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FLUID MASS SOURCES AND POINT FORCES IN LINEAR ELASTIC DIFFUSIVE SOLIDS

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Solutions for the point force and line load suddenly applied in a linear elastic fluid-infiltrated porous solid are rederived in a manner that emphasizes the relation to solutions for the homogeneous diffusion equation. Specifically, the stress, displacement, and pore pressure due to instantaneous and continuous injection of fluid mass are obtained. These are differentiated to yield solutions for fluid mass dipoles. Finally, the desired solutions are obtained by adding a constant multiple of the continuous dipole solution to the pure elasticity solution based on the undrained (short-time) moduli.

Introduction

Although the response of near surface earth materials is seldom precisely linear, the theory of linear fluid infiltrated elastic solids has proven to be very useful in studying a variety of geotechnical and geophysical problems. These include soil consolidation (McNamee and Gibson, 1960; Biot, 1941), hydraulic fracture (Rice and Cleary, 1976; Ruina, 1978, Cleary, 1979) and various aspects of earth faulting (Booker, 1974; Rice and Cleary, 1976; Rice and Simons, 1976; Rice et al., 1978; Rice and Rudnicki, 1979; Rudnicki, 1979; Rice, 1980; Roeloffs and Rudnicki, 1984/85; Rudnicki, 1985, 1986).

The governing equations of a linear fluid infiltrated elastic porous solid are identical to those for the fully coupled theory of linear thermoelasticity. However, the coupling which can justifiably be neglected in thermoelasticity must be retained in the porous media equations. This coupling greatly magnifies the difficulty of analysis and, as a consequence, there are a very limited number of analytical solutions for these equations. This situation was greatly improved by Cleary (1977) who, in an elegant and extensive analysis, established the three dimensional fundamental solutions and outlined their use in modelling embedded regions of inelasticity. (Rudnicki (1981) corrected a minor algebraic error in Cleary's (1977) results for the continuous fluid mass source solution).

This paper rederives Cleary's (1977) result for the suddenly applied point force emphasizing the relation of this solution to fluid mass source solutions of the diffusion equation. We also obtain the result for the line load in plane strain again emphasizing the connection to fluid mass source solutions. Rice and Cleary (1976) have given this solution in the form of complex potentials but expressions for the displacement and stresses are recorded here for the first time. We also record the displacements and stress for fluid mass sources and dipoles. Although the results obtained here are not new, the approach is fresh: the point force and line load solutions are obtained by superposing the dipole solutions of appropriate strength with the pure elasticity results. This approach exploits a feature of the governing equations noted by Cleary (1977) and provides useful insight into the character of solutions. Although the approach has limited application to problems in bounded bodies, it has proven useful in development of a boundary element method for numerical solution of these problems (P.K. Bannerjee, personal communication).

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Review of governing equations

The most direct formulation of the governing equations for a linear fluid-infiltrated porous elastic solid, which were first established by Biot (1941), has been given by Rice and Cleary (1976), and this treatment will be followed here. If the deformation occurs so slowly that the infiltrating fluid mass has sufficient time to diffuse from material elements, there will be no alteration of pore fluid pressure. In this limit of longtime deformation, the response is labelled *drained* and the total stress σ_{ij} is related to the displacement gradient $u_{i,j}$ $(=\partial u_i/\partial x_j)$ as in an ordinary linearly elastic solid:

$$\sigma_{ij}=\mu(u_{i,j}+u_{j,i})+\lambda u_{k,k}\delta_{ij} \tag{1}$$

where λ and μ are the Lamé moduli appropriate for drained response and δ_{ij} is the Kronecker delta. More generally, a term proportional to the pore fluid pressure must be included in (1):

$$\sigma_{ij}=\mu(u_{i,j}+u_{j,i})+\lambda u_{k,k}\delta_{ij}-\zeta p\delta_{ij} \tag{2}$$

where $\zeta=1-K/K_s'$, K $(=\lambda+\frac{2}{3}\mu)$ is the drained bulk modulus and K_s' is an empirical constant which, under circumstances stated precisely by Rice and Cleary (1976), can be identified with the bulk modulus of the solid constituents. A second constitutive relation is needed for the alteration in m, the fluid mass content per unit volume of porous solid:

$$m-m_0=\zeta\rho_0[u_{k,k}+\zeta p/(\lambda_u-\lambda)] \tag{3}$$

where m_0 is the reference value of m, ρ_0 is the mass density of homogeneous pore fluid, and λ_u is the Lamé modulus for undrained response. Undrained response occurs when load alterations are too rapid to allow time for fluid mass diffusion from material elements. Hence, $m=m_0$ for undrained response. Solving for p from (3) and substituting into (2) recovers the form of (1) with λ replaced by λ_u. These constants satisfy $\lambda<\lambda_u<\infty$ where the corresponding relation in terms of Poisson's ratio ν $(=\frac{1}{2}\lambda/(\lambda+\mu))$ is $\nu<\nu_u<\frac{1}{2}$. In both relations the upper limit is attained for separately incompressible constituents and the lower for highly compressible pore fluid. Setting $m=m_0$ in (3) and using (2) to eliminate $u_{k,k}$ reveals that for undrained response the alteration of pore fluid pressure is given by

$$p=-\tfrac{1}{3}B\sigma_{kk} \tag{4}$$

where $B=(\lambda_u-\lambda)/\zeta(\lambda_u+\frac{2}{3}\mu)$ is Skempton's pore pressure coefficient (Rice and Cleary, 1976).

The constitutive formulation is completed by Darcy's law which, in the absence of body force, is given by

$$q_i=-\rho_0\kappa\,\partial p/\partial x_i \tag{5}$$

where q_i is the mass flow rate per unit area in the x_i direction and κ is a permeability. The permeability is often expressed as $\kappa=k/\gamma$ where k is measured in units of area and γ is the fluid viscosity.

The constitutive equations (2), (3), and (5) must be combined with field equations expressing equilibrium and fluid mass conservation. The equilibrium equation is

$$\sigma_{ij,i}+F_j=0 \tag{6}$$

where F_j is the component of the body force per unit volume of porous solid (including both solid and fluid phases). Substituting (2) into (6) yields

$$(\lambda+\mu)u_{i,ij}+\mu\,\nabla^2u_j-\zeta p_{,j}+F_j=0 \tag{7}$$

where $\nabla^2(\cdot)=\partial^2(\cdot)/\partial x_k\,\partial x_k$. If (3) is used to eliminate p in favour of m in (7), the result is

$$(\lambda_u+\mu)u_{i,ij}+\mu\,\nabla^2u_j-(\lambda_u-\lambda)(\zeta\rho_0)^{-1}m_{,j}+F_j=0. \tag{8}$$

Another useful equation is obtained by forming the divergence of (8):

$$\nabla^2\left[(\lambda_u + 2\mu)u_{k,k} - (\lambda_u - \lambda)(\zeta\rho_0)^{-1}m\right] + F_{k,k} = 0. \tag{9}$$

The equation of fluid mass conservation is

$$\nabla \cdot \boldsymbol{q} + \partial m/\partial t = Q(\boldsymbol{x}, t) \tag{10}$$

where $Q(\boldsymbol{x}, t)$ is the fluid mass source. Substituting Darcy's law (5) into (10) and using (3) and (9) yield the following diffusion equation for m (Cleary, 1977):

$$\partial m/\partial t = c\,\nabla^2 m + \left\{Q(\boldsymbol{x}, t) + \left[\rho_0\kappa(\lambda_u - \lambda)/\zeta(\lambda_u + 2\mu)\right]F_{k,k}\right\} \tag{11}$$

where the diffusivity is

$$c = \kappa(\lambda_u - \lambda)(\lambda + 2\mu)/\zeta^2(\lambda_u + 2\mu). \tag{12}$$

The solution method to be described here exploits the observation of Cleary (1977) that the body force F_k contributes to the right hand side of (11) a term corresponding to a distribution of fluid mass dipoles. To see this, consider a source term of the form $Q_0 f(\boldsymbol{x}, t)$. A distribution of fluid mass dipoles is generated by superposing a source of strength Q_0/ϵ at $\boldsymbol{x} - \epsilon\boldsymbol{\xi}$ and a sink of strength $-Q_0/\epsilon$ at $\boldsymbol{x}$ and letting $\epsilon \to 0$. The result is

$$-h_k \frac{\partial f}{\partial x_k}(\boldsymbol{x}, t) = \lim_{\epsilon \to 0} Q_0\epsilon^{-1}\left[f(\boldsymbol{x} - \epsilon\boldsymbol{\xi}, t) - f(\boldsymbol{x}, t)\right] \tag{13}$$

where $h_k = Q_0\xi_k$. This term has the same form as that contributed by a body force distribution $P_j f(\boldsymbol{x}, t)$. Thus, the solution to (11) for a body force distribution $P_j f(\boldsymbol{x}, t)$ is that for a distribution of dipoles h_j where

$$h_j = -\rho_0\kappa P_j(\lambda_u - \lambda)/\zeta(\lambda_u + 2\mu). \tag{14}$$

The displacement obtained from the dipole distribution will also satisfy (8) with $F_k = 0$. To this displacement must be added the solution of (8) with $m = 0$; that is, the pure elasticity solution with undrained moduli. Thus, the complete solution for a body force distribution $P_j f(\boldsymbol{x}, t)$ is the sum of the solution for a distribution of dipoles of strength given by (14) and the pure elasticity solution based on the undrained moduli.

In the following section we will use this approach to rederive the point force solution obtained by Cleary (1977) and the line load solution for which the complex potentials were given by Rice and Cleary (1976). In both cases, we will begin by presenting the fluid mass point source solutions and then use these to get the dipole solutions. The point force and line load solutions are obtained by adding to these the appropriate elasticity solutions.

Three-dimensional solutions

Fluid mass source

The instantaneous injection of an amount of fluid mass Q_0 at the origin at $t = 0$ corresponds to choosing $\boldsymbol{F} = 0$ and

$$Q(\boldsymbol{x}, t) = Q_0\delta(\boldsymbol{x})\delta(t) \tag{15}$$

in (11) where $\delta(\cdot)$ is the Dirac singular function. The solution of (11) with (15) is well-known to be (e.g., Carslaw and Jaeger, 1959)

$$m(x, t) = Q_0(4\pi ct)^{-3/2} \exp(-r^2/4ct) \tag{16}$$

where $r = (x_k x_k)^{1/2}$. Cleary (1977) uses the procedure of Rice and Cleary (1976) for spherically symmetric problems to determine the displacements. An alternative approach, analogous to that used by Cleary (1977) to obtain the point force solution, is to recognize that dimensional analysis, linearity, spherical symmetry and isotropy require the displacement components to have the following form

$$u_i(x, t) = (Q_0/\rho_0)(x_i/r^3)U(\xi) \tag{17}$$

where $\xi = r/(ct)^{1/2}$. The dimensionless function U can be determined by substitution of (16) and (17) into (8) with $F_j = 0$. Multiplying by x_j to obtain the radial component yields an ordinary differential equation for U. The solution is

$$U(\xi) = [(\lambda_u - \lambda)/4\pi\zeta(\lambda_u + 2\mu)]\, g(\xi) \tag{18}$$

where

$$g(\xi) = (2\pi^{1/2})^{-1} \int_0^{\xi} s^2 \exp(-\tfrac{1}{4}s^2)\, ds = \operatorname{erf}(\tfrac{1}{2}\xi) - \pi^{-1/2}\xi \exp(-\tfrac{1}{4}\xi^2) \tag{19}$$

and

$$\operatorname{erf}(z) = 2\pi^{-1/2} \int_0^z e^{-t^2}\, dt$$

is the error function (Abramowitz and Stegun, 1964). Note that $g = 0$ at $t = \infty$ ($\xi = 0$) and $g = 1$ at $t = 0$ ($\xi = \infty$). Thus, at $t = 0$, the displacement has the form of a center of dilatation (e.g. Love, 1944). The pore pressure can be determined from (3) and the result is

$$p(x, t) = (Q_0/\rho_0)[(\lambda_u - \lambda)(\lambda + 2\mu)/\zeta^2(\lambda_u + 2\mu)](4\pi ct)^{-3/2} \exp(-r^2/4ct) \tag{20}$$

The stress, computed from (2), is

$$\sigma_{ij}(x, t) = \frac{Q_0}{2\pi\rho_0 r^3} \frac{\mu(\lambda_u - \lambda)}{\zeta(\lambda_u + 2\mu)} \left[\delta_{ij}(g - \xi g') + (x_i x_j/r^2)(\xi g' - 3g)\right] \tag{21}$$

where the prime denotes differentiation with respect to the argument. Cleary (1977) gives the polar component form of the stresses, as well as the pore pressure, and (20) and (21) agree with his results.

The solutions for continuous injection of fluid mass at a constant rate q can be obtained by the usual superposition procedure: replace Q_0 by $q\, dt'$ and t by $t - t'$ in (16) and (17) and integrate from $t' = 0$ to $t' = t$. The results are as follows:

$$m(x, t) = (q/4\pi cr) \operatorname{erfc}(\tfrac{1}{2}\xi) \tag{22}$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$ and

$$u_i(x, t) = \frac{q}{\rho_0 c} \frac{x_i}{r} \left[\frac{\lambda_u - \lambda}{8\pi\zeta(\lambda_u + 2\mu)}\right] u(\xi) \tag{23}$$

where

$$u(\xi) = \operatorname{erfc}(\tfrac{1}{2}\xi) + 2\xi^{-2} g(\xi). \tag{24}$$

The function $u(\xi)$ is zero at $t=0$ ($\xi=\infty$) and unity at $t=\infty$ ($\xi=0$). The pore fluid pressure and stress can then be calculated from (3) and (2). These are

$$p(x, t)=\frac{q}{\rho_0 c}\frac{1}{4\pi r}\left[\frac{(\lambda_u-\lambda)(\lambda+2\mu)}{\zeta^2(\lambda_u+2\mu)}\right]\operatorname{erfc}(\tfrac{1}{2}\xi) \tag{25}$$

and

$$\sigma_{ij}=-\frac{q}{\rho_0 c}\frac{(\lambda_u-\lambda)\mu}{4\pi r\zeta(\lambda_u+2\mu)}\left\{\delta_{ij}\left[\operatorname{erfc}(\tfrac{1}{2}\xi)-2\xi^{-2}g(\xi)\right]+\left(x_ix_j/r^2\right)\left[\operatorname{erfc}(\tfrac{1}{2}\xi)+6\xi^{-2}g(\xi)\right]\right\}. \tag{26}$$

The last expression (26) corrects that given in (39b) of Cleary (1977) as discussed by Rudnicki (1981).

Fluid mass dipoles

The solution for fluid mass dipoles can be obtained from the source solutions of the last section by the usual technique: if a particular field quantity is given by $qF(x, t)$ for a source of strength q, then the corresponding quantity for a dipole of strength q and direction λ_k is given by

$$F_{\text{dipole}}(x, t)=-q\lambda_k\frac{\partial}{\partial x_k}F(x, t). \tag{27}$$

Applying this procedure to the fluid mass solution for a continuous source given by (22) yields

$$m(x, t)=(h_k/4\pi c)(x_k/r^3)[1-g(\xi)] \tag{28}$$

where $h_k=q\lambda_k$. Similarly, the solutions for the pore fluid pressure, displacement, and stress follow from (24), (25) and (26) respectively. These are recorded below:

$$p(x, t)=(h_k/\rho_0\kappa)(x_k/4\pi r^3)[1-g(\xi)], \tag{29}$$

$$u_i(x, t)=\frac{-h_k\zeta}{\rho_0\kappa(\lambda+2\mu)}\frac{1}{8\pi r}\left\{\delta_{ik}u(\xi)+\left(x_ix_k/r^2\right)[\xi u'(\xi)-u(\xi)]\right\} \tag{30}$$

where $u(\xi)$ is given by (24); and

$$\sigma_{ij}(x, t)=\frac{h_k\zeta\mu}{\rho_0\kappa(\lambda+2\mu)}\frac{1}{4\pi r^3}\left\{x_k\delta_{ij}(\xi\Sigma_1'-\Sigma_1)+(\delta_{ik}x_j+\delta_{jk}x_i)\Sigma_2(\xi)-3\left(x_ix_jx_k/r^2\right)\left(\Sigma_2-\tfrac{1}{3}\xi\Sigma_2'\right)\right\} \tag{31}$$

where

$$\Sigma_1(\xi)=\operatorname{erfc}(\tfrac{1}{2}\xi)-2\xi^{-2}g(\xi),\qquad \Sigma_2(\xi)=\operatorname{erfc}(\tfrac{1}{2}\xi)+6\xi^{-2}g(\xi)$$

Alternatively, the functions Σ_1 and Σ_2 can be expressed as follows in terms of $u(\xi)$:

$$\Sigma_1(\xi)=u(\xi)+\xi u'(\xi)\quad\text{and}\quad\Sigma_2(\xi)=u(\xi)-\xi u'(\xi).$$

(The expression given by Cleary (1977) for the stresses due to a fluid mass dipole is incorrect because of the minor error mentioned above.)

Point force

The sudden application of a point force with components P_j at the origin corresponds to setting $Q=0$ and

$$F_j = P_j\delta(x)H(t) \tag{32}$$

in (8) and (11) where $H(t)$ is the unit step function. From the earlier discussion, the solution for the displacements is obtained from the dipole solution (30) by substituting (14) and adding the solution of (8) with $m=0$ and F_j given by (32). The latter is simply the Kelvin solution (e.g., Love, 1944) for which the displacements are as follows:

$$u_i = \frac{P_j}{8\pi r\mu(\lambda_u+2\mu)}\left[(\lambda_u+3\mu)\delta_{ij} + (\lambda_u+\mu)\left(x_i x_j/r^2\right)\right] \tag{33}$$

where the undrained value of the Lamé constant λ has been used. The corresponding stress components are

$$\sigma_{ij} = \frac{P_k}{4\pi r^2(\lambda_u+2\mu)}\left\{\mu\left[(x_k/r)\delta_{ij} - \left(x_i\delta_{jk}+x_j\delta_{ik}\right)/r\right] - 3(\lambda_u+\mu)x_i x_j x_k/r^3\right\}. \tag{34}$$

Thus, the point force solution is as follows

$$\begin{aligned}\mu u_i = {} & \frac{P_j}{8\pi r(\lambda_u+2\mu)}\left[(\lambda_u+3\mu)\delta_{ij} + (\lambda_u+\mu)\left(x_i x_j/r^2\right)\right] \\ & + \frac{P_j}{8\pi r}\left\{\left[\frac{\lambda+3\mu}{\lambda+2\mu} - \frac{\lambda_u+3\mu}{\lambda_u+2\mu}\right]u(\xi)\delta_{ij} + \left[\frac{\lambda+\mu}{\lambda+2\mu} - \frac{\lambda_u+\mu}{\lambda_u+2\mu}\right]\frac{x_i x_j}{r^2}\left[u(\xi)-\xi u'(\xi)\right]\right\}\end{aligned} \tag{35}$$

where the first term is from (33) and the second from (30) with (14). The fluid mass change for the point force solution is given by (28) after substituting from (14) and (12):

$$m(x,t) = \frac{-\rho_0\zeta P_k}{4\pi(\lambda+2\mu)}\frac{x_k}{r^3}\left[1-g(\xi)\right]. \tag{36}$$

The pore pressure can be obtained by computing σ_{kk} from (34), using (4) and adding (29) after substituting (14). The result is

$$p(x,t) = \frac{(\lambda_u-\lambda)}{\zeta(\lambda_u+2\mu)}\frac{P_k x_k}{4\pi r^3}g(\xi). \tag{37}$$

The stress components are recorded below:

$$\begin{aligned}4\pi r^3\sigma_{ij} = {} & P_k\left\{\frac{\mu}{(\lambda_u+2\mu)}\left[x_k\delta_{ij} - \left(x_i\delta_{jk}+x_j\delta_{ik}\right)\right] - \frac{3(\lambda_u+\mu)}{(\lambda_u+2\mu)}\frac{x_i x_j x_k}{r^2}\right\} \\ & + P_k\left\{\left[\frac{\mu}{\lambda+2\mu} - \frac{\mu}{\lambda_u+2\mu}\right]\left[x_k\delta_{ij}(\Sigma_1-\xi\Sigma_1') - \left(\delta_{ik}x_j+\delta_{jk}x_i\right)\Sigma_2\right]\right. \\ & \left. - \left[\frac{(\lambda+\mu)}{(\lambda+2\mu)} - \frac{(\lambda_u+\mu)}{(\lambda_u+2\mu)}\right]\frac{3x_i x_j x_k}{r^2}\left(\Sigma_2 - \tfrac{1}{3}\xi\Sigma_2'\right)\right\}.\end{aligned} \tag{38}$$

Plane strain solutions

In plane strain $u_3=0$ and $\partial(\cdot)/\partial x_3=0$, but, otherwise, the governing equations (2), (3), (8), (11) remain the same. As in the three-dimensional solutions, we will begin with fluid mass (line) sources, progress to fluid mass dipoles, and, finally, obtain the line load solution by simple superposition.

Fluid mass source

The fluid mass change induced by instantaneous introduction of an amount of fluid mass Q_0' per length in the z direction is (Carslaw and Jaeger, 1959)

$$m(x,\,t)=Q_0'(4\pi ct)^{-1}\exp(-R^2/4ct) \tag{39}$$

where $R=(x_1^2+x_2^2)^{1/2}$. By the same arguments used in the three-dimensional solution, the displacements must have the form

$$u_\alpha(x,\,t)=(Q_0'/\rho_0)(x_\alpha/R^2)U_0(\xi) \tag{40}$$

where $\xi=R/(ct)^{1/2}$. (Here, and in the following, the Greek subscripts take on values 1 and 2). Substituting into equilibrium (8) (with $F_j=0$) and taking the radial component yields the following equation for the dimensionless function $U_0(\xi)$:

$$\xi^2U_0''(\xi)-\xi U_0'(\xi)+\left[(\lambda_u-\lambda)/8\pi\zeta(\lambda_u+2\mu)\right]\xi^4\exp(-\tfrac{1}{4}\xi^2)=0. \tag{41}$$

The solution subject to bounded displacements at $R=\infty$ and no initial displacement is

$$U_0(\xi)=\frac{(\lambda_u-\lambda)}{2\pi s(\lambda_u+2\mu)}\left[1-\exp(-\tfrac{1}{4}\xi^2)\right]. \tag{42}$$

The displacements are given by

$$u_\alpha=\frac{Q_0'(\lambda_u-\lambda)}{2\pi\rho_0\zeta(\lambda_u+2\mu)}\frac{x_\alpha}{R^2}\left[1-\exp(-R^2/4ct)\right]. \tag{43}$$

The stress components and pore pressure are calculated from (3) and (2) and are given by

$$p=(Q_0'/\rho_0)(4\pi\kappa t)^{-1}\exp(-R^2/4ct) \tag{44}$$

and

$$\sigma_{\alpha\beta}=\frac{\mu Q_0'(\lambda_u-\lambda)}{\pi\zeta(\lambda_u+2\mu)R^2}\left\{\left(\delta_{\alpha\beta}-\frac{2x_\alpha x_\beta}{R^2}\right)\left[1-\exp(-R^2/4ct)\right]\right.$$
$$\left.-\left[\delta_{\alpha\beta}-x_\alpha x_\beta/R^2\right](R^2/2ct)\exp(-R^2/4ct)\right\}. \tag{45}$$

Solutions for continuous fluid mass injection at a rate q' per unit length can be generated by the superposition procedure described earlier. The result for the fluid mass change is

$$m=(q'/4\pi c)E_1(R^2/4ct) \tag{46}$$

where (Abramowitz and Stegun, 1964)

$$E_1(z)=\int_z^\infty s^{-1}\exp(-s)\,\mathrm{d}s. \tag{47}$$

For small values of its argument

$$E_1(z) \sim -\gamma - \ln z \tag{48}$$

where $\gamma = 0.57721$ is Euler's constant. Therefore, for large times, the fluid mass change is given by

$$m \sim (q'/4\pi c)\ln(1/R) - q'\gamma/4\pi c + (q'/4\pi c)\ln(4ct). \tag{49}$$

As explained by Carslaw and Jaeger (1959), the first term gives the steady state response and, although the last term in (49) becomes unbounded as $t \to \infty$, it is independent of position and can be discarded. The expressions for the displacement, pore pressure, and stress are

$$u_\alpha = \frac{q'\zeta x_\alpha}{8\pi\kappa\rho_0(\lambda + 2\mu)}\left\{(4ct/R^2)\left[1 - \exp(-R^2/4ct)\right] + E_1(R^2/4ct)\right\}, \tag{50}$$

$$p = (q'/4\pi\kappa\rho_0)E_1(R^2/4ct), \tag{51}$$

$$\sigma_{\alpha\beta} = \frac{q'\zeta\mu}{4\pi\rho_0\kappa(\lambda + 2\mu)}\left\{(\delta_{\alpha\beta} - 2x_\alpha x_\beta/R^2)(4ct/R^2)\left[1 - \exp(-R^2/4ct)\right] - \delta_{\alpha\beta}E_1(R^2/4ct)\right\}. \tag{52}$$

Fluid mass dipoles

Solutions for a continuous fluid mass dipole with strength h_α are obtained by applying (27) to (46) and (50)–(52). The results are as follows

$$m = (h_\alpha x_\alpha/2\pi c R^2)\exp(-R^2/4ct), \tag{53}$$

$$u_\alpha = \frac{-\zeta h_\beta}{8\pi\kappa\rho_0(\lambda + 2\mu)}\left\{(4ct/R^2)\left[1 - \exp(-R^2/4ct)\right]\left[\delta_{\alpha\beta} - 2x_\alpha x_\beta/R^2\right] + \delta_{\alpha\beta}E_1(R^2/4ct)\right\}, \tag{54}$$

$$p = (h_\alpha x_\alpha/2\pi R^2\rho_0\kappa)\exp(-R^2/4ct), \tag{55}$$

$$\begin{aligned}\sigma_{\alpha\beta} = \frac{-\mu\zeta h_\gamma}{2\pi\rho_0\kappa R^2(\lambda + 2\mu)}&\left\{2\left[x_\gamma\delta_{\alpha\beta} - x_\alpha x_\beta x_\gamma/R^2\right]\exp(-R^2/4ct)\right.\\ &\left.- (4ct/R^2)\left[1 - \exp(-R^2/4ct)\right]\left[x_\alpha\delta_{\gamma\beta} + x_\beta\delta_{\alpha\gamma} + x_\gamma\delta_{\alpha\beta} - 4x_\alpha x_\beta x_\gamma/R^2\right]\right\}.\end{aligned} \tag{56}$$

Point force

The remaining ingredient needed to obtain the solution for sudden application of a line load with components P_γ is the undrained solution. This is simply the pure elasticity solution with undrained moduli. The displacement and stress components are (e.g., Love, 1944)

$$u_\alpha = \frac{-P_\beta(\lambda_u + \mu)}{8\pi\mu(\lambda_u + 2\mu)}\left\{\frac{2(\lambda_u + 3\mu)}{(\lambda_u + \mu)}\delta_{\alpha\beta}\ln R + \delta_{\alpha\beta} - 2x_\alpha x_\beta/R^2\right\}, \tag{57}$$

$$\sigma_{\alpha\beta} = \frac{-P_\gamma}{2\pi R^2(\lambda_u + 2\mu)}\left\{\mu(x_\alpha\delta_{\beta\gamma} + x_\beta\delta_{\alpha\gamma} - x_\gamma\delta_{\alpha\beta}) + 2(\lambda_u + \mu)x_\alpha x_\beta x_\gamma/R^2\right\}. \tag{58}$$

The pore pressure induced by undrained response can be calculated be setting $m = m_0$ in (3) and computing $u_{\alpha,\alpha}$ from (57). Substituting (14) in (53)–(56) and adding (57)–(58) and the undrained pore pressure yield the suddenly applied line load solution:

$$u_\alpha = \frac{-P_\beta(\lambda_u+\mu)}{8\pi\mu(\lambda_u+2\mu)}\left\{\frac{2(\lambda_u+3\mu)}{(\lambda_u+\mu)}\delta_{\alpha\beta}\ln R + \delta_{\alpha\beta} - 2x_\alpha x_\beta/R^2\right\}$$
$$+\frac{P_\beta(\lambda_u-\lambda)}{8\pi(\lambda_u+2\mu)(\lambda+2\mu)}\left\{\frac{4ct}{R^2}\left[1-\exp(-R^2/4ct)\right]\left(\delta_{\alpha\beta}-2x_\alpha x_\beta/R^2\right)+\delta_{\alpha\beta}E_1(R^2/4ct)\right\}, \tag{59}$$

$$p = \frac{c\zeta}{\kappa(\lambda+2\mu)}\frac{P_\beta x_\beta}{2\pi R^2}\left[1-\exp(-R^2/4ct)\right], \tag{60}$$

$$\sigma_{\alpha\beta} = \frac{-P_\gamma}{2\pi R^2(\lambda_u+2\mu)}\left\{\mu\left(x_\alpha\delta_{\beta\gamma}+x_\beta\delta_{\alpha\gamma}-x_\gamma\delta_{\alpha\beta}\right)+2(\lambda_u+\mu)x_\alpha x_\beta x_\gamma/R^2\right\}$$
$$+\frac{P_\gamma(\lambda_u-\lambda)\mu}{2\pi R^2(\lambda_u+2\mu)(\lambda+2\mu)}\left\{2\left[x_\gamma\delta_{\alpha\beta}-x_\alpha x_\beta x_\gamma/R^2\right]\exp(-R^2/4ct)\right.$$
$$\left.-(4ct/R^2)\left[1-\exp(-R^2/4ct)\right]\left[x_\alpha\delta_{\gamma\beta}+x_\beta\delta_{\alpha\gamma}+x_\gamma\delta_{\alpha\beta}-4x_\alpha x_\beta x_\gamma/R^2\right]\right\}. \tag{61}$$

The expressions for the displacement (59) and the stress components (61) can easily be shown to reduce to the proper drained limit as $t \to \infty$; that is, they reduce to (57) and (58) with λ_u replaced by λ. These expressions can also be shown to agree with the results given by Rice and Cleary (1976). Although they do not give expressions for the stress and displacement for the line load, Rice and Cleary (1976) do give the complex potentials from which they can be derived. (Note, however, that the sign of the second term in their expression for the complex constant B (their (40)) should be changed from minus to plus).

Conclusion

This note has rederived the solutions for the suddenly applied point force and line load in an elastic fluid-infiltrated solid using fluid mass source and dipole solutions, and superposition of the undrained pure elasticity solution. This approach emphasizes the relation of the point force and line load solutions and the solutions for continuous fluid mass dipoles, a feature noted by Cleary (1977), but not exploited by him in his solution technique. The clarity of hindsight suggests that present approach is simpler, though, perhaps, less direct.

The rearrangement of the point force and line load solutions into two components, a solution of the classical elasticity equations with the undrained moduli and a solution to the homogeneous diffusion equation, provides insight into the structure of the governing equations and the character of the solutions. This rearrangement has also proven advantageous for implementation in a numerical boundary element technique.

Unfortunately, because the boundary conditions are typically expressed in terms of the pore pressure and tractions or displacement, rather than fluid mass, it is generally not possible to solve the diffusion equation independently of the elasticity equations. Consequently, the technique described here may be of limited usefulness in obtaining analytical solutions for bounded bodies. However, one practically important exception is problems involving fluid mass injection or withdrawal from embedded regions. In this case the diffusion equation can be solved for the fluid mass and the result inserted into (8) where it plays the role of an effective body force. This procedure has been used by Segall (1985) to predict surface subsidence and stresses due to fluid withdrawal from a narrow layer. The calculated surface displacements agree well with observations of subsidence due to oil extraction in Coalinga, California.

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Appendix

We record here the relations between the functions used in Cleary's (1977) solution for the point force and those used here.

$$f_1(\xi) = \left[(\lambda_u - \lambda)/4\pi\xi(\lambda_u + 2\mu)\right] g(\xi),$$

$$f_2(\xi) = \frac{(\lambda_u + \mu)}{8\pi(\lambda_u + 2\mu)}\left\{1 - \frac{\mu(\lambda_u - \lambda)}{(\lambda + 2\mu)(\lambda_u + \mu)}(u - \xi u')\right\},$$

$$f_3(\xi) = \frac{(\lambda_u + 3\mu)}{8\pi(\lambda_u + 2\mu)}\left\{1 + \frac{\mu(\lambda_u - \lambda)}{(\lambda + 2\mu)(\lambda_u + 3\mu)}u(\xi)\right\},$$

$$F_1(\xi) = \frac{-3(\lambda_u + \mu)}{4\pi(\lambda_u + 2\mu)}\left\{1 - \left[1 - \frac{(\lambda_u + 2\mu)(\lambda + \mu)}{(\lambda + 2\mu)(\lambda_u + \mu)}\right](\Sigma_2 - \tfrac{1}{3}\xi\Sigma_2')\right\},$$

$$F_2(\xi) = \frac{-\mu}{4\pi(\lambda_u + 2\mu)}\left\{1 - \left[1 - \frac{(\lambda_u + 2\mu)}{(\lambda + 2\mu)}\right]\Sigma_2(\xi)\right\},$$

$$F_3(\xi) = \frac{\mu}{4\pi(\lambda_u + 2\mu)}\left\{1 - \left[1 - \frac{(\lambda_u + 2\mu)}{(\lambda + 2\mu)}\right](\Sigma_1 - \xi\Sigma_1')\right\},$$

where $g(\xi)$ is given by (19), $u(\xi)$ by (24) and Σ_1 and Σ_2 are given following (31). Because Σ_1 and Σ_2 can be expressed in terms of $u(\xi)$, F_1, F_2, and F_3 can all be expressed in terms of $u(\xi)$.

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