Name: ....
Student Number: ....

Software Engineering/Mechatronics 3DX4

DAY CLASS — Dr. Mark Lawford
DURATION OF EXAMINATION: 2.5 Hours
McMaster University Final Examination — April 2018

The following figure (Fig. 1) is the basis of many of the questions.

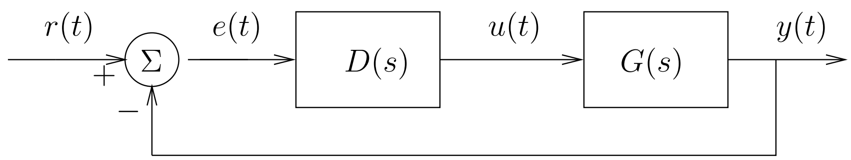
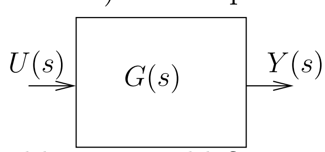


Figure 1: Closed loop system model

Here we can consider $G(s)$ as physical plant we want to control and $D(s)$ is the controller. In each question we will provide the transfer function for $G(S)$ and state or ask what type of control should be used for $D(s)$.

1. **Stability** (25 marks)

   a) (5 marks) Is the open loop plant

   where

   $$G(s) = \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600}$$

   stable or unstable? Justify your answer.

   Construct a Routh table to get:

   $$\begin{pmatrix} s^4 & 1 & 182 & 600 \\ s^3 & 23 & 580 & 0 \\ s^2 & \frac{3606}{23} & 600 & 0 \\ s & \frac{295680}{601} & 0 & 0 \\ 1 & 600 & 0 & 0 \end{pmatrix}$$

   Since all of the values in the first column are > 0 we know that all of the poles are in the Left Half Plane (LHP). Therefore $G(s)$ is stable.

   4 marks for constructing the Routh table (1 mark/row)
   1 mark for correctly interpreting it and saying the system is stable.

   b) (5 marks) Compute the closed loop transfer function of the system in Fig. 1 in the case when

   $$D(s) = K \quad \text{and} \quad G(s) = \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600}$$

$$\begin{aligned}
G_{cl}(s) &= \frac{D(s)G(s)}{1+D(s)G(s)} \\
&= \frac{K\frac{s+3}{s^4+23s^3+182s^2+580s+600}}{1+\left(\frac{s+3}{s^4+23s^3+182s^2+580s+600}\right)} \\
&= \frac{K\,(s+3)}{s^4+23s^3+182s^2+580s+600+K\,(s+3)} \\
&= \frac{K\,(s+3)}{s^4+23s^3+182s^2+(580+K)s+600+3K}
\end{aligned}$$

2 marks for 1st line
1 mark for each of the following lines roughly

c) (10 marks) For what values of K is the closed loop system stable?
Use Routh Hurwitz criteria on the denominator polynomial of the closed loop plant calculated in the previous part to determine the closed loop stability.
The first 3 rows give us:

$$\begin{bmatrix} s^4 & 1 & 182 & 3\,K+600 \\ s^3 & 23 & K+580 & 0 \\ s^2 & \frac{3606}{23}-\frac{K}{23} & 3\,K+600 & 0 \end{bmatrix}$$

Multiplying the 3rd row by 23 to simplify things gives:

$$\begin{bmatrix} s^4 & 1 & 182 & 3\,K+600 \\ s^3 & 23 & K+580 & 0 \\ s^2 & 3606-K & 69\,K+13800 & 0 \end{bmatrix}$$

Continuing on we get:

$$\begin{bmatrix} s^4 & 1 & 182 & 3\,K+600 \\ s^3 & 23 & K+580 & 0 \\ s^2 & 3606-K & 69\,K+13800 & 0 \\ s & \frac{-K^2+1439\,K+1774080}{3606-K} & 0 & 0 \\ 1 & 69\,K+13800 & 0 & 0 \end{bmatrix}$$

We need all first row elements to be > 0 for stability.

(i) From s^2 row we get $3606-K>0 \Rightarrow K<3606$

(ii) From s^1 row we get $-K^2+1439\,K+1774080>0$
Solving the quadratic equation for the case when

$$K^2-1439\,K-1774080=0$$

we get

$$\begin{aligned}
K &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-1439) \pm \sqrt{(-1439)^2 - 4(1)(-1774080)}}{2(1)} \\
&= \frac{1439}{2} \pm \frac{\sqrt{2070721 + 7096320}}{2} \\
&= \frac{1439}{2} \pm \frac{\sqrt{9167041}}{2} \\
&= 719.5 \pm 1513.9
\end{aligned}$$

or $K = -794.4$ or $K = 2233.4$ We know from the typical shape of such a parabola that it is greater than zero for $-794.4 < K < 2233.4$. We ignore the negative root in this case since we are considering the root locus for $K > 0$. Thus this row gives us $K < 2233.4$ for stability.

(iii) From s^0 row we get $69K + 13800 > 0 \Rightarrow K > \frac{-13800}{69}$

So we conclude $K < 2233.4$ for stability.

2 marks for knowing to use Routh-Hurwitz.

4 marks for Routh table (1 for each row)

3 mark for analysis in items (i)-(iii) above.

1 mark for final answer.

d) (5 marks) Calculate the $j\omega$-axis crossings (frequency). From your answer to (1c) you know the value of K where this occurs.

Two possible solutions:

Solution 1:

Take the solution of the polynomial corresponding to the s^2 row of the Routh table.

$$(3606 - K)s^2 + 69K + 13800 = 0$$

From the part (1c), we know that the system becomes unstable at

$$K = \frac{1439}{2} + \frac{\sqrt{9167041}}{2} \approx 2233.4$$

Substituting this into the s^2 row polynomial for K we get:

$$\begin{aligned}
(3606 - K)s^2 + 69K + 13800 &\approx (3606 - 2233.4)\, s^2 + (69(2233.4) + 13880) \\
&= 1372.6s^2 + 167900
\end{aligned}$$

Seeting this equation equal to zero and solving for the roots we get:

$$\begin{aligned}
1372.6s^2 + 167900 &= 0 \\
s^2 &= -\frac{167900}{1372.6)} \\
s = &= \pm j\sqrt{\frac{167900}{1372.6)}} \\
&\approx \pm j11.0598
\end{aligned}$$

2 marks for getting the polynomial right
1 mark for subbing correct value of K
1 mark for simplification
1 mark for final answer

Solution 2:

From the part (1c), we know that the system becomes stable at

$$K = \frac{1439}{2} + \frac{\sqrt{9167041}}{2} \approx 2233.4$$

Substituting this into the denominator polynomial for K we get:

$$\begin{aligned} D(s) &= s^4 + 23s^3 + 182s^2 + (K + 580)s + 3K + 600 \\ &= s^4 + 23s^3 + 182s^2 + (2233.4 + 580)s + 3(2233.4) + 600 \\ &= s^4 + 23s^3 + 182s^2 + 2813.4s + 7300.1 \end{aligned}$$

If we have imaginary roots we have poles $s = -j\omega$ and $s = j\omega$ and 3rd and 4th real valued poles at $s = -\sigma_3$ and $s = -\sigma_4$.

Substituting $s = j\omega$ gives us a polynomial of the form:

$$\begin{aligned} D(j\omega) &= (j\omega)^4 + 23(j\omega)^3 + 182(j\omega)^2 + 2813.4(j\omega) + 7300.1 \\ &= \omega^4 - 182\omega^2 + 7300.1 + j(2813.4\omega - 23\omega^3) = 0 + j0 \end{aligned}$$

This must equal $0 + j0$ since it is a closed loop pole so we have two equations to solve to get the value of ω and the same value of ω should work in both so we can use the second as a check.

Starting with the real part:

$$\omega^4 - 182\omega^2 + 7300.1 = 0$$

Let $\alpha = \omega^2$, then we want to solve

$$\begin{aligned} \alpha^2 - 182\alpha + 7300.1 &= 0 \\ \alpha &= \frac{182}{2} \pm \frac{1}{2}\sqrt{(-182)^2 - 4(1)(7300.1)} \\ &= 122.32 \text{ or } 59.68 \\ & \quad \text{But } \alpha = \omega^2 \text{ so,} \\ \omega^2 &= 122.32 \text{ or } 59.68 \\ \omega &= 11.06 \text{ or } 7.725 \end{aligned}$$

Checking with the imaginary part:

$$\begin{aligned} 2813.4\omega - 23\omega^3 &= 0 \\ (2813.4 - 23\omega^2)\omega &= 0 \\ \Rightarrow \omega^2 &= \frac{2813.4}{23} = 122.32 \\ \omega &= \pm\sqrt{} = \pm 11.06 \end{aligned}$$

Therefore the imaginary axis crossing occurs for $s = \pm j\omega = \pm j11.06$ since $\omega = 7.725$ does not make the imaginary part 0.
1 mark for stating $j\omega$ axis crossing occurs at

$$K = \frac{1439}{2} + \frac{\sqrt{9167041}}{2} \approx 2233.4$$

and hence that is where we get imaginary axis crossing.
1 mark for subbing in $K = 2233.4$ and simplifying to get polynomial
1 marks for getting to symbolic polynomial with ω and setting it to $0 + j0$
1 mark for simplifications
1 mark for final answer

2. **Root Locus, Time Response and Controller Design** (25 marks)

Consider Fig. 1 in the case when:

$$D(s) = K \;\text{ and }\; G(s) = \frac{s+3}{(s+2)(s+5)(s+6)(s+10)}$$

a) (5 marks) Sketch the root locus for the control system. You do not need to calculate the exact breakaway/breakin points.
So the open loop gain is:

$$D(s)G(s) == \frac{K(s+3)}{(s+2)(s+5)(s+6)(s+10)}$$

We have n=1 open loop zeros at:

$$s = -3$$

We have p=4 open loop poles at:

$$s = -2, -5, -6, -10$$

So we have $p - n = 3$ asymptotes going to $|s| = \infty$ as $K \to \infty$ at angles of:

$$\theta_k = \frac{(2k+1)\pi}{p-n} = \left(\frac{2k+1}{3}\right)\pi$$

i.e. 3 poles going to $s = -\infty$ at angles of $\pm\frac{pi}{3}$ and π.
The asymptotes intersect the real axis at:

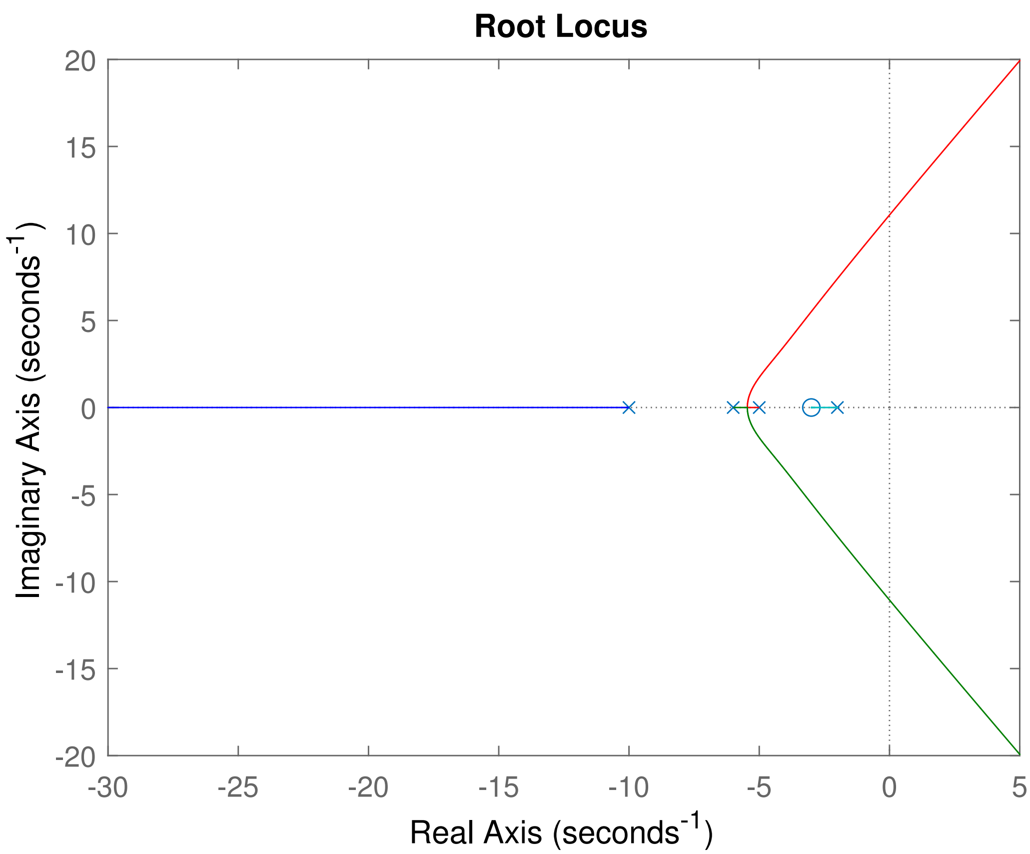
$$\begin{aligned}\sigma_a &= \frac{\Sigma\text{finite poles} - \Sigma\text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \\ &= \frac{(-2)+(-5)+(-6)+(-10)-(-3)}{4-1} \\ &= -\frac{20}{3}\end{aligned}$$

3 poles go to ∞ approaching the 3 asymptotes. The remaining pole winds up at $s = -3$ as $K \to \infty$.

Since we know that there must be an odd number of poles and zeros to the right on the imaginary axis we know that the locus occupies

$$s \in (-3, -2] \text{ and } s \in [-5, -6] \text{ and } s \in (-\infty, -20)$$

Putting all this information together with the $j\omega$ axis crossing values from (1d) we get the picture below.



1 marks for identifying open loop poles and zeros

2 marks for correct asymptotes

2 marks for final correct sketch (1 for real axis, 1 for the rest).

b) (5 marks) For $K = 430$ the system is stable and the closed loop poles are approximately located at

$$s = -2.7 \pm j6.1, -2.9, -14.7$$

Assuming that the closed loop pole at $s = -2.9$ cancels the system zero at $s = -3$, what is the damping ratio of the dominant poles? Assuming ideal 2nd order behaviour, what do you expect the resulting percent overshoot to be for the system?

In this case for the pole location $s = -\sigma_d \pm j\omega_d = -2.7 \pm j6.1$ we have

$$\begin{aligned}
\zeta &= \frac{\sigma_d}{\sqrt{\sigma_d^2 + \omega_d^2}} \\
&= \cos(\tan^{-1}(\frac{\omega_d}{\sigma_d})) \\
&= \cos(\tan^{-1}(\frac{6.1}{2.7})) = 0.405
\end{aligned}$$

$$
\begin{aligned}
\%\text{OS} &= e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 \\
&= e^{-\left(0.405\pi/\sqrt{1-(0.405)^2}\right)} \times 100 \\
&= 24.8\%
\end{aligned}
$$

2 marks for calculation method to obtain ζ (there's more than one way to do this!)

1 mark for correct ζ answer

1 marks for formula for % overshoot

1 marks for correct % overshoot answer

c) (5 marks) Your manager said that she is happy with % overshoot of your controller design for $K = 430$, but she would like you to design a PD compensator to cut the settling time in half while keeping the same % overshoot. Assuming ideal 2nd order behaviour, what would be the new location for the dominant poles?

Let $s = -\sigma \pm j\omega = -2.7 \pm j12.2$ be the current pole location for a proportional controller with $K = 430$.

Let $s_d = -\sigma_d \pm j\omega_d$ be the new, desired pole location. We want the new settling time T_s^{new} to be $\frac{1}{2}$ of its current value T_s

$$
\begin{aligned}
T_s &\approx \frac{4}{\zeta\omega_n} \\
&= \frac{4}{\sigma} \\
T_s^{new} &= \frac{T_s}{2} \\
&= \frac{4}{2\sigma} \\
&= \frac{4}{\sigma_d}
\end{aligned}
$$

So now we know $\sigma_d = 2\sigma = 2(2.7) = 5.4$.

Now we have to find the value of ω_d.

For the same 24.8% overshoot we want the same value of $\zeta = \cos(\tan^{-1}\frac{\omega}{\sigma}))$. This means keeping the same angle $\theta = \cos^{-1}(\zeta)$ from the origin, since radial lines from the origin represent lines of constant damping ratio. Thus if $\sigma_d = 2\sigma$ we must have $\omega_d = 2\omega = 2(6.1) = 12.2$.

Alternatively

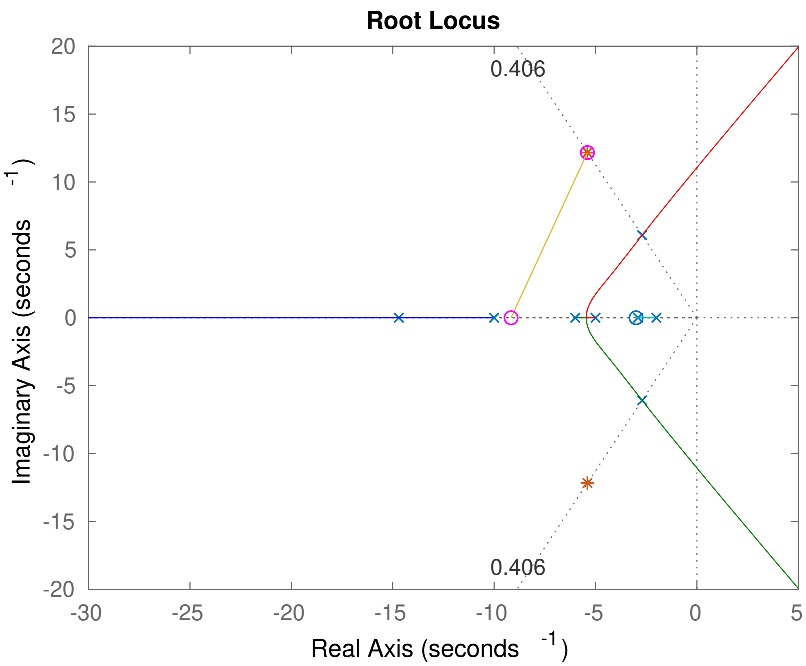
$$
\begin{aligned}
\omega_d &= \sigma_d \tan(\theta) \\
&= \sigma_d \left(\frac{\omega}{\sigma}\right) \\
&= 2\cancel{\sigma}\left(\frac{\omega}{\cancel{\sigma}}\right) \\
&= 2\omega = 2(6.1) = 12.2
\end{aligned}
$$

so the new pole location for the dominant poles is:

$$s = -\sigma_d \pm j\omega_d = -5.4 \pm j12.2$$

1 mark for equation for equation for T_s o
2 marks for reasoning/equation to get new ω_d
1 mark for correct σ_d value
1 mark for correct ω_d value and resulting new dominant pole location

d) (5 marks) Design a PD compensator to place the dominant poles at the location you computed in part (2c).



$$
\begin{aligned}
G(s) &= = \frac{s+3}{(s+2)(s+5)(s+6)(s+10)} \\
G(-5.4+j12.2) &= (-1.4894 + j4.8255i) \times 10^{-4} \\
&= 5.0501 \times 10^{-4} \angle 1.8702 rad \\
&= 5.05 \times 10^{-4} \angle 107.15^\circ
\end{aligned}
$$

Want to find zero location $-z_c$ for PD controller $D(s) = K(s+z_c)$ such that

$$
\begin{aligned}
\pi &= \angle(D(-\sigma_d + \jmath\omega_d) + G(-\sigma_d + \jmath\omega_d)) \\
&= \angle D(-\sigma_d + \jmath\omega_d) + \angle G(-\sigma_d + \jmath\omega_d) \\
\angle D(-\sigma_d + j\omega_d) &= \pi - \angle G(-\sigma_d + \jmath\omega_d) \\
\angle D(-5.4 + j12.2) &= \pi - \angle G(-5.4 + j12.2) \\
&= \pi - 1.87 \\
&= 1.27 \text{ rad}
\end{aligned}
$$

But

$$\begin{aligned}
D(-\sigma_d + j\omega_d) &= z_c - \sigma_d + j\omega_d \\
\tan(\angle D(-\sigma_d + j\omega_d)) &= \frac{\omega_d}{z_c - \sigma_d} \\
z_c - \sigma_d &= \frac{\omega_d}{\tan(\angle D(-\sigma_d + j\omega_d))} \\
z_c &= \frac{\omega_d}{\tan(\angle D(-\sigma_d + j\omega_d))} + \sigma_d \\
&= \frac{12.2}{\tan(1.2714)} + 5.4 \\
&= 9.1656
\end{aligned}$$

To find the gain K of the controller we know that for the desired pole location $s_d = -5.4 + j12.2$ to be on the root locus (i.e. a pole of the closed loop system $\frac{KD(s)G(s)}{1+KD(s)G(s)}$)

$$\begin{aligned}
1 + KD(s_d)G(s_d) &= 0 \\
&\text{So} \\
K &= \frac{-1}{D(s_d)G(s_d)} \\
&= \frac{1}{|D(-5.4 + j12.2))||G(-5.4 + j12.2)|} \\
&= \frac{1}{|-5.4 + 9.1656 - j12.2||5.05 \times 10^{-4}|} \\
&= \frac{1}{(12.8)(5.05 \times 10^{-4})} \\
&= 154.26
\end{aligned}$$

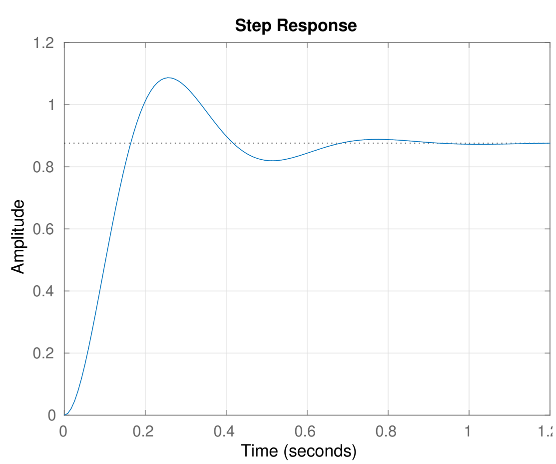
1 mark knowing PD controller form $D(s) = K(s + z_c)$

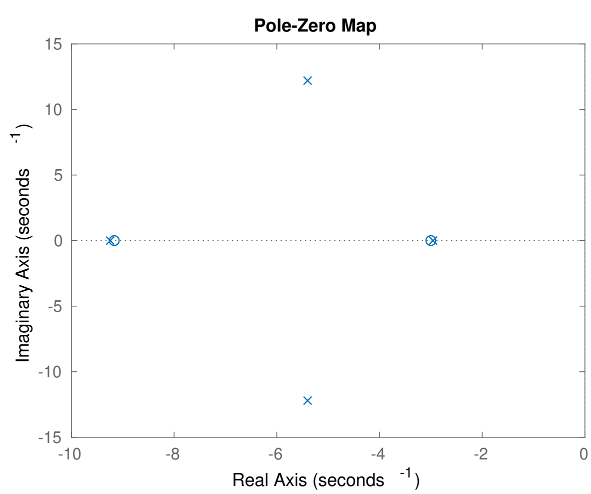
2 mark for computing $\angle D(-5.2098 + j15.59)$

1 mark for computing z_c

1 mark for gain

e) (5 marks) The actual step response of the compensated closed loop system (i.e. $D(s) = K(s + z_c)$) is shown below on the left and the closed loop pole zero map is shown on the right.





Does the percent overshoot and settling time appear to correspond to the ideal second order behaviour your manager wanted for the compensated system? Does the closed loop pole zero map indicate that this should be the case?

Yes

Plant zero at -3 and the closed loop pole just to the right of -3 seem to be cancelling.

Also, the compensator zero at -9.1656 is almost cancelling the closed loop pole that started at the open loop pole of -10 so the effects of the higher order pole and zero are probably cancelling out.

Also, looking at the step response, you can sort of eyeball the final value at about 0.87 and the peak value at about 1.08. This would give us a % overshoot of approximately:

$$\begin{aligned}
\%\ overshoot &= \frac{y(t_{peak}) - y(\infty)}{y(\infty)} \times 100\% \\
&\approx \frac{1.08 - 0.87}{0.87} \times 100\% \\
&= 24.14\%
\end{aligned}$$

The originial (uncompenstaed) settling time would have been approximately $T_s = \frac{4}{\sigma} = \frac{4}{2.7} = 1.48$.

Looking at the step response you could eyeball the 2% settling time to be about $0.8 \approx \frac{1.48}{2} = 0.74$ so "Looking good Lewis!"

You would validate this be doing a step response simulation of $G_{cl}(s)$ and a step response simulation of the second order system:

$$\begin{aligned}
G_{ideal}(s) &= |G_{cl}(0)| \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{3(430)}{3(430) + 600} \left(\frac{|s_d|^2}{(s - s_d)(s - s_d')} \right. \\
&= \frac{4.738}{s^2 + 1.446s + 4.738}
\end{aligned}$$

Plotting both of these in matlab shows virtually no difference until you zoom in substantially - Good approximation to the 2nd order system as we suspected!

1 mark for calculating approximate % overshoot and/or settling time

2 marks for saying it should be a good a second order approximation

2 mark for talking about closed loop poles cancelling zeros,

3. **Steady State Error** (15 marks)

Assume that

$$G(s) = \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600} = \frac{s+3}{(s+2)(s+5)(s+6)(s+10)}$$

in Fig.1.

a) (5 marks) Assuming the feedback configuration shown in Fig. 1 and $D(s) = K$ is chosen so that the closed loop system is stable, what is the type of the system? What is its static error constant for a given gain K? What is the steady state error of the closed loop system in response to a unit step input?

$$D(s)G(s) = K\frac{(s+3)}{s^4+23s^3+182s^2+580s+600}$$

System is type 0.

$$\begin{aligned} K_p &= \lim_{s\to 0} D(s)G(s) \\ &= \frac{K(s+3)}{s^4+23s^3+182s^2+580s+600}|_{s=0} \\ &= \frac{3K}{600} \\ &= \frac{K}{200} \end{aligned}$$

Steady state error due to a step is:

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\frac{K}{200}} = \frac{200}{200+K}$$

So for $K = 430$ we have $e_{step}(\infty) = \frac{200}{200+430} = \frac{20}{63} \approx 0.3175$

1 for correct type

1 for K_p position constant formula

1 for correct value of K_p

1 for steady state error formula

1 for correct value of steady state error.

NOTE: Do not have to sub in $K = 430$ for full marks.

b) (5 marks) The system requires the elimination of the steady state error and robustness with respect to step disturbances. You colleague (who graduated from McMaster) suggests changing the controller from $D(s) = K$ to an integral controller $D(s) = \frac{K}{s}$. Is this a good choice? Does it eliminate the steady state error and still have the same % overshoot for an appropriate choice of K? What might have changed? Justify your answer.

First we need to consider if the system would be stable some values of $K > 0$.

How do we know this?

Try sketching a root locus or doing a Routh Table. Root locus is probably the quicker option.

So the open loop gain is:

$$D(s)G(s) = \frac{K(s+3)}{s(s+2)(s+5)(s+6)(s+10)}$$

We have n=1 open loop zeros at:

$$s = -3$$

We have p=5 open loop poles at:

$$s = 0, -2, -5, -6, -10$$

So we have $p - n = 4$ asymptotes going to $|s| = \infty$ as $K \to \infty$ at angles of:

$$\theta_a = \frac{(2k+1)\pi}{p-n} = \left(\frac{2k+1}{4}\right)\pi$$

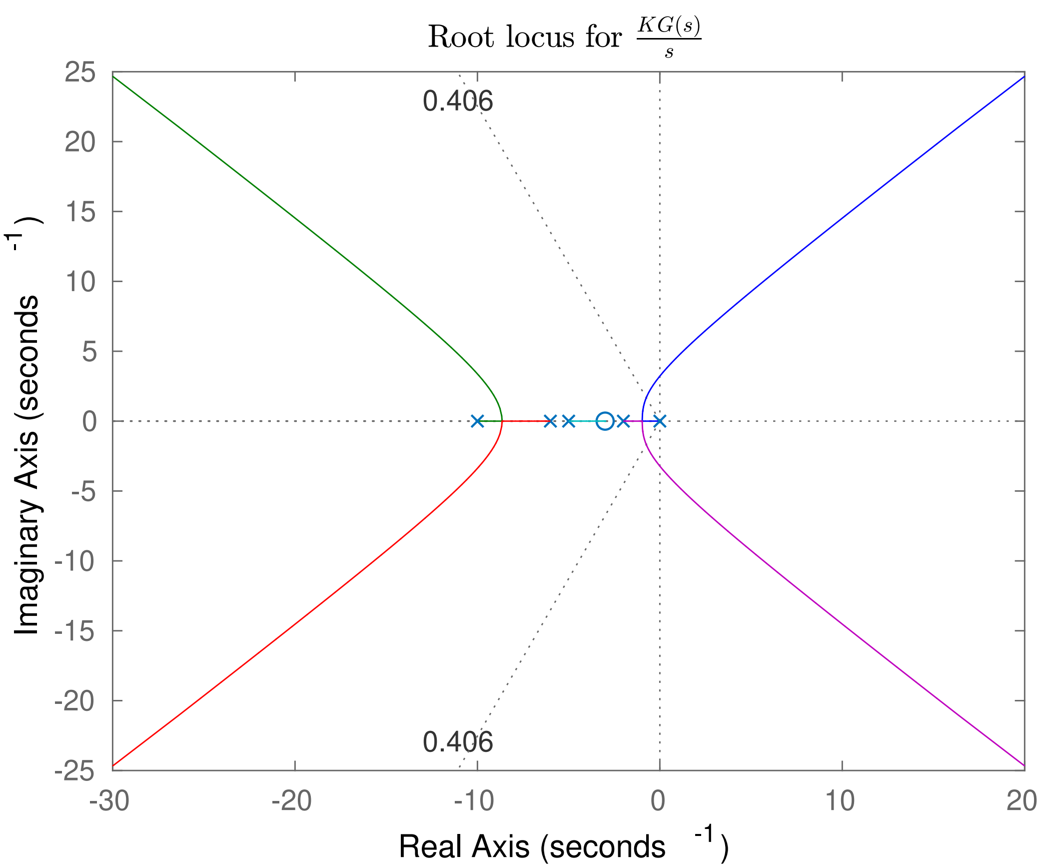
i.e. 4 poles going to $s = -\infty$ at angles of $\pm\frac{pi}{4}$ and $\pm\frac{5\pi}{4}$.
The asymptotes intersect the real axis at:

$$\begin{aligned}\sigma_a &= \frac{\Sigma\text{finite poles} - \Sigma\text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \\ &= \frac{(-2)+(-5)+(-6)+(-10)-(-3)}{5-1} \\ &= -\frac{20}{4} = 5\end{aligned}$$

4 poles go to ∞ approaching the 4 asymptotes. The remaining pole winds up at $s = -3$ as $K \to \infty$.
Since we know that there must be an odd number of poles and zeros to the right on the imaginary axis we know that the locus occupies

$$s \in [-2, -0] \text{ and } s \in [-5, -3) \text{ and } s \in [-6, -10]$$

Putting all this information together with the $j\omega$ axis crossing values from (1d) we get the picture below.



Similar to the proportional controller, it is stable for K sufficiently small.
In this case $D(s)G(s) = \frac{K}{s}G(s) = \frac{K(s+3)}{s(s^4+23s^3+182s^2+580s+600)}$ is type 1.

$$\begin{aligned} K_p &= \lim_{s\to 0} D(s)G(s) \\ &= \frac{K(s+3)}{s\left(s^4+23s^3+182s^2+580s+600\right)}\Big|_{s=0} \\ &= \infty \end{aligned}$$

So

$$e_{step}(\infty) = \frac{1}{1+K_p} = 0$$

Therefore it tracks steps with 0 error.

It looks like it could be a good choice since it is possible for the dominant poles to hit all values of $\zeta \in [0,1]$, including $\zeta = 0.405$ so we should be able to get the same % overshoot, but the higher order poles and plant zero might cause some problems.

Also, the real part of the dominant poles is now smaller so the settling time T_s might be larger than desired.

1 marks for knowing that you need check stability.

2 marks for stability justification either by a root locus or a Routh table.

1 mark for saying zero steady state error.

1 mark for remaining discussion.

c) (5 marks) For the control $D(s) = \frac{K}{s}$ with the plant $G(s)$ as above, what type of input (step, ramp, parabola) now results in a steady state error? What is the static error constant of the system and control in this case?

As discussed above

$$D(s)G(s) = \frac{K(s+3)}{s(s+2)(s+5)(s+6)(s+10)} = \frac{K(s+3)}{s\left(s^4+23s^3+182s^2+580s+600\right)}$$

is a type 1 system so it tracks steps with zero error and tracks ramps with a constant error. The error constant in this case is:

$$\begin{aligned} K_v &= \lim_{s\to 0} sD(s)G(s) \\ &= \lim_{s\to 0} \cancel{s}\left(\frac{K(s+3)}{\cancel{s}\left(s^4+23s^3+182s^2+580s+600\right)}\right) \\ &= \frac{3K}{600} = \frac{K}{200} \end{aligned}$$

1 marks for realizing that you need to find the type of $D(s)G(s)$ not just $G(s)$

1 mark for saying it's type 1

1 mark for saying it therefore tracks ramps with a constant error

1 marks for formula for $K_v$tion that now we get a stable type 1 system (for appropriately chosen K) so 0 error

1 marks for final value of K_v

4. **State Space Control** (20 marks)

   Assume that the plant is given by:

$$G(s) = \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600}$$

   a) (5 marks) Given $G(s)$ what is the controller canonical (phase variable) state space representation of the plant $G(s)$?

   Consider the following general 4$^{\text{th}}$ order transfer function:

$$G(s) = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$$

   The state-space Controller Canonical (Phase Variable) Form for the transfer function is:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \mathbf{x}(t)
\end{aligned}$$

   In our case we have:

$$\begin{aligned}
G(s) &= \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600} \\
&= \frac{0s^3 + 0s^2 + s + 3}{s^4 + 23s^3 + 182s^2 + 580s + 600}
\end{aligned}$$

   Substituting into the above we get:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -600 & -580 & -182 & -23 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} 3 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)
\end{aligned}$$

   2 mark for the form

   3 marks for correct A, B, C

   b) (5 marks) A state feedback control $u = -\mathbf{K}\mathbf{x}$ is to be used to:

   - place one of the closed loop poles is to be placed at $s = -3$ to exactly cancel the system zero,
   - place the dominant poles at $s = -5.4 \pm j12.2$ to get the desired % overshoot and settling time, and
   - place the final pole at $s = -54$ so it won't interfere.

What would the desired characteristic equation of the designed closed loop system be?

$$
\begin{aligned}
(s+5.4-j12.2)(s+5.4+j12.2)(s+3)(s+54) &= (s^2+10.8s+178)(s^2+57s+162) \\
&= s^4+67.8s^3+955.6s^2+11895.6s+28836
\end{aligned}
$$

2 marks for LHS of first equation

2 marks for simplification

1 mark for final correct answer

c) (5 marks) Compute a state feedback control matrix K to place the poles at the desired locations.

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \\
u &= -\mathbf{K}\mathbf{x}+r \\
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}+\mathbf{B}(-\mathbf{K}\mathbf{x}+r) \\
\dot{\mathbf{x}}(t) &= (\mathbf{A}-\mathbf{B}\mathbf{K})\mathbf{x}+\mathbf{B}r) \\
\mathbf{K} &= \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \\
\dot{\mathbf{x}}(t) &= \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -600 & -580 & -182 & -23 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \right) \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{r}(t) \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(600+k_1) & -(580+k_2) & -(182+k_3) & -(23+k_4) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{r}(t)
\end{aligned}
$$

Desired closed loop transfer function is

$$
\begin{aligned}
G(s) &= \frac{b_3 s^3_+ b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \\
&= \frac{0s^3+0s^2+s+3}{s^4+67.8s^3+955.6s^2+11895.6s+5090}
\end{aligned}
$$

This would have a state equation

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}_{\mathbf{cl}}\mathbf{x}+\mathbf{B}\mathbf{r} \\
\dot{\mathbf{x}}(t) &= (\mathbf{A}-\mathbf{B}\mathbf{K})\mathbf{x}+\mathbf{B}r \\
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -28836 & -11895.6 & -955.6 & -67.8 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{r}(t)
\end{aligned}
$$

Equating $\mathbf{A}-\mathbf{BK}=\mathbf{A_{cl}}$ we get

$$\begin{aligned}
-(600+k_1) &= -28836 \Rightarrow k_1 = 28236\\
-(580+k_2) &= -11895.6 \Rightarrow k_2 = 11315.6\\
-(182+k_3) &= -955.6 \Rightarrow k_3 = 773.6\\
-(23+k_4) &= -67.8 \Rightarrow k_4 = 44.8\\
\mathbf{K} &= \begin{bmatrix} 28236 & 11315.6 & 773.6 & 44.8 \end{bmatrix}
\end{aligned}$$

1 mark for getting dimension of $\mathbf{K}$ right

2 mark for setting up $\mathbf{A}-\mathbf{BK}$ to solve for k_i's

2 marks for correct $\mathbf{K}$

d) (5 marks) Is the state space representation from part (4a) observable so that we can implement our state feedback controller given our current sensor outputs? Justify your answer.

The *observability matrix*, $\mathbf{O}_M$, for a state space system is

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{C\,A} \\ \vdots \\ \mathbf{C\,A}^{n-1} \end{bmatrix} \tag{1}$$

In this case $n=4$ so we need to find

$$\begin{aligned}
\mathbf{O}_M &= \begin{bmatrix} \mathbf{C} \\ \mathbf{C\,A} \\ \mathbf{C\,A}^2 \\ \mathbf{C\,A}^3 \end{bmatrix}\\
&= \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ -600 & -580 & -182 & -20 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{O}_M &= \det \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ -600 & -580 & -182 & -20 \end{bmatrix}\\
&= -42 \neq 0
\end{aligned}$$

Therefore $\mathbf{O}_M$ has full rank the the state space representation from part (4a) is observable.

2 marks for correct formula for $\mathbf{O}_M$

2 marks for correctly computing $\mathbf{O}_M$

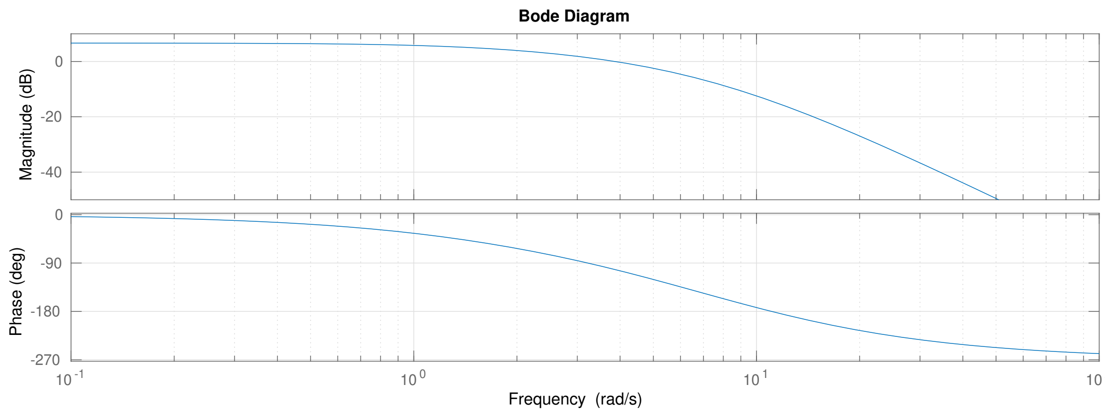
1 mark for checking rank and concluding system is observable.

5. **Bode Plot Margins, Bandwidth & Digital Control** (15 marks)

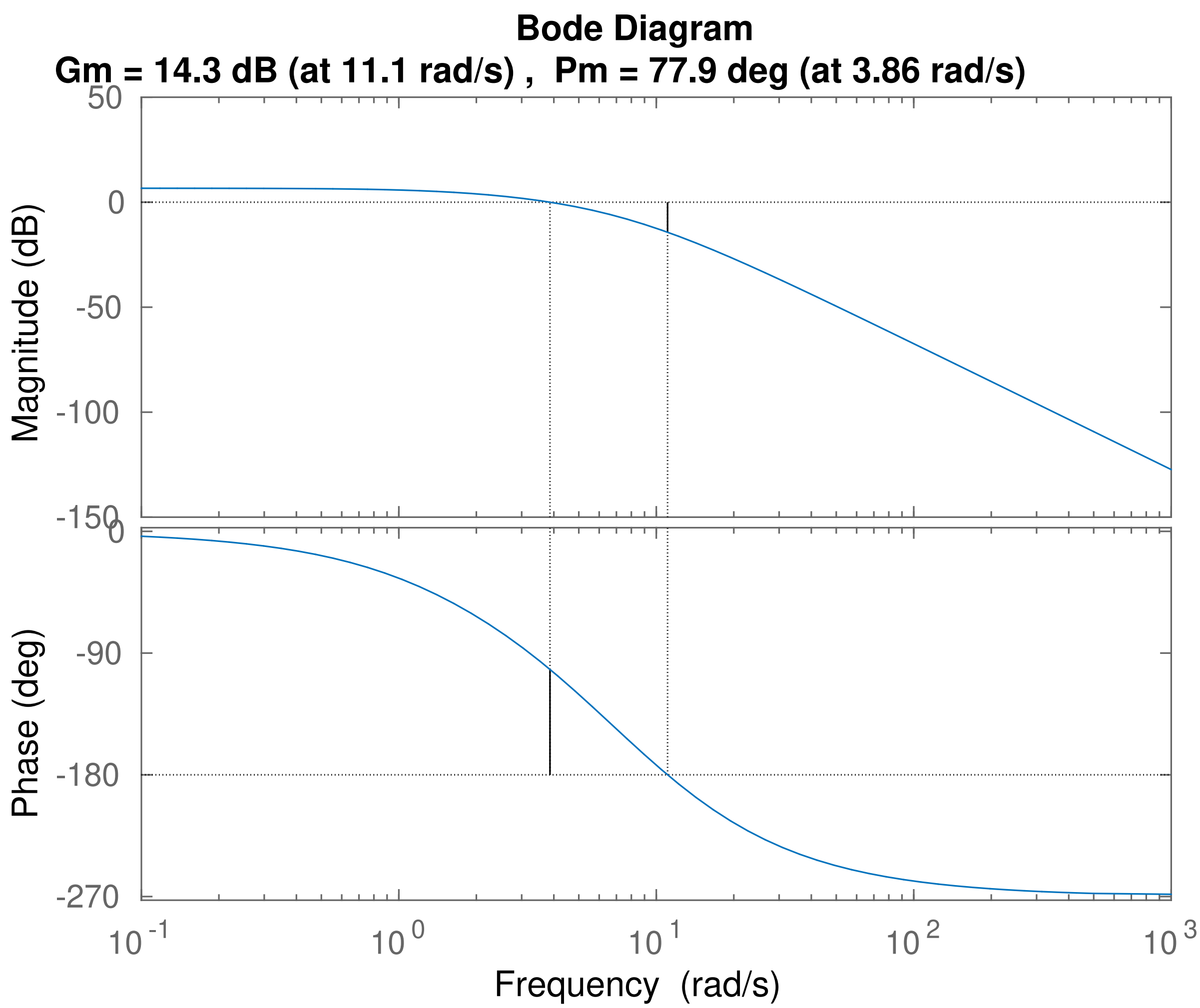
The open loop bode plot of $D(s)G(s)$ for:

$$D(s) = K = 430, G(s) = \frac{s+3}{s^4 + 23s^3 + 182s^2 + 580s + 600}$$

is shown below:



a) (5 marks) What is the gain margin for the system? What is the phase margin? How much could the gain change by before the systems would be unstable? (Note your answer to (1c) might help with this question.)



Gain margin

$$G_M = 0 - M\text{dB}$$

where M is the magnitude of the open loop transfer function when the phase is an odd multiple of 180°.

In this case the odd multiple is $-180°$

In this case

$$G_M = 0 - (-14.3) = 14.3\text{dB}$$

at 11.1 rad/s

For the phase margin the 0 dB magnitude point occurs at 3.86 rad/s. Here the phase is $\phi = -102.1°$ so

$$\begin{aligned} P_M &= |\phi - (-180°)| \\ &= -102.1° + 180° \\ &= 77.9° \end{aligned}$$

The gain margin is 14.3 dB so the gain could be increased by a factor of X times where

$$\begin{aligned} 14.3 \text{ dB} &= 20\log_{10}(X) \\ \log_{10}(X) &= \frac{14.3}{20} \\ X &= 10^{\frac{14.3}{20}} \\ &= 5.19 \end{aligned}$$

Note that we currently have $K = 430$ and we know that the system becomes unstable at

$$\begin{aligned} K_{stable} &= 2233.4 \\ \frac{K}{K_{stable}} &= \frac{2233.4}{430} \\ &= 5.19 \end{aligned}$$

Coincidence? I think not!

1 mark for gain formula

1 mark for approximately correct gain (they are reading it off a graph!)

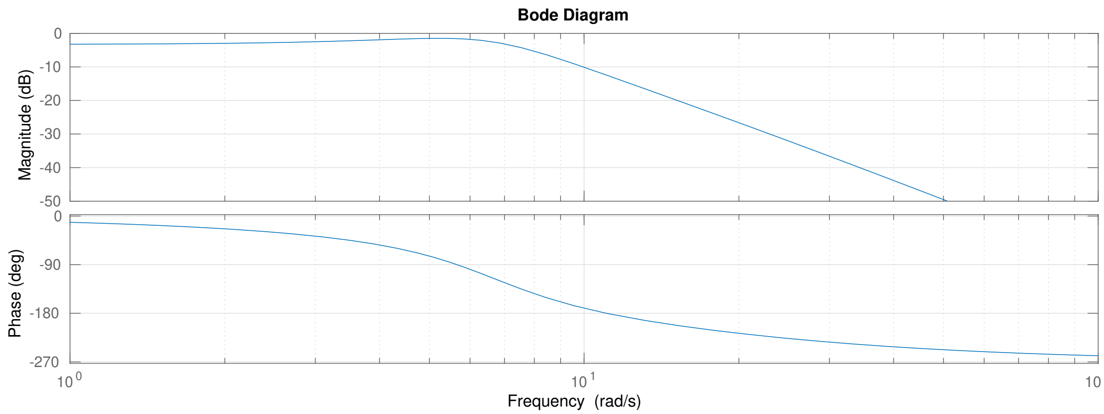
1 mark for phase margin formula

1 mark for approximately correct phase margin

1 mark for figuring out the factor

NOTE: Students are eyeballing it off of the graph, not running the matlab `margin(K*G)` command so it is expected that their answers are approximately correct, not the exact values above.

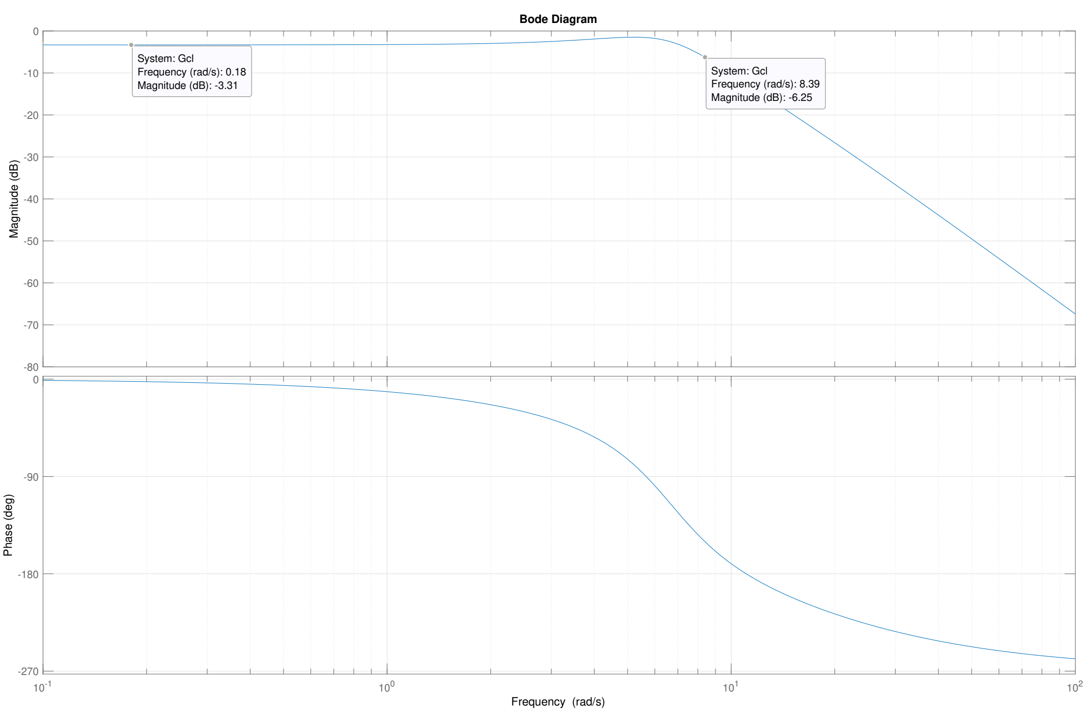
b) (5 marks) The Bode plot for the closed loop system with $D(s) = K = 430$ is shown below.



What is the bandwidth of the closed loop control system? What is the minimum sampling frequency that could be used to implement a digital version of the system?

The bandwidth of the system in this case we'll take to be the frequency that is -3 dB less than the magnitude of the close loop system at low frequency which is around -3dB so we want to find the place where it is -6 dB.

In this case it occurs between 8 and 9 rad/s.



It actually occurs at $\omega_{BW} = 8.42$ rad/s

The Nyquist frequency $\frac{f_s}{2}$ must be larger tha the system bandwidth. This will give you the minimum sampling frequency in theory (in practice you want to typically sample be 5 to 10 times faster than the system bandwidth).

$$\begin{aligned}
\omega_s &> 2\omega_{BW} \\
&> 2(8.42) = 16.84 \text{ rad/s} \\
&= 16.84\text{rad/s} \times \frac{1}{2\pi} Hz/rad \\
&= 2.68 \text{ Hz or samples per second}
\end{aligned}$$

2 marks for knowing to look for -3 dB frequency

1 marks for relatively correct estimate of ω_{BW} from plot

1 marks for knowing Nyquist frequency is must be larger than bandwidth

1 mark for correct minimum sampling frequency

c) (5 marks) What is the formula to compute the Zero Order Hold Equivalent of the plant $G(s)$? For a sampling rate of 10 Hz, where would the pole corresponding to $s = -6$ be located in the z-plane? (Note: You do not have to solve for the complete Zero Order Hold equivalent of $G(s)$, just give the general formula and compute the z-plane pole location for $s = -6$).

$$G_{ZOH}(z) = \frac{z-1}{z} \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

From line 4 of table we know that

$$F(s) = \frac{1}{s+a} \Rightarrow F(z) = \frac{z}{z - e^{-aT}}$$

In our case $a = -6$ so we get

$$\frac{z}{z - e^{-aT}} = \frac{z}{z - e^{-6(0.1)}}$$

so the pole goes to $z = e^T = e^{-0.6} = 0.5488$ since $T = \frac{1}{10}$

2 marks for formula

2 marks for knowing that pole at $s = -a$ goes to pole at $z = e^{-aT}$

1 mark for computing new pole location correctly

The End