The VC dimension of

$$\mathcal{A} = \{ A_\alpha = \{x \in \mathbb{R} : \sin(\alpha x) > 0\} : \alpha > 0 \}$$

is <u>infinite</u>. To see this, it suffices to prove that for all $n = 1, 2, \ldots$, there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that these n points are shattered. This means that for all possible label assignments $y_1, \ldots, y_n \in \{0, 1\}$ there exists $\alpha > 0$ such that

$$\sin(\alpha x_i) > 0 \text{ if } y_i = 1 \text{ and } \sin(\alpha x_i) < 0 \text{ if } y_i = 0.$$

We show that this is the case for

$$x_1 = 1,\ x_2 = \tfrac{1}{2}, \ldots, x_n = 2^{-n}.$$

Indeed, regardless of what $y_1, \ldots, y_n$ are, if

$$\alpha = \pi\left(1 + \sum_{j=1}^{n} y_j \cdot 2^j\right),$$

then for all i,

$$\alpha x_i = \pi\left(2^{-i} + \sum_{j=1}^{i} 2^{j-i} y_j\right) + \underbrace{2\pi\left(\sum_{j=i+1}^{n} 2^{j-i-1} y_j\right)}_{\text{integer multiple of } 2\pi, \text{ does not change the value of sin.}}$$

so $\sin(\alpha x_i)$

$$= \sin\left(\pi\left(y_i + \underbrace{\sum_{j=1}^{i-1} 2^{j-i} y_j + 2^{-i}}_{>0,\ <1}\right)\right)$$

> 0 if $y_i = 0$

< 0 if $y_i = 1$