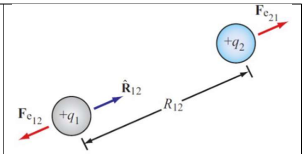
L2:

Force exerted on charge 2 by charge 1:

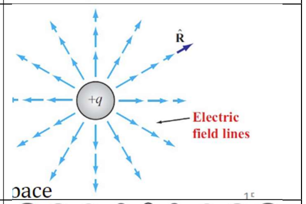
$$\mathbf{F}_{e21} = \hat{\mathbf{R}}_{12}\frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2} \quad \text{[N]}$$

$$\mathbf{F}_{e21} = \mathbf{E}_1 q_2$$



$$\mathbf{E} = \hat{\mathbf{R}}\frac{q}{4\pi\varepsilon_0 R^2} \quad \text{[V/m] (in free space)}$$

$\varepsilon_0 \equiv$ permittivity of free space

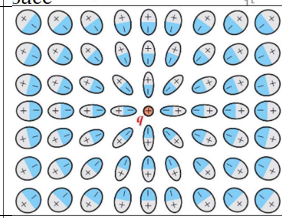


Field inside a dielectric:

$$\mathbf{E} = \hat{\mathbf{R}}\frac{q}{4\pi\varepsilon R^2}$$

Another quantity that will show up in our EM "toolbox" is **D**, the electric flux density:

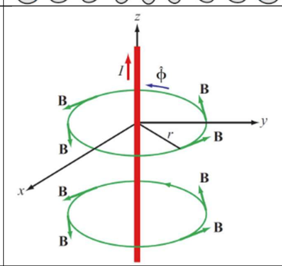
$$\mathbf{D} = \varepsilon\mathbf{E} = \hat{\mathbf{R}}\frac{q}{4\pi R^2} \quad \text{[C/m}^2\text{]}$$



The magnetic field induced by a long current carrying wire:

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{2\pi r} \quad \text{[T]}$$

$\mu_0 \equiv$ magnetic permeability of free space $= 4\pi \times 10^{-7}$ H/m



$$\mathbf{B} = \mu\mathbf{H}$$

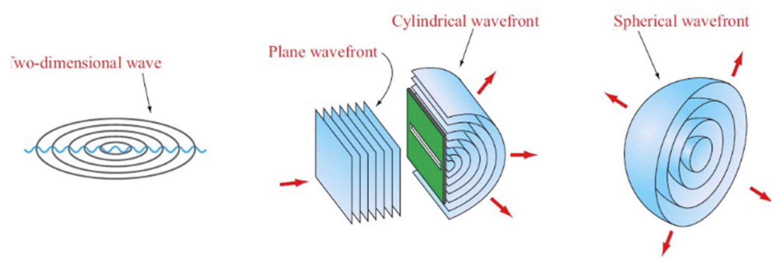
Magnetic flux density (T) — Magnetic field intensity (A/m)

Both are called "magnetic field"

Branch	Condition	Field Quantities [Units]
Electrostatics	Stationary charges $\partial q/\partial t = 0$	Electric field intensity **E** [V/m] Electric flux density **D** [C/m²] $\mathbf{D} = \varepsilon\mathbf{E}$
Magnetostatics	Steady currents $\partial I/\partial t = 0$	Magnetic field intensity **H** [A/m] Magnetic flux density **B** [T] $\mathbf{B} = \mu\mathbf{H}$
Dynamics (Time-varying fields)	Time-varying currents $\partial I/\partial t \neq 0$	**E**, **D**, **H**, and **B** (**E**, **D**) coupled to (**H**, **B**)

Waves:

Types of Waves



(a) Circular waves — 2D

(b) Plane and cylindrical waves — 3D

(c) Spherical wave — 3D

Phase velocity: velocity of the wave (or wave pattern) as it moves through a medium

- $u_p = \frac{dx}{dt} = \frac{\lambda}{T}$

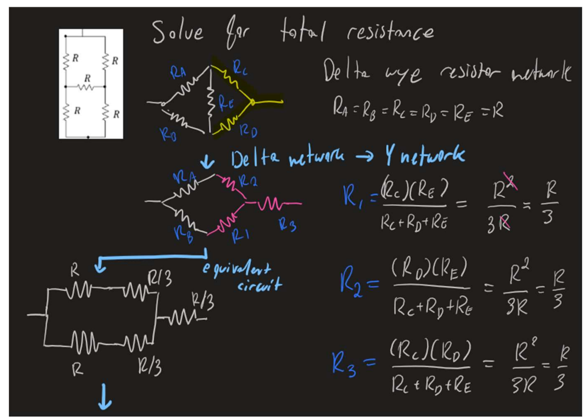
Wave equation: $y(x,t) = A\cos(\omega t - \beta x + \phi_0), \omega = 2\pi f, \beta = \frac{2\pi}{\lambda}$

**cos(x-pi/2) == sin(x)

Direction of wave travel: +x if t and x have opposite signs, -x if t and x have same sign

Phase lead and lag: when $\phi_0 > 0$, phase lead. When $\phi_0 < 0$, phase lag.

@steady-state DC: inductor is a short, capacitor is an open



Batteries:

- The maximum total power output of the battery is:

$$P_{max} = I\Delta V = IV_{emf}$$

- The power delivered to the external load resistor:

$$P = I^2R = \frac{V^2}{R}$$

- Things to note:
  - battery is a supply of **constant emf** but does not supply a constant terminal voltage because of its internal resistance
  - the battery **does not** supply a constant current (depends on the load resistor)

Capacitors and inductors:

A **capacitor** is a device that stores charge separation $C = \frac{Q}{V_{emf}}$

An **inductor** is a device which creates a strong magnetic flux for a given current, and therefore opposes changes in current flow $L = -\frac{V_{emf}}{\frac{di}{dt}}$

RMS current: avg power delivered to the resistor is determined by Irms

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max}$$

-

Average Power and RMS Voltage

- It follows that the average power delivered to the resistor is:

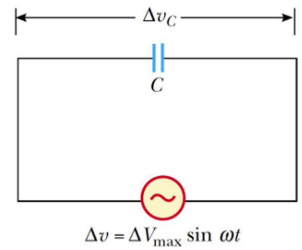
$$P_{avg} = I^2_{rms} R$$

- And we typically discuss voltage in an AC circuit in terms of the **rms voltage**:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707 \Delta V_{max}$$

Note: for an AC outlet rated at 120 V, we are referring to the rms voltage. Therefore the AC voltage has a maximum of 170 V.

Capacitors in AC Circuits



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$$\Delta v + \Delta v_c = 0$$

$$\Delta v - \frac{q}{C} = 0$$

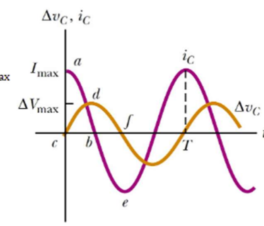
$$q = C\Delta V_{max} \sin \omega t$$

$$i_c = \frac{dq}{dt} = \omega C \Delta V_{max} \cos \omega t$$

$$i_c = \omega C \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

- The current and voltage are **90 degrees out of phase**.
- The current leads the voltage across the capacitor by 90°.

$$I_{max} = \omega C \Delta V_{max}$$



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Capacitive Reactance

- The current reaches its maximum value when cosωt = ±1:

$$I_{max} = \omega C \Delta V_{max} = \frac{\Delta V_{max}}{1/\omega C}$$

- Note that 1/ωC plays a similar role to the resistance.
- We define the **capacitive reactance, X_C**:

$$X_c = \frac{1}{\omega C} \qquad \text{[Ohms]}$$

Z=x+jy

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Get rectangular from polar: $x = |z|\cos\theta$, $y = |z|\sin\theta$

Get polar from rectangular: $|z| = \sqrt[+]{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

Euler's identity:	**Euler's Identity:** $e^{j\theta} = \cos\theta + j\sin\theta$	
	$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
Rectangular and polar form:	$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
	$x = \Re\mathfrak{e}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2+y^2}$
	$y = \Im\mathfrak{m}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
	$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
Complex algebra:	$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
	$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
	$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
Useful relations:	$-1 = e^{j\pi} = e^{-j\pi} = 1\angle{\pm 180^\circ}$	
	$j = e^{j\pi/2} = 1\angle{90^\circ}$	$-j = e^{-j\pi/2} = 1\angle{-90^\circ}$
	$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

Time Domain		Phasor Domain
$v(t) = V_0\cos\omega t$	$\leftrightarrow$	$\mathbf{V} = V_0$
$v(t) = V_0\cos(\omega t + \phi)$	$\leftrightarrow$	$\mathbf{V} = V_0 e^{j\phi}$.

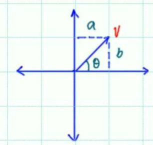
If $\phi = -\pi/2$,

$$v(t) = V_0\cos(\omega t - \pi/2) \leftrightarrow \mathbf{V} = V_0 e^{-j\pi/2}.$$

$z(t)$		$\widetilde{Z}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t + \phi_0)$	$\leftrightarrow$	$Ae^{j\phi_0}$
$A\cos(\omega t + \beta x + \phi_0)$	$\leftrightarrow$	$Ae^{j(\beta x + \phi_0)}$
$Ae^{-\alpha x}\cos(\omega t + \beta x + \phi_0)$	$\leftrightarrow$	$Ae^{-\alpha x}e^{j(\beta x + \phi_0)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}$
$A\sin(\omega t + \phi_0)$	$\leftrightarrow$	$Ae^{j(\phi_0 - \pi/2)}$
$\dfrac{d}{dt}(z(t))$	$\leftrightarrow$	$j\omega\widetilde{Z}$
$\dfrac{d}{dt}[A\cos(\omega t + \phi_0)]$	$\leftrightarrow$	$j\omega Ae^{j\phi_0}$
$\int z(t)\,dt$	$\leftrightarrow$	$\dfrac{1}{j\omega}\widetilde{Z}$
$\int A\sin(\omega t + \phi_0)\,dt$	$\leftrightarrow$	$\dfrac{1}{j\omega}Ae^{j(\phi_0 - \pi/2)}$

Rectangular → polar

Rectangular form, $V = a + jb$



Phasor magnitude, $V = \sqrt{a^2 + b^2}$

Angle w.r.t real axis, $\theta = \tan^{-1}(b/a)$

AC resistors

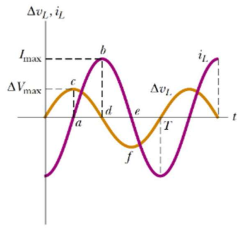
$$i_R = \frac{\Delta v}{R} = \frac{\Delta V_m}{R}\sin\omega t$$
$$i_R = I_m\cos(\omega t + \phi)$$

$$\Delta v_R = i_R R = \Delta V_m \sin\omega t$$
$$\Delta v_R = RI_m\cos(\omega t + \phi)$$

Inductors in AC Circuits

- The current and voltage are **90 degrees out of phase**.
- The current lags the voltage across the inductor by 90°.

$$I_{\max} = \frac{\Delta V_{\max}}{\omega L}$$



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- The current reaches its maximum value when cosωt = ±1:

$$I_{\max} = \frac{\Delta V_{\max}}{\omega L}$$

- Note that ωL plays a similar role to the resistance.
- We define the **inductive reactance, X_L**:

$$X_L = \omega L \qquad \text{[Ohms]}$$

Resistors	Phasor Domain $\mathbf{I_R} = I_m e^{j\phi}$ $\mathbf{V_R} = R\mathbf{I_R} = RI_m e^{j\phi}$
Capacitors	Phasor Domain $\mathbf{V}_C = V_m e^{j\phi}$ $\mathbf{I}_C = j\omega C V_m e^{j\phi} = j\omega C \mathbf{V}_C$
Inductors	Phasor Domain $\mathbf{I}_L = I_m e^{j\phi}$ $\mathbf{V}_L = j\omega L I_m e^{j\phi} = j\omega L \mathbf{I}_L = jX_L \mathbf{I}_L$

Resistor: $\mathbf{V}_R = R\mathbf{I}_R \qquad \frac{\mathbf{V}_R}{\mathbf{I}_R} = R$

Capacitor: $\mathbf{V}_C = \frac{1}{j\omega C}\mathbf{I}_C \qquad \frac{\mathbf{V}_C}{\mathbf{I}_C} = \frac{1}{j\omega C}$

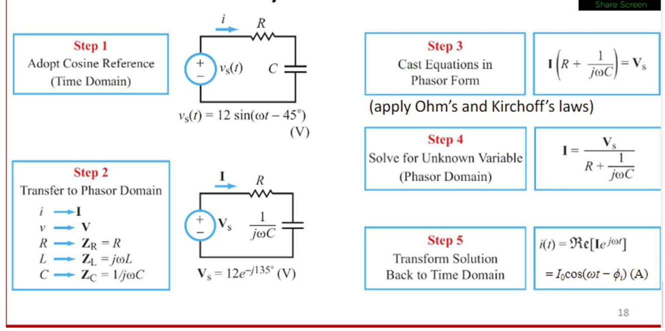
Inductor: $\mathbf{V}_L = j\omega L\mathbf{I}_L \qquad \frac{\mathbf{V}_L}{\mathbf{I}_L} = j\omega L$

Resistor: $Z_R = R$

Capacitor: $Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C} = -jX_C$

Inductor: $Z_L = j\omega L = jX_L$

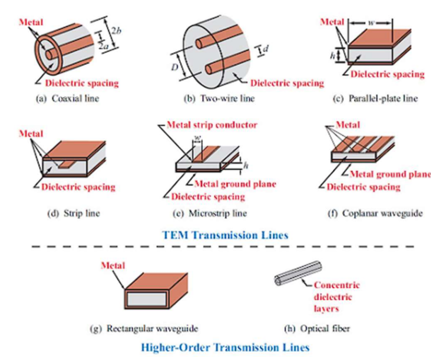
ac Phasor Analysis: General Procedu



Transmission lines:

Types of Transmission Modes

TEM (Transverse Electromagnetic): Electric and magnetic fields are orthogonal to one another, and both are orthogonal to direction of propagation



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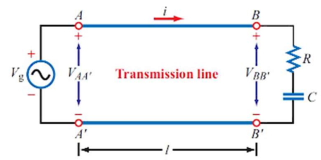
Transmission Line Effects

$$\phi_0 = \frac{2\pi f l}{c} = \frac{2\pi l}{\lambda}$$



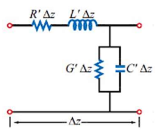
- When l/λ is very small, ignore
- When $l/\lambda >$ 0.01, need to account for phase delay and possibly reflections
- When l/λ > 0.25, definitely need to account for phase delay and reflections

$$\lambda = \frac{c}{f} = \frac{3x10^8 m/s}{1000\ Hz} = 300{,}000\ m$$

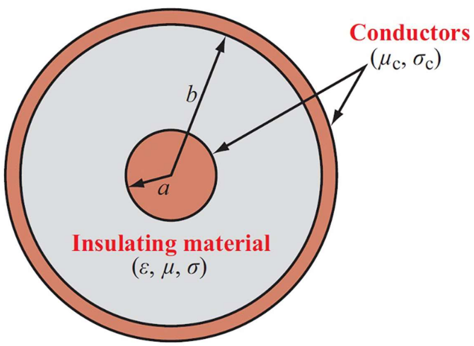
$$\frac{0.05m}{3x10^5} = 1.67x10^{-7} \ll 0.01$$

$$\frac{20\ km}{300\ km} = 0.067 > 0.01$$

Transmission Line Model



- R': The combined *resistance* of both conductors per unit length, in Ω/m,
- L': The combined *inductance* of both conductors per unit length, in H/m,
- C': The *capacitance* of the two conductors per unit length, in F/m.
- G': The *conductance* of the insulation medium between the two conductors per unit length, in S/m, and



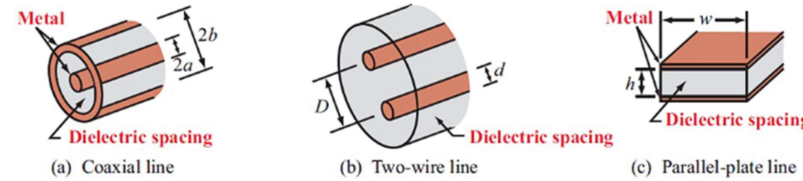


Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

Play around!
Interactive
Module 2.1-2.2

$$R_S = \sqrt{\pi f\, ^{\mu_C}/_{\sigma_C}}$$

μ_c = permittivity of dielectric, σ_c = conductivity of conductor (i.e. gold)

$$\sigma = 3.2 * 10^{-6}\ S/m$$

$$\varepsilon_0 = 8.854 * 10^{-12}\ F/m$$

$$\left.\begin{aligned} L'C' &= \mu\varepsilon \\ \frac{G'}{C'} &= \frac{\sigma}{\varepsilon} \end{aligned}\right\}\ \text{All TEM transmission lines}$$

Wave equations

$$\boxed{\begin{aligned} \frac{d^2\tilde{V}}{dz^2} - \gamma^2\tilde{V}(z) &= 0 \\ \frac{d^2\tilde{I}}{dz^2} - \gamma^2\tilde{I}(z) &= 0 \end{aligned}}$$

Complex propagation constant

$$\boxed{\begin{aligned} \gamma &= \sqrt{(R'+j\omega L')(G'+j\omega C')} \\ &= \alpha + j\beta \end{aligned}}$$

Air line: $\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12}$ F/m
$\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m
$\sigma = 0$
$G' = 0$

$\boxed{\alpha = \Re\mathrm{e}(\gamma)}$ Attenuation constant (Np/m)

$\boxed{\beta = \mathrm{Im}(\gamma)}$ Phase constant (rad/m)

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless ($R' = G' = 0$)	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (60/\sqrt{\varepsilon_r})\ln(b/a)$
Lossless two-wire	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120/\sqrt{\varepsilon_r}) \cdot \ln[(D/d) + \sqrt{(D/d)^2 - 1}]$ $Z_0 \simeq (120/\sqrt{\varepsilon_r})\ln(2D/d)$, if $D \gg d$
Lossless parallel-plate	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = (120\pi/\sqrt{\varepsilon_r})(h/w)$

$$u_p = \frac{\omega}{\beta} = f\lambda$$

Phase velocity — Guide wavelength

$$\lambda = \frac{2\pi u_p}{\omega}$$

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}}$

The Lossless Transmission Line

- Often in practice, R' and G' are negligible and we can simplify our wave solutions

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

$\gamma = \alpha + j\beta$ Lossless ($\alpha = 0$) $\longrightarrow$

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \qquad \tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

$$\alpha = 0, \beta = \omega\sqrt{L'C'} = \frac{\omega\sqrt{\varepsilon_r}}{c}, u_p = \frac{c}{\sqrt{\varepsilon_r}}, Z_0 = \sqrt{\frac{L'}{C'}}$$

Voltage Reflection Coefficient, Γ

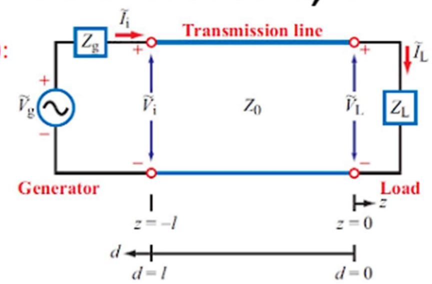
Define Γ (voltage reflection coefficient):

$$\Gamma = \frac{V_0^+}{V_0^-} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$z_L = \frac{Z_L}{Z_0}$$

Normalized load impedance

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

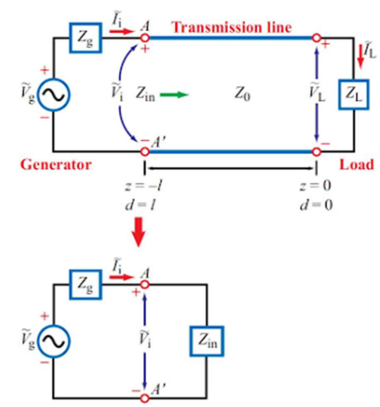


Input Impedance, Z_{in}

We can complete our solution for the lossless transmission line by looking at the input impedance, Z_{in}, the wave impedance $Z(d)$ at the source end of the line, $d = l$.

At input, $d = l$:

$$Z_{in} = Z_0 \left[\frac{z_L + j\tan\beta l}{1 + j z_L \tan\beta l}\right]$$



$z_L = Z_L/Z_0$, Tan(BL), sometimes take L == 1

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}$$

$$V_{input} = \frac{V_g Z_{input}}{Z_g + Z_i}$$

Solving for V_0^+: Forward Voltage

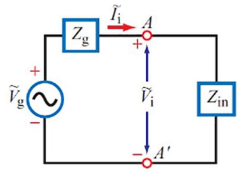
The phasor voltage across Z_{in} (using the voltage divider rule):

$$\tilde{V}_i = \tilde{I}_i Z_{in}$$

Using the equation for phasor voltage at *z=-l*

$$\tilde{V}(-l) = V_0^+\left[e^{j\beta l} + \Gamma e^{-j\beta l}\right] = \tilde{V}_i$$

Equivalent circuit:



We can find an equation to describe the forward voltage term, V_0^+, our last unknown!:

$$\boxed{V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left[\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right]}$$

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Wave Solutions: Summary

Full final equation for phasor voltage and phasor current on the lossless line:

$$\tilde{V}(z) = |V_0^+|e^{j\phi^+}\left[e^{-j\beta z} + |\Gamma|e^{j\theta_r}e^{j\beta z}\right]$$

$$\tilde{I}(z) = \frac{|V_0^+|e^{j\phi^+}}{|Z_0|e^{j\phi_z}}\left[e^{-j\beta z} - |\Gamma|e^{j\theta_r}e^{j\beta z}\right]$$

phasor solutions

Full final equation for instantaneous voltage and current on the line:

$$\begin{aligned} v(z) &= \mathrm{Re}\{\tilde{V}(z)e^{j\omega t}\} \\ &= |V_0^+|\{[\cos(\omega t - \beta z + \phi^+)] + |\Gamma|[\cos(\omega t + \beta z + \theta_r + \phi^-)]\} \\ i(z) &= \mathrm{Re}\{\tilde{I}(z)e^{j\omega t}\} \\ &= \frac{|V_0^+|}{|Z_0|}\{\cos(\omega t - \beta z + \phi^+ - \phi_z) + |\Gamma|\cos(\omega t + \beta z + \phi^- - \phi_z)\} \end{aligned}$$

time domain solutions

$$\boxed{V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left[\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right]}$$

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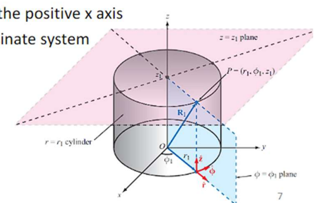
Cylindrical Coordinate System

$$\hat{x}, \hat{y}, \hat{z} \Leftrightarrow \hat{r}, \hat{\phi}, \hat{z}$$

- Coordinates:
  - r: radial distance in the *x-y* plane
  - ϕ: azimuthal angle measured from the positive x axis
  - z: same old z as the Cartesian coordinate system

Useful for solving problems involving structures with cylindrical symmetry (e.g. calculating *C'* of a coaxial cable)



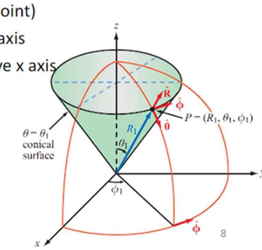
7

Spherical Coordinate System

$$\hat{x}, \hat{y}, \hat{z} \Leftrightarrow \hat{R}, \hat{\theta}, \hat{\phi}$$

- Coordinates:
  - R: range coordinate (distance from origin to point)
  - θ: zenith angle measured from the positive z axis
  - ϕ: azimuthal angle measured from the positive x axis

Useful for solving problems involving structures with spherical symmetry (e.g. calculating charge on a sphere)



8

Cylindrical: $Surface\ area = \iint r dr d\theta$

Spherical: $SA = \iint R^2 \sin(\theta)\, d\theta d\varphi$, $V = \iiint R^2 \sin(\theta)\, dR d\theta d\varphi$

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

$V = 4\pi r^2 dr$ $\left(= \frac{4}{3}\pi r^3\right)$ for sphere

Vector calculus: differential variables

Differential length vector

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

Differential area vectors

$$ds_x = \hat{x}dxdy$$

Differential volume vectors

$$dV = dxdydz$$

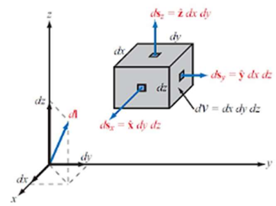


Table 3-1: Summary of vector relations.

Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$ds_x = \hat{\mathbf{x}}\,dy\,dz$ $ds_y = \hat{\mathbf{y}}\,dx\,dz$ $ds_z = \hat{\mathbf{z}}\,dx\,dy$	$ds_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $ds_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $ds_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$ds_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $ds_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $ds_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$ds_x = \hat{\mathbf{x}}\,dy\,dz$ $ds_y = \hat{\mathbf{y}}\,dx\,dz$ $ds_z = \hat{\mathbf{z}}\,dx\,dy$	$ds_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $ds_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $ds_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$ds_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $ds_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $ds_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Distance Between 2 Points

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \quad (3.66)$$

$$\begin{aligned} d &= \left[(r_2\cos\phi_2 - r_1\cos\phi_1)^2 + (r_2\sin\phi_2 - r_1\sin\phi_1)^2 + (z_2 - z_1)^2\right]^{1/2} \\ &= \left[r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2} \quad \text{(cylindrical)}. \quad (3.67) \end{aligned}$$

$$d = \left\{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2\cos\theta_1 + \sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1)]\right\}^{1/2} \quad \text{(spherical)}. \quad (3.68)$$

Gradient Operator (del), ∇

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

- The symbol ∇ is called the del or gradient operator
- It has no physical meaning by itself
- When it operates on a scalar quantity it attains physical meaning
- The result of the operation is a vector with:
  - magnitude equal to the maximum rate of change of the physical quantity per unit distance and
  - direction pointing in the direction of maximum increase

Divergence operator, $\nabla \cdot \vec{A}$

$$\nabla \cdot \vec{A} = \operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- When it operates on a vector quantity, the divergence results in a scalar quantity
- It can be considered as the dot product of del with a vector
- It is a measure of the quantity of outward flux from any point of the vector field
  - magnitude describes net flux, sign convention indicates source or sink of field

Electric Field Lines

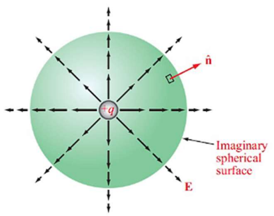


Figure 3-20: Flux lines of the electric field **E** due to a positive charge q.

- Coulomb's Law gives vector E-field around point charge:

$$\vec{E} = \hat{R}\frac{q}{4\pi\varepsilon_0 R^2}$$

- The flux density is amount of outward flux crossing a unit surface *ds*:

$$\text{Flux density of } \vec{E} = \frac{\vec{E}\cdot d\vec{s}}{|d\vec{s}|} = \frac{\vec{E}\cdot\hat{n}ds}{ds} = \vec{E}\cdot\hat{n}$$

- Total flux crossing closed surface *S*:

$$\text{Total flux } = \oint_S \vec{E}\cdot d\vec{s}$$

Divergence

- By shrinking the volume element (enclosed by the surface) to zero, we define the divergence of **E** at a point as the net outward flux per unit volume over a closed incremental surface:

$$\nabla\cdot\vec{E} = \lim_{\Delta V\to 0}\frac{\oint_S \vec{E}\cdot d\vec{s}}{\Delta V}$$

- Solving for this limit, yields

$$\nabla\cdot\vec{E} = \text{div }\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

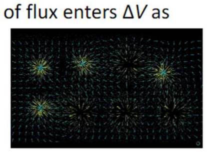
see derivation pgs. 153-155

Interpretation of the Divergence

- If the field **E** has positive divergence then the net flux out of surface *S* is positive, which can be viewed as if volume ΔV contains a source of field lines.
- If the field **E** has negative divergence then ΔV may be viewed as containing a sink of field lines because the net flux is into ΔV.
- If the field **E** is uniform the same amount of flux enters ΔV as leaves it, hence its divergence is zero.

Play around!
Interactive Module 3.3

Divergence visualized



Divergence Theorem

$$\int_V \nabla\cdot\vec{E}dV = \oint_S \vec{E}\cdot d\vec{s}$$

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa (e.g. Gauss's Law -> determine **E** given enclosed charge)

Curl operator, $\nabla\times\vec{A}$

Share Screen

$$\boxed{\nabla\times\vec{A} = \text{curl }\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}}$$

- When it operates on a vector quantity, the curl results in a vector quantity
- It can be considered as the cross product of del with a vector
- It is a measure of how much a field rotates about a given point of the vector field
  - magnitude is the circulation, per unit area, at a point
  - direction is determined by the right-hand rule applied to the associated path of integration

Curl

For a vector field: $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$

The curl is given by:

$$\nabla\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

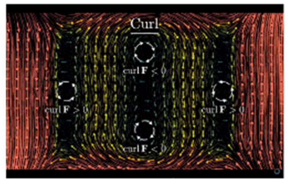
$$= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{\mathbf{z}}$$

Interpretation of the Curl

- The curl of a vector field describes its rotational property (circulation).
- Curl $\vec{B}$ is the circulation of $\vec{B}$ per unit area, with the area Δs of the contour C being oriented such that the circulation is maximum.
- E.g. curl of uniform field is zero and curl of magnetic field circulating a wire is non-zero
- Taking the curl of a vector field gives a vector which follows the right hand rule (points normal to the surface)

[Play around!](#)
Interactive
Module 3.4

[Curl visualized](#)



Stoke's Theorem

$$\int_S \nabla\times\vec{B}\cdot ds = \oint_C \vec{B}\cdot d\vec{l}$$

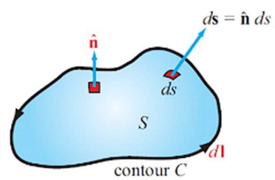


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along th thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

Useful tool for converting the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S
(e.g. Ampere's Law -> current through closed magnetic field loop)

see verification pgs. 161

[Gradient, dive](#)

Cartesian

(x, y, z): Scalar function F; Vector field $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

- gradient : $\nabla F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$
- divergence : $\nabla\cdot\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$
- curl : $\nabla\times\mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$
- Laplacian : $\Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$

Cylindrical

(r, θ, z): Scalar function F; Vector field $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$

- gradient : $\nabla F = \frac{\partial F}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial F}{\partial\theta}\mathbf{e}_\theta + \frac{\partial F}{\partial z}\mathbf{e}_z$
- divergence : $\nabla\cdot\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$
- curl : $\nabla\times\mathbf{f} = \left(\frac{1}{r}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial\theta}\right)\mathbf{e}_z$
- Laplacian : $\Delta F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} + \frac{\partial^2 F}{\partial z^2}$

Spherical

(ρ, θ, φ): Scalar function F; Vector field $\mathbf{f} = f_\rho\mathbf{e}_\rho + f_\theta\mathbf{e}_\theta + f_\varphi\mathbf{e}_\varphi$

- gradient : $\nabla F = \frac{\partial F}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}\frac{\partial F}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial\varphi}\mathbf{e}_\varphi$
- divergence : $\nabla\cdot\mathbf{f} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 f_\rho) + \frac{1}{\rho}\sin\varphi\frac{\partial f_\theta}{\partial\theta} + \frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi f_\theta)$
- curl : $\nabla\times\mathbf{f} = \frac{1}{\rho\sin\varphi}\left(\frac{\partial}{\partial\varphi}(\sin\varphi f_\theta) - \frac{\partial f_\varphi}{\partial\theta}\right)\mathbf{e}_\rho + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho f_\varphi) - \frac{\partial f_\rho}{\partial\varphi}\right)\mathbf{e}_\theta + \left(\frac{1}{\rho\sin\varphi}\frac{\partial f_\rho}{\partial\theta} - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\theta)\right)\mathbf{e}_\varphi$
- Laplacian : $\Delta F = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial F}{\partial\rho}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 F}{\partial\theta^2} + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial F}{\partial\varphi}\right)$

Coulomb's law → Find electric field given charge

Gauss's law → Find charge given a field

Coulomb's Law

Electric field at point P due to single charge

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}) = \hat{\boldsymbol{R}}\frac{q}{4\pi\varepsilon R^2} = \frac{q\vec{\boldsymbol{R}}}{4\pi\varepsilon\left|\vec{\boldsymbol{R}}\right|^3}} \quad \text{(V/m)}$$

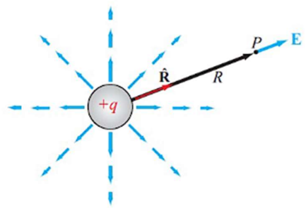


Figure 4-3: Electric-field lines due to a charge q.

Electric force on a test charge placed at P

$$\vec{\boldsymbol{F}} = q'\vec{\boldsymbol{E}} \quad \text{(N)}$$

Electric Field Due to Multiple Charges

Linear superposition

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}) = \frac{1}{4\pi\varepsilon}\sum_{i=0}^{N}\frac{q_i(\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}_i)}{\left|\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}_i\right|^3}} \quad \text{(V/m)}$$

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$
$$= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right]. \qquad (4.18)$$

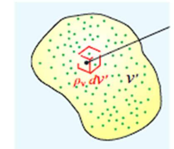
$$\boxed{\vec{\boldsymbol{E}} = \pm\hat{\boldsymbol{z}}\frac{\rho_s}{2\varepsilon}}$$

Infinite sheet of charge

Charge Distributions

Volume charge density:

$$\rho_v = \lim_{\Delta\mathcal{V}\to 0}\frac{\Delta q}{\Delta\mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3)$$



Total Charge in a Volume

$$Q = \int_{\mathcal{V}'}\rho_v(\vec{r}')\,d\mathcal{V}'$$

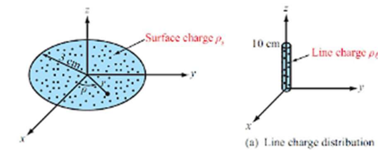
$x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}}$

$dx'dy'dz'$

Surface and Line Charge Densities

$$\rho_s = \lim_{\Delta s\to 0}\frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2) \qquad Q = \int_{s'}\rho_s\,ds'$$

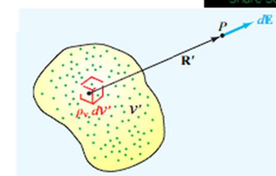
$$\rho_\ell = \lim_{\Delta l\to 0}\frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m}) \qquad Q = \int_{l'}\rho_l\,dl'$$



Electric Field Due to Charge Distributions

E-field at point P due to:

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}) = \int_{\mathcal{V}'}\frac{\rho_v(\vec{\boldsymbol{R}}')(\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}')}{4\pi\varepsilon\left|\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}_i\right|^3}d\mathcal{V}'}$$



Differential E-field

$$d\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}) = \hat{\boldsymbol{R}}\frac{dq}{4\pi\varepsilon R^2} = \frac{d\mathcal{V}'\rho_v(\vec{\boldsymbol{R}}')(\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}')}{4\pi\varepsilon\left[(\vec{\boldsymbol{R}}-\vec{\boldsymbol{R}}')^2\right]^{3/2}}$$

$$\vec{E}(\vec{R}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{R}')(\vec{R}-\vec{R}')}{4\pi\varepsilon|\vec{R}-\vec{R}_i|^3} d\mathcal{V}'$$ In a volume

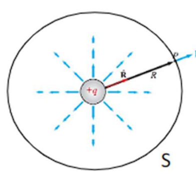
$$\vec{E}(\vec{R}) = \int_{s'} \frac{\rho_s(\vec{R}')(\vec{R}-\vec{R}')}{4\pi\varepsilon|\vec{R}-\vec{R}_i|^3} ds'$$ Over a surface

$$\vec{E}(\vec{R}) = \int_{l'} \frac{\rho_l(\vec{R}')(\vec{R}-\vec{R}')}{4\pi\varepsilon|\vec{R}-\vec{R}_i|^3} dl'$$ On a line

Gauss's Law (Integral Form)

- Coulomb's law → Find electric field given charge
- Gauss's law → Find charge given a field

Electric flux density $\vec{D} = \varepsilon\vec{E}$ (C/m²)



$$\vec{E}(@P) = \hat{R}\frac{q}{4\pi\varepsilon R^2}$$

$$\vec{D}(r,\theta,\phi) = \hat{R}\frac{q}{4\pi R^2}$$

$$\oint_S \vec{D}\cdot ds = q$$ Gauss's Law

ε - permittivity

$$\boxed{\oint_S \vec{D}\cdot ds' = q = \int_{\mathcal{V}'} \rho_v d\mathcal{V}'}$$

<u>**Gauss's law**</u> : a convenient method for <u>determining the electric flux density $\vec{D}$ (or field intensity $\vec{E} = {}^{\vec{D}}/_{\varepsilon_0}$) when charge distribution possesses symmetry properties</u>

Electric Scalar Potential

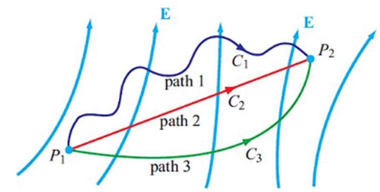
$$V = -\int_{l'} \vec{E}\cdot d\hat{l}'$$

- Voltage = voltage potential = electric scalar potential = V
- Voltage difference between <u>**2 points**</u> in a circuit represents the amount of work or potential energy required to move a unit of charge from one point to another

Electric field is conservative

<u>Potential difference</u> between P_1 & P_2

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E}\cdot d\hat{l}'$$



In electrostatics, for any closed path, the line integral is 0

$$\oint_C \vec{E}\cdot d\hat{l}' = 0$$

Electric Potential at a Point

- In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage.
- In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point P

"zero reference at infinity"

$$V = -\int_{\infty}^{\vec{P}} \vec{E}\cdot d\hat{l}'$$

$$\vec{P} = x\hat{x} + y\hat{y} + z\hat{z}$$

Voltage due to Point Charges

For a point charge, V at range R is:

$$V(\vec{\boldsymbol{R}}) = \frac{q}{4\pi\varepsilon|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}_i|}$$

$\vec{\boldsymbol{R}}$ is the position of observation
$\vec{\boldsymbol{R}}_i$ is the location of the charge

Because voltage is linear, superposition holds for multiple point charges:

$$V(\vec{\boldsymbol{R}}) = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}_i|}$$

Voltage Due to Charge Distributions

$$V(\vec{\boldsymbol{R}}) = \int_{v'} \frac{\rho_v(\vec{\boldsymbol{R}}')}{4\pi\varepsilon|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'|}\,dv'$$ In a volume

$$V(\vec{\boldsymbol{R}}) = \int_{s'} \frac{\rho_s(\vec{\boldsymbol{R}}')}{4\pi\varepsilon|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'|}\,ds'$$ Over a surface

$$V(\vec{\boldsymbol{R}}) = \int_{l'} \frac{\rho_l(\vec{\boldsymbol{R}}')}{4\pi\varepsilon|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'|}\,dl'$$ On a line

Electric Field $\vec{\boldsymbol{E}}$ as a function of V

$$\Delta V = V_B - V_A = \int_A^B dV = -\int_A^B \vec{\boldsymbol{E}} \cdot \mathrm{d}\hat{\boldsymbol{l}}'$$

$$\boxed{\vec{\boldsymbol{E}} = -\nabla V}$$

Dipoles

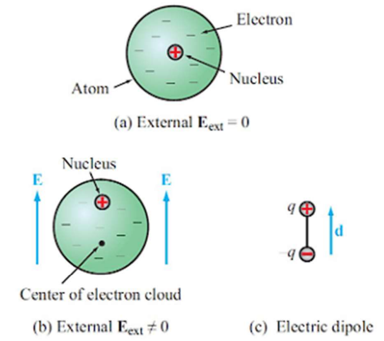


Figure 4-16: In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

- An electric dipole consists of two point charges of equal magnitude but opposite polarity, separated by a distance d.
- Applications: dielectrics, molecular bonds, antennas
- We want to determine V and **E** at any point P at a distance R from the dipole center, where $R >> d$ (in free space).

Potential and Field of Dipole

Electric dipole moment, $\vec{\mathbf{p}}$:

$$\boxed{\vec{\mathbf{p}} = q\vec{\mathbf{d}}}$$

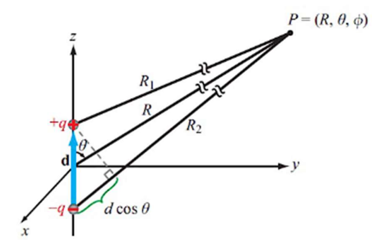
Potential:

$$V(\vec{\boldsymbol{R}}) = \frac{qd\cos\theta}{4\pi\varepsilon_o R^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\boldsymbol{R}}}{4\pi\varepsilon_o|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'|^2}$$

Electric field:

$$\vec{\mathrm{E}}(\vec{\boldsymbol{R}}) = \frac{qd}{4\pi\varepsilon_o|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'|^3}(\hat{\boldsymbol{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta)$$ V/m

Only when $R >> d$



(a) Electric dipole

Field of an Electric Dipole

- If you are very far from a localized charge distribution it "looks" like a point charge.

$$\vec{E}(\vec{R}) = \frac{q_{\text{tot}}(\vec{R} - \vec{R}')}{4\pi\varepsilon|\vec{R} - \vec{R}'|^3} \qquad V(\vec{R}) = \frac{q_{\text{tot}}}{4\pi\varepsilon|\vec{R} - \vec{R}'|}$$

- We will see that the potential and field at large distances depends almost entirely on a quantity called the dipole moment

Electric Field Inside Dielectric

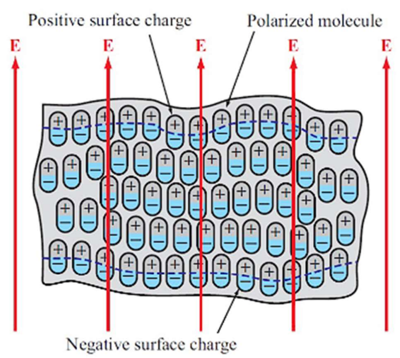


Figure 4-17: A dielectric medium polarized by an external electric field **E**.

- Each atomic or molecular dipole sets up a small electric field, pointing from the positively charged nucleus to the center of the equally but negatively charged electron cloud.
- This *induced electric field in the material*, or polarization field, $\vec{\mathbf{P}}$, is weaker than and opposite in direction to the applied electric field $\vec{\mathbf{E}}$.
- Result: the net electric field in the material is smaller than $\vec{\mathbf{E}}$.

Polarization, $\vec{\mathbf{P}}$

In free space: $\vec{D} = \varepsilon_o \vec{E}$

In a dielectric material: $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$

$\vec{P}$ = the electric flux density induced by the applied field $\vec{E}$

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

Where χ_e is the electric susceptibility (χ_e is unitless)

$$\vec{D} = \varepsilon_o \vec{E} + \varepsilon_o \chi_e \vec{E}$$

$$= \varepsilon_o \underbrace{(1 + \chi_e)}_{\varepsilon_r \text{ relative permittivity}} \vec{E}$$

Electric Breakdown

The dielectric strength E_{ds}: largest magnitude of $\vec{E}$ (i.e. charging) that can be sustained without breakdown

Table 4-2: Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Note: $\varepsilon_r \approx 1$ for most conductors

Current Density

Total current

$$\boxed{I = \int_S \vec{J} \cdot d\vec{s}}$$

Where current density $\boxed{\vec{J} = \rho_v \vec{u}}$ (A/m²)

ρ_v is charge density
($\rho_v = qN_e \rightarrow N_e$: number of charges per unit volume (m⁻³))

$\vec{u}$ is velocity

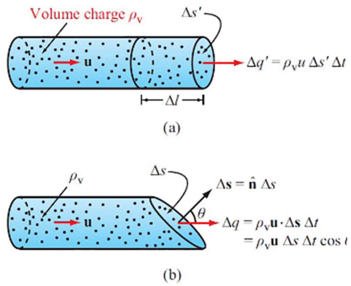


Figure 4-2: Charges with velocity **u** moving through a cros section $\Delta s'$ in (a) and Δs in (b).

Drift Velocity and Mobility

Drift Velocity, $\vec{u}$: Steady state average velocity of the electrons determined by the balance between the accelerating force of the applied electric field and the scattering effect of the collisions in the lattice.

Mobility, μ: The mobility accounts for the effective mass of a charged particle and the average distance over which the applied electric field can accelerate it before it is stopped by colliding with an atom and then starts accelerating all over again.

Conduction current density

Drift velocity of electrons, $\vec{\mathbf{u}}_e$:

$$\vec{\mathbf{u}}_e = -\mu_e \vec{\mathbf{E}} \qquad \text{(m/s)}$$

Drift velocity of holes, $\vec{\mathbf{u}}_h$:

$$\vec{\mathbf{u}}_h = \mu_h \vec{\mathbf{E}} \qquad \text{(m/s)}$$

Total conduction current density ($\vec{J} = \rho_v \vec{u}$):

$$\begin{aligned} \vec{\mathbf{J}} &= \vec{\mathbf{J}}_e + \vec{\mathbf{J}}_h \\ &= \rho_{ve} \vec{\mathbf{u}}_e + \rho_{vh} \vec{\mathbf{u}}_h \\ &= (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\vec{\mathbf{E}} \longrightarrow \vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \qquad \text{(A/m}^2\text{)} \end{aligned}$$

μ_e = electron mobility (m²/V·s)
μ_h = hole mobility (m²/V·s)
ρ_{ve} = volume charge density of electrons (C/m³)
ρ_{vh} = volume charge density of holes (C/m³)
N_e = number of electrons per unit volume (m⁻³)
N_h = number of holes per unit volume (m⁻³)

Conduction current density

The conductivity of a material is a measure of how easily electrons can travel through the material *under the influence of an externally applied electric field*.

Conduction current density:

$$\boxed{\vec{J} = \sigma \vec{E}} \qquad \text{A/m}^2 \qquad \text{(Ohm's Law)}$$

Perfect dielectric: $\sigma = 0$
Perfect conductor: $\sigma = \infty$

Units: S/m = Ω⁻¹m⁻¹ = A/Vm

Table 4-1: Conductivity of some common materials at 20°C

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Note how wide the range is, over 24 orders of magnitude!

Conductivity, σ

Semiconductor/dielectric

$$\begin{aligned} \sigma &= -\rho_{ve}\mu_e + \rho_{vh}\mu_h \qquad \text{(S/m)} \\ &= -N_e q \mu_e + N_h q \mu_h \end{aligned}$$

Conductor

$$\sigma = -\rho_{ve}\mu_e = eN_e\mu_e \qquad \text{(S/m)}$$

For a perfect dielectric,

$$N_e = 0, \sigma = 0 \longrightarrow \boxed{\vec{J} = 0}$$

For a perfect conductor,

$$\mu_e = \infty, \sigma = \infty, \vec{E} = \frac{\vec{J}}{\sigma} = 0 \longrightarrow \boxed{\vec{E} = 0}$$

μ_e = electron mobility (m²/V·s)
μ_h = hole mobility (m²/V·s)
ρ_{ve} = volume charge density of electrons (C/m³)
ρ_{vh} = volume charge density of holes (C/m³)
N_e = number of electrons per unit volume (m⁻³)
N_h = number of holes per unit volume (m⁻³)
e = charge on an electron (or hole)

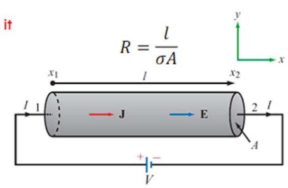
Resistance

Longitudinal Resistor with potential applied across it

$$R = \frac{V}{I} \qquad V = -\int_{l'} \vec{E} \cdot d\vec{l} = -\int_{x_2}^{x_1} \hat{x}E_x \cdot \hat{x}dl = E_x l$$

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_A \hat{x}\sigma E_x \cdot \hat{x}dA = \sigma E_x A$$

$$R = \frac{l}{\sigma A}$$



For any conductor:

$$\boxed{R = \frac{V}{I} = \frac{-\int_{l'} \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{-\int_{l'} \vec{E} \cdot d\vec{l}}{\int_S \sigma\vec{E} \cdot d\vec{s}}}$$

$$R = \frac{l}{\sigma A}$$

- The resistance R (and conductance G) of an element can be found by:
  - determining the potential across the element in terms of the applied electric field $V = -\int_{l'} \vec{E} \cdot d\vec{l}$
  - determining the current flowing through the cross section in terms of the current density and applying Ohm's law $I = \int_S \vec{J} \cdot d\vec{s} = \int_S \sigma\vec{E} \cdot d\vec{s}$
  - using the relationship *R = V/I (G=I/V)*

Electrostatic boundary conditions

From our general boundary conditions for electrostatic case:

Conservative property of $\vec{E}$ leads to continuous tangential component across a boundary

$$\nabla\times\vec{E} = 0 \qquad \oint_C \vec{E}\cdot d\vec{l} = 0 \longrightarrow \boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$

Divergent property of $\vec{D}$ leads to discontinuous normal component across a boundary

$$\nabla\cdot\vec{D} = \rho_v \qquad \oint_S \vec{D}\cdot d\vec{s} = Q \longrightarrow \boxed{\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s}$$

outside inside

If $\rho_s = 0, \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

$$if\ E = < x, y, z >, \qquad E_t = < x, y >, \qquad E_n = |z| = |E_2|\cos\theta$$

$$\vec{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y} + E_{1z}\hat{z}$$

Apply boundary conditions:

1) $E_{2t} = E_{1t}$ (*x* and *y* components equal)

2) $E_{2n} = \frac{\varepsilon_1}{\varepsilon_2}E_{1n}$ (no surface charge, ρ_s)

$$\vec{E}_2 = E_{2x}\hat{x} + E_{2y}\hat{y} + E_{2z}\hat{z}$$
$$= E_{1x}\hat{x} + E_{1y}\hat{y} + \frac{\varepsilon_1}{\varepsilon_2}E_{1z}\hat{z}$$

$$\frac{\tan\theta_2}{\varepsilon_2} = \frac{\tan\theta_1}{\varepsilon_1}$$

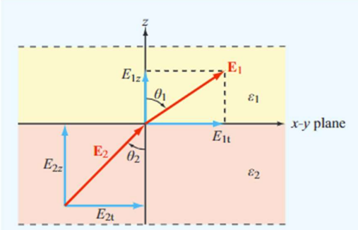


Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

See Example 4-10

Dielectric-Conductor Boundary

In a perfect conductor $\vec{E}_2 = \vec{D}_2 = 0$

In a very good conductor $\vec{E}_2 \approx \vec{D}_2 \approx 0$

Material 1: Dielectric

Material 2: Conductor

From our general boundary conditions:

Tangential components: $D_{1t} = E_{1t} = 0$

Normal components: $D_{1n} = \varepsilon_1 E_{1n} = \rho_s$

Combining the above:

$\vec{D}_1 = \varepsilon_1\vec{E}_1 = \rho_s\hat{n}$ (electric field points normal to surface at conductor surface)

Conductors

Net electric field $\vec{E}$ inside a conductor is zero

Net voltage V inside a conductor is constant

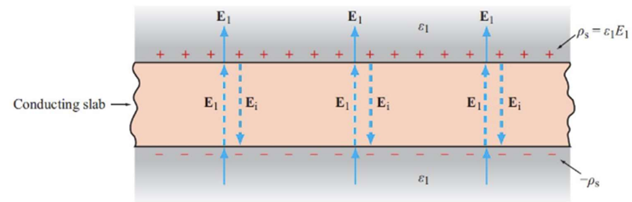
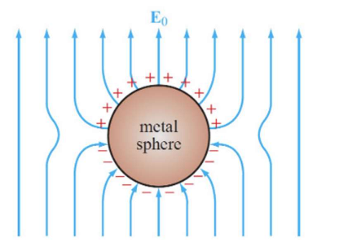


Figure 4-20: When a conducting slab is placed in an external electric field E_1, charges that accumulate on the conductor surfaces induce an internal electric field $E_i = -E_1$. Consequently, the total field inside the conductor is zero.

Field Lines at Conductor Boundary



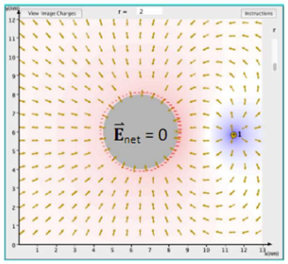


Figure 4-21: Metal sphere placed in an external electric field E_0.

At conductor boundary, $\vec{E}$ field direction is always perpendicular to conductor surface

Play around!
Interactive Module 4.2-4.4

(Semi)Conductor-(Semi)Conductor Boundary

From our general boundary conditions:

$\boxed{\vec{\mathbf{E}}_{1t} = \vec{\mathbf{E}}_{2t}}$ $\boxed{\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s}$

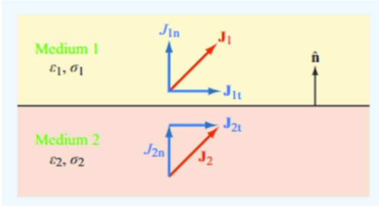


Figure 4-22 Boundary between two conducting media.

Since $\vec{J}_1 = \sigma_1 \vec{\mathbf{E}}_1$ and $\vec{J}_2 = \sigma_2 \vec{\mathbf{E}}_2$:

$$\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2} \rightarrow \vec{J}_{1t} = \frac{\sigma_1}{\sigma_2}\vec{J}_{2t}$$

Tangential currents flow parallel to boundary -> no tangential transfer of charge

$$\varepsilon_1 \frac{J_{1n}}{\sigma_1} - \varepsilon_2 \frac{J_{2n}}{\sigma_2} = \rho_s$$

$$J_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s \text{ (electrostatics)}$$

Normal components must satisfy conditions of electrostatics -> for constant ρ_s, charge arriving at boundary must equal charge leaving it, therefore $J_{1n} = J_{2n}$

Capacitance and Resistance

What if we don't have a perfect dielectric?

Answer: some current flows. For any resistor (see lecture 19):

$$R = \frac{V}{I} = \frac{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}{\int_S \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{s}}} = \frac{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}{\int_S \sigma \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{s}}}$$

If a dielectric material has uniform σ and ε,

$$RC = \frac{V}{I}\frac{Q}{V} = \frac{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}{\int_S \sigma \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{s}}} * \frac{\int_S \varepsilon \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{s}}}{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}$$

$$RC = \frac{\varepsilon}{\sigma}$$

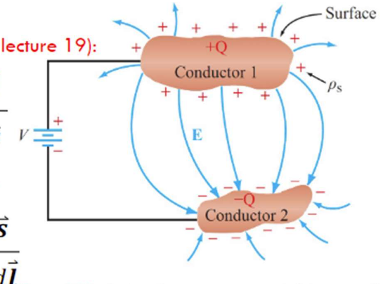


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

*RC time constant for charging or discharging the capacitor

Capacitance

For any two-conductor configuration:

$$C = \frac{Q}{V} \qquad Q = \int_S \varepsilon \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} \quad \leftarrow \text{Gauss's Law, } Q$$

$$V = -\int_{P_2}^{P_1} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} \quad \leftarrow \text{Potential, } V$$

$$\boxed{C = \frac{Q}{V} = \frac{\int_S \varepsilon \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}}{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}}$$

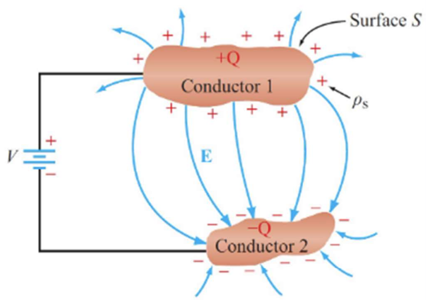
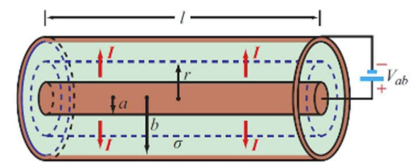


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Remember:

- C is positive
- S is the $+Q$ surface (Conductor 1)
- P_1 and P_2 are any points on Conductor 1 and Conductor 2, respectively

Coaxial Capacitor



$$C' = \frac{Q}{Vl} = \frac{2\pi\varepsilon}{\ln\left(b/a\right)} \quad \text{(F/m)}$$

-Q is total charge on inside of outer cylinder, and +Q is on outside surface of inner cylinder

Electrostatic Potential Energy

- Work done in piling up charge onto plates of capacitor.
- No charge flows and negligible dissipation as heat – where does energy go?
- Energy stored in the electric field of the capacitor

Energy stored in a capacitor

$$W_e = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \text{ (J)}$$

Energy Stored in Parallel Plate Capacitor

$C = \frac{\varepsilon A}{d}$

$V = Ed$

$$\Rightarrow W_e = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon E^2 (Ad)$$

Electrostatic potential energy density, w_e (Joules/volume):

$$w_e = \frac{1}{2}\varepsilon E^2 \qquad \text{(J/m}^3\text{)}$$

Valid for any medium containing an electric field $\vec{E}$

Total electrostatic energy stored in <u>any volume</u>:

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 d\mathcal{V} \qquad \text{(J)}$$

Electric and Magnetic Forces

Magnetic force

$$\vec{F}_m = q\vec{u} \times \vec{B} \qquad \text{(N)}$$

Electromagnetic (Lorentz) force

$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B}$$

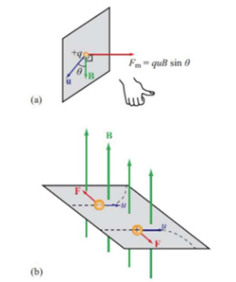


Figure 5-1: The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both B and u and (b) depends on the charge polarity (positive or negative).

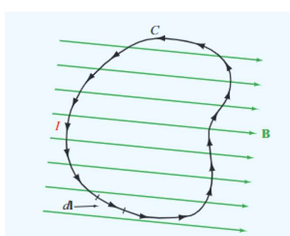
Magnetic Force on a Current Loop

If differential force $d\vec{F}_m$ on a differential length of wire $d\vec{l}$ with current I is:

$$d\vec{F}_m = I d\vec{l} \times \vec{B}$$

If $\vec{B}$ is uniform and constant

$$\boxed{\vec{F}_m = I \oint_C d\vec{l} \times \vec{B} = 0}$$



Total magnetic force on any <u>closed current loop in a uniform magnetic field</u> is zero.

Magnetic Field of a Loop

At height z, along the axis of the loop,

$$\vec{B} = \frac{\mu I a^2 \cdot N}{2(a^2 + z^2)^{3/2}}\hat{z}$$

At z = 0: $\vec{B} = \frac{\mu I}{2a}\hat{z} \cdot N$

At points far away: $\vec{B} = \frac{\mu I a^2}{2z^3}\hat{z}$

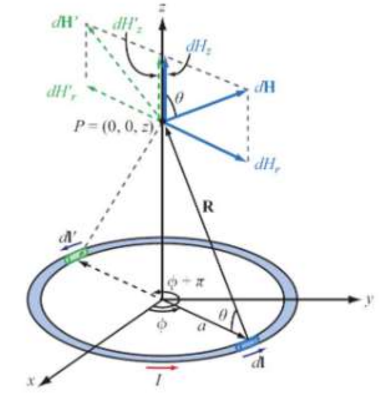


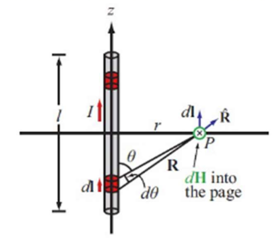
Figure 5-12: Circular loop carrying a current I (Example 5-3).

Magnetic Field of a Linear Conductor

$$\vec{B} = \frac{\mu I a}{2\pi r\sqrt{4r^2 + a^2}}\hat{\phi} \quad \text{(T)}$$

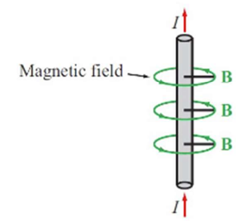
r = distance from center to point P

a = length of wire segment



For an infinitely long wire (length of wire>>r):

$$\boxed{\vec{B} = \frac{\mu I}{2\pi r}\hat{\phi}}$$



See Example 5-2

Forces on Parallel Conductors

Parallel wires *attract* if their currents are in the same direction, and *repel* if currents are in opposite directions

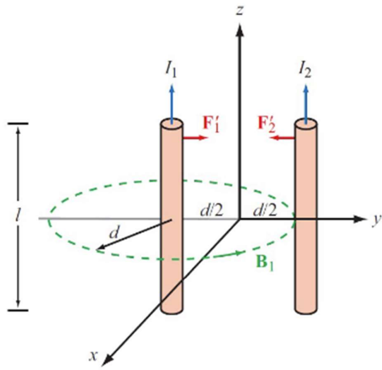
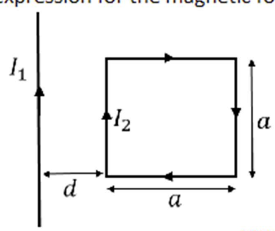


Figure 5-14: Magnetic forces on parallel current-carry conductors.

- The force between two parallel currents I_1 and I_2, separated by a distance r, has a magnitude per unit length given by

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

The wire on the left carries a current I_1, while the loop on the right carries a current of I_2. The loop is positioned a distance d from the wire, and has dimensions $a \times a$. Find a simplified expression for the magnetic force acting on the loop – and remember, this is a vector quantity.



$$\boldsymbol{F} = \hat{\boldsymbol{x}}\frac{\mu_0 I_1 I_2 a}{2\pi}\left(\frac{1}{d+a} - \frac{1}{d}\right) = -\hat{\boldsymbol{x}}\frac{\mu_0 I_1 I_2 a^2}{2\pi d(d+a)}$$

Magnetic Torque on an Inclined Current Loop

For a loop with N turns and whose surface normal is at angle θ relative to B direction:

$$|\vec{T}| = NIAB\sin\theta$$

Magnetic moment, m

$$\vec{m} = \hat{n}NIA$$

Right hand rule: four fingers curl in the circulation direction of current, thumb points in the surface normal $\hat{n}$ direction

$$\boxed{\vec{T} = \vec{m} \times \vec{B}} \quad \text{(N m)}$$

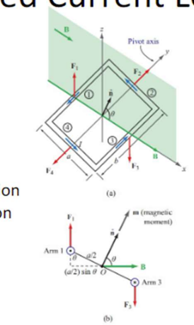


Figure 5-7: Rectangular loop in a uniform magnetic field with

Gauss's Law for Magnetism

For electricity (see lecture 16): $\boxed{\nabla \cdot \vec{D} = \rho_V(x,y,z) \longleftrightarrow \oint_S \vec{D} \cdot d\vec{s}' = Q}$

For magnetism: $\boxed{\nabla \cdot \vec{B} = 0 \longleftrightarrow \oint_S \vec{B} \cdot d\vec{s}' = 0}$

Gauss's Law for Magnetism, The Law of Nonexistence of Isolated Monopoles, The Law of Conservation of Magnetic Flux

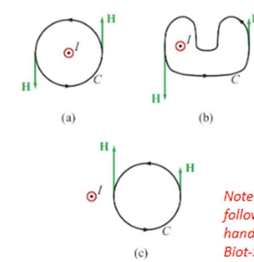
↑ Differential form

↑ Integral form

Ampere's Law in Words

Ampere's law states that the line integral of $\vec{\boldsymbol{H}}$ around a closed path is equal to the current I traversing the surface bounded by that path.

$$\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}} = I$$



Note: directions follow right-hand rule of Biot-Savart Law

Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol ⊙) is not enclosed by the contour C.

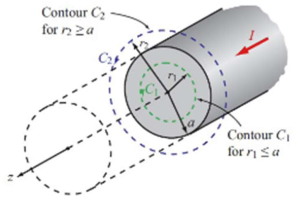
Example 1: H Field of Long Wire

For $r_1 \le a$

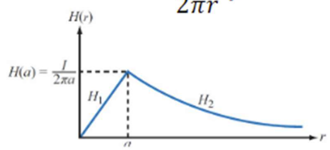
$$\vec{H} = H_1\hat{\boldsymbol{\phi}} = \frac{r_1 I}{2\pi a^2}\hat{\boldsymbol{\phi}}$$

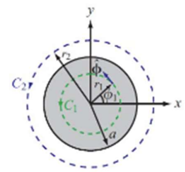
For $r_2 \ge a$

$$\vec{H} = H_2\hat{\boldsymbol{\phi}} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$$



(a) Cylindrical wire





(b) Wire cross section

Example 2: H Field inside Toroidal Coil

For $r < a$ $\quad \vec{H} = \mathbf{0}$

For $r > b$ $\quad \vec{H} = \mathbf{0}$

For $a < r < b$

$$\vec{H} = -H\hat{\boldsymbol{\phi}} = -\frac{NI}{2\pi r}\hat{\boldsymbol{\phi}}$$

(N loops)

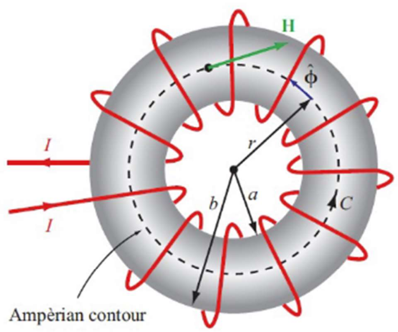


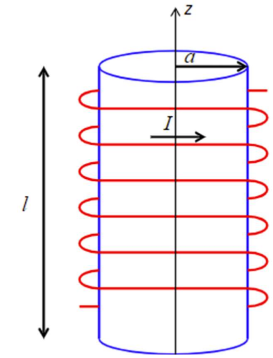
Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

Example 3: H Field inside Long Solenoid

For $r > a$ $\quad \vec{H} \approx \mathbf{0}$

For $r < a$ $\quad \vec{H} = \frac{NI}{l}\hat{\mathbf{z}}$

(N loops)



Example 4: H Field of Current Sheet

For $z > 0$ $\quad \vec{H} = -\frac{J_s}{2}\hat{\mathbf{y}}$

For $z < 0$ $\quad \vec{H} = \frac{J_s}{2}\hat{\mathbf{y}}$

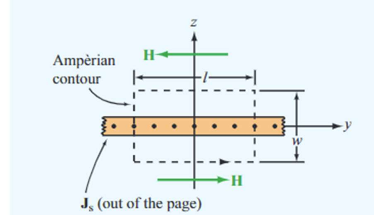


Figure 5-19 A thin current sheet in the x–y plane carrying a surface current density $\mathbf{J}_s = \hat{\mathbf{x}}J_s$ (Example 5-6).

Magnetic Vector Potential

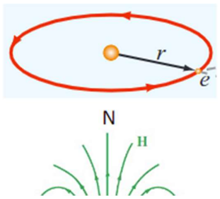
- Gauss's law for magnetism : $\quad \nabla \cdot \vec{B} = 0$
- Vector identity for any vector $\vec{V}$: $\quad \nabla \cdot (\nabla \times \vec{V}) = 0$
- We define a magnetic vector potential $\vec{A}$ such that:

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

$\vec{A}$ is in units of Wb/m

Magnetic Properties of Materials

Consider the planetary model of the atom:



Electrons circulating around the nucleus analogous to a current loop

For a loop with $N=1$ turn:
Magnetic moment, $\vec{m}$

$$\vec{m} = \hat{n} I A$$

Orbital Magnetic Moment

<u>Classical treatment:</u>

Current: $I = \dfrac{-e}{T} = \dfrac{-ev}{2\pi r}$

speed, v

period for one revolution, T

Angular momentum: $L_e = m_e v r$

Orbital magnetic moment, m_o:

$$m_o = IA = \left(\frac{-ev}{2\pi r}\right)(\pi r^2) = -\frac{eL_e}{2m_e}$$

<u>Quantum mechanics extension:</u>
L_e is always quantized,
i.e. $L_e = 0, \hbar, 2\hbar, \ldots,$
$\hbar = h/2\pi = 6.626\times10^{-34}/2\pi$ [Js]

Therefore, smallest possible m_o:

$$m_o = -\frac{e\hbar}{2m_e}$$

All electrons contribute, but for most materials are oriented randomly and they exhibit zero net orbital magnetic moment.

Polarization, $\vec{P}$ and Magnetization, $\vec{M}$,

Electric flux

In <u>free space</u>: $\vec{D} = \varepsilon_o \vec{E}$

In a <u>dielectric material</u>: $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$

$\vec{P}$ = the electric flux density induced by the applied field $\vec{E}$

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

Where χ_e is the electric susceptibility

$$\vec{D} = \varepsilon_o \vec{E} + \varepsilon_o \chi_e \vec{E}$$
$$= \varepsilon_o \underbrace{(1 + \chi_e)}_{\varepsilon_r} \vec{E}$$

relative permittivity

$$\vec{D} = \varepsilon \vec{E}$$

($\chi_{e,m}$ is unitless)

Magnetic flux

In <u>free space</u>: $\vec{B} = \mu_o \vec{H}$

In a <u>magnetic material</u>: $\vec{B} = \mu_o \vec{H} + \mu_o \vec{M}$

$\vec{M}$ = vector sum of magnetic dipole moments in medium

$$\vec{M} = \chi_m \vec{H}$$

Where χ_m is the magnetic susceptibility

$$\vec{B} = \mu_o \vec{H} + \mu_o \chi_m \vec{H}$$
$$= \mu_o \underbrace{(1 + \chi_m)}_{\mu_r} \vec{H}$$

relative permeability

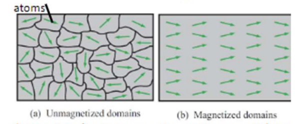
$$\vec{B} = \mu \vec{H}$$

Macroscopic Magnetic Properties

- diamagnetic, paramagnetic, or ferromagnetic.
  E.g. iron, nickel, cobalt

$\vec{H}$ applied:

- diamagnetic: each atom acquires small additional orbital dipole moment, *antiparallel* to $\vec{H}$
- paramagnetic: spin magnetic dipoles tend to align *parallel* to $\vec{H}$
- ferromagnetic: neighbouring atom spins aligned (*domains*), domains align to $\vec{H}$



domains (microscopic regions within which all the magnetic moments of the atoms are permanently aligned)

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis (see Fig. 5-22)
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

e.g. for pure iron $|\mu_r| \sim 2\times10^5$

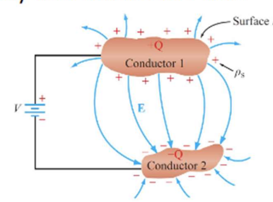
Inductance

- Remember that the capacitance for any two conductors was:

$$C = \frac{Q}{V} = \frac{\int_S \varepsilon \vec{E} \cdot d\vec{s}}{-\int_{l'} \vec{E} \cdot d\vec{l}}$$

← Gauss's Law, Q
← Potential, V

$$\boxed{C = \frac{Electric\ Flux}{Voltage}}$$



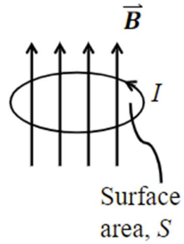
- The inductance for any two conductors

$$\boxed{L = \frac{Magnetic\ Flux}{Current}}$$

Magnetic Flux and Self-inductance

Total magnetic flux through a surface:

$$I=\int_S \vec{J}\cdot d\vec{s}=\int_S \sigma\vec{E}\cdot d\vec{s} \qquad \boxed{\Phi_m=\int_S \vec{B}\cdot d\vec{s}} \text{ (Wb)}$$



Self inductance :

$$\boxed{L=\frac{\Phi_m}{I}}=\frac{\int_S \vec{B}\cdot d\vec{s}}{\int_{l'} \vec{H}\cdot d\vec{l}} \quad \begin{matrix}\leftarrow \text{Flux, } \Phi_m \\ \leftarrow \text{Ampere's law, } I\end{matrix} \qquad \text{(H or Wb/A)}$$

The *self-inductance* represents the ratio of the *total flux through the structure* to the *current*.

Self-inductance in a Solenoid

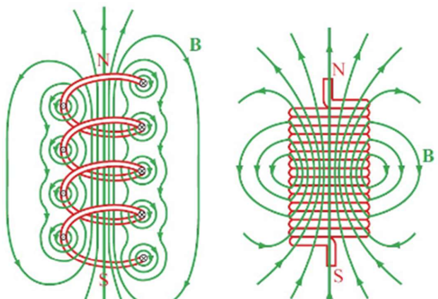
Outside a solenoid $\quad \vec{H}\approx \mathbf{0}$

Inside a solenoid When $l \gg a$ $\quad \vec{H}=\frac{NI}{l}\hat{z}$ (N loops)

For a single loop $\quad \boxed{\Phi_m=\frac{\mu NI}{l}S}$

For all loops $\quad \underbrace{N\Phi_m}_{\Lambda}=\frac{\mu N^2 I}{l}S$

$$\boxed{L=\frac{\mu N^2}{l}S}$$



S= area of one loop

(a) Loosely wound solenoid (b) Tightly wound solenoid

See section 5.7.1 for derivation and general case

Magnetostatic Potential Energy

Magnetostatic case:

- Work done in building up the current from 0 to I in the inductor
- Steady current and negligible dissipation as heat – where does energy go?
- Energy stored in the magnetic field of the inductor

Energy stored in an inductor: $\quad W_e=\int_0^I i v dt = L\int_0^I i di=\frac{1}{2}LI^2$ (J)

$$v=L\frac{di}{dt}$$

Energy Stored in Solenoid

$$L=\frac{\mu N^2 A}{l} \qquad I=\frac{Bl}{\mu N} \quad \Rightarrow \quad W_m=\frac{1}{2}LI^2=\frac{1}{2}\left(\frac{\mu N^2 A}{l}\right)\left(\frac{Bl}{\mu N}\right)^2=\frac{1}{2}\frac{B^2}{\mu}(lA)$$

Magnetic energy density, w_m *(Joules/volume):* $\quad w_m=\frac{1}{2}\mu H^2$ (J/m³)

Valid for any medium containing a magnetic field $\vec{H}$

Total magnetic energy stored in any volume: $\quad W_m=\frac{1}{2}\int_{\mathcal{V}}\mu H^2 d\mathcal{V}$ (J)

Faraday's Law

Remember L4!

Electromotive force (a voltage, V_{emf}) induced by time-varying magnetic flux:

$$V_{emf}=-N\frac{d\Phi_m}{dt}=-N\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \quad \text{(V)}$$

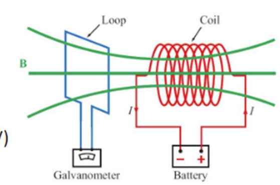


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

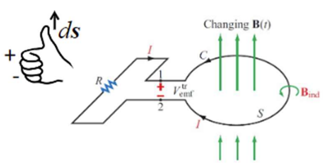
Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time. The key to the induction process is change.

3 Conditions for Generation of V_{emf}

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the *transformer emf*, $V^{\text{tr}}_{\text{emf}}$.
2. A moving loop with a time-varying surface area (relative to the normal component of $\vec{B}$) in a static field $\vec{B}$; the induced emf is then called the *motional emf*, $V^{\text{m}}_{\text{emf}}$.
3. A moving loop in a time-varying field **B**.
   The total emf is given by: $V_{\text{emf}} = V^{\text{tr}}_{\text{emf}} + V^{\text{m}}_{\text{emf}}$

Lenz's Law

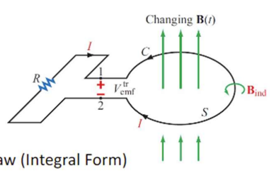
The direction of the current I is governed by Lenz's Law, which states that "the current in the loop is always in a direction that opposes the change of magnetic flux $\Phi(t)$ that produced I."



Relating $\vec{E}$ and $\vec{B}$

For contour C:

$$V^{tr}_{emf} = \oint_C \vec{E} \cdot d\vec{l}$$



$$\boxed{\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}}$$ Faraday's Law (Integral Form)

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Faraday's Law (Differential Form)

Applies in any point in space, not just where there's a circuit.

Motional EMF

In general, if any segment of a closed circuit with contour C moves with a velocity $\vec{u}$ across a static magnetic field $\vec{B}$, then the induced motional emf is given by

$$\boxed{V^{m}_{emf} = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}}$$

Only those segments of the circuit that cross magnetic field lines contribute to V^{m}_{emf}.

Take home points

- Moving conductor in a static field sets up motional EMF

$$V^{m}_{emf} = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

- Combining both transformer and motional EMF

$$V_{emf} = -\int_S \frac{d\vec{B}}{dt} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

- Can always use the general expression of Faraday's law for any EMF

$$V_{emf} = -N\frac{d\Phi_m}{dt} = -N\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

Mutual Inductance

Mutual inductance describes the magnetic coupling between two conducting structures (e.g. 2 current loops or 2 solenoids)

Current flows through first solenoid (N_1 turns)
No current flows through 2nd solenoid (N_2 turns)

Total Flux through solenoid 2 due to solenoid 1: $$N_2\Phi_{12} = N_2 \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

Mutual inductance: $$L_{12} = \frac{N_2}{I_1}\int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

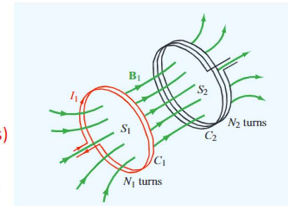


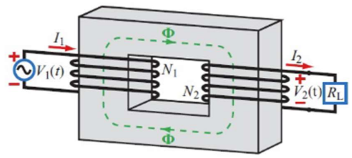
Figure 5-29 Magnetic field lines generated by current I_1 in loop 1 linking surface S_2 of loop 2.

Ideal Transformer

Assumptions:
Core has infinite permeability
Coupling is perfect – i.e. all flux in N1 is shared to N2 (flux confined to core)
No power is lost within the core

$$V_1^{tr} = -N_1 \frac{d\Phi_m}{dt}$$

$$V_2^{tr} = -N_2 \frac{d\Phi_m}{dt}$$



$$P_1 = I_1V_1 = P_2 = I_2V_2$$

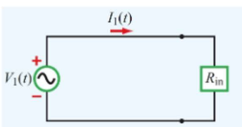
$$\boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}}$$

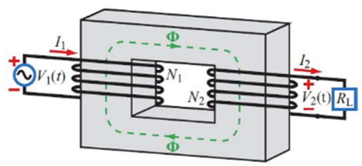
$N_2 > N_1$: "step-up" transformer
$N_2 < N_1$: "step-down" transformer

$$\boxed{\frac{I_1}{I_2} = \frac{N_2}{N_1}}$$

where N_1/N_2 = turns ratio

Transformers





$$R_{in} = \frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

$$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

Transformers make load impedances "look" bigger or smaller depending on their turns ratio.

EM Generator EMF

Apply Faraday's Law to find V_{emf}:

$$\vec{B} = B_0\hat{z} \qquad \Phi_m = \int_S \vec{B} \cdot d\vec{s}$$

$$= \int_S (B_0\hat{\mathbf{z}}) \cdot (ds\hat{\mathbf{n}})$$

$$= B_0 A \cos\alpha$$

$$= B_0 A \cos(\omega t + C_0)$$

$$V_{emf} = -\frac{d\Phi_m}{dt} = A\omega B_0 \sin(\omega t + C_0)$$

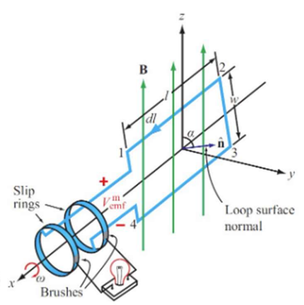
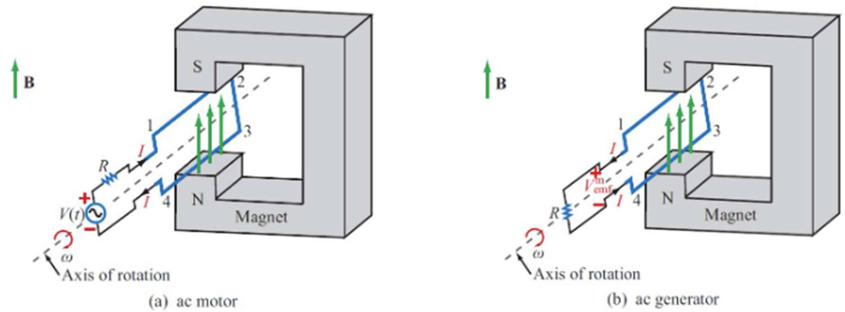


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Can also use motional emf equation to get same result: see p. 283-284

EM Motor/ Generator Reciprocity

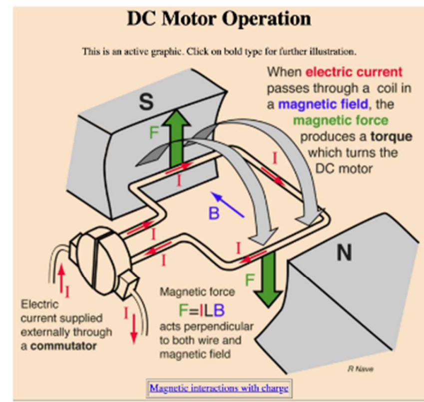


Motor: Electrical to mechanical energy conversion

Generator: Mechanical to electrical energy conversion

e.g. regenerative braking

DC motor



Hyperphysics – DC motor

Ampere's Law: Integral Form

Integrating both sides of Ampere's law over surface S with contour C:

$$\oint_C \vec{\boldsymbol{H}} \cdot d\boldsymbol{l}' \ ' = \int_S \vec{\boldsymbol{J}} \cdot d\boldsymbol{s}' + \int_S \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \cdot d\boldsymbol{s}'$$

$$\oint_C \vec{\boldsymbol{H}} \cdot d\boldsymbol{l}' = \int_S \nabla \times \vec{\boldsymbol{H}} \cdot d\boldsymbol{s}'$$

Stoke's law

$$\boxed{\oint_C \vec{\boldsymbol{H}} \cdot d\boldsymbol{l}' = I_C + I_D = I}$$

Conduction current

Displacement current

Total current

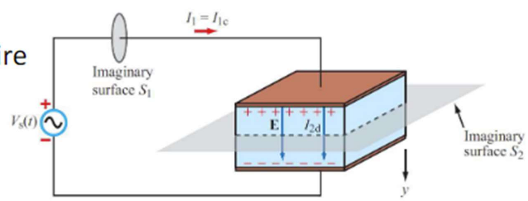
Capacitor in an AC circuit

In perfect conducting wire

$$I_1 = I_{1c} + I_{1d} = -CV_0 \omega \sin \omega t$$

In perfect capacitor

$$I_2 = I_{2c} + I_{2d} = -CV_0 \omega \sin \omega t = I_1$$



✓ Continuity of current flow through the circuit

Physical Interpretation of displacement current

- In the capacitor example, the displacement current does not carry free charges.
- Nonetheless, it behaves like a real current.
- The displacement current accounts for polarization in the medium.
- We considered a perfect wire with infinite conductivity. If it has finite conductivity, $\sigma \neq \infty$, then $\vec{\boldsymbol{D}}$ in the wire would be non-zero and I_1 would consist of both conduction and displacement currents. Likewise for I_2 when considering an imperfect dielectric, $\sigma \neq 0$, in the capacitor).

See Example 6-7 and Exercise 6-5

The Continuity Equation

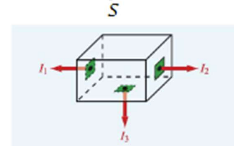
$$\nabla \cdot \vec{\boldsymbol{J}} = -\frac{\partial \rho}{\partial t}$$

Continuity equation (mathematical expression of conservation of charge)

For steady currents, the divergence of $\vec{\boldsymbol{J}}$ is zero

$$\oint_S \vec{\boldsymbol{J}} \cdot d\boldsymbol{s}' = 0$$

This is just a reformulation of Kirchoff's current law:



$$\oint_S \vec{\boldsymbol{J}} \cdot d\boldsymbol{s}' = \sum_i I_i = 0$$

Maxwell's Equations in Free Space

In free space: $\rho_V = 0$ and $\vec{J} = 0$ (no charges and no currents)

1) $\nabla \cdot \vec{D} = \rho_V(x,y,z) \longrightarrow \nabla \cdot \vec{D} = 0; \boxed{\nabla \cdot \vec{E} = 0}$

2) $\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

3) $\boxed{\nabla \cdot \vec{B} = 0}$

4) $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}; \boxed{\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$

Wave Equations in a Conductor

$$\boxed{\begin{aligned} \frac{\partial^2 \mathbf{E}}{\partial z^2} &= \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} \\ \frac{\partial^2 \mathbf{B}}{\partial z^2} &= \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t} \end{aligned}}$$

Solutions: $\vec{E} = E_0 e^{i(\omega t - \gamma z)}$

$\vec{B} = B_0 e^{i(\omega t - \gamma z)}$

Where γ is the complex propagation constant:

$$\gamma^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

Have	Want	Used	Eqn
$\rho_v(\vec{R}')$	$\vec{E}(\vec{R})$	Coulomb's law	$\vec{E}(\vec{R}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{R}')(\vec{R}-\vec{R}')}{4\pi\varepsilon\lvert\vec{R}-\vec{R}'\rvert^3} d\mathcal{V}'$
$\vec{E}(\vec{R})$	$\vec{D}(\vec{R})$	Material properties	$\vec{D} = \varepsilon\vec{E}$
$\vec{D}(\vec{R})$	$\rho_v(\vec{R}')$	Gauss's law	$\nabla \cdot \vec{D} = \rho_V(x,y,z)$
$V(\vec{R})$	$\vec{E}(\vec{R})$	Scalar potential function	$\vec{E} = -\nabla V$
$\vec{E}(\vec{R})$	$V(\vec{R})$	Line integral	$V = -\int_{l'} \vec{E} \cdot d\vec{l}'$
$\rho_v(\vec{R}')$	$V(\vec{R})$	Integral	$V(\vec{R}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{R}')}{4\pi\varepsilon\lvert\vec{R}-\vec{R}'\rvert} d\mathcal{V}'$

EM Toolbox

Electrostatics

Used	Eqn
Coulomb's law	$\vec{E}(\vec{R}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{R}')(\vec{R}-\vec{R}')}{4\pi\varepsilon\lvert\vec{R}-\vec{R}'\rvert^3} d\mathcal{V}'$
Material properties	$\vec{D} = \varepsilon\vec{E}$
Gauss's law	$\nabla \cdot \vec{D} = \rho_V(x,y,z)$, $\int_S \varepsilon\vec{E} \cdot d\vec{s} = \rho_V(x,y,z)$
Scalar potential function	$\vec{E} = -\nabla V$, $V(\vec{R}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{R}')}{4\pi\varepsilon\lvert\vec{R}-\vec{R}'\rvert} d\mathcal{V}'$
Energy density	$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 d\mathcal{V}$

Magnetostatics

Used	Eqn
Biot-Savart law	$\vec{B}(\vec{R}) = \int_{\mathcal{V}'} \frac{\mu\vec{J}(\vec{R}')\times(\vec{R}-\vec{R}')}{4\pi\lvert\vec{R}-\vec{R}'\rvert^3} d\mathcal{V}'$
Material properties	$\vec{B} = \mu\vec{H}$
Ampere's law	$\nabla \times \vec{H} = \vec{J}$, $\int_C \vec{H} \cdot dl' = I$
Vector potential function	$\vec{B} = \nabla \times \vec{A}$, $\vec{A}(\vec{R}) = \int_{\mathcal{V}'} \frac{\mu\vec{J}(\vec{R}')}{4\pi\lvert\vec{R}-\vec{R}'\rvert} d\mathcal{V}'$
Energy density	$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 d\mathcal{V}$

EM Constitutive Parameters of Materials

Parameter	Units	Free-Space Value	
Electrical permittivity, ε	F/m	$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m $\simeq 1/36\pi \times 10^{-9}$ F/m	Now (4.7)
Magnetic permeability, μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$ H/m	Ch. 5
Conductivity, σ	S/m	0	Next (4.6)

Electric and magnetic fields are connected through the speed of light:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

Resistance, Capacitance

For any resistor:

$$\boxed{R = \frac{V}{I} = \frac{-\int_{l'} \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{-\int_{l'} \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}}}$$

For any two-conductor configuration:

$$\boxed{C = \frac{Q}{V} = \frac{\int_S \varepsilon \vec{E} \cdot d\vec{s}}{-\int_{l'} \vec{E} \cdot d\vec{l}}}$$

Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Remember $\vec{\mathbf{E}} = 0$ in a good conductor

The Three Branches of Electricity & Magnetism

Branch	Condition	Field Quantities [Units]
Electrostatics	Stationary charges $\partial q / \partial t = 0$	Electric field intensity $\vec{E}$ [V/m] Electric flux density $\vec{D}$ [C/m²] $\vec{D} = \varepsilon \vec{E}$
Magnetostatics	Steady currents $\partial I / \partial t = 0$	Magnetic field intensity $\vec{H}$ [A/m] Magnetic flux density $\vec{B}$ [T] $\vec{B} = \mu \vec{H}$
Dynamics (Time-varying fields)	Time-varying currents $\partial I / \partial t \neq 0$	$\vec{E}$, $\vec{D}$, $\vec{H}$, and $\vec{B}$ ($\vec{E}$, $\vec{D}$) coupled to ($\vec{H}$, $\vec{B}$)

For most materials, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

Maxwell's Equations

	Differential Form	Integral Form	
1)	$\nabla \cdot \vec{\boldsymbol{D}} = \rho_V(x,y,z)$	$\oint_S \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}}' = Q$	Gauss's law
2)	$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$	$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}' = -\int_S \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot d\vec{\boldsymbol{s}}'$	Faraday's law (stationary surface S)
3)	$\nabla \cdot \vec{\boldsymbol{B}} = 0$	$\oint_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}}' = 0$	Gauss's law, magnetism (no magnetic charges)
4)	$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$	$\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}}' = \int_S \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{s}}' + \int_S \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \cdot d\vec{\boldsymbol{s}}'$	Ampere's law

Weird relationships:
$E = V/l$ (for a cable of finite length)
$E = V/d$ (for a parallel plate capacitor) (parallel capacitors have the same voltage)
$F = E * q$ (electric field force)