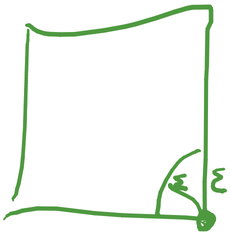
Let $\varepsilon > 0$. Then

$$P(\|X_{(k)} - X\| > \varepsilon) = \mathbb{E}\, P(\|X_{(k)} - X\| > \varepsilon \,|\, X)$$

$$= \mathbb{E}\, P(\mathrm{Bin}(n, q_x^{\varepsilon}) < k \,|\, X)$$

where $q_x^{\varepsilon} = P(\|X' - x\| < \varepsilon \,|\, X) \geq \varepsilon^d \cdot \frac{V_1}{2^d} = q_{\varepsilon}$




$X'$ independent, uniform in $[0,1]^d$

where $V_1$ is the volume of a ball of radius 1 in $\mathbb{R}^d$.

$$\leq P(\mathrm{Bin}(n, q_{\varepsilon}) < k)$$

$$= \sum_{i=1}^{k-1} \binom{n}{i} q_{\varepsilon}^{k} (1-q_{\varepsilon})^{n-k} = \sum_{i=1}^{k-1} \left(\frac{q_{\varepsilon}}{1-q_{\varepsilon}}\right)^{k} \cdot \underbrace{\binom{n}{i}(1-q_{\varepsilon})^{n}}_{\leq n^k e^{-n q_{\varepsilon}}} \longrightarrow 0$$