LAST (family) NAME: __________________________ **Test # 2**

FIRST (given) NAME: __________________________ **Math 2ZZ3**

SAMPLE TEST 2-b: SOLUTIONS

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NAME: ______________________ **ID #:**______________________

Part I: Enter your answer in the appropriate box. **Provide all details and fully justify your answer in order to receive credit.**

1. (a) *(4 pts.)* Find the equation of the plane tangent to the surface with equation $x^2 + y^2 - z^2 = 0$ at the point $P = (3, 4, 5)$.

Solution. The surface is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = x^2 + y^2 - z^2$. We have

$$\nabla F(x, y, z) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \langle 2\,x, 2\,y, -2\,z \rangle$$

and

$$\nabla F(3, 4, 5) = \langle 6, 8, -10 \rangle$$

The equation of the tangent plane at $P = (3, 4, 5)$ is thus

$$\boxed{6\,(x-3) + 8\,(y-4) - 10\,(z-5) = 0 \quad \text{or} \quad 3\,x + 4\,y - 5\,z = 0.}$$

(b) *(4 pts.)* Prove that any plane tangent to the surface $x^2 + y^2 - z^2 = 0$ passes through the origin.

Solution. If (x_0, y_0, z_0) is a point on the surface, we have $x_0^2 + y_0^2 - z_0^2 = 0$. Since $\nabla F(x_0, y_0, z_0) = \langle 2\,x_0, 2\,y_0, -2\,z_0 \rangle$ and the tangent plane to the surface at that point has equation

$$x_0\,(x - x_0) + y_0\,(y - y_0) - z_0\,(z - z_0) = 0.$$

For the point $(0, 0, 0)$ to satisfy the equation of the tangent plane, we need

$$-x_0^2 - y_0^2 + z_0^2 = 0,$$

which is always true since (x_0, y_0, z_0) is a point on the surface.

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NAME: ______________________ **ID #:**______________________

2. *(6 pts.)* Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$$

by changing the order of integration.

Solution. We have

$$I := \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy = \iint_D \sqrt{x^3+1}\, dA$$

where D is the region of type II

$$D = \{(x,y),\ 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}.$$

Expressing D as a region of type I, we have

$$D = \{(x,y),\ 0 \le x \le 1,\ 0 \le y \le x^2\}.$$

Therefore,

$$\begin{aligned} I &= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 x^2\sqrt{x^3+1}\, dx \\ &= \left[\frac{2}{9}\,(x^3+1)^{3/2}\right]_0^1 = \frac{2}{9}\,(2^{3/2}-1). \end{aligned}$$

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NAME: ______________________ **ID #:**______________________

3. (a) *(4 pts.)* Find the values of A and B for which the vector field

$$\mathbf{F}(x, y, z) = \langle B\,y + y\,z, x + z^3 + x\,z, A\,y\,z^2 + x\,y\rangle$$

is conservative.

Solution. We have

$$\begin{aligned}\nabla \times \mathbf{F}(x, y, z) &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B\,y + y\,z & x + z^3 + x\,z & A\,y\,z^2 + x\,y \end{bmatrix} \\ &= (A\,z^2 + x - 3\,z^2 - x)\,\mathbf{i} + (y - y)\,\mathbf{j} + (1 + z - B - z)\,\mathbf{k} \\ &= (A - 3)\,z^2\,\mathbf{i} + (1 - B)\,\mathbf{k} = \mathbf{0}\end{aligned}$$

if and only if $A = 3$ and $B = 1$.

(b) *(4 pts.)* Find a potential function for the conservative vector field $\mathbf{F}$ found in part (a).

Solution. We have

$$\frac{\partial f}{\partial x} = y + y\,z, \quad \frac{\partial f}{\partial y} = x + z^3 + x\,z, \quad \frac{\partial f}{\partial z} = 3\,y\,z^2 + x\,y. \quad (1)$$

Integrating both sides of the first equation yields $f(x, y, z) = x\,y + x\,y\,z + C(y, z)$. Differentiating with respect to y, we obtain

$$\frac{\partial f}{\partial y} = x + x\,z + \frac{\partial C}{\partial y} = x + z^3 + x\,z,$$

showing that $\frac{\partial C}{\partial y} = z^3$ and $C(y, z) = y\,z^3 + D(z)$. We have thus

$$f(x, y, z) = x\,y + x\,y\,z + y\,z^3 + D(z)$$

and

$$\frac{\partial f}{\partial z} = x\,y + 3\,y\,z^2 + D'(z) = 3\,y\,z^2 + x\,y.$$

Thus $D(z) = D$, a constant. Taking $D = 0$, we find thus the potential function

$$\boxed{f(x, y, z) = x\,y + x\,y\,z + y\,z^3.}$$

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NAME: ______________________ **ID #:**______________________

(c) *(3 pts.)* Let $\mathbf{F}$ be the conservative vector field found in part (a). Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any path starting at $(0,0,0)$ and ending at $(1,1,1)$.

Solution. We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 3 - 0 = 3$$

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NAME: ______________________ **ID #:**______________________

4. Let D be the region in the first quadrant of the x, y plane bounded by the curves $y = x^2$ and $x = y^2$ and let C be the positively oriented boundary of D.

(a) *(5 pts.)* Compute the line integral

$$\oint_C (x - y)\,dx + (x + y)\,dy$$

directly after parametrizing both parts of C appropriately.

Solution. The 2 curves intersect at $(0, 0)$ and $(1, 1)$. The curve C consists of two parts:

- C_1: the part of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$,
- C_2: the part of the parabola $x = y^2$ from $(1, 1)$ back to $(0, 0)$.

We can parametrize C_1 by letting $x = t, y = t^2$, where $0 \le t \le 1$. We have $x'(t) = 1$, $y'(t) = 2\,t$ and

$$\int_{C_1} (x - y)\,dx + (x + y)\,dy = \int_0^1 (t - t^2)(1) + (t + t^2)\,(2\,t)\,dt$$
$$= \int_0^1 2\,t^3 + t^2 + t\,dt = \left[\frac{t^4}{2} + \frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 = \frac{4}{3}.$$

The curve traversing C_2 in the opposite direction (i.e. from $(0, 0)$ to $(1, 1)$ can be parametrized by letting $y = t, x = t^2$, where $0 \le t \le 1$. We have $x'(t) = 2\,t$, $y'(t) = 1$ and

$$\int_{C_2} (x - y)\,dx + (x + y)\,dy = -\int_0^1 (t^2 - t)(2\,t) + (t^2 + t)\,(1)\,dt$$
$$= -\int_0^1 2\,t^3 - t^2 + t\,dt = \int_0^1 -2\,t^3 + t^2 - t\,dt = \left[-\frac{t^4}{2} + \frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = -\frac{2}{3}.$$

Thus,

$$\oint_C (x-y)\,dx+(x+y)\,dy = \int_{C_1} (x-y)\,dx+(x+y)\,dy+\int_{C_2} (x-y)\,dx+(x+y)\,dy = \frac{4}{3}-\frac{2}{3} = \frac{2}{3}.$$

NAME: ______________________ **ID #:**______________________

(b) *(5 pts.)* Compute the line integral in part (a) using Green's theorem.

Solution. The region D bounded by C can be expressed as the type I region

$$D = \{(x, y), 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\}.$$

Using Green's theorem,

$$\oint_C (x - y)\, dx + (x + y)\, dy = \iint_D \frac{\partial(x + y)}{\partial x} - \frac{\partial(x - y)}{\partial y}\, dA$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2\, dy\, dx = 2 \int_0^1 \sqrt{x} - x^2\, dx = 2 \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3\right]_0^1$$
$$= \frac{2}{3}.$$

NAME: ______________________ **ID #:**______________________

PART II: Multiple choice. Indicate your choice very clearly. There is only one correct answer in each multiple-choice problem. Circle the letter (A,B,C,D or E) corresponding to your choice. Ambiguous answers will be marked as wrong. You don't need to justify your answers and no negative marks are given for a wrong answer.

5. *(3 pts.)* Let $g = \operatorname{div}(\mathbf{F} + \operatorname{curl}(\mathbf{F}))$ where

$$\mathbf{F}(x, y, z) = -x\,y\,\mathbf{i} + x\,z^2\,\ln(y^2+1)\,\mathbf{j} + e^{xyz}\,\mathbf{k}.$$

The value of a for which $g(a, 2, 0) = 0$ is

(A) -2

(B) -1

(C) 0

$\rightarrow$ **(D)** 1

(E) 2

Solution. Since $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$, we have

$$\begin{aligned} g(x, y, z) = \operatorname{div}(\mathbf{F} + \operatorname{curl}(\mathbf{F})) = \operatorname{div}(\mathbf{F}) &= \frac{\partial(-x\,y)}{\partial x} + \frac{\partial(x\,z^2\,\ln(y^2+1))}{\partial y} + \frac{\partial(e^{xyz})}{\partial z} \\ &= -y + \frac{2\,x\,y\,z^2}{1+y^2} + x\,y\,e^{xyz}. \end{aligned}$$

Thus, $g(a, 2, 0) = -2 + 2\,a = 0$ if and only if $a = 1$.

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NAME: ______________________ **ID #:**______________________

6. *(3 pts.)* Let

$$\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + x\,e^y\,\mathbf{j} + (z+1)\,e^z\,\mathbf{k}$$

and suppose C is path parametrized by

$$\mathbf{r}(t) = t\,\sin^3(\pi t/2)\mathbf{i} + t^2\,e^{t-1}\,\mathbf{j} + t^{100}\,\mathbf{k}, \quad 0 \le t \le 1.$$

Which of the following is equal to $\int_C \mathbf{F} \cdot d\mathbf{r}$?
(**Hint.** Is the vector field conservative?)

(A) 0

→ **(B)** $2\,e$

(C) $4\,e$

(D) $-2\,e$

(E) $-4\,e$

Solution. By inspection, a potential function for $\mathbf{F}$ is given by

$$f(x, y, z) = x\,e^y + \int (z+1)\,e^z\,dz = x\,e^y + z\,e^z$$

Since $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ and $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 2\,e.$$

NAME: ______________________ **ID #:**______________________

7. *(3 pts.)* Evaluate the line integral

$$I = \int_C x\, ds,$$

where C is the arc of parabola $y = x^2$ fron $(0,0)$ to $(1,1)$.

Solution.

(A) $I = 1$

(B) $I = \dfrac{2\sqrt{2} - 1}{6}$

(C) $I = 0$

(D) $I = \dfrac{\sqrt{5}}{2}$

$\rightarrow$ **(E)** $I = \dfrac{5\sqrt{5} - 1}{12}$

Solution. We can parametrize C using $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$. We have

$$\mathbf{r}'(t) = \langle 1, 2\,t \rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 4\,t^2}.$$

Thus,

$$I = \int_0^1 t\sqrt{1 + 4\,t^2}\, dt = \left[\frac{1}{12}\,(1 + 4\,t^2)^{3/2}\right]_0^1 = \frac{5^{3/2} - 1}{12}.$$

NAME: ______________________ **ID #:**______________________

8. *(3 pts.)* Let R be the solid region inside the cylinder $x^2 + y^2 = x$, above the x, y plane and below the paraboloid $z = 2 + x^2 + y^2$. Then the volume of R can be obtained by computing the double integral:

Solution.

(A) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} 2\,r + r^3\,dr\,d\theta$

→**(B)** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} 2\,r + r^3 dr d\theta$

(C) $\displaystyle\int_0^1 \int_0^{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}} 2 + x^2 + y^2\,dy\,dx$

(D) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} 2 + r^2\,dr\,d\theta$

(E) $\displaystyle\int_0^{\frac{1}{2}} \int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} 2 + x^2 + y^2\,dy\,dx$

Solution. R is the region above the region D in the x, y plane inside the circle $x^2 + y^2 = x$ and below the graph of $z = 2 + x^2 + y^2$. Thus,

$$\text{Vol}(R) = \iint_D 2 + x^2 + y^2\,dA$$

The circle $x^2 + y^2 = x$ can be expressed as the polar curve $r = \cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and the region D can be expressed as the region

$$D^* = \{(r, \theta),\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \cos\theta\}$$

in polar coordinates. Thus,

$$\text{Vol}(R) = \iint_{D^*} (2 + r^2)\,r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} 2\,r + r^3 dr d\theta.$$

NAME: ______________________ **ID #:**______________________

9. *(3 pts.)* Compute the integral $\frac{1}{2}\oint_C x\,dy - y\,dx$ where C is the polar curve $r = 3 + \cos\theta$, $0 \le \theta \le 2\,\pi$.
(**Hint.** Use Green's theorem.)

→**(A)** $\frac{19\,\pi}{2}$

(B) $\frac{15\,\pi}{2}$

(C) 0

(D) $-\frac{7\,\pi}{2}$

(E) $-\frac{3\,\pi}{2}$

Solution. By Green's theorem, we have

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \iint_D 1\,dA$$

where D is the region inside the curve C. This region can be expressed in polar coordinates as

$$D^* = \{(r,\theta),\ 0 \le \theta \le 2\,\pi, 0 \le r \le 3 + \cos\theta\}.$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\oint_C x\,dy - y\,dx &= \int_0^{2\pi}\int_0^{3+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(3+\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 9 + 6\,\cos\theta + \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} 9 + 6\,\cos\theta + \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \left[\frac{19}{4}\theta + 3\,\sin\theta + \frac{\sin(2\theta)}{8}\right]_0^{2\pi} = \frac{19\,\pi}{2}
\end{aligned}$$

NAME: ______________________ **ID #:**______________________

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NAME: ______________________ **ID #:**______________________

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Some formulas you may use:

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t), \qquad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t).$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta), \quad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad \cosh t = \frac{e^t + e^{-t}}{2} \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

THE END