CLASS QUESTIONS

1) Two drunks start out together at the origin, having equal probability of taking a left or right step along x-axis. Find the probability that they meet again after N steps. The two men take their steps simultaneously.

Ans: Let L_1 and R_1 denote a left or right move for the first drunk
Let L_2 and R_2 denote a left or right move for the second drunk.
For this problem, we can assume that all steps are random (equal prob.)
The sequence of moves until N steps can be as follows,

$$L_1 L_1 R_1 R_1 L_1 R_1 L_1 R_1 R_1 \ldots L_1$$

$$L_2 L_2 L_2 R_2 R_2 R_2 L_2 R_2 L_2 \ldots R_2$$

So, the condition for the two drunks to meet after N steps is that no. of L_1 + no. of R_2 = N
Or, we can say that the no. of left steps should be equal, for both the drunks.

Let 'k' be the no. of steps to the left for the first drunk. So, for the given condition to be met, the second drunk must also take 'k' steps to the left.

$\therefore$ The probability is $\boxed{\sum_{k=0}^{N} \left({}^{N}C_{k}\right)^{2} \left(\frac{1}{2}\right)^{2N}} = \boxed{{}^{2N}C_{N} \left(\frac{1}{2}\right)^{2N}}$

This is because total possibility of steps = $\left(\frac{1}{2}\right)^{2N}$.
$\because$ Random walk, $\frac{1}{2}$ is the probability of each step and $2N$ can be the maximum distance covered by two drunks

Related Questions:

1) What is the probability for a drunk to be at the origin after taking N steps?

Let 'p', 'q' be the probabilities of a drunk to take a step to the right or left respectively.

Let n_1, n_2 be the steps taken to the right and left respectively by the drunk.

We understand that,

$$p+q = 1 \qquad n_1 + n_2 = N \qquad \text{(total no. of steps = N) taken}$$

Now, the probability of taking exactly n_1 steps to the right out of N steps, is

$$P(n_1) = {}^{N}C_{n_1}\, p^{n_1} (1-p)^{n_2} = {}^{N}C_{n_1}\, p^{n_1} q^{n_2}$$

Here, due to random walk of drunks, $p = q = \frac{1}{2}$;

Also, in order to return to the origin, $n_1 = n_2 = \frac{N}{2}$.

i.e. no. of left steps taken = no. of right steps taken.

$\therefore$ P(drunk returning to origin after N steps)

$$= {}^{N}C_{N/2} \left(\frac{1}{2}\right)^{\frac{N}{2}} \left(\frac{1}{2}\right)^{\frac{N}{2}}$$

$$= \boxed{{}^{N}C_{N/2} \left(\frac{1}{2}\right)^{N}}$$

2) Mean displacement of the drunk?

For any one complete walk of the drunk, we can model it as, $d = a_1 + a_2 + \dots + a_N$ 'd' is total distance covered. a_i ($i = 1$ to N) are the steps.

As the drunk starts from the origin,

$$a_i = +1 \text{ (right step)} \quad \text{or} \quad a_i = -1 \text{ (left step)}$$

$\langle d \rangle$ (average or mean displacement of drunk)

$$= \langle (a_1 + a_2 + \dots + a_N) \rangle = \langle a_1 \rangle + \langle a_2 \rangle + \dots + \langle a_N \rangle$$

However, $\langle a_i \rangle = 0$ because if we repeat the experiment many times and that a_i can only take values ± 1.

So, we can write,

$$\langle d\rangle = \langle a_1\rangle + \langle a_2\rangle + \dots + \langle a_N\rangle = 0 + \dots + 0 = \underline{\underline{0}}$$

$\therefore$ Mean displacement of drunk $= \underline{\underline{0}}$

3) Mean square displacement of the drunk?

We saw from last question,

$$d = a_1 + a_2 + \dots + a_N$$

$\langle d^2\rangle$ (mean square displacement of drunk)

$$= \langle (a_1 + a_2 + \dots + a_N)^2\rangle = \langle (a_1 + a_2 + \dots + a_N)(a_1 + a_2 + \dots + a_N)\rangle$$

$$= \langle a_1^2\rangle + \langle a_2^2\rangle + \langle a_3^2\rangle + \dots + \langle a_N\rangle^2 + 2\big(\langle a_1 a_2\rangle + \langle a_1 a_3\rangle + \dots + \langle a_1 a_N\rangle + \langle a_2 a_3\rangle + \dots + \langle a_2 a_N\rangle + \dots\big)$$

Now,

a_i^2 will always be 1 $(a_i = \pm 1)$ $(i = 1 \text{ to } N)$

$\therefore \langle a_i^2\rangle = 1$

Consider $\langle a_1 a_2\rangle$. There are four possible combinations for $a_1 a_2$.

a_1	a_2	$a_1 a_2$
-1	-1	1
-1	1	-1
1	-1	-1
1	1	1

As $a_1 a_2$ can only be either 1 or -1, repeating the experiment several times, makes $\langle a_1 a_2\rangle = 0$. Same goes for the other multiplication terms $\langle a_1 a_3\rangle, \langle a_2 a_N\rangle, \dots$

$$\therefore \langle d^2\rangle = \underbrace{1 + 1 + \dots + 1}_{N \text{ terms}} + 2(0 + \dots + 0 + 0 + \dots 0 + \dots) = \underline{\underline{N}}$$