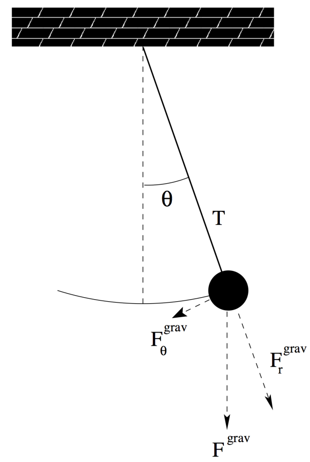
ASTR 119 Final Project Option #1
Forced, Damped Pendulum

One of the most basic systems exhibiting oscillatory behavior is the simple pendulum. This is a mass *m* connected by a massless, rigid string to an immovable support. In the simple pendulum case, there are only two forces acting on the mass: the tension in the string and gravity. Later we will include drag and driving forces.



The equations of motion for this system are most easily expressed in terms of the polar coordinates *r* and θ, where we define θ=0 to be when the pendulum is vertical.

Since we are assuming that the string is taut and inelastic, the gravitational force will be balanced by the tension and there will be no motion in the *r* direction (*dr/dt=0).* We then only need to worry about the component of the force in the θ direction.

From this figure, it is easy to see that the total force in the θ direction is

$$F_\theta^{\mathrm{grav}} = -mg\sin\theta$$

where *m* is the mass and *g* is the local gravitational acceleration.

To obtain an equation of motion, we need to determine the acceleration in polar coordinates. The acceleration only depends on θ, and we can use the relationship between the displacement along the arc and the angle, which is given by

$$s = l\theta$$

We set the gravitational acceleration in the θ direction equal to d^2s/dt^2 to obtain the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

where *l* is the length of the string.

Small Angle Approximation

If we consider only oscillations of small amplitude (say less than 5 degrees), we may approximate sin θ ~ θ. So we obtain

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

This differential equation may be familiar to you. It has the analytical solution

$$\theta(t) = \theta_0\sin(\Omega t + \phi)$$

where the natural period of oscillation is $\Omega = (g/l)^{1/2}$, the maximum amplitude is θ_0, and its phase is ϕ. The oscillations are sinusoidal with a frequency Ω, and will continue forever without decaying.

Damped, Driven Pendulum

We now add a dissipative force (e.g., air friction) to damp the motion, and a driving force (e.g., a varying electric field acting on a charged pendulum mass) to add energy to our pendulum. While an analytical solution is not generally possible for this damped driven pendulum, it is in fact relatively simple to deal with numerically.

How would we write a differential equation for this system?

First, we need to parameterize the frictional and driving forces. If we consider some type of resistance to be our dissipative force then, in general, the force is proportional to some power of the velocity.

For low velocities, F ~ *v*, while for higher velocities F ~ v^2. We'll limit ourselves to the low velocity case. We can describe the velocity in polar coordinates in terms of the angular velocity $\omega = v / l$. We will therefore write

$$F_{\text{friction}} = -mq\omega$$

where $\omega = d\theta/dt$ and q is a measure of the dissipative force into which we have absorbed the 1/l dependence. The force is in the direction opposite to the pendulum motion (hence the negative sign).

The driving force for our system will not depend on position or velocity of the pendulum, but is only a function of time that we take to be sinusoidal in dependence. In particular, we consider

$$F_{\text{driver}} = mA_d \sin(\Omega_d t)$$

where A_d is the amplitude of the driving force and Ω_d is the angular frequency of this force.

Using our equations for the forces applied to the pendulum, we can write the equation of motion for the pendulum as

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\omega + A_d \sin(\Omega_d t)}$$

note that we are no longer making the small angle approximation now, so that the differential equation is now *nonlinear*. This is difficult to deal with analytically, but is not a significant challenge numerically.

Comments about the Verlet Scheme

One complication arises when solving our damped driven pendulum equation using a Verlet scheme. Since our acceleration now depends on velocity as well as position, we have broken our assumptions.

Here is the Verlet scheme in polar coordinates:

$$\theta_{i+1} = \theta_i + \omega_i \Delta t + \frac{1}{2} a_i \Delta t^2 + O(\Delta t^3)$$
$$\omega_{i+1} = \omega_i + \frac{1}{2}(a_i + a_{i+1})\Delta t + O(\Delta t^3)$$

the second equation is now implicit, since the acceleration a_{i+1} depends on the velocity ω_{i+1}.

There are two routes to a solution.

The first solution is that we can use this equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\omega + A_d \sin(\Omega_d t)$$

in the velocity Verlet equation for a_i and a_{i+1}, and then collect the ω_{i+1} terms on the left side of this equation:

$$\omega_{i+1} = \omega_i + \frac{1}{2}(a_i + a_{i+1})\Delta t + O(\Delta t^3)$$

The other approach is to use a Runge-Kutta like approach to take a first-order guess at ω_{i+1} that maintains the second order guess for a_{i+1}.

By including the full nonlinear expression for the gravitational force, we have implicitly allow for motion at large amplitudes. We want to allow the pendulum to reach the full range of θ, rather than restricting |θ| to be less than π. We'll have to wrap 2π -> 0.

The behavior of the system will depend strongly on the parameters of our system, namely q, A_d, and Ω_d.

If there is no driving (A_d=0), then the amplitude of motion will decay. In this case, if we restricted ourselves to small θ, we would again have an analytical solution available to us.

If the driving and natural frequencies of the system are very different, then the system will be stable since the two sources of energy (driving and gravity) will be out of phase. If instead the driving and natural frequencies are comparable, there can be a very complicated motion.

Final Project #1 Requirements

1) Write a differential equation solver to evolve the coupled differential equations for θ and ω.

2) Starting from rest and initial angle of 1 radian with respect to vertical, and over the domain t = [0,100], integrate the evolution of four different pendula with the following parameters:

Case A: $(g/l)^{0.5} = 1$, $q = 0$, $A_d = 0$, $\Omega_d = 2/3$

Case B: $(g/l)^{0.5} = 1$, $q = 1/2$, $A_d = 0$, $\Omega_d = 2/3$

Case C: $(g/l)^{0.5} = 1$, $q = 1/2$, $A_d = 0.5$, $\Omega_d = 2/3$

Case D: $(g/l)^{0.5} = 1$, $q = 1/2$, $A_d = 1.46$, $\Omega_d = 2/3$

3) Produce 3-panel plots that show:

1st panel : θ vs. t, t=[0,100]
2nd panel: ω vs. t, t = [0, 100]
3rd panel: ω vs. θ, $\theta = [-\pi, \pi]$

For each Case A-D above.

4) Extra credit:

Produce four animations containing each of the 3 panels where the lines advance with time, plus an illustration of the pendulum with time in a 4th panel (make the plots 2x2 panels).