2A04 Equations and Explanations

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Waves, Circuits and Phasors

General Wave Equations

A general expression for a wave is:

$$y(x,t) = A_0 \cos(kx - \omega t + \phi)$$

A_0 is the amplitude of the wave.

The k term is known as the wave number, and is related to the wavelength of the wave through the following expressions, that have been rearranged:

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k}$$

The ω term is the angular frequency. It is related to the frequency of the wave through the following relation:

$$\omega = 2\pi f$$

Also note that the period of a wave T is $\frac{1}{f}$. Noting this, we now have:

$$\omega = \frac{2\pi}{T}$$

The ϕ term is the reference phase, also referred to as the phase shift. It ranges from $0\ to\ 2\pi$. It effectively changes where the wave "starts".

To convert between sine waves and cosine waves, recall that:

$$\cos(\theta) = \sin\left(\theta - \frac{\pi}{2}\right)$$

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

Which is to state that to convert a sine wave to a cosine wave, one must subtract a phase of $\frac{\pi}{2}$. Inversely, to convert a cosine wave to a sine wave, one must add a phase of $\frac{\pi}{2}$. For this phase term, it being positive signifies a phase lead in time, whereas it being negative signifies a phase lag in time.

Phase velocity is defined as:

$$u = \frac{\lambda}{T} = \lambda f = \omega k$$

Lossy Media

Waves may have an attenuation factor, which will result in the wave decaying over time. Media without this factor is called a lossless medium. Once it is added in, the attenuation constant α looks like this:

$$y(x,t) = A_0 e^{-\alpha} \cos(kx - \omega t + \phi)$$

DC VS AC

Most of these next few sections is going to be review, but on the off chance you get amnesia and forget how circuits work, this section should prove useful. Firstly, starting off with DC:

DC – Direct Current

The current has a constant direction.

The current/voltage may vary with time.

Typically assumed to be steady state with constant current magnitude and direction.

Wires have negligible resistance.

AC – Alternating Current

The current has alternating direction.

The current/voltage supplied by the source is sinusoidal with time – positive for half a cycle and negative for the other half.

Typically assumed to have varying current magnitude and direction.

Wires have negligible resistance.

Kirchhoff's Laws apply in both cases, but phase differences must be accounted for in the case of AC.

Batteries and EMF

EMF, the electromotive force, is the max possible voltage that a battery can provide through its terminals. In real batteries, the terminal voltage is not equal to the EMF, due to internal resistance r. In this case, there should be a resistor in series with the cell that considers the internal resistance.

$$\Delta V = V_{emf} - Ir$$

Expressions for Power

Power is expressed by:

$$P = I^2 R = \frac{V^2}{R}$$

KVL and KCL

Kirchhoff's Voltage Law dictates that for any closed loop, the sum of potential differences across all elements must be zero.

$$\sum \Delta V = 0$$

Kirchhoff's Current Law dictates that for any junction, the sum of currents into a junction must equal the sum of currents out of a junction. This is to state that if all currents at a junction are added, one should get zero as the sum.

$$\sum \mathrm{I} = 0$$

Capacitors and Inductors

Capacitors are added as detailed below:

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

Capacitors store charge based on a potential difference across the two plates. The capacitance is defined as the ratio between the charge stored and the potential difference across the plates.

$$C = \frac{Q}{V}$$

Inductors are added as detailed below:

$$L_{eq} = L_1 + L_2$$

Inductors create a strong magnetic flux for a given current and oppose changes in current flow. The current is opposed due to the magnetic field that exists – reversing current flow would cause an opposing magnetic field, which is what causes the inductor to oppose current flow. This defines the inductance:

$$L = -\frac{V}{\frac{di}{dt}}$$

RMS Values

RMS, root mean squared values represent the average value of the current or voltage. This is due to how the average of sinusoids are defined. For example:

$$I_{rms} = \sqrt{\langle i^2 \rangle}$$

$$I_{rms} = \sqrt{I_{max}^2 \sin(\omega t)}$$

$$I_{rms} = \sqrt{I_{max}^2 \left(\frac{1}{2}\right)}$$

Thus:

$$I_{rms} = \frac{I_{max}}{\sqrt{2}}$$

$$V_{rms} = \frac{V_{max}}{\sqrt{2}}$$

Phasor Domain

You know what a complex number is. So, knowing that complex numbers can be easily used to represent sinusoids while also maintaining phase information, using complex numbers for purposes such as this makes sense. A phasor is a complex expression of a current or voltage, which does not have time dependence. Examples of which are seen below:

$$v = \Re\left[\, V_0 e^{j\phi} \right]$$

$$i = \Re\left[\, I_0 e^{j\phi} \right]$$

Time Domain

Multiplying in time dependence, and then converting back to a sinusoid gives a more interpretable expression of a phasor, with respect to time.

$$v(t) = \Re\left[\, V_0 e^{j\phi} e^{\omega t} \right] = V_0 \cos(\omega t + \phi)$$

$$i(t) = \Re\left[\, I_0 e^{j\phi} e^{\omega t} \right] = I_0 \cos(\omega t + \phi)$$

Complex Impedance

Like how resistors have a real component of impedance, which affects values of magnitude, inductors and capacitors have a complex component of impedance which affects phase. They also have another term, called reactance – this is simply the complex impedance without multiplying in the imaginary number j.

$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C} = -jX_C$$

$$Z_L = j\omega L = jX_L$$

Transmission Lines

Transmission Lines

Transmission lines are all structures that can be used to transfer information between two points. These are things such as nerve fibres, mechanical pressure waves, and optical fibres. A transmission line in electricity and magnetism transfers electromagnetic signals from one place to another.

Accounting for Transmission Line Effects

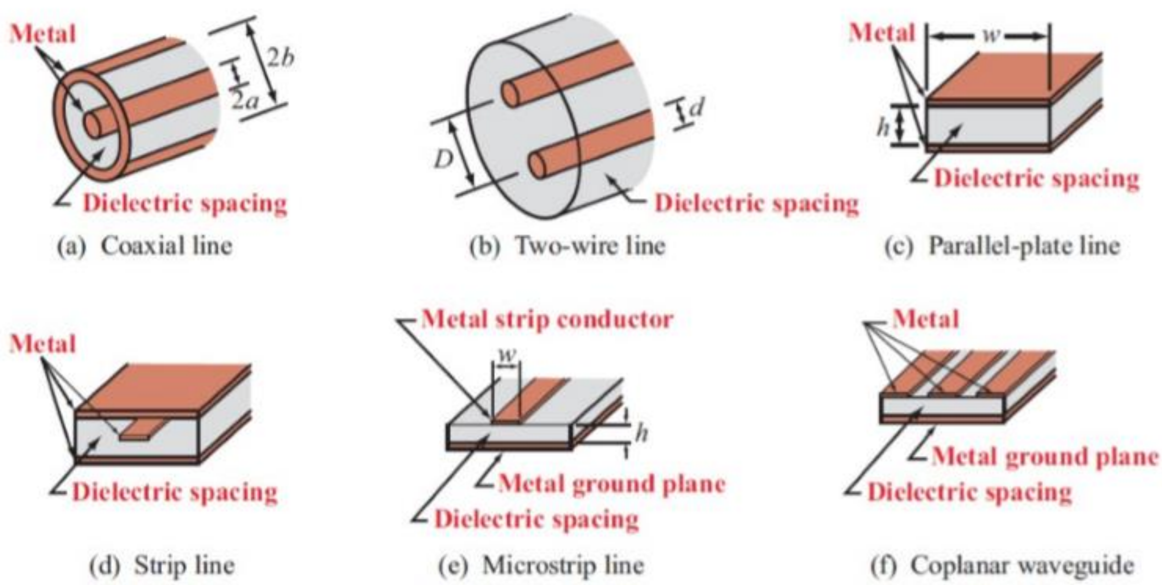
$$if\ \frac{l}{\lambda}\ is\ very\ small, ignore\ transmission\ line\ effects$$

$$if\ \frac{l}{\lambda} > 0.01, need\ to\ account\ for\ phase\ delay\ and\ possibly\ reflections$$

$$if\ \frac{l}{\lambda} > 0.25, need\ to\ account\ for\ phase\ delay\ and\ reflections$$

Types of Transmission Modes

The following are examples of different modes of transmission through transverse electromagnetic waves.



TEM Transmission Lines

Parameters of Transmission Lines

$$R' = \textit{the combined resistance of both conductors per unit length}$$

$$L' = \textit{the combined inductance of both conductors per unit length}$$

$$C' = \textit{the capacitance of both conductors per unit length}$$

$$G' = \textit{the conductance of the MEDIUM between the two condictions per unit length}$$

The calculation of these parameters is dependant on the structure of the transmission line. For this, we have a useful screenshot from lecture.

Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

With R_S, the surface resistance of the conductors, being defined as the following:

$$R_S = \sqrt{\pi f \frac{\mu_C}{\sigma_C}}$$

We also have the following useful relations:

$$\left.\begin{aligned} L'C' &= \mu\varepsilon \\ \frac{G'}{C'} &= \frac{\sigma}{\varepsilon} \end{aligned}\right\} \text{All TEM transmission lines}$$

Air line: $\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12}$ F/m
$\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m
$\sigma = 0$
$G' = 0$

Telegrapher's Equations

These are the telegrapher's equations. I have no clue if they will be actually utilized in any way, but here they are.

Telegraphers equations (time domain):

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C' \frac{\partial v(z,t)}{\partial t}$$

Telegraphers equations (phasor domain):

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z)$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z)$$

Complex Propagation Constant

In attempting to solve the telegrapher's equations, we end up with the following:

$$\boxed{\begin{aligned} \frac{d^2\tilde{V}}{dz^2} - \gamma^2 \tilde{V}(z) = 0 \\ \frac{d^2\tilde{I}}{dz^2} - \gamma^2 \tilde{I}(z) = 0 \end{aligned}}$$

The gamma term here, is the complex propagation constant. This constant is represented by the following:

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

Which when expressed in rectangular form, we have:

$$\gamma = \alpha + j\beta$$

Such that:

$$\alpha = \Re(\gamma) = Attenuation\ Constant\ (yes, for\ attenuated\ waves)\ \left[\frac{Np}{m}\right]$$

$$\beta = \Im(\gamma) = Phase\ Constant\ \left[\frac{Rad}{m}\right]$$

Characteristic Impedance

This term, Z_0, represents the ratio of the forward travelling voltage term to the forward current term. It is NOT Equal to the total voltage to total current on a line at any position. It is defined similarly to the propagation constant gamma, as seen below.

$$Z_0 = \frac{\sqrt{(R' + j\omega L')}}{\sqrt{(G' + j\omega C')}} = \frac{R' + j\omega L'}{\gamma}$$

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

Phase Velocity and Guide Wavelength

The propagation velocity of the wave through the transmission line is defined below, based on the complex component of the propagation constant. Note that this phase velocity is less than the speed of light.

$$u_p = \frac{\omega}{\beta}$$

The guide wavelength can be determined from this. This wavelength is defined below:

$$u_p = f\lambda \therefore$$

$$\lambda = \frac{u_p}{f}$$

Lossless Transmission Lines

A lossless transmission line is one in which no attenuation occurs. In this case, the attenuation constant is zero. When this happens, we can state a few things as simplifications:

$$\alpha = 0$$

$$\beta = \sqrt{L'C'} = \frac{\omega\sqrt{\varepsilon_r}}{c}$$

$$u_p = \frac{c}{\sqrt{\varepsilon_r}}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

Voltage Reflection Coefficient

The voltage reflection coefficient is defined as the ratio of the input wave and reflected wave in terms of amplitude and phase. It is defined as capital gamma, and dependant on the Load Impedance, Z_L:

$$\Gamma = \frac{Z_L - 1}{Z_L + 1}$$

Input Impedance

Input impedance is dependant on the end of the transmission line, l as a distance. It is dependant on the phase constant β and both the load impedance Z_L and the characteristic impedance Z_0. This is seen below:

$$Z_{in} = Z_0 \left[\frac{Z_L + j\tan\beta l}{1 + jZ_L \tan\beta l}\right]$$

Forward Voltage

One can determine the forward voltage term, V_0^+, by first determining the phasor voltage across the transmission line. This can be done with the following process:

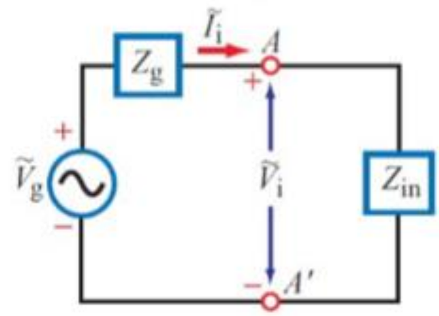
The phasor voltage across Z_{in} (using the voltage divider rule):

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \left(\tilde{V}_g \frac{Z_{in}}{Z_g + Z_{in}}\right)$$

Using the equation for phasor voltage at $z=-l$

$$\tilde{V}(-l) = V_0^+\left[e^{j\beta l} + \Gamma e^{-j\beta l}\right] = \tilde{V}_i$$

Equivalent circuit:



We can find an equation to describe the forward voltage term, V_0^+, our last unknown!:

$$\boxed{V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left[\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right]}$$

Vector Analysis

Cylindrical Coordinates

Picture a cylinder with a face on the xy plane, extending in the z-dimension infinitely. We have the following definitions:

$$r = radial\ distance\ in\ the\ xy\ plane$$

$$\phi = angle\ measured\ from\ postive\ x\ axis$$

$$z = same\ as\ cartesian\ z$$

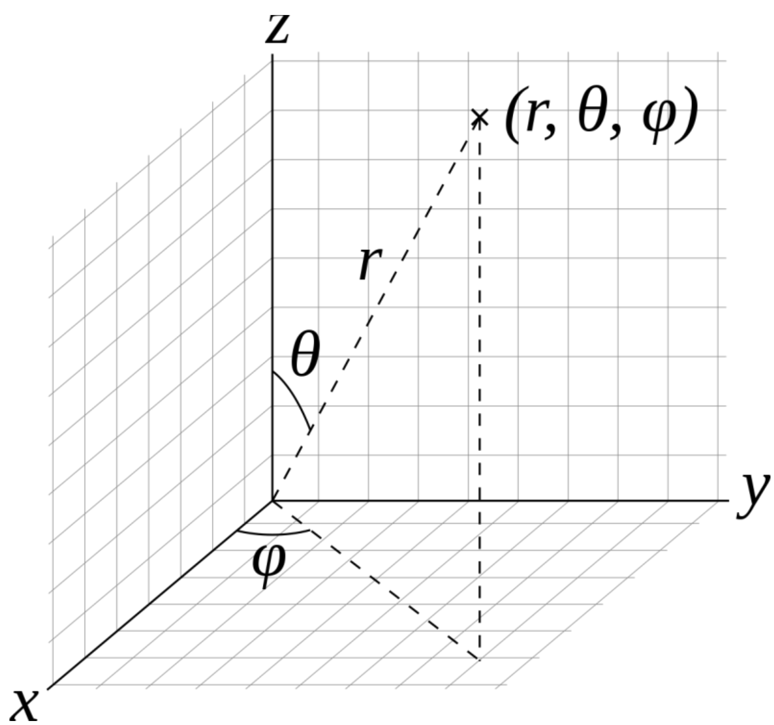
Spherical Coordinates

Similarly to cylindrical coordinates, spherical coordinates has a few differences, described below for each of the three coordinates:

$$r = distance\ from\ origin\ to\ point$$

$$\theta = angle\ downwards\ from\ positive\ z\ axis$$

$$\phi = angle\ from\ positive\ x\ axis$$



Coordinate Conversions

We have the following table, which should be useful for converting between different coordinate systems:

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Vector Operators Review

In vector calculus, there are three operators of note: Gradient, Divergence, and Curl.

Gradient

The differential change over a differential distance. When applied to a scalar quanitiy, it will obtain physical meaning. Otherwise, the operator – del – has no physical meaning. It is the partial derivatives with respect to each coordinate variable, in vector form. The operation applied to a vector will give a magnitude equal to the maximum rate of change per unit distance. This vector will be pointing in the direction of the maximum rate of change.

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Divergence

If we dot the del operator with a vector, or vector function, we obtain what is effectively the derivative of the vector field. We lose any directionality of the quantity and obtain a scalar value. This is a measure of quantity of outward flux from any point on the vector field. The magnitude of this scalar quantity is the magnitude of flow, and the sign convection dictates if it is a source or sink. The divergence of a field at any point as the net outward flux per unit volume over a closed incremental surface.

$$\nabla \cdot \vec{E} = div\ \vec{E}$$

Curl

The rotational property of a vector field. It describes how rotational the field is, and is a vector quantity obtained by crossing the del operator with a vector field. This results in the following expression:

$$\nabla \times \vec{E} = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

Or, more intuitively:

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{\mathbf{z}}$$

Divergence Theorem Review

This theorem is useful for converting integration over a volume to integration over a surface. For a given vector field, $\vec{E}$, we have the following:

$$\int_V \nabla \cdot \vec{E} dV = \oint_S \vec{E} \cdot d\vec{s}$$

This is saying that the surface integral of a given vector field can be determined using the divergence of a volume integral of the same vector field, and vice versa.

Stoke's Theorem Review

This theorem is useful for converting the surface integral of the curl of a vector over an open surface S into a line integral along the contour C bounding the surface S. The theorem is as follows, for a given magnetic field B:

$$\int_S \nabla \times \vec{B} \cdot ds = \oint_C \vec{B} \cdot d\vec{l}$$

Electrostatics

Coulomb's Law

The electric field at a given point due to a single point charge, is given by Coulombs law:

$$\vec{E}(\vec{R}) = \hat{R}\frac{q}{4\pi\varepsilon R^2} = \frac{\vec{R}}{|\vec{R}|}\frac{q}{4\pi\varepsilon R^2} = \frac{q\vec{R}}{4\pi\varepsilon|\vec{R}|^3}$$

As the electric field at a given point due to multiple charges can be determined using superposition, for any number of charges in space, we have the following expression for the resultant electric field:

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\varepsilon}\sum_{i=0}^{n}\frac{q_i(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3}$$

Such that R is the distance to the point, and R_i is the distance of the charge to the point.

Volumetric Charge Distributions

For a charged volume, we have the volumetric charge density given by:

$$\rho_v = \lim_{\Delta V\to 0}\frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

Where V is the volume. For the total charge in a volume, we have the following:

$$Q = \int_{\mathrm{V}'}\rho_v\left(\overrightarrow{R'}\right)dV'$$

The prime dictates the location inside of the charge density.

This means that the total electric field given by a charged volume is given by:

$$\vec{E}(\vec{R}) = \int_{\mathrm{V}'}\rho_v\left(\overrightarrow{R'}\right)\frac{\left(\vec{R}-\overrightarrow{R'}\right)}{4\pi\varepsilon\left|\vec{R}-\overrightarrow{R'}\right|^3}dV$$

Gauss's Law

Electric flux density is defined below:

$$\vec{D} = \varepsilon\vec{E}$$

$$For\ free\ space: \varepsilon = \varepsilon_0 = 8.85\cdot 10^{-12}\frac{F}{m}, For\ other\ materials: \varepsilon = \varepsilon_0\varepsilon_r$$

D is the displacement field and represents being able to determine the electric field based off of how much a given charge is displaced. This is used to find the charge given a specific electric field. This can be done using the integral form of Gauss's Law, which can be seen below:

$$\oint_S \vec{D} \cdot ds = q$$

This is the mathematic expression stating that the electric field coming from a given volume is equal to the charge enclosed by that volume. In addition, keep in mind the application of the divergence theorem:

$$q = \oint_S \vec{D} \cdot ds = \int_V \nabla \cdot \vec{D} dV$$

Electric Scalar Potential

Voltage, voltage potential, electric potential, electric scalar potential – whatever you call it, it represents the required amount of work to move a unit of charge from one point to another in an electric field. In other words, it is the POTENTIAL DIFFERENCE at two different points in an electric field. Mathematically, this is expressed as the following:

$$V = -\int_l \vec{E} \cdot d\hat{l}$$

Electric fields are conservative, so taking the difference between two points is independent of path. This means, that for a closed path, we start and end at the same potential. Thus:

$$\oint_C^0 \vec{E} \cdot d\hat{l} = 0$$

Electric Potential as a Function of Voltage

By reversing the definition of voltage, we can define the electric field as the negative gradient of voltage:

$$\vec{E} = -\nabla V$$

Electric Dipoles and Moments

An electric dipole will form when two equal and opposite charges are separated by a distance. The electric dipole moment is equal to the magnitude of the charge, multiplied by the direction vector pointing to the positive charge. This is expressed as:

$$\vec{p} = q\vec{d}$$

The potential from a dipole is expressed as the following:

$$V(\vec{R}) = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{\vec{p} \cdot \hat{R}}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R'}\right|^2}$$

The electric field is given by:

$$\vec{E}(\vec{R}) = \frac{qd}{4\pi\varepsilon_0\left|\vec{R}-\vec{R'}\right|^3}(\hat{R}2\cos\theta + \hat{\theta}\sin\theta)$$

Conductivity and Current Density

The conductivity of a material is how easily electrons can travel through a material under the influence of an electric field. This is represented by the conductivity of a material, σ. This gives:

$$\vec{J} = \sigma\vec{E}$$

Such that the current density, J, is equal to:

$$J = \rho_V\vec{u}$$

Multiplying the volumetric charge density by the velocity of the charges. In a perfect dielectric, there will be no conductivity. Thus, there will be no current density:

$$\vec{J} = 0$$

In a perfect conductor, there will be an infinite amount of conductivity, caused by the idealized infinite amount of electrons. This results in an electric field that must be zero in a conductor to fulfill the equality:

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{\vec{J}}{\infty} = 0$$

Drift Velocity and Mobility

Drift velocity is the steady state average velocity of the electrons determined by the balance between the accelerating force of the applied electric field and the scattering effect of the collisions in the lattice. It is calculated according to:

$$u_e = -\mu_e\vec{E}$$

Holes also have drift velocity:

$$u_H = \mu_H\vec{E}$$

Mobility accounts for the mass of a charged particle and the average distance over which the applied electric field can accelerate it before it is stopped by colliding with an atom, and then starts accelerating all over again. It is defined as μ.

Electric Field Boundary Conditions

At the interface of two different materials, such as the dielectric/conductor interface in a capacitor, we want to know what will happen to the electric field. The boundary conditions will specify how the components of fields will relate to one another and change. There could also be some surface charge density that could effect these fields.

Tangential components are equal:

$$\vec{E}_{1t} = \vec{E}_{2t}$$

Which carries through to the electric charge displacement field:

$$\frac{\vec{D}_{1t}}{\varepsilon_1} = \frac{\vec{D}_{2t}}{\varepsilon_2}$$

Normal components are varied according to the charge density:

$$D_{1n} - D_{2n} = \rho_s$$

Which carries through to electric field:

$$\varepsilon_1 \vec{E}_{1n} - \varepsilon_2 \vec{E}_{2n} = \rho_s$$

At Dielectric Boundaries:

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$
Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.			

Remember $\vec{\mathbf{E}} = 0$ in a good conductor 21

Capacitors

We know that for a capacitor, the capacitance is defined as:

$$C = \frac{Q}{V}$$

This extends to the integral expressions of both of these quantities:

$$C = \frac{\int_S \varepsilon \vec{E} d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}} = \frac{\varepsilon A}{-(-d)} \cdot \frac{\vec{E}}{\vec{E}} = \frac{\varepsilon A}{d} = \frac{Q}{V}$$

The energy stored in any volume containing an electric field, including a capacitor:

$$W_e = \frac{1}{2}\int_V \varepsilon E^2 dV$$

Magnetostatics

Magnetic Force

Acting only on moving charged particles, the magnetic force is given by:

$$\vec{F}_m = q\vec{u} \times \vec{B}$$

Unlike electric fields, magnetic fields do not do work on particles. It acts in the direction perpendicular to the direction of motion, and the magnetic field – hence the use of the cross product. The total magnetic force on a closed current loop in a uniform magnetic field is zero.

Magnetic Torque

For a loop with N turns whose surface normal is at an angle theta, relative to the direction of magnetic field B, we have the following expression for the torque on the loop:

$$\left|\vec{T}\right| = NIAB\sin\theta$$

Magnetic moment is given by:

$$\vec{m} = \hat{n}NIA$$

Which gives the expression for torque as:

$$\vec{T} = \vec{m} \times \vec{B}$$

Biot-Savart Law

We know that the magnetic field is defined as:

$$\vec{B} = \mu_0\vec{H}$$

The Biot-Savart law states that the magnetic field induced at point P can be determined by integrating across the length:

$$\vec{B} = \int_l \frac{\mu I\, dl \times \left(\vec{R} - \vec{R}'\right)}{4\pi\left|\vec{R} - \vec{R}'\right|^3}$$

Magnetic Field of a Loop

For a loop of wire with radius r and height z:

$$\vec{B} = \frac{\mu I r^2}{2(r^2 + z^2)^{\frac{3}{2}}}\hat{z}$$

So, at z = 0:

$$\vec{B} = \frac{\mu I}{2r}\hat{z}$$

And at points such that $z \gg r$

$$\vec{B} = \frac{\mu I r^2}{2z^3}$$

Magnetic Field of a Linear Conductor

For the length of wire segment, a at a radial distance

$$\vec{B} = \frac{\mu I a}{2\pi r\sqrt{4r^2 + a^2}}\hat{\phi}$$

For an infinite amount of wire:

$$\vec{B} = \frac{\mu I}{2\pi r}\hat{\phi}$$

For two separate parallel wires, they will attract if their currents are in the same direction and repel if they are in a different direction.

Gauss's Law for Magnetism

Gauss's law for magnetism states the nonexistence of isolated magnetic monopoles. This law is known as the conservation of magnetic flux. This is different than Gauss's law for electricity, which defines the electric flux through a surface as the charge enclosed by the surface. Expressed differentially, we have:

$$\nabla \cdot \vec{B} = 0$$

The integral form:

$$\oint_S \vec{B} \cdot ds = 0$$

No matter how a magnet is cut up, the magnet will always be a dipole. There is zero net magnetic flux through any closed surface. This is different than the law for electricity, where this is not usually true.

Ampere's Law

Ampere's law states that the line integral of $\vec{H}$ around a closed path is equal to the current I traversing the surface bounded by that path. In integral form, this means:

$$\oint_C \vec{H} \cdot dl = I$$

Converting to differential form, this is:

$$\nabla \times \vec{H} = \vec{J}$$

Stating that the curl of the H field is given by the current density, J.

H Field of Long Wire

For a wire with radius a, we have the following while inside the wire:

$$\vec{H} = \frac{r_1 I}{2\pi a^2}\hat{\phi}$$

Outside the wire at radius r:

$$\vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

H Field Inside Toroidal Coil

We have a toroidal core with outer radius b and inner radius a. The core is wrapped with wire, coiled around it N times.

For any region not within the coil:

$$\vec{H} = 0$$

For a region within the core, at distance r from the centre of the core:

$$\vec{H} = -\frac{NI}{2\pi r}\hat{\phi}$$

H Field Inside Long Solenoid

We have a solenoid coil of wire of length l, coiled N times, with a cylinder-shaped cavity inside with radius a. When the radius is greater than the radius of the core:

$$\vec{H} \approx 0$$

When the radius of evaluation is less than a:

$$\vec{H} = \frac{NI}{l}\hat{z}$$

H Field of Plane of Current

We have a sheet of current, located in the x-y plane. The current is flowing out of the page. Directionality can be easily determined through the right-hand rule.

For z greater than zero:

$$\vec{H} = -\frac{J_s}{2}\hat{y}$$

For z less than zero:

$$\vec{H} = \frac{J_s}{2}\hat{y}$$

Magnetic Vector Potential

The magnetic version of voltage, the magnetic vector potential is represented by the following:

$$\vec{B} = \nabla \times \vec{A}$$

Such that $\vec{A}$ is in units of Wb/m.

Inductors

The capacitance of any two conductors is defined as:

$$C = \frac{Electric\ Flux}{Voltage}$$

So similarly, the inductance of any two conductors is given by:

$$L = \frac{Magnetic\ Flux}{Current}$$

Self inductance is given by:

$$L = \frac{\Phi_m}{I}$$

Dynamics

Faraday's Law

The basis of Faraday's law is that an EMF is created from a time varying magnetic flux. Magnetic fields can produce a current in a closed loop, but only if the magnetic flux linked to the surface area of the loop changes with time. Mathematically, we have:

$$V_{emf} = -N\frac{d\Phi_m}{dt}$$

Generations of EMF

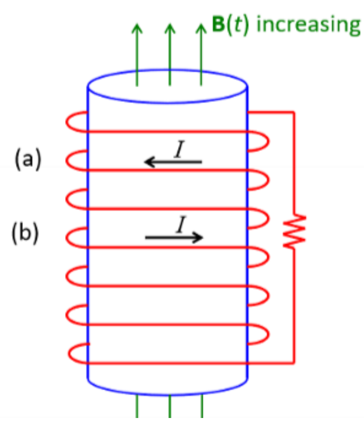
A time varying magnetic field linking a stationary loop, the induced EMF is called the transformer EMF. If the surface normal of the loop itself is changing, then it is referred to as a motional EMF. The total EMF is the summation of these two types of EMF.

Transformer EMF:

$$V_{emf}^{tr} = -N\int_S \frac{d\vec{B}}{dt}\cdot d\vec{s}$$

Lenz's Law

The direction of current I is governed by Lenz's Law. This states that the current in the loop is in the direction that opposes the change of magnetic flux. For example, for the below image, the magnetic field is increasing in the positive z direction. This means that the current would be in a direction to oppose this change. By RHR, we have that the current must be flowing in the clockwise direction, or a).



EM Generators and Motors

Ideal Transformers

The working principle of transformers is the concept of mutual inductance. This concept describes the magnetic coupling between two current conducting structures, such as two loops or two solenoids. The total flux from one solenoid from the other solenoid is given by:

$$N_2\Phi_{12} = N_2 \int_{S_2} \overrightarrow{B_1} \cdot ds$$

Which gives the mutual inductance as:

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \overrightarrow{B_1} \cdot ds$$

In an ideal transformer, the two sides of the core are coupled, and produce voltage dependant on the change of flux in the core:

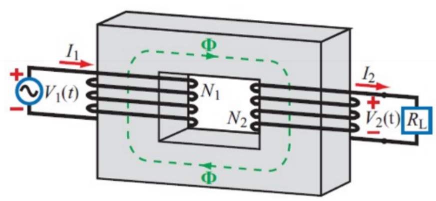
Ideal Transformer

Assumptions:

Core has infinite permeability

Coupling is perfect – i.e. all flux in N1 is shared to N2 (flux confined to core)

No power is lost within the core



$$V_1^{tr} = -N_1 \frac{d\Phi_m}{dt}$$

$$V_2^{tr} = -N_2 \frac{d\Phi_m}{dt}$$

$$P_1 = I_1 V_1 = P_2 = I_2 V_2$$

$$\boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}}$$

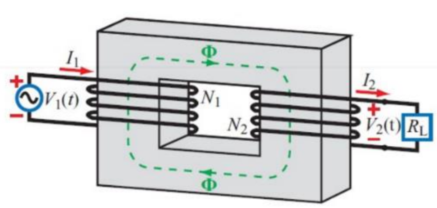
$N_2>N_1$: "step-up" transformer

$N_2<N_1$: "step-down" transformer

$$\boxed{\frac{I_1}{I_2} = \frac{N_2}{N_1}}$$

where N_1/N_2 = turns ratio

Transformers



$$R_{in} = \frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

$$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

Transformers make load impedances "look" bigger or smaller depending on their turns ratio.

EM Generator EMF

EM Generator EMF

Apply Faraday's Law to find V_{emf}:

$$\vec{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}} \qquad \Phi_m = \int_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}}$$

$$= \int_S (B_0\hat{\boldsymbol{z}}) \cdot (ds\hat{\boldsymbol{n}})$$

$$= B_0 A \cos\alpha$$

$$= B_0 A \cos(\omega t + C_0)$$

$$V_{emf} = -\frac{d\Phi_m}{dt} = A\omega B_0 \sin(\omega t + C_0)$$

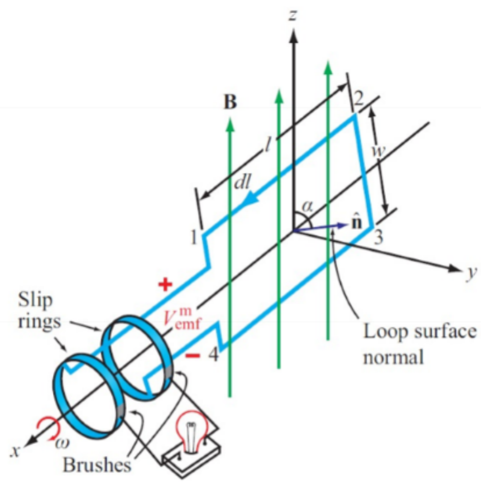


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Can also use motional emf equation to get same result: see p. 283-284

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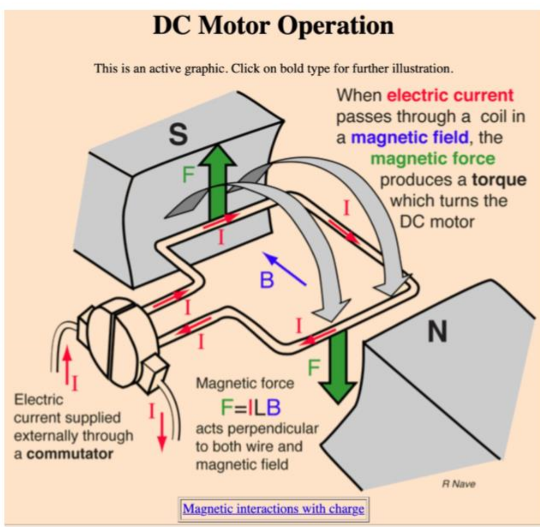
DC Motor Operation

DC motor



Hyperphysics – DC motor

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Maxwells Equations

Equations Summary

Maxwell's Equations

	Differential Form	Integral Form	
1)	$\nabla \cdot \vec{\boldsymbol{D}} = \rho_V(x,y,z)$	$\oint_S \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}}' = Q$	Gauss's law
2)	$\nabla \times \vec{\boldsymbol{E}} = -\dfrac{\partial \vec{\boldsymbol{B}}}{\partial t}$	$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}' = -\int_S \dfrac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot d\vec{\boldsymbol{s}}'$	Faraday's law (stationary surface *S*)
3)	$\nabla \cdot \vec{\boldsymbol{B}} = 0$	$\oint_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}}' = 0$	Gauss's law, magnetism (no magnetic charges)
4)	$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \dfrac{\partial \vec{\boldsymbol{D}}}{\partial t}$	$\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}}' = \int_S \left(\vec{\boldsymbol{J}} + \dfrac{\partial \vec{\boldsymbol{D}}}{\partial t}\right) \cdot d\vec{\boldsymbol{s}}'$	Ampere's law

- An electric field can be produced either by charges or changing magnetic fields.
- A magnetic field can be produced either by currents or changing electric fields.

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Displacement Current

Constants

Constitutive Parameters of Materials

Parameter	Units	Free-Space Value
Electrical permittivity, ε	F/m	$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m $\simeq 1/36\pi \times 10^{-9}$ F/m
Magnetic permeability, μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$ H/m
Conductivity, σ	S/m	0

Electric and magnetic fields are connected through the speed of light:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

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Mass of Electron: $9.1093837015 \times 10^{-31}\, kg$

Mass of Proton: $1.67262192369 \times 10^{-27}\, kg$

Elementary Charge: $1.602176634 \times 10^{-19}\, C$