# Block Preconditioning for Navier-Stokes

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# Multiphase Clotting Model

$$
\begin{aligned}
\frac{\partial \phi_{\mathsf{u}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{f}} \phi_{\mathsf{u}}) &= D \Delta \phi_{\mathsf{u}} - f^{\mathsf{u} \to \mathsf{a}} \\
\frac{\partial \phi_{\mathsf{a}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{f}} \phi_{\mathsf{a}}) &= D \Delta \phi_{\mathsf{a}} + f^{\mathsf{u} \to \mathsf{a}} + f^{\mathsf{b} \to \mathsf{a}} - f^{\mathsf{a} \to \mathsf{b}} \\
\frac{\partial \phi_{\mathsf{b}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{b}} \phi_{\mathsf{b}}) &= f^{\mathsf{a} \to \mathsf{b}} - f^{\mathsf{b} \to \mathsf{a}} \\
\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{f}} c) &= D_{\mathsf{c}} \Delta c + A f^{\mathsf{u} \to \mathsf{a}} - kc
\end{aligned}
$$

# Multiphase Clotting Model

$$\begin{aligned}
\rho\left(\frac{\partial\theta_{\mathsf{b}}\mathbf{u}_{\mathsf{b}}}{\partial t} + \nabla\cdot(\theta_{\mathsf{b}}\mathbf{u}_{\mathsf{b}}\mathbf{u}_{\mathsf{b}})\right) &= \nabla\cdot\boldsymbol{\sigma}_{\mathsf{b}} + \xi\theta_{\mathsf{b}}\theta_{\mathsf{f}}(\mathbf{u}_{\mathsf{f}} - \mathbf{u}_{\mathsf{b}}) + \nabla\cdot\boldsymbol{\tau}_{\mathsf{b}} + \mathbf{f}_{\mathsf{b}} \\
\rho\left(\frac{\partial\theta_{\mathsf{f}}\mathbf{u}_{\mathsf{f}}}{\partial t} + \nabla\cdot(\theta_{\mathsf{f}}\mathbf{u}_{\mathsf{f}}\mathbf{u}_{\mathsf{f}})\right) &= \nabla\cdot\boldsymbol{\sigma}_{\mathsf{f}} + \xi\theta_{\mathsf{b}}\theta_{\mathsf{f}}(\mathbf{u}_{\mathsf{b}} - \mathbf{u}_{\mathsf{f}}) + \mathbf{f}_{\mathsf{f}} \\
0 &= \nabla\cdot(\theta_{\mathsf{b}}\mathbf{u}_{\mathsf{b}} + \theta_{\mathsf{f}}\mathbf{u}_{\mathsf{f}}) \\
\theta_{\mathsf{b}} &= v_{\mathsf{pl}}\phi_{\mathsf{b}} \\
\theta_{\mathsf{f}} &= 1 - \theta_{\mathsf{b}}
\end{aligned}$$

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\frac{\partial \phi_{\mathsf{a}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{f}} \phi_{\mathsf{a}}) &= D \Delta \phi_{\mathsf{a}} + f^{\mathsf{u} \to \mathsf{a}} + f^{\mathsf{b} \to \mathsf{a}} - f^{\mathsf{a} \to \mathsf{b}} \\
\frac{\partial \phi_{\mathsf{b}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{b}} \phi_{\mathsf{b}}) &= f^{\mathsf{a} \to \mathsf{b}} - f^{\mathsf{b} \to \mathsf{a}} \\
\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{f}} c) &= D_{\mathsf{c}} \Delta c + A f^{\mathsf{u} \to \mathsf{a}} - kc \\
\frac{\partial z_{\mathsf{b}}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{b}} z_{\mathsf{b}}) &= \alpha(\phi_{\mathsf{a}}, \phi_{\mathsf{b}}) - \beta z_{\mathsf{b}} \\
\frac{\partial \boldsymbol{\sigma}}{\partial t} + \nabla \cdot (\mathbf{u}_{\mathsf{b}} \boldsymbol{\sigma}) &= \boldsymbol{\sigma} \nabla \mathbf{u}_{\mathsf{b}} + (\boldsymbol{\sigma} \nabla \mathbf{u}_{\mathsf{b}})^{\mathsf{T}} + C_1 \alpha(\phi_{\mathsf{a}}, \phi_{\mathsf{b}}) \mathbb{I} - \beta \boldsymbol{\sigma}
\end{aligned}
$$

# Multiphase Clotting Model

Insert visualization of two phase model

# Multiphase Clotting Model

$$
\begin{aligned}
\rho \frac{\partial \theta_{\mathsf{b}} \mathbf{u}_{\mathsf{b}}}{\partial t} &= -\theta_{\mathsf{b}} \nabla p + \nabla \cdot \boldsymbol{\sigma}_{\mathsf{b}} \\
&\quad + \xi \theta_{\mathsf{b}} \theta_{\mathsf{f}} (\mathbf{u}_{\mathsf{f}} - \mathbf{u}_{\mathsf{b}}) \\
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\end{aligned}
$$

- Discretization:

$$
\begin{pmatrix} \mathcal{A}_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} & \theta_{\mathsf{b}} \nabla_h \\ -\boldsymbol{\xi} & \mathcal{A}_{\mathsf{f}} + \boldsymbol{\xi} & \theta_{\mathsf{f}} \nabla_h \\ -(\nabla_h \cdot)\theta_{\mathsf{b}} & -(\nabla_h \cdot)\theta_{\mathsf{f}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{\mathsf{b}} \\ \mathbf{u}_{\mathsf{f}} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{\mathsf{b}} \\ \mathbf{b}_{\mathsf{f}} \\ \mathbf{b}_{\mathsf{p}} \end{pmatrix}
$$

- Solve with a Krylov method with a geometric multigrid preconditioner.

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0 &= \nabla \cdot (\theta_{\mathsf{b}} \mathbf{u}_{\mathsf{b}} + \theta_{\mathsf{f}} \mathbf{u}_{\mathsf{f}}) \\
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$$

- Solve with a Krylov method with a geometric multigrid preconditioner.
- Use a box relaxation smoother: on each cell, we use a direct solver for one pressure and eight velocity dofs.

# Multiphase Clotting Model

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
                                                              Timer Name       Total
                                    IBTK::HierarchyIntegrator::advanceHierarchy()   7.67058 (99%)
           multiphase::MultiphaseStaggeredHierarchyIntegrator::integrateHierarchy()   7.50069 (97%)
                                     IBTK::PETScKrylovLinearSolver::solveSystem()   7.28467 (94%)
                                       IBTK::FullFACPreconditioner::solveSystem()    6.1696 (80%)
       multiphase::MultiphaseStaggeredStokesBoxRelaxationFACOperator::smoothError()   5.80681 (75%)
                                                 xfer::RefineSchedule::fillData()  0.954041 (12%)
                                             xfer::RefineSchedule::recursive_fill  0.859446 (11%)
                         multiphase::MultiphaseStaggeredStokesOperator::apply()   0.377646 (4%)
                                        IBTK::PETScSAMRAIVectorReal::VecMDot()   0.233488 (3%)
   multiphase::MultiphaseStaggeredStokesBoxRelaxationFACOperator::solveCoarsestLevel()   0.216332 (2%)
                                   IBTK::CCPoissonHypreLevelSolver::solveSystem()   0.216323 (2%)
                            IBTK::CCPoissonHypreLevelSolver::solveSystem()[hypre]    0.18325 (2%)
                                       IBTK::PETScSAMRAIVectorReal::VecMAXPY()   0.172895 (2%)
                  multiphase::MultiphaseStaggeredHierarchyIntegrator::preprocess()   0.151408 (1%)
                               IBTK::HierarchyGhostCellInterpolation::fillData()   0.144828 (1%)
                                                   tbox::Schedule::communicate()   0.144628 (1%)
                                         xfer::RefineSchedule::refineScratchData()  0.0746266 (0%)
                       IBTK::HierarchyGhostCellInterpolation::fillData()[refine]  0.0729131 (0%)
             IBTK::HierarchyGhostCellInterpolation::fillData()[set_physical_bcs]  0.0689686 (0%)
                                               IBTK::CCLaplaceOperator::apply()  0.0524488 (0%)
                                           xfer::CoarsenSchedule::coarsenData()  0.0515759 (0%)
                                        IBTK::PETScSAMRAIVectorReal::VecNorm()  0.0451969 (0%)
                          IBTK::PETScKrylovLinearSolver::initializeSolverState()  0.0325276 (0%)
                IBTK::HierarchyGhostCellInterpolation::initializeOperatorState()  0.0253217 (0%)
                                       IBTK::PETScSAMRAIVectorReal::VecScale()  0.0149808 (0%)
                 multiphase::MultiphaseStaggeredHierarchyIntegrator::postprocess()  0.0140977 (0%)
             IBTK::HierarchyGhostCellInterpolation::resetTransactionComponents()  0.0130245 (0%)
                           IBTK::CCPoissonHypreLevelSolver::initializeSolverState()  0.0123116 (0%)
                               IBTK::FullFACPreconditioner::initializeSolverState()  0.0108996 (0%)
                                     xfer::RefineSchedule::generate_comm_schedule 0.00889866 (0%)
                                     IBTK::PETScSAMRAIVectorReal::VecDuplicate() 0.00885765 (0%)
           multiphase::MultiphaseStaggeredStokesOperator::initializeOperatorState() 0.00837345 (0%)
                                                               TOTAL RUN TIME:   7.68787 (100%)
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

# Alternatives to Multigrid

- Consider the single phase Navier-Stokes equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla \cdot \boldsymbol{\tau}(\mathbf{u})$$
$$\nabla \cdot \mathbf{u} = 0$$

- Discretized system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{G} \\ -\mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{\mathsf{u}} \\ \mathbf{b}_{\mathsf{p}} \end{pmatrix}$$
$$\mathbf{A} = \omega \boldsymbol{\rho} - \mathbf{L}_{\mu}$$

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$$\mathbf{A} = \omega \boldsymbol{\rho} - \mathbf{L}_\mu$$

- Can solve using the inverse of the Schur complement:

$$\mathbf{S}^{-1} = (-\mathbf{D}\mathbf{A}^{-1}\mathbf{G})^{-1}$$
$$p = -\mathbf{S}^{-1}(\mathbf{D}\mathbf{A}^{-1}\mathbf{b}_{\mathsf{u}} + \mathbf{b}_{\mathsf{p}})$$
$$\mathbf{u} = \mathbf{A}^{-1}(\mathbf{b}_{\mathsf{u}} - \mathbf{G}p) = \mathbf{A}^{-1}\mathbf{b}_{\mathsf{u}} + \mathbf{A}^{-1}\mathbf{G}\mathbf{S}^{-1}(\mathbf{D}\mathbf{A}^{-1}\mathbf{b}_{\mathsf{u}} + \mathbf{b}_{\mathsf{p}})$$

- Build a preconditioner based on approximations to $\mathbf{S}^{-1}$

# Block Preconditioners for Stokes

- For constant coefficients

$$
\begin{aligned}
\mathbf{A} &= \omega\rho_0\mathbf{I} - \mu_0\mathbf{L} \\
\mathbf{S}^{-1} &= (-\mathbf{D}\mathbf{A}^{-1}\mathbf{G})^{-1} \\
&\approx ((-\mathbf{D}\mathbf{G})(\omega\rho_0\mathbf{I} - \mu_0\mathbf{L}_p)^{-1})^{-1} \\
&= -\omega\rho_0\mathbf{L}_p^{-1} + \mu_0\mathbf{I}
\end{aligned}
$$

  with $\mathbf{L}_p = \mathbf{D}\mathbf{G}$.
- Assumes $\mathbf{L}\mathbf{G} \approx \mathbf{G}\mathbf{L}_p$. Exact for certain boundary conditions.

# Block Preconditioners for Stokes

- For constant coefficients

$$\begin{aligned} \mathbf{A} &= \omega\rho_0\mathbf{I} - \mu_0\mathbf{L} \\ \mathbf{S}^{-1} &= (-\mathbf{D}\mathbf{A}^{-1}\mathbf{G})^{-1} \\ &\approx ((-\mathbf{D}\mathbf{G})(\omega\rho_0\mathbf{I} - \mu_0\mathbf{L}_p)^{-1})^{-1} \\ &= -\omega\rho_0\mathbf{L}_p^{-1} + \mu_0\mathbf{I} \end{aligned}$$

  with $\mathbf{L}_p = \mathbf{D}\mathbf{G}$.
- Assumes $\mathbf{L}\mathbf{G} \approx \mathbf{G}\mathbf{L}_p$. Exact for certain boundary conditions.
- For variable coefficient:
  - $\omega\rho_0\mathbf{L}_p^{-1}$ is the inviscid terms, replace with $\omega\mathbf{L}_{\boldsymbol{\rho}}^{-1}$ with $\mathbf{L}_{\boldsymbol{\rho}} = \mathbf{D}\boldsymbol{\rho}^{-1}\mathbf{G}$ is the variable coefficient Laplacian.
  - $\mu_0\mathbf{I}$ is the viscous term, replace with $2\boldsymbol{\mu}$.
- Use approximate Schur complement for preconditioning: $\mathcal{S}^{-1} = -\omega\mathbf{L}_{\boldsymbol{\rho}}^{-1} + 2\boldsymbol{\mu}$

# Block Preconditioners for Multiphase Stokes

- Multiphase system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{G} \\ -\mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathbf{u}_{\mathsf{b}} \\ \mathbf{u}_{\mathsf{f}} \end{pmatrix} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{\mathsf{u}} \\ \mathbf{b}_{\mathsf{p}} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \omega\theta_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} \\ -\boldsymbol{\xi} & \omega\theta_{\mathsf{f}} + \boldsymbol{\xi} \end{pmatrix} + \begin{pmatrix} \mathbf{L}_{\mu,\mathsf{b}} & 0 \\ 0 & \mathbf{L}_{\mu,\mathsf{f}} \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \theta_{\mathsf{b}}\nabla_h \\ \theta_{\mathsf{f}}\nabla_h \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} (\nabla_h\cdot)\theta_{\mathsf{b}} & (\nabla_h\cdot)\theta_{\mathsf{f}} \end{pmatrix}$$

# Block Preconditioners for Multiphase Stokes

- Multiphase system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{G} \\ -\mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathbf{u}_{\mathsf{b}} \\ \mathbf{u}_{\mathsf{f}} \end{pmatrix} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{\mathsf{u}} \\ \mathbf{b}_{\mathsf{p}} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \omega\theta_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} \\ -\boldsymbol{\xi} & \omega\theta_{\mathsf{f}} + \boldsymbol{\xi} \end{pmatrix} + \begin{pmatrix} \mathbf{L}_{\mu,\mathsf{b}} & 0 \\ 0 & \mathbf{L}_{\mu,\mathsf{f}} \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \theta_{\mathsf{b}}\nabla_h \\ \theta_{\mathsf{f}}\nabla_h \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} (\nabla_h\cdot)\theta_{\mathsf{b}} & (\nabla_h\cdot)\theta_{\mathsf{f}} \end{pmatrix}$$

- Schur complement:

$$\begin{aligned} \mathbf{S}^{-1} &= (-\mathbf{D}\mathbf{A}^{-1}\mathbf{G})^{-1} \\ &= \left(-\mathbf{D}\left[\begin{pmatrix} \omega\theta_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} \\ -\boldsymbol{\xi} & \omega\theta_{\mathsf{f}} + \boldsymbol{\xi} \end{pmatrix} + \begin{pmatrix} \mathbf{L}_{\mu,\mathsf{b}} & 0 \\ 0 & \mathbf{L}_{\mu,\mathsf{f}} \end{pmatrix}\right]^{-1} \mathbf{G}\right)^{-1} \end{aligned}$$

# Block Preconditioners for Multiphase Stokes

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- For constant coefficient single phase:

$$\begin{aligned} \mathbf{A} &= \omega\rho_0\mathbf{I} - \mu_0\mathbf{L} \\ \mathbf{S}^{-1} &= (-\mathbf{D}\mathbf{A}^{-1}\mathbf{G})^{-1} \\ &\approx ((-\mathbf{D}\mathbf{G})(\omega\rho_0\mathbf{I} - \mu_0\mathbf{L}_p)^{-1})^{-1} \\ &= -\omega\rho_0\mathbf{L}_p^{-1} + \mu_0\mathbf{I} \end{aligned}$$

- Multiphase:
  - $\omega\rho_0\mathbf{L}_p^{-1}$ is the inviscid terms, replace with
  $\mathbf{L}_{\boldsymbol{\rho}}^{-1} = \left[\mathbf{D}\begin{pmatrix} \omega\theta_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} \\ -\boldsymbol{\xi} & \omega\theta_{\mathsf{f}} + \boldsymbol{\xi} \end{pmatrix}^{-1}\mathbf{G}\right]^{-1}.$
  - $\mu_0\mathbf{I}$ is the viscous term, replace with $\boldsymbol{\theta}_{\boldsymbol{\mu}} = 2\begin{pmatrix} \theta_{\mathsf{b}}\boldsymbol{\mu} & 0 \\ 0 & \theta_{\mathsf{f}}\boldsymbol{\mu} \end{pmatrix}.$
- Use approximate Schur complement for preconditioning: $\mathcal{S}^{-1} = -\omega\mathbf{L}_{\boldsymbol{\rho}}^{-1} + \boldsymbol{\theta}_{\boldsymbol{\mu}}$

# Block Preconditioners for Multiphase Stokes

- Proposed approximate Schur compliment:

$$\mathcal{S}^{-1} = -\omega \mathbf{L}_{\boldsymbol{\rho}}^{-1} + \boldsymbol{\theta}_{\boldsymbol{\mu}}$$

$$\mathbf{L}_{\boldsymbol{\rho}} = \mathbf{D} \begin{pmatrix} \omega\theta_{\mathsf{b}} + \boldsymbol{\xi} & -\boldsymbol{\xi} \\ -\boldsymbol{\xi} & \omega\theta_{\mathsf{f}} + \boldsymbol{\xi} \end{pmatrix}^{-1} \mathbf{G}$$

$$\boldsymbol{\theta}_{\boldsymbol{\mu}} = 2 \begin{pmatrix} \theta_{\mathsf{b}}\boldsymbol{\mu} & 0 \\ 0 & \theta_{\mathsf{f}}\boldsymbol{\mu} \end{pmatrix}$$

- Then solution is:

$$p = \mathcal{S}^{-1} \left( \mathbf{b}_{\mathsf{p}} + \mathbf{D}\mathbf{A}^{-1} \begin{pmatrix} \mathbf{b}_{\mathsf{b}} \\ \mathbf{b}_{\mathsf{f}} \end{pmatrix} \right)$$

$$\begin{pmatrix} \mathbf{u}_{\mathsf{b}} \\ \mathbf{u}_{\mathsf{f}} \end{pmatrix} = \mathbf{A}^{-1} \left( -\mathbf{G}p + \begin{pmatrix} \mathbf{b}_{\mathsf{b}} \\ \mathbf{b}_{\mathsf{f}} \end{pmatrix} \right)$$

- Relies on fast approximate solvers for $\mathbf{L}_{\boldsymbol{\rho}}^{-1}$ and $\mathbf{A}^{-1}$.