

$$\frac{n_g(\hat{n})}{\bar{n}_g} = 1 + \delta_g^{(2D)}(\hat{n}) = \int d\mathbf{r} \frac{H(z(r))}{c} \left[\bar{\phi}(r) + \Delta\phi(r\hat{n}) \right] \left[\frac{N_g^{(3D)}(r\hat{n})}{\bar{N}_g^{(3D)}} \right]$$

where $\bar{\phi}(r) = \langle \frac{dn}{dz} \rangle_r$. We can rewrite it as

$$\frac{n_g(\hat{n})}{\bar{n}_g} = \int d\mathbf{r} \frac{H(z(r))}{c} \left[\bar{\phi}(r) + \Delta\phi(r\hat{n}) \right] \left[\delta_g^{(3D)}(r\hat{n}) + 1 \right]$$

$$= \int d\mathbf{r} \frac{H(z(r))}{c} \left[\bar{\phi}(r) + \Delta\phi(r\hat{n}) \right] \delta_g^{(3D)}(r\hat{n}) \\ + \int d\mathbf{r} \frac{H(z(r))}{c} \bar{\phi}(r) + \int d\mathbf{r} \frac{H(z(r))}{c} \Delta\phi(r\hat{n})$$

but, by convention, $\int d\mathbf{r} \frac{H(z(r))}{c} \bar{\phi}(r) = \int dz \bar{\phi}(z) = 1$, so

$$\delta_g^{(2D)}(\hat{n}) = \int d\mathbf{r} \frac{H(z(r))}{c} \left[\bar{\phi}(r) + \Delta\phi(r\hat{n}) \right] \delta_g^{(3D)}(r\hat{n})$$

$$+ \int d\mathbf{r} \frac{H(z(r))}{c} \Delta\phi(r\hat{n})$$

We want to expand this projected field in spherical harmonics, such that

$$f_g^{(2D)}(\hat{n}) = \sum_{\ell m} (f_g^{(2D)})_{\ell m} Y_{\ell m}(\hat{n}).$$

Multiplying both sides by $Y_{\ell m}^*$ and integrating over the solid angle $d\Omega$ corresponding to \hat{n} :

$$\begin{aligned} \int d\Omega f_g^{(2D)}(\hat{n}) Y_{\ell m}^* &= \sum_{\ell m} (f_g^{(2D)})_{\ell m} \int d\Omega Y_{\ell m}^*(\hat{n}) Y_{\ell m}(\hat{n}) \\ \Rightarrow (f_g^{(2D)})_{\ell m} &= \int d\Omega f_g^{(2D)}(\hat{n}) Y_{\ell m}^*(\hat{n}) \\ &= \int dr d\Omega [\bar{\Phi}(r) + \Delta\phi(r\hat{n})] f_g^{(3D)}(r\hat{n}) Y_{\ell m}^*(\hat{n}) \\ &\quad + \int dr d\Omega \Delta\phi(r\hat{n}) Y_{\ell m}^*(\hat{n}) \\ &= \int dr \frac{H(r)}{c} \bar{\Phi}(r) \int d\Omega f_g^{(3D)}(r\hat{n}) Y_{\ell m}^*(\hat{n}) \quad \text{Term A} \\ &\quad + \int dr \frac{H(r)}{c} \int d\Omega \Delta\phi(r\hat{n}) f_g^{(3D)}(r\hat{n}) Y_{\ell m}^*(\hat{n}) \quad \text{Term B} \\ &\quad + \int ds \frac{H(s)}{c} \int d\Omega \Delta\phi(s\hat{n}) Y_{\ell m}^*(\hat{n}) \quad \text{Term C} \end{aligned}$$

|

This product in real space is equivalent to a convolution in real space. It will make the effects of the bias be non-local in multipole space

Let's expand out the second term:

$$B_{em} = \int d\mathbf{r} \frac{H(r)}{c} \int d\Omega \Delta\phi(\mathbf{r}\hat{n}) S_g^{(3D)}(\mathbf{r}\hat{n}) Y_{em}^*(\hat{n})$$

$$= \int d\mathbf{r} \frac{H(r)}{c} \int d\Omega Y_{em}^*(\hat{n}) \\ \times \int \frac{d^3 \underline{k}_1}{(2\pi)^3} \Delta\phi(\underline{k}_1, z(r)) e^{i \underline{k}_1 \cdot \hat{\mathbf{r}} \underline{k}_1 \cdot \hat{n}}$$

$$\times \int \frac{d^3 \underline{k}_2}{(2\pi)^3} S_g^{(3D)}(\underline{k}_2, z(r)) e^{i \underline{k}_2 \cdot \hat{\mathbf{r}} \underline{k}_2 \cdot \hat{n}}$$

(in the Fourier convection where $f(\underline{k}) = \int d^3x f(x) e^{-i \underline{k} \cdot \underline{x}}$)

Now, using Rayleigh's plane wave expansion,

$$e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_l i^l (2l+1) j_l(kr) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})$$

$$= 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{n}}),$$

we get

$$B_{em} = \int d\mathbf{r} \frac{H(r)}{c} \int d\Omega \Delta\phi(r\hat{n}) \oint_g^{(3D)}(r\hat{n}) Y_{em}^*(\hat{n})$$

$$= \int d\mathbf{r} \frac{H(r)}{c} \int d\Omega Y_{em}^*(\hat{n})$$

$$\times 4\pi \int \frac{d^3 K_1}{(2\pi)^3} \Delta\phi(K_1, z(r)) \sum_{\ell_1 m_1} i^{\ell_1} j_{\ell_1}(K_1 r) Y_{\ell_1 m_1}^*(\hat{k}_1) Y_{\ell_1 m_1}(\hat{n})$$

$$\times 4\pi \int \frac{d^3 K_2}{(2\pi)^3} \oint_g^{(3D)}(K_2, z(r)) \sum_{\ell_2 m_2} i^{\ell_2} j_{\ell_2}(K_2 r) Y_{\ell_2 m_2}^*(\hat{k}_2) Y_{\ell_2 m_2}(\hat{n})$$

$$= (4\pi)^2 \int d\mathbf{r} \frac{H(r)}{c} \int \frac{d^3 K_1}{(2\pi)^3} \frac{d^3 K_2}{(2\pi)^3} \Delta\phi(K_1, z(r)) \oint_g^{(3D)}(K_2, z(r))$$

$$\times \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} i^{\ell_1 + \ell_2} j_{\ell_1}(K_1 r) j_{\ell_2}(K_2 r) Y_{\ell_1 m_1}^*(\hat{k}_1) Y_{\ell_2 m_2}^*(\hat{k}_2)$$

$$\times (-1)^m \int d\Omega Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_2 m_2}(\hat{n})$$

The last integral is related to the Gauvut integral

$$G_{mm_1m_2}^{l_1l_2} = \int d\Omega \gamma_{1m}(\hat{u}) \gamma_{1,m_1}(\hat{u}) \gamma_{1,m_2}(\hat{u}).$$

Simplifying:

$$\beta_{lm} = (-1)^m \int d\Omega \frac{H(r)}{c} \sum_{l_1 m_1} \sum_{l_2 m_2} G_{mm_1m_2}^{l_1l_2} F_{1,m_1}[\Delta\phi](r) F_{1,m_2}[S^{(3D)}](r)$$

where

$$\begin{aligned} F_{1m}[g](r) &= 4\pi i^l \left\{ \frac{d^3 k}{(2\pi)^3} g(k, z(r)) j_l(kr) Y_{1m}^*(\hat{k}) \right. \\ &= \frac{4\pi}{(2\pi)^3} \int dk k j_l(kr) \left[k i^l \int \frac{d\hat{k}}{(2\pi)^3} Y_{1m}^*(\hat{k}) g(k, z(r)) \right] \\ &= \sqrt{\frac{2}{\pi}} \int dk k j_l(kr) g_{1m}(k, z(r)) \quad \text{in SF\beta language} \end{aligned}$$

Proceeding similarly for terms A and C, we can show that

$$\left(S_g^{(2D)} \right)_{lm} = \int d\Omega \frac{H(r)}{c} \left\{ \phi(r) F_{1m}[S_g^{(3D)}](r) + F_{1m}[\Delta\phi](r) \right. \\ \left. + (-1)^m \sum_{l_1 m_1} \sum_{l_2 m_2} G_{mm_1m_2}^{l_1l_2} F_{1,m_1}[\Delta\phi](r) F_{1,m_2}[S^{(3D)}](r) \right\}$$

- Projection of 3D fields onto 2D spherical shell

- Then sum over all shells

- Coupling of angular momenta of 3D fields projected onto sphere

How's the observed power, T_e , related to that expected in the fiducial model where $dN/d\theta$ is the same in every direction?

$$T_e = \frac{\sum_m \langle |(\delta_g^{(3D)})_{em}|^2 \rangle}{2l+1}$$

$$\begin{aligned}
 &= \sum_m \frac{1}{2l+1} \int d\sigma_1 d\sigma_2 \frac{H(\sigma_1)}{c} \frac{H(\sigma_2)}{c} \\
 &\times \left\langle \bar{\phi}(\sigma_1) F_{em} [S_g^{(3D)}](\sigma_1) + F_{em} [\Delta\phi](\sigma_1) + (-1)^m \sum_{l_1 m_1} \sum_{l_2 m_2} G_{mm_1 m_2}^{l_1 l_2} F_{l_1 m_1} [\Delta\phi](\sigma_1) F_{l_2 m_2} [S_g^{(3D)}](\sigma_1) \right. \\
 &\quad \left. \left(\bar{\phi}(\sigma_2) F_{em}^* [S_g^{(3D)}](\sigma_2) + F_{em}^* [\Delta\phi](\sigma_2) + (-1)^m \sum_{l_1' m_1'} \sum_{l_2' m_2'} G_{mm_1' m_2'}^{l_1' l_2'} F_{l_1' m_1'}^* [\Delta\phi](\sigma_2) F_{l_2' m_2'}^* [S_g^{(3D)}](\sigma_2) \right) \right\rangle \\
 &= \sum_m \frac{1}{2l+1} \int d\sigma_1 d\sigma_2 \frac{H(\sigma_1)}{c} \frac{H(\sigma_2)}{c} \\
 &\times \left\{ \bar{\phi}(\sigma_1) \bar{\phi}(\sigma_2) \langle F_{em} [S_g^{(3D)}](\sigma_1) F_{em}^* [S_g^{(3D)}](\sigma_2) \rangle \right. \\
 &\quad + \langle F_{em} [\Delta\phi](\sigma_1) F_{em}^* [\Delta\phi](\sigma_2) \rangle \\
 &\quad + 2(-1)^m \bar{\phi}(\sigma_1) \sum_{l_1' m_1'} \sum_{l_2' m_2'} G_{mm_1' m_2'}^{l_1' l_2'} \langle F_{em} [S_g^{(3D)}](\sigma_1) F_{l_1' m_1'}^* [\Delta\phi](\sigma_2) F_{l_2' m_2'}^* [S_g^{(3D)}](\sigma_2) \rangle \\
 &\quad \left. + \sum_{l_1 m_1} \sum_{l_2 m_2} \sum_{l_1' m_1'} \sum_{l_2' m_2'} G_{mm_1 m_2}^{l_1 l_2} G_{mm_1' m_2'}^{l_1' l_2'} \langle F_{l_1 m_1} [\Delta\phi](\sigma_1) F_{l_2 m_2} [S_g^{(3D)}](\sigma_1) F_{l_1' m_1'}^* [\Delta\phi](\sigma_2) F_{l_2' m_2'}^* [S_g^{(3D)}](\sigma_2) \rangle \right\} \\
 &\quad \text{Note that the Gaunt integral is real}
 \end{aligned}$$

We take the $\Delta\phi$ field to be fixed rather than stochastic, so it comes out of the ensemble average, and we take δ_g to be zero-mean. (We already used this above to drop terms).

$$\begin{aligned}
 &= \sum_m \frac{1}{2l+1} \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \\
 &\times \left\{ \bar{\phi}(\zeta_1) \bar{\phi}(\zeta_2) \langle F_{em} [S^{(3D)}](\zeta_1) F_{em}^* [S^{(3D)}](\zeta_2) \rangle + \langle F_{em} [\Delta\phi](\zeta_1) F_{em}^* [\Delta\phi](\zeta_2) \rangle \right. \\
 &+ 2(-1)^m \bar{\phi}(\zeta_1) \sum_{l_1 m_1} \sum_{l_2 m_2} G_{mm_1 m_2}^{ll_1 l_2} F_{l_1 m_1}^* [\Delta\phi](\zeta_2) \langle F_{em} [S^{(3D)}](\zeta_1) F_{l_2 m_2}^* [S^{(3D)}](\zeta_2) \rangle \\
 &\left. + \sum_{l_1 m_1} \sum_{l_2 m_2} \sum_{l_1' m_1'} \sum_{l_2' m_2'} G_{mm_1 m_2}^{ll_1 l_2} G_{mm_1' m_2'}^{ll_1' l_2'} F_{l_1 m_1} [\Delta\phi](\zeta_1) F_{l_1' m_1'} [\Delta\phi](\zeta_2) \langle F_{em} [S^{(3D)}](\zeta_1) F_{l_2 m_2}^* [S^{(3D)}](\zeta_2) \rangle \right\}
 \end{aligned}$$

Before can can proceed further, we must calculate

$$\begin{aligned}
 \langle F_{l_1 m_1} [S_g^{(3D)}](\zeta_1) F_{l_2 m_2}^* [S_g^{(3D)}](\zeta_2) \rangle &= (4\pi)^2 i^{l_1 - l_2} \int \frac{d^3 \underline{k}_1}{(2\pi)^3} \frac{d^3 \underline{k}_2}{(2\pi)^3} j_{l_1}(\underline{k}_1, \zeta_1) j_{l_2}(\underline{k}_2, \zeta_2) Y_{l_1 m_1}^*(\hat{\underline{k}}_1) Y_{l_2 m_2}(\hat{\underline{k}}_2) \\
 &\quad \times \underbrace{\langle \delta_g^{(3D)}(\underline{k}_1, z(\zeta_1)) \delta_g^{(3D)*}(\underline{k}_2, z(\zeta_2)) \rangle}_{(2\pi)^3 \delta(\underline{k}_1 - \underline{k}_2)} P(\underline{k}_1, z(\zeta_1), z(\zeta_2)) \\
 &= (4\pi)^2 i^{l_1 - l_2} \int \frac{d^3 \underline{k}_1}{(2\pi)^3} j_{l_1}(\underline{k}_1, \zeta_1) j_{l_2}(\underline{k}_1, \zeta_2) Y_{l_1 m_1}^*(\hat{\underline{k}}_1) Y_{l_2 m_2}(\hat{\underline{k}}_1) P(\underline{k}_1, z(\zeta_1), z(\zeta_2)) \\
 &= (4\pi)^2 i^{l_1 - l_2} \int \frac{d \underline{K}_1}{(2\pi)^3} K_1^2 j_{l_1}(\underline{k}_1, \zeta_1) j_{l_2}(\underline{k}_1, \zeta_2) P(\underline{k}_1, z(\zeta_1), z(\zeta_2)) \\
 &\quad \underbrace{\int d \hat{\underline{k}}_1 Y_{l_1 m_1}^*(\hat{\underline{k}}_1) Y_{l_2 m_2}(\hat{\underline{k}}_1)}_{\delta_{l_1 l_2} \delta_{m_1 m_2}}
 \end{aligned}$$

$$= \int_{\ell_1, \ell_2} \int_{m_1, m_2} \frac{2}{\pi} \int dK_i K_i^2 j_{\ell_1}(K_i r_1) j_{\ell_2}(K_i r_2) P_g(K_i; z(r_1), z(r_2))$$

$$= \int_{\ell_1, \ell_2} \int_{m_1, m_2} C_\ell(r_1, r_2)$$

Therefore,

$$T_\ell = \frac{1}{2\ell+1} \left\{ \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \right. \\ \times \overline{\phi(\zeta_1)} C_\ell(\zeta_1, \zeta_2) \underbrace{\sum_m \left[\bar{\phi}(\zeta_2) + 2(-1)^{\ell} \sum_{l_1' m_1'} G_{-mm, m_1' m_1} F_{l_1' m_1'}^* [\Delta \phi](\zeta_2) \right]}_{\equiv \beta} \\ + \underbrace{\sum_{l_1 m_1} \sum_{l_2 m_2} \sum_{l_1' m_1'} \sum_m G_{-mm, m_2}^{l_1 l_2} G_{-mm, m_2}^{l_1' l_2} C_{l_2}(\zeta_1, \zeta_2) F_{l_1 m_1}^* [\Delta \phi](\zeta_1) F_{l_1' m_1'}^* [\Delta \phi](\zeta_2)}_{\equiv \alpha} \\ \left. + (2\ell+1) C_\ell^{\Delta \phi}(\zeta_1, \zeta_2) \right\},$$

where we have defined

$$C_\ell^{\Delta \phi}(\zeta_1, \zeta_2) \equiv \frac{\sum_m F_{l_1 m_1}^* [\Delta \phi](\zeta_1) F_{l_1' m_1'}^* [\Delta \phi](\zeta_2)}{2\ell+1}.$$

Let us try to simplify this α term:

$$\alpha = \sum_{l_1, l_2, l_1', m_1, m_1', m_2} C_{l_2}(\zeta_1, \zeta_2) F_{l_1 m_1}^* [\Delta \phi](\zeta_1) F_{l_1' m_1'}^* [\Delta \phi](\zeta_2) \sum_{m_1, m_2} G_{-mm, m_2}^{l_1 l_2} G_{-mm, m_2}^{l_1' l_2},$$

but

$$\sum_{m_1, m_2} G_{-mm, m_2}^{l_1 l_2} G_{-mm, m_2}^{l_1' l_2} = \sum_{m_1, m_2} \sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)(2\ell_1'+1)(2\ell_2'+1)}{(4\pi)^2}} \binom{\ell_1, l_2}{m_1, m_2} \binom{\ell_1', l_2'}{m_1', m_2'} \binom{l_1, l_2}{0, 0, 0} \binom{l_1', l_2'}{0, 0, 0}$$

$$\begin{aligned}
&= \frac{(2l+1)(2l_2+1)}{4\pi} \sqrt{(2l_1+1)(2l_1'+1)} \underbrace{\left(\begin{matrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{matrix} \right) \left(\begin{matrix} l & l_1' & l_2 \\ 0 & 0 & 0 \end{matrix} \right)}_{m'm_1} \underbrace{\left(\begin{matrix} l_2 & l & l_1 \\ m_2 - m & m_1 & m_1' \end{matrix} \right) \left(\begin{matrix} l_2 & l & l_1 \\ m_2 - m & m_1 & m_1' \end{matrix} \right)}_{\delta_{l,l'} \delta_{m,m_1} / (2l+1)} \\
&= \frac{(2l+1)(2l_2+1)}{4\pi} \left(\begin{matrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{matrix} \right)^2 \delta_{l,l'} \delta_{m,m_1} ,
\end{aligned}$$

so

$$\begin{aligned}
\alpha &= \sum_{\substack{l_1, l_2 \\ m_1}} C_{l_2}(r_1, r_2) F_{l_1, m_1} [\Delta\phi](r_1) F_{l_1, m_1} [\Delta\phi](r_2) - \frac{(2l+1)(2l_2+1)}{4\pi} \left(\begin{matrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{matrix} \right)^2 , \\
&= \frac{1}{4\pi} \sum_{l_1, l_2} (2l+1)(2l_1+1)(2l_2+1) \left(\begin{matrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{matrix} \right)^2 C_{l_1}^{\Delta\phi}(r_1, r_2) C_{l_2}(r_1, r_2)
\end{aligned}$$

But this is analogous to the mode-coupling induced by convolving with a mask (see, e.g., Hirou et al 01). Defining

$$M_{l_1 l_2}^{\Delta\phi}(r_1, r_2) = \frac{(2l_2+1)}{4\pi} \sum_{l_1} (2l_1+1) \left(\begin{matrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{matrix} \right)^2 C_{l_1}^{\Delta\phi}(r_1, r_2)$$

gives

$$\alpha = (2l+1) \sum_{l_2} M_{l_1 l_2}^{\Delta\phi}(r_1, r_2) C_{l_2}(r_1, r_2)$$

We can also simplify the β term by noticing that

$$G_{-m m_1' m}^{l l_1' l} = G_{m-m m_1' m}^{l l_1' l} = (2l+1) \sqrt{\frac{(2l_1'+1)}{4\pi}} \left(\begin{matrix} l & l_1' & l \\ m-m & m_1' & m \end{matrix} \right) \left(\begin{matrix} l & l & l \\ 0 & 0 & 0 \end{matrix} \right) ,$$

is non-vanishing if and only if

- $|l-l'| \leq l_1' \leq |l+l'|$

$$\Rightarrow 0 \leq l_1' \leq 2l$$

- $m-m+m_1'=0 \Rightarrow m_1'=0$

so that

$$G_{-m m' m}^{\ell \ell' \ell'} = (2\ell+1) \sqrt{\frac{(2\ell'+1)}{4\pi}} \begin{pmatrix} \ell & \ell & \ell' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell' \\ 0 & 0 & 0 \end{pmatrix} \delta_{m'0}.$$

With this, term β becomes

$$\begin{aligned} \beta &= \sum_m \left[\bar{\phi}_{(1)} + 2(-1)^m \sum_{l_1' m_1'} G_{-m m' m}^{\ell \ell' \ell'} F_{l_1' m_1'}^* [\Delta\phi]_{(12)} \right] \\ &= \left[(2\ell+1) \bar{\phi}_{(1)} + 2 \sum_{l_1'} \sum_m (-1)^m G_{m-m' 0}^{\ell \ell' \ell'} F_{l_1' 0}^* [\Delta\phi]_{(12)} \right] \\ &= (2\ell+1) \bar{\phi}_{(1)} + 2 \sum_{l_1'} F_{l_1' 0}^* [\Delta\phi]_{(12)} \begin{pmatrix} \ell & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix} \sqrt{\frac{(2\ell'+1)}{4\pi}} (2\ell+1) \\ &\quad \times \sum_m (-1)^m \begin{pmatrix} \ell & \ell & \ell' \\ m & -m & 0 \end{pmatrix} \end{aligned}$$

We can simplify further using the identities

$$\sum_m (-1)^m \begin{pmatrix} \ell & \ell & L \\ m & -m & 0 \end{pmatrix} = (-1)^\ell \sqrt{2\ell+1} \delta_{L0}$$

$$\Rightarrow \beta = (2\ell+1) \bar{\phi}_{(1)} + \frac{1}{\sqrt{4\pi}} (2\ell+1) \sqrt{2\ell+1} (-1)^\ell F_{00}^* [\Delta\phi]_{(12)} \begin{pmatrix} \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, noting that

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \frac{1}{\sqrt{2j+1}} (-1)^{j-m}. \Rightarrow \begin{pmatrix} \ell & \ell & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^\ell}{\sqrt{2\ell+1}}$$

we get

$$\beta = (2\ell+1) \left[\bar{\phi}_{(1)} + \frac{1}{\sqrt{4\pi}} F_{00}^* [\Delta\phi]_{(12)} \right],$$

where we have also used the fact that $\tilde{Y}_{\ell\ell}(\hat{n})$ is real to write $F_{00}^*[\Delta\phi](r_2) = F_{00}[\Delta\phi](r_2)$.

All in all :

$$T_\ell = \int d\zeta_1 d\zeta_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c} \times \left\{ \phi(r_1) C_L(r_1, r_2) \left[\phi(r_2) + \frac{1}{\sqrt{\pi}} F_{00}[\Delta\phi](r_2) \right] + C_e^{\Delta\phi}(r_1, r_2) + \sum_L M_{eL}^{\Delta\phi}(r_1, r_2) C_L(r_1, r_2) \right\}$$

The structure of the last term is analogous to the mode coupling induced by a mask.

- Let's do some of those out. We'll start with the fiducial term:

$$\begin{aligned}
 T_e &> \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) \phi(\zeta_2) C_e(\zeta_1, \zeta_2) \\
 &= \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) \phi(\zeta_2) \\
 &\quad \times \frac{2}{\pi} \int dK K^2 j_\ell(K\zeta_1) j_\ell(K\zeta_2) P(K; z(\zeta_1), z(\zeta_2))
 \end{aligned}$$

Using the leading-order Limber approximation:

$$\begin{aligned}
 T_e &> \int dr \left[\frac{H(r)}{c} \right]^2 \frac{1}{r} \left[\frac{\phi(r)}{\sqrt{r}} \right]^2 P_g \left(\frac{1 + \frac{1}{2}}{r}; z(r) \right) \\
 &= \int dr \left[\frac{H(r)}{c} \right]^2 \frac{1}{r^2} \left[\frac{\phi(r)}{r} \right]^2 P_g \left(\frac{1 + \frac{1}{2}}{r}; z(r) \right).
 \end{aligned}$$

This is the fiducial term. Everything else are biases.

Before continuing, let us define

$$C_e^{\Delta\phi}(\zeta_1, \zeta_2) \equiv \frac{\sum_m F_{im}[\Delta\phi](\zeta_1) F_{im}^*[\Delta\phi](\zeta_2)}{2l+1} \equiv \frac{2}{\pi} \int dK_1 dK_2 K_1 K_2 j_\ell(K_1 \zeta_1) j_\ell(K_2 \zeta_2) C_e^{\Delta\phi}(K_1, K_2)$$

where we have defined the SFB power spectrum

$$C_e^{\Delta\phi}(K_1, K_2) \equiv \frac{1}{2l+1} \sum_m \Delta\phi_{em}(K_1) \Delta\phi_{im}^*(K_2)$$

• Let's do out the additive correction:

$$T_e \supset \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c} C_e^{\Delta\phi}(\mathbf{r}_1, \mathbf{r}_2)$$

$$= \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c} \underbrace{\sum_m F_{lm}[\Delta\phi](\mathbf{r}_1) F_{lm}^*[\Delta\phi](\mathbf{r}_2)}_{2l+1}.$$

$$= \left(\frac{2}{\pi}\right)^2 \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c}$$

$$\times \int dk_r K_r^2 j_e(Kr_r) \int dr'_1 r'^2 j_e(Kr'_1)$$

$$\times \int dk_r K_r^2 j_e(Kr_r) \int dr'_2 r'^2 j_e(Kr'_2)$$

$$\times \int d\hat{n}_1 \Delta\phi(r_1 \cdot \hat{n}_1) \int d\hat{n}_2 \Delta\phi(r_2 \cdot \hat{n}_2) \underbrace{\sum_m Y_{lm}^*(\hat{n}_1) Y_{lm}(\hat{n}_2)}_{2l+1}$$

But the addition theorem for spherical harmonics states that

$$\underbrace{\sum_m Y_{lm}^*(\hat{n}_1) Y_{lm}(\hat{n}_2)}_{2l+1} = \frac{1}{4\pi} P_l(\hat{n}_1 \cdot \hat{n}_2),$$

so

$$T_e \supset \frac{1}{\pi^3} \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} \\ \times \int dk_i K_i^2 j_\ell(K_i r_i) \int dr'_i r'^2 j_\ell(K_i r'_i) \int d\hat{n}_i \Delta\phi(\epsilon_i; \hat{n}_i) \\ \times \int dk_i K_i^2 j_\ell(K_i r'_i) \int dr'_i r'^2 j_\ell(K_i r'_i) \int d\hat{n}_2 \Delta\phi(\epsilon_i; \hat{n}_i) P_e(\hat{n}_1 \cdot \hat{n}_2)$$

Let us rearrange terms in order to more clearly realize the applicability of the Limber approximation

$$= \frac{1}{\pi^3} \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} \\ \times \int dr'_i r'^2 \int d\hat{n}_i \Delta\phi(\epsilon_i; \hat{n}_i) \int dk_i K_i^2 j_\ell(K_i r'_i) j_\ell(K_i r'_i) \\ \times \int dr'_i r'^2 \int d\hat{n}_2 \Delta\phi(\epsilon_i; \hat{n}_i) P_e(\hat{n}_1 \cdot \hat{n}_2) \int dk_i K_i^2 j_\ell(K_i r'_i) j_\ell(K_i r'_i)$$

From the orthogonality relation:

$$\delta^D(k - k') = \frac{2kk'}{\pi} \int_0^\infty dr r^2 j_\ell(kr) j_\ell(k'r),$$

$$T_e \supset \frac{1}{\pi^3} \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} \\ \times \int dr'_i r'^2 \int d\hat{n}_i \Delta\phi(\epsilon_i; \hat{n}_i) \left\{ \frac{\pi}{2\epsilon_i} \delta_D(r_i - r'_i) \right\} \\ \times \int dr'_i r'^2 \int d\hat{n}_2 \Delta\phi(\epsilon_i; \hat{n}_i) P_e(\hat{n}_1 \cdot \hat{n}_2) \left\{ \frac{\pi}{2\epsilon_i \epsilon_2} \delta_D(r_i - r'_i) \right\}$$

$$= \frac{1}{4\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c}$$

$$\times \int d\hat{\mathbf{n}}_1 \int d\hat{\mathbf{n}}_2 \Delta\phi(\mathbf{r}, \hat{\mathbf{n}}_1) \Delta\phi(\mathbf{r}_2, \hat{\mathbf{n}}_2) P_\ell(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2)$$

Equivalently, we could've used the definition of the SFB power spectrum to write (directly from the 1st line)

$$T_\ell \geq \frac{2}{\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c} \int d\mathbf{k}_1 d\mathbf{k}_2 k_1 k_2 j_\ell(k_1 r_1) j_\ell(k_2 r_2) C_\ell^{\Delta\phi}(k_1, k_2)$$

if we had statistical isotropy, these k 's
would be forced to be equal (but we don't)

In the (leading-order) Limber approximation,

$$j_\ell(kx) \rightarrow \sqrt{\frac{\pi}{2v}} \delta_D(v - kx) = \frac{1}{x} \sqrt{\frac{\pi}{2v}} \delta_D(\frac{v}{x} - k)$$

in the radial integrals, where $v = \ell + \frac{1}{2}$.

$$\approx v \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c} \frac{1}{r_1^2} \frac{1}{r_2^2} C_\ell^{\Delta\phi}(\frac{v}{r_1}, \frac{v}{r_2})$$

!

- As per the simpler of the two multiplicative corrections:

$$\begin{aligned}
 T_e &\geq \frac{1}{\pi^2} \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) C_l(\zeta_1, \zeta_2) F_{00}[\Delta\phi](\zeta_2) \\
 &= 2 \frac{\pi}{\pi^2} \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) F_{00}[\Delta\phi](\zeta_2) \\
 &\times \int dK K^2 j_l(K\zeta_1) j_l(K\zeta_2) P_g(K; z(\zeta_1), z(\zeta_2)).
 \end{aligned}$$

Note: we can use Limber irrespective of the smoothness of $\Delta\phi$ as long as $P(k)$ is smooth w.r.t. the spherical Bessel functions. Question: is this a more robust Limber than that in Lavares & Alshordi? A special, extra robust case of it? That's what Gehardt and Dore seem to imply.

In the leading-order Limber approximation,

$$\begin{aligned}
 &\approx 2 \frac{\pi}{\pi^2} \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) F_{00}[\Delta\phi](\zeta_2) \\
 &\times \int dK K^2 \frac{1}{r_i} \sqrt{\frac{\pi}{2v}} \delta_0\left(\frac{v}{r_i} - K\right) \frac{1}{r_i} \sqrt{\frac{\pi}{2v}} \delta_0\left(\frac{v}{r_i} - K\right) P_g(K; z(\zeta_1), z(\zeta_2)) \\
 &= \frac{\pi}{\pi v} \int d\zeta_1 d\zeta_2 \frac{H(\zeta_1)}{c} \frac{H(\zeta_2)}{c} \phi(\zeta_1) F_{00}[\Delta\phi](\zeta_2) \\
 &\quad \times \frac{1}{r_i r_2} \left(\frac{v}{r_2}\right)^2 P_g\left(\frac{v}{r_i}; z(\zeta_1), z(\zeta_2)\right) \underbrace{\delta_0\left(\frac{v}{r_i} - \frac{v}{r_2}\right)}_{= \delta_0\left(\frac{r_1 r_2}{v} (r_2 - r_1)\right)} \\
 &= \frac{1}{\pi v} \int dr \left[\frac{H(r)}{c}\right]^2 \frac{\phi(r)}{r^2} F_{00}[\Delta\phi](r) P_g\left(\frac{v}{r}; z(r)\right).
 \end{aligned}$$

but

$$\begin{aligned}
 F_{\text{sum}}[\Delta\phi](r) &= \sqrt{\frac{2}{\pi}} \int dk K J_l(Kr) \Delta\phi_{\text{sum}}(K, 2cr) \\
 &= \sqrt{\frac{2}{\pi}} \int dk K \left[\sqrt{\frac{\pi}{2Kr}} J_{l+\frac{1}{2}}(kr) \right] \Delta\phi_{\text{sum}}(K, 2cr) \\
 &= \frac{1}{\sqrt{r}} \int dk \sqrt{K} J_{l+\frac{1}{2}}(kr) \Delta\phi_{\text{sum}}(K, 2cr) \\
 &= \int dk K J_{l+\frac{1}{2}}(kr) \left[\frac{\Delta\phi_{\text{sum}}(K, 2cr)}{\sqrt{Kr}} \right] \\
 &= \frac{1}{\sqrt{r}} \mathcal{H}_{l+\frac{1}{2}} \left[\frac{\Delta\phi_{\text{sum}}(K, 2cr)}{\sqrt{K}} \right] (r)
 \end{aligned}$$

↑ i.e., the $(l+\frac{1}{2})^{\text{th}}$ -order Hankel
 transform of this function

Consequently,

$$T_e^{\text{mult}} \approx \frac{1}{\sqrt{\pi}} \int dr \left[\frac{H(r)}{c} \right]^2 r^{-\frac{5}{2}} \phi(r) P\left(\frac{v}{r}; z(r)\right) \mathcal{H}_{\frac{1}{2}} \left[\frac{\Delta\phi_{\text{sum}}(K, 2cr)}{\sqrt{K}} \right] (r)$$

- And finally, the multiplicative term that resembles mask-convolution:

$$\begin{aligned}
 T_e &\supset \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} \lesssim_L M_{1L}^{\Delta\phi}(\epsilon_1, \epsilon_2) C_L(\epsilon_1, \epsilon_2) \\
 &= \lesssim_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1) \left(\begin{smallmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{smallmatrix} \right)^2 \\
 &\times \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} C_L(\epsilon_1, \epsilon_2) C_{l_1}^{\Delta\phi}(\epsilon_1, \epsilon_2) \\
 &= \lesssim_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1) \left(\begin{smallmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{smallmatrix} \right)^2 \\
 &\times \int d\epsilon_1 d\epsilon_2 \frac{H(\epsilon_1)}{c} \frac{H(\epsilon_2)}{c} \\
 &\quad \text{Series-expand here as in Liddle & Alford: } j_\ell(kx) \rightarrow \sqrt{\frac{\pi}{2x}} \delta_D(x-kx) \\
 &\quad \text{This is the weaker version of Limber.} \\
 &\times \frac{2}{\pi} \int dk_1 dk_2 K_1 K_2 j_{l_1}(K_1 \epsilon_1) j_{l_1}(K_2 \epsilon_2) C_{l_1}^{\Delta\phi}(K_1, K_2)
 \end{aligned}$$

$$\times \frac{2}{\pi} \int dk \quad K^2 j_L(K \epsilon_1) j_L(K \epsilon_2) P_s(K; z(\epsilon_1), z(\epsilon_2))$$

assume $P(k)$ is slowly varying s.t.

$$\begin{aligned}
 \int dk \quad K^2 j_L(K \epsilon_1) j_L(K \epsilon_2) P(k) &\propto P\left(\frac{L}{\epsilon_1}\right) \int dk \quad K^2 j_L(K \epsilon_1) j_L(K \epsilon_2) \\
 &\propto P\left(\frac{L}{\epsilon_1}\right) \frac{\pi}{2\epsilon_1 \epsilon_2} \delta(\epsilon_1 - \epsilon_2)
 \end{aligned}$$

(actually, you can probably recover this from applying the version above twice)

Important insight: the smoothness of $P(k)$ relative to the Bessel functions enforces $\epsilon_1 \approx \epsilon_2$ irrespective of $C^{\Delta\phi}(\epsilon_1, \epsilon_2)$. This is the more robust version of Limber.

We now use two versions of Limber, as highlighted in red.

$$\approx \sum_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1) \left(\begin{smallmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{smallmatrix} \right)^2$$

$$\int dr_1 dr_2 \frac{H(r_1)}{c} \frac{H(r_2)}{c}$$

$$\times \frac{v_i}{r_1^2 r_2^2} C_{l_1}^{\Delta\phi} \left(\frac{l_1 + l_2}{r_1}, \frac{l_1 + l_2}{r_2} \right)$$

$$\times \frac{1}{r_1^2} \int (r_1 - r_2) P_g \left(\frac{l + l_2}{r_1}; z^{(1)}, z^{(2)} \right)$$

$$= \frac{1}{2} \sum_L \frac{(2L+1)}{4\pi} \sum_{l_1} (2l_1+1)^2 \left(\begin{smallmatrix} l & l_1 & L \\ 0 & 0 & 0 \end{smallmatrix} \right)^2$$

$$\int dr \left[\frac{H(r)}{c} \right]^2 r^{-6} P_g \left(\frac{l + l_2}{r}; z^{(1)} \right) C_{l_1}^{\Delta\phi} \left(\frac{l_1 + l_2}{r}, \frac{l_1 + l_2}{r} \right)$$

*What if there's no fluctuation power below a certain scale
 $l_1 > l_{\max}$?*