

Comparative anatomy of geophysical flow models and modeling assumptions using uncertainty quantification

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Abstract

Complex systems (e.g., volcanoes, debris flows, climate) commonly have many models advocated by different modelers and incorporating different modeling assumptions. Limited and sparse data on the modeled phenomena does not permit a clean discrimination among models for fitness of purpose, and, heuristic choices are usually made, especially for critical predictions of behavior that has not been experienced. We advocate here for characterizing models and the modeling assumptions they represent using a statistical approach over the full range of applicability of the models. Such a characterization may then be used to decide the appropriateness of a model for use, and, perhaps as needed weighted compositions of models for better predictive power. We use the example of dense granular representations of natural mass flows in volcanic debris avalanches, to illustrate our approach.

1 Introduction

This paper presents a systematic approach to the study of models of complex systems, applied to the case of geophysical mass flows modeling.

A simple though not necessarily comprehensive definition of a model is that:

A model is a representation of a postulated relationship among inputs and outputs of a system usually informed by observation and a hypothesis that best explains them.

The definition captures two of the most important characteristics

- models depend on a *hypothesis*, and,
- models use the *data from observation* to validate and refine the hypothesis.

Errors and uncertainty in the data and limitations in the hypothesis (usually a tractable and computable mathematical construct articulating beliefs like proportionality, linearity etc.) are immediate challenges that must be overcome to construct useful and credible models.

A model is most useful in predicting the behavior of a system for unobserved inputs and interpretability or explainability of the system's behavior. Since, models require a hypotheses implies that the model is a formulation of a belief about the data. The immediate consequence of this that the model may be very poor about such prediction even when sufficient care is taken to use all the available data and information since the *subjectivity of the belief* can never be completely eliminated. Secondly, the data at hand may not provide enough information about the system to characterize its behavior at the desired prediction. What makes this problem even more acute is that we are often interested in modeling outcomes that are not observed and perhaps sometimes not observable.

The consequence of this lack of knowledge and limited data is the multiplicity of beliefs about the complex system being modeled and a profusion of models based on different modeling assumptions and data use. These competing models lead to much debate among scientists. Principles like "Occam's razor" and Bayesian statistics [Farrell et al. \(2015\)](#) provide some guidance but simple robust approaches that allow the testing of models for fitness need to be developed. We present in this paper a simple data driven approach to discriminate among models and the modeling assumptions implicit in each model, given a range of phenomena to be studied. We illustrate the approach by work on granular flow models of large mass flows.

1.1 Models and assumptions

An assumption is a simple concept – any atomic postulate about relationships among quantities under study e.g., a linear stress strain relationship $\sigma = E\epsilon$ or neglecting some quantities in comparison to larger quantities $\theta \approx \sin(\theta)$ for small θ . Models are compositions of many such assumptions. The study of models is thus implicitly a study of these assumptions and their composability and applicability in a particular context. Sometimes a good model contains a useless assumption that may be removed, sometimes a good assumption should be implemented inside a different model - these are usually subjective choices, not data driven. Moreover, the correct assumptions may change through time, making *model choice* more difficult.

The rest of the paper will define our approach and a simple illustration using 3 models for large scale mass flows incorporated in our large scale mass flow simulation framework TITAN2D [Patra et al. \(2005, 2006\); Yu et al. \(2009\); Aghakhani et al. \(2016\)](#). The 4th release of TITAN2D offers multiple rheology options in the same cyber-infrastructure. The availability of 3 distinct models for similar phenomena in the same tool provides us the ability to directly compare inputs, outputs and internal variables in all the 3 models.

So far, TITAN2D achieved many successful applications in the simulation of different geophysical mass flows with specific characteristics ([Sheridan et al., 2005; Rupp et al., 2006; Norini et al., 2009; Charbonnier and Gertisser, 2009; Procter et al., 2010; Sheridan et al., 2010; Sulpizio et al., 2010; Capra et al., 2011](#)). Several studies involving TITAN2D were recently directed towards a statistical study of geophysical flows, focusing on uncertainty quantification and propagation ([Dalbey et al., 2008; Dalbey, 2009; Stefanescu et al., 2012a,b](#)), or on the more efficient production of hazard maps ([Bayarri et al., 2009; Spiller et al., 2014; Bayarri et al., 2015; Ogburn et al., 2016](#)).

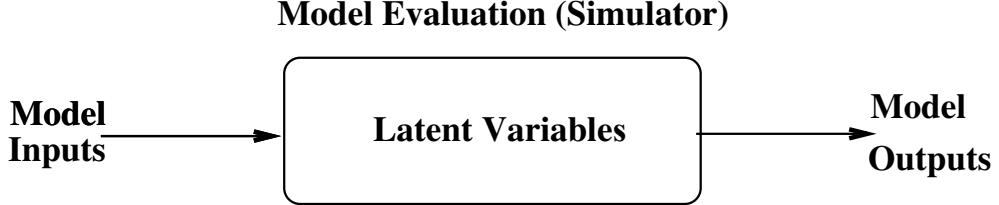


Figure 1: Models and variables

1.2 Analysis of Modeling assumptions and models

Let us define $(M(A), P_{M(A)})$, where A is a set of assumptions, $M(A)$ is the model which combines those assumptions, and P_M is a probability distribution in the parameter space of M . For the sake of simplicity we assume P_M to be uniformly distributed on selected parameter ranges. While the support of P_M can be restricted to a single value by solving an inverse problem for the optimal reconstruction of a particular flow, this is not possible if we are interested in the general predictive capabilities of the model, where we are interested in the outcomes over a whole range.

Stage 1: Parameter Ranges In this study, we always assume

$$P_M \sim \bigotimes_{i=1}^{N_M} \text{Unif}(a_{i,M}, b_{i,M}),$$

where N_M is the number of parameters of M . These parameter ranges will be chosen using information gathered from the literature about the physical meaning of those values together with a preliminary testing for physical consistency of model outcomes and range of inputs/outcomes of interest.

Stage 2 Simulations and Data Gathering The simulation algorithms can be represented as:

The *model inputs* are the parameters of M , the *latent variables* include quantities in the model evaluation that are ascribable to specific assumptions A_i . These are usually not observed as outputs from the model. For example in momentum balances of complex flow calculations these could be values of different source terms, dissipation terms and inertia terms. Finally, the *model outputs* include explicit outcomes e.g., for flow calculations these could be flow height, lateral extent, area, velocity, acceleration, and derived quantities such as Froude number Fr . In general, for each quantity of interest (QoI), we use a Monte Carlo simulation, sampling the input variables and obtaining a family of graphs plotting their expectation, and their 5th and 95th percentiles. Our sampling technique of the input variables is based on the Latin Hypercube Sampling (LHS) idea, and in particular, on the improved space-filling properties of the orthogonal array-based Latin Hypercubes (see Appendix A).

Stage 3 Results Analysis These and other statistics can now be compared to determine the need for different modeling assumptions and the relative merits of different models. Thus, analysis of the data gathered over the entire range of flows for the state variables and outcomes leads to a quantitative basis for accepting or rejecting particular assumptions or models for specific outcomes.

2 Modeling of geophysical mass flows

Dense large scale granular avalanches are a complex class of flows with physics that has often been poorly captured by models that are computationally tractable. Sparsity of actual flow data (usually only a posteriori deposit information is available), and large uncertainty in the mechanisms of initiation and flow propagation, make the modeling task challenging, and a subject of much continuing interest. Models that appear to represent the physics well in certain flows, may turn out to be poorly behaved in others, due to intrinsic mathematical or numerical issues. Nevertheless, given the large implications on life and property, many models with different modeling assumptions have been proposed. For example in Iverson (1997); Iverson and Denlinger (2001); Denlinger and Iverson (2001); Pitman et al. (2003); Denlinger and

Iverson (2004); Iverson et al. (2004), the depth-averaged model was applied in the simulation of test geophysical flows in large scale experiments. Several studies were specifically devoted to the modeling of volcanic mass flows (Bursik et al., 2005; Kelfoun and Druitt, 2005; Charbonnier and Gertisser, 2009; Kelfoun et al., 2009; Procter et al., 2010; Kelfoun, 2011; Charbonnier et al., 2013). In fact, volcanoes are great sources for a rich variety of geophysical flow types and provide field data from past flow events.

Modeling in this case proceeds by first assuming that the laws of mass and momentum conservation hold for properly defined system boundaries. The scale of these flows – very long and wide with small depth led to the first most generally accepted assumption – shallowness Savage and Hutter (1989). This allows an integration through the depth to obtain simpler and more computationally tractable equations. This is the next of many assumptions that have to be made. Both of these are fundamental assumptions which can be tested in the procedure we established above. Since, there is a general consensus and much evidence in the literature of the validity of these assumptions we defer analysis of these to future work.

The depth-averaged Saint-Venant equations that result are:

$$\begin{aligned}\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) &= 0 \\ \frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}\left(h\bar{u}^2 + \frac{1}{2}kg_zh^2\right) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) &= S_x \\ \frac{\partial}{\partial t}(h\bar{v}) + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(h\bar{v}^2 + \frac{1}{2}kg_zh^2\right) &= S_y\end{aligned}\quad (1)$$

Here the Cartesian coordinate system is aligned such that z is normal to the surface; h is the flow height in the z direction; $h\bar{u}$ and $h\bar{v}$ are respectively the components of momentum in the x and y directions; and k is the coefficient which relates the lateral stress components, $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$, to the normal stress component, $\bar{\sigma}_{zz}$. The definition of this coefficient depends on the constitutive model of the flowing material we choose. Note that $\frac{1}{2}kg_zh^2$ is the contribution of hydrostatic pressure to the momentum fluxes. S_x and S_y are the sum local stresses: they include the gravitational driving forces, the basal friction force resisting to the motion of the material, and additional forces specific of rheology assumptions.

The final class of assumptions are the assumptions on the rheology of the flows – in particular in this context assumptions used to model different dissipation mechanisms embedded in S_x, S_y that lead to a plethora of models with much controversy on the most suitable model.

2.1 Overview of the models

In the three following sections, we briefly describe *Mohr-Coulomb* (MC), *Pouliquen-Forsterre* (PF) and *Voellmy-Salm* (VS) models. Models based on additional heterogeneous assumptions are possible, either more complex (Pitman and Le, 2005) or more simple (Dade and Huppert, 1998), but we are focusing on those because of their historical relevance and their comparable degree of complexity.

2.1.1 Mohr-Coulomb

Based on the long history of studies in soil mechanics (Rankine, 1857; Jaeger et al., 1989), the Mohr-Coulomb rheology (MC) was developed and used to represent the behavior of geophysical mass flows Savage and Hutter (1989).

Shear and normal stress are assumed to obey Coulomb friction equation, both within the flow and at its boundaries. In other words,

$$\tau = \sigma \tan \phi, \quad (2)$$

where τ and σ are respectively the shear and normal stresses on failure surfaces, and ϕ is a friction angle. This relationship does not depend on the flow speed.

We can summarize the MC rheology assumptions as:

- *Basal Friction* based on a constant friction angle.
- *Internal Friction* based on a constant friction angle.
- *Earth pressure coefficient* formula depends on the Mohr circle.
- *Velocity based curvature effects* are included into the equations.

Under the assumption of symmetry of the stress tensor with respect to the z axis, the earth pressure coefficient $k = k_{ap}$ can take on only one of three values $\{0, \pm 1\}$. The material yield criterion is represented by the two straight lines at angles $\pm\phi$ (the internal friction angle) relative to horizontal direction. Similarly, the normal and shear stress at the bed are represented by the line $\tau = -\sigma \tan(\delta)$ where δ is the bed friction angle.

MC equations As a result, we can write down the source terms of the Eqs. (1):

$$\begin{aligned} S_x &= g_x h - \frac{\bar{u}}{\|\bar{\mathbf{u}}\|} \left[h \left(g_z + \frac{\bar{u}^2}{r_x} \right) \tan(\phi_{bed}) \right] - h k_{ap} \operatorname{sgn} \left(\frac{\partial \bar{u}}{\partial y} \right) \frac{\partial(g_z h)}{\partial y} \sin(\phi_{int}) \\ S_y &= g_y h - \frac{\bar{v}}{\|\bar{\mathbf{u}}\|} \left[h \left(g_z + \frac{\bar{v}^2}{r_y} \right) \tan(\phi_{bed}) \right] - h k_{ap} \operatorname{sgn} \left(\frac{\partial \bar{v}}{\partial x} \right) \frac{\partial(g_z h)}{\partial x} \sin(\phi_{int}) \end{aligned} \quad (3)$$

Where, $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$, is the depth-averaged velocity vector, r_x and r_y denote the radii of curvature of the local basal surface. The inverse of the radii of curvature is usually approximated with the partial derivatives of the basal slope, e.g., $1/r_x = \partial \theta_x / \partial x$, where θ_x is the local bed slope.

2.1.2 Pouliquen-Forterre

The scaling properties for granular flows down rough inclined planes led to the development of the Pouliquen-Forterre rheology (PF), assuming a variable frictional behavior as a function of flow regime (i.e. Froude Number, Fr) and flow depth (Pouliquen, 1999; Forterre and Pouliquen, 2002; Pouliquen and Forterre, 2002; Forterre and Pouliquen, 2003). They allowed for an improved modeling of front propagation, mass spreading, surface wave-propagation and vortices-evolution in the flows on a rough terrain (Forterre, 2006; Jop et al., 2006; Forterre and Pouliquen, 2008).

PF rheology assumptions can be summarized as:

- *Basal Friction* is based on an interpolation of two different friction angles, based on the flow regime and depth.
- *Internal Friction* is neglected.
- *Earth pressure coefficient* is equal to one.
- Normal stress is modified by a *hydrostatic pressure force* related to the flow height gradient.
- Velocity based *curvature effects* are included into the equations.

Two critical slope inclination angles are defined as functions of the flow thickness, namely $\phi_{start}(h)$ and $\phi_{stop}(h)$. The function $\phi_{stop}(h)$ gives the slope angle at which a steady uniform flow leaves a deposit of thickness h , while $\phi_{start}(h)$ is the angle at which a layer of thickness h is mobilized. They define two different basal friction coefficients.

$$\mu_{start}(h) = \tan(\phi_{start}(h)) \quad (4)$$

$$\mu_{stop}(h) = \tan(\phi_{stop}(h)) \quad (5)$$

An empirical friction law $\mu_b(\|\bar{\mathbf{u}}\|, h)$ is then defined in the whole range of velocity and thickness. The expression changes depending on two flow regimes, according to a parameter β and the Froude number $Fr = \|\bar{\mathbf{u}}\| / \sqrt{hg_z}$.

Dynamic friction regime - $Fr \geq \beta$

$$\mu(h, Fr) = \mu_{stop}(h\beta/Fr) \quad (6)$$

Intermediate friction regime - $0 \leq Fr < \beta$

$$\mu(h, Fr) = \left(\frac{Fr}{\beta} \right)^\gamma [\mu_{stop}(h) - \mu_{start}(h)] + \mu_{start}(h), \quad (7)$$

where γ is the power of extrapolation, assumed equal to 10^{-3} in the sequel (Pouliquen and Forterre, 2002).

The functions μ_{stop} and μ_{start} are defined by:

$$\mu_{stop}(h) = \tan \phi_1 + \frac{\tan \phi_2 - \tan \phi_1}{1 + h/\mathcal{L}} \quad (8)$$

and

$$\mu_{start}(h) = \tan \phi_3 + \frac{\tan \phi_2 - \tan \phi_1}{1 + h/\mathcal{L}} \quad (9)$$

The critical angles ϕ_1 , ϕ_2 and ϕ_3 and the parameters \mathcal{L}, β are the parameters of the model.

In particular, \mathcal{L} is the characteristic depth of the flow over which a transition between the angles ϕ_1 to ϕ_2 occurs, in the μ_{stop} formula. In practice, if $h \ll \mathcal{L}$, then $\mu_{stop}(h) \approx \tan \phi_2$, and if $h \gg \mathcal{L}$, then $\mu_{stop}(h) \approx \tan \phi_1$. The effect of the topographic local curvatures is also taken into account.

PF equations The depth-averaged Eqs. (1) source terms thus take the following form:

$$\begin{aligned} S_x &= g_x h - \frac{\bar{u}}{\|\tilde{\mathbf{u}}\|} \left[h \left(g_z + \frac{\bar{u}^2}{r_x} \right) \mu_b(\|\tilde{\mathbf{u}}\|, h) \right] + g_z h \frac{\partial h}{\partial x} \\ S_y &= g_y h - \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \left[h \left(g_z + \frac{\bar{v}^2}{r_y} \right) \mu_b(\|\tilde{\mathbf{u}}\|, h) \right] + g_z h \frac{\partial h}{\partial y} \end{aligned} \quad (10)$$

2.1.3 Voellmy-Salm

The theoretical analysis of dense snow avalanches led to the VS rheology (VS) (Voellmy, 1955; Salm et al., 1990; Salm, 1993; Bartelt et al., 1999). Dense snow or debris avalanches consist of mobilized, rapidly flowing ice-snow mixed to debris-rock granules (Bartelt and McArdell, 2009). The VS rheology assumes a velocity dependent resisting term in addition to the traditional basal friction, ideally capable of including an approximation of the turbulence-generated dissipation. Many experimental and theoretical studies were developed in this framework (Gruber and Bartelt, 2007; Kern et al., 2009; Christen et al., 2010; Fischer et al., 2012).

The following relation between shear and normal stresses holds:

$$\tau = \mu \sigma + \frac{\rho \|\mathbf{g}\|}{\xi} \|\tilde{\mathbf{u}}\|^2, \quad (11)$$

where, σ denotes the normal stress at the bottom of the fluid layer and $\mathbf{g} = (g_x, g_y, g_z)$ represents the gravity vector. The VS rheology adds a velocity dependent *turbulent* friction to the traditional velocity independent basal friction term which is proportional to the normal stress at the flow bottom. The two parameters of the model are the bed friction coefficient μ and the turbulent friction coefficient ξ .

We can summarize VS rheology assumptions as:

- *Basal Friction* is based on a constant coefficient, similarly to the MC rheology.
- *Internal Friction* is neglected.
- *Earth pressure coefficient* is equal to one.
- Additional *turbulent friction* is based on the local velocity by a quadratic expression.
- Velocity based *curvature effects* are included into the equations, following an alternative formulation.

The effect of the topographic local curvatures is again taken into account by adding the terms containing the local radii of curvature r_x and r_y . In this case the formula is considering the modulus of velocity instead than the scalar component Pudasaini and Hutter (2003); Fischer et al. (2012).

VS equations Therefore, the final source terms take the following form:

$$\begin{aligned} S_x &= g_x h - \frac{\bar{u}}{\|\tilde{\mathbf{u}}\|} \left[h \left(g_z + \frac{\|\tilde{\mathbf{u}}\|^2}{r_x} \right) \mu + \frac{\|\mathbf{g}\|}{\xi} \|\tilde{\mathbf{u}}\|^2 \right], \\ S_y &= g_y h - \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \left[h \left(g_z + \frac{\|\tilde{\mathbf{u}}\|^2}{r_y} \right) \mu + \frac{\|\mathbf{g}\|}{\xi} \|\tilde{\mathbf{u}}\|^2 \right]. \end{aligned} \quad (12)$$

2.2 Latent variables

For analysis of modeling assumptions we need to record and classify the results of different modeling assumptions. In our case study, we focus on the right-hand side terms in the momentum equation, we call them RHS forces. These terms are explored in detail in the next sections.

$$\mathbf{RHS}_1 = [g_x h, g_y h], \quad (13)$$

it is the gravitational force term, it has the same formulation in all models.

The formula of **basal friction force** \mathbf{RHS}_2 depends on the model:

$$\begin{aligned} \mathbf{RHS}_2 &= -hg_z \tan(\phi_{bed}) \left[\frac{\bar{u}}{\|\tilde{\mathbf{u}}\|}, \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \right], \text{ in MC model.} \\ \mathbf{RHS}_2 &= -hg_z \mu_b(\|\tilde{\mathbf{u}}\|, h) \left[\frac{\bar{u}}{\|\tilde{\mathbf{u}}\|}, \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \right], \text{ in PF model.} \\ \mathbf{RHS}_2 &= -hg_z \mu \left[\frac{\bar{u}}{\|\tilde{\mathbf{u}}\|}, \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \right], \text{ in VS model.} \end{aligned} \quad (14)$$

The formula of the force related to the **topography curvature**, \mathbf{RHS}_3 , also depends on the model:

$$\begin{aligned} \mathbf{RHS}_3 &= -h \tan(\phi_{bed}) \left[\frac{\bar{u}^3}{r_x \|\tilde{\mathbf{u}}\|}, \frac{\bar{v}^3}{r_y \|\tilde{\mathbf{u}}\|} \right], \text{ in MC model.} \\ \mathbf{RHS}_3 &= -h \mu_b(\|\tilde{\mathbf{u}}\|, h) \left[\frac{\bar{u}^3}{r_x \|\tilde{\mathbf{u}}\|}, \frac{\bar{v}^3}{r_y \|\tilde{\mathbf{u}}\|} \right], \text{ in PF model.} \\ \mathbf{RHS}_3 &= -h \mu \left[\frac{\bar{u} \|\tilde{\mathbf{u}}\|}{r_x}, \frac{\bar{v} \|\tilde{\mathbf{u}}\|}{r_y} \right], \text{ in VS model.} \end{aligned} \quad (15)$$

All the three models have an additional force term, having a different formula and meaning in the three models:

$$\begin{aligned} \mathbf{RHS}_4 &= -hk_{ap} \sin(\phi_{int}) \left[\operatorname{sgn}\left(\frac{\partial \bar{u}}{\partial y}\right) \frac{\partial(g_z h)}{\partial y}, \operatorname{sgn}\left(\frac{\partial \bar{v}}{\partial x}\right) \frac{\partial(g_z h)}{\partial x} \right], \text{ in MC model.} \\ \mathbf{RHS}_4 &= g_z h \left[\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right], \text{ in PF model.} \\ \mathbf{RHS}_4 &= -\frac{\|\mathbf{g}\|}{\xi} \|\tilde{\mathbf{u}}\|^2 \left[\frac{\bar{u}}{\|\tilde{\mathbf{u}}\|}, \frac{\bar{v}}{\|\tilde{\mathbf{u}}\|} \right], \text{ in VS model.} \end{aligned} \quad (16)$$

These latent variables can be analyzed locally and globally for discriminating among the different modeling assumption.

2.3 Monte Carlo Process and Statistical Analysis

For our study, the flow range is defined by establishing boundaries for inputs like flow volume and rheology coefficients. Optionally, these could include also flow initiation site and geometry, and the digital elevation map. The Latin Hypercube Sampling is performed over $[0, 1]^3$ for the MC and VS input parameters, and $[0, 1]^4$ for PF input parameters. Those dimensionless samples are linearly mapped to fill the required intervals.

Following the simulations, we generate data for each sample run and each outcome and latent variable $f(\underline{x}, t)$ calculated as a function of time on the elements of the computational grid. This analysis generates tremendous volume of data which must then be analyzed using statistical methods for summative impact. The latent variables in this case are the mass and force terms in the conservation laws defined above. In more detail, local sampling is performed by considering the elements of the adapting mesh which are found to contain the sample points. Instead, spatial integrals are defined by $F(t) = \int_{\mathbb{R}^k} f(\underline{x}, t) d\underline{x}$. In

the most of the cases $k = 2$, and $d\mathbf{x}$ is given by the area of the mesh elements. Sometimes $k = 3$, e.g. concerning speed, and $d\mathbf{x}$ is the element of volume corresponding to the mesh elements.

We devise many statistical measures for analyzing the data. For instance, let $(F_i(x, t))_{i=1,\dots,4}$ be an array of force components, where $x \in \mathbf{R}^2$ is a spatial location, and $t \in T$ is a time instant. The degree of contribution of those force terms can be significantly variable in space and time, and we define the *dominance factors* $(p_j)_{j=1,\dots,k}$, i.e., the probability of each F_j to be the dominant force at (x, t) . Those probabilities provide insight into the dominance of a particular source or dissipation (identified with a particular modeling assumption) term on the model dynamics. The dominance factors can be adopted to define a statistical decomposition of the contributions of the forces, as detailed in Appendix B.

2.4 Overview of the case studies

The first case study assumes very simple boundary conditions, and corresponds to an experiment fully described in Webb (2004); Bursik et al. (2005); Webb and Bursik (2016). It is a classical flow down an inclined plane set-up, including a change in slope to an horizontal plane (Fig. 2). Modeling flow of granular material down an inclined plane was explored in detail by several studies, both theoretically and experimentally (Ruyer-Quil and Manneville, 2000; Silbert et al., 2001; Da Cruz et al., 2005).

In our setting, four locations are selected among the center line of the flow to accomplish local testing. These are: the initial pile location $L_1 = (-0.70, 0)$ m, the middle of the inclined plane $L_2 = (-0.35, 0)$ m, the change in slope $L_3 = (0, 0)$ m, the middle of the flat plane $L_4 = (0.15, 0)$ m (see Section 3).

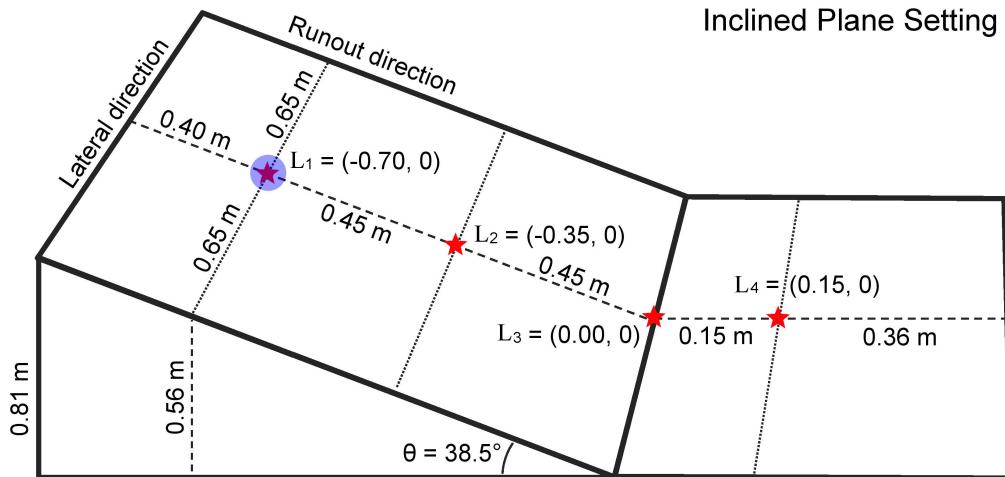


Figure 2: Inclined plane description, including local samples sites (red stars). Pile location is marked by a blue dot.

The second case study is a block and ash flow down the slope of Volcán de Colima (MX) - an andesitic stratovolcano that rises to 3,860 m above sea level, situated in the western portion of the Trans-Mexican Volcanic Belt (Fig. 3).

Volcán de Colima has historically been the most active volcano in México (la Cruz-Reyna, 1993; Zobin et al., 2002; González et al., 2002). The modeling of pyroclastic flows generated by explosive eruptions and lava dome collapses of Volcán de Colima is a well studied problem (Martin Del Pozzo et al., 1995; Sheridan and Macías, 1995; Saucedo et al., 2002, 2004, 2005; Sarocchi et al., 2011; Capra et al., 2015). The presence of a change in slope followed by multiple ravines characterize this framework. The volcano has been already used as a case study in several research involving the Titan2D code (Rupp, 2004; Rupp et al., 2006; Dalbey et al., 2008; Yu et al., 2009; Sulpizio et al., 2010; Capra et al., 2011; Aghakhani et al., 2016). During July 10th-11th, 2015, the volcano underwent its most intense eruptive phase since

its Subplinian-Plinian 1913 AD eruption (Saucedo et al., 2010; Zobin et al., 2015; Reyes-Dávila et al., 2016; Capra et al., 2016; Macorps et al., 2018).

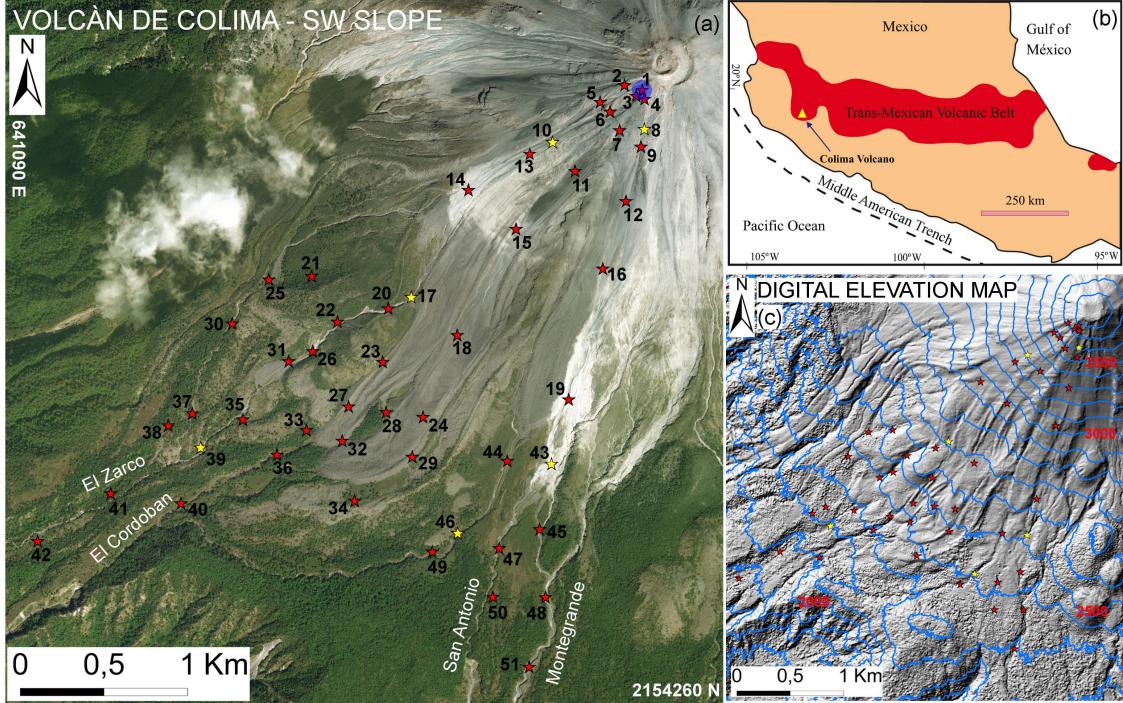


Figure 3: (a) Volcán de Colima (México) overview, including 51 numbered local sample sites (stars) and four labeled major ravines channeling the flow. Pile location is marked by a blue dot. Reported coordinates are in UTM zone 13N. Background is a satellite photo. (b) Regional geology map. (c) Digital elevation map. Six points that are adopted as preferred locations are highlighted in yellow. Elevation isolines are included in blue, elevation values in red.

We assume the flow to be generated by the gravitational collapse of a material pile placed close to the summit area. The volcano already produced either pyroclastic flows generated by lava dome collapse, called Merapi style flows, or by an eruptive column partial collapse, i.e. Soufrière style flows (Macorps et al., 2018). A dome collapse occurs when there is a significant amount of recently-extruded highly-viscous lava piled up in an unstable configuration around a vent. Further extrusion and/or external forces can cause the still hot dome of viscous lava to collapse, disintegrate, and avalanche downhill (Bursik et al., 2005; Wolpert et al., 2016; Hyman and Bursik, 2018). Eruptive column collapse can occur during explosive eruptions, when the eruption column can no longer sustain the weight of material due to loss of pressure, and hence it partially collapses down. The hot, dense blocks in this “block and ash” flow (BAF) will typically range from centimeters to a few meters in size. The matrix is composed of fine ash from the comminuted blocks. Computations were performed on a DEM with 5m-pixel resolution, obtained from LiDAR data acquired in 2005 (Davila et al., 2007; Sulpizio et al., 2010). We placed 51 locations along the flow inundated area to accomplish local testing. Six of them are then adopted as preferred locations, being representative of different flow regimes.

2.5 Preliminary consistency testing of the input ranges

In general, the three rheologies considered in this study, MC, PF, and VS, have totally different parameters. If the total frictions of the models were not covering a similar span, the statistical comparison would be dominated by trivial macroscopic differences, and could not focus on the rheology details. Some degree of parameter consistency is hence required between the models.

We can find several instances of parameter choices in literature, and in general these choices appear to depend on the case study. In all the models it is defined a basal friction stress (see RHS_2 in 3). However, PF interpolates different effective friction angles, while VS includes a speed dependent additional term. We found that a direct consistency of effective friction angles is problematic to impose globally. Hence, we did not follow such approach. Instead, we made a preliminary testing on the extreme values of the parameter space, i.e. matching the runouts of ***max volume – min resistance***, and ***min volume – max resistance***.

General definitions We assume what follows to simplify testing. Except when differently specified, parameters are uniformly sampled in linear scale.

- Material Volume V is an additional input parameter in all the models, on a fixed range.
- In MC, sampled input parameters are ϕ_{bed} , and $\Delta\phi := \phi_{int} - \phi_{bed}$. In particular, $\Delta\phi \in [2^\circ, 10^\circ]$ (Dalbey et al., 2008).
- In PF, sampled input parameters are ϕ_1 , $\Delta\phi_{12} := \phi_2 - \phi_1$, and β . In particular, $\Delta\phi_{12} \in [10^\circ, 15^\circ]$, and $\beta \in [0.1, 0.85]$. Moreover, $\phi_3 = \phi_1 + 1^\circ$, and \mathcal{L} is equal to 1 dm and 1 mm in the two case studies, respectively (Pouliquen and Forterre, 2002; Forterre and Pouliquen, 2003; Gray and Edwards, 2014; Barker et al., 2015).
- In VS, sampled input parameters are μ , and ξ . In detail, ξ uniform sampling is accomplished in log-scale. In fact, values of ξ between 250 and 4,000 m/s^2 have been described for snow avalanches (Salm, 1993; Bartelt et al., 1999; Gruber and Bartelt, 2007).

In summary, MC and VS have three-dimensional parameter spaces, while PF a four-dimensional parameter space.

Flow down an inclined plane In this case study, Dalbey et al. (2008) assumed $\phi_{bed} = [15^\circ, 30^\circ]$, while Webb and Bursik (2016) performed a series of laboratory experiments and found $\phi_{bed} = [18.2^\circ, 34.4^\circ]$. We relied on those published parameter choices to decide a comprehensive parameter range. Figure 4 displays the maps of max flow height and max velocities observed in the extreme cases tested. Simulation options are - $\text{max_time} = 2$ s, $\text{height/radius} = 1.34$, $\text{length_scale} = 1$ m, $\text{number_of_cells_across_axis} = 10$, $\text{order} = \text{first}$, $\text{geoflow_tiny} = 1e4$ (Patra et al., 2005; Aghakhani et al., 2016). Initial pile geometry is cylindrical.

- **Material Volume:** $[449.0, 607.0] \text{ cm}^3$, i.e. average of 528.0 cm^3 and uncertainty of $\pm 15\%$.
- **Rheology models' parameter space:**

The parameter ranges adopted in this case study are:

$$\text{MC} - \phi_{bed} \in [18^\circ, 30^\circ].$$

$$\text{PF} - \phi_1 \in [10^\circ, 22^\circ].$$

$$\text{VS} - \mu \in [0.22, 0.45], \quad \log(\xi) \in [3, 4].$$

Even if maximum and minimum runout are both matching, the shape and lateral extent of the flow is remarkably different in the maximum runout case, according to the three models. In particular, MC model can produce the largest lateral extent, and VS model displays an accentuated bow-like shape, due to the increased friction in the lateral margins.

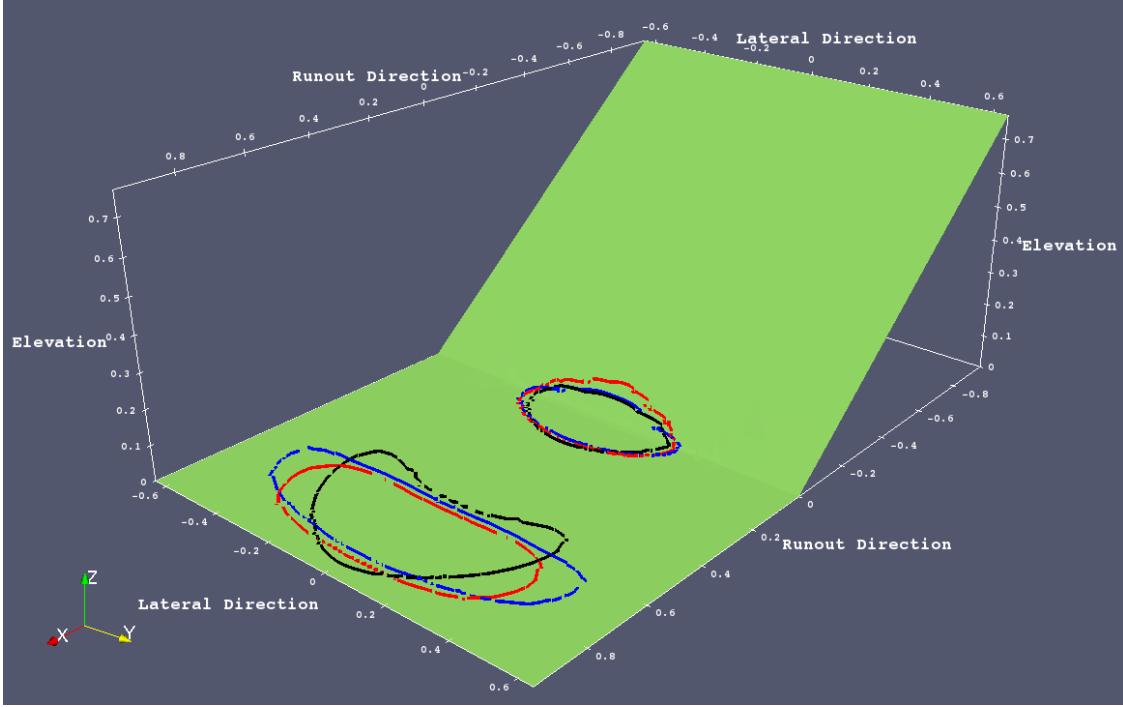


Figure 4: Inclined plane setup. Contours of $h = 1.0$ mm at last simulated snapshot ($t = 1.5$ sec) for simulated flows with *minimum runout obtained from min volume – max resistance*, and *maximum runout obtained from max volume – min resistance*. — : MC, — : PF, — : VS.

Volcán de Colima block and ash flow In this case study, [Dalbey et al. \(2008\)](#) assumed $\phi_{bed} = [15^\circ, 35^\circ]$, while ([Capra et al., 2011](#)) adopted $\phi_{bed} = 30^\circ$ in the simulation of a BAF in that same setting. Moreover, [Spiller et al. \(2014\)](#); [Bayarri et al. \(2015\)](#); [Ogburn et al. \(2016\)](#) found a statistical correlation between flow size and effective basal friction inferred from field observation of geophysical flows. The size of the BAF of this study would have $\phi_{bed} = [13^\circ, 18^\circ]$ according to their estimates. Figure 5 displays the maps of max flow height and max velocities observed in the extreme cases tested. Simulation options are - `max_time = 7200 s` (2 hours), `height/radius = 0.55`, `length_scale = 4e3 m`, `number_of_cells_across_axis = 50`, `order = first`, `geoflow_tiny = 1e4` ([Patra et al., 2005](#); [Aghakhani et al., 2016](#)). Initial pile geometry is paraboloid.

- **Material Volume:** $[2.08, 3.12] \times 10^5 \text{ m}^3$, i.e. average of $2.6 \times 10^5 \text{ m}^3$ and uncertainty of $\pm 20\%$.
- **Rheology models' parameter space:**

The parameter ranges adopted in this case study are:

MC - $\phi_{bed} \in [10^\circ, 25^\circ]$.

PF - $\phi_1 \in [8^\circ, 18^\circ]$.

VS - $\mu \in [0.15, 0.45]$, $\log(\xi) \in [1.7, 4]$.

The models possess different features, but the maximum runuout after channelization in the ravines is matching. In particular, VS lateral spreading is significant lower and the material reaches higher thickness, whereas PF model seems to stop more gradually than MC with a more complex inundated area boundary lines. These features are the macroscopic consequence of the rheology differences.

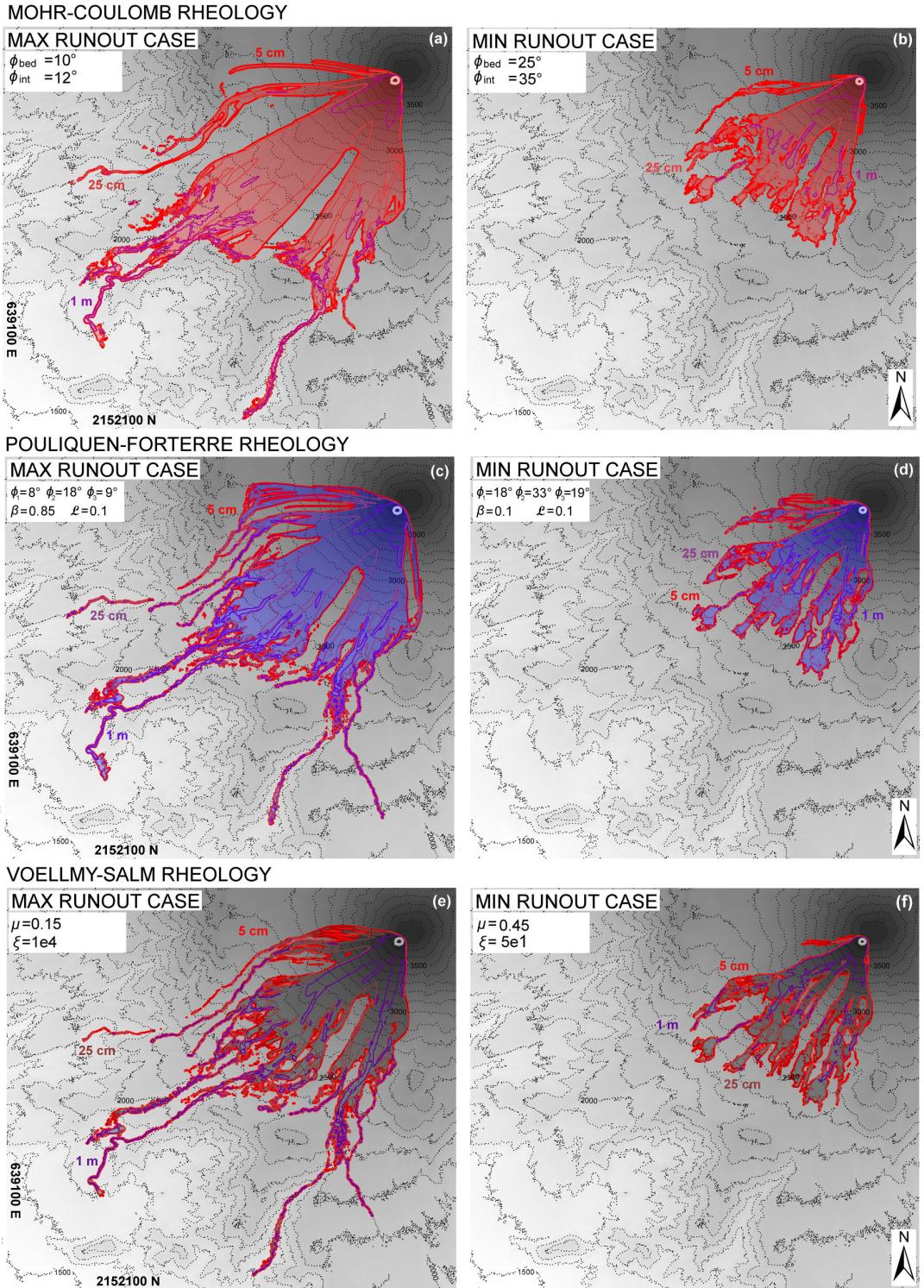


Figure 5: Volcán de Colima BAF. Comparison between *max flow height* maps of simulated flow, assuming Mohr-Coulomb (a),(b), Pouliquen-Forterre (c),(d), and Voellmy-Salm (e),(f) rheology. Extreme cases producing maximum and minimum runout, i.e. (a),(c),(e) ***max volume – min resistance*** and (b),(d),(f) ***min volume – max resistance***.

3 Results of small scale flows on inclined plane and flat runway

This case study is a small scale experimental setting. First we describe the observable outputs (Section 3.1), and then the latent variables (Section 3.2). In many plots we note the effect of cutting the flow when height is < 1 mm, which is at the scale of the smallest granular size. In fact, continuity assumption would not be valid below this scale. In particular, in Fig. 6 both the 5th and 95th percentile plots are vertically cut to zero when they decrease over that threshold. The mean plot is not cut to zero but it is dulled by this cutoff.

3.1 Observable outputs

Observable outputs include the flow height and acceleration as a function of time, measured in the four locations L_1, \dots, L_4 displayed in Fig. 2. In addition, lateral extension, flow area, and spatially averaged speed and Fr are displayed. Uncertainty quantification (UQ) is always performed, accordingly to the parameter ranges described in Section 2.5. Locally measured Froude Number is included in Supporting Information S1.

3.1.1 Flow height

Figure 6 shows the flow height, $h(L, t)$, at the points $(L_i)_{i=1,\dots,4}$, for the three rheology models. It clearly shows the differences in the statistics of the flow outcomes induced by the different choices of rheology at different locations in the plane. Availability of data allows us to subject the data to tests of reasonability both for the means and extremal values. Given a particular type of flow and collected data we can clearly distinguish model skill in capturing not only that flow but also possible flows. Past work [Webb \(2004\)](#) allows us to conclude that MC rheology is adequate for modeling simple dry granular flows.

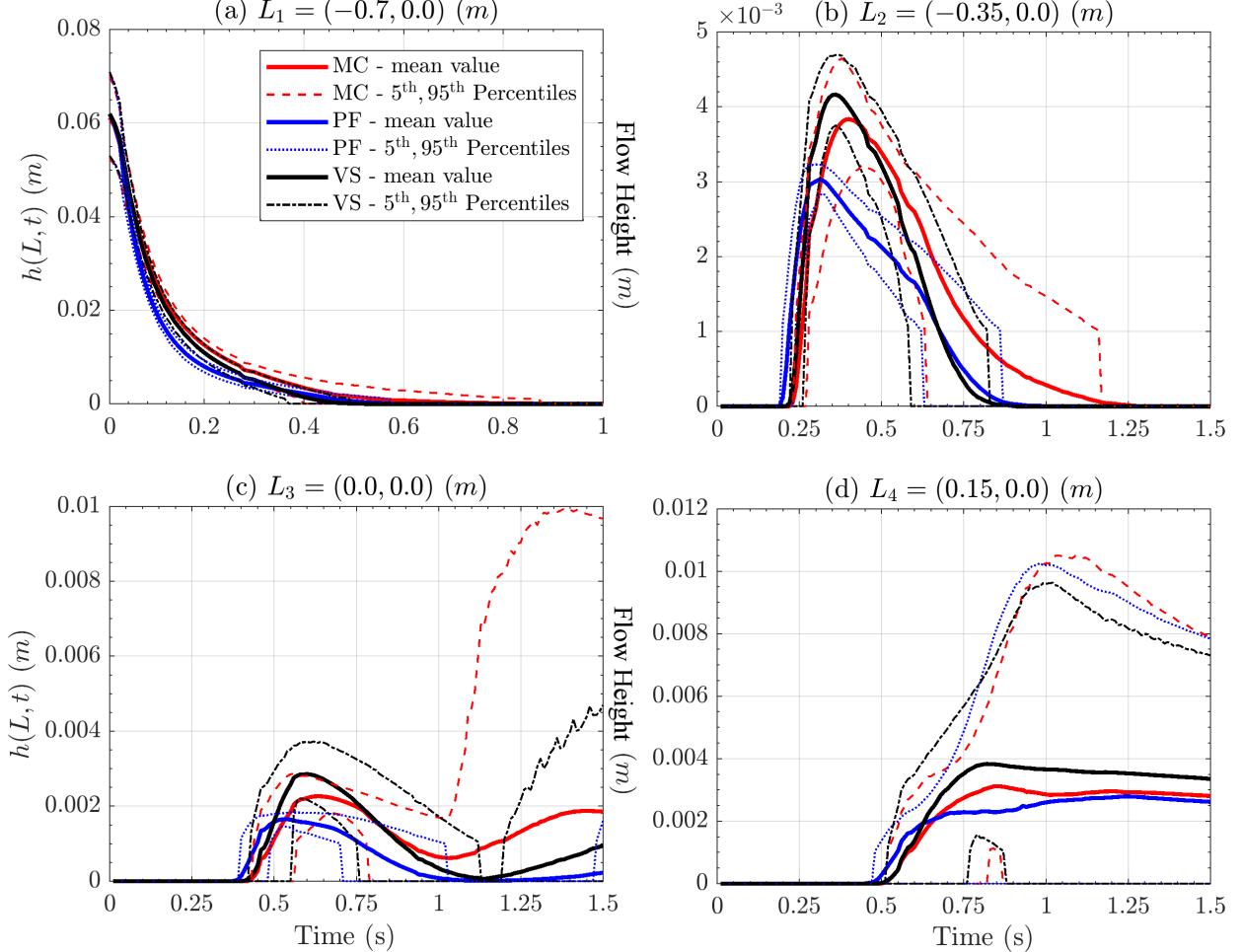


Figure 6: Records of flow height at four spatial locations of interest. Bold line is mean value, dashed/dotted lines are 5th and 95th percentile bounds. Different rheology models are displayed with different colors. Plots are at different scale, for simplifying lecture.

In plot 6a, related to point L_1 placed on the initial pile, the initial values of $\sim 6 \pm 1$ cm are the equal and express the assigned pile height. The flow height decreases slightly faster in PF model, and slower in MC, compared to VS. Differences are more significant in plot 6b, related to point L_2 , placed in the middle of the slope. Maximum flow height on average is greater in VS, 4.1 ± 0.2 mm, but more uncertain in MC, 3.9 ± 0.4 mm, and generally smaller in PF model, 3.0 ± 0.1 mm. After the peak, PF decreases significantly slower than the other models. These height values are about 15 times smaller than initial pile height. None of the models leaves a significant material deposit in L_1 or L_2 , and hence the 95th percentile of the height is null at the ending-time. In contrast, a deposit is left at points L_3 and L_4 , i.e.

plot 6c placed at the change in slope, and plot 6d in the middle of the flat runout. At L_3 MC's deposit, 2 mm with uncertainty [-2,+8] mm, is higher than the other models'. The plot profile is bimodal, showing a first peak at ~ 0.6 s, and then a reduction until 1 s, before the final accumulation. At L_4 ,deposit it is not significantly different between the three models. It measures ~ 3 mm on average, slightly more than this in VS, with uncertainty [-3,+7] mm.

3.1.2 Flow acceleration

Figure 7 shows the flow speed, $\|\bar{\mathbf{a}}\|(L, t)$, at the points $(L_i)_{i=1,\dots,4}$, for the three rheology models.

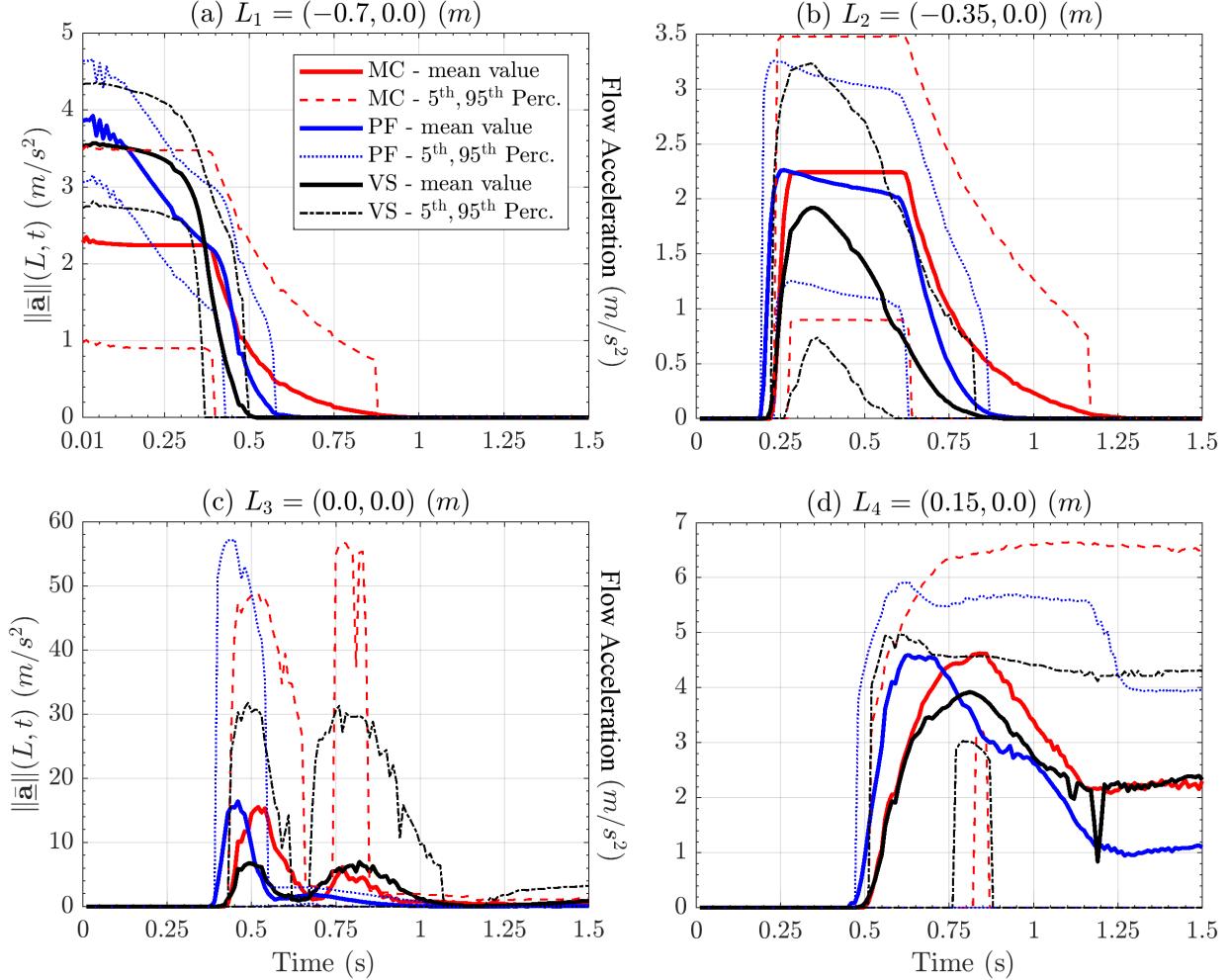


Figure 7: Records of flow acceleration magnitude (computed from LHS). Bold line is mean value, dashed lines are 5th and 95th percentile bounds. Different rheology models are displayed with different colors. Plots are at different scale.

Acceleration is the link between force terms and observable motion. We calculated it from the LHS of the dynamical equation, but using the RHS terms produces very similar results, although not identical due to the numerical approximations in solving the equations. In plot 7a, related to point L_1 , MC and VS show a plateau before ~ 0.4 sec, at $\sim 2.5 \text{ m/s}^2$ and $\sim 3.5 \text{ m/s}^2$, respectively, while PF linearly decreases between those same values. In plot 7b, related to point L_2 , are MC and PF to show a plateau, at $\sim 2.2 \text{ m/s}^2$, while VS has a more bell-shaped profile. UQ tells us that PF has a smaller uncertainty

than the other models. In plot 7c, related to point L_3 , all the models show a bimodal profile, with peaks at ~ 0.5 sec and 0.8 sec. This is more accentuated in MC and VS, whereas the second peak is almost absent from PF's profile. The second peak is motivated by an increase of velocity in the down-slope direction after its reduction due to lateral spreading of material. At the first peak, acceleration values are significant, with average peaks in MC and PF both at $\sim 15 \text{ m/s}^2$, and 95th percentile plot reaching $\sim 50 \text{ m/s}^2$ and $\sim 55 \text{ m/s}^2$, respectively. VS shows about halved acceleration peak values. At the second peak, average acceleration values are similar in MC and VS, at $\sim 5 \text{ m/s}^2$. In contrast, 95th percentile plot is $> 50 \text{ m/s}^2$ for MC, while $\sim 30 \text{ m/s}^2$ in VS. In plot 7d, related to point L_4 , the acceleration has a first peak at $\sim 4 \text{ m/s}^2$, and a final asymptote at $\sim 2 \text{ m/s}^2$ for MC and VS, $\sim 1 \text{ m/s}^2$ for PF. These values generally mean flow deceleration, and uncertainty is more relevant in MC and PF than in VS.

3.1.3 Flow extents and spatial integrals

Figure 8 shows the spatial average of speed and Froude Number, in the three rheology models. Moreover, it shows the (maximum) lateral extent and inundated area of flow, as a function of time. Spatial averages and global quantities as maximum lateral extent and inundated area have smoother plots than local measurements. Most of the details observed in local measurements are not easy to discern.

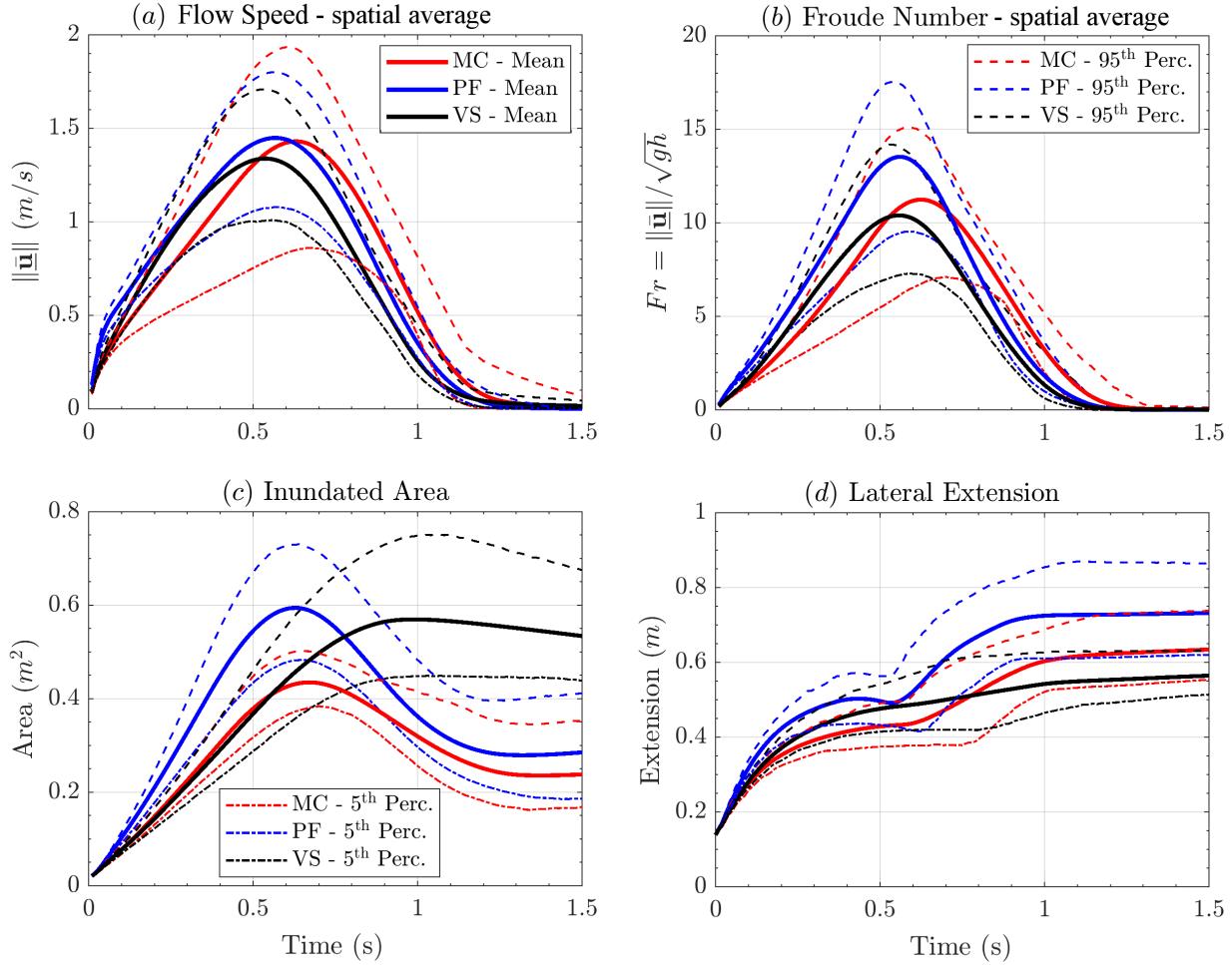


Figure 8: Comparison between spatial averages of (a) flow speed, and (b) Froude Number in addition to the flow (c) lateral extent, and (d) inundated area, as a function of time.

In plot 8a the speed shows a bell-shaped profile in all the models, with an average peak at $\sim 1.4 \text{ m/s}$

and uncertainty range, given by 5th and 95th percentiles, of $\pm 0.4m/s$ for PF and VS. VS is slightly slower, reaching $\sim 1.3m/s$ in the average. MC shows a larger uncertainty range, of $\pm 0.6m/s$. The maximum speed is reached first by VS and PF at $\sim 0.55s$, and last by MC at $\sim 0.65s$. In plot 8b, also Froude Numbers have a bell-shaped profile. Fr peaks are temporally aligned with speed peaks, and are ~ 10 in VS, ~ 11 in MC, ~ 13.5 in PF, on average. Uncertainty range is about ± 4 in all models. In plot 8c inundated area shows similar max values in PF and VS, at $\sim 0.6m^2$ on average, and uncertainty of $\pm 0.15m^2$. MC is lower, at $\sim 0.45m^2$ on average, and less uncertain, $\pm 0.10m^2$. VS does not decrease significantly after reaching the peak, whereas the other models contract their area, to approximately half of the maximum extent. In plot 8d the lateral extent starts equal to the pile diameter $15cm$, and then rises in two stages in MC and PF, the second and greater rise starting at $\sim 0.6s$, corresponding to the time of arrival at the change in slope L_3 (see Fig. 6c). VS rises without showing two phases. After the first phase, average lateral extent is at $\sim 50cm$ in PF and VS, while it is $\sim 43cm$ in MC. Uncertainty range is $\pm 7cm$ for all models at that time. Final extent is $\sim 75cm$ in PF, $\sim 65cm$ in MC, $\sim 55cm$ in VS. Uncertainty range is $\pm 5cm$ in VS, but rises to $\pm 10cm$ in MC and PF.

3.2 Statistical analysis of latent variables

In this section we focus on the force terms and related powers, as well as their statistical analysis. The decomposition of force terms follows the definitions in section 2.2. In particular, all the estimates assume a material density of the flow $\rho = 805kg/m^3$. This a fixed scaling factor, and the plots aspect is not affected by its value.

3.2.1 Power terms

Figure 9 shows the spatial integral of powers in the three rheology models. The spatial integration is performed on half spatial domain, due to the symmetry with respect to the flow central axis. Corresponding plots of the force terms are included in Supporting Information S2. The scalar product with velocity imposes a bell-shaped profile, as observed in Fig. 8a. In plot 9a the power of \mathbf{RHS}_1 represents the effect of the gravity in all the models. It starts from zero and rises up to $\sim 1.5W$ at $\sim 0.55s$, then decreases to zero after the material crosses the change in slope. Uncertainty range of $\pm 0.5W$ on the peak values. MC decreases slower, and has a more significant uncertainty, after the change in slope. PF decreases faster. In plot 9b the power of \mathbf{RHS}_2 represents the friction at the base of the flow. It is negative and peaks to $\sim 1.1 \pm 0.2W$ in MC, $\sim 1.0 \pm 0.2W$ PF, $\sim 0.7 \pm 0.3W$ in VS. A similar bell-shaped profile is shared by the three models. In contrast, the force plots in Figure S2 show an initial ripple before $\sim 0.1s$ and a plateau until $\sim 1s$. Basal friction power becomes negligible at the end, whereas the basal friction force in MC can be still significantly large. In plot 9c the power of \mathbf{RHS}_3 is related to the curvature effects, and is not null only at the change in slope. It is always dissipative, i.e. opposing flow velocity, indeed it is equivalent to the friction due to the additional weight generated by centrifugal forces. It is weaker than $-0.1W$ on average, ten times smaller than the previous powers, although MC lower percentile reaches $\sim -0.25W$. VS displays a bimodal profile, with a second and weaker peak at $\sim 0.75s$. In plot 9d the power of \mathbf{RHS}_4 is related to the additional forces of the models, differently characterized. This term is really relevant in VS, although also in PF has a very short lasting positive peak up to $0.3W$ before to become null at $\sim 0.1s$. This power in VS is a speed dependent term, always dissipative. It is bell shaped and null before $\sim 0.1s$ and after $\sim 1s$. At the time of change in slope it is $\sim -0.7W$, $\pm 0.3W$. In plot 9e the power of the total force $\sum_{i=1}^4 \mathbf{RHS}_i$ summarizes the energy accumulation and dissipation, following a sinusoid profile. The profile is characterized by a positively valued stage before $\sim 0.55s$ before the change in slope, and by a negatively valued stage after that, with bell-shaped profile, and a negative peak at $\sim 0.75s$. In the first stage MC increases more linearly, while PF and VS have a concave shape. In the second stage MC is affected by a larger uncertainty and the decrease occurs later in time of $\sim 0.1s$. PF and VS are remarkably similar, but VS wanes slightly faster.

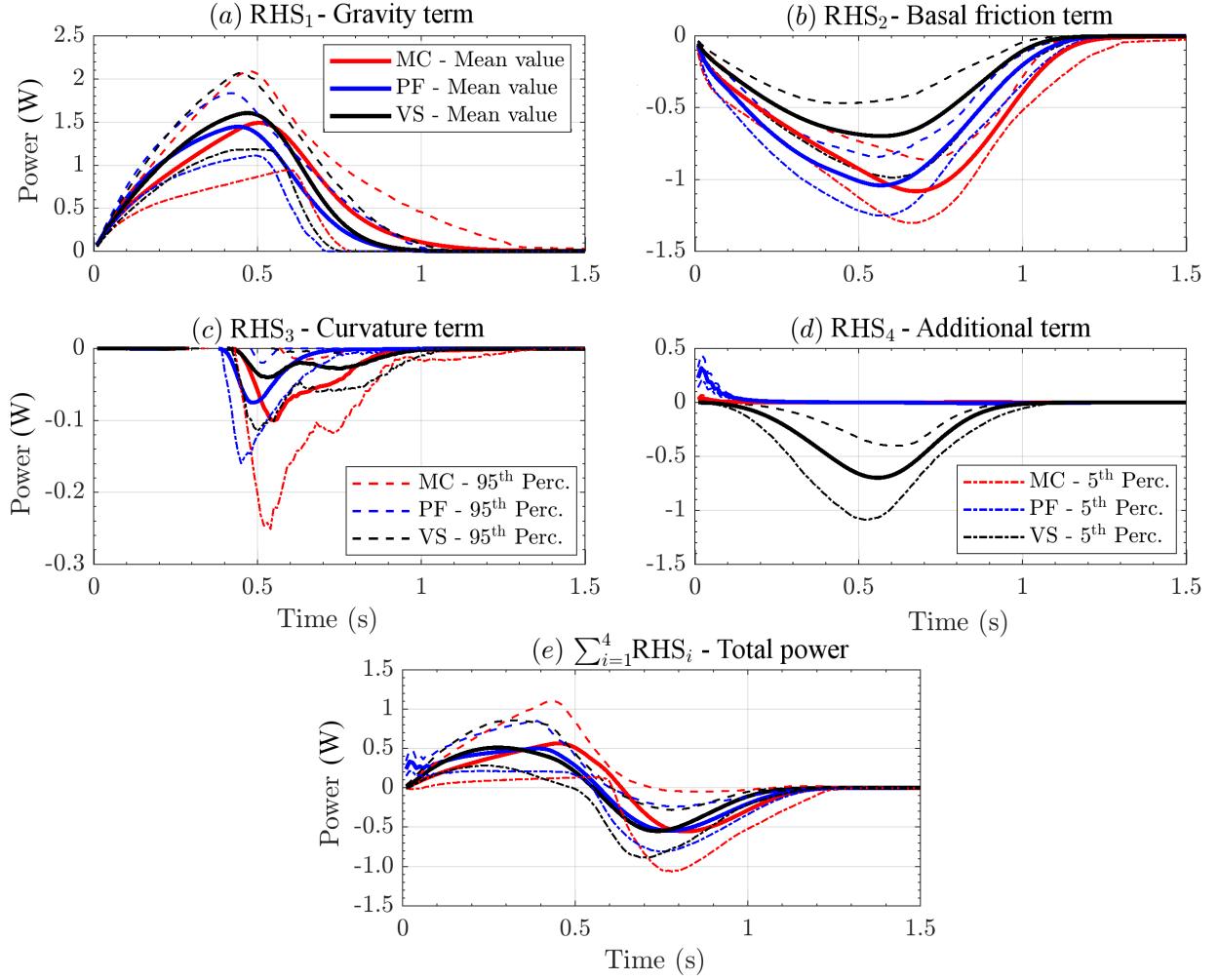


Figure 9: Spatial sum of the RHS powers. Bold line is mean value, dashed lines are 5th and 95th percentile bounds. Model comparison on the mean value is also displayed.

3.2.2 Force dominance factors

Figure 10 shows the Dominance Factors (P_i) $_{i=1,\dots,4}$ in the three rheology models, focusing on the RHS terms in the slope direction. A dominance factor is the probability of a force term to be the greatest one. Because these values are probabilities, they always belong to $[0, 1]$. The plots include also the probability of no-flow being observed at the considered point. The different models are plotted separately: 10a,d,g,j assume MC; 10b,e,h,k assume PF; 10c,f,i,l assume VS. The plots 10a,b,c are related to point L_1 , placed on the initial pile. Only \mathbf{RHS}_1 can be the dominant force, and no-flow probability is $(1 - P_1)$. Same thing in the plots 10d,e,f related to point L_2 , placed in the middle of the slope. Then, plots 10g,h,i are related to point L_3 , placed at the change in slope. In L_3 , \mathbf{RHS}_3 can be the dominant term for a short time, with a peak probability of $\sim 30\%$. Plots 10j,k,l are related to point L_4 , placed in the middle of the flat runout. In L_4 only \mathbf{RHS}_2 can be the dominant term, except in PF where there is $\sim 10\%$ that \mathbf{RHS}_4 is the dominant term at then ending-time.

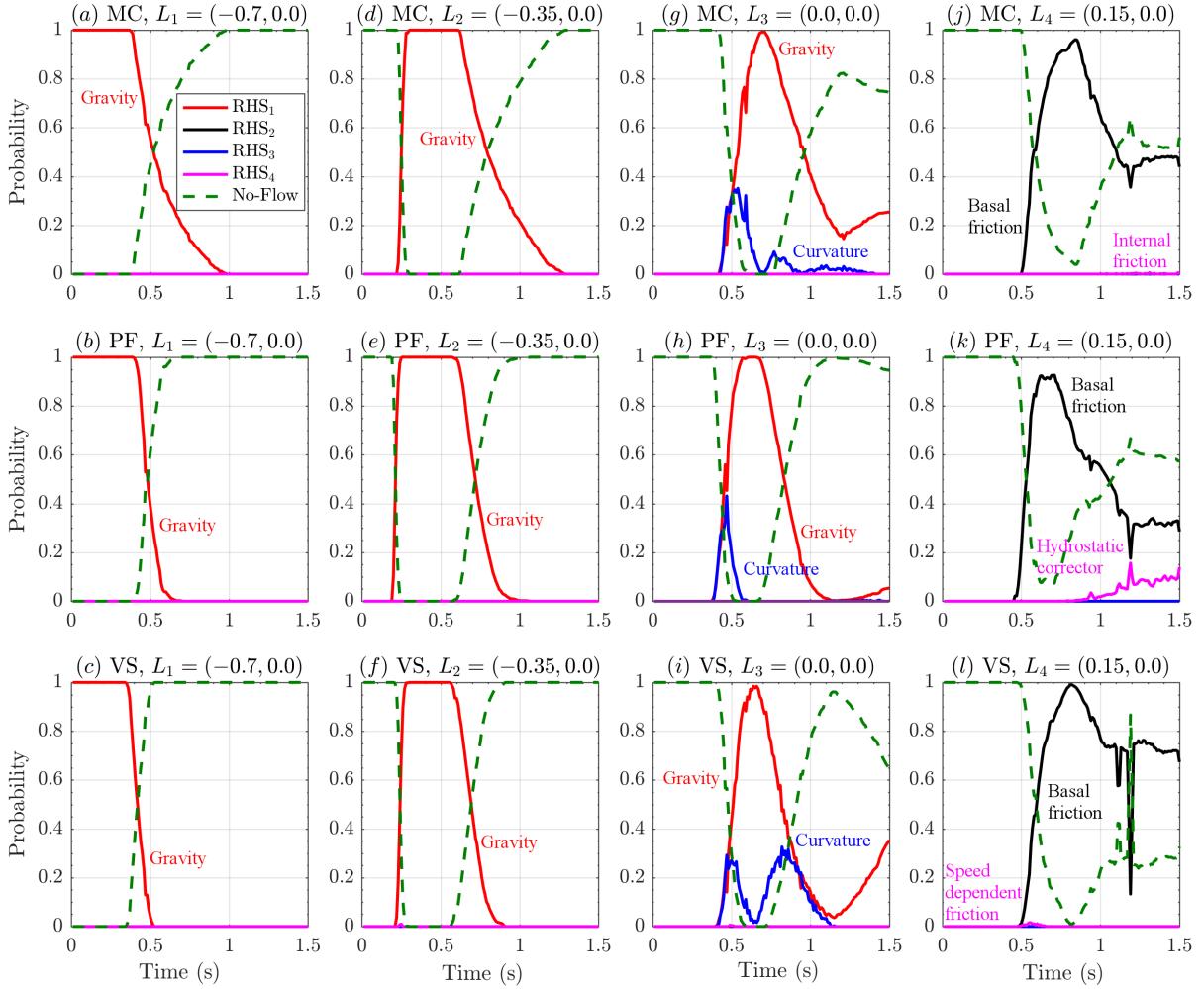


Figure 10: Records of dominance factors of **RHS** forces, in the slope direction, in four spatial locations of interest. Different rheology models are displayed with different colors. No-flow probability is also displayed with a green dashed line.

4 Results of large scale flows on the SW slope of Volcán de Colima (MX)

4.1 Flow height overview - Mohr-Coulomb model

In our second case study, the number of spatial locations is significantly higher than what previously assumed. We placed 51 points to span the entire inundated area, in search of different flow regimes, as displayed in Fig. 3. First we show the average flow height in all the locations, then, based on that, we select six points which we find representative of interesting flow regimes. Figure 11 assumes MC model, while the results related to PF and VS models are detailed on the six selected points.

Figure 11 shows the mean flow height, $h(L, t)$, at the 51 spatial locations of interest, according to MC.

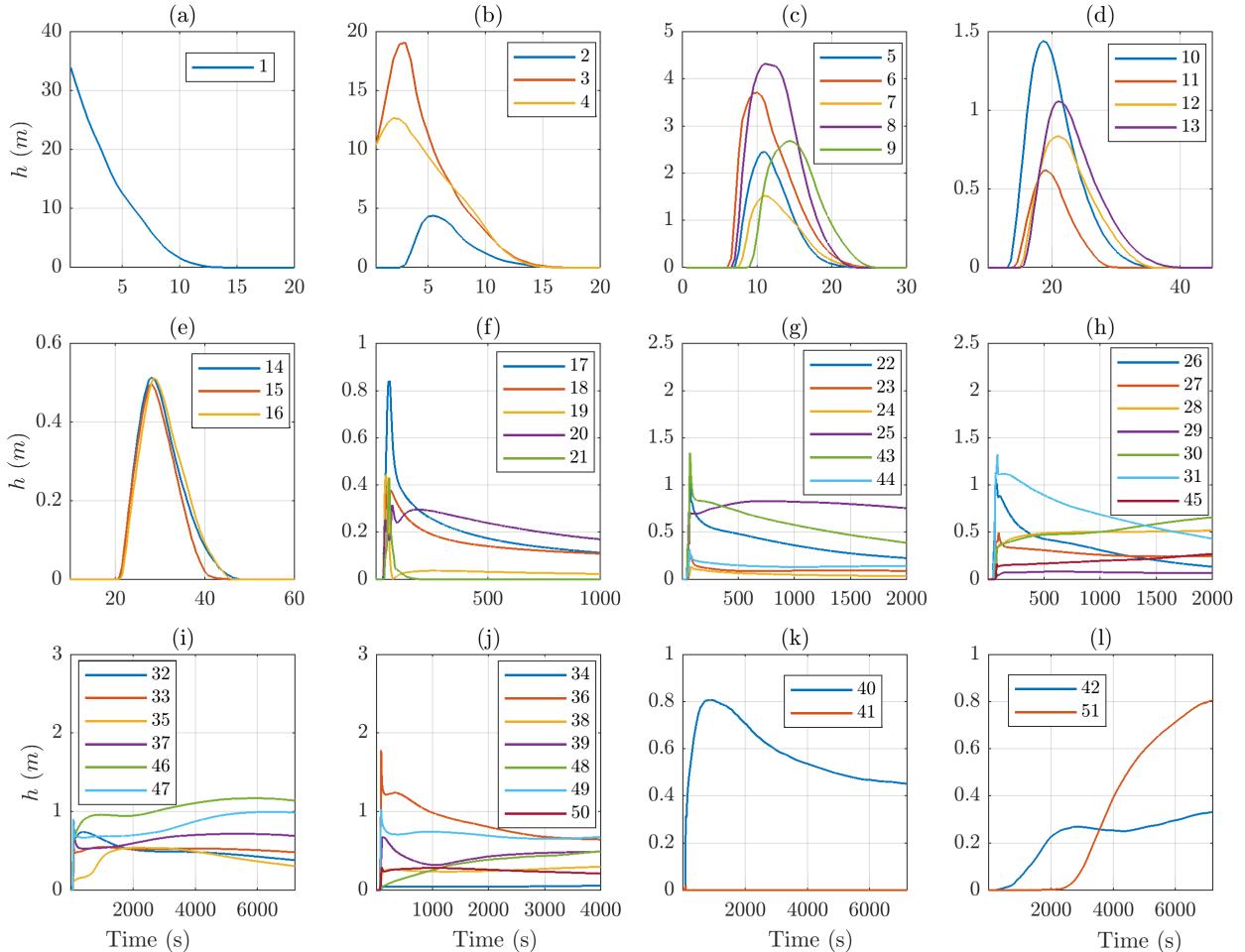


Figure 11: MC model, records of average flow height, $h(L, t)$, in 51 spatial locations of interest (Fig. 3).

Different plots have different scales on either time and space axes. In plot 11a, the only location is set on the center of the initial pile, and the profile is similar to what observed in point L_1 of the inclined plane case study, in Fig.6a. In particular, the height decreases from the initial value to zero in ~ 15 s. In plots 11b,c,d,e, the locations are set at less than ~ 1 km radius from the initial pile (projected distance, without considering slope). Their profiles are similar to point L_2 in Fig.6b. The height profile is bell-shaped, starting from zero and then waning back to zero in ~ 20 s. All the dynamics occurs during the first minute. Plot (b) shows transitional features, and focuses on points at the boundary of

the initial pile. In plots 11f,g,h,i,j, points are set where the slope reduces, and the flow can channelize and leave a deposit. Projected distance from the initial pile is $\sim 2 - 3$ km. The profiles are sometimes similar to L_3 of Fig.6c, other times to L_4 of Fig.6d, in a few cases showing intermediate aspects. In general is either observed an initial short-lasting bulge followed by a slow decrease lasting minutes and asymptotically tending to a positive height, or a steady increase of material height tending to a positive height. In both cases it is sometimes observed a bimodal profile in the first 5 minutes. Finally, plots 11k,l focus on three points set at about the runout distance of the flow, in the most important ravines, at $\sim 4 - 5$ km projected distance from the initial pile. Profiles are similar to what observed in point L_4 of Fig.6d. Plots of the Froude Number at the 51 points are included in Supporting Information S4 - strongly bimodal profiles and sharp changes are observed due to the interplay between flow height and flow speed.

4.2 Observable outputs - UQ on six selected locations

The six selected locations are $[L_8, L_{10}, L_{17}, L_{39}, L_{43}, L_{46}]$, displayed in Figure 12). First two points, L_8 and L_{10} are both significantly close to the initiation pile. Points L_{17} and L_{43} are placed where the slope is reducing and the ravine channels start, and L_{39} and L_{46} are placed in the channels, further down-slope. Moreover, L_8 , L_{43} , and L_{46} are placed at the western side of the inundated area, whereas L_{10} , L_{17} , and L_{39} are placed at the eastern side.

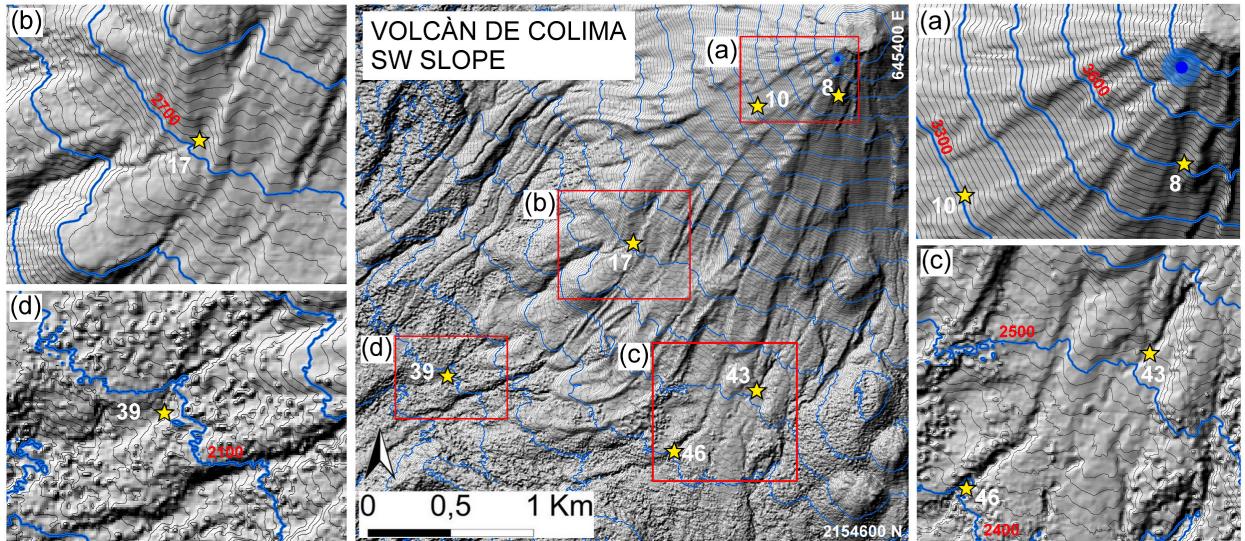


Figure 12: Volcán de Colima (Méjico) overview, including six selected sample sites (stars). In (a), (b), (c), (d) are enlarged the topographic features in proximity of those locations. Pile location is marked by a blue dot. Reported coordinates are in UTM zone 13N. Elevation isolines are included, at intervals of 10m in black, and 100m in bold blue. Elevation values in red.

Observable outputs include the flow height and acceleration as a function of time. In addition, flow area, and spatially averaged speed and F_r are displayed. Locally measured Froude Number and flow acceleration are included in Supporting Information S5 and S6.

4.2.1 Flow height

Figure 13 shows the flow height, $h(L, t)$, at the points $(L_i)_{i=8,10,17,39,43,46}$, for the three rheology models. In plots 13a,b, we show the flow height in points L_8 and L_{10} , $\sim 200m$ and $\sim 500m$ from the initial pile, respectively. L_8 is on the east side, and L_{10} on the west side of the flow. Models MC and PF display similar profiles, positive for less than 15s and bell-shaped. VS requires a significantly longer time to decrease, particularly in point L_{10} , where the average flow height is still positive after $\sim 200s$. Peak average values in L_8 are 3.4m in PF, 4.3m in MC, 4.7m in VS. Uncertainty is $\sim \pm 2m$, halved on the lower side in MC, and PF. In L_{10} , models MC and PF are almost indistinguishable, with peak height at 1.4m and uncertainty $\pm 0.5m$. Model VS, in contrast, has a maximum height of 1.1m lasting for 50s, and 95th percentile reaching 3.7m. In plots 13c,e, we show the flow height in points L_{17} and L_{43} , both at $\sim 2km$ from the initial pile, on the west and east side of the flow, respectively. All the three models show in both the points a fast spike during the first minute, followed by a slow decrease, still showing a positive average height after 30 minutes. Again, VS is significantly different from MC and PF, and has a secondary rise peaking at $\sim 450s$, which is not observed in the other models. This produces higher values for the most of the temporal duration, but converges to similar deposit thickness after more than 1 hour. Maximum values are 1m for MC, 2m for PF, and 1.5m for VS, in both locations. The 5th percentile is zero in all the models, meaning that the parameter range does not always allow the flow to reach those locations. The 95th percentile is always above 5m in the models, except in VS, point L_{17} . In plots 13d,f, we show the flow height in points L_{39} and L_{46} , both placed at more than 3km from the initial pile, on the west and east side of the flow, respectively. The three models all show an increasing profile, except in MC, in point L_{39} , which display an initial spike and a decrease before to rise again. A similar decreasing profile can be also observed in the 95th percentiles of all the models. It is significant that the 5th percentile of PF becomes positive after $\sim 5400s$, meaning that the flows almost surely have reached that location. Deposit thickness is $\sim 0.5m$ in all the models in point L_{39} , and 1.7m in VS, 1.6m in PF, 1.2m in MC, in L_{46} .

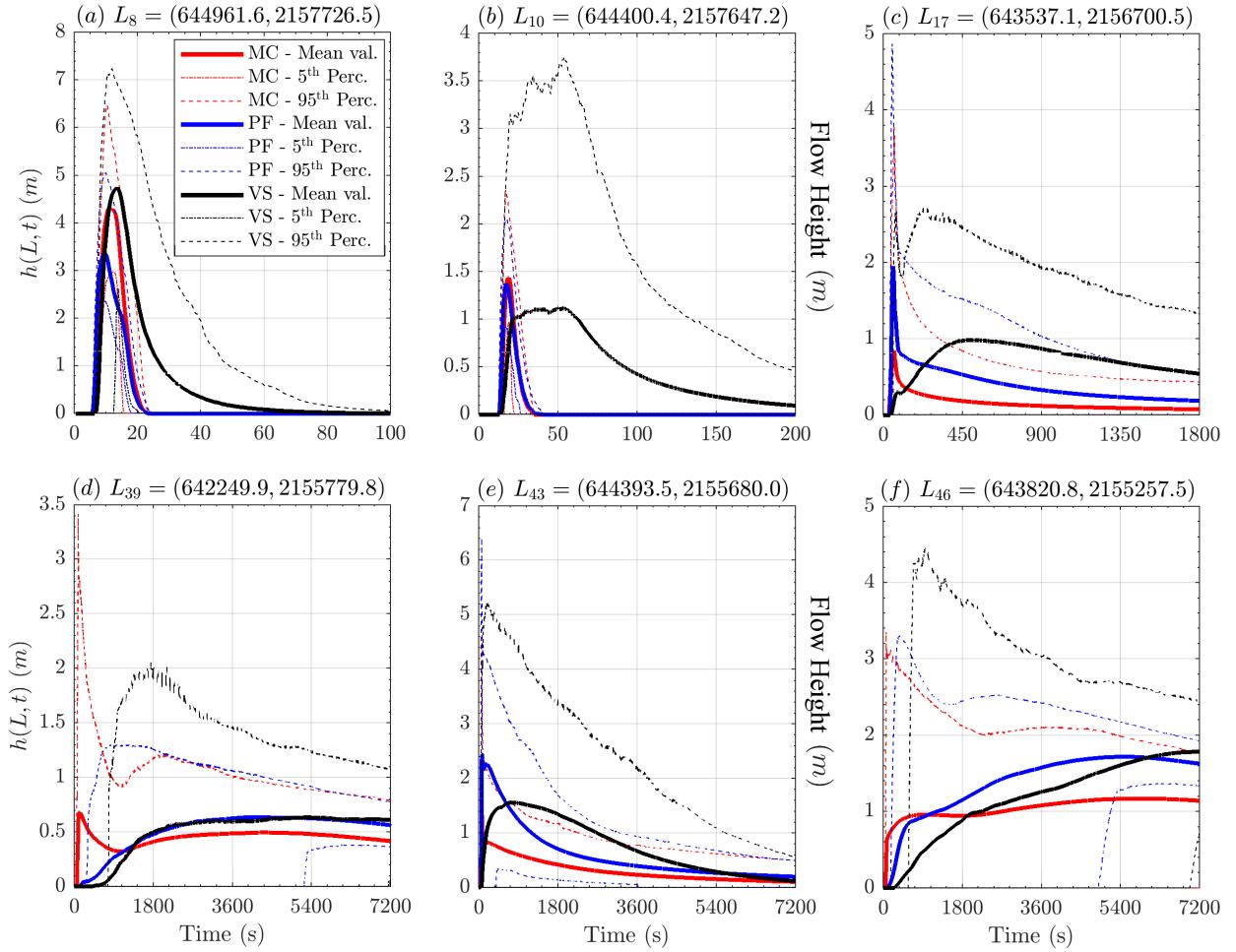


Figure 13: Records of flow height at six selected locations. Bold line is mean value, dashed/dotted lines are 5th and 95th percentile bounds. Different rheology models are displayed with different colors. Plots are at different scale, for simplifying lecture.

4.2.2 Flow area and spatial integrals

Figure 14 shows the spatial average of speed and Froude Number, in the three rheology models. It also shows the inundated area of flow, as a function of time. Analysis of model suitability can be conducted here given recorded deposits. In past work Patra et al. (2005), we have tuned MC rheology to match deposits for known block and ash flows but *a priori* predictive ability was limited by inability to tune without knowledge of flow character.

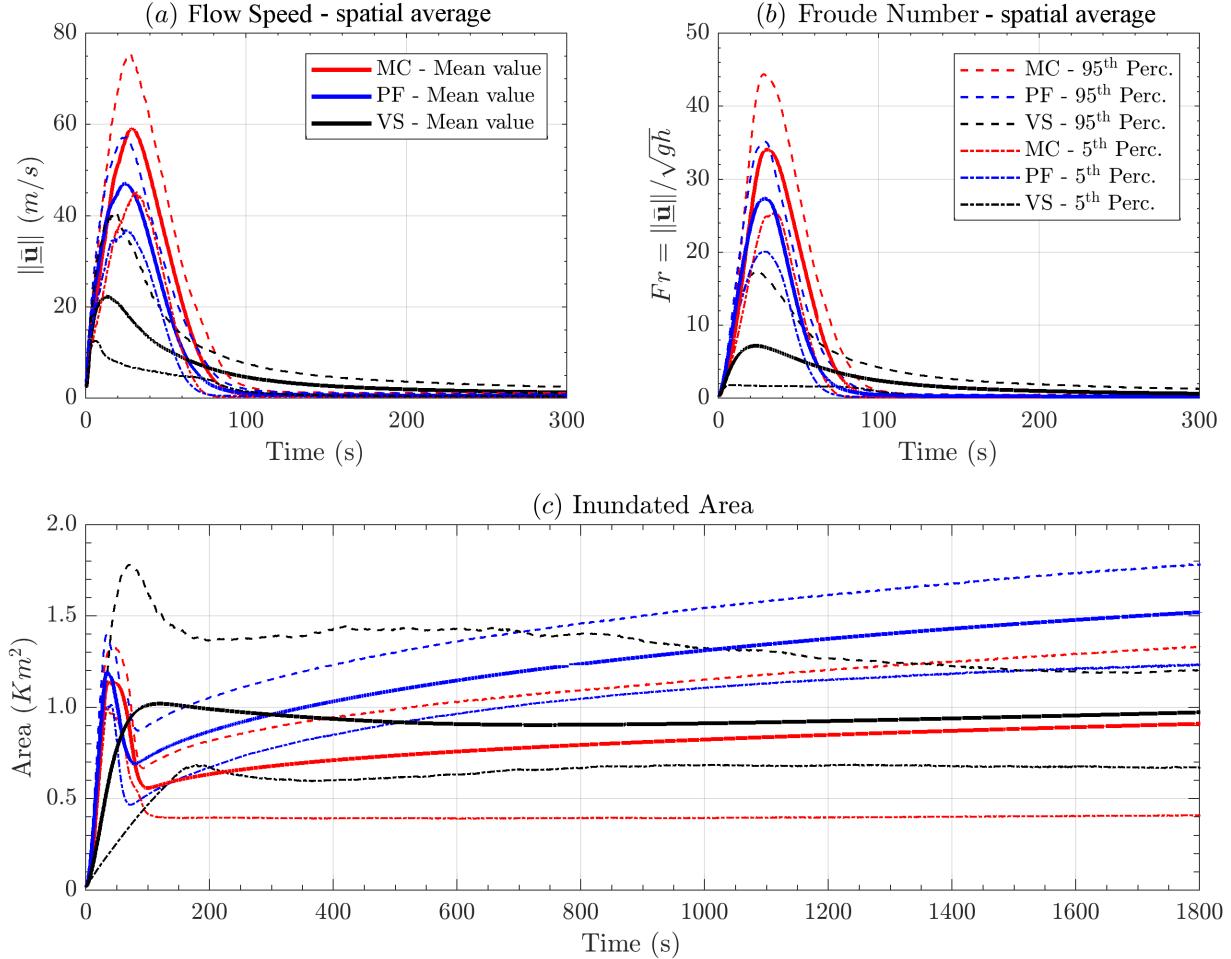


Figure 14: Comparison between spatial averages of (a) flow speed, and (b) Froude Number, in addition to the (c) inundated area, as a function of time.

Like in Fig.8, spatial averages and inundated area have smoother plots than local measurements, and most of the details observed in local measurements are not easy to discern. In plot 14a the speed shows a bell-shaped profile in all the models, but whereas the values were significantly similar in the inclined plane experiment, in this case the maximum speed is $\sim 60\text{m/s}$ in MC, $\sim 50\text{m/s}$ in PF, $\sim 20\text{m/s}$ in VS, on average. Uncertainty is $\pm 18\text{m/s}$ in MC, similar, but skewed on the larger values in VS, $\pm 10\text{m/s}$ in PF. In plot 14b, the Froude profile is very similar to the speed, but the difference between VS and the other models is accentuated. Maximum values are ~ 50 in MC, ~ 38 in PF, ~ 5 in VS, whereas uncertainty is ± 10 in MC, ± 7 in PF, and skewed $[-5, +10]$ in VS. In plot 14c, inundated area has a first peak in MC and PF, both at $\sim 1.15km^2$, followed by a decrease to $0.55km^2$ and $0.7km^2$, respectively, and then a slower increase up to a flat plateau at $0.9km^2$ and $1.5km^2$, respectively. Uncertainty is $\sim \pm 0.2km^2$ in both MC and PF until $\sim 100\text{s}$, and then it increases at $\pm 0.3km^2$ and $[-0.5, +0.4]km^2$, respectively.

In MC the increase of uncertainty is concentrated at ~ 100 s, while it is more gradual in PF. VS has a different profile. The initial peak is significant only in the 95th percentile values, and occurs significantly later, i.e. at ~ 100 s against ~ 50 s in MC and PF. It is of $\sim 1\text{km}^2$ on the average, but up to $\sim 1.8\text{km}^2$ in the 95th percentile. The decrease after the peak is very slow and the average inundated area is never below 0.85km^2 , and eventually reaches back to $\sim 1\text{km}^2$. Uncertainty is $[-0.3, +0.2]\text{km}^2$.

4.3 Statistical analysis of latent variables

All the estimates in this section assume the material density of the flow $\rho = 1800\text{kg/m}^3$. This a fixed scaling factor, and the plots aspect is not affected by its value.

4.3.1 Power terms

Figure 15 shows the spatial sum of the powers, for the three rheology models. Corresponding plots of the force terms are included in Supporting Information S7.

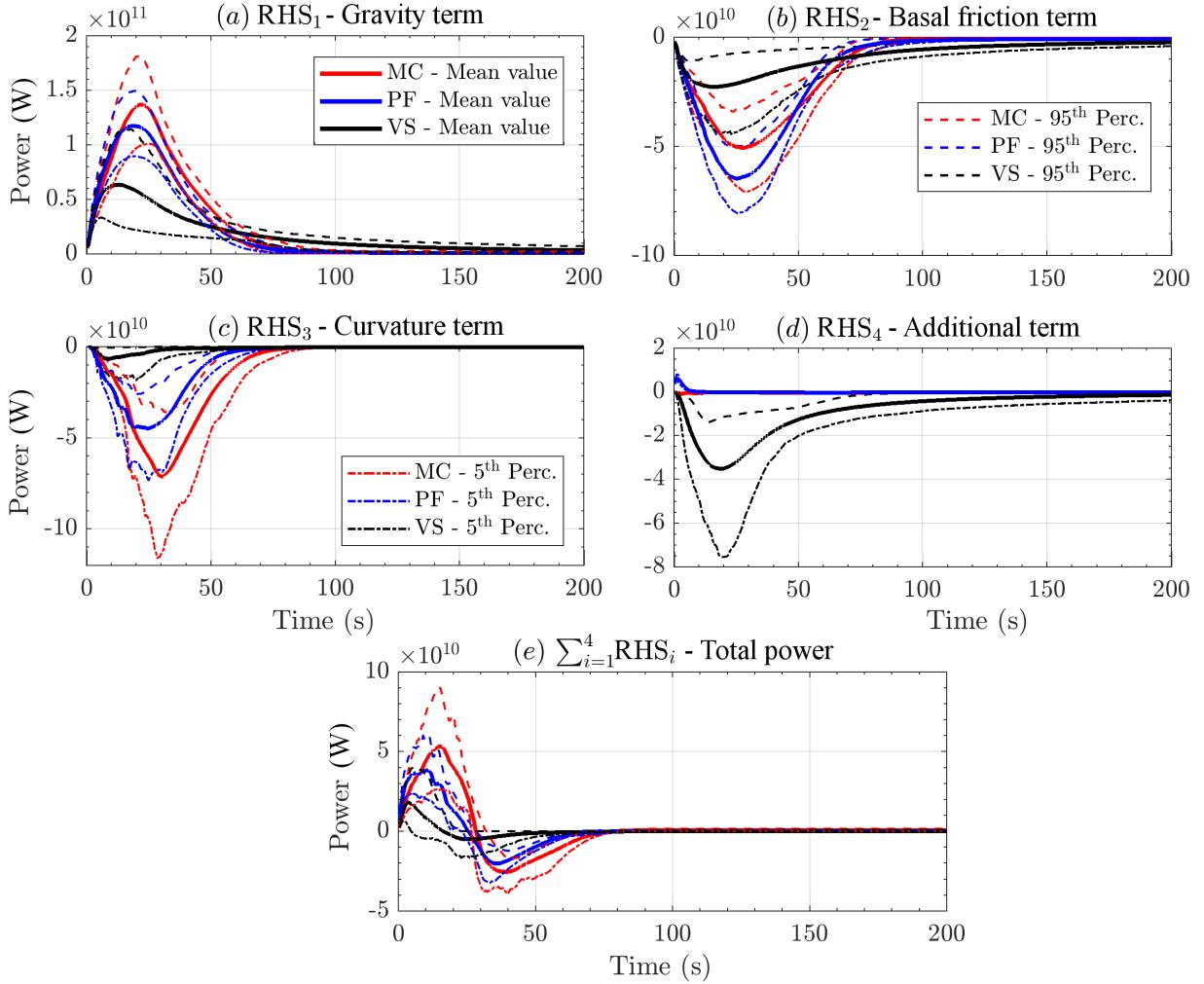


Figure 15: Spatial sum of the RHS powers. Bold line is mean value, dashed lines are 5th and 95th percentile bounds. Model comparison on the mean value is also displayed.

The scalar product with velocity imposes the bell-shaped profile already observed in Fig. 9a. In plot

15a the power of \mathbf{RHS}_1 represents the effect of the gravity in all the models. It starts from zero and rises up to $\sim 1.4e11W$ in MC, $\sim 1.2e11W$ in PF, $\sim 6.5e10W$ in VS. Uncertainty is $\pm 4e10W$ in MC, $\pm 3e10W$ in PF, $[-4e10, +5e10]W$ in VS. The decrease of gravitational forces is related to the slope reduction, and this decrease is more gradual in VS than in the other models. In Figure S5, at $\sim 10s$ the forces in all the models show a slowing down of the decrease, which is permanent in VS, and lasting $\sim 10s$ in MC and PF. This is not visible in the power plots. In plot **15b** the power of \mathbf{RHS}_2 represents the effect of the basal friction in all the models. It is negative and peaks to $\sim -6.5e10W$ in MC, $\sim -5e10W$ in PF, $\sim -2e10W$ in VS. In VS this dissipative power is significantly more flat than in the other models. MC and PF show negligible powers after $\sim 100s$, VS after $\sim 200s$. Uncertainty is $\pm 2e10W$ in MC, $\pm 1.5e10W$ in PF, $[-2e10, +1e10]W$ in VS. In Figure S2, the basal friction force in MC shows a bimodal profile with a peak at $\sim 5s$ and a second one at $\sim 60s$. This is not visible in powers. In PF, the plot starts from stronger values than in the other models, but it is also the faster to wane. In plot **15c** the power of \mathbf{RHS}_3 represents the effect of the curvature of terrain. It has a similar profile to the corresponding force, peaking to $\sim -7e10W$ in MC, $\sim -4.5e10W$ in PF, $\sim 5e9W$ in VS. The rise of this dissipative power has a similar profile, but a different duration between the models. The decrease is more gradual in VS. Uncertainty on the peak value is $[-4.5e10, +3.5e10]W$ in MC, $[-2.5e10, +2e10]W$ in PF, $[-1e10, +5e9]W$ in VS. The three models all show a bell-shaped profile, MC and PF waning to zero at $90s$, VS at $\sim 30s$. In plot **15d** the power of \mathbf{RHS}_4 has a different meaning in the three models. In MC it is the internal friction term, and it only has almost negligible ripple visible in the first second. In PF it is a depth averaged correction in the hydrostatic pressure, and has an almost negligible effect only during the first second of simulation, at $5e9W$. It becomes null at $\sim 10s$. In VS, instead, it is a speed dependent term, and has a very relevant effect. The plot shows a bell-shaped profile, with a peak of $\sim -3.5e10W$, $[-2e10, +1e10]W$. After that, this dissipative power gradually decreases, and becomes negligible at $200s$. In plot **15e** the power of the total force $\sum_{i=1}^4 \mathbf{RHS}_i$ summarizes the energy accumulation and dissipation, following, like in the inclined plane case study, a sinusoid profile. First is observed a steep increase due to the domination of the gravity upon the friction, then a negative part when the friction stops the dynamics. As displayed in Fig. 9e, MC profile is delayed compared to the others, and is affected by a larger uncertainty.

4.3.2 Force dominance factors

Figure 16 shows the Dominance Factors $(P_i)_{i=1,\dots,4}$, in the three rheology models and focusing on the RHS terms moduli, in the three selected points L_8 , L_{10} , and L_{17} , closer than 1 km to the initial pile (in horizontal projection).

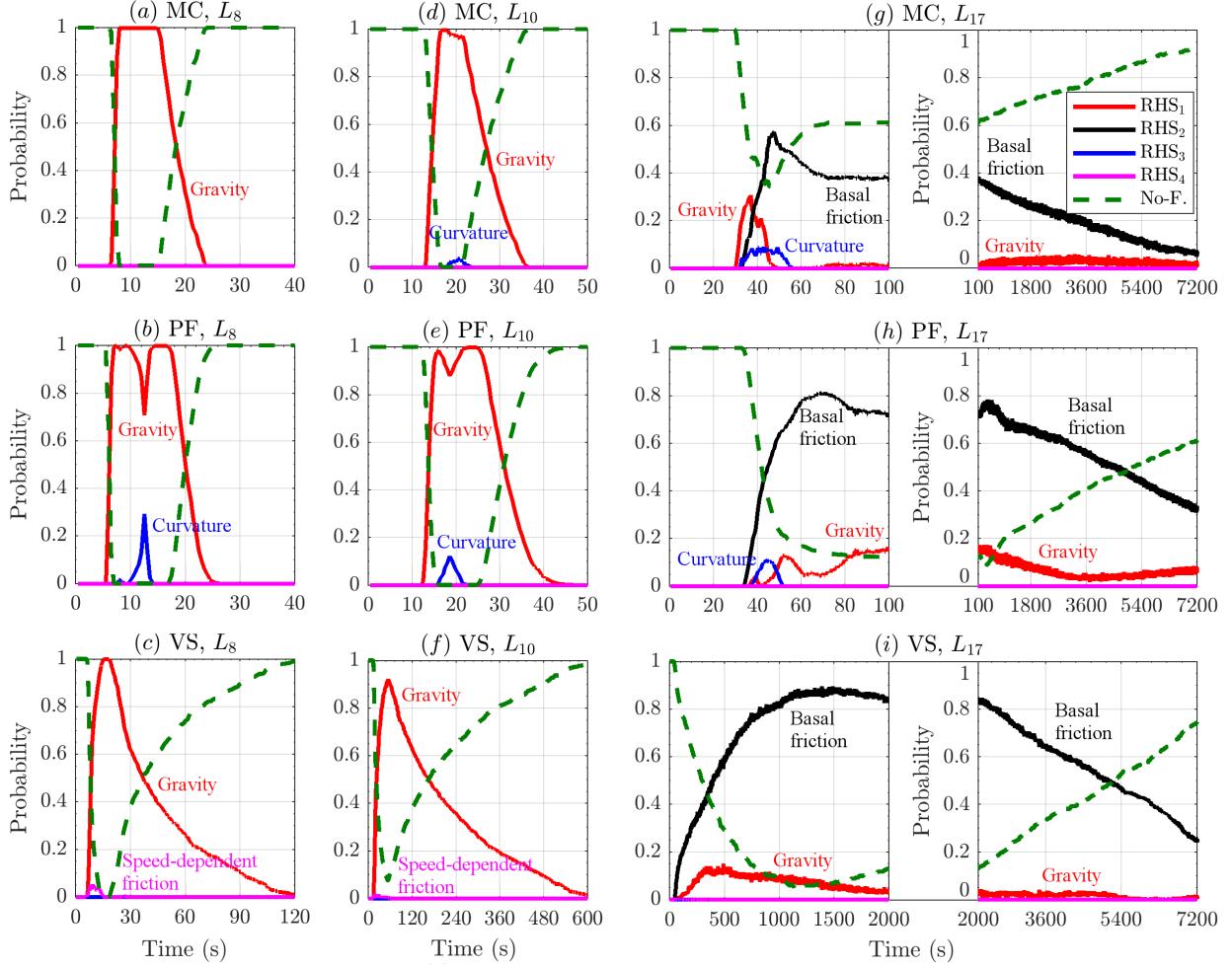


Figure 16: Records of dominance probabilities of **RHS** force moduli, at three spatial locations of interest, in the first km of runout. Bold line is mean value, dashed/dotted lines are 5th and 95th percentile bounds. Different rheology models are displayed with different colors. No-flow probability is also displayed.

The plots 16a,b,c and 16d,e,f are related to point L_8 and L_{10} , respectively. They are significantly similar. **RHS**₁ related to the gravitational force is the dominant force with a very high chance, $P_1 > 90\%$. In MC and PF there is a small probability, i.e. $P_3 = 5\% - 30\%$ at most, of **RHS**₃ being the dominant force for a short amount of time, i.e. $\sim 5s$. This occurs in the middle of the time interval in which the flow is almost surely inundating the points being observed. In VS it is observed a $P_4 = 5\%$ chance of **RHS**₄ being dominant, for a few seconds, anticipating the minimum of no-flow probability. Plots 16g,h,i, are related to point L_{17} , and the plots are split in two sub-frames, following different temporal scales. In all the models, **RHS**₂ is the most probable dominant force, and its dominance factor has a bell-shaped profile, similar to the complementary of no-flow probability. In all the models, **RHS**₁ has a small chance of being the dominant force. In MC, this is more significant, at most $P_1 = 30\%$, for $\sim 20s$ after the flow arrival, and has again about $P_1 = 2\%$ chance to be dominant in $[100, 7200]s$. In PF, the chance is $P_1 = 15\%$ at most, and has two maxima, one short lasting at about $55s$, and the second in $[100, 500]s$. Also in VS, the chance is at most $P_1 = 15\%$, reached at $[300, 500]s$, but its profile is unimodal in time, and becomes lower than $P_1 = 2\%$ after $2000s$. In MC and PF, **RHS**₃ has a chance of $P_3 = 10\%$ of being the dominant force, for a short amount of time $[30, 50]s$ and $[40, 50]s$, respectively.

Figure 17 shows the Dominance Factors (P_i) $_{i=1,\dots,4}$, in the three rheology models, in the three selected points L_{39} , L_{43} , and L_{46} , over 2 km far away from the initial pile.

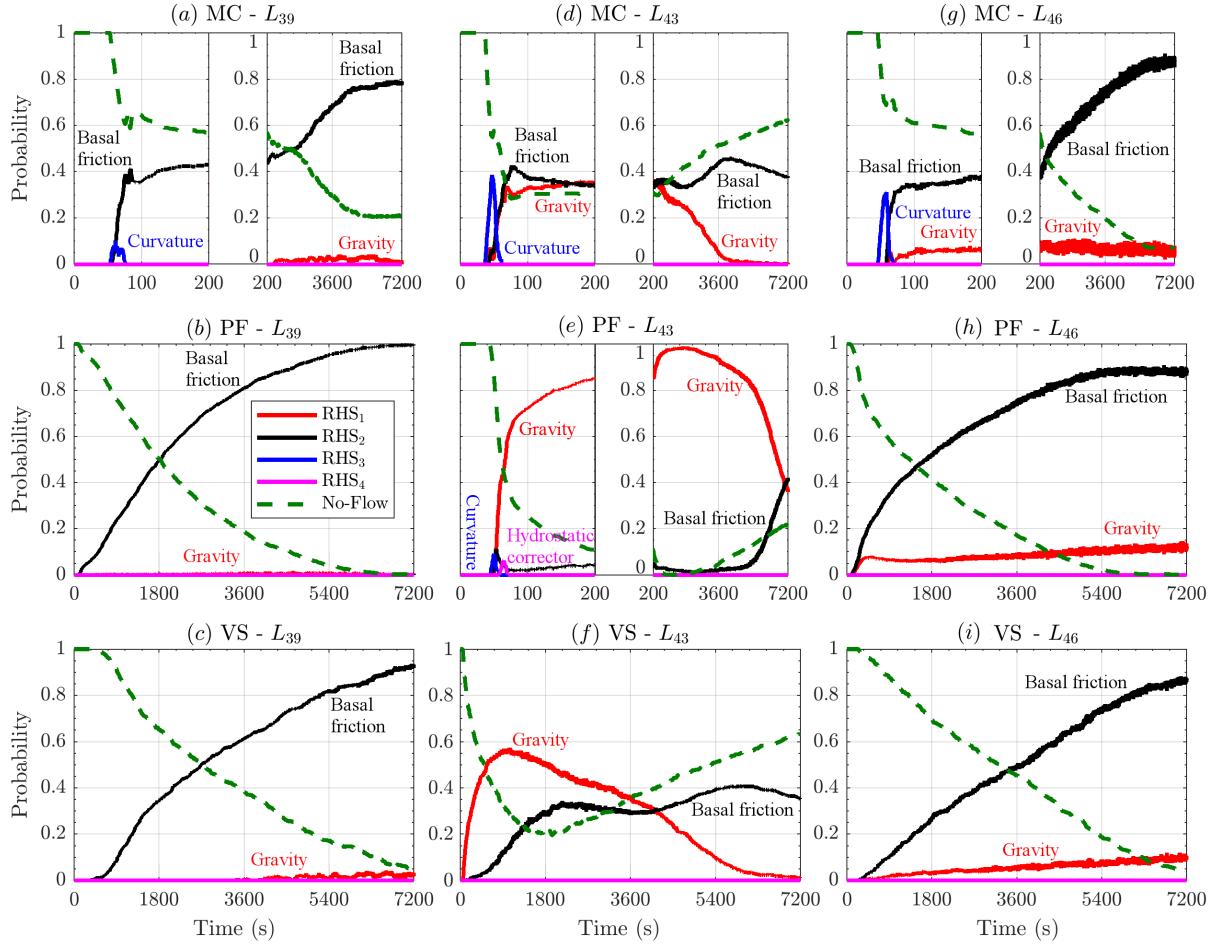


Figure 17: Records of dominance probabilities of **RHS** force moduli, at three spatial locations of interest, after 2 km of runout. Bold line is mean value, dashed/dotted lines are 5th and 95th percentile bounds. Different rheology models are displayed with different colors. No-flow probability is also displayed.

The plots 17a,b,c and 17g,h,i are related to point L_{39} and L_{46} , respectively, and they are significantly similar. Those points are both in the most distal part of the ravines, on the western and eastern side of the inundated area, respectively. They are also similar to the dominance factors at point L_{17} , shown in plots 16g,h,i, but the no-flow probability never becomes negligible in this case. The probability of **RHS₂** being the dominant force is $P_2 > 90\%$ till the end of the temporal window. Moreover, **RHS₁** does not show any initial peak in P_2 , and generally increases slowly, reaching at most $P_2 = 10\%$ chance of being the dominant force in L_{46} , after more than 3600s, in all the models. Plots 17d,e,f, are related to point L_{43} , are remarkably different. In MC, either **RHS₁** and **RHS₂** can be dominant forces, both with chances of $\sim 35\%$ in the first 200s. Then, **RHS₂** increases its chance, and becomes the only dominant force after 3600s. The no-flow probability is never below 30%. **RHS₃** has a chance P_3 up to 35% for a short amount of time, i.e. [40, 60]s. In PF, **RHS₁** is the dominant force with chance above $P_1 = 90\%$ until 3600s. **RHS₂** chance rises only in the very last amount of time, reaching $P_2 = 40\%$ at 7200s, at the same level of P_1 . The no-flow probability is very low during the most of the temporal window, rising at 20% only at 7200s. Both **RHS₃** and **RHS₄** have a short peak of dominance factor, at about 10% for a few seconds, at [50, 60]s. Finally, in VS the no-flow probability is never below 20%, and the profile is similar to MC, although **RHS₁** is more relevant, up to 4000s. The other force are almost never dominant.

5 Discussion on the comparative anatomy of geophysical flow models

In this section we summarize the general features which differ between the models, with the purpose of depicting a thorough identikit of them, and breaking down the effects of the modeling assumptions listed in section 2.1. The main focus is on the average plots, but the uncertainty percentiles are also considered, and often they exalt the differences between the models.

5.1 Characteristic features and their motivations

Even in the process of the preliminary alignment of the parameter ranges, the differences features in the models already affected the shape of the inundated area, both in the small flow on the inclined plane, and in the large scale geophysical flow. In Fig. 4 the max runouts display a larger lateral extent in PF, and a bent shape in VS where the lateral wings remain behind the central section of the flow. In Fig. 5, the three models look clearly different both in the min-runout and in the max-runout cases. Even if the maximum runout is matched, MC displays a further distal spread before entering the ravines, PF shows a larger angle of lateral spread at the initiation pile, VS is less laterally extended and generally looks significantly channelized, and displays several not-inundated spots due to minor topographical coulées.

5.1.1 Flow height

Flow height gives additional insight on the model features. In Fig. 6, MC looks more distally stretched, but starts to deposit material earlier and closer to the initial pile. PF height is generally shorter, and displays a small temporal anticipation in its arrival at the sample points. These features are probably due to the hydrostatic correction term, which pushes forward the material at the very beginning, during the pile collapse. Also, the interpolation between the minimum ϕ_1 and maximum ϕ_2 friction angles, as a function of flow height and speed, plays a role in the linear cut in the flow height profile when the flow thins on the slope. VS tends to be higher than the other models, because of the reduced lateral spreading of material. In Fig. 13 the large scale flow height plots allow us to classify the points according to their similarity with the four points of the small scale flow. There is a significant feature which was not observed in the small scale flow - VS is temporally stretched, and material arrives later and stays longer in the sample points. This is clearly a consequence of the speed dependent term.

5.1.2 Flow acceleration

Flow acceleration brings our analysis from the mechanics to the dynamics of the flow. In Fig. 7, at the point L_2 on the slope, MC has a flat plateau, while PF is linearly decreasing, and VS is bell shaped. Those differences are a consequence of the assumptions behind the models - double angle in PF, and speed dependent term in VS. At the slope change point L_3 , VS and MC display a bimodal profile in the acceleration. This is not a statistical effect, but it is physically observed in single simulations. The first maximum is when the head of the flow hitting the ground, while the second maximum is when the accumulating material in the tail arrives. In VS the maxima are equal, because its tail is not laterally spread and hits the ground compactly. In contrast, PF does not show such a second peak, due to the accentuated lateral spreading in the tail. In point L_4 on the flat runway, the average acceleration in PF is lower than in the other models at the end of the simulation time, probably because the material stops more suddenly, and the probability of no deposit is greater.

5.1.3 Flow extents and spatial integral

In Fig. 8 the average speed and Froude Number of the different models are significantly similar, and this points to the fact that the differing features are mostly localized in space. VS is confirmed to be significantly slower than the other models, after the initial collapse. Moreover, it is the only model which presents a significant amount of long lasting and slowly moving material. Inundated area in PF has a greater maximum value, because of the accentuated lateral spread. In VS the inundated area almost does not decrease from its peak, because of the strictly increasing lateral spread. Vice versa, lateral spread in MC and PF has a temporary stop when the bulk of the flow hits the ground. This is a consequence of the

interplay of accumulating material and the push of new material, which is stronger in the middle than in the lateral wings. In PF lateral extent is even decreasing slightly for a short time. The hydrostatic correction term may generate the pull reducing the lateral extent. In Fig. 14 average speed and Froude Number display that VS slower speed is not a local feature. Inundated area is again significantly larger in PF.

5.1.4 Power and force terms

Power terms have several features in common with the corresponding forces, and provide the decomposition of the acceleration sources. Main dissimilarity between forces and powers is that gravitational and basal friction powers have a profile starting from zero when the flow initiates, because the flow speed starts from zero. In Fig. 9 the difference between models is particularly relevant in term \mathbf{RHS}_4 . Speed dependent power in VS is at least one order of magnitude larger than the maximum values of the corresponding terms in MC and PF. Those are decreasing to zero after a short time from the initiation. Hydrostatic correction in PF is clearly positive in the speed direction, and hence contributing to push the flow ahead, at the beginning. The effects of internal friction in MC are almost negligible, and initially positive, then negative. This is motivated by an initial compression of the material during the pile collapse, followed by its stretching. It is worth remarking that \mathbf{RHS}_2 and \mathbf{RHS}_3 are both smaller in VS, due to the lower basal friction angles involved. In Fig. 15, the differences are accentuated, because of the topography complexity. Gravity term is larger in VS, because a portion of the flow lingers on the higher slopes for a long time. Basal friction has a higher peak in PF, due to the effect of the interpolation of the friction angle to the larger value ϕ_2 in several portions of the flow.

5.1.5 Force dominance factors

The force dominance factors illustrate the predominant force term through time and the probability of no-flow. In Fig. 10 the differences between the models are minor in the inclined plane case study. In general, there is only a single dominant force, and its profile is complementary with the no-flow probability. More details can be found in the force contributions analysis (Appendix B and Supporting Information S3 and S8). Only in the slope change point L_3 the differences are remarkable. Curvature term dominance probability is bimodal in MC and VS, and in MC the second peak is smaller. The profile is similar to the spatial average of the force term. In the flat runway, at point L_4 , in PF the hydrostatic correction can be the dominant force with a small chance. In Fig. 16 and 17, the complexity in the dominance factors is greater. At the points L_8 and L_{10} , proximal to the initiation, gravitational force is dominant with a high probability until the no-flow probability becomes predominant. Curvature force can be dominant in MC and PF, when the no-flow change is zero, and for a short time. Gravitational force is dominant for a much larger time, because of the longer presence of the flow. The speed dependent friction can be dominant with a small probability, at the beginning of the dynamics, when flow is faster. At the points L_{17} , L_{39} , and L_{46} , the basal friction is dominant with high probability. L_{17} is left by the flow, whereas L_{39} , and L_{46} have a deposit at the end with a high chance. In general, in MC the no-flow probability tends to be larger than in the other models, because part of the parameter spaces stops earlier, or completely leaves the site. Again, curvature can have a small chance to be dominant in MC and PF, particularly when the speed is high. Point L_{43} is particularly interesting. It is not close to the initiation, but it is placed right after a local hill. This location shows an intermediate regime between the proximal points and the distal ones. The no-flow probability is increasing at the end, meaning that the material is leaving the site, like in L_{17} . Moreover, the gravity can be dominant like in the proximal points, but also the bed friction can be dominant, and in MC and VS both the two forces have similar chances to be dominant for the most of the time of the simulation. In PF, only the gravitational force is dominant with a high chance, and the no-flow probability is almost null, meaning that there is accelerating material for the most of time. This is probably because point L_{43} is placed in a place right downhill of a place where a significant amount of material stops according to that model.

6 Conclusions

In this study, we have introduced a simple, robust statistically driven method for analyzing complex models. We have used 3 different models arising from different rheology assumptions. The data shows

unambiguously the performance of the models across a wide range of possible flow regimes and topographies. We analyze local and global quantities and latent variables. The analysis of latent variables is particularly illustrative of the impact of modeling assumption. Knowledge of which assumptions dominate, and, by how much, at the level of assumptions will allow us to construct efficient models for desired inputs. Such model composition is the subject of ongoing and future work.

Our new method enabled us to break down the effects of the different physical assumptions in the dynamics, providing an improved understanding of what characterizes each model. We considered a Monte Carlo simulation on a range of parameter inputs, and estimated the mean and the uncertainty range of the corresponding outputs. Fitting the results on a specific flow was thus not necessary, and the procedure unfolded the physics of the models in a general framework. We also developed the concept of force dominance factor, in the study of the force terms in the different regions of space and time. This paves the way to future research aimed at designing a new generation of algorithms, which could completely bypass the search for a unique best model. They would go beyond the concept of alternative models, first decomposing them into their characterizing features, and then focusing instead on a more flexible selection of the assumptions which are locally suitable for each specific part of the simulation.

In particular, we presented:

- a short review of the assumptions characterizing three commonly used rheologies of Mohr-Coulomb, Poliquenne-Forsterre, Voellmy-Salm. This included a qualitative list of such assumptions, and the breaking down of the different terms in the differential equations.
- a preliminary procedure for selecting the portions of the parameter ranges of the models which produced consistent outputs. This was fundamental because the model comparison cannot focus on the modeling differences if the scales of the stress tensor were on different scales.
- an articulated statistical framework, processing the mean, and the uncertainty range of many mechanical and latent variables in the flow simulation. A new function was featured in the TITAN2D solver, capable of producing local measurements of those quantities at selected sites. Spatial integrals and spatial averages were also performed, illustrating the characteristics of the entire flow under the different models. The new concept of force dominance factors enabled a simplified description of the local dynamics.
- the detailed results of the procedure applied to two different case studies: an inclined plane with a flat runway, and the real DEM on the SW slope of Volcán de Colima (MX). All the differences were quantitatively commented.
- a final discussion, explaining all the observed features in the results on the light of the known physical assumptions of the models, and the evolving flow regime in space and time.

Our statistical analysis based on UQ depicted three very different models, due to the different assumptions underlying to them. Compared to a classical MC model, PF lacks of internal friction and this produces an accentuated lateral spread. This is increased by the hydrostatic correction, which briefly pushes the flow ahead and laterally during the initial collapse. That force can also have some minor effects in the final deposit accumulation. The interpolation of the smaller bed friction angle ϕ_1 with the larger value ϕ_2 , suddenly stops the flow if it thins compared to its speed. This mechanism does not allow for large speed peaks. Instead in VS, the speed dependent friction has a great effect in reducing lateral spread and producing channeling features even due to minor ridges in topography. The flow tends to be significantly slower and more stretched in the slope direction. The effects of the different formulation of the curvature term are less impacting than the effects of the lower basal friction and speed.

The procedure may be applied to other models, or modeling assumptions. Moreover, the two case studies considered had the purpose of exploring two very different scale of flows, the first on a controlled setting, the second on a real topography. Additional research concerning other case studies, and different parameter ranges, might reveal other flow regimes, and hence differences in the consequences of the modeling assumptions under new circumstances.

7 Appendix A: Latin Hypercubes and orthogonal arrays

The Latin Hypercube Sampling (LHS) is a well established procedure for defining pseudo-random designs of samples in \mathbb{R}^d , with good properties with respect to the uniform probability distribution on an hypercube $[0, 1]^d$ (McKay et al., 1979; Owen, 1992a; Stein, 1987; Ranjan and Spencer, 2014; Ai et al., 2016). In

particular, compared to a random sampling, a LHS: (i) enhances the capability to fill the d-dimensional space with a finite number of points, (ii) in case $d > 1$, avoids the overlapping of point locations in the one dimensional projections, (iii) reduces the dependence of the number of points necessary on the dimensionality d .

Definition 1 (Latin hypercube sampling) Let $\Xi = \{\xi_i : i = 1, \dots, N\}$ be a set of points inside the d -dimensional hypercube $C = [0, 1]^d$. Let $[0, 1] = \bigcup_{j=1}^N I_j$, where $I_j = [\frac{(j-1)}{N}, \frac{j}{N}]$. Let $\xi_i = (\xi_i^1, \dots, \xi_i^d)$, and for each $k \in \{1, \dots, d\}$, let $\Xi^k = \{\xi_i^k : i = 1, \dots, N\}$. Let λ^d be the uniform probability measure supported inside C , called Lebesgue measure. Then Ξ is a latin hypercube w.r.t. $\lambda^d \iff \forall j \in \{1, \dots, N\}, \forall k \in \{1, \dots, d\}, |\mathcal{I}_j \cap \Xi^k| = 1$.

The procedure is simple: once the desired number of samples $N \in \mathbb{N}$ is selected, and $[0, 1]$ is divided in N equal bins, then each bin will contain one and only one projection of the samples over every coordinate. The LHS definition is trivially generalized over $C = \prod_i^d [a_i, b_i]$, i.e. the cartesian product of d arbitrary intervals. That will be applied in this study, defining LHS over the parameter domain of the flow models.

There are a large number of possible designs, corresponding the number of permutations of the bins in the d -projections, i.e. $d \cdot N!$. If the permutations are randomly sampled there is a high possibility that the design will have good properties. However, this is not assured, and clusters of points or regions of void space may be observed in C . For this reason, we base our design on the orthogonal arrays (OA) (Owen, 1992b; Tang, 1993).

Definition 2 (Orthogonal arrays) Let $S = \{1, \dots, s\}$, where $s \geq 2$. Let $Q \in S^{n \times m}$ be a matrix of such integer values. Then Q is called an $OA(n, m, s, r) \iff$ each $n \times r$ submatrix of Q contains all possible $1 \times r$ row vectors with the same frequency $\lambda = n/s^r$, which is called the index of the array. In particular, r is called the strength, n the size, ($m \geq r$) the constrains, and s the levels of the array.

Orthogonal arrays are very useful for defining latin hypercubes which are also forced to fill the space (or its r -dimensional subspaces) in a more robust way, at the cost of potentially requiring a larger number of points than a traditional LHS.

Proposition 3 Let Q be an $OA(n, m, s, r)$. Then let $U \in \mathbb{R}^{n \times m}$ be defined as follows:

$$\forall k \in \{1, \dots, s\}, \forall j \in \{1, \dots, m\}, \{Q[\cdot, j] : Q[i, j] = k\} = \Pi((k-1)\lambda s^{r-1}, \dots, k\lambda s^{r-1}),$$

where Π is a random permutation of λs^{r-1} elements. Then $\Xi = \{\xi_i = U[i, \cdot] : i = 1, \dots, n\}$ is a LHS w.r.t to λ^m over $C = [0, 1]^m$. Moreover, let $[0, 1]^r = \bigcup_{(h_1, \dots, h_r)=1}^s I_{(h_i)}$, where $I_{(h_i)} = \prod_i^r [\frac{(h_i-1)}{s}, \frac{h_i}{s}]$. Then $\forall D = (d_1, \dots, d_r) \subseteq \{1, \dots, m\}$, let $\Xi^D = \{(\xi^{d_1}, \dots, \xi^{d_r}) : i = 1, \dots, n\}$. We have that

$$\forall k \in \{1, \dots, s\}, \forall (h_i : i = 1, \dots, r) \in \{1, \dots, d\}^r, |I_{(h_i)} \cap \Xi^D| = \lambda.$$

For each column of Q we are replacing the λs^{r-1} elements with entry k by a random permutation of $((k-1)\lambda s^{r-1} + h)_{h \in 1, \dots, \lambda s^{r-1}}$. After the replacement procedure is done, the newly obtained matrix U is equivalent to a LHS which inherits from Q the property of fully covering s^r equal r -dimensional hypercubes in every r -dimensional projection. Each hypercube contains λ points. In other words, inside each r -dimensional projection, the design associated to U fills the space like a regular grid at the scale of those s^r hypercubes, but it is still an LHS at a finer scale, i.e. the λs^{r-1} one dimensional bins. A complete proof can be found in Tang (1993) and it is a straightforward verification of the required properties.

However, even in an LHS based on an $OA(n, m, s, r)$, if $r < m$ what happens in the projections with dimension $r' > r$ is not controlled, and randomizing procedures are made more difficult by the additional structure imposed by the OA. Moreover, the total number of points necessary to achieve a full design increases with r , and hence is affected by dimensionality issues.

Dealing with relatively small d , i.e. $d \in \{3, 4\}$, we adopt a LHS U created by a $OA(s^d, d, s, d)$. The strength is equal to the dimension d , hence the design fills the entire space like a d dimensional grid, but it is a LHS as well. In this case there is one point in each hypercube, and $\lambda = 1$. We take $s = 8$ for the 3-dimensional designs over the parameter space of Mohr-Coulomb and Voellmy-Salm models, i.e. 512 points; we took $s = 6$ for the 4-dimensional designs over the more complex parameter space of the Pouliquen-Forterre model, i.e. 1296 points.

8 Appendix B: Force contribution coefficients, and their conditional decomposition

The *contribution coefficients* are normalized quantities related to the force terms. They are obtained dividing the force terms described in section 2.2 by a dominant function Φ . This is a tool to compare the different force terms, scaling the plots by the dominant function Φ - in this case in $[-1, 1]$. It also represent the degree of relevance of the assumptions behind the force terms, changing as a function of time.

In general we focus on the moduli of the forces, or their projection on the slope direction. Hence, in the following the forces are scalar and not vectorial terms. It is important to remark that all the forces depend on the input variables, and they are thus considered as random variables. The definitions are not depending on the location, but all the results will significantly depend of that choice. Next notation will assume to be in a selected location $x = L_k$, where $k \geq 1$.

Definition 4 (Contribution coefficients) Let $(F_i)_{i \in I}$ be random variables on (Ω, \mathcal{F}, P) , representing the considered force components in location x at time t . Then, for each component i , the contribution coefficient is defined as:

$$C_i := \mathbb{E} \left[\frac{F_i}{\Phi} \right],$$

where Φ is a dominating function, i.e. $\Phi \geq |F_i|, \forall i \in I$.

The *total force*, i.e $\sum_i F_i$ excluding the inertial terms, is not a good candidate for a dominating function. Indeed, the terms often have opposite signs, and their sum can be really small. Another issue is given by the existence of a subset of times Θ characterized by the absence of flow in the selected location x . In Θ the dominant force is null, and cannot be the denominator of a fraction.

We have tested two alternative Φ choices: Φ_1 is based on the l^1 norm, and defined as:

$$\Phi_1 := \begin{cases} \sum_i \frac{|F_i|}{2}, & \text{if not null;} \\ 1, & \text{otherwise.} \end{cases}$$

In contrast Φ_2 is a dominant force based on the l^∞ norm:

$$\Phi_2 := \begin{cases} \max_i |F_i|, & \text{if not null;} \\ 1, & \text{otherwise.} \end{cases}$$

In particular, for a particular location x , time t , and parameter sample ω , we have $C_i = 0$ if there is no flow or all the forces are null. The expectation of C_i is reduced by the chance of F_i being small compared to the other terms, or by the chance of having no flow in (x, t) . Moreover, $\mathbb{E}[C_i] \in [-1, 1], \forall i$.

Proposition 5 Let $\Phi_k, k = 1, 2$ be the functions described above. Then they are well defined dominating function, i.e. $\Phi_k \geq |F_j|, \forall j, k$.

Proof. If $k = 1$, the Newton equation states $\sum_{j \in J} F_j = \sum_{i \in I} F_i$, where $(F_j)_{j \in J}$ are inertial forces, and $(F_i)_{i \in I}$ the local forces. Then, $\forall h \in I$

$$F_h = \sum_{i \neq h, i \in I} F_i - \sum_{j \in J} F_j.$$

And hence

$$|F_h| \leq |F_h| + \sum_{i \neq h, i \in I} |F_i| - \sum_{j \in J} |F_j| = 2\Phi_1.$$

A similar equation is valid $\forall k \in J$. If $k = 2$, or if the inertial forces are not included, the proof is trivial.

□

Furthermore, assuming $\Phi = \Phi_2$, there is a useful result explaining the meaning of those coefficients through the conditional expectation.

Proposition 6 Let $(F_i)_{i \in I}$ be random variables on (Ω, \mathcal{F}, P) , representing the considered force components in location x at time t . For each i , let C_i be the contribution coefficient of force F_i , assuming $\Phi = \Phi_2$. Then we have the following expression:

$$C_i = \sum_j p_j \mathbb{E} \left[\frac{F_i}{|F_j|} \mid \Phi = |F_j| \right],$$

where $p_j := P\{\Phi = |F_j|\}$.

Proof. Let Z be a discrete random variable such that, for each $j \in \mathbb{N}$, $(Z = j) \iff (\Phi = |F_j|)$. Then, by the rule of chain expectation:

$$\begin{aligned} C_i &= \mathbb{E}\left[\frac{F_i}{\Phi}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{F_i}{\Phi} \mid Z = j\right]\right] = \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{F_i}{|F_j|} \mid Z = j\right]\right] = \sum_j P\{Z = j\} \mathbb{E}\left[\frac{F_i}{|F_j|} \mid Z = j\right]. \end{aligned}$$

Moreover, by definition, $p_j = P\{Z = j\}$. This completes the proof. \square

The last proposition brings to a conditional decomposition of the contribution coefficients, taking advantage of the *dominance factors* $(p_j)_{j=1,\dots,k}$, i.e. the probability of each F_j to be the dominant force in (x, t) .

Definition 7 (Conditional contributions and dominance factors) *Let $(F_i)_{i=1,\dots,k}$ be random variables on (Ω, \mathcal{F}, P) , representing arbitrary force components in location x at time t . Then, for each pair of components (i, j) , the conditional contribution $C_{i,j}$ is defined as:*

$$C_{i,j} := \mathbb{E}\left[\frac{F_i}{|F_j|} \mid \Phi = |F_j|\right],$$

where $\Phi = \Phi_2$ is the dominant force. In particular, for each component i , the dominance factor is defined as:

$$p_j := P\{\Phi = |F_j|\}.$$

Example of the Contribution Coefficients in the cases studies of this paper are reported in the Supporting Information S3 and S8. They represent an additional tool to compare the different force terms, following a less restrictive approach than the Dominance Factors.

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