5. DYNAMICS

5.1 Introduction

So far we have studied the equations of motion for robots without consideration of forces (*i.e.* the kinematics equations). In this chapter we will consider the problem of relating the joint torques (for revolute joints) and joint forces (for prismatic joints) to the resulting motion of the robot. The equations describing this relationship are the "dynamics equations" for the robot. Applications of these equations are discussed in section 5.4.

5.2 Lagrangian Mechanics

Robots are spatial multiple degrees-of-freedom mechanisms with distributed masses. This makes their dynamics very complex. It is much easier to solve for the dynamics equations of a complex system using "Lagrangian mechanics" than the "Newtonian mechanics" we are more familiar with.

Lagrangian mechanics is based on an analysis of the energy of the system. There are three fundamental equations. First we have

$$L = K - P \tag{5.1}$$

where L is known as the Lagrangian, K is the kinetic energy of the system and P is the potential energy of the system. Second, if the ith joint is a prismatic joint

$$F_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \tag{5.2}$$

where x_i is the joint variable and F_i is the joint force. Third, if the ith joint is a revolute joint

$$\tau_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} \tag{5.3}$$

where θ_i is the joint variable and τ_i is the joint torque.

The next issue is understanding the kinetic and potential energy of the robot. The kinetic energy of link j can be written

$$K_j = \tfrac{1}{2} v_{cj}{}^T m_j v_{cj} + \tfrac{1}{2} \omega_j{}^T I_j \omega_j \tag{5.4}$$

where m_j is the mass of the link, v_{cj} is the linear velocity vector for the centre of mass, ω_j is the angular velocity vector for the link and I_j is the inertia matrix of link j about its centre of mass. For planar motion this may be simplified to

$$K_j = \tfrac{1}{2} m_j v_{cj}{}^2 + \tfrac{1}{2} I_j \omega_j{}^2 \tag{5.5}$$

where v_{cj} is the magnitude of the linear velocity for the centre of mass, ω_j is the angular velocity (scalar quantity) for the link and I_j is the inertia of link j about its centre of mass (scalar quantity).

The potential energy of link j is given by

$$P_j = -m_j G^T p_{cj} \tag{5.6}$$

where G is the gravity vector and p_{cj} is the position vector for the centre of mass of the link. The standard datum for zero potential energy is the configuration when all of the links are horizontal.

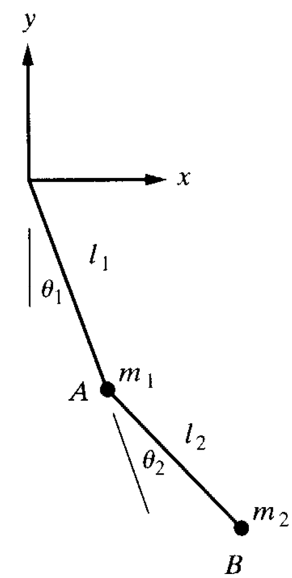

Figure 5.1

Example 5.1 (based on Niku's example 4.3)

A simplified model of a 2R planar robot is shown in Figure 5.1. The robot operates in the vertical plane. The masses of the links are modelled as two concentrated masses.

The centre of mass of link 1 is located at point A, so we have:

$$v_{c1} = v_A = l_1\dot{\theta} \tag{5.7}$$

and the kinetic energy of the first link is

$$\begin{aligned} K_1 &= \tfrac{1}{2} m_1 v_{c1}^{\ 2} + \tfrac{1}{2} I_1 \omega_1^{\ 2} \\ &= \tfrac{1}{2} m_1 v_A^{\ 2} \\ &= \tfrac{1}{2} m_1 l_1^{\ 2} \dot{\theta}_1^{\ 2} \end{aligned} \tag{5.8}$$

where I_1=0 since the mass is concentrated at a point.

The centre of mass of link 2 is located at point B. The velocity of point B can be obtained by differentiating its position equations with respect to time as follows

$$x_B = l_1 S\theta_1 + l_2 S\theta_{12} \tag{5.9}$$

$$y_B = -l_1 C\theta_1 - l_2 C\theta_{12} \tag{5.10}$$

$$\dot{x}_B = \dot{\theta}_1 l_1 C\theta_1 + (\dot{\theta}_1 + \dot{\theta}_2) l_2 C\theta_{12} \tag{5.11}$$

$$\dot{y}_B = \dot{\theta}_1 l_1 S\theta_1 + (\dot{\theta}_1 + \dot{\theta}_2) l_2 S\theta_{12} \tag{5.12}$$

Since $v_B^{\ 2} = \dot{x}_B^{\ 2} + \dot{y}_B^{\ 2}$, the kinetic energy of link 2 (from equation 5.5) is

$$\begin{aligned} K_2 &= \tfrac{1}{2} m_2 v_{c2}^{\ 2} + \tfrac{1}{2} I_2 \omega_2^{\ 2} \\ &= \tfrac{1}{2} m_2 v_{c2}^{\ 2} \\ &= \tfrac{1}{2} m_2 (\dot{x}_B^{\ 2} + \dot{y}_B^{\ 2}) \\ &= \tfrac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) C\theta_2 \end{aligned} \tag{5.13}$$

The potential energy for link 1 (from equation 5.6) is

$$\begin{aligned} P_1 &= -m_1 G^T p_{c1} \\ &= -m_1 \begin{bmatrix} 0 & -g \end{bmatrix} \begin{bmatrix} x_A \\ y_A \end{bmatrix} \\ &= -m_1(-g) y_A \\ &= -m_1(-g)(-l_1 C\theta_1) \\ &= -m_1 g l_1 C\theta_1 \end{aligned} \tag{5.14}$$

The potential energy for link 2 is

$$\begin{aligned} P_2 &= -m_2 G^T p_{c2} \\ &= -m_2 \begin{bmatrix} 0 & -g \end{bmatrix} \begin{bmatrix} x_B \\ y_B \end{bmatrix} \\ &= -m_2(-g) y_B \\ &= -m_2(-g)(-l_1 C\theta_1 - l_2 C\theta_{12}) \\ &= -m_2 g(l_1 C\theta_1 + l_2 C\theta_{12}) \end{aligned} \tag{5.15}$$

So the kinetic energy of the system is $K = K_1 + K_2$, the potential energy of the system is $P = P_1 + P_2$ and the Lagrangian is

$$L = K - P = K_1 + K_2 - P_1 - P_2 \tag{5.16}$$

From equation (5.3) the equations for the joint torques are:

$$\tau_1 = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \tag{5.17}$$

$$\tau_2 = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} \tag{5.18}$$

After calculating the derivatives these equations may be written in the matrix form:

$$\begin{aligned} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} (m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 C\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 C\theta_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & -m_2 l_1 l_2 S\theta_2 \\ m_2 l_1 l_2 S\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 S\theta_2 & -m_2 l_1 l_2 S\theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} \\ &+ \begin{bmatrix} (m_1+m_2) g l_1 S\theta_1 + m_2 g l_2 S\theta_{12} \\ m_2 g l_2 S\theta_{12} \end{bmatrix} \end{aligned} \tag{5.19}$$

These are the dynamics equations for this robot.

Example 5.2 (based on Niku's example 4.4)

A more realistic model of a 2R planar robot is shown in Figure 5.2. The robot operates in the vertical plane. The links are assumed to have uniformly distributed mass. With this assumption the centre of mass will be located at the midpoint of the link. If we also assume that the cross section of the links are much smaller than their lengths then:

$$I_1 = \tfrac{1}{12} m_1 l_1^2 \quad \text{and} \tag{5.20}$$

$$I_2 = \tfrac{1}{12} m_2 l_2^2 \tag{5.21}$$

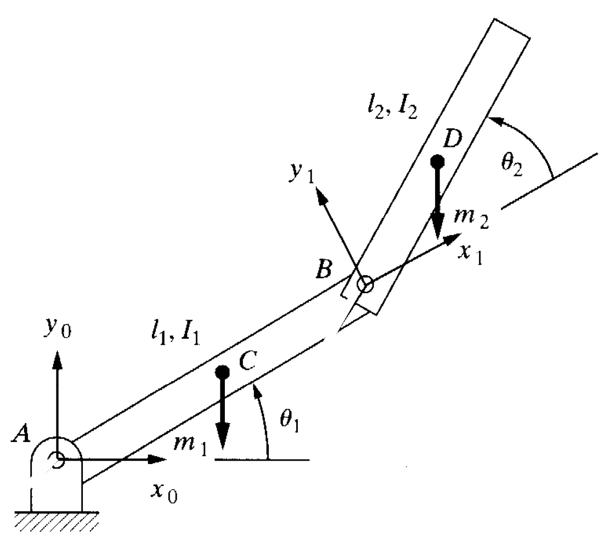

Figure 5.2

Applying equation (5.5) gives for the first link

$$\begin{aligned} K_1 &= \tfrac{1}{2} m_1 v_{c1}^{\ 2} + \tfrac{1}{2} I_1 \omega_1^{\ 2} \\ &= \tfrac{1}{2} m_1 v_C^{\ 2} + \tfrac{1}{2}\left(\tfrac{1}{12} m_1 l_1^{\ 2}\right)\dot{\theta}_1^{\ 2} \\ &= \tfrac{1}{2} m_1 \left(\tfrac{1}{2} l_1 \dot{\theta}_1\right)^2 + \left(\tfrac{1}{24} m_1 l_1^{\ 2}\right)\dot{\theta}_1^{\ 2} \\ &= \left(\tfrac{1}{8} m_1 l_1^{\ 2}\right)\dot{\theta}_1^{\ 2} + \left(\tfrac{1}{24} m_1 l_1^{\ 2}\right)\dot{\theta}_1^{\ 2} \\ &= \left(\tfrac{1}{6} m_1 l_1^{\ 2}\right)\dot{\theta}_1^{\ 2} \end{aligned} \tag{5.22}$$

and for the second link

$$\begin{aligned} K_2 &= \tfrac{1}{2} m_2 v_{c2}^{\ 2} + \tfrac{1}{2} I_2 \omega_2^{\ 2} \\ &= \tfrac{1}{2} m_2 v_D^{\ 2} + \tfrac{1}{2}\left(\tfrac{1}{12} m_2 l_2^{\ 2}\right)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \tfrac{1}{2} m_2 v_D^{\ 2} + \left(\tfrac{1}{24} m_2 l_2^{\ 2}\right)(\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned} \tag{5.23}$$

where v_D may be calculated in a similar way to the previous example.

The potential energy for link 1 (from equation 5.6) is

$$\begin{aligned} P_1 &= -m_1 G^T p_{c1} \\ &= -m_1 \begin{bmatrix} 0 & -g \end{bmatrix} \begin{bmatrix} x_C \\ y_C \end{bmatrix} \\ &= -m_1(-g) y_C \\ &= -m_1(-g)(\tfrac{1}{2} l_1 S\theta_1) \\ &= \tfrac{1}{2} m_1 g l_1 S\theta_1 \end{aligned} \tag{5.24}$$

The potential energy for link 2 is

$$\begin{aligned} P_2 &= -m_2 G^T p_{c2} \\ &= -m_2 \begin{bmatrix} 0 & -g \end{bmatrix} \begin{bmatrix} x_D \\ y_D \end{bmatrix} \\ &= -m_2(-g) y_D \\ &= -m_2(-g)(l_1 S\theta_1 + \tfrac{1}{2} l_2 S\theta_{12}) \\ &= m_2 g (l_1 S\theta_1 + \tfrac{1}{2} l_2 S\theta_{12}) \end{aligned} \tag{5.25}$$

Equations (5.16)-(5.18) apply again since this is also a 2R robot. After calculating the derivatives, the dynamic equations may be written in the matrix form:

$$\begin{aligned} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} \tfrac{1}{3} m_1 l_1^{\ 2} + m_2 l_1^{\ 2} + \tfrac{1}{3} m_2 l_2^{\ 2} + m_2 l_1 l_2 C\theta_2 & \tfrac{1}{3} m_2 l_2^{\ 2} + \tfrac{1}{2} m_2 l_1 l_2 C\theta_2 \\ \tfrac{1}{3} m_2 l_2^{\ 2} + \tfrac{1}{2} m_2 l_1 l_2 C\theta_2 & \tfrac{1}{3} m_2 l_2^{\ 2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & -\tfrac{1}{2} m_2 l_1 l_2 S\theta_2 \\ \tfrac{1}{2} m_2 l_1 l_2 S\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^{\ 2} \\ \dot{\theta}_2^{\ 2} \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 S\theta_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} \\ &+ \begin{bmatrix} (\tfrac{1}{2} m_1 + m_2) g l_1 C\theta_1 + \tfrac{1}{2} m_2 g l_2 C\theta_{12} \\ \tfrac{1}{2} m_2 g l_2 C\theta_{12} \end{bmatrix} \end{aligned} \tag{5.26}$$

Not surprisingly, this final result is similar to the previous example.

5.3 Dynamics Equations for More Complicated Robots

Systematic methods exist for deriving the dynamics equations for robots with several DOF. One method is presented in section 4.4 of Niku's textbook [1]. This is beyond the scope of this course.

The dynamics equations for more complex robots all have the common matrix form that we have already seen in the examples. The dynamics equations for an *n*-DOF robot may be written [3]:

$$\tau = M(q)[\ddot{q}] + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(q) \tag{5.27}$$

where q are the joint variables, M(q) is an $n \times n$ matrix termed the mass matrix, B(q) is an $n \times n(n-1)/2$ matrix of Coriolis coefficients, C(q) is an $n \times n$ matrix of centripetal coefficients, G(q) is an $n \times 1$ vector of gravity related terms, $[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2 \quad \dot{q}_1\dot{q}_3 \quad \cdots \quad \dot{q}_{n-1}\dot{q}_n]^T$, $[\ddot{q}] = [\ddot{q}_1 \quad \ddot{q}_2 \quad \cdots \quad \ddot{q}_n]^T$, $[\dot{q}^2] = \left[\dot{q}_1^{\,2} \quad \dot{q}_2^{\,2} \quad \cdots \quad \dot{q}_n^{\,2}\right]^T$, and $[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2 \quad \dot{q}_1\dot{q}_3 \quad \cdots \quad \dot{q}_{n-1}\dot{q}_n]^T$. τ is an $n \times 1$ vector of joint torques (for each revolute joint) and/or joint forces (for each prismatic joint). For example, $\tau = \begin{bmatrix} F_1 \\ \tau_2 \end{bmatrix}$ for a PR robot.

Examining equation (5.27) we can observe that:

- It is a matrix of 2nd order differential equations.
- The differential equations are non-linear since they involve products of derivatives (we can't use Laplace transform-based methods).
- Since there are many terms off the main diagonal of the matrices the differential equations are highly coupled. This means the state of one of the links affects all of the others.
- The matrices and the gravity related terms are all functions of the joint variables so the coefficients of the differential equations will vary with time as the robot arm moves.

Just when you think it can't get worse... The dynamics model of equation (5.27) in fact leaves out several important dynamic elements of a real-world robot. These include:

- Motor dynamics.
- Friction in the joints.
- Backlash in gears and other transmission elements such as leadscrews.
- Flexibility of gears and other transmission elements.
- Flexibility of the links.
- Forces/torques resulting from contact with the environment, *e.g.* during robotic grinding or robotic assembly.

We will revisit some of these dynamic elements later on in the course.

5.4 Applications of the Dynamics Equations

The dynamics equations have several uses when designing a new robot, including:

- Allows the development of computer-aided design tools for simulating the robot ("virtual prototyping").
- Helps to properly choose the motor size required to provide sufficient torque/force.

- Shows the significance of motor placement and other design parameters.
- Can be used to predict the maximum accelerations and velocities.
- Can be used to design and simulate robot control algorithms.
- Can be used to develop advanced model-based control algorithms.

5.5 A Brief Introduction to the Servo Control of Robots

Since the topic of control systems is covered in other courses, only a few points will be mentioned here. Since we are interested in commanded the robot to follow a changing setpoint, robot motion control is an example of a servo control problem. From the previous sections we have seen that the dynamics of a robot are very complex. This makes the servo control of a robot difficult. Two approaches are normally used.

If a robot has motors connected directly to the joints it is known as a “direct-drive robot”. With this type of robot, the control algorithm must be based on the dynamics equations for stable and precise control to be produced. The disadvantages are the difficulty and cost associated with deriving a model-based control algorithm and implementing the complex calculations involved. The advantage is that direct-drive robots can move faster than conventional robots and are more precise since backlash is not an issue.

Most industrial robots use a gearbox or other mechanical transmission between each motor and joint. This produces a mechanical advantage so that a smaller motor may be used. The mechanical advantage also reduces the varying loads from the robot dynamics that the motor feels. As long as this mechanical advantage is fairly large (gear ratio of 300:1 or more) a standard PID control algorithm may be used to control each motor (*i.e.* six PID algorithms for a six DOF robot). Note that the best set of PID gains for the robot arm fully outstretched will be different that the best PID gain settings for the arm close-in, so an average set of gains must be used. The simplicity and cost of this conventional approach are superior but the motion performance is inferior.

References

1. S.B. Niku, “Introduction to Robotics”, Pearson Education, 2001.
2. L.-W. Tsai, “Robot Analysis”, John Wiley & Sons, 1999.
3. J.J. Craig, “Introduction to Robotics”, Addison Wesley Longman, 1989.
4. M.W. Spong and M. Vidyasagar, “Robot Dynamics and Control”, John Wiley & Sons, 1989.