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The effective bandwidth of a multitaper spectral estimator

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SUMMARY

The bandwidth of a spectral estimator is a measure of the minimum separation in frequency between approximately uncorrelated spectral estimates. We determine an effective bandwidth measure for multitaper spectral estimators, a relatively new and very powerful class of spectral estimators proving to be very valuable whenever the spectrum of interest is detailed and/or varies rapidly with a large dynamic range. The multitaper spectral estimator is the average of several direct spectral estimators, each of which uses one of a set of orthogonal tapers. We show that the equivalent width of the autocorrelation of the overall spectral window is a suitable measure of the effective bandwidth of a multitaper spectral estimator and illustrate its use in the case of both Slepian and sinusoidal orthogonal tapers. This measure allows a unified treatment of bandwidth for the class of quadratic spectral estimators. Hence, for example, it is now possible properly to compare multitaper spectral estimators with traditional lag window spectral estimators, by assigning a fixed and equal effective bandwidth to both methods. An application is given to the spectral analysis of ocean wave data.

Some key words: Bandwidth; Lag window; Multitapering; Quadratic estimator; Sinusoidal taper; Slepian taper; Spectral estimation; Tapering.

1. INTRODUCTION

The bandwidth of a spectral estimator is a measure of the minimum separation in frequency between approximately uncorrelated spectral estimates. Priestley (1965) discusses the role of bandwidth in spectral analysis. In order to compare one spectrum analysis method with another, to see for example which best picks up certain details of the true spectrum, the same effective bandwidth should be used in each case. The main purpose of this paper is to determine a practical and useful effective bandwidth measure for multitaper spectral estimators, a relatively new and very powerful class of spectral estimators used for analysing many physical time series, including terrestrial free oscillations, the relationship between carbon dioxide and global temperature, oxygen isotope ratios, and the rotation of the earth. In fact, the proposed bandwidth measure is suitable for other quadratic spectral estimators, including lag window estimators.

In § 2 we introduce quadratic spectral estimators, which motivates multitaper spectral estimation. We discuss the two types of orthogonal tapers in use, namely the Slepian and sinusoidal tapers. The effective bandwidth of the estimator can be regarded as determining the cut-off point of the correlation of the spectral estimator, so in § 3 we look at the correlation of direct spectral estimators, those using a single taper but no smoothing over frequencies. We show that the equivalent width of the autocorrelation of the spectral window is a better measure of the cut-off point than the equivalent width of the approximate correlation. In § 4 we show that, when applied to the overall spectral window for multitapering, this is very satisfactory as a measure of the effective bandwidth of a multitaper spectral estimator for both Slepian and sinusoidal tapers. In particular, for the Slepian tapers, as the number of contributing tapers increases, the bandwidth measure increases monotonically, and, when the maximum number of suitable tapers is used, the above measure is very close to the design bandwidth, as required. It can also be used for measuring the effective bandwidth of a lag window spectral estimator, another quadratic spectrum estimator, as shown in § 5. Hence, it becomes possible to compare directly multitaper spectral estimators with lag window spectral estimators by using the same effective bandwidth for both methods. An illustration of such use is given in § 6.

2. QUADRATIC ESTIMATORS AND MULTITAPERING

2·1. *General*

Suppose that $X_1, \ldots, X_N$ is a segment of length N of a real-valued stationary process with zero mean and spectral density function $S(.)$. For a fixed frequency f such that $0 \leqslant f \leqslant \frac{1}{2}$, let $Z_t = X_t \exp(-i2\pi ft)$. A quadratic spectral estimator takes the form

$$\hat{S}(f) = \sum_{s=1}^{N} \sum_{t=1}^{N} Z_s^* Q_{s,t} Z_t,$$

where the asterisk denotes complex conjugation and $Q_{s,t}$ is the (s, t) element of the $N \times N$ matrix Q of weights. To ensure that the spectral estimate is nonnegative, we require that Q is positive semidefinite.

As demonstrated by, for example, Percival & Walden (1993, § 7.3), any quadratic estimator with a real-valued, symmetric, positive semidefinite matrix Q of weights can be written as an average of K direct spectral estimators, K being the rank of Q:

$$\hat{S}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \hat{S}_k(f), \tag{1}$$

with

$$\hat{S}_k(f) \equiv \left| \sum_{t=1}^{N} h_{t,k} X_t e^{-i2\pi ft} \right|^2,$$

where $\{h_{t,k}\}$ is the data taper for the kth direct spectral estimator $\hat{S}_k(.)$. Moreover, the tapers are orthogonal so that $\sum_{t=1}^{N} h_{t,j} h_{t,k} = 0$ for all $j \neq k$. With the normalisation $\sum_{t=1}^{N} h_{t,k}^2 = 1$, the estimator is unbiased in the case of white noise. The tapers are hence orthonormal. An estimator of the form (1) was first proposed by Thomson (1982) and called a multitaper spectral estimator, denoted by $\hat{S}^{(mt)}(.)$, with associated direct spectral estimators denoted by $\hat{S}_k^{(mt)}(.)$.

An associated spectral window $\mathscr{H}_k(f) = |\sum_{t=1}^{N} h_{t,k} \exp(-i2\pi ft)|^2$ is defined for each data taper. The first moment of $\hat{S}_k^{(mt)}(.)$ is given by

$$E\{\hat{S}_k^{(mt)}(f)\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}_k(f-f')S(f')\,df',$$

so that

$$E\{\hat{S}^{(mt)}(f)\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\mathscr{H}}(f-f')S(f')\,df',$$

with

$$\bar{\mathscr{H}}(f) = \frac{1}{K}\sum_{k=0}^{K-1} \mathscr{H}_k(f). \tag{2}$$

The function $\bar{\mathscr{H}}(.)$ is the spectral window for the estimator $\hat{S}^{(mt)}(.)$. For the periodogram the equivalent spectral window is Fejér's kernel which has very high sidelobes giving rise to severe sidelobe leakage. The design of $\bar{\mathscr{H}}(.)$ can be used to control sidelobe leakage. The averaging of the K different eigenspectra produces an estimator of $S(f)$ with smaller variance than that of any individual $\hat{S}_k^{(mt)}(f)$. Approximately (Walden, McCoy & Percival, 1994) the variance of $\hat{S}^{(mt)}(f)$ is smaller than that of $\hat{S}_k^{(mt)}(f)$ by a factor of $1/K$, so that var $\{\hat{S}^{(mt)}(f)\} \propto S^2(f)/K$ as $N \to \infty$ where K increases with N, so that the multitaper estimator is consistent. This is in contrast to the periodogram, or to a single direct spectral estimator, both of which are inconsistent. It is also to be appreciated that while multitaper spectrum estimates have about $2K$ degrees of freedom, 2 for each of the K spectra averaged, they have the same frequency resolution as single-tapered spectral estimates with 2 degrees of freedom before the latter are smoothed across frequencies. An extensive discussion of multitapering with references is given by Percival & Walden (1993, Ch. 7).

The two main approaches to designing the set of orthogonal tapers currently in use result in the Slepian tapers and the sinusoidal tapers, discussed next. Note though that as long as a lag window estimator is positive semidefinite, for example a Bartlett or Parzen estimator, then it too can be expressed as a multitaper spectral estimator, as can also estimators formed via segment averaging (Percival & Walden, 1993).

2·2. *Slepian tapers*

This design concentrates on sidelobe suppression. The first taper sequence, $\{h_{t,0}, t=1,\ldots,N\}$, is chosen such that its corresponding spectral window $\mathscr{H}_0(f)$ maximises the concentration ratio over the chosen interval $[-W, W]$ with design bandwidth $2W$:

$$\int_{-W}^{W} \mathscr{H}_0\,df \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}_0\,df = \lambda_0(N, W).$$

The second taper sequence, $\{h_{t,1}, t=1,\ldots,N\}$, is chosen to maximise this concentration ratio defined subject to being orthogonal to the first, yielding $\lambda_1(N, W)$. The third taper sequence similarly maximises the concentration ratio, but subject to being orthogonal to the first two, etc. Maximisation of the concentration ratio ensures that the sidelobes are minimised in this sense. The kth maximised value λ_k must be close to unity for $\{h_{t,k}\}$ to be a decent taper, but this holds only for the tapers of order $k = 0,\ldots,2NW-2$, and hence at most this number $K = 2NW-1$ of tapers should be used. These tapers do not have simple analytic forms: an algorithm for the numerical calculation of the Slepian tapers is given by Bell, Percival & Walden (1993), along with instructions for obtaining code.

2·3. *Sine tapers*

Another important aspect of multitapering is its variance reduction ability, without the need to lose resolution by smoothing across frequencies. Hence, if the spectrum is not changing too rapidly, so that sidelobe bias is not severe, alternative and computationally simpler designs are possible which will still lead to variance reduction compared to the periodogram. Riedel & Sidorenko (1995) suggested a much simpler set of orthogonal tapers, known as sinusoidal tapers, the kth of which is given by

$$h_{t,k} = \left(\frac{2}{N+1}\right)^{\frac{1}{2}} \sin\left\{\frac{(k+1)\pi t}{N+1}\right\} \quad (t = 1, \ldots, N).$$

The first term is a standardisation factor to ensure that these tapers are orthonormal. These tapers achieve a smaller 'local' bias, namely the bias due to the smoothing by the main lobe of $\bar{\mathscr{H}}(f)$, than the Slepian tapers, at the expense of sidelobe suppression. Riedel & Sidorenko (1995) showed that the kth sinusoidal taper has its spectral energy concentrated in the frequency bands

$$\frac{k}{2(N+1)} \leqslant |f| \leqslant \frac{k+2}{2(N+1)} \quad (k = 0, \ldots, N-1). \tag{3}$$

Note that, while there is no auxiliary design parameter W in this scheme, use of K sine tapers yields a multitaper estimator with spectral window concentrated in the interval $[-W', W']$, where $W' = (K+1)/\{2(N+1)\}$; see § 4·3. Hence a design bandwidth can be implicitly achieved by setting K appropriately.

2·4. *Taper examples*

By way of example we look at the tapers and spectral windows for $N = 32$. The Slepian tapers $\{h_{t,k}\}$, for $NW = 4$ and orders $k = 0$, 1 and 6 are given in Fig. 1 along with their corresponding spectral windows $\mathscr{H}_k(.)$. Also given is the overall, or mean, spectral window $\bar{\mathscr{H}}(.)$ using the single taper $k = 0$ only, then using the two tapers $k = 0, 1$, and finally, in Fig. 1(i), using the $2NW - 1 = 7$ tapers $k = 0, \ldots, 6$, the mean spectral window $\bar{\mathscr{H}}(f) \equiv \frac{1}{7}\sum_{k=0}^{6} \mathscr{H}_k(f)$. As the number of spectral windows contributing to $\bar{\mathscr{H}}(\cdot)$ increases, it becomes more and more rectangular, so that $2W$ becomes a more accurate measure of its bandwidth. The position of W is marked by the vertical line. However, for a small number of contributing tapers, $2W$ is not a good measure of the effective bandwidth of $\bar{\mathscr{H}}(.)$. The equivalent results for the sinusoidal tapers are given in Fig. 2; the sidelobes are much higher, but the average spectral window again becomes rectangular as more tapers are averaged in.

3. BANDWIDTH AND CORRELATION

3·1. *General*

There are two main approaches to estimating the bandwidth of a spectral estimator. We look at both, firstly for the case of a direct spectral estimator $\hat{S}^{(d)}(f)$ using a single taper $\{h_t\}$, for which

$$E\{\hat{S}^{(d)}(f)\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}(f-f')S(f')\,df'.$$

3·2. *Equivalent width of the approximate autocorrelation*

The first measure is the 'equivalent width' of the approximate autocorrelation $R(.)$ between $\hat{S}^{(d)}(f)$ and $\hat{S}^{(d)}(f+\eta)$. Equivalent width is often used to measure the width of 'signals' that are real-valued, positive everywhere, and peaked about 0. This measure takes the form (Percival & Walden, 1993, p. 252)

$$\text{width}_e\,\{R(.)\} = \sum_{t=1}^{N} h_t^4. \tag{4}$$

From (4) it can be argued that the variance inflation factor following tapering is $N\sum h_t^4$, reproducing an important approximation obtained by Hannan (1970, p. 271) using different reasoning; see also Brillinger (1975, p. 150). However, the limitations of (4) for our purposes will soon become apparent.

3·3. *Autocorrelation width of the spectral window*

Another way to define the bandwidth of a direct spectral estimator is as the equivalent width of the deterministic autocorrelation of the spectral window $\mathscr{H}(.)$. Such a measure is known simply as the autocorrelation width (Bracewell, 1978, p. 154), denoted here by width$_a$. This takes the form

$$\begin{aligned}\text{width}_a\,\{\mathscr{H}(.)\} &= \left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}(f)\,df\right\}^2 \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}^2(f)\,df \\ &= \left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}^2(f)\,df\right\}^{-1}.\end{aligned}$$

By Parseval's theorem

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{H}^2(f) = \sum_{\tau=-(N-1)}^{(N-1)} c_\tau^2,$$

where $\{c_\tau\} \leftrightarrow \mathscr{H}(.)$; that is, they form a Fourier transform pair. But

$$\mathscr{H}(f) = \left|\sum_{t=1}^{N} h_t e^{-i2\pi f t}\right|^2 = \sum_{\tau=-(N-1)}^{(N-1)} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right) e^{-i2\pi f t},$$

and hence $c_\tau = \sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}$. Thus

$$\text{width}_a\,\{\mathscr{H}(.)\} = \left(\sum_{\tau=-(N-1)}^{(N-1)} c_\tau^2\right)^{-1} = \left\{\sum_{\tau=-(N-1)}^{(N-1)} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right)^2\right\}^{-1}. \tag{5}$$

Just as for the smoothing window bandwidth, width$_a$ $\{\mathscr{H}(.)\}$ is a measure of the cut-off point of the correlation of $\hat{S}^{(d)}(f)$ and $\hat{S}^{(d)}(f+\eta)$. Because the outer summation in the last part of (5) has a $\tau = 0$ term that is unity and all other terms are nonnegative, the equivalent width of the autocorrelation of $\mathscr{H}(.)$ is well defined and can easily be computed in a numerically stable way.

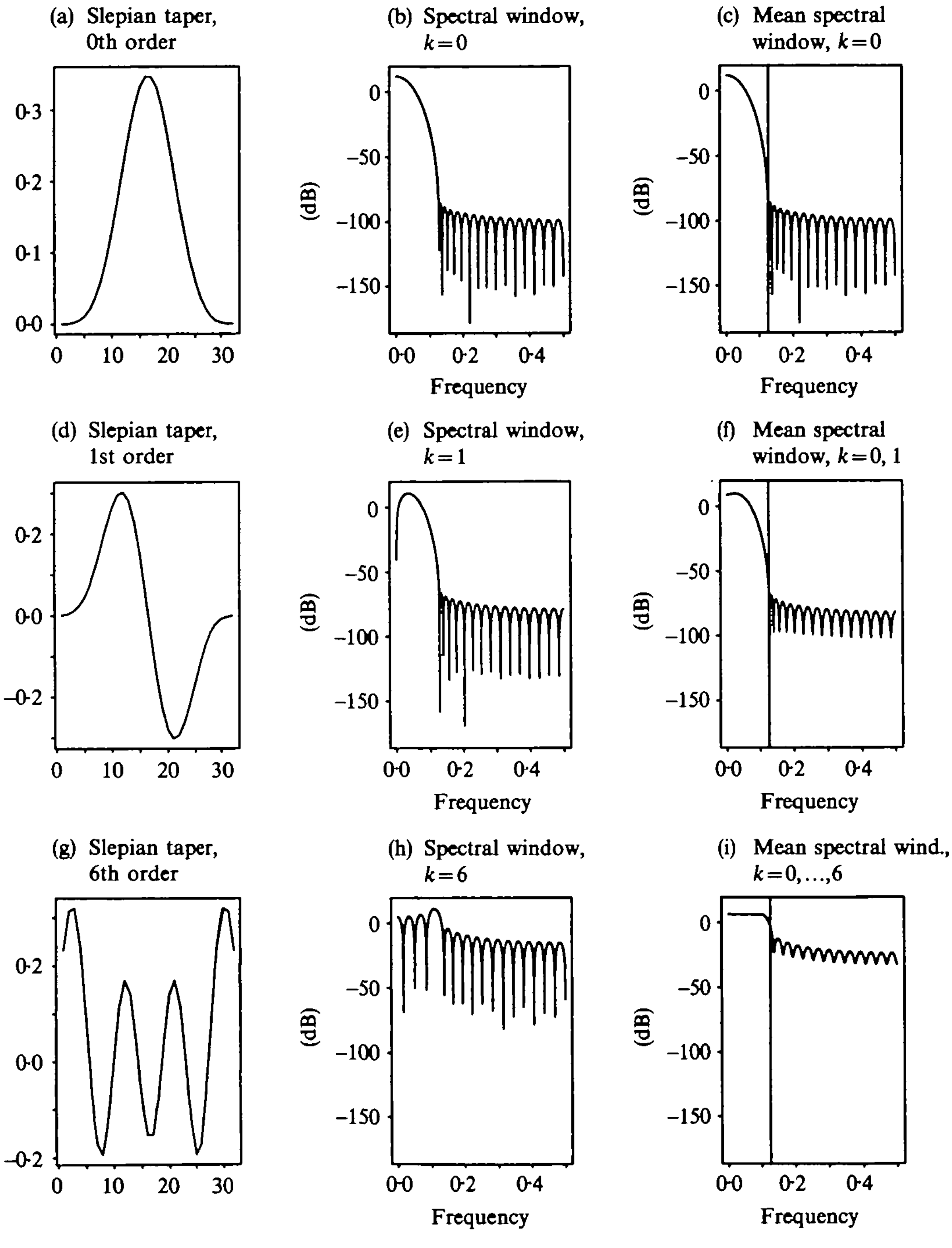

Fig. 1. Slepian tapers and spectral windows for $N=32$, $NW=4$ and order k. (c), (f) and (i) show the mean spectral windows for all orders up to the given order; the vertical line shows the size of W. The mean spectral window becomes squarer as more tapers are averaged in. All the spectral windows are symmetric about zero frequency but the plots show only the positive frequency portion.

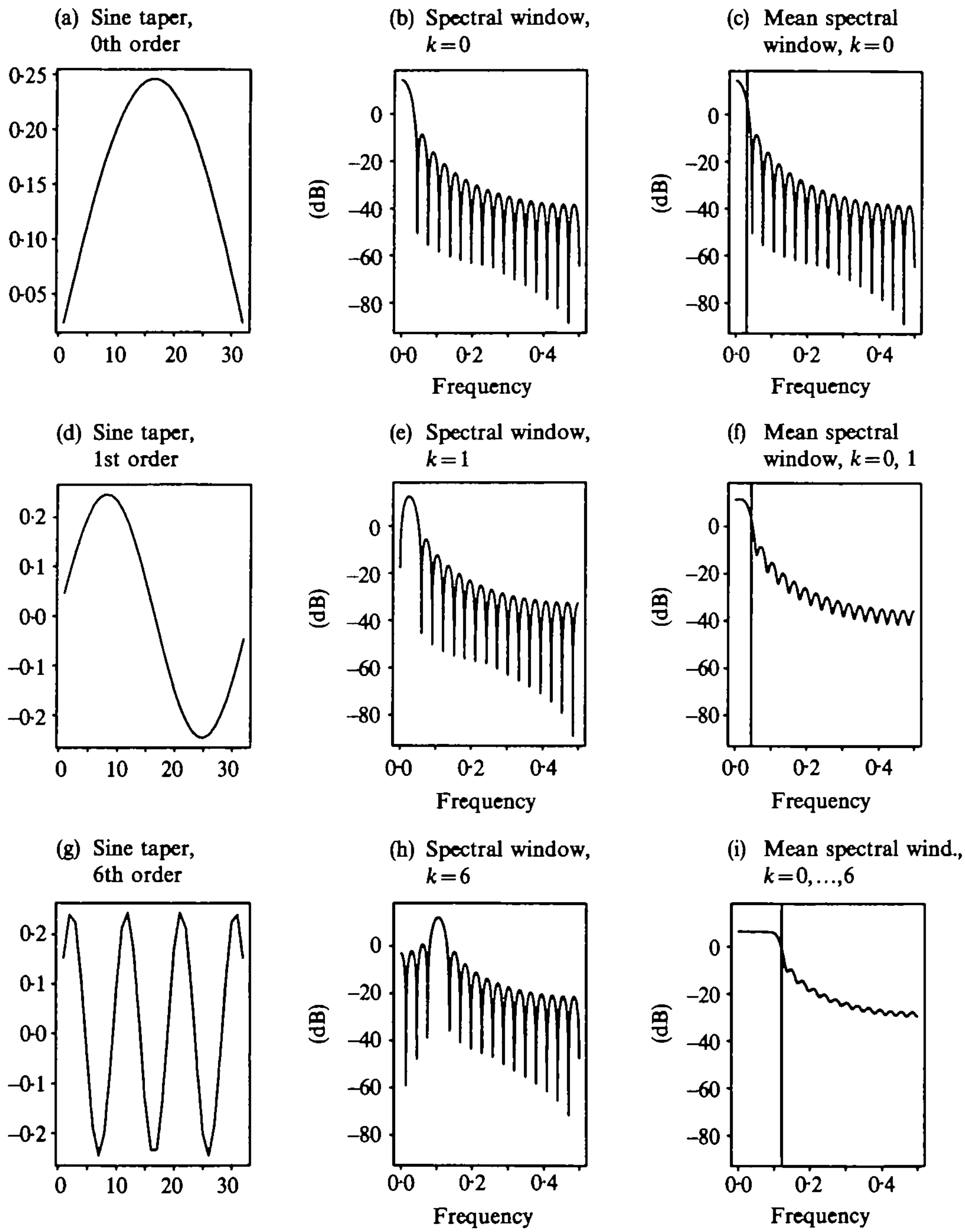

Fig. 2. Sinusoidal tapers and spectral windows for $N=32$, $NW=4$ and order k. (c), (f) and (i) show the mean spectral windows for all orders up to the given order; the vertical line in the plots of mean spectral window indicate $(K+1)/\{2(N+1)\}$; see § 4·3. The mean spectral window becomes squarer as more tapers are averaged in. All the spectral windows are symmetric about zero frequency but the plots show only the positive frequency portion.

3·4. *Comparison of width measures*

Both width$_e$ $\{R(.)\}$ and width$_a$ $\{\mathscr{H}(.)\}$ are compared in Fig. 3, for the Slepian multitaper scheme for $NW = 2, 3, 4$ with $N = 32$. Basically, width$_e$ $\{R(.)\}$ shrinks with increasing taper order, because the measure is dominated by the main lobe of $R(.)$ and fails to 'see' the sidelobes, while width$_a$ $\{\mathscr{H}(.)\}$ increases with increasing taper order because the measure is calculated from the autocorrelation of $\mathscr{H}(.)$, which has much more prominent sidelobes. Note that the last one or two of the $2NW - 1$ tapers used may not clearly satisfy the low sidelobe condition for inclusion in the Slepian multitapering scheme, leading to erratic behaviour; see Fig. 2 and $k = 6$ with $NW = 4$ in particular.

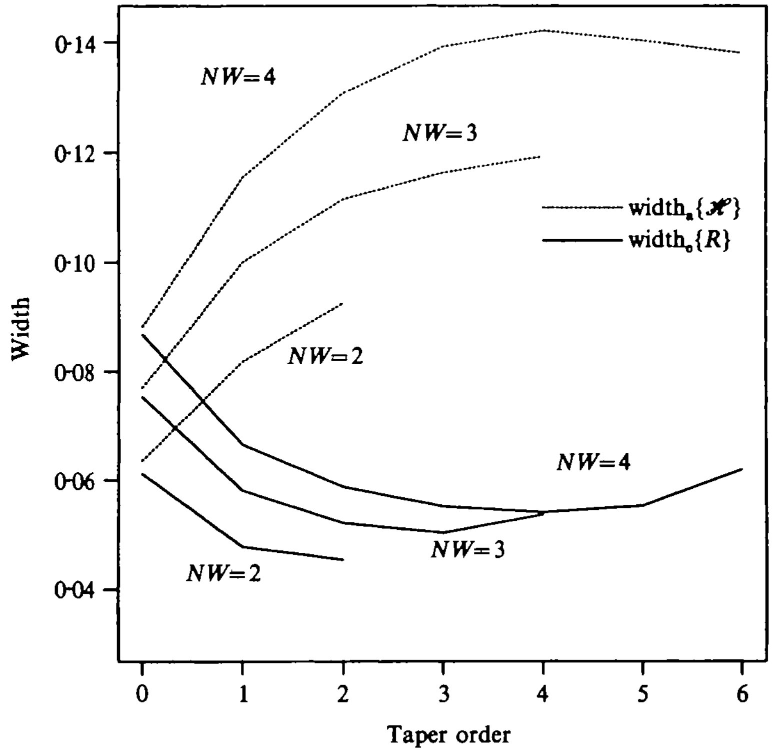

Fig. 3. Comparison of width$_e$ $\{R(.)\}$, solid lines, and width$_a$ $\{\mathscr{H}(.)\}$, dotted lines, with taper order; $N = 32$.

For the sinusoidal tapers width$_e$ $\{R(.)\}$ remains constant with increasing taper order, while width$_a$ $\{\mathscr{H}(.)\}$ again increases with increasing taper order.

3·5. *Cut-off of exact spectral correlation*

To illustrate our above comments, we computed the form of the exact correlation for direct spectral estimators for white noise at frequency $f = f_0 = 0{\cdot}25$. Note that in practice we would not need to taper if the series was white noise!

For white noise the exact correlation of a direct spectral estimator can be calculated (Walden et al., 1994). This is plotted as the solid curve in Fig. 4 for the Slepian tapers $\{h_{t,k}\}$ for $NW = 4$, $N = 32$ and taper orders $k = 1$, 2 and 3. The number of lobes in the exact correlation grows linearly with the taper order, and hence the cut-off point of the correlation increases with increasing taper order. The two estimates of the cut-off point of the exact correlation, solid curve, have been marked. The vertical solid line marks the

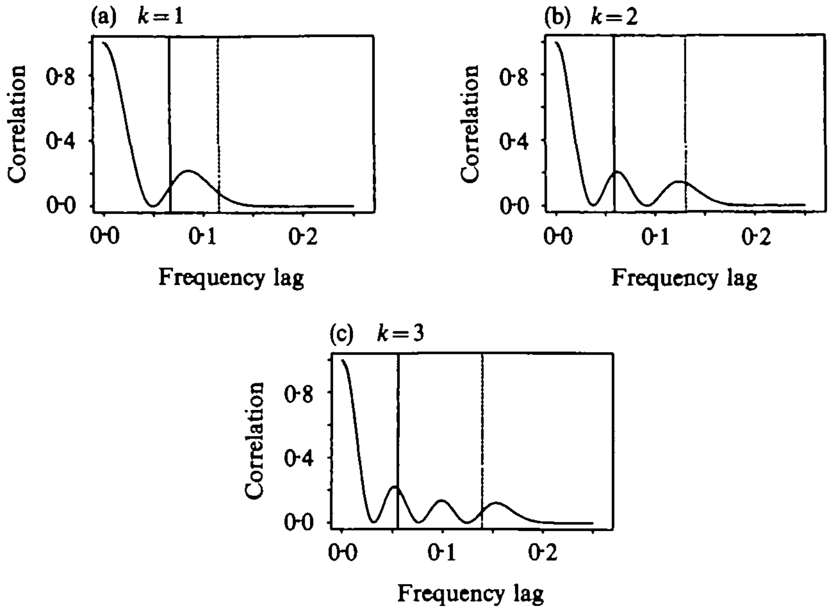

Fig. 4. Exact correlation, solid curve, of direct spectral estimators for white noise using Slepian tapers ($N=32$, $NW=4$) and corresponding width or cut-off measures: $\text{width}_e\{R(.)\}$, solid vertical lines, and $\text{width}_a\{\mathscr{H}(.)\}$, dotted vertical lines, for taper orders, k, of 1, 2 and 3.

value $\text{width}_e\{R(.)\}$ given in (4). The vertical dotted line marks $\text{width}_a\{\mathscr{H}(.)\}$ given in (5). The measure (5) clearly gives a more sensible measure of the cut-off. Similar results are obtained for the sinusoidal tapers.

4. Effective bandwidth of multitaper estimators

4·1. *General*

The bandwidth measure (5) for a direct spectral estimator using a single taper extends readily to the multitaper estimator (1). The associated spectral window is given in (2). The equivalent width of its autocorrelation is given by

$$\text{width}_a\{\bar{\mathscr{H}}(.)\} = \left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\mathscr{H}}^2(f)\,df\right\}^{-1}.$$

By Parseval's theorem

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\mathscr{H}}^2(f) = \sum_{\tau=-(N-1)}^{(N-1)} \bar{c}_\tau^2,$$

where $\{\bar{c}_\tau\} \leftrightarrow \bar{\mathscr{H}}(.)$. It readily follows that

$$\bar{c}_\tau = \frac{1}{K}\sum_{k=0}^{K-1}\sum_{t=1}^{N-|\tau|} h_{t,k}h_{t+|\tau|,k}.$$

Thus

$$\text{width}_a\{\bar{\mathscr{H}}(.)\} = \left\{\sum_{\tau=-(N-1)}^{(N-1)}\left(\frac{1}{K}\sum_{k=0}^{K-1}\sum_{t=1}^{N-|\tau|} h_{t,k}h_{t+|\tau|,k}\right)^2\right\}^{-1}. \tag{6}$$

Hence for multitapering expression (6) can be used to measure the effective bandwidth of the associated spectral window $\bar{\mathscr{H}}(.)$.

4·2. *Bandwidth for Slepian tapers*

We saw in Fig. 1 how, as the number of spectral windows, or tapers, contributing to $\bar{\mathscr{H}}(.)$ increases, it changes from bell-shaped, for a single taper, to approximately rectangular, when the maximum number of suitable tapers, $K = 2NW - 1$, is used. In the latter case we would expect the effective bandwidth to be only a little less than the design bandwidth of $2W$. Hence we would expect the measure in (6) to (i) reflect a monotonic

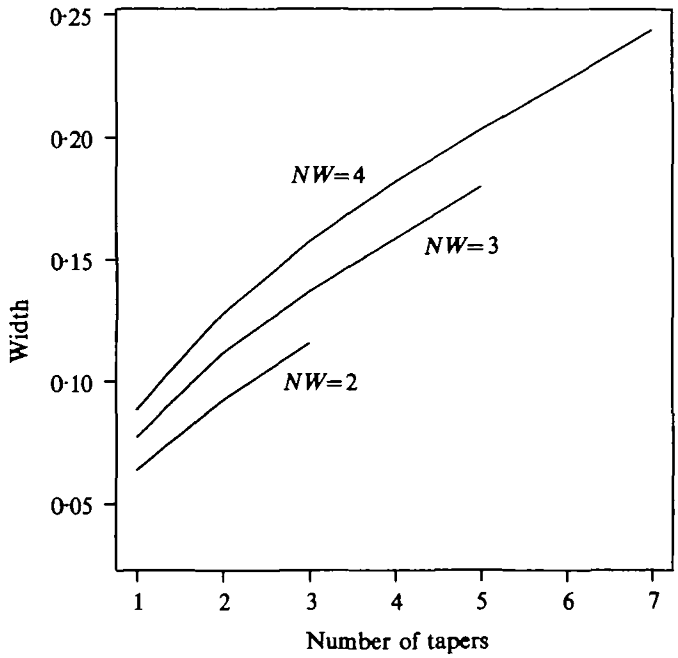

Fig. 5. Plot of $\text{width}_a\{\bar{\mathscr{H}}(.)\}$ for $K = 1, \ldots, 2NW - 1$ Slepian tapers, for $N = 32$ and $NW = 2$ (lower), 3 (middle), and 4 (upper). The design bandwidths $2W$ are 0·125, 0·1875 and 0·25, respectively. These are almost attained by the ends of the curves, corresponding to the maximum number of suitable tapers.

increase in bandwidth as the number of contributing tapers increases, and (ii) be only slightly less than $2W$ when all $2NW - 1$ tapers are used. Figure 5 shows this to be the case. For $N = 32$ and each of $NW = 2, 3, 4$ we plot $\text{width}_a\{\bar{\mathscr{H}}(.)\}$ for $K = 1, \ldots, 2NW - 1$. The design bandwidth $2W$ is 0·125, 0·1875 and 0·25, and $2NW - 1$ is 3, 5 and 7 for $NW =$ 2, 3 and 4, respectively. For example, for $NW = 3$ equation (6) gives an effective bandwidth of 0·18 when 5 tapers are used, compared to the nominal or design bandwidth $2W$ of 0·1875. An important point to emerge from Fig. 5 is the steepness of the slopes of the curves; i.e. if using only a small number of the maximum possible of $2NW - 1$ tapers, the effective bandwidth is only a fraction of the design bandwidth. This is a useful result for enabling fair comparisons to be made between Slepian multitapering and other spectral analysis methods, such as lag windowing; see § 6.

4·3. *Bandwidth for sinusoidal tapers*

For the sinusoidal tapers $\text{width}_a\,\{\bar{\mathscr{H}}(.)\}$ is plotted as a function of the number of tapers in Fig. 6. It grows linearly with the number of tapers. The availability of analytical

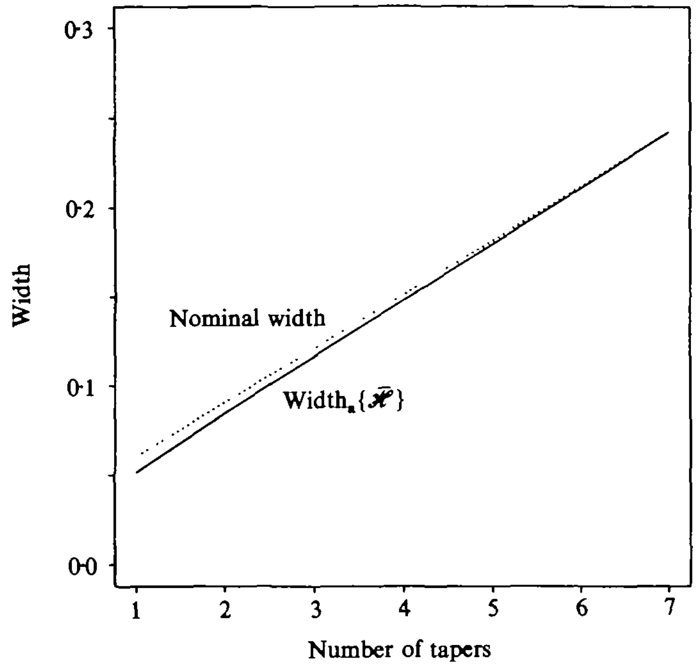

Fig. 6. Plot of $\text{width}_a\,\{\bar{\mathscr{H}}(.)\}$, solid line, for the sinusoidal tapers for $K = 1, \ldots, 8$ tapers, compared to the nominal width $(K+1)/(N+1)$, dotted.

expressions for the sinusoidal tapers allowed the spectral concentration for each taper to be defined as in (3); this is equivalent to saying that $\{\bar{\mathscr{H}}(.)\}$ should have a width of approximately $(K+1)/(N+1)$ when K sinusoidal tapers are used. This is compared to width_a in Fig. 6, where it is seen that the two expressions are very close, being virtually indistinguishable for several tapers.

5. Effective bandwidth of lag window estimators

Another member of the class of quadratic spectral estimators is the simple lag window spectral estimator

$$\hat{S}^{(lw)}(f) = \sum_{\tau=-(N-1)}^{N-1} w_{\tau,m}\hat{s}_\tau e^{-i2\pi f\tau},$$

where $\{w_{\tau,m}\}$ is a lag window, $Q_{s,t} = w_{t-s,m}$, and

$$\hat{s}_\tau = \begin{cases} (1/N)\sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|} & \text{for } \tau = 0, \pm 1, \ldots, \pm(N-1), \\ 0 & \text{otherwise.} \end{cases}$$

As for multitaper estimators, we can write the expectation of the spectral estimator as the convolution of a spectral window with the true spectrum (Percival & Walden, 1993, p. 243):

$$E\{\hat{S}^{(lw)}(f)\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathscr{U}_m(f-f')S(f')\,df',$$

where the spectral window is given by

$$\mathcal{U}_m(f) = \sum_{\tau=-(N-1)}^{N-1} w_{\tau,m}(1 - |\tau|/N)e^{-i2\pi f\tau}.$$

We can calculate $\text{width}_a\,\{\mathcal{U}_m(.)\}$ as in (5) by noting that, with $\{c_\tau\} \leftrightarrow \mathcal{U}_m(.)$, the Fourier coefficients are given by $c_\tau = w_{\tau,m}(1 - |\tau|/N)$, so that

$$\text{width}_a\,\{\mathcal{U}_m(.)\} = \left(\sum_{\tau=-(N-1)}^{(N-1)} c_\tau^2\right)^{-1} = \left\{\sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m}^2(1 - |\tau|/N)^2\right\}^{-1}. \tag{7}$$

6. EXAMPLE

To illustrate the use of our proposed bandwidth measure, we consider a time series of $N = 4096$ observations of heights of ocean waves recorded every 1/30th of second over a 136·5 second interval by a wire wave gauge. Jessup, Melville & Keller (1991) and Percival (1994) also look at this series. A multitaper spectral estimate was computed using $K = 6$ Slepian tapers with design bandwidth W chosen such that $NW = 4$ and is plotted in decibel units in Fig. 7(b) for frequencies ranging from 0 to 4 Hz, that is, 0 to 4 cycles/second. The

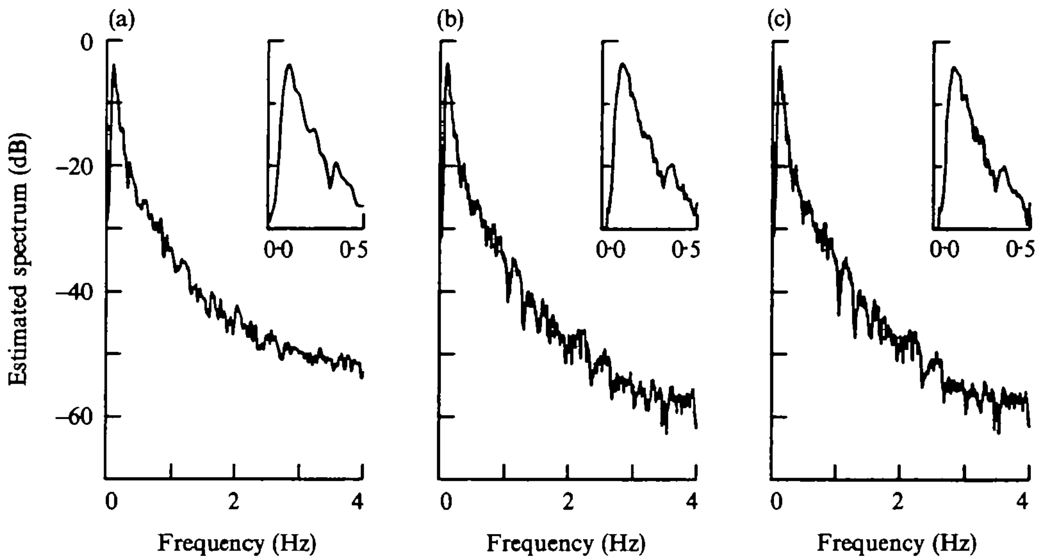

Fig. 7. Spectral estimates of ocean wave data, measured in meters²/Hz and plotted in dB, for (a) Parzen lag window, (b) multitaper method with Slepian tapers, and (c) multitaper method using sine tapers. All methods use essentially the same effective bandwidth of about 0·05 Hz.

small subplot shows an expanded view of the multitaper estimate for frequencies from 0 to 0·5 Hz. The width of the criss-cross in the subplot depicts the bandwidth for this estimate, 0·0524 Hz, which was computed by dividing the bandwidth measure (6) by the sampling time of 1/30th second. The height of the criss-cross shows the length of a 95% confidence interval based on $2K = 12$ equivalent degrees of freedom. To compute a comparable lag window estimate, we used a Parzen lag window with parameter $m = 1172$ (Percival & Walden, 1993, p. 265). Using (7), this yields the same bandwidth as before. This estimate is shown in Fig. 7(a) and has approximately 13 equivalent degrees of freedom. The most striking difference between the two estimates is that the lag window estimate is elevated in comparison to the multitaper estimate at higher frequencies, by about 5 dB at $f = 4$ Hz. This elevation is undoubtedly a bias in the lag window estimate due to leakage. Leakage might also explain why the lag window estimate seems to be less variable at the higher frequencies than the multitaper estimate. The subplots in Fig. 7(a) and (b) indicate that the two estimates are quite comparable at low frequencies, with the multitaper estimate being slightly choppier in appearance. Thus comparison of the Slepian multitaper and lag window estimates with identical bandwidths shows roughly equivalent results at low frequencies, but markedly different results in regions in which the lag window estimate suffers from leakage that multitapering successfully suppresses.

Figure 7(c) shows a sinusoidal multitaper estimate using $K = 6$ tapers. The bandwidth for this estimate is 0·0508 Hz, which differs by about 3% from the bandwidth for the Slepian multitaper estimate. The equivalent degrees of freedom, namely 12, are the same as for the Slepian estimate. Comparison of Fig. 7(b) and 7(c) indicate that the sinusoidal estimate is virtually identical to the Slepian estimate, with the former being choppier in appearance than the latter. We can attribute the close correspondence here to the fact that the underlying spectrum is not changing too rapidly.

7. Comments and conclusions

The measure of the bandwidth of a spectral estimator defined as width_a of the appropriate spectral window, as used in (5) and (7), enables a unified treatment of bandwidth for different quadratic spectral estimators, such as multitaper, single taper or direct spectral estimators, lag window, and weighted overlapped segment averaged (Percival & Walden, 1993, p. 289).

It might be suspected that there is a simple inverse relationship between the variance of the spectral estimator and width_a of the spectral window. This does not appear to be the case. For multitapering, var $\{\hat{S}^{(mt)}(f)/S(f)\} \propto 1/K$, while for Slepian tapers (Fig. 5) the increase in the bandwidth measure is not simply linearly increasing in K. The lag window bandwidth measure in (7) is not simply the inverse of the variance due to the $(1-|\tau|/N)^2$ term. The weighted overlapped segment averaged case is particularly interesting: the effective bandwidth would be given by (5) with N replaced by N_S, the size of data block being averaged (Percival & Walden, 1993, p. 291). In this case the effective bandwidth is decoupled from the variance of the weighted overlapped segment averaged estimator which depends mainly on N_B, the number of blocks being averaged.

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