Stochastic Modeling and Optimization Problemset 3

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Problem 1

Problem variables

$$
\begin{aligned}
P_l &= \text{ probability of large demand} \\
P_s &= \text{ probability of small demand} \\
x_k &= \text{ current inventory level} \\
w_k &= \text{ demand drawn from } P_l \text{ or } P_s \\
q &= \text{ apriori probability of large demand}
\end{aligned}
$$

In the standard inventory control problem, for every overrage or underage in stock given demand, there are associated holding and shortage costs:

$$r(x) = p(max(0, -x)) + h(max(0, x))$$

If there is a non-negative x, h denotes the unit holding cost. If there x is negative, there is an opportunity cost of p per unit of demand not met.

The minimized cost function is:

$$min_{u \geq 0} \Big\{ cu + \mathop{\mathbb{E}}_{w} \{ pmax(0, w - x - u) + hmax(0, x + u - w) \| y \} \Big\}$$

where c is the cost of ordering level u. E.g. it is optimized by the u minimizing the holding cost plus the overrage cost and y is the indicator in $\{1, 2\}$ indicating the realization of large or small demand.

Part 1 Solution

Let $y_i \in \{1, 2\}$ denote the realization of large or small demand and D^i the level of demand. The single-period optimal choice is the same for either large or small demand:

$$\mu^*(x, y) == \begin{cases} D^i - x, & \text{if } y = i \text{ and } x < D^i, \\ 0, & \text{otherwise.} \end{cases}$$

Part 2 Solution

The augmented system for the multi-period problem is:

$$
\begin{aligned}
x_{k+1} &= x_k + u_k - w_k \\
y_{k+1} &= \psi_k
\end{aligned}
$$

Where ψ_k\$ takes on values 1 and 2 indicating a realization of demand from P_l and P_s respectively and taking on value 1 with probability q and 2 with probability $1 - q$

The DP algorithm is:

1. Terminal cost is:

$$J_N(x_N, y_N) = 0$$

2. Choose the u_k minimizing the cost function at time k given by:

$$costfunction = cu_k + \{pmax(0, w_k - x_k - u_k) + hmax(0, x_k + u_k - w_k)\}$$
$$J_k(x_k, y_k) = min_{u \geq 0} \underset{w_k}{\{\ \mathbb{E}\}} \Big\{ costfunction + qJ_{k+1}(x_k + u_k - w_k, 1) + (1-q)J_{k+1}(x_k + u_k - w_k, 2) | y_k \Big\}$$

$$\mu_k^*(x_k, y_k) == \begin{cases} D_k^i - x_k, & \text{if } y_k = i, x_k < D_k^i, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5

$$p_i : \text{probability of answering question } i \text{ correctly}$$
$$r_i : \text{reward for answering question } i \text{ correctly}$$
$$F_i : \text{cost for answering question } i \text{ incorrectly}$$

The expected value of question i:

$$R_i = \mathbb{E}[q_i] = p_i R_i - (1 - p_i) F_i$$

Part 1 Solution

Claim the optimal ordering of the set of questions is L and questions i and j are the k^{th} and $(k+1)^{st}$ questions:

$$L = \Big\{ i_0, ..., i_{k-1}, i, j, i_{k+2}, ..., i_{N-1} \Big\}$$

If we consider the list with i and j interchanged:

$$L' = \Big\{ i_0, ..., i_{k-1}, j, i, i_{k+2}, ..., i_{N-1} \Big\}$$

The expectation of the reward for L is greater than L' by definition, so it follows that:

$$p_i R_i - (1 - p_i) F_i / (1 - p_i) \geq p_j R_j - (1 - p_j) F_j / (1 - p_j)$$

Part 2 Solution

If there is a no cost option to stop answering questions, the contestant will stop answering questions whenever the expected value of answering the next question is negative.

The game stops at period $k - 1$, with k satisfying:

$$p_k R_k < (1 - p_k) F_k$$

That is, the game stops whenever the expected value of answering the next question becomes negative.