Spherical Harmonic Transform Notes

I. SPHERICAL HARMONICS

A scalar field $f(\hat{n})$ on the sphere can be represented using spherical harmonic coefficients a_{lm} given by

$$a_{lm} = \int Y^*_{lm}(\hat{n}) f(\hat{n}) \quad (1)$$

$$f(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}). \quad (2)$$

The harmonics Y_{lm} are eigenfunctions of the Laplacian on the sphere, and the decomposition above is analogous to a Fourier expansion on the plane.

The harmonics obey the relation $Y_{lm} = (-1)^m Y^*_{l(-m)}$ and so for $f(\hat{n})$ which is real we have $a_{lm} = (-1)^m a_{l(-m)}$. In this case, we can alternatively decompose $f(\hat{n})$ using the real spherical harmonics R_{lm}, defined by

$$R_{lm} = \begin{cases} \frac{1}{\sqrt{2}}(Y_{lm} + Y^*_{lm}) & \text{if } m > 0 \\ Y_{l0} & \text{if } m = 0 \\ \frac{(-1)^m}{\mathrm{i}\sqrt{2}}(Y^*_{lm} - Y_{lm}) & \text{if } m < 0\,; \end{cases} \quad (3)$$

The corresponding decomposition of $f(\hat{n})$ is then

$$r_{lm} = \int R_{lm}(\hat{n}) f(\hat{n}) \quad (4)$$

$$f(\hat{n}) = \sum_{lm} r_{lm} R_{lm}(\hat{n}). \quad (5)$$

This representation is often simpler to work with on a computer, as it means that the coefficients and their covariance matrices are real.

II. SPIN-S SPHERICAL HARMONICS

Complex quantities on the sphere are often spin fields. Under rotation of the coordinate system by an angle α, the representation of these fields at the origin of the rotation must acquire an opposing phase of $e^{-is\alpha}$ to remain physically consistent, where s is referred to as the spin, or spin-weight. Both the (Q,U) representation of polarization (with $s = 2$) as well as lensing deflection vectors (with $s = 1$) are spin fields. To work with such spin-s quantities it is useful to represent them in a basis which absorbs this coordinate-dependent phase. Such a basis may be constructed from the standard (spin-0) spherical harmonics by the operation of spin raising and lowering operators. On the plane these operators are intuitively given by

$$\eth_\pm = \left[\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right]. \quad (6)$$

On the sphere, the situation is complicated by the connection which mixes the coordinate basis vectors under differentiation. The spin operators become [1]

$$\eth_\pm = -(\sin\theta)^{\pm s}\left[\frac{\partial}{\partial\theta} \pm \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right](\sin\theta)^{\mp s}. \quad (7)$$

We can then define the spin-s spherical harmonics as a natural extension of the standard spherical harmonics. Using the convention

$${}_sY_{lm} = \begin{cases} \sqrt{\frac{(l-s)!}{(l+s)!}}\eth^s_+ Y_{lm} & (0 \le s \le l) \\ \sqrt{\frac{(l+s)!}{(l-s)!}}(-1)^s \eth^{-s}_- Y_{lm} & (-l \le s \le 0) \\ 0 & (l < |s|). \end{cases} \quad (8)$$

The normalization factor ensures that the magnitude $|{}_sY_{lm}|^2$ integrates to unity over the sphere. A complex spin-s quantity ${}_sf(\hat{n}) = [\alpha_1 + i\alpha_2](\hat{n})$ may then be decomposed into harmonic coefficients ${}_sv_{lm}$ as

$${}_sv_{lm} = \int {}_sf(\hat{n})\, {}_sY^*_{lm}(\hat{n}). \quad (9)$$

The spin-s harmonics obey the parity relation

$${}_sY_{lm}(\pi - \theta, \phi + \pi) = (-1)^l\, {}_{-s}Y_{lm}(\theta, \phi). \quad (10)$$

This often makes it useful, instead of ${}_sv_{lm}$ to work with the gradient/curl modes defined by

$$\int {}_{\pm|s|}Y^*_{lm}(\hat{n})[\alpha_1 \pm i\alpha_2](\hat{n}) = -(\pm)^{|s|}[G_{lm} \pm iC_{lm}]. \quad (11)$$

In the spin-2 case G/C are usually referred to as E/B modes. The overall negative sign is designed for consistency with [2, 3] and the $(\pm)^{|s|}$ ensures that the gradient of a scalar function (a spin-1 quantity) is entirely G-mode. By construction, the G and C modes are parity eigenstates. The G_{lm} acquire a sign of $(-1)^l$ and the C_{lm} acquire a sign of $(-1)^{l+1}$ under a parity transformation. In terms of the fundamental transform of Eq. (9) we have

$$G_{lm} = -\frac{1}{2}[{}_{|s|}v_{lm} + (-1)^m\, {}_{|s|}v^*_{l(-m)}] \quad (12)$$

$$C_{lm} = -\frac{1}{2i}[{}_{|s|}v_{lm} - (-1)^m\, {}_{|s|}v^*_{l(-m)}]. \quad (13)$$

A. Recursion relations

The spin-s spherical harmonics are separable in θ, ϕ, and can be written as

$${}_sY_{lm}(\theta, \phi) = {}_s\lambda_{lm}(\theta) e^{im\phi}. \quad (14)$$

The ${}_s\lambda_{lm}$ are real, and the only complex contribution to ${}_sY_{lm}$ comes from $e^{im\phi}$.

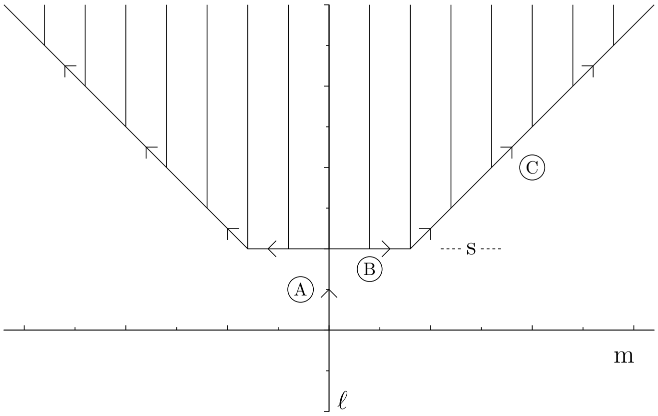

FIG. 1: Diagram illustrating the recursions in the (l,m) plane to determine ${}_s\lambda_{lm}$.

To calculate the ${}_s\lambda_{lm}$ the following recursion relation is useful [4]:

$$
{}_s\lambda_{lm} = \left[\left(\cos(\theta) + \frac{sm}{l(l-1)} \right) {}_s\lambda_{(l-1)\ m} - {}_s\lambda_{(l-2)\ m} / r_{s(l-1)m} \right] r_{slm}, \quad (15)
$$

where

$$
r_{slm} = \sqrt{\frac{l^2(4l^2-1)}{(l^2-m^2)(l^2-s^2)}}. \quad (16)
$$

To determine starting points for the recursion we can use an explicit formula for ${}_s\lambda_{lm}$ from Goldberg et. al. [5]: [1]

$$
{}_s\lambda_{lm}(\theta) = \sqrt{\frac{(l+m)!(l-m)!}{(l+s)!(l-s)!} \frac{(2l+1)}{4\pi}} \sin^{2l}\left(\frac{\theta}{2}\right) \times \sum_r \binom{l-s}{r} \binom{l+s}{r+s-m} \times (-1)^{l+m-r-s} \cot^{2r+s-m}\left(\frac{\theta}{2}\right). \quad (17)
$$

This leads to three relations which can be used to move through the (l,m,s) space:

$$
{}_s\lambda_{s0}(\theta) = +\sqrt{\frac{(2s+1)}{2s}} \sin(\theta)_{(s-1)}\lambda_{(s-1)0} \quad (A)
$$

$$
{}_s\lambda_{s(\pm|m|)}(\theta) = -\sqrt{\frac{(s-|m|+1)}{s+|m|}} \tan^{\pm 1}\left(\frac{\theta}{2}\right) {}_s\lambda_{s(\pm(|m|-1))} \quad (B)
$$

$$
{}_s\lambda_{l(\pm l)} = \mp\sqrt{\frac{l(2l+1)}{2(l+s)(l-s)}} \sin(\theta)_s\lambda_{(l-1)(\pm(l-1))}. \quad (C)
$$

Fig. 1 illustrates the use of these relations to move into a starting location for the l recursion of Eq. (15) from the origin at ${}_0Y_{00} = 1/\sqrt{4\pi}$.

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