bar{\phi}(r) = 1$, so

$$\delta_g^{(2D)}(\hat{n}) = \int dr \left[\bar{\phi}(r) + \Delta\phi(r\hat{n})\right] \delta_g^{(3D)}(r\hat{n}) + \int dr\, \Delta\phi(r\hat{n})$$

We want to expand this projected field in spherical harmonics, such that

$$\delta_g^{(2D)}(\hat{n}) = \sum_{\ell m} (\delta_g^{(2D)})_{\ell m} Y_{\ell m}(\hat{n}) .$$

Multiplying both sides by $Y^*_{\ell' m'}$ and integrating over the solid angle $d\Omega$ corresponding to $\hat{n}$:

$$\int d\Omega \; \delta_g^{(2D)}(\hat{n}) \; Y^*_{\ell' m'} = \sum_{\ell m} (\delta_g^{(2D)})_{\ell m} \int d\Omega \; Y^*_{\ell' m'}(\hat{n}) \, Y_{\ell m}(\hat{n})$$

$$\Rightarrow \; (\delta_g^{(2D)})_{\ell m} = \int d\Omega \; \delta_g^{(2D)}(\hat{n}) \; Y^*_{\ell m}(\hat{n})$$

$$= \int dr \, d\Omega \, \left[ \bar{\Phi}(r) + \Delta\phi(r\hat{n}) \right] \delta_g^{(3D)}(r\hat{n}) \, Y^*_{\ell m}(\hat{n})$$

$$+ \int dr \, d\Omega \, \Delta\phi(r\hat{n}) \, Y^*_{\ell m}(\hat{n})$$

$$= \int dr \, \bar{\Phi}(r) \int d\Omega \;\; \delta_g^{(3D)}(r\hat{n}) \, Y^*_{\ell m}(\hat{n}) \qquad \text{Term A}$$

$$+ \int dr \int d\Omega \;\; \Delta\phi(r\hat{n}) \;\; \delta_g^{(3D)}(r\hat{n}) \, Y^*_{\ell m}(\hat{n}) \qquad \text{Term B}$$

$$+ \int dr \int d\Omega \; \Delta\phi(r\hat{n}) \; Y^*_{\ell m}(\hat{n}) . \qquad \text{Term C}$$

This product in real space is equivalent to a convolution in real space. It will make the effects of the bias be non-local in multipole space

Let's expand out the second term:

$$B_{\ell m} = \int dr \int d\Omega \, \Delta\phi(r\hat{n}) \, \delta_g^{(3D)}(r\hat{n}) \, Y^*_{\ell m}(\hat{n})$$

$$= \int dr \int d\Omega \, Y^*_{\ell m}(\hat{n})$$

$$\times \int \frac{d^3\underline{k}_1}{(2\pi)^3} \, \Delta\phi(\underline{k}_1, z(r)) \, e^{i k_1 r \hat{k}_1 \cdot \hat{n}}$$

$$\times \int \frac{d^3\underline{k}_2}{(2\pi)^3} \, \delta_g^{(3D)}(\underline{k}_2, z(r)) \, e^{i k_2 r \hat{k}_2 \cdot \hat{n}}$$

(in the Fourier convention where $f(\underline{k}) = \int d^3\underline{x} \, f(\underline{x}) \, e^{-i\underline{k}\cdot\underline{x}}$)

Now, using Rayleigh's plane wave expansion,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_l i^l (2l+1) j_l(kr) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})$$
$$= 4\pi \sum_{lm} i^l j_l(kr) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{n}}),$$

we get

$$B_{\ell m} \doteq \int dr \int d\Omega \, \Delta\phi(r\hat{n}) \, \delta_g^{(3D)}(r\hat{n}) \, Y^*_{\ell m}(\hat{n})$$

$$= \int dr \int d\Omega \, Y^*_{\ell m}(\hat{n})$$

$$\times 4\pi \int \frac{d^3\underline{k}_1}{(2\pi)^3} \Delta\phi(\underline{k}_1, z(r)) \sum_{\ell_1 m_1} i^{\ell_1} j_{\ell_1}(k_1 r) Y^*_{\ell_1 m_1}(\hat{k}_1) Y_{\ell_1 m_1}(\hat{n})$$

$$\times 4\pi \int \frac{d^3\underline{k}_2}{(2\pi)^3} \delta_g^{(3D)}(\underline{k}_2, z(r)) \sum_{\ell_2 m_2} i^{\ell_2} j_{\ell_2}(k_2 r) Y^*_{\ell_2 m_2}(\hat{k}_2) Y_{\ell_2 m_2}(\hat{n})$$

$$= (4\pi)^2 \int dr \int \frac{d^3\underline{k}_1}{(2\pi)^3} \frac{d^3\underline{k}_2}{(2\pi)^3} \Delta\phi(\underline{k}_1, z(r)) \, \delta_g^{(3D)}(\underline{k}_2, z(r))$$

$$\times \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} i^{\ell_1+\ell_2} j_{\ell_1}(k_1 r) j_{\ell_2}(k_2 r) Y^*_{\ell_1 m_1}(\hat{k}_1) Y^*_{\ell_2 m_2}(\hat{k}_2)$$

$$\times (-1)^m \int d\Omega \, Y_{\ell\, -m}(\hat{n}) Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n})$$

The last integral is related to the Gaunt integral

$$\mathcal{G}^{\ell \ell_1 \ell_2}_{m m_1 m_2} \equiv \int d\Omega \, Y_{\ell m}(\hat{n}) \, Y_{\ell_1 m_1}(\hat{n}) \, Y_{\ell_2 m_2}(\hat{n}) \, .$$

Simplifying:

$$B_{\ell m} = (-1)^m \int dr \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \mathcal{G}^{\ell \ell_1 \ell_2}_{-m m_1 m_2} \, F_{\ell_1 m_1}[\Delta\phi](r) \, F_{\ell_2 m_2}[\delta^{(3D)}](r)$$

where

$$F_{\ell m}[g](r) \equiv 4\pi \, i^\ell \int \frac{d^3\underline{k}}{(2\pi)^3} \, g(\underline{k}, z(r)) \, j_\ell(kr) \, Y^*_{\ell m}(\hat{k})$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \int dk \, k \, j_\ell(kr) \left[ k \, i^\ell \int \frac{d\hat{k}}{(2\pi)^{3/2}} \, Y^*_{\ell m}(\hat{k}) \, g(\underline{k}, z(r)) \right]$$

$$= \sqrt{\frac{2}{\pi}} \int dk \, k \, j_\ell(kr) \, g_{\ell m}(k, z(r)) \quad \text{in SFB language}$$

Proceeding similarly for terms A and C, we can show that

$$\boxed{\begin{aligned}(\delta^{(2D)}_g)_{\ell m} = \int dr \Big\{ & \phi(r) \, F_{\ell m}[\delta^{(3D)}_g](r) + F_{\ell m}[\Delta\phi](r) \\ & + (-1)^m \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \mathcal{G}^{\ell \ell_1 \ell_2}_{-m m_1 m_2} \, F_{\ell_1 m_1}[\Delta\phi](r) \, F_{\ell_2 m_2}[\delta^{(3D)}](r) \Big\}\end{aligned}}$$

- Projection of 3D fields onto 2D spherical shell
- Then sum over all shells

- Coupling of angular momenta of 3D fields projected onto sphere

How's the observed power, $T_\ell$, related to that expected in the fiducial model where $dN/dz$ is the same in every direction?

$$T_\ell = \frac{\sum_m \langle |(\delta_g^{(2D)})^2_{\ell m}| \rangle}{2\ell+1}$$

$$= \sum_m \frac{1}{2\ell+1} \int dr_1\, dr_2$$

$$\times \Big\langle \Big( \bar{\phi}(r_1)\, F_{\ell m}[\delta_g^{(3D)}](r_1) + F_{\ell m}[\Delta\phi](r_1) + (-1)^m \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} G^{\ell \ell_1 \ell_2}_{-m m_1 m_2} F_{\ell_1 m_1}[\Delta\phi](r_1)\, F_{\ell_2 m_2}[\delta_g^{(3D)}](r_1) \Big)$$

$$\Big( \bar{\phi}(r_2)\, F^*_{\ell m}[\delta_g^{(3D)}](r_2) + F^*_{\ell m}[\Delta\phi](r_2) + (-1)^m \sum_{\ell_1' m_1'} \sum_{\ell_2' m_2'} G^{\ell \ell_1' \ell_2'}_{-m m_1' m_2'} F^*_{\ell_1' m_1'}[\Delta\phi](r_2)\, F^*_{\ell_2' m_2'}[\delta_g^{(3D)}](r_2) \Big) \Big\rangle$$

$$= \sum_m \frac{1}{2\ell+1} \int dr_1\, dr_2$$

$$\times \Big\{ \bar{\phi}(r_1)\, \bar{\phi}(r_2) \langle F_{\ell m}[\delta_g^{(3D)}](r_1)\, F^*_{\ell m}[\delta_g^{(3D)}](r_2) \rangle$$

$$+ \langle F_{\ell m}[\Delta\phi](r_1)\, F^*_{\ell m}[\Delta\phi](r_2) \rangle$$

$$+ 2(-1)^m \bar{\phi}(r_1) \sum_{\ell_1' m_1'} \sum_{\ell_2' m_2'} G^{\ell \ell_1' \ell_2'}_{-m m_1' m_2'} \langle F_{\ell m}[\delta_g^{(3D)}](r_1)\, F^*_{\ell_1' m_1'}[\Delta\phi](r_2)\, F^*_{\ell_2' m_2'}[\delta_g^{(3D)}](r_2) \rangle$$

$$+ \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \sum_{\ell_1' m_1'} \sum_{\ell_2' m_2'} G^{\ell \ell_1 \ell_2}_{-m m_1 m_2}\, G^{\ell \ell_1' \ell_2'}_{-m m_1' m_2'} \langle F_{\ell_1 m_1}[\Delta\phi](r_1)\, F_{\ell_2 m_2}[\delta_g^{(3D)}](r_1)\, F^*_{\ell_1' m_1'}[\Delta\phi](r_2)\, F^*_{\ell_2' m_2'}[\delta_g^{(3D)}](r_2) \rangle \Big\}$$

Note that the Gaunt integral is real

We take the $\Delta\phi$ field to be fixed rather than stochastic, so it comes out of the ensemble averages, and we take $\delta_g$ to be zero-mean. (We already used this above to drop terms).

$$= \sum_m \frac{1}{2l+1} \int dr_1 dr_2$$

$$\times \Big\{ \bar{\phi}(r_1)\bar{\phi}(r_2) \langle F_{lm}[\delta^{(3D)}](r_1) F^*_{lm}[\delta^{(3D)}](r_2) \rangle + F_{lm}[\Delta\phi](r_1) F^*_{lm}[\Delta\phi](r_2)$$

$$+ 2(-1)^m \bar{\phi}(r_1) \sum_{l_1' m_1'} \sum_{l_2' m_2'} G^{l\, l_1' l_2'}_{-m m_1' m_2'} F^*_{l_1' m_1'}[\Delta\phi](r_2) \langle F_{lm}[\delta^{(3D)}](r_1) F^*_{l_2' m_2'}[\delta^{(3D)}](r_2) \rangle$$

$$+ \sum_{l_1 m_1} \sum_{l_2 m_2} \sum_{l_1' m_1'} \sum_{l_2' m_2'} G^{l\, l_1 l_2}_{-m m_1 m_2} G^{l\, l_1' l_2'}_{-m m_1' m_2'} F_{l_1 m_1}[\Delta\phi](r_1) F^*_{l_1' m_1'}[\Delta\phi](r_2) \langle F_{l_2 m_2}[\delta^{(3D)}](r_1) F^*_{l_2' m_2'}[\delta^{(3D)}](r_2) \rangle \Big\}$$

Before can can proceed further, we must calculate

$$\langle F_{l_1 m_1}[\delta_g^{(3D)}](r_1) F^*_{l_2 m_2}[\delta_g^{(3D)}](r_2) \rangle = (4\pi)^2 i^{l_1 - l_2} \int \frac{d^3\underline{k}_1}{(2\pi)^3} \frac{d^3\underline{k}_2}{(2\pi)^3} j_{l_1}(k_1 r_1) j_{l_2}(k_2 r_2) Y^*_{l_1 m_1}(\hat{k}_1) Y_{l_2 m_2}(\hat{k}_2)$$

$$\times \underbrace{\langle \delta_g^{(3D)}(\underline{k}_1, z(r_1)) \delta_g^{(3D)*}(\underline{k}_2, z(r_2)) \rangle}_{(2\pi)^3 \delta^{(3)}(\underline{k}_1 - \underline{k}_2) P(k_1; z(r_1), z(r_2))}$$

$$= (4\pi)^2 i^{l_1 - l_2} \int \frac{d^3\underline{k}_1}{(2\pi)^3} j_{l_1}(k_1 r_1) j_{l_2}(k_1 r_2) Y^*_{l_1 m_1}(\hat{k}_1) Y_{l_2 m_2}(\hat{k}_1) P_g(k_1; z(r_1), z(r_2))$$

$$= (4\pi)^2 i^{l_1 - l_2} \int \frac{dk_1\, k_1^2}{(2\pi)^3} j_{l_1}(k_1 r_1) j_{l_2}(k_1 r_2) P_g(k_1; z(r_1), z(r_2))$$

$$\underbrace{\int d\hat{k}_1\, Y^*_{l_1 m_1}(\hat{k}_1) Y_{l_2 m_2}(\hat{k}_1)}_{\delta_{l_1 l_2} \delta_{m_1 m_2}}$$

$$= \delta_{l_1 l_2}\,\delta_{m_1 m_2}\,\frac{2}{\pi}\int dk_1\, k_1^2\, j_{l_1}(k_1 r_1)\, j_{l_1}(k_1 r_2)\, P_g(k_1; z(r_1), z(r_2))$$

$$\equiv \delta_{l_1 l_2}\,\delta_{m_1 m_2}\, C_{l_1}(r_1, r_2)$$

Therefore,

$$T_l = \frac{1}{2l+1}\int dr_1\, dr_2$$

$$\times \Bigg\{ \bar{\phi}(r_1)\, C_l(r_1, r_2)\, \overbrace{\sum_m \Big[ \bar{\phi}(r_2) + 2(-1)^m \sum_{l_1' m_1'} G^{l\, l_1'\, l}_{-m\, m_1'\, m}\, F^*_{l_1' m_1'}[\Delta\phi](r_2) \Big]}^{\equiv \beta}$$

$$+ \underbrace{\sum_{l_1 m_1}\sum_{l_2 m_2}\sum_{l_1' m_1'}\sum_{m} G^{l\, l_1\, l_2}_{-m\, m_1\, m_2}\, G^{l\, l_1'\, l_2}_{-m\, m_1'\, m_2}\, C_{l_2}(r_1, r_2)\, F_{l_1 m_1}[\Delta\phi](r_1)\, F^*_{l_1' m_1'}[\Delta\phi](r_2)}_{\equiv \alpha}$$

$$+ (2l+1)\, C_l^{\Delta\phi}(r_1, r_2) \Bigg\},$$

where we have defined

$$C_l^{\Delta\phi}(r_1, r_2) \equiv \frac{\sum_m F_{lm}[\Delta\phi](r_1)\, F^*_{lm}[\Delta\phi](r_2)}{2l+1}.$$

Let us try to simplify this $\alpha$ term:

$$\alpha = \sum_{\substack{l_1, l_2, l_1' \\ m_1, m_1', m}} C_{l_2}(r_1, r_2)\, F_{l_1 m_1}[\Delta\phi](r_1)\, F^*_{l_1' m_1'}[\Delta\phi](r_2) \sum_{m_1, m_2} G^{l\, l_1\, l_2}_{-m\, m_1\, m_2}\, G^{l\, l_1'\, l_2}_{-m\, m_1'\, m_2},$$

but

$$\sum_{m_1, m_2} G^{l\, l_1\, l_2}_{-m\, m_1\, m_2}\, G^{l\, l_1'\, l_2}_{-m\, m_1'\, m_2} = \sum_{m_1, m_2} \sqrt{\frac{(2l+1)(2l_1+1)(2l_2+1)(2l+1)(2l_1'+1)(2l_2+1)}{(4\pi)^2}} \begin{pmatrix} l & l_1 & l_2 \\ -m & m_1 & m_2 \end{pmatrix} \begin{pmatrix} l & l_1' & l_2 \\ -m & m_1' & m_2 \end{pmatrix} \begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l_1' & l_2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{(2l+1)(2l_2+1)}{4\pi}\sqrt{(2l_1+1)(2l_1'+1)}\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l_1' & l_2 \\ 0 & 0 & 0 \end{pmatrix}\underbrace{\sum_{mm'}\begin{pmatrix} l_2 & l & l_1 \\ m_2 & -m & m_1 \end{pmatrix}\begin{pmatrix} l_2 & l & l_1' \\ m_2 & -m & m_1' \end{pmatrix}}_{\delta_{l_1 l_1'}\delta_{m_1 m_1'}/(2l_1+1)}$$

$$= \frac{(2l+1)(2l_2+1)}{4\pi}\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \delta_{l_1 l_1'}\delta_{m_1 m_1'} ,$$

so

$$\alpha = \sum_{\substack{l_1, l_2 \\ m_1}} C_{l_2}(r_1, r_2)\, F_{l_1 m_1}[\Delta\phi](r_1)\, F^*_{l_1 m_1}[\Delta\phi](r_2)\, \frac{(2l+1)(2l_2+1)}{4\pi}\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 ,$$

$$= \frac{1}{4\pi}\sum_{l_1, l_2}(2l+1)(2l_1+1)(2l_2+1)\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 C^{\Delta\phi}_{l_1}(r_1, r_2)\, C_{l_2}(r_1, r_2)$$

But this is analogous to the mode-coupling induced by convolving with a mask (see, e.g., Hivon et al 01). Defining

$$M^{\Delta\phi}_{l\, l_2}(r_1, r_2) \doteq \frac{(2l_2+1)}{4\pi}\sum_{l_1}(2l_1+1)\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 C^{\Delta\phi}_{l_1}(r_1, r_2)$$

gives

$$\alpha = (2l+1)\sum_{l_2} M^{\Delta\phi}_{l\, l_2}(r_1, r_2)\, C_{l_2}(r_1, r_2)$$

We can also simplify the $\beta$ term by noticing that

$$G^{l\, l_1'\, l}_{-m\, m_1'\, m} = G^{l\, l\, l_1'}_{m\, -m\, m_1'} = (2l+1)\sqrt{\frac{(2l_1'+1)}{4\pi}}\begin{pmatrix} l & l & l_1' \\ m & -m & m_1' \end{pmatrix}\begin{pmatrix} l & l & l_1' \\ 0 & 0 & 0 \end{pmatrix},$$

is non-vanishing if and only if

- $|l - l| \leq l_1' \leq |l + l|$
  $\Rightarrow 0 \leq l_1' \leq 2l$
- $m - m + m_1' = 0 \Rightarrow m_1' = 0$ ,.

so that

$$G^{\ell\ l_1'\ l}_{-m\ m_1'\ m} = (2l+1)\sqrt{\frac{(2l_1'+1)}{4\pi}} \begin{pmatrix} \ell & \ell & l_1' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l_1' \\ 0 & 0 & 0 \end{pmatrix} \delta_{m_1' 0}.$$

With this, term $\beta$ becomes

$$\beta = \sum_m \left[ \bar{\phi}(r_1) + 2(-1)^m \sum_{l_1' m_1'} G^{l\ l_1'\ l}_{-m\ m_1'\ m} F^*_{l_1' m_1'}[\Delta\phi](r_2) \right]$$

$$= \left[ (2\ell+1)\,\bar{\phi}(r_1) - + 2\sum_{l_1'} \sum_m (-1)^m G^{\ell\ \ell\ l_1'}_{m\ -m\ 0} F^*_{l_1' 0}[\Delta\phi](r_2) \right]$$

$$= (2\ell+1)\,\bar{\phi}(r_1) + 2\sum_{l_1'} F^*_{l_1' 0}[\Delta\phi](r_2) \begin{pmatrix} \ell & \ell & l_1' \\ 0 & 0 & 0 \end{pmatrix} \sqrt{\frac{(2l_1'+1)}{4\pi}}\,(2l+1)$$
$$\times \sum_m (-1)^m \begin{pmatrix} \ell & \ell & l_1' \\ m & -m & 0 \end{pmatrix}$$

We can simplify further using the identity

$$\sum_m (-1)^m \begin{pmatrix} \ell & \ell & L \\ m & -m & 0 \end{pmatrix} = (-1)^\ell \sqrt{2\ell+1}\,\delta_{L0}$$

$$\Rightarrow \beta = (2l+1)\,\bar{\phi}(r_1) + \frac{1}{\sqrt{\pi}}(2l+1)\sqrt{2l+1}\,(-1)^\ell F^*_{00}[\Delta\phi](r_2) \begin{pmatrix} \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, noting that

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \frac{1}{\sqrt{2j+1}}(-1)^{j-m}. \quad \Rightarrow \begin{pmatrix} \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^\ell}{\sqrt{2l+1}}$$

we get

$$\beta = (2l+1)\left[ \bar{\phi}(r_1) + \frac{1}{\sqrt{\pi}} F_{00}[\Delta\phi](r_2) \right],$$

where we have also used the fact that $Y_{00}(\hat{n})$ is real to write $F_{00}^{*}[\Delta\phi](r_2) = F_{00}[\Delta\phi](r_2)$.

All in all:

$$\boxed{\begin{aligned} T_\ell = &\int dr_1 dr_2 \\ &\times \Big\{ \phi(r_1) C_\ell(r_1, r_2) \Big[ \phi(r_2) + \frac{1}{\sqrt{\pi}} F_{00}[\Delta\phi](r_2) \Big] \\ &\quad + C_\ell^{\Delta\phi}(r_1, r_2) + \sum_L M_{\ell L}^{\Delta\phi}(r_1, r_2)\, C_L(r_1, r_2) \Big\} \end{aligned}}$$

The structure of the last term is analogous to the mode coupling induced by a mask.

- Let's do some of those out. We'll start with the fiducial term:

$$T_\ell \supset \int dr_1 dr_2 \, \phi(r_1) \phi(r_2) \, C_\ell(r_1, r_2)$$

$$= \int dr_1 dr_2 \, \phi(r_1) \phi(r_2) \frac{2}{\pi} \int dk \, k^2 j_\ell(k r_1) j_\ell(k r_2) P(k; z(r_1), z(r_2))$$

Using the leading-order Limber approximation:

$$T_\ell \supset \int dr \frac{1}{r} \left[\frac{\phi(r)}{\sqrt{r}}\right]^2 P_g\left(\frac{l + 1/2}{r}; z(r)\right)$$

$$= \int dr \frac{1}{r^2} \left[\phi(r)\right]^2 P_g\left(\frac{l+1/2}{r}; z(r)\right).$$

This is the fiducial term. Everything else are biases.

Before continuing, let us define

$$C_\ell^{\Delta\phi}(r_1, r_2) \equiv \frac{\sum_m F_{lm}[\Delta\phi](r_1) \, F^*_{lm}[\Delta\phi](r_2)}{2l+1} \equiv \frac{2}{\pi} \int dk_1 dk_2 \, k_1 k_2 \, j_\ell(k_1 r_1) j_\ell(k_2 r_2) \, C_\ell^{\Delta\phi}(k_1, k_2)$$

where we have defined the SFB power spectrum

$$C_\ell^{\Delta\phi}(k_1, k_2) \equiv \frac{1}{2l+1} \sum_m \Delta\phi_{\ell m}(k_1) \, \Delta\phi^*_{\ell m}(k_2)$$

- Let's do at the additive correction:

$$T_\ell \supset \int dr_1 dr_2 \, C_\ell^{\Delta\phi}(r_1, r_2)$$

$$= \int dr_1 dr_2 \, \frac{\sum_m F_{\ell m}[\Delta\phi](r_1) \, F^*_{\ell m}[\Delta\phi](r_2)}{2\ell+1}$$

$$= \left(\frac{2}{\pi}\right)^2 \int dr_1 dr_2$$

$$\times \int dk_1 \, k_1^2 \, j_\ell(k r_1) \int dr_1' \, r_1'^2 \, j_\ell(k_1 r_1')$$

$$\times \int dk_2 \, k_2^2 \, j_\ell(k_2 r_2) \int dr_2' \, r_2'^2 \, j_\ell(k_2 r_2')$$

$$\times \int d\hat{n}_1 \, \Delta\phi(r_1' \hat{n}_1) \int d\hat{n}_2 \, \Delta\phi(r_2' \hat{n}_2) \, \frac{\sum_m Y^*_{\ell m}(\hat{n}_1) \, Y_{\ell m}(\hat{n}_2)}{2\ell+1}$$

But the addition theorem for spherical harmonics states that

Legendre polynomials

$$\frac{\sum_m Y^*_{\ell m}(\hat{n}_1) \, Y_{\ell m}(\hat{n}_2)}{2\ell+1} = \frac{1}{4\pi} P_\ell(\hat{n}_1 \cdot \hat{n}_2),$$

So

$$T_\ell \supset \frac{1}{\pi^3} \int dr_1 dr_2$$

$$\times \int dk_1\, k_1^2\, j_\ell(k_1 r_1) \int dr_1'\, r_1'^2\, j_\ell(k_1 r_1') \int d\hat{n}_1\, \Delta\phi(r_1' \hat{n}_1)$$

$$\times \int dk_2\, k_2^2\, j_\ell(k_2 r_2) \int dr_2'\, r_2'^2\, j_\ell(k_2 r_2') \int d\hat{n}_2\, \Delta\phi(r_2' \hat{n}_2)\, P_\ell(\hat{n}_1 \cdot \hat{n}_2)$$

Let us rearrange terms in order to more clearly realize the applicability of the Limber approximation

$$= \frac{1}{\pi^3} \int dr_1 dr_2$$

$$\times \int dr_1'\, r_1'^2 \int d\hat{n}_1\, \Delta\phi(r_1' \hat{n}_1) \int dk_1\, k_1^2\, j_\ell(k_1 r_1)\, j_\ell(k_1 r_1')$$

$$\times \int dr_2'\, r_2'^2 \int d\hat{n}_2\, \Delta\phi(r_2' \hat{n}_2)\, P_\ell(\hat{n}_1 \cdot \hat{n}_2) \int dk_2\, k_2^2\, j_\ell(k_2 r_2)\, j_\ell(k_2 r_2')$$

From the orthogonality relation:

$$\delta^D(k-k') = \frac{2kk'}{\pi} \int_0^\infty \mathrm{d}r\, r^2\, j_\ell(kr)\, j_\ell(k'r)\,,$$

$$T_\ell \supset \frac{1}{\pi^3} \int dr_1 dr_2$$

$$\times \int dr_1'\, r_1'^2 \int d\hat{n}_1\, \Delta\phi(r_1' \hat{n}_1) \left\{ \frac{\pi}{2 r_1 r_1'} \delta_D(r_1 - r_1') \right\}$$

$$\times \int dr_2'\, r_2'^2 \int d\hat{n}_2\, \Delta\phi(r_2' \hat{n}_2)\, P_\ell(\hat{n}_1 \cdot \hat{n}_2) \left\{ \frac{\pi}{2 r_2 r_2'} \delta_D(r_2 - r_2') \right\}$$

$$= \frac{1}{4\pi} \int dr_1 dr_2$$

$$\times \int d\hat{n}_2 \int d\hat{n}_1 \, \Delta\phi(r_1 \hat{n}_1) \, \Delta\phi(r_2 \hat{n}_2) \, P_\ell(\hat{n}_1 \cdot \hat{n}_2)$$

Equivalently, we could've used the definition of the SFB power spectrum to write (directly from the 1st line)

If we had statistical isotropy, these k's would be forced to be equal (but we don't)

$$T_\ell \supset \frac{2}{\pi} \int dr_1 dr_2 \int dk_1 dk_2 \, k_1 k_2 \, j_\ell(k_1 r_1) j_\ell(k_2 r_2) \, C_\ell^{\Delta\phi}(k_1, k_2)$$

In the (leading-order) Limber approximation,

$$j_\ell(k\chi) \to \sqrt{\frac{\pi}{2\nu}} \, \delta_D(\nu - k\chi) = \frac{1}{\chi} \sqrt{\frac{\pi}{2\nu}} \, \delta_D\left(\frac{\nu}{\chi} - k\right)$$

in the radial integrals, where $\nu = \ell + \frac{1}{2}$.

$$\approx \nu \int dr_1 dr_2 \, \frac{1}{r_1^2} \frac{1}{r_2^2} \, C_\ell^{\Delta\phi}\left(\frac{\nu}{r_1}, \frac{\nu}{r_2}\right)$$

!

- As per the simpler of the two multiplicative corrections:

$$T_\ell \supset \frac{1}{\sqrt{\pi}} \int dr_1 dr_2\, \phi(r_1)\, C_\ell(r_1, r_2)\, F_{00}[\Delta q](r_2)$$

$$= \frac{2\sqrt{\pi}}{\pi^2} \int dr_1 dr_2\, \phi(r_1)\, F_{00}[\Delta q](r_2)$$

$$\times \int dK\, K^2\, j_\ell(K r_1)\, j_\ell(K r_2)\, P_g(K; z(r_1), z(r_2)).$$

Note: we can use Limber irrespective of the smoothness of $\Delta q$ as long as $P(K)$ is smooth w.r.t. the spherical Bessel functions. Question: is this a more robust Limber than that in Loverde & Afshordi? A special, extra robust case of it? That's what Gebhardt and Doré seem to imply.

In the leading-order Limber approximation,

$$\approx \frac{2\sqrt{\pi}}{\pi^2} \int dr_1 dr_2\, \phi(r_1)\, F_{00}[\Delta q](r_2)$$

$$\times \int dK\, K^2\, \frac{1}{r_1}\sqrt{\frac{\pi}{2\nu}}\, \delta_D\!\left(\frac{\nu}{r_1} - K\right) \frac{1}{r_2}\sqrt{\frac{\pi}{2\nu}}\, \delta_D\!\left(\frac{\nu}{r_2} - K\right) P_g(K; z(r_1), z(r_2))$$

$$= \frac{\sqrt{\pi}}{\pi \nu} \int dr_1 dr_2\, \phi(r_1)\, F_{00}[\Delta q](r_2)$$

$$\times \frac{1}{r_1 r_2} \left(\frac{\nu}{r_2}\right)^2 P_g\!\left(\frac{\nu}{r_1}; z(r_1), z(r_2)\right) \underbrace{\delta_D\!\left(\frac{\nu}{r_1} - \frac{\nu}{r_2}\right)}_{= \delta_D\left(\frac{r_1 r_2}{\nu}(r_2 - r_1)\right) = \frac{r_1 r_2}{\nu}\delta_D(r_2 - r_1)}$$

$$= \frac{1}{\sqrt{\pi}} \int dr\, \frac{\phi(r)}{r^2}\, F_{00}[\Delta q](r)\, P_g\!\left(\frac{\nu}{r}; z(r)\right).$$

but

$$F_{\ell m}[\Delta\phi](r) = \sqrt{\frac{2}{\pi}} \int dk\, k\, j_\ell(kr)\, \Delta\phi_{\ell m}(k, z(r))$$

$$= \sqrt{\frac{2}{\pi}} \int dk\, k \left[\sqrt{\frac{\pi}{2kr}}\, J_{\ell+1/2}(kr)\right] \Delta\phi_{\ell m}(k, z(r))$$

$$= \frac{1}{\sqrt{r}} \int dk\, \sqrt{k}\, J_{\ell+1/2}(kr)\, \Delta\phi_{\ell m}(k, z(r))$$

$$= \int dk\, k\, J_{\ell+1/2}(kr) \left[\frac{\Delta\phi_{\ell m}(k, z(r))}{\sqrt{kr}}\right]$$

$$= \frac{1}{\sqrt{r}}\, \mathcal{H}_{\ell+1/2}\left[\frac{\Delta\phi_{\ell m}(k, z(r))}{\sqrt{k}}\right](r)$$

i.e., the $(\ell+1/2)^{th}$-order Hankel transform of this function

Consequently,

$$T_\ell^{mult} \approx \frac{1}{\sqrt{\pi}} \int dr\, r^{-5/2}\, \phi(r)\, P_g\!\left(\frac{\nu}{r}; z(r)\right) \mathcal{H}_{1/2}\left[\frac{\Delta\phi_{00}(k, z(r))}{\sqrt{k}}\right](r)$$

- And finally, the multiplicative term that resembles mask-convolution:

$$T_\ell \supset \int dr_1 dr_2 \sum_L M^{\Delta\phi}_{\ell L}(r_1, r_2)\, C_L(r_1, r_2)$$

$$= \sum_L \frac{(2L+1)}{4\pi} \sum_{\ell_1} (2\ell_1+1) \begin{pmatrix} \ell & \ell_1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\times \int dr_1 dr_2\, C_L(r_1, r_2)\, C^{\Delta\phi}_{\ell_1}(r_1, r_2)$$

$$= \sum_L \frac{(2L+1)}{4\pi} \sum_{\ell_1} (2\ell_1+1) \begin{pmatrix} \ell & \ell_1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\times \int dr_1 dr_2$$

Series-expand these as in Loverde & Afshordi: $j_\ell(k\chi) \to \sqrt{\frac{\pi}{2\nu}}\, \delta_D(\nu - k\chi)$

This is the weaker version of Limber.

$$\times \frac{2}{\pi} \int dk_1 dk_2\, k_1 k_2\, j_{\ell_1}(k_1 r_1)\, j_{\ell_1}(k_2 r_2)\, C^{\Delta\phi}_{\ell_1}(k_1, k_2)$$

$$\times \frac{2}{\pi} \int dk\, k^2\, j_L(k r_1)\, j_L(k r_2)\, P_g(k; z(r_1), z(r_2))$$

assume $P(k)$ is slowly varying s.t.

$$\int dk\, k^2 j_L(kr_1) j_L(kr_2) P(k) \approx P\left(\frac{L+\frac{1}{2}}{r_1}\right) \int dk\, k^2 j_L(kr_1) j_L(kr_2)$$

$$\approx P\left(\frac{L}{r_1}\right) \frac{\pi}{2 r_1 r_2} \delta(r_1 - r_2)$$

(actually, you can probably recover this from applying the version above twice)

Important insight: the smoothness of $P(k)$ relative to the Bessel functions enforces $r_1 \approx r_2$ irrespective of $C^{\Delta\phi}(r_1, r_2)$. This is the more robust version of Limber.

We now use two versions of Limber, as highligthed in red.

$$\approx \sum_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1) \begin{pmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\int dr_1 dr_2$$

$$\times \frac{\nu_1}{r_1^2 r_2^2} C_{l_1}^{\Delta\phi}\left(\frac{l_1+1/2}{r_1}, \frac{l_1+1/2}{r_2}\right)$$

$$\times \frac{1}{r_1^2} \delta(r_1 - r_2) P_g\left(\frac{L+1/2}{r_1}; z(r_1), z(r_2)\right)$$

$$= \frac{1}{2} \sum_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1)^2 \begin{pmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\int dr\, r^{-6} P_g\left(\frac{L+1/2}{r}; z(r)\right) C_{l_1}^{\Delta\phi}\left(\frac{l_1+1/2}{r}, \frac{l_1+1/2}{r}\right)$$

what if there's no fluctuation power below a certain scale $l_1 > l_{max}$?