LAST (family) NAME: ____________________________ **Test # 1**

FIRST (given) NAME: ____________________________ **Math 2ZZ3**

SAMPLE TEST 1 B:SOLUTIONS

Test duration: 75 **min.**

Instructions: You **must** use permanent ink. Tests submitted in pencil will not be considered later for remarking. This exam consists of 11 problems on 14 pages (make sure you have all 14 pages). The last two pages are for scratch or overflow work. The total number of points is 50. Do not add or remove pages from your test. No books, notes, or "cheat sheets" allowed. No calculator or other electronic devices are allowed. There is a formula that might be useful on the last page. **GOOD LUCK!**

SOLUTIONS

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NAME: ______________________ **ID #:**______________________

Part I: Enter your answer in the appropriate box. **Provide all details and fully justify your answer in order to receive credit.**

1. *(6 pts.)* Suppose that the function

$$f(x) = \begin{cases} 1, & -1 < x < 0, \\ x, & 0 < x < 1, \end{cases}$$

is expanded as a 2-periodic Fourier series $f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\}$, then,

$a_0 = \boxed{\frac{3}{2}}$ $a_3 = \boxed{-\frac{2}{9\pi^2}}$ $b_3 = \boxed{-\frac{1}{3\pi}}$

Solution. We have $p = 1$. Thus

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$\begin{aligned} a_3 &= \int_{-1}^{1} f(x)\,\cos(3\pi x)\,dx = \int_{-1}^{0} \cos(3\pi x)\,dx + \int_{0}^{1} x\,\cos(3\pi x)\,dx \\ &= \underbrace{\left[\frac{\sin(3\pi x)}{3\pi}\right]_{-1}^{0}}_{0} + \int_{0}^{1} x\left(\frac{\sin(3\pi x)}{3\pi}\right)'\,dx \\ &= \underbrace{\left[\frac{x\,\sin(3\pi x)}{3\pi}\right]_{0}^{1}}_{0} - \int_{0}^{1} \frac{\sin(3\pi x)}{3\pi}\,dx = \left[\frac{\cos(3\pi x)}{(3\pi)^2}\right]_{0}^{1} = -\frac{2}{9\pi^2}. \end{aligned}$$

$$\begin{aligned} b_3 &= \int_{-1}^{1} f(x)\,\sin(3\pi x)\,dx = \int_{-1}^{0} \sin(3\pi x)\,dx + \int_{0}^{1} x\,\sin(3\pi x)\,dx \\ &= \left[-\frac{\cos(3\pi x)}{3\pi}\right]_{-1}^{0} - \int_{0}^{1} x\left(\frac{\cos(3\pi x)}{3\pi}\right)'\,dx \\ &= -\frac{2}{3\pi} + \left[\frac{-x\,\cos(3\pi x)}{3\pi}\right]_{0}^{1} + \int_{0}^{1} \frac{\cos(3\pi x)}{3\pi}\,dx = -\frac{1}{3\pi} + \underbrace{\left[\frac{\sin(3\pi x)}{(3\pi)^2}\right]_{0}^{1}}_{0} = -\frac{1}{3\pi}. \end{aligned}$$

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NAME: ______________________ **ID #:**______________________

2. The $2\,\pi$-periodic function given on the interval $(-\pi, \pi)$ by $f(x) = e^x + e^{-x}$ has the Fourier series expansion

$$f(x) \simeq c\left(1 + \sum_{k=1}^{\infty} \frac{2\,(-1)^k}{1+k^2}\cos(kx)\right)$$

(a) *(3 pts.)* Determine the value of the constant c.

$$c = \boxed{\frac{e^{\pi} - e^{-\pi}}{\pi}}$$

Solution. We have $p = \pi$ and thus

$$c = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x + e^{-x}\,dx = \frac{1}{\pi}\int_0^{\pi} e^x + e^{-x}\,dx = \frac{1}{\pi}\left[e^x - e^{-x}\right]_0^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi}.$$

(b) *(3 pts.)* Determine the sum of the series $s = \displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

$$s = \boxed{\frac{\pi\,(e^{\pi} + e^{-\pi})}{2\,(e^{\pi} - e^{-\pi})} - \frac{1}{2}}$$

Solution. Since $\lim_{x\to\pi^-} f(x) = \lim_{x\to-\pi^+} f(x) = e^{\pi} + e^{-\pi}$, the function $f(x)$ is continuous at $x = \pi$ with $f(\pi) = e^{\pi} + e^{-\pi}$. Its Fourier series evaluated at $x = \pi$ converges thus to that value, i.e.

$$c\left(1 + \sum_{k=1}^{\infty} \frac{2\,(-1)^k}{1+k^2}\cos(k\pi)\right) = c\left(1 + 2\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right) = e^{\pi} + e^{-\pi}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{1+k^2} = \frac{\pi\,(e^{\pi} + e^{-\pi})}{2\,(e^{\pi} - e^{-\pi})} - \frac{1}{2}.$$

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NAME: ______________________ **ID #:**______________________

3. *(4 pts.)* For what value of t is the tangent line to the curve

$$\mathbf{r}(t) = \frac{1}{12}t^3\ \mathbf{i} + \frac{1}{2}t^2\ \mathbf{j} + \ln t\ \mathbf{k}, \quad t > 0$$

parallel to the vector $\langle 6, 12, 3\rangle$?

$$t = \boxed{2}.$$

Solution. A direction vector for the tangent line to the curve at the point P with $\overrightarrow{OP} = \mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \frac{1}{4}\,t^2\ \mathbf{i} + t\ \mathbf{j} + \frac{1}{t}\ \mathbf{k}.$$

The tangent line is parallel to vector $\langle 6, 12, 3\rangle$ if there is a number $\lambda \neq 0$ such that

$$\frac{1}{4}\,t^2\ \mathbf{i} + t\ \mathbf{j} + \frac{1}{t}\ \mathbf{k} = \lambda\ (6\,\mathbf{i} + 12\ \mathbf{j} + 3\ \mathbf{k})$$

or

$$\frac{1}{4}\,t^2 = 6\,\lambda, \quad t = 12\,\lambda, \quad \frac{1}{t} = 3\,\lambda.$$

Using the last 2 equations, we get

$$t = 12\,\lambda = 12\,\frac{1}{3\,t} = \frac{4}{t}, \text{so } t^2 = 4 \text{ and } t = \pm 2.$$

Since $\lambda > 0$ from the first equation, $t > 0$ and thus $t = 2$ and all the equations hold for $\lambda = \frac{1}{6}$.

NAME: ______________________ **ID #:**______________________

4. *(4 pts.)* Find the curvature κ of curve C parametrized by

$$\mathbf{r}(t) = \langle -t^2 - 1,\, -t^2 - t - 2,\, 2\,t\rangle, \quad -\infty < t < \infty,$$

at the point $(-2, -2, -2)$.

$$\kappa = \boxed{\frac{2}{9}}$$

Solution. Note that $\langle -2, -2, -2\rangle = \mathbf{r}(-1)$. We have

$$\mathbf{r}'(t) = \langle -2\,t,\, -2\,t - 1,\, 2\rangle \quad \text{and} \quad \mathbf{r}''(t) = \langle -2,\, -2,\, 0\rangle$$

and

$$\mathbf{r}'(-1) = \langle 2,\, 1,\, 2\rangle \quad \text{and} \quad \mathbf{r}''(-1) = \langle -2,\, -2,\, 0\rangle$$

We have

$$\|\mathbf{r}'(-1)\| = \sqrt{4+1+4} = \sqrt{9} = 3$$

and

$$\mathbf{r}'(-1) \times \mathbf{r}''(-1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ -2 & -2 & 0 \end{vmatrix} = \langle 4,\, -4,\, -2\rangle.$$

Thus, $\|\mathbf{r}'(-1) \times \mathbf{r}''(-1)\| = 6$ and

$$\kappa(-1) = \frac{\|\mathbf{r}'(-1) \times \mathbf{r}''(-1)\|}{\|\mathbf{r}'(-1)\|^3} = \frac{6}{3^3} = \frac{2}{9}.$$

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NAME: ______________________ **ID #:**______________________

5. Consider the function

$$f(x, y, z) := x^2 + y^2 - z^2$$

and its level surface S defined by $f(x, y, z) = 1$.

(a) *(2 pts.)* Compute the tangent plane of S at the point $Q = (1, 1, -1)$. If the equation is written in the form $a\,x + b\,y + c\,z = 1$, then

$$a = \boxed{\ 1\ } \qquad b = \boxed{\ 1\ } \qquad c = \boxed{\ 1\ }$$

Solution. We have $\nabla f(x, y, z) = \langle 2\,x, 2\,y,\ -2\,z\rangle$. A normal vector to S at Q is thus $\langle 2, 2,\ 2\rangle$. The equation of the tangent plane of S at Q is thus

$$2\,(x - 1) + 2\,(y - 1) + 2\,(z + 1) = 0 \quad \text{or} \quad x + y + z = 1.$$

(b) *(3 pts.)* Compute the normal line to S at the point $R = (1, 2, 2)$. Find the point of intersection P of the normal line with the tangent plane computed in part (a).

$$P = \boxed{\text{(-3, -6, 10)}}$$

Solution. The normal to S at R is $\langle 2, 4,\ -4\rangle$ or $\langle 1, 2,\ -2\rangle$ and the normal line to S at R can be parametrized by

$$x = 1 + t, \quad y = 2 + 2\,t \quad z = 2 - 2\,t.$$

The line intersect the plane in part (a) when

$$x + y + z = 1 + t + 2 + 2\,t + 2 - 2\,t = t + 5 = 1, \quad \text{i.e. when } t = -4.$$

The point of intersection is thus $(-3, -6, 10)$.

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NAME: ______________________ **ID #:**______________________

6. Consider the function

$$f(x, y, z) := \log(1 + x^2) - 3\, e^{y^2}\, x - z^2.$$

(a) *(3 pts.)* Compute d, the directional derivative of $f(x, y, z)$ at the point $(1, 0, -4)$ in the direction of the vector $\langle -6, 0, 1\rangle$.

$$d = \boxed{\frac{20}{\sqrt{37}}}$$

Solution. We have

$$\nabla f(x, y, z) = \left\langle \frac{2x}{1 + x^2} - 3\, e^{y^2},\ -6\, y\, e^{y^2}\, x, -2\, z \right\rangle$$

and

$$\nabla f(1, 0, -4) = \langle -2,\ 0, 8\rangle\,.$$

A unit vector with the same direction as $\langle -6, 0, 1\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{37}}\langle -6, 0, 1\rangle$. Thus

$$d = (D_{\mathbf{u}} f)\,(1, 0, -4) = \nabla f(1, 0, -4) \cdot \mathbf{u} = \frac{20}{\sqrt{37}}.$$

(b) *(2 pts.)* Compute m, the maximal rate of change of $f(x, y, z)$ at the point $(1, 0, -4)$

$$m = \boxed{2\sqrt{17}}$$

Solution.

$$m = \|\nabla f(1, 0, -4)\| = \|\langle -2,\ 0, 8\rangle\| = 2\sqrt{17}.$$

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NAME: ______________________ **ID #:**______________________

PART II: Multiple choice. Indicate your choice very clearly. There is only one correct answer in each multiple-choice problem. Circle the letter (A,B,C,D or E) corresponding to your choice. Ambiguous answers will be marked as wrong. **Provide all details and fully justify your answer in order to receive credit. No marks will be given if the answer circled is incorrect or if it is correct but not justified.**

7. *(4 pts.)* A 2π-periodic function $f(x)$ has the sine Fourier series expansion

$$f(x) \simeq \sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin(nx).$$

Then, $f(x)$ can also be expanded as the complex Fourier series

(A) $f(x) \simeq \sum_{n=-\infty}^{\infty} \frac{n}{1+n^2} e^{inx}$

(B) $f(x) \simeq \sum_{n=-\infty}^{\infty} \frac{n}{i+n^2} e^{inx}$

(C) $f(x) \simeq \sum_{n=-\infty}^{\infty} \frac{2i}{1+n^2} e^{inx}$

(D) $f(x) \simeq \sum_{n=-\infty}^{\infty} \frac{1}{1+in} e^{inx}$

$\rightarrow$ **(E)** $f(x) \simeq \sum_{n=-\infty}^{\infty} \frac{n}{(2i)(1+n^2)} e^{inx}$

Solution. Since $\sin(nx) = \dfrac{e^{inx} - e^{-inx}}{2i}$,

$$\sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin(nx) = \sum_{n=1}^{\infty} \frac{n}{1+n^2} \frac{e^{inx} - e^{-inx}}{2i} = \sum_{n=1}^{\infty} \frac{n}{1+n^2} \frac{e^{inx}}{2i} + \sum_{n=1}^{\infty} \frac{-n}{1+n^2} \frac{e^{-inx}}{2i}$$
$$= \sum_{n=-\infty}^{\infty} \frac{n}{(2i)(1+n^2)} e^{inx}$$

NAME: ______________________ **ID #:**______________________

8. *(4 pts.)* Let $f(x)$ be the 2π-periodic function $f(x)$ such that $f(x) = x$, for $-\pi < x < \pi$. Then, a periodic solution $y(x)$ of the differential equation $y'' + 5\,y = f(x)$ has the Fourier series expansion

(A) $y(x) \simeq \displaystyle\sum_{n=1}^{\infty} \frac{2\,(-1)^n}{(5-n^2)} \sin(nx)$

(B) $y(x) \simeq \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,(1-5\,n^2)} \sin(nx)$

→**(C)** $y(x) \simeq \displaystyle\sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\,(5-n^2)} \sin(nx)$

(D) $y(x) \simeq \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\,(5-n^2)} \sin(nx)$

(E) $y(x) \simeq \displaystyle\sum_{n=1}^{\infty} \frac{2\,(-1)^n}{(1+n^2)\,(n^2-5)} \sin(nx)$

Solution. The function $f(x)$ has the Fourier series expansion $f(x) \simeq \sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} x\,\frac{-(\cos(nx))'}{n}\,dx \\ &= \frac{2}{\pi} \left[-x\,\frac{\cos(nx)}{n}\right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \frac{\cos(nx)}{n}\,dx \\ &= \frac{2\,(-1)^{n+1}}{n} + \frac{2}{\pi}\,, \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi} = \frac{2\,(-1)^{n+1}}{n}. \end{aligned}$$

Since a particular solution of $y'' + 5\,y = \sin(nx)$ is given by $\dfrac{\sin(nx)}{5-n^2}$, a periodic solution of the differential equation $y'' + 5\,y = f(x)$ is given by the Fourier series

$$y(x) \simeq \sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\,(5-n^2)} \sin(nx).$$

NAME: ______________________ **ID #:**______________________

9. *(4 pts.)* A particle starts moving from the origin at time $t = 0$ with initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$. Its acceleration at time t is given by

$$\mathbf{a}(t) = 2\,\mathbf{i} + (6\,t - 2)\,\mathbf{j} - 6\,t\,\mathbf{k}.$$

At which time t does it cross the plane $x + y + z = 6$?

(A) $t = 1$

(B) $t = 2$

→**(C)** $t = 3$

(D) $t = 4$

(E) $t = 5$

Solution. The particle's velocity at time t is given by

$$\mathbf{v}(t) = (2\,t + C_1)\,\mathbf{i} + (3\,t^2 - 2\,t + C_2)\,\mathbf{j} + (-3\,t^2 + C_3)\,\mathbf{k}.$$

and, using the condition $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$, we obtain

$$\mathbf{v}(t) = (2\,t + 1)\,\mathbf{i} + (3\,t^2 - 2\,t)\,\mathbf{j} + (-3\,t^2 + 1)\,\mathbf{k}.$$

The position at time t is thus

$$\mathbf{r}(t) = (t^2 + t + D_1)\,\mathbf{i} + (t^3 - t^2 + D_2)\,\mathbf{j} + (-\,t^3 + t + D_3)\,\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, we have thus

$$\mathbf{r}(t) = (t^2 + t)\,\mathbf{i} + (t^3 - t^2)\,\mathbf{j} + (-\,t^3 + t)\,\mathbf{k}.$$

The particle crosses the plane $x + y + z = 6$ when

$$(t^2 + t) + (t^3 - t^2) + (-\,t^3 + t) = 6, \quad \text{i.e.} \quad 2\,t = 6 \quad \text{or} \quad t = 3.$$

NAME: ______________________ **ID #:**______________________

10. *(4 pts.)* Suppose $z = f(u, v, w)$, $u = x + 2y$, $v = x + yx$, $w = x^2 + y^3$, and

$$\frac{\partial f}{\partial u}(1,2,2) = -1 \qquad \frac{\partial f}{\partial u}(3,2,2) = 1 \qquad \frac{\partial f}{\partial u}(1,1,1) = -3$$

$$\frac{\partial f}{\partial v}(1,2,2) = 2 \qquad \frac{\partial f}{\partial v}(3,2,2) = 0 \qquad \frac{\partial f}{\partial v}(1,1,1) = 2$$

$$\frac{\partial f}{\partial w}(1,2,2) = 2 \qquad \frac{\partial f}{\partial w}(3,2,2) = 0 \qquad \frac{\partial f}{\partial w}(1,1,1) = 2$$

The value of the partial derivative $\frac{\partial z}{\partial x}$ at the point $(x, y) = (1, 1)$ is

→**(A)** 1 **(D)** 4

(B) 2 **(E)** 5

(C) 3

Solution. Note that, when $(x, y) = (1, 1)$, we have $(u, v, w) = (3, 2, 2)$. By the chain rule

$$\begin{aligned}\frac{\partial z}{\partial x}(1,1) &= \frac{\partial f}{\partial u}(3,2,2)\,\frac{\partial u}{\partial x}(1,1) + \frac{\partial f}{\partial v}(3,2,2)\,\frac{\partial v}{\partial x}(1,1) + \frac{\partial f}{\partial w}(3,2,2)\,\frac{\partial w}{\partial x}(1,1) \\ &= (1)\,(1) + (0)\,(1) + (0)\,(2) = 1.\end{aligned}$$

NAME: ____________________ **ID #:**____________________

11. *(4 pts.)* Find a function $g(z)$ such that the vector field

$$\mathbf{F}(x, y, z) := \langle 4y,\, 4x + g(z),\, 6yz^2 \rangle$$

satisfies $\operatorname{curl}(\mathbf{F}) = \langle 1, 0, 0 \rangle$.

(A) $g(z) = z^3 - z + 3$

→**(B)** $g(z) = 2z^3 - z + 3$

(C) $g(z) = 2z^3 + 3$

(D) $g(z) = 2z^3$

(E) $g(z) = z^3 - 1$

Solution. We have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 4x + g(z) & 6yz^2 \end{vmatrix} = (6\,z^2 - g'(z))\,\mathbf{i} + (0)\,\mathbf{i} + (0)\,\mathbf{k} = \mathbf{i}$$

if and only if $6\,z^2 - g'(z) = 1$ or $g'(z) = 6\,z^2 - 1$. Integrating yields $g(z) = 2\,z^3 - z + C$. Choosing $C = 3$ yields the solution $g(z) = 2\,z^3 - z + 3$.

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NAME: ______________________ **ID #:**______________________

SCRATCH

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NAME: ______________________ **ID #:**______________________

SCRATCH

Some formulas you may use:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}, \qquad a_N = \kappa v^2 = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta), \quad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta), \quad \cosh t = \frac{e^t+e^{-t}}{2}, \quad \sinh t = \frac{e^t-e^{-t}}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right\}, \quad -p < x < p,$$

where,

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx.$$

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