2A04 Tutorial 10

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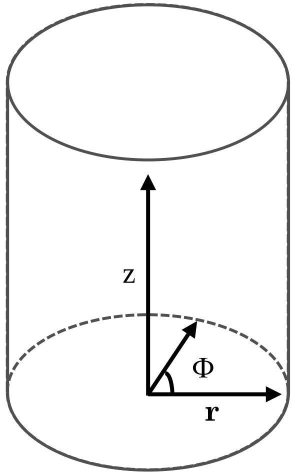
1. Problem 5.24

In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\boldsymbol{H} = \widehat{\boldsymbol{\Phi}}\frac{4}{r}[1 - (1 + 3r)e^{-3r}]$$

Find the current density **J**.



What equation relates **J** to **H**?

$$\boldsymbol{J} = \nabla \times \boldsymbol{H}$$

Curl of a system in cylindrical coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

H_r and H_z are zero, so the curl simplifies to:

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\boldsymbol{H}_{\boldsymbol{\varphi}}\right)\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(4[1 - (1 + 3r)e^{-3r}])\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}[4 - 4(1 + 3r)e^{-3r}]\right]$$

1. Problem 5.24

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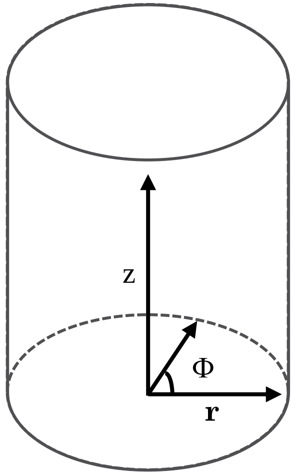
Find the current density **J**.



H_r and H_z are zero, so the curl simplifies to:

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(r\boldsymbol{H_\varphi})\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(4[1-(1+3r)e^{-3r}])\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}[4-4(1+3r)e^{-3r}]\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}\left[\frac{\partial}{\partial r}[(1+3r)e^{-3r}]\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}[3e^{-3r} - 3e^{-3r}(1+3r)]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}[3e^{-3r}(1-(1+3r))]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}[3e^{-3r}(-3r)]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}36e^{-3r}\ \ A/m^2$$

2. Problem 5.27

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

a) Determine **B**.

b) Use Eq 5.66 to calculate the magnetic flux passing through a square loop with 0.25-m long edges if the loop is in the x-y plane, its center is at the origin, and its edges are parallel to the x and y axes.

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

2. Problem 5.27a

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}} 5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

a) Determine **B**.

What equation relates **A** to **B**?

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

- There is no component in y ($A_y = 0, \frac{\partial A_y}{\partial anything} = 0$)
- The x component only depends on y ($\frac{\partial A_x}{\partial z} = 0$)
- The z component only depends on x ($\frac{\partial A_z}{\partial y} = 0$)

$$\boldsymbol{B} = \hat{\boldsymbol{x}}(0 - 0) + \hat{\boldsymbol{y}}\left(0 - \frac{\partial A_z}{\partial x}\right) + \hat{\boldsymbol{z}}\left(0 - \frac{\partial A_x}{\partial y}\right)$$

$$\boldsymbol{B} = -\hat{\boldsymbol{y}}\frac{\partial A_z}{\partial x} - \hat{\boldsymbol{z}}\frac{\partial A_x}{\partial y} = -\hat{\boldsymbol{y}}\frac{\partial}{\partial x}(2 + \sin\pi x) - \hat{\boldsymbol{z}}\frac{\partial}{\partial y}5\cos\pi y$$

$$\boldsymbol{B} = -\hat{\boldsymbol{y}}\pi\cos\pi x + \hat{\boldsymbol{z}}5\pi\sin\pi y$$

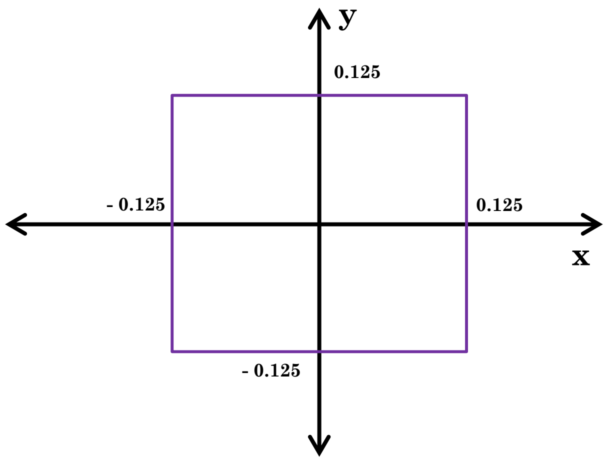
2. Problem 5.27b

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

b) Use Eq 5.66 to calculate the magnetic flux passing through a square loop with 0.25-m long edges if the loop is in the x-y plane, its center is at the origin, and its edges are parallel to the x and y axes.

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad (\text{Wb})$$



$$\Phi = \int_{-0.125}^{0.125}\int_{-0.125}^{0.125} (-\hat{\boldsymbol{y}}\pi\cos\pi x + \hat{\boldsymbol{z}}5\pi\sin\pi y)\hat{\boldsymbol{z}}dxdy$$

$$\Phi = \int_{-0.125}^{0.125}\int_{-0.125}^{0.125} (5\pi\sin\pi y)\hat{\boldsymbol{z}}dxdy = \hat{\boldsymbol{z}}\int_{-0.125}^{0.125} x5\pi\sin\pi y\,\Big|_{x=-0.125}^{x=0.125} dy$$

$$\Phi = -\hat{\boldsymbol{z}}x5\pi\frac{\cos\pi y}{\pi}\Big|_{x=-0.125}^{x=0.125}\Big|_{y=-0.125}^{y=0.125} = -\hat{\boldsymbol{z}}5(0.25)\left(\cos\frac{\pi}{8} - \cos\frac{-\pi}{8}\right)$$

$$\Phi = 0$$

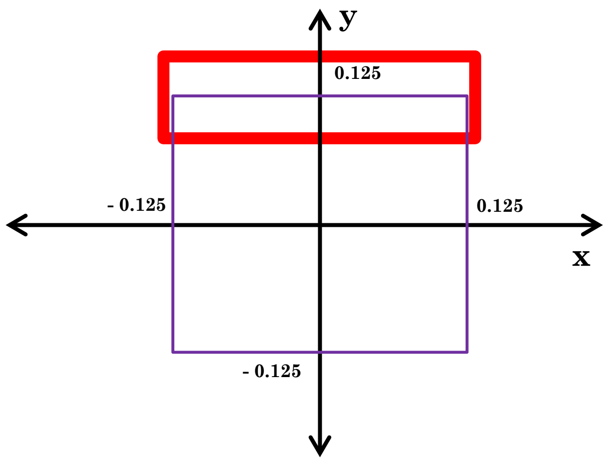
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along y=0.125:

$$S_{top} = \int_{-0.125}^{0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x)\cdot\hat{\boldsymbol{x}}dx$$

$$S_{top} = \int_{-0.125}^{0.125} 5\cos(0.125\pi)\,dx = 4.6194x\Big|_{x=-0.125}^{x=0.125}$$

S_{top} =1.15485

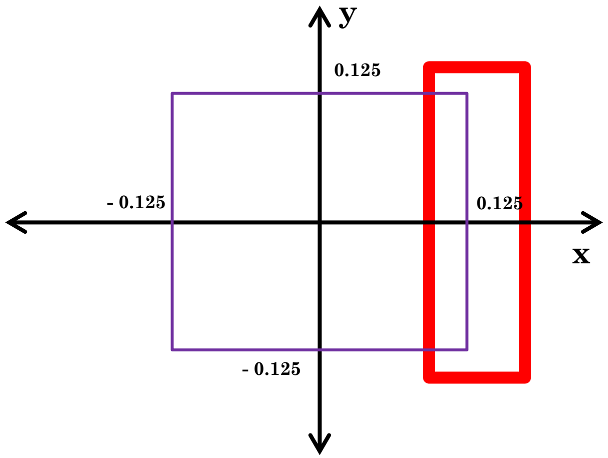
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along x=0.125:

$$S_{right} = \int_{0.125}^{-0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x)\cdot\hat{\boldsymbol{y}}dy$$

$$S_{right} = \int_{0.125}^{-0.125} 0dx = 0$$

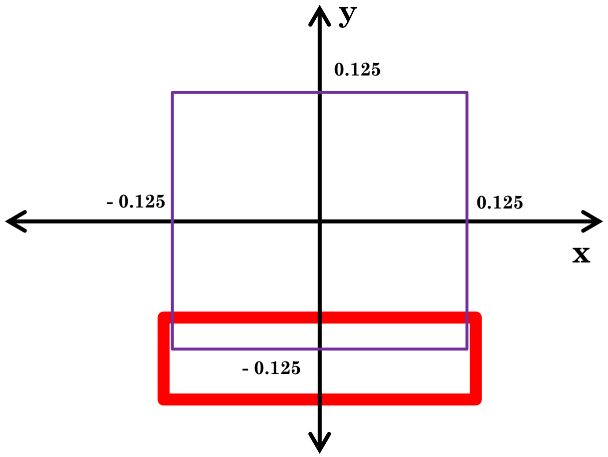
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along y=-0.125:

$$S_{bottom} = \int_{0.125}^{-0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x)\cdot\hat{\boldsymbol{x}}dx$$

$$S_{bottom} = \int_{0.125}^{-0.125} 5\cos(0.125\pi)\,dx = 4.6194x\Big|_{x=0.125}^{x=-0.125}$$

$$S_{bottom} = -1.15485$$

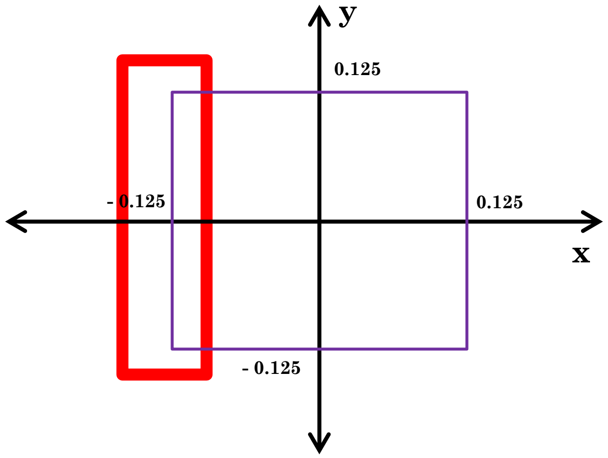
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along x=-0.125:

$$S_{left} = \int_{-0.125}^{0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x)\cdot\hat{\boldsymbol{y}}dy$$

$$S_{left} = \int_{0.125}^{-0.125} 0dx = 0$$

$$\Phi = 1.15485 + 0 - 1.15485 + 0$$
$$\Phi = 0$$

3. Problem 5.31

Iron contains $8.5 \times 10^{28} atoms/m^3$. At saturation, the alignment of the electrons' spin magnetic moments in iron can contribute 1.5T to the total magnetic flux density **B**. If the spin magnetic moment of a single electron is $9.27 \times 10^{-24}\ (A \cdot m^2)$, how many electrons per atom contribute to the saturated field?

The number of electrons contributing to the bulk magnetization *per atom* is n_e, the total number of contributing electrons is N_e, and the number of atoms is N_{atoms}.

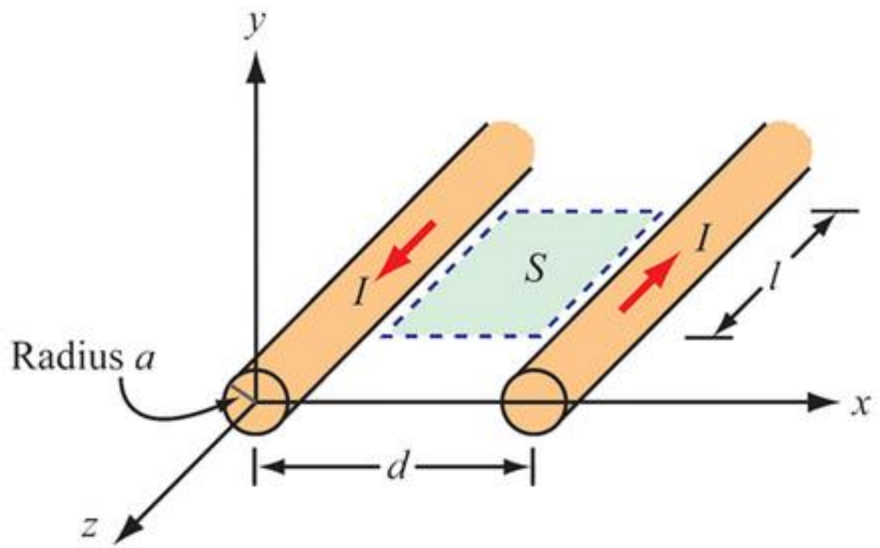
So, $n_e = \frac{N_e}{N_{atoms}}$

From $B_m = \mu_0 M = \mu_0 N_e m_s \;\rightarrow\; N_e = \frac{B_m}{\mu_0 m_s}$

$$n_e = \frac{N_e}{N_{atoms}} = \frac{B_m}{\mu_0 m_s N_{atoms}} = \frac{1.5}{\mu_0 (9.27 \times 10^{-24})(8.5 \times 10^{28})} = 1.5\ electrons/atom$$

4. Problem 5.37

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.

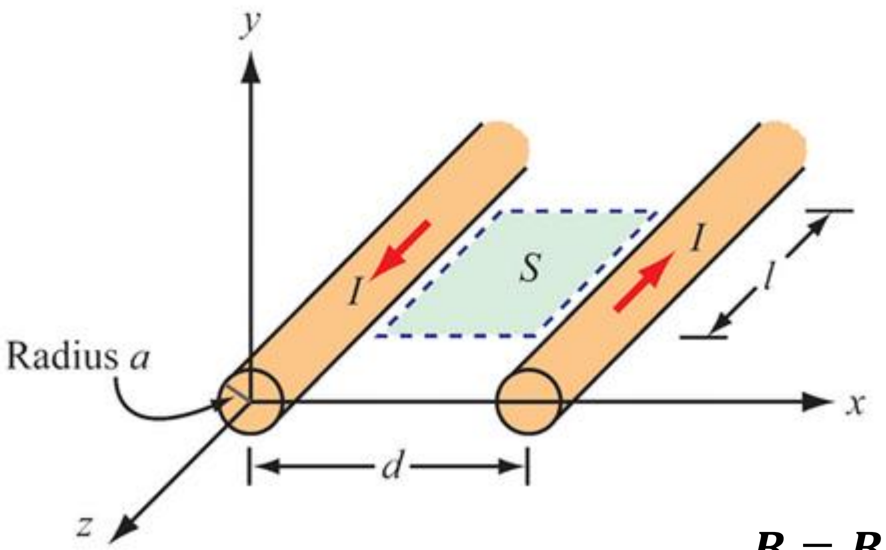


Notes:
- Self-inductance is based on magnetic flux
- Magnetic flux is based on magnetic field strength
- ASSUMPTION: radii are much smaller than the separation distance

Plan:
- Calculate the magnetic field caused by each wire
- Calculate the magnetic flux
- Calculate the inductance

4. Problem 5.37 – step 1

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic field caused by each wire

Currents are both I, the magnetic field at a position x for long wires is:

$$\boldsymbol{B_1} = \hat{y}\frac{\mu I}{2\pi x} \qquad \boldsymbol{B_2} = \hat{y}\frac{\mu I}{2\pi(d-x)}$$

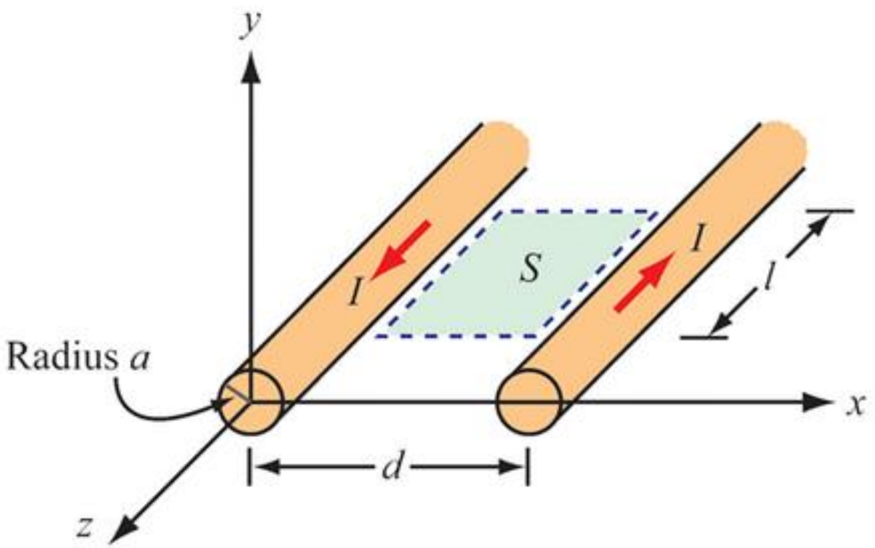
They're both positive, because the currents are in opposite directions.

$$\boldsymbol{B} = \boldsymbol{B_1} + \boldsymbol{B_2} = \hat{\boldsymbol{y}}\frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) = \hat{\boldsymbol{y}}\frac{\mu I}{2\pi}\left(\frac{d-x}{x(d-x)} + \frac{x}{x(d-x)}\right)$$

$$\boldsymbol{B} = \hat{\boldsymbol{y}}\frac{\mu I d}{2\pi x(d-x)}$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

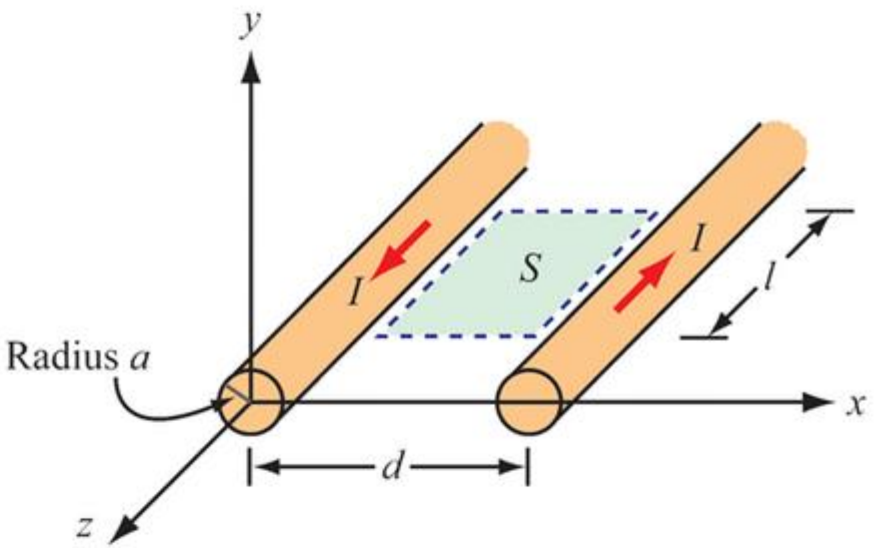
$$\boldsymbol{\Phi} = \int\int \boldsymbol{B} \cdot d\boldsymbol{s}$$

$$\boldsymbol{\Phi} = \int_{z=0}^{l} \int_{x=a}^{d-a} \hat{\boldsymbol{y}} \frac{\mu I d}{2\pi x(d-x)} \cdot \hat{\boldsymbol{y}} dx dz$$

$$\boldsymbol{\Phi} = \frac{\mu I d}{2\pi} \int_{z=0}^{l} \int_{x=a}^{d-a} \frac{1}{x(d-x)} dx dz$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

$$\boldsymbol{\Phi} = \frac{-\mu I d}{2\pi} \int_{z=0}^{l} \int_{x=a}^{d-a} \frac{1}{x(x-d)}\, dx dz$$

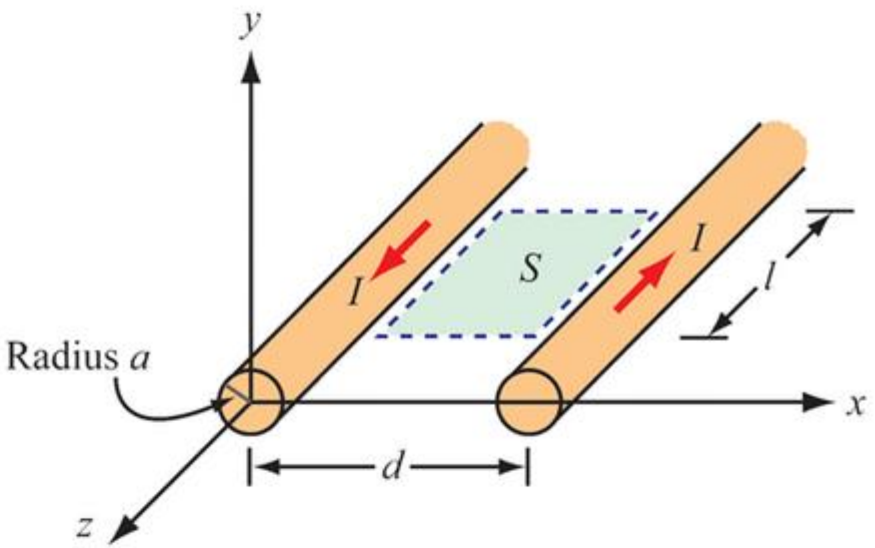
Use common integral:

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x},\ a \neq b$$

$$\boldsymbol{\Phi} = \frac{-\mu I d}{2\pi} \int_{z=0}^{l} \left(\frac{1}{-d-0} \ln \frac{x}{x-d} \Big|_{x=a}^{d-a} \right) dz = \frac{\mu I l}{2\pi} \left(\ln \frac{d-a}{-a} - \ln \frac{a}{a-d} \right)$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

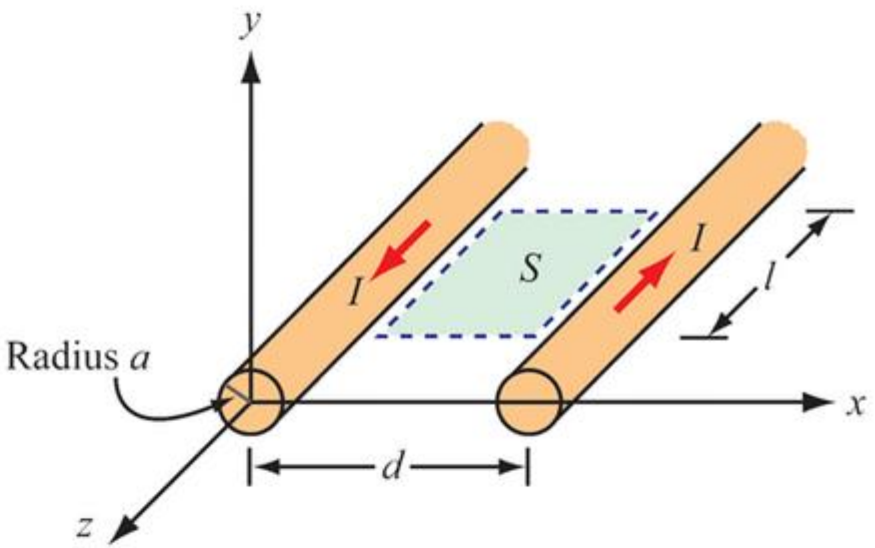
$$\boldsymbol{\Phi} = \frac{\mu I l}{2\pi}\left(\ln\frac{d-a}{-a} - \ln\frac{a}{a-d}\right)$$

$$\boldsymbol{\Phi} = \frac{\mu I l}{2\pi}\left(\ln\frac{(d-a)(a-d)}{-a^2}\right) = \frac{\mu I l}{2\pi}\left(\ln\frac{(d-a)^2}{a^2}\right)$$

$$\boldsymbol{\Phi} = \frac{\mu I l}{\pi}\ln\frac{d-a}{a}$$

4. Problem 5.37

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the inductance

There is only one 'loop', so inductance is:

$$L = \frac{\Phi}{I} = \frac{\mu I l}{\pi} \ln \frac{d-a}{a} \Big/ I = \frac{\mu l}{\pi} \ln \frac{d-a}{a}$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\mu l}{\pi} \ln \frac{d-a}{a}$$

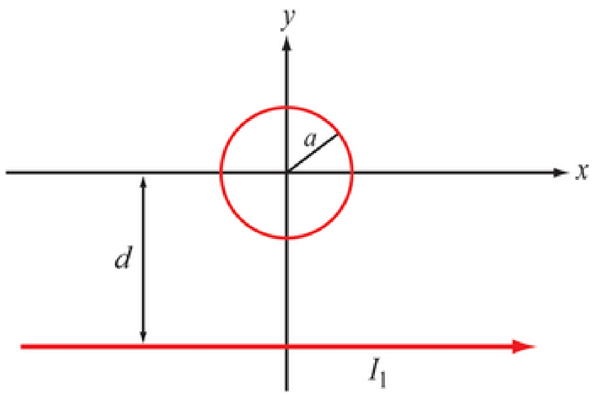
Assuming a is very small relative to d:

$$L' \approx \frac{\mu l}{\pi} \ln \frac{d}{a}$$

5. Problem 5.40

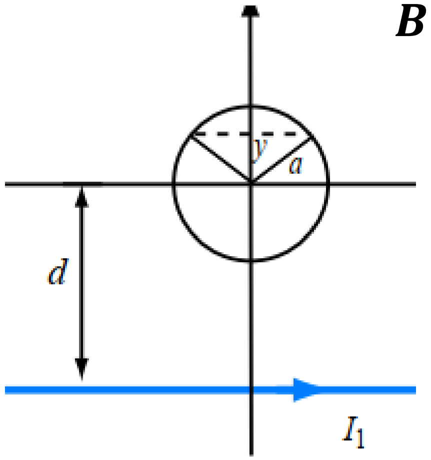
Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).



Find the magnetic field induced by the current on the linear conductor, for any line of constant y:

$$\boldsymbol{B} = \hat{\boldsymbol{z}}\frac{\mu_0 I}{2\pi(d+y)}$$



Find the flux through each infinitesimal strip of constant y inside the loop.

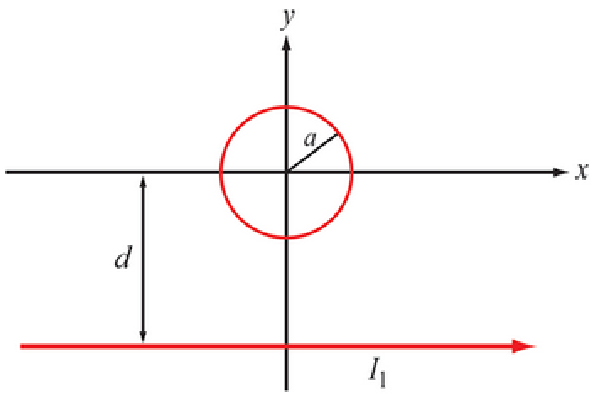
$$d\Phi = \boldsymbol{B}(y)\cdot d\boldsymbol{s} = \hat{\boldsymbol{z}}\frac{\mu_0 I}{2\pi(d+y)}\cdot\hat{\boldsymbol{z}}2\sqrt{(a^2-y^2)}dy$$

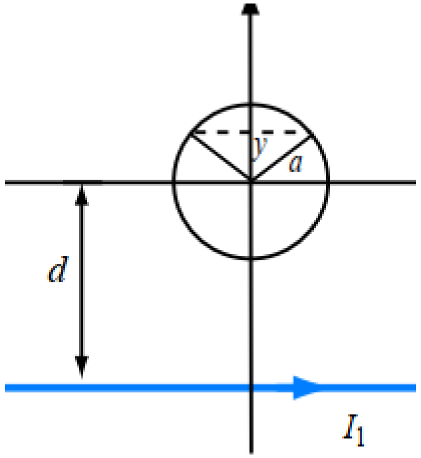
5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$d\Phi = \boldsymbol{B}(y)\cdot d\boldsymbol{s} = \hat{\boldsymbol{z}}\frac{\mu_0 I}{2\pi(d+y)}\cdot\hat{\boldsymbol{z}}2\sqrt{(a^2-y^2)}dy$$

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).





$$d\Phi = \frac{\mu_0 I\sqrt{(a^2-y^2)}}{\pi(d+y)}dy$$

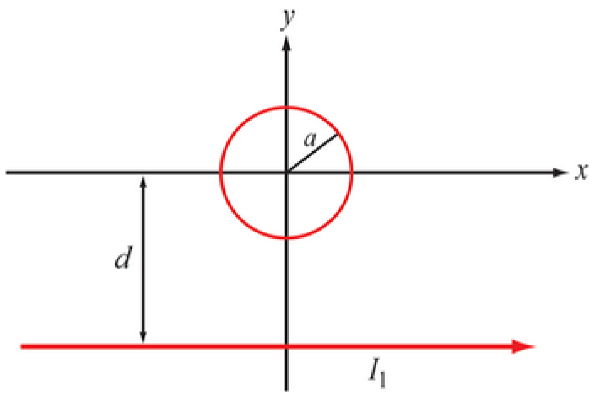
$$L = \frac{\Phi}{I} = \frac{1}{I}\int_S d\Phi = \frac{1}{I}\int_{y=-a}^{a}\frac{\mu_0 I\sqrt{(a^2-y^2)}}{\pi(d+y)}dy$$

$$L = \frac{\mu_0}{\pi}\int_{y=-a}^{a}\frac{\sqrt{(a^2-y^2)}}{(d+y)}dy$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).



Solving the following integral:

$$L = \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{\sqrt{(a^2 - y^2)}}{(d+y)} dy$$

Let $z = d + y$: $L = \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{(a^2 - (z-d)^2)}}{z} dz$

$$L = \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{(a^2 - d^2) + 2dz - z^2}}{z} dz$$

Let $R = a_0 + b_0 z + c_0 z^2$ be the polynomial $(a^2 - d^2) + 2dz - z^2$:

From a table of integrals: $\int \frac{\sqrt{R}}{z} dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

So now we're solving *these* integrals:

Let $R = a_0 + b_0 z + c_0 z^2$ be the polynomial $(a^2 - d^2) + 2dz - z^2$:

$$\int \frac{\sqrt{R}}{z} dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

Evaluating the first term over $d - a$ to $d + a$:

$$\sqrt{R}\Big|_{d-a}^{d+a} = \sqrt{(a^2 - d^2) + 2d(d+a) - (d+a)^2} - \sqrt{(a^2 - d^2) + 2d(d-a) - (d-a)^2} = 0$$

So our integral is now $a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

From a table of integrals:

$$a_0 \int \frac{dz}{z\sqrt{R}} = a_0 \left[\frac{1}{\sqrt{-a_0}} \sin^{-1} \left(\frac{2a_0 + b_0 z}{z\sqrt{{b_0}^2 - 4a_0 c_0}} \right) \right]_{z=d-a}^{d+a}$$

$$= -\sqrt{d^2 - a^2} \left[\sin^{-1} \left(\frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a}$$

$$= -\pi \sqrt{d^2 - a^2}$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

From a table of integrals:

$$\frac{b_0}{z} \int \frac{dz}{\sqrt{R}} = \frac{b_0}{2} \left[\frac{-1}{\sqrt{-c_0}} \sin^{-1} \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right]_{z=d-a}^{d+a}$$

$$= -d \left[\sin^{-1} \left(\frac{d-z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Last step!

$$L = \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{\sqrt{(a^2 - y^2)}}{(d + y)} dy$$

$$L = \frac{\mu_0}{\pi} \left(\pi d - \pi\sqrt{d^2 - a^2} \right)$$

$$L = \mu_0 \left(d - \sqrt{d^2 - a^2} \right)$$

Reminders

- Assignment 10 is out, and is due at 8AM on April 4th.
- Good luck!