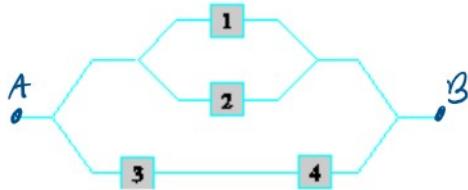


**Problem #1:** Consider the system of components connected as in the figure below. Components 1 and 2 are connected in parallel, so that subsystem works if and only if either 1 or 2 works. Since 3 and 4 are connected in series, that subsystem works if and only if both 3 and 4 work. Suppose that  $P(1 \text{ works}) = 0.7$ ,  $P(2 \text{ works}) = 0.75$ ,  $P(3 \text{ works}) = 0.86$ , and  $P(4 \text{ works}) = 0.83$ . Find the probability that the system works.



$$P(1) = 0.7 \quad P(3) = 0.86$$

$$P(2) = 0.75 \quad P(4) = 0.83$$

$$P(A \rightarrow B) = P(\text{UP} \cup \text{DOWN})$$

$$\begin{aligned} P(\text{UP}) &= P(1 \cup 2) = P(1) + P(2) - P(1 \cap 2) \\ &= P(1) + P(2) - P(1)P(2) \text{ independence} \\ &= 0.7 + 0.75 - 0.7 \cdot 0.75 \\ &= 0.925 \end{aligned}$$

$$\begin{aligned} P(\text{DOWN}) &= P(3 \cap 4) = P(3)P(4) \\ &= 0.86 \cdot 0.83 \\ &= 0.7138 \end{aligned}$$

$$\begin{aligned} P(A \rightarrow B) &= P(\text{UP} \cup \text{DOWN}) \\ &= P(\text{UP}) + P(\text{DOWN}) - P(\text{UP})P(\text{DOWN}) \\ &= 0.925 + 0.7138 - 0.925 \cdot 0.7138 \\ &= 0.978535 \end{aligned}$$

$$= 0.978535$$

**Problem #2:** Consider purchasing a system of audio components consisting of a receiver, a pair of speakers, and a CD player. Let  $A_1$  be the event that the receiver functions properly throughout the warranty period. Let  $A_2$  be the event that the speakers function properly throughout the warranty period. Let  $A_3$  be the event that the CD player functions properly throughout the warranty period. Suppose that these events are (mutually) independent with  $P(A_1) = 0.93$ ,  $P(A_2) = 0.84$ , and  $P(A_3) = 0.71$ .

- (a) What is the probability that at least one component needs service during the warranty period?
- (b) What is the probability that exactly one of the components needs service during the warranty period?

$$P(A_1) = 0.93 \quad P(A_2) = 0.84 \quad P(A_3) = 0.71$$

$$P(\bar{A}_1) = 0.07 \quad P(\bar{A}_2) = 0.16 \quad P(\bar{A}_3) = 0.29$$

a) what is the probability of one or more failure?

$$\begin{aligned} P(1 \text{ or more fail}) &= 1 - P(\text{all working}) \\ &= 1 - P(A_1)P(A_2)P(A_3) \\ &= 1 - 0.554652 \\ &= 0.445348 \end{aligned}$$

b) Probability exactly one needs service

$$\begin{aligned} P(\text{exactly 1 fail}) &= P(\bar{A}_1 \text{ only fails}) \cup P(\bar{A}_2 \text{ only}) \cup P(\bar{A}_3 \text{ only}) \\ &= P(\bar{A}_1 \text{ only}) + P(\bar{A}_2 \text{ only}) + P(\bar{A}_3 \text{ only}) \\ &= P(\bar{A}_1)P(A_2)P(A_3) \\ &\quad + P(A_1)P(\bar{A}_2)P(A_3) \\ &\quad + P(A_1)P(A_2)P(\bar{A}_3) \end{aligned}$$

$$\begin{aligned}
 &= 0.97 \cdot 0.84 \cdot 0.71 \\
 &+ 0.93 \cdot 0.16 \cdot 0.71 \\
 &+ 0.93 \cdot 0.84 \cdot 0.29 \\
 &= 0.373944
 \end{aligned}$$

**Problem #3:** A box consists of 9 components, 6 of which are defective.

- (a) Components are selected and tested one at a time, without replacement, until a non-defective component is found. Let  $X$  be the number of tests required. Find  $P(X = 3)$ .
- (b) Components are selected and tested, one at a time without replacement, until two consecutive non defective components are obtained. Let  $X$  be the number of tests required. Find  $P(X = 5)$ .

9 components, 6 defective, 3 working

a)

$\frac{X}{6/9}$	$\frac{X}{5/8}$	$\frac{\checkmark}{3/7}$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
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what is the chance of 2 failures & then one success?

$$P(X=3) = \frac{6}{9} \times \frac{5}{8} \times \frac{3}{7} = 0.178571$$

b)

1)

$\frac{X}{6/9}$	$\frac{X}{5/8}$	$\frac{X}{4/7}$	$\frac{\checkmark}{3/6}$	$\frac{\checkmark}{2/5}$	$\dots$	$\dots$	$\dots$
-----------------	-----------------	-----------------	--------------------------	--------------------------	---------	---------	---------

2)

$\frac{X}{6/9}$	$\frac{\checkmark}{3/8}$	$\frac{X}{5/7}$	$\frac{\checkmark}{2/6}$	$\frac{\checkmark}{1/5}$	$\dots$
-----------------	--------------------------	-----------------	--------------------------	--------------------------	---------

3)

$\frac{\checkmark}{3/9}$	$\frac{X}{6/8}$	$\frac{X}{5/7}$	$\frac{\checkmark}{2/6}$	$\frac{\checkmark}{1/5}$	$\dots$
--------------------------	-----------------	-----------------	--------------------------	--------------------------	---------

$$P(X=5) = P(1 \cup 2 \cup 3), \quad P(\checkmark \cap X \cap X \cap \checkmark \cap \checkmark)$$

$$\begin{aligned}
 P(X=5) &= P(1 \cup 2 \cup 3) && P(\text{✓}/\text{✗}/\text{✗}/\text{✗}/\text{✓}) \\
 &= P(1) + P(2) + P(3) \\
 &= \left(\frac{6}{9}\right)\left(\frac{5}{8}\right)\left(\frac{4}{7}\right)\left(\frac{3}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{6}{9}\right)\left(\frac{3}{8}\right)\left(\frac{5}{7}\right)\left(\frac{2}{6}\right)\left(\frac{1}{5}\right) \\
 &\quad + \left(\frac{3}{9}\right)\left(\frac{6}{8}\right)\left(\frac{5}{7}\right)\left(\frac{2}{6}\right)\left(\frac{1}{5}\right) \\
 &= 0.071428
 \end{aligned}$$

**Problem #4:** An assembly consists of two mechanical components. Suppose that the probabilities that the first and second components meet specifications are 0.99 and 0.85, respectively. Let  $X$  be the number of components in the assembly that meet specifications.

A      B

- (a) Find the mean of  $X$ .
- (b) Find the variance of  $X$ .

$$P(A) = 0.49 \quad P(B) = 0.85$$

$$P(\bar{A}) = 0.01 \quad P(\bar{B}) = 0.15$$

$$\text{range}(X) = \{0, 1, 2\} \leftarrow \text{posses}$$

$$P(X=\emptyset) = P(\bar{A}) \cap P(\bar{B}) = 0.01 \times 0.15 = 0.0015$$

$$\begin{aligned}
 P(X=1) &= P(\bar{A} \cap B) + P(A \cap \bar{B}) \\
 &= P(\bar{A})P(B) + P(A)P(\bar{B}) \\
 &= (0.01)(0.85) + (0.49)(0.15)
 \end{aligned}$$

$$P(X=2) = P(A)P(B) = 0.49 \cdot 0.85 = 0.8415$$

$$\begin{aligned}
 \text{a)} \quad E[X] &= \sum x P(x) \\
 &= (\emptyset)(0.0015) + (1)(0.157) + (2)(0.8415) \\
 &= 1.84
 \end{aligned}$$

$$\begin{aligned}
 b) \text{var}(X) &= \left( \sum_x x^2 P(X=x) \right) - \mu^2 \\
 &= ((\emptyset)^2 P(X=\emptyset) + (1)^2 P(X=1) \\
 &\quad + (2)^2 P(X=2)) - (1.84)^2 \\
 &= (0.157 + (4)(0.8415)) - (3.3856) \\
 &= 0.1374
 \end{aligned}$$

**Problem #5:** A manufacturing process has 45 customer orders to fill. Each order requires one component part that is purchased from a supplier. However, typically 4% of the components are identified as defective, and the components can be assumed to be independent.

- (a) If the manufacturer stocks 47 components, what is the probability that the 45 orders can be filled without reordering components?
- (b) Let  $X$  be the number of good (i.e., non-defective) components among the 47 in stock. Find the mean of  $X$ .
- (c) Find the variance of  $X$  [from part (b)].

45 orders, 4% are defective

a) 47 components

What are the odds you choose 45 components without choosing a defective one?

at least the first 45 of the total 47 must be working

$$P(X=45) = \binom{47}{45} (0.96)^{45} (0.04)^2$$

$$P(X=45) = 0.275519$$

$$P(X=46) = \binom{47}{46} (0.96)^{46} (0.04) = 0.287498$$

$$P(X=47) = \binom{47}{47} (0.96)^{47} (1) = 0.146807$$

$$\begin{aligned}
 P(\text{at least 45 working}) &= P(X=45) \cup P(X=46) \cup P(X=47) \\
 &= 0.275519 + 0.287498 + 0.146807
 \end{aligned}$$

$$= 0.275519 + 0.287498 + 0.146802 \\ = 0.709824$$

b)  $E[X] = np$       Binomial( $n, p$ )  
 $= 47(0.96)$   
 $= 45.12$

c)  $\text{Var}(X) = np(1-p)$   
 $= 47(0.96)(0.04)$   
 $= 1.8048$

**Problem #6:** The probability that a randomly selected box of a certain type of cereal has a particular prize is 0.11. Suppose that you purchase box after box until you have obtained 2 of these prizes.

- (a) What is the probability that you purchase exactly 7 boxes?
- (b) What is the probability that you purchase at least 10 boxes?
- (c) How many boxes would you expect to purchase, on average?

$$P(\text{prize}) = 0.11$$

a)  
1) ✓ ✗ ✗ ✗ ✗ ✗ ✓

2) ✗ ✓ ✗ ✗ ✗ ✗ ✓

3) ✗ ✗ ✓ ✗ ✗ ✗ ✓

4) ✗ ✗ ✗ ✓ ✗ ✗ ✓

5)   X  X  X  X  ✓  X  ✓

6)   X  X  X  X  X  ✓  ✓

negative binomial distribution

$$P(X=7) = \binom{7-1}{2-1} (0.11)^2 (0.89)^5$$

$$= 0.040540$$

$$b) P(X \geq 10) = 1 - P(X < 10)$$

$$= 1 - P(X=9) - P(X=8) \dots$$

$$= 1 - \binom{9-1}{2-1} (0.11)^2 (0.89)^7 - \binom{8-1}{2-1} (0.11)^2 (0.89)^6 \dots$$

$$= 0.740079$$

$$c) E[X] = r/p = 2 / 0.11 = 18.1818\dots$$

**Problem #7:** A geologist has collected 21 specimens of basaltic rock and 17 specimens of granite. The geologist instructs a laboratory assistant to randomly select 8 of the specimens for analysis.

- (a) What is the probability that at least 4 of the selected specimens are granite?
- (b) What is the expected number of granite specimens in the sample?
- (c) If this same process is repeated every day, how many days (on average) will it take before getting a sample consisting entirely of granite?

21 aF b 8 19 aF g

choose 8 for analysis

2) How many are granite?

— — — — — — — → 4 granite, 4 - b rock

hypergeometric

1 1 1 1 1 1 1 1

hypergeometric

$$\text{range}(X) = \{0, 1, 2, 3, \overbrace{4, 5, 6, 7, 8}\}$$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$$

$$P(X=4) = \binom{17}{4} \binom{21}{4} / \binom{17+21}{8}$$

$$= 0.291273698$$

$$P(X=5) = \binom{17}{5} \binom{21}{3} / \binom{17+21}{8}$$

$$= 0.168291458$$

$$P(X=6) = \binom{17}{6} \binom{21}{2} / \binom{17+21}{8}$$

$$= 0.053144671$$

$$P(X=7) = \binom{17}{7} \binom{21}{1} / \binom{17+21}{8}$$

$$= 0.008351305$$

$$P(X=8) = \binom{17}{8} \binom{21}{0} / \binom{17+21}{8}$$

$$= 0.000497101$$

$$P(X \geq 4) = 0.52158213$$

- (b) What is the expected number of granite specimens in the sample?

$$E[X] = nle/N$$

$$= 8 \cdot 17 / (17+21)$$

$$= 68/19$$

$$= 68/19$$

- (c) If this same process is repeated every day, how many days (on average) will it take before getting a sample consisting entirely of granite?

$$P(X=8) = 0.000497101$$

$$\sqrt{P(X=8)} \approx 2.012 \text{ days}$$

## Problem 7

Poisson's dist.

$$\text{in } 225g = 19.4$$

$$\text{in } 43g = 2.75$$

$$\lambda = 2.75, \text{ unit length} = 43g$$

$$P(X=x) = \frac{(\lambda T)^x}{x!} e^{-\lambda T}$$

$$T=1$$

$$= \frac{(2.75)^x e^{-2.75}}{x!}$$

a)  $P(X \geq 3) = ?$

no maximum  $\therefore \rightarrow 1 - P$

$$P(X \geq 3) = 1 - P(X < 3)$$

$$P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$P(X=0) = \frac{(2.75)^0 e^{-2.75}}{0!}$$

$$= 0.063927861$$

$$P(X=1) = \frac{2.75 e^{-2.75}}{1!}$$

$$= 0.175801618$$

$$P(X=2) = \frac{(2.75)^2 e^{-2.75}}{2!}$$

$$= 0.241727225$$

$$P(X < 3) = 0.481956704$$

$$P(X \geq 3) = 1 - P(X < 3)$$

$$= 0.518543295$$

b) 13 weeks, 0 growth  
or EXACTLY 6 weeks.

binomial RV

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(\text{exactly 6}) = \binom{13}{6} P(X=0)^6 P(X>0)^{9-6}$$

$$P(\text{exactly 6}) = \binom{13}{6} P(X=0)^6 P(X>0)^{9-6}$$

$$P(X>0) = 1 - P(X=0)$$

$$P(X=0) = 0.063927861$$

$$P(X>0) = 0.936072139$$

$$P(X=6) = \binom{13}{6} (0.063927861)^6 (0.936072139)^3$$

$$= 0.000096069$$

**Problem #9:** Let  $X$  denote the vibratory stress (psi) on a wind turbine blade at a particular wind speed in a wind tunnel. Suppose that  $X$  has the following *Rayleigh* pdf.

$$f(x) = \begin{cases} (x/\theta^2) e^{-x^2/(2\theta^2)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) If  $\theta = 91$ , find the probability that the vibratory stress is between 81 and 314.
- (b) If  $\theta = 91$ , then 78% of the time the vibratory stress is greater than what value?

$$\begin{aligned} \text{(a)} \quad P(81 \leq X \leq 314) &= \int_{81}^{314} f(x) dx \\ &= \int_{81}^{314} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx \\ &= \left[ \frac{1}{2\theta^2} x^2 \left( \frac{2\theta^2}{-x^2} \right) e^{-x^2/2\theta^2} \right]_{81}^{314} \\ &= \left( -e^{-x^2/2\theta^2} \right)_{81}^{314}, \quad \theta = 91 \end{aligned}$$

$$= \left( -e^{-x^2/2\theta^2} \right)_{\theta=91}$$

$$= 0.670309571$$

b)  $P(X \geq x) = 0.78 = \int_x^\infty f(x) dx \leftarrow \text{no}$   
 $P(X \leq x) = 1 - 0.78$

$$1 - 0.78 = \int_0^x f(x) dx$$

$$0.78 = -e^{-\frac{x^2}{2\theta^2}} \pm$$

$$-0.78 = -e^{-x^2/2\theta^2}$$

$$0.78 = e^{-x^2/2\theta^2}$$

$$\ln(0.78) = \frac{-x^2}{2\theta^2}$$

$$\sqrt{-2\theta^2 \ln(0.78)} = x$$

$$x = 64.14839852$$