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The Response of Well-Aquifer Systems to Seismic Waves

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Abstract. The degree to which the water level in an open well fluctuates in response to a seismic wave is determined by the dimensions of the well, the transmissibility, storage coefficient, and porosity of the aquifer, and the type, period, and amplitude of the wave. The water level responds to pressure-head fluctuations due to dilatation of the aquifer and to vertical motion of the well-aquifer system; hence a wave that produces either of these can cause the water level to fluctuate. However, the response to dilatation is much larger than the response to vertical motion. A solution is derived for the nonsteady drawdown in the aquifer due to a harmonic motion of the water level. This solution is then used in the equation of motion of the water column to derive expressions for the amplification, which is defined as the ratio x_0/h_0 (for oscillation due to dilatation) or the ratio x_0/a (for oscillation due to vertical motion of the well-aquifer system), where x_0 is the amplitude of water-level fluctuation, h_0 is the amplitude of pressure-head fluctuation, and a is the amplitude of vertical motion of well-aquifer system. Amplification curves are given for differing well dimensions and aquifer constants.

NOTATION

$A = x_0/h_0$,	amplification of pressure-head fluctuation (dimensionless).
$A' = x_0/a$,	amplification of motion of well-aquifer system (dimensionless).
a,	amplitude of land-surface motion (L).
b,	effective proportion of aquifer area that responds elastically (dimensionless).
c,	phase velocity (L/T).
D,	constant in Rayleigh equation (L).
d,	thickness of aquifer and length of screen (L).
E_s,	bulk modulus of elasticity of aquifer (M/LT^2).
E_w,	bulk modulus of elasticity of water (M/LT^2).
f,	coefficient of roughness (dimensionless).
g,	gravity field strength (L/T^2).
H,	height of water column in well casing (L).
$H_e = H + 3d/8$,	effective height of water column (L).
h,	pressure head (L).
h_L,	head loss in pipe (L).
h_0,	amplitude of pressure-head fluctuation (L).
k,	restoring force per unit displacement (M/T^2).
Ker and Kei,	Kelvin functions (dimensionless).
K,	permeability (hydraulic conductivity) of aquifer (L/T).
l,	damping force per unit velocity (M/T).
l_0,	critical damping force per unit velocity (M/T).
m,	mass of a mechanical system (M).
$M = x_0/\omega_0$,	magnification of vertical component of land-surface motion in Rayleigh wave (dimensionless).
n,	porosity of aquifer (dimensionless).
p,	pressure (M/LT^2).
p_c,	pressure at bottom of casing (M/LT^2).
p_f,	fluctuating pressure in aquifer (M/LT^2).
p_m,	pressure at midpoint of screen (M/LT^2).

p_0, amplitude of pressure fluctuation (M/LT^2).
$P(t)$, forcing function (ML/T^2).
Q, flow into well (L^3/T).
q, flow per unit area through aquifer (L/T).
$R = h_0/w_0$, (dimensionless).
r, radial distance from center of well (L).
r_w, radius of well (L).
S, storage coefficient of aquifer (dimensionless).
s, drawdown in aquifer (L).
s_m, drawdown outside midpoint of screen (L).
s_w, drawdown outside screen (L).
t, time (T).
T, transmissibility of aquifer (L^2/T).
u, horizontal component of particle displacement in Rayleigh wave (L).
V_z, velocity of water in well (L/T).
V, volume of effective cavity (L^3).
v, drawdown caused by constant unit discharge from well (L).
w, vertical component of particle displacement in Rayleigh wave (L).
w_0, amplitude of vertical component of particle displacement in Rayleigh wave (L).
x, displacement of water level in well (L).
x_0, amplitude of water-level fluctuation (L).
x_1, displacement of well-aquifer system (L).
x_2, displacement of water level in well (L).
y, distance along direction of travel of Rayleigh wave (L).
z, depth below bottom of casing or depth below surface in Rayleigh wave (L).
$\alpha = r(\omega S/T)^{1/2}$, (dimensionless).
$\alpha_w = r_w(\omega S/T)^{1/2}$, (dimensionless).
β, function defined in equation 22 (dimensionless).
η, phase angle (radians).
θ, dilatation (dimensionless).
$\kappa = 2\pi/\lambda$, wave number (radians/L).
λ, length of seismic wave (L).
ξ, phase angle (radians).
ρ, density of water (M/L^3).
τ, period of seismic wave (T).
$\omega = 2\pi/\tau$, angular frequency of seismic wave (radians/T).
ω_n, natural circular frequency of a mechanical system (radians/T).
ω_w, function defined in equation 21 (dimensionless).

Introduction

To the groundwater hydrologist earthquake-induced fluctuations of water levels in wells may provide a means of obtaining significant information on aquifer characteristics, once the mechanics governing them are adequately understood. To the seismologist they imply a possibility of using wells as seismographs. *Rexin et al.* [1962, p. 18] have suggested, for example, that the response of some wells to long-period waves may be superior to the response of conventional long-wave seismographs.

The effect of earthquakes on water levels in some open artesian wells is remarkably large. For example, the water level in an open artesian well near Perry, Florida (U.S. Geological Survey well Taylor 35), fluctuated over a double amplitude of as much as 4.6 meters in response to the Alaskan earthquake of March 27, 1964. On the other hand, the response of other wells that penetrate the same aquifer was only a small fraction of a meter. That the response to a given seismic wave is much larger in some wells than in others has interested hydrologists and seismologists alike. An obvious explanation is that well-aquifer systems have the essential features of a seismograph and hence magnify the seismic disturbance by a degree that depends not only on the characteristics of the well and aquifer but also on the period of the disturbing wave. Like the seismograph, the well-aquifer system consists of a mass (the column of water in the well plus some part of the water in the aquifer), a restoring force (the difference between the pressure head in the aquifer and the displaced water level in the well), and a damping force (the friction that accompanies the flow of water through the well and aquifer).

That the column of water in some wells can oscillate very much like the classic spring-mass system is demonstrated by the results of a series of field tests in wells in Florida and Georgia in July 1964. Figure 1 shows the motion of the water surface in the well near Perry, Florida, as recorded by a pressure transducer suspended below the water surface during one of these tests. The free oscillation of the water surface was characteristic of free vibration with viscous damping. The well is cased with 30.5-cm (12-inch) pipe to a depth of 57.6 meters and was completed as a 30.5-cm (12-inch) open hole in the Floridan aquifer (cavernous limestone) between depths of 57.6 and 74.7 meters. The total depth was measured in 1945 to be 70.1 meters. The aquifer has a transmissibility of about 0.13 m²/sec.

A theory to account for the magnification of longitudinal earth movements as registered in a well was presented by *Blanchard and Byerly* [1935]. They assumed that an aquifer can be represented by a finite open cavity that contracts and expands during the passage of seismic waves that produce dilatation. According to this theory the ratio of the amplitude of the water-level oscillation to that of the longitudinal displacement of the earth is

$$M = 2\pi \Big/ \lambda\left(\frac{\pi r_w^{\ 2}}{V} + \frac{\rho g}{E_w}\right)$$

where V is the volume of the effective cavity. The inertia of water in the well and aquifer and the friction that would accompany flow through the aquifer are neglected. Also, as Blanchard and Byerly point out, the effective volume V would be a function of the length of the seismic wave.

Eaton and Takasaki [1959] established an empirical relationship between the maximum amplitude of water-level fluctuations in an open well at Honolulu, Hawaii, and the type, wavelength, and amplitude of the earthquake wave producing the fluctuations. They pointed out that the relation between the response of a well and the response of a Wood-Anderson seismograph can be expressed as $H = CB$, where H is the double amplitude of maximum fluctuation in the well, B is the maximum Wood-Anderson response, and C is a 'constant of proportionality, which accounts for the differences in magnifications between the well and Wood-Anderson . . . .' The word 'constant' can be misleading, because they clearly recognized C to be a function of the wavelength; they stated that the response of the well at Honolulu diminishes rapidly as the wavelength decreases.

By a different procedure, *Rexin et al.* [1962] established a relationship between the response of a well at Milwaukee, Wisconsin, and distant earthquakes. Their method of analysis of well and earthquake data led to a plot of points representing the magnification of the 'well seismograph' versus the period of the wave.

Our objective was to derive an analytic solution for the degree to which earthquake-induced fluctuations of artesian pressure and vertical vibrations of the land surface are amplified in an open well. Advances that have been made in the theory of aquifer mechanics since the paper by Blanchard and Byerly was published permit the treatment of a more realistic model than the one they assumed. The solution to be derived will relate the amplification to aquifer characteristics (the transmissibility and storage coefficients of the aquifer), the dimensions of the well (the diameter, height of water column, and length of screen), and the period of the seismic wave.

Analysis

Response to Harmonic Fluctuations of Artesian Pressure

Consider an uncapped nonflowing artesian well cased to the top of a homogeneous isotropic aquifer and screened (or open) throughout the thickness d of the aquifer (Figure 2). Suppose that the artesian pressure starts to fluctuate (as in response to dilatational waves due to a seismic shock) according to

$$p_f = \rho g(H + z) + p_0 \sin(\omega t - \eta) \qquad (1)$$

causing the water level in the well to oscillate. One can show that the oscillation of the water level would consist of a transient component

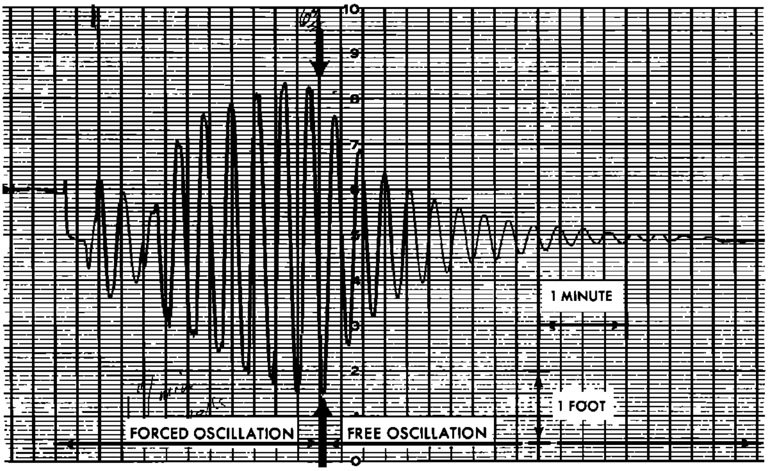

Fig. 1. Hydrograph of a well near Perry, Florida.

plus a steady-periodic component having the same frequency as the pressure p_f. To simplify the analysis we assume that sufficient time has elapsed for the transient component to die away, leaving only the steady-periodic component.

Let us suppose that p_0 and η in (1) are such that the steady oscillation of the water is described by

$$x = x_0 \sin \omega t \tag{2}$$

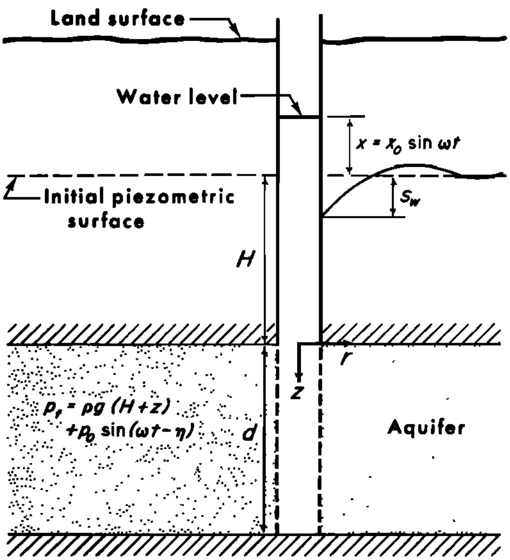

Fig. 2. Idealized representation of an open well in which water level fluctuations are caused by oscillation of artesian pressure.

Our objective is to find an expression for the amplification factor of the well-aquifer system which we define to be the ratio of the amplitude of the oscillation of the water level in the well to the amplitude $h_0 = p_0/\rho g$ of the pressure head fluctuation in the aquifer. Accordingly, we define the amplification factor as

$$A = x_0/h_0 = \rho g x_0/p_0 \tag{3}$$

To develop the expression for A, we must find the pressure amplitude p_0 in terms of the well dimensions, the aquifer constants, and the period τ.

We assume that the velocity of the water in the casing is everywhere vertical and uniform, so that it is described by $dx/dt = \omega x_0 \cos \omega t$. If we consider only situations for which the product ωx_0 is so small that the 'pipe' friction due to the movement of water through the well is negligible in relation to the drawdown in the aquifer, the momentum equation for the water in the casing can be written

$$\frac{d}{dt}\left[\rho \pi r_w^2 (H + x) \frac{dx}{dt}\right] - \rho \pi r_w^2 \left(\frac{dx}{dt}\right)^2 + \rho g \pi r_w^2 (H + x) = \pi r_w^2 p_c$$

The first term is the time rate of increase of the momentum of the water column, the second is the influx of momentum at the bottom of the casing, and the third is the weight of the column. The equation of momentum reduces to

$$\rho(H + x) \frac{d^2x}{dt^2} + \rho g(H + x) = p_c \tag{4}$$

We seek next to evaluate p_c. The motion of the column of water creates a discharge from the aquifer into the well described by

$$Q = \pi r_w^2 \, dx/dt = \pi r_w^2 \omega x_0 \cos \omega t \qquad (5)$$

where Q is positive toward the well. The discharge produces in the aquifer a drawdown s (positive downward) which is superposed on the fluctuating pressure head represented by $p_f/\rho g$. Hence the pressure at a given depth z below the top of the aquifer is

$$p = p_f - \rho g s = \rho g(H + z - s) + p_0 \sin (\omega t - \eta) \qquad (6)$$

Inertia within screened interval. The inertia of water within the screened part of the well causes the pressure head within the screen to vary with depth. This causes the discharge through the screen to be slightly nonuniform. To estimate p_c we assume that s is symmetric about the midpoint of the screen. We let s_m represent the drawdown in the aquifer immediately outside the midpoint of the screen and let d be the length of the screen, which is also the thickness of the aquifer. Thus, if friction losses through the screen are neglected, the unit pressure within the screen at the midpoint is

$$p_m = p_f - \rho g s_m = \rho g\left(H + \frac{d}{2} - s_m\right) + p_0 \sin (\omega t - \eta) \qquad (7)$$

To compute the pressure difference $p_m - p_c$ we develop the momentum equation for the control volume consisting of the upper half of the interior of the well screen. The total force acting on the water in this volume will be equal to the rate of efflux of momentum across its surface plus the rate of change of momentum in its interior.

We assume that the flow from the aquifer per unit length of screen is uniform and that the velocity within the screen is vertical and uniform across a horizontal section. Then the velocity V_z of a particle at depth z is described by

$$V_z = (1 - z/d) \, dx/dt \qquad (8)$$

The velocity at the midpoint of the screen is $(\frac{1}{2})(dx/dt)$, and hence the upward momentum flux across the bottom surface of the control volume is $(\rho\pi r_w^2/4)(dx/dt)^2$. The upward flux across the top surface is $\rho\pi r_w^2(dx/dt)^2$. Hence the net efflux of momentum across the control surface in the z direction is

$$(3/4)\rho\pi r_w^2 (dx/dt)^2 \qquad (9)$$

The time rate of change of momentum inside the volume is

$$\frac{d}{dt}\int_0^{d/2} \rho\pi r_w^2\left(1 - \frac{z}{d}\right)\frac{dx}{dt}\,dz = \tfrac{3}{8}\rho\pi r_w^2 d \frac{d^2x}{dt^2} \qquad (10)$$

The body force is $\rho g\pi r^2 d/2$. Hence the momentum equation in the z direction is

$$p_m - p_c - \frac{\rho g d}{2} = \frac{3\rho d}{8}\frac{d^2x}{dt^2} + \frac{3\rho}{4}\left(\frac{dx}{dt}\right)^2 \qquad (11)$$

The last term on the right will be negligible in relation to the other terms when the product ωx_0 is small. Neglecting this term, we have for the pressure at the bottom of the casing

$$p_c = p_m - \frac{3\rho d}{8}\frac{d^2x}{dt^2} - \frac{\rho g d}{2} \qquad (12)$$

As we observed above, the discharge through the screen is not uniform, and hence the velocity V_z is not a linear function of z as assumed in (8). We observe, however, that the inertia that produces the nonuniformity is zero when the velocity in the casing is maximum, and vice versa. Also, if we restrict the analysis to small values of the product $\omega^2 x_0$, the inertia-produced nonuniformity will be small. Therefore, for the purpose of evaluating the inertia the linear relationship represented by (8) constitutes a satisfactory approximation.

Substituting from (7) into (12) yields

$$p_c = \rho g(H - s_m) + p_0 \sin (\omega t - \eta) - \frac{3\rho d}{8}\frac{d^2x}{dt^2} \qquad (13)$$

When (13) is substituted into (4) we find the equation of motion of the water in the well to be

$$\rho\left(H + \frac{3d}{8} + x\right)\frac{d^2x}{dt^2} + \rho g(x + s_m) = p_0 \sin (\omega t - \eta) \qquad (14)$$

We let $H_e = (H + 3d/8)$ be a quantity which we shall call the 'effective column height.' Then, since we are restricting our attention to displacements x that are much smaller than H_e, the equation of motion (14) is approximated by

$$\frac{d^2x}{dt^2} + \frac{g}{H_e}(x + s_m) = \frac{p_0}{\rho H_e}\sin(\omega t - \eta) \quad (15)$$

Drawdown in the aquifer. The drawdown s in the aquifer consists of two components, a frictional component due to the resistance to flow through the aquifer and an inertial component due to the time rate of change of the harmonically varying velocity through the aquifer. To develop an expression for the drawdown we must assume that Darcy's law applies, which is equivalent to assuming that the inertial component of the drawdown is negligible. Darcy's law can be written as $q = K\nabla h$.

Where Darcy's law applies, an expression for the drawdown due to the oscillation of water in the well can be developed from the *Theis* [1935] equation for the nonsteady drawdown produced in an infinite aquifer by a constant discharge from a well. For a unit discharge the Theis equation can be written as

$$v(r, t) = \frac{1}{4\pi T}\int_{r^2S/4Tt}^{\infty} \frac{e^{-u}}{u}\,du \quad (16)$$

where v is the drawdown (positive downward) at distance r from the center of the well at time t after the discharge begins. The transmissibility T is defined as the flow per unit width of aquifer per unit hydraulic gradient and the storage coefficients S as the volume of water released from or taken into storage per unit surface area per unit change in head [*Theis,* 1935]. If the flow is parallel to the individual layers that make up an aquifer, the transmissibility can be expressed as

$$T = \sum_{i=1}^{n} K_i d_i$$

where K_i is the permeability of the ith layer, d_i is the thickness of the ith layer, and n is the number of layers.

The drawdown produced by the harmonically varying discharge can be obtained by using Q from (5) and v from (16) in Duhamel's formula [*Sneddon,* 1951, p. 164]. Thus

$$\begin{aligned} s &= \frac{\partial}{\partial t}\int_0^t Q(t')v(r, t - t')\,dt' \\ &= \int_0^t [\pi r_w^2 \omega x_0 \cos \omega t']\left[\frac{e^{-r^2S/4T(t-t')}}{4\pi T(t - t')}\right] dt' \\ &= \frac{r_w^2\omega x_0}{4T}\int_0^t (\cos \omega t')e^{-r^2S/4\pi T(t-t')}\frac{dt'}{t - t'} \end{aligned} \quad (17)$$

When the integral is evaluated we obtain

$$s = \frac{r_w^2\omega x_0}{2T}\left[(\mathrm{Ker}\,\alpha)(\cos \omega t) - (\mathrm{Kei}\,\alpha)(\sin \omega t) - \int_0^\infty \frac{e^{-\omega t u^2/\alpha^2} u^3 J_0(u)}{\alpha^4 + u^4}\,du\right] \quad (18)$$

where $\alpha = r(\omega S/T)^{1/2}$. The functions Ker α and Kei α, sometimes called Kelvin functions, are the real and imaginary parts of $K_0(\alpha i^{1/2})$, which is the modified Bessel function of the second kind of order zero [*McLachlan,* 1955, p. 139]. The function $J_0(\beta)$ is the Bessel function of the first kind of order zero.

The integral in (18) is a transient term which dies away as t becomes large. This integral can be developed into a series by expanding $J_0(\beta)$ and integrating term by term. When this is done, we find that for $\alpha \leq 2$, corresponding to ranges of r_w, ω, S, and T that are likely to be of interest, the transient term dies away rapidly, becoming negligibly small within one oscillation, as shown in Figure 3. The transient term dies away more rapidly as α becomes larger.

Using (2) we can write the trigonometric terms in (18) as

$$\sin \omega t = x/x_0 \qquad \cos \omega t = (1/\omega x_0)(dx/dt)$$

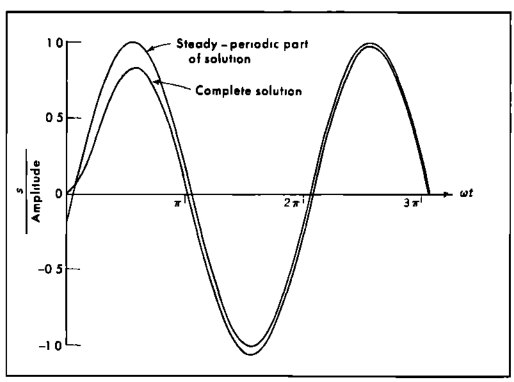

Fig. 3. Graph showing rapid decay of the transient motion for conditions in which $\alpha = 2$.

Hence, within one oscillation the drawdown immediately outside the screen is closely described by

$$s_w = \frac{r_w^2}{2T}\left(\frac{dx}{dt}\,\mathrm{Ker}\,\alpha_w - \omega x\,\mathrm{Kei}\,\alpha_w\right) \qquad (19)$$

where $\alpha_w = r_w(\omega S/T)^{1/2}$. This result is valid when the discharge Q is distributed uniformly over the screened interval. As mentioned previously, the inertia of the water in the screen would cause a slight nonuniformity. We have already assumed, however, that the drawdown would be symmetrical about the midpoint of the aquifer, so that s_m, the drawdown at the midpoint, will be very nearly equal to s_w. Taking $s_m = s_w$ and substituting from (19) into (15) we find the equation of motion of the water in the well to be

$$\frac{d^2x}{dt^2} + \frac{r_w^2 g}{2TH_e}\,\mathrm{Ker}\,\alpha_w\,\frac{dx}{dt} + \frac{g}{H_e}\left(1 - \frac{r_w^2\omega}{2T}\,\mathrm{Kei}\,\alpha_w\right)x = \frac{p_0}{\rho H_e}\sin(\omega t - \eta) \qquad (20)$$

If we make the designations

$$\omega_w^2 = \frac{g}{H_e}\left(1 - \frac{r_w^2\omega}{2T}\,\mathrm{Kei}\,\alpha_w\right) \qquad (21)$$

$$\beta = \frac{r_w^2 g}{4\omega_w T H_e}\,\mathrm{Ker}\,\alpha_w \qquad (22)$$

the equation of motion can be written

$$\frac{d^2x}{dt^2} + 2\beta\omega_w\frac{dx}{dt} + \omega_w^2 x = \frac{p_0}{\rho H_e}\sin(\omega t - \eta) \qquad (23)$$

Equation 23 has the form of the differential equation of motion of a mechanical system subjected to forced vibration with viscous damping. Most seismographs are examples of such a system. For a mechanical system the equation is (see, for example, *Den Hartog* [1956, p. 47])

$$m\frac{d^2x}{dt^2} + l\frac{dx}{dt} + kx = P(t)$$

which can be written

$$\frac{d^2x}{dt^2} + 2\frac{l}{l_0}\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{P(t)}{m} \qquad (24)$$

Hence ω_w plays the role of the natural angular frequency $\omega_n = (k/m)^{1/2}$ and β that of the damping index l/l_0. There is, however, one important difference between a mechanical system and a well-aquifer system: whereas in the mechanical system l/l_0 and ω_n are constants, in the well-aquifer system β and ω_w are both functions of the forcing frequency ω. Strictly speaking, therefore, there is no such thing as a resonant frequency of a well. However, if $(r_w^2\omega/2T)$ Kei $\alpha_w \ll 1$, (21) is approximated by

$$\omega_w \approx (g/H_e)^{1/2} \qquad (25)$$

so that under this condition $(g/H_e)^{1/2}$ can be treated as the resonant frequency of the well.

When (2) is substituted into (20) the pressure amplitude p_0 which will cause the water level to oscillate according to $x = x_0 \sin \omega t$ is found to be

$$p_0 = x_0\rho H_e\left[\left(\frac{g}{H_e}\left(1 - \frac{r_w^2\omega}{2T}\,\mathrm{Kei}\,\alpha_w\right) - \omega^2\right)^2 + \left(\frac{\omega r_w^2 g}{2TH_e}\,\mathrm{Ker}\,\alpha_w\right)^2\right]^{1/2} \qquad (26)$$

and the phase angle is found to be

$$\eta = \arctan\,[2\beta\omega_w\omega/(\omega^2 - \omega_w^2)] \qquad (27)$$

Finally, substituting from (26) into (3) and using the relation $\omega = 2\pi/\tau$, we find the amplification factor to be

$$A = \left[\left(1 - \frac{\pi r_w^2}{T\tau}\,\mathrm{Kei}\,\alpha_w - \frac{4\pi^2 H_e}{\tau^2 g}\right)^2 + \left(\frac{\pi r_w^2}{T\tau}\,\mathrm{Ker}\,\alpha_w\right)^2\right]^{-1/2} \qquad (28)$$

Response to Harmonic Vertical Motion of Well-Aquifer System

Let us consider a vertical vibration of the well-aquifer system described by $x_1 = a \sin(\omega t - \xi)$ measured positive upward (Figure 4). We let x_2 represent the displacement of the water level above its initial position. The position of the water level as recorded by an instrument moving with the land surface is then $x = x_2 - x_1$. The motion of the water level is again $x = x_0 \sin \omega t$. In this case we define the amplification factor to be

$$A' = x_0/a \qquad (29)$$

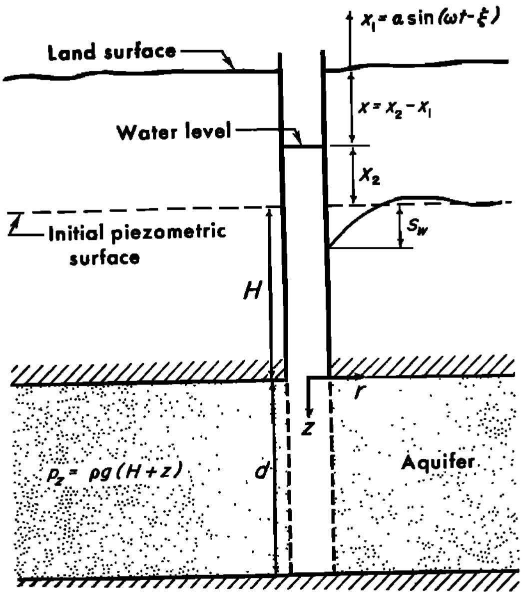

Fig. 4. Idealized representation of an open well in which water level fluctuates owing to vertical motion of well-aquifer system.

and hence we need to find a in terms of the physical constants of the system. By a procedure similar to that used in developing (4) it can be shown that the equation of motion of the water in the casing is

$$\rho(H + x)\frac{d^2x_2}{dt^2} + \rho g(H + x) = p_c \qquad (30)$$

By a procedure similar to that used in developing (13), the pressure p_c is shown to be

$$p_c = \rho g(H - s_m) - \frac{3\rho d}{8}\frac{d^2x}{dt^2} - \frac{3\rho}{4}\left(\frac{dx}{dt}\right)^2 \qquad (31)$$

Neglecting the last term as before and substituting from (31) into (30) yields

$$H_e\frac{d^2x_2}{dt^2} + g(x + s_m) = 0 \qquad (32)$$

The discharge from the well caused by the oscillation $x = x_0 \sin \omega t$ will be represented by (5), so that the drawdown will again be represented by (19). Taking $s_m = s_w$, using $x_2 = x + x_1$, and substituting from (19) into (32), we obtain the final form of the equation of motion:

$$\frac{d^2x}{dt^2} + \frac{r_w^2 g}{2TH_e}\,\mathrm{Ker}\,\alpha_w\,\frac{dx}{dt} + \frac{g}{H_e}\left(1 - \frac{r_w^2\omega}{2T}\,\mathrm{Kei}\,\alpha_w\right)x = -\frac{d^2x_1}{dt^2} \qquad (33)$$

By substituting $x = x_0 \sin \omega t$ and $x_1 = a \sin (\omega t - \xi)$ into (33), we find the amplification factor for this case to be

$$A' = \frac{4\pi^2 H_e}{\tau^2 g}A \qquad (34)$$

where A is as given by (28). The phase angle ξ is found to be

$$\xi = \arctan\,[2\beta\omega_w\omega/(\omega^2 - \omega_w^2)] \qquad (35)$$

Relation between Fluctuations of Pressure Head and Vertical Motion of Land Surface

It is apparent from the foregoing that any type of earthquake wave that produces dilatation of the aquifer or vertical vibration of the well-aquifer system can cause the water level in a well to fluctuate. Rayleigh waves cause larger fluctuations in wells than any other wave that has been identified. Since a Rayleigh wave produces both dilatation and vertical motion, a comparison of the fluctuation of pressure head with the vibration of the land surface due to a Rayleigh wave is of interest.

For a Poisson's ratio of 0.25 the particle motion due to Rayleigh waves in a semi-infinite elastic solid is described [*Jeffreys*, 1929, p. 91] by

$$u = D(e^{-0.8475\kappa z} - 0.5773e^{-0.3933\kappa z})\cdot\sin\kappa(y - ct)$$
$$w = D(0.8475e^{-0.8475\kappa z} - 1.4679e^{-0.3933\kappa z})\cdot\cos\kappa(y - ct) \qquad (36)$$

For $z = 0$, the latter of equations 36 gives the vertical motion of the land surface:

$$w|_{z=0} = -0.6204D\cos\kappa(y - ct) = -w_0\cos\kappa(y - ct) \qquad (37)$$

where

$$w_0 = 0.6204D \qquad (38)$$

The dilatation due to the Rayleigh wave is

$$\theta = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} = 1.7700 \frac{D}{\lambda} e^{-5.3250z/\lambda} \cdot \cos \kappa(ct - y) \quad (39)$$

For $z < \lambda/100$ the dilatation is closely approximated by

$$\theta|_{z<\lambda/100} = 1.7 \frac{D}{\lambda} \cos \kappa(ct - y) = \theta_0 \cos \kappa(ct - y) \quad (40)$$

where

$$\theta_0 = 1.7D/\lambda \quad (41)$$

In treatments of the properties of aquifers it is commonly assumed (see, for example, *Jacob* [1940, p. 583]) that when the aquifer is compressed, the change in volume of the solid material due to deformation of the individual particles is small in comparison with the change in volume of the water. This assumption is apparently valid for granular aquifers but, depending on the pore geometry and Poisson's ratio, may not be valid for such aquifers as limestone and basalt. To the extent that the above assumption holds for a given aquifer, nearly all the volume change of a dilated aquifer goes into a volume change of the water in its pores. Hence, the dilatation of the water is nearly equal to that of the aquifer divided by the porosity. The amplitude of the fluctuation of the pressure head due to the dilatation is, therefore,

$$h_0 = p_0/\rho g = (\theta_0 E_w/n)/\rho g \quad (42)$$

Substituting from (41) into (42), we obtain

$$h_0 = 1.7DE_w/\rho g n\lambda \quad (43)$$

The ratio of the change in pressure head to the vertical motion of the well-aquifer system is, from (38) and (43),

$$R = h_0/w_0 = 2.7E_w/\rho g n\lambda \quad (44)$$

Water at 18°C has a bulk modulus of elasticity E_w of about 22×10^8 newtons/m^2 and a specific weight ρg of about 98×10^2 newtons/m^3. The porosity n of aquifers commonly ranges from 0.03 to 0.30. Assuming that the length λ of the Rayleigh wave ranges from 15,000 to 150,000 meters, we find that the range of R would be

$$\frac{2.7 \times 22 \times 10^8}{98 \times 10^2 \times 0.30 \times 150{,}000} \leq R \leq \frac{2.7 \times 22 \times 10^8}{98 \times 10^2 \times 0.03 \times 15{,}000}$$

or

$$13 \leq R \leq 1300 \quad (45)$$

Since the maximum amplifications for pressure-head fluctuations and vertical motions of the well-aquifer system (given by equations 28 and 34) are very nearly equal, we can conclude from (45) that fluctuations due to Rayleigh waves having periods near or greater than the 'resonant frequency' of the well are due almost entirely to fluctuations of artesian head. In other words, a well acts essentially as a dilatation meter.

Magnification of Vertical Land-Surface Motion

Using (28) and (44) makes it possible to define a magnification for a well which, for Rayleigh waves, is comparable to the magnification of a vertical-component seismograph. Thus we define M to be the ratio of the amplitude x_0 of the water-level oscillation to the amplitude w_0 of the land-surface oscillation, and from (28) and (44)

$$M = x_0/w_0 = (x_0/h_0)(h_0/w_0) = AR \quad (46)$$

The amplification curves (Figure 5) and equation 45 indicate that in extreme cases A and R can be such that the magnification M at 'resonant frequency' would be as much as several hundred thousand.

Water-Level Oscillation in Terms of Rayleigh-Wave Parameters

In the analysis we supposed that a pressure oscillation in the aquifer was described by (1) and found expressions for p_0 and η such that the water-level oscillation $x = x_0 \sin \omega t$ would result. Using the relations that have already been developed, we can now find an expression for the water-level fluctuation in terms of the parameters of a Rayleigh wave described by (36). The vertical component of the land-surface motion, as would be recorded by a seismograph, is given by (37). A dilatation described by (40) would occur, producing a head fluctuation of the same phase. Hence a head fluctua-

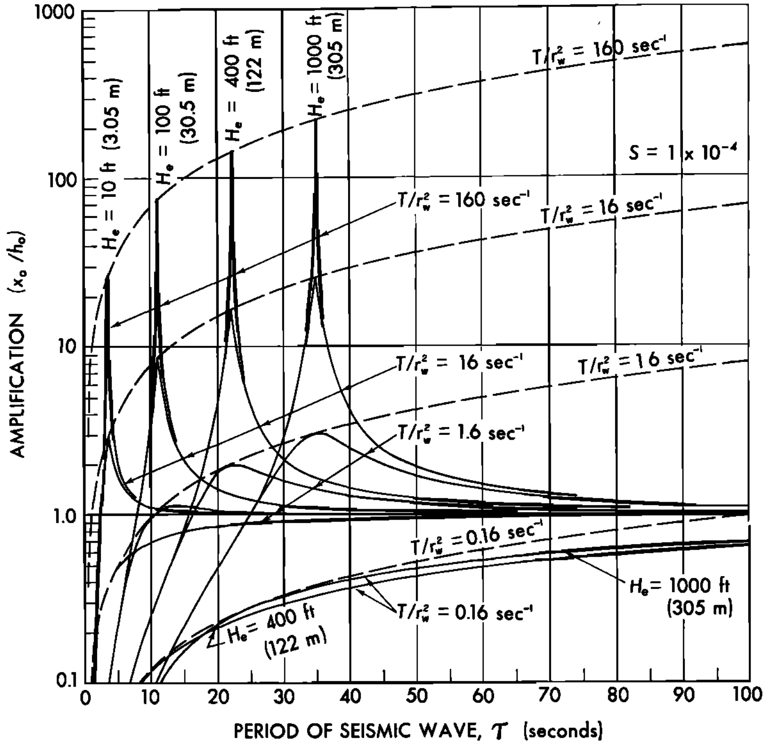

Fig. 5. Amplification of pressure-head fluctuation.

tion described by

$$h = h_0 \cos \kappa(y - ct) = h_0 \cos(\omega t - \kappa y) \quad (47)$$

would be produced. It is apparent from (1) and (2) that this would produce a water-level oscillation described by

$$x = x_0 \cos(\omega t + \eta - \kappa y) \quad (48)$$

From (46), $x_0 = ARw_0$, so that in terms of the parameters of the Rayleigh wave the oscillation would be

$$x = ARw_0 \cos(\omega t + \eta - \kappa y)$$

$$= 0.6204ARD \cos(\omega t + \eta - \kappa y) \quad (49)$$

where η, A, and R are given by (27), (28), and (44), respectively.

Computations and Discussion

The amplification curve for a given open well can be established empirically if the motion of the water level, the fluctuations of pressure in the aquifer, and the period of the oscillation caused by seismic waves are observed. A shut-in well near the open well would provide a direct means of observing the fluctuation of pressure in the aquifer.

Pressure-Head Amplification Curves

Figure 5 shows curves computed from (28) for the amplification of pressure-head fluctuations that would be registered in the response of water levels in open wells. The figure includes four families of curves corresponding to column heights of 3.05, 30.5, 122, and 305 meters. Each family (except for the 3.05-meter column height) includes curves corresponding to values of T/r_w^2 of 0.16, 1.6, 16, and 160 sec^{-1}. The dashed curves are envelopes representing, for the indicated values of T/r_w^2, the maximum amplification that can occur in a well having an optimum column height for a given frequency.

A storage coefficient of 0.0001 was assumed for all curves. According to *Jacob* [1940, p. 576] the storage coefficient of an elastic artesian aquifer can be represented by

$$S = \rho g d\left(\frac{n}{E_w} + \frac{b}{E_s}\right)$$

In an aquifer composed of uncemented granular material the value of b is unity; in a solid aquifer, such as a limestone having tubular channels, b is apparently equal to the porosity. For a typical sandstone having a bulk modulus of about 2×10^{10} newtons/m^2 [*Birch*, 1942, p. 61], a thickness of 60 meters, and a porosity of 0.30, Jacob's equation shows the storage coefficient to be about 0.0001.

The curves show that the amplification decreases rapidly as frequency increases at frequencies greater than the 'resonant frequency' of the well. For large transmissibilities the maximum amplification occurs at a frequency which depends almost entirely upon the height of the column of water in the well—that is, at $\omega = (g/H_e)^{1/2}$ (see equation 25).

Ground-Motion Amplification Curves

The amplification curves for vertical motion of the well-aquifer system (Figure 6), computed from (34), show that the amplification decreases rapidly for frequencies less than the resonant frequency. For large transmissibilities the maximum amplification occurs at very nearly the same frequency as the maximum pressure-head amplification.

Total Amplification

The total amplification is the sum of the pressure-head amplification plus the ground-surface amplification; however, for Rayleigh waves the fluctuation of pressure head is 13 to 1300 times that of the land-surface motion, as indicated in (45). Hence, for Rayleigh waves, the pressure-head amplification curves should closely approximate the total amplification.

Assumptions

Several of the simplifying assumptions deserve comment:

1. *Assumption that the aquifer is homogeneous and isotropic and that the well is screened throughout the thickness of the aquifer.* The purpose of this is to ensure two-dimensional radial flow. With complete screening it is necessary that the aquifer be isotropic only in

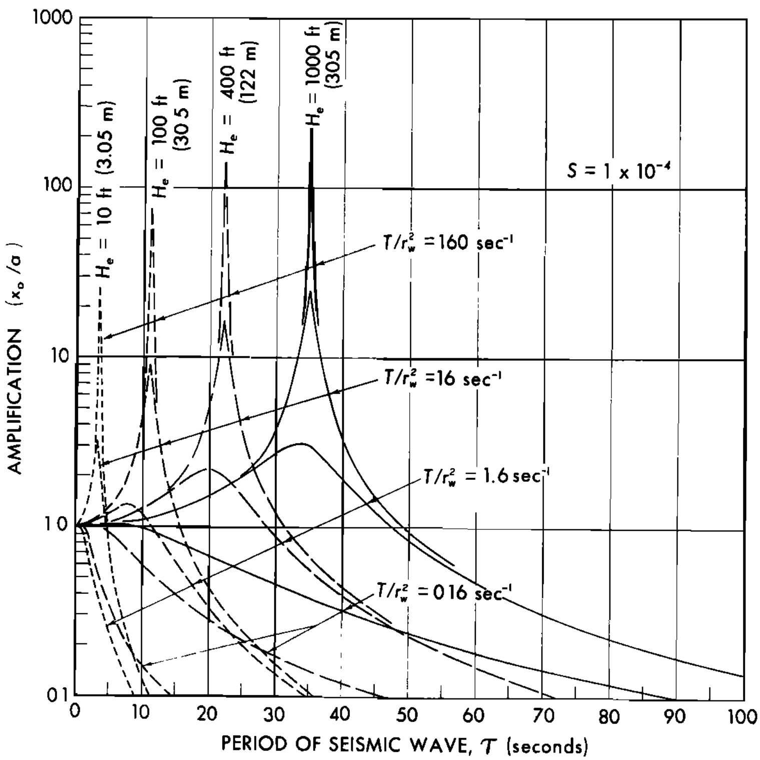

Fig. 6. Amplification of vertical motion of well-aquifer system.

the horizontal plane. Few wells are screened throughout an aquifer. However, even with partial screening the induced flow within the few hundred meters in which the water-level oscillations produce significant drawdown is likely to be essentially two dimensional, because in most aquifers the cross-bed permeabilities are only a small fraction of that along the beds. For most aquifers, therefore, the purpose of the assumption may be satisfied even though the well is partially screened.

2. *Neglect of friction loss in the well.* Head loss due to friction in the well casing is given by the Darcy-Weisbach formula:

$$h_L = fHV_s^2/D2g$$

To include the pipe friction in the analysis would require introducing into (20) the term

$$\pm\frac{\rho g h_L}{\rho H} = \pm\frac{f}{2r_w} V_s^2 \qquad (50)$$

which results in a nonlinear equation. The pipe friction can be neglected if the integral of the pipe friction over one-quarter cycle of the oscillation is less than about one-tenth that of the second term in (20)—that is, if

$$\int_0^{\tau/2} 10\frac{f}{2r_w} V_s^2\, dt < \int_0^{\tau/2} \frac{r_w^2 g}{2TH_e} \operatorname{Ker}\alpha_w \, V_s dt \qquad (51)$$

where

$$v = dx/dt = \omega x_0 \cos\omega t$$

Integrating and simplifying reduces (51) to

$$\omega x_0 < \frac{\pi r_w^3 g}{10fTH_e} \operatorname{Ker}\alpha_w \qquad (52)$$

which is the criterion for neglecting pipe-friction losses.

3. *Neglect of inertia of water in the aquifers.* In the absence of a theory in which the effect of inertia in unsteady groundwater motion is included, the magnitude of the inertia effect cannot be computed. However, the inertial force would be larger than it will be in reality if at any instant all the water within the radius of influence r_e were moving in the same direction, and it can be shown by an analysis which will be omitted that under this condition the component of drawdown at the well due to the inertia would be approximately

$$s_w' = \frac{r_w^2}{2gdn} \ln\frac{r_e}{r_w}\frac{d^2x}{dt^2}$$

When this component is introduced into the equation of motion, we find that the effect is the same as increasing H_e by gs_w'. We can conclude, therefore, that the inertial effect will be negligible whenever the quantity $(r_w^2/2dn)\cdot\ln(r_e/r_w)$ is small in comparison with H_e.

Since friction losses and velocity head in the well can be significant for large values of the products $\omega^2 x_0$ and ωx_0, the amplification curves can be considered valid only for small amplitudes or large periods or both.

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