# Stochastic Modeling and Optimization Problemset 3

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## Problem 1

Problem variables

$$
\begin{aligned}
P_l &= \text{ probability of large demand} \\
P_s &= \text{ probability of small demand} \\
x_k &= \text{ current inventory level} \\
w_k &= \text{ demand drawn from } P_l \text{ or } P_s \\
q &= \text{ apriori probability of large demand}
\end{aligned}
$$

In the standard inventory control problem, for every overrage or underage in stock given demand, there are associated holding and shortage costs:

$$r(x) = p(max(0, -x)) + h(max(0, x))$$

If there is a non-negative $x$, $h$ denotes the unit holding cost. If there $x$ is negative, there is an opportunity cost of $p$ per unit of demand not met.

The minimized cost function is:

$$min_{u \geq 0} \Big\{ cu + \mathop{\mathbb{E}}_{w} \{ pmax(0, w - x - u) + hmax(0, x + u - w) \| y \} \Big\}$$

where $c$ is the cost of ordering level $u$. E.g. it is optimized by the $u$ minimizing the holding cost plus the overrage cost and $y$ is the indicator in $\{1, 2\}$ indicating the realization of large or small demand.

*Part 1 Solution*

Let $y_i \in \{1, 2\}$ denote the realization of large or small demand and $D^i$ the level of demand. The single-period optimal choice is the same for either large or small demand:

$$\mu^*(x, y) = \begin{cases} D^i - x, & \text{if } y = i \text{ and } x < D^i, \\ 0, & \text{otherwise.} \end{cases}$$

*Part 2 Solution*

The augmented system for the multi-period problem is:

$$
\begin{aligned}
x_{k+1} &= x_k + u_k - w_k \\
y_{k+1} &= \psi_k
\end{aligned}
$$

Where $\psi_k$\$ takes on values 1 and 2 indicating a realization of demand from $P_l$ and $P_s$ respectively and taking on value 1 with probability $q$ and 2 with probability $1 - q$

The DP algorithm is:

1. Terminal cost is:

$$J_N(x_N, y_N) = 0$$

2. Choose the $u_k$ minimizing the cost function at time $k$ given by:

$$costfunction = cu_k + \{pmax(0, w_k - x_k - u_k) + hmax(0, x_k + u_k - w_k)\}$$
$$J_k(x_k, y_k) = min_{u \geq 0} \underset{w_k}{\{\mathbb{E}\}} \Big\{costfunction + qJ_{k+1}(x_k + u_k - w_k, 1) + (1-q)J_{k+1}(x_k + u_k - w_k, 2)|y_k\Big\}$$

$$\mu_k^*(x_k, y_k) = \begin{cases} D_k^i - x_k, & \text{if } y_k = i, x_k < D_k^i, \\ 0, & \text{otherwise.} \end{cases}$$

## Problem 2

a) First we consider demands as correlated $w_k = \epsilon_k - \gamma\epsilon_{k-1}$. Considering the relation given in the problem, the values of $\epsilon_{k-1}$ can be known apriori, as long as $\epsilon_{-1}$ is provided. This means that we can redefine the state variable as $z_k = x_k + \gamma\epsilon_{k-1}$. To show that this converts the problem to an uncorrelated one, consider the dynamics for inventory control

$$x_{k+1} = x_k + u_k - w_k,$$

which we can now expand and make a change of variable

$$x_{k+1} = x_k + \gamma\epsilon_{k-1} + u_k - \epsilon_k = z_k + u_k - \epsilon_k.$$

And so now the problem is an uncorrelated one.

b) Now considering a one period delay, this is essentially the same problem except instead of $u_k$, we have $u_{k-1}$. Then, continuing on from part (a), we have

$$x_{k+1} = z_k + u_{k-1} - \epsilon_k.$$

Again we can make the change in variable to $y_k = z_k + u_{k-1}$, so that we are left with

$$x_{k+1} = y_k - \epsilon_k$$

, and again, we have the same result.

## Problem 3

If we consider the case of covering each demand from a separate inventory repository, with optimal order quantity $Q^*$, then the optimal amount for a centralised inventory is $Q_p^* = \Sigma Q_i^*$. This is true because all locations are independent and follow the same distribution. The ideal order amount is equal to the expected demand, $Q^* = E(D)$, so the optimal amount to order is $\Sigma E(D_i) = E(\Sigma D_i)$. Then, using the distribution of the demands,

$$E(\Sigma D_i) = E(\sqrt{n}D_1 + \mu(n - \sqrt{n})) = \sqrt{n}E(D_1) + \mu(n - \sqrt{n}) = \sqrt{n}Q^* + \mu(n - \sqrt{n})$$

For the cost of the centralised system, we have

$$C_p = E[h(Q_p^* - \Sigma D_i)^+ + p(\Sigma D_i - Q_p^*)^+] = E[h(\sqrt{n}Q^*)^+ + p(\sqrt{n}Q^*)^+] = \sqrt{n}E[h(Q^*)^+ + p(Q^*)^+] = \sqrt{n}C$$

where $C$ is the cost of an individual repository and we have used the relation for the sum of demands as before. Therefore, $\frac{nC}{C_p} = \frac{nC}{\sqrt{n}C} = \sqrt{n}C$, as desired.

## Problem 4

- $x_k$ the state is the maximum offer received so far,
- $u_k$ the control of the problem is to accept or not accept offer at period $k$,
- $w_k$ offer at period $k$ from set $w_k$ which is from the set of $n$ possible offers $w_j \in \{w_1, ..., w_n\}$ each having probability from the set $p_j \in \{p_1, ..., p_n\}$

The optimal policy is to "stop" e.g. to accept the offer $x_k$ whenever $x_k$ satisifies the condition:

$$h(x_k) = -c + \sum_{i=1}^{n} p_i max(x_k, w_i)$$

In words: we stop whenever the probability of recovering the maintenance cost from a higher offer is greater than the current maximum. The one-step stopping set is:

$$T_{N-1} = \left\{ x | x \geq -c + \sum_{i=1}^{n} p_i max(x, w_i) \right\}$$

## Problem 5

$$p_i : \text{probability of answering question } i \text{ correctly}$$

$$r_i : \text{reward for answering question } i \text{ correctly}$$

$$F_i : \text{cost for answering question } i \text{ incorrectly}$$

The expected value of question $i$:

$$R_i = \mathbb{E}[q_i] = p_i R_i - (1 - p_i) F_i$$

*Part 1 Solution*

Claim the optimal ordering of the set of questions is $L$ and questions $i$ and $j$ are the $k^{th}$ and $(k+1)^{st}$ questions:

$$L = \left\{ i_0, ..., i_{k-1}, i, j, i_{k+2}, ..., i_{N-1} \right\}$$

Than the expectation of the reward of L is given by the expectation of the set $\{i_0, ..., i_{k-1}\}$ plus the joint probability of getting all $i_0, ..., i$ questions correct times the expected reward of the $i - th$ question plus the joint probability of getting all $i_0, ..., j$ questions correct times the expected reward of the $j - th$ question plus the joint probabilities for $k + 2$ to $N - 1$. E.g.:

$$\mathbb{L} = \mathbb{E}\Big\{ \text{ reward of } \{i_0, ..., i_{k-1}\} \Big\}$$
$$+ p_{i_0} ... p_{i_{k-1}} \Big\{ p_i R_i - (1 - p_i) F_i + p_i (p_j R_j - (1 - p_j) F_j) \Big\}$$
$$+ p_{i_0} ... p_{i_{k-1}} p_i p_j \mathbb{E}\Big\{ \text{ reward of} \{i_{k+2}, ..., i_{N-1}\} \Big\}$$

If we consider the list with $i$ and $j$ interchanged:

$$L' = \left\{ i_0, ..., i_{k-1}, j, i, i_{k+2}, ..., i_{N-1} \right\}$$

The expectation of the reward for $L$ is greater than $L'$ by definition, so it follows that:

$$\frac{p_i R_i - (1-p_i)F_i}{(1-p_i)} \geq \frac{p_j R_j - (1-p_j)F_j}{(1-p_j)}$$

*Part 2 Solution*

If there is a no cost option to stop answering questions, the contestant will stop answering questions whenever the expected value of answering the next question is negative.

The game stops at period $k-1$, with $k$ satisfying:

$$p_k R_k < (1-p_k)F_k$$

That is, the game stops whenever the expected value of answering the next question becomes negative.