A1

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1) Discrete Systems

The System S is given by:

$$y(n) = y(n-1) - x(n-1) + x(n)$$

We assume that $y(-1) = 0$.

1. Compute the output to the input $x(n) = \delta(n)$.
2. Compute the output to the input $x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$.
3. Compute the output to the input $x(n) = \cos(\pi n)$.

1) $x(n) = (1 \quad 0 \quad 0 \quad 0 \quad 0)$

x	1	0	0	0	0
y	1	0	0	0	0

2) $x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$
$= (1 \quad 1 \quad 1 \quad 0 \quad 0)$

x	1	1	1	0	0
y	1	1	1	0	0

3) $x(n) = \cos(\pi n) = (1 \quad -1 \quad 1 \quad -1 \quad 1)$

x	1	-1	1	-1	1
y	1	-1	1	-1	1

2) Difference equations to State Space equation

Given the difference equation

$$y(n) = y(n-1) - y(n-2) + x(n) - x(n-2)$$

Determine the $[A, B, C, D]$ representation of the system.

$y(n) = y(n-1) - y(n-2) + x(n) - x(n-2)$

$S_1(n) = x(n-2)$	$S_1(n+1) = x(n-1) = S_2(n)$
$S_2(n) = x(n-1)$	$S_2(n+1) = x(n)$
$S_3(n) = y(n-2)$	$S_3(n+1) = y(n-1) = S_4(n)$
$S_4(n) = y(n-1)$	$S_4(n+1) = y(n) = S_4(n) - S_3(n) + x(n) - S_1(n)$

$$s_3(n) = y(n-2) \quad \Big| \quad s_3(n+1) = y(n-1) = s_4(n)$$
$$S_4(n) = y(n-1) \quad \Big| \quad S_4(n+1) = y(n) = S_4(n) - S_3(n) + x(n) - S_1(n)$$

$$S(n+1) = \overset{\alpha}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 \end{pmatrix}} S(n) + \overset{\beta}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}} x(n)$$

$$y(n) = \underset{C}{\begin{pmatrix} -1 & 0 & -1 & 1 \end{pmatrix}} S(n) + \underset{D}{\begin{pmatrix} 1 \end{pmatrix}} x(n)$$

3) $[A, B, C, D]$ **representation**

Given the system

$$y(n) = x(n) + x(n-2) - y(n-2)$$

1. Compute the $[A, B, C, D]$ representation
2. Use the state space equation to compute the output (5 values) to the input $x(n) = \delta(n)$.

$$y(n) = x(n) + x(n-2) - y(n-2)$$

$$S_1(n) = x(n-2) \rightarrow S_1(n+1) = x(n-1) = S_2(n)$$
$$S_2(n) = x(n-1) \rightarrow S_2(n+1) = x(n)$$

$$S_3(n) = y(n-2) \rightarrow S_3(n+1) = y(n-1) = S_4(n)$$
$$S_4(n) = y(n-1) \rightarrow S_4(n+1) = y(n)$$

$$S_4(n+1) = x(n) + S_1(n) - S_3(n)$$

$$S(n+1) = \overset{\alpha}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}} S(n) + \overset{\beta}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}} x(n)$$

$$y(n) = \underset{C}{\begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}} S(n) + \underset{D}{\begin{pmatrix} 1 \end{pmatrix}} x(n)$$

2. Use the state space equation to compute the output (5 values) to the input $x(n) = \delta(n)$.

$$S(\emptyset) = A\,\cancel{S(-1)}^{\vec{\emptyset}} + \beta\,\cancel{x(-1)}^{\vec{\emptyset}} = \vec{\emptyset}$$

$$S(1) = A\,\cancel{S(\emptyset)}^{\vec{\emptyset}} + \beta\,x(\emptyset) = \beta\,\delta(\emptyset)$$

$$S(1) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(1) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S(2) = A S(1) + \beta\,\cancel{x(1)}^{\vec{\emptyset}} = A\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S(2) = \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\1&0&-1&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$S(3) = \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\1&0&-1&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$S(4) = \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\1&0&-1&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{array}{cccc} & C & S(\emptyset) & D\ \ x(\emptyset) \end{array}$$

$$y(\emptyset) = (1\ 0\ {-1}\ 0)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)(1)$$

$$y(\emptyset) = 1 \checkmark$$

$$y(1) = (1\ 0\ {-1}\ 0)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + (1)(\emptyset)$$

$$y(1) = \emptyset \checkmark$$

$$y(2) = (1\ 0\ {-1}\ 0)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + (1)(\emptyset)$$

$$y(2) = \emptyset$$

$$y(3) = (1\ 0\ {-1}\ 0)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \emptyset \checkmark$$

$$y(4) = (1\ 0\ {-1}\ 0)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \emptyset \checkmark$$

4) Compound Interest

A bank account is just a system; $x(n)$ be the amount deposited or withdrawn (x negative) at day n, and α be the daily interest rate. Then the account balance is given by

$$y(n) = y(n-1) + \alpha y(n-1) + x(n)$$

How much money do you have after 10 days if you initially deposit \$100 ? (α is used as a symbolic parameter).

$$y(n) = y(n-1) + \alpha y(n-1) + x(n)$$

find $y(10)$

$$S_1(n) = y(n-1)$$

$$S_1(n+1) = y(n) = S_1(n) + \alpha S_1(n) + x(n)$$

$$S(n+1) = \underset{A}{(\alpha+1)}\,S(n) + \underset{B}{(1)}\,x(n)$$

$$y(n) = \underset{C}{(\alpha+1)}\,S(n) + \underset{D=1}{x(n)}$$

$$h(n) = C A^{n-1} B$$

$n(n) = C A B$

$h(10) = (\alpha+1)^{10}(1)$

money $= h(10) \cdot$ initial deposit

$= (\alpha+1)^{10} \cdot \$100$

A2

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1) Impulse Response

Given the following system:

$$y(n) = y(n-1) - y(n-2) + x(n)$$

1. Determine the $[A, B, C, D]$ representation of the system
2. Compute the impulse response

$$y(n) = y(n-1) - y(n-2) + x(n)$$

$$S_1(n) = y(n-2) \quad \Big| \quad S_1(n+1) = y(n-1) = S_2$$

$$S_2(n) = y(n-1) \quad \Big| \quad S_2(n+1) = y(n) = S_2 - S_1 + x(n)$$

$$S(n+1) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} S(n) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x(n)$$

$$y(n) = \begin{pmatrix} -1 & 1 \end{pmatrix} S(n) + (1)\, x(n)$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 1 \end{pmatrix} \qquad D = 1$$

2) impulse response

$$h(n) = \begin{cases} \emptyset & n < \emptyset \\ 2 & n = \emptyset \\ (-1\ \ 1)\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} & n > \emptyset \end{cases}$$

2) Convolution Systems

Given is a LTI system by its impulse response

$$h(n) = \delta(n) + \delta(n-2)$$

Determine the output of the system to the input

$$x(n) = \delta(n) + \delta(n-2) + \delta(n-4)$$

To obtain full credit you have to give **all** non zero values of y.

$$h(n) = \delta(n) + \delta(n-2)$$

$$x(n) = \delta(n) + \delta(n-2) + \delta(n-4)$$

$$y(n) = \sum h(n)\, x(n-k)$$

$h(\emptyset) = 1$

$h(1) = \emptyset$

$h(2) = 1$

$h(3) = \emptyset$

...

$$y(\emptyset) = \sum_{k=\emptyset}^{\emptyset} h(\emptyset)\, x(\emptyset) = 1$$

$$y(1) = \emptyset$$

$$y(2) = \sum_{0}^{2} \ldots = (1)\, x(2-\emptyset) + (\emptyset) + (1)\, x(2-2)$$

$$y(2) = 1 + 1 = 2$$

...

n	0	1	2	3	4	5	6
$\delta(n)$	1	0	0	0	0	0	0
$h(n)$	1	0	1	0	0	0	0
$\delta(n-2)$	0	0	1	0	0	0	0
$h(n)$	0	0	1	0	1	0	0
$\delta(n-4)$	0	0	0	0	1	0	0
$h(n)$	0	0	0	0	1	0	1

$y(n) = (1 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 1)$

3) Convolution

Given the impulse response

$$h(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 3 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine the difference equation and the $[A, B, C, D]$ representation of the system. Use convolution to compute the output to the input

$$x(n) = \delta(n) + \delta(n-1) + \delta(n-3)$$

$$h(n) = \begin{cases} 1 & n = \emptyset \\ 2 & n = 1 \\ 3 & n = 2 \\ \emptyset & \end{cases}$$

$$h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

$$y(n) = x(n) + 2x(n-1) + 3x(n-2)$$

$$s_1(n) = x(n-2) \to s_1(n+1) = x(n-1) = s_2(n)$$

$$s_2(n) = x(n-1) \to s_2(n+1) = x(n)$$

$$y(n) = x(n) + 2S_2(n) + 3S_1(n)$$

$$S(n+1) = \overset{A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} S(n) + \overset{B}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} x(n)$$

$$y(n) = \underset{C}{\begin{pmatrix} 3 & 2 \end{pmatrix}} S(n) + \underset{D}{\begin{pmatrix} 1 \end{pmatrix}} x(n)$$

	0	1	2	3	4	5	6
$\delta(n)$	1	0	0	0	0	0	0
$h(n)$	1	2	3	0	0	0	0
$\delta(n-1)$	0	1	0	0	0	0	0
$h(n)$	0	1	2	3	0	0	0
$\delta(n-3)$	0	0	0	1	0	0	0
$h(n)$	0	0	0	1	2	3	0

$$y(n) = (1 \;\; 3 \;\; 5 \;\; 4 \;\; 2 \;\; 3)$$

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Kenzie was here

1) System representations

Given the following impulse response:

$$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

1. Determine the corresponding difference equation
2. Determine the $[A, B, C, D]$ representation of the system
3. Compute the output to the input $x_n = \delta(n) + \delta(n-2)$ using convolution, the difference equations and the $[A, B, C, D]$ representation.

$$h(n) = \delta(n) + \tfrac{1}{2}\delta(n-1) + \tfrac{1}{4}\delta(n-2)$$

1) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$

$S_1(n) = x(n-2) \longrightarrow S_1(n+1) = x(n-1) = S_2$

$S_2(n) = x(n-1) \longrightarrow S_2(n+1) = x(n)$

$y(n) = x(n) + \frac{1}{2}S_2(n) + \frac{1}{4}S_1(n)$

2) $S(n+1) = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} S(n) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} x(n)$

$y(n) = \underbrace{\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \end{pmatrix}}_{C} S(n) + \underbrace{(1)}_{D} x(n)$

3) $x(n) = \delta(n) + \delta(n-2)$

$S(n+1) = S(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\cancel{\begin{pmatrix} S_1(0) \\ S_2(0) \end{pmatrix}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cancel{x(0)}^{1}$

$S(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S(2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 1 \end{pmatrix}x(1)}^{0}$

$S(2) = \begin{pmatrix} 0\cdot 0 + 1\cdot 1 \\ 0\cdot 0 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S(3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cancel{x(2)}^{1}$

$S(3) = \cancel{\begin{pmatrix} 0\cdot 1 + 1\cdot 0 \\ 0 + 0 \end{pmatrix}}^{0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S(4) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varnothing$

$S(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S(5) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y(\varnothing) = \left(\frac{1}{4}, \frac{1}{2}\right)\cancel{S(0)}^{0} + \cancel{x(0)}^{1} = 1$

$y(1) = \left(\frac{1}{4}\ \frac{1}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cancel{x(1)}^{0} = \frac{1}{2}$

$y(2) = \left(\frac{1}{4}\ \frac{1}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cancel{x(2)}^{1} = \frac{5}{4}$

$y(3) = \left(\frac{1}{4}\ \frac{1}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cancel{x(3)}^{0} = \frac{1}{2}$

$y(4) = \left(\frac{1}{4}\ \frac{1}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varnothing = \frac{1}{4}$

$y(5) = \varnothing$

$y(n) = \left\{1 \quad \frac{1}{2} \quad \frac{5}{8} \quad \frac{1}{2} \quad \frac{1}{4}\right\}$

	0	1	2	3	4	5	6
$\delta(n)$	1	0	0	0	0	0	0
$h(n)$	1	1	1	-	-	-	-

	0	1	2	3	4	5	6
$\delta(n)$	1	0	0	0	0	0	0
$h(n)$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0
$\delta(n-2)$	0	0	1	0	0	0	0
$h(n)$	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	0
$y(n)$	1	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		

2) Find the system

Given is the following system:

$$y(n) = \alpha_0 x(n) + \alpha_1 x(n-1) + \beta_1 y(n-1)$$

1. Determine $\alpha_0, \alpha_1, \beta_1$ such that the impulse response $h(n)$ is

$$h(n) = \delta(n-1)$$

2. Determine $\alpha_0, \alpha_1, \beta_1$ such that the impulse $h(n)$ response is

$$h(n) = \frac{1}{3^n}, \quad n \geq 0$$

3. Determine $\alpha_0, \alpha_1, \beta_1$ such that the impulse response $h(n)$ is $h(n) = 1$.

1) $x(n) = \delta(n)$

n	0	1	2	3	4
x	1	0	0	0	0
y	0	1	0	0	0

$$y = \underbrace{\cancel{\alpha_0 x(n)}}_{0} + \alpha_1 x(n-1) + \underbrace{\cancel{\beta_1 y(n-1)}}_{0}$$

$$y = x(n-1)$$

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \beta_1 = 0$$

2) $h(n) = \frac{1}{3^n}, \quad n \geq 0$

n	0	1	2	3	4
x	1	0	0	0	0
y	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{27}$	...

$$y = \alpha_0 x(n) + \alpha_1 x(n-1) + \beta_1 y(n-1)$$

$1 = y(0) = \alpha_0(1) + \alpha_1(0) + \beta_1(0)$

$\alpha_0 = 1$

$\frac{1}{3} = y(1) = 1\,\cancel{x(n)} + \alpha_1(1) + \beta_1(1)$

$\frac{1}{3} = \alpha_1 + \beta_1 \longrightarrow \alpha_1 = 0$

$\frac{1}{9} = y(2) = \beta_1\left(\frac{1}{3}\right)$

$\frac{1}{27} = y(3) = \beta_1\left(\frac{1}{9}\right)$

$\beta_1 = \frac{1}{3}$

3)

n	0	1	2	3	4	5
x	1	0	0	0	0	0
y	1	1	1	1	1	1

$1 = y(0) = \alpha_0 = 1$

$1 = y(1) = \alpha_1 + \beta_1 \longrightarrow \alpha_1 = 0$

$1 = y(2) = \beta_1$

$1 = y(3) = \beta_1$

$\beta_1 = 1$

3) Old Mid-term questions

- What is the normalized frequency, frequency and period of the discrete signal $x(n) = \cos(\frac{3}{4}\pi n)$?
- What is the period and discrete frequency of the signal $x(n) = \sin(\frac{2\pi}{3}n + \frac{\pi}{3})$?
- Compute the following:

  convert to $re^{i\phi}$: $-1-i=$

  compute $|e^{i\frac{\pi}{2}}| =$

  convert to $re^{i\phi}$: $-2=$

  compute the **argument** of $1-i=$

  convert to $re^{i\phi}$: $-i=$

1) $x(n) = \cos\left(\frac{3}{4}\pi n\right)$

$k \cdot 2\pi = p\left(\frac{3\pi}{4}\right)$

$\boxed{p = 8} \longrightarrow (6\pi)$

seconds/cycle

$f = \frac{1}{8}$ cycles/second

$\omega = \frac{3\pi}{4} \longrightarrow p = \frac{2\pi k}{\omega} = \frac{8}{3}k \rightarrow k = 3$

$w = \frac{3\pi}{4} \longrightarrow p = \frac{2\pi k}{w} = \frac{8}{3}k \rightarrow k = 3$

$p = 3$ samples/cyc

$f = \frac{1}{3}$ cycles/sample

2) • What is the period and discrete frequency of the signal $x(n) = \sin(\frac{2\pi}{3}n + \frac{\pi}{3})$?

$x(n) = \sin(\frac{2\pi}{3}n + \frac{\pi}{3})$?

$\frac{2\pi}{3}p = 2\pi k$

$p = 3 \quad f = \frac{1}{3}$

$\frac{2\pi k}{\frac{2\pi}{3}} = 3k$

$p = 3$ samples/cycle

$f = \frac{1}{3}$ cycles/sample

3) • Compute the following:

convert to $re^{i\phi}$: $-1-i=$

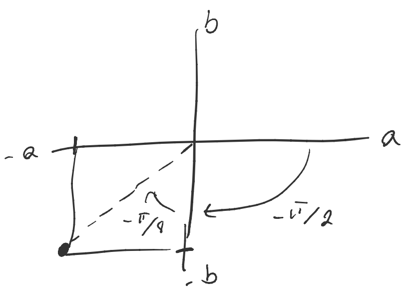
compute $|e^{i\frac{\pi}{2}}| =$

convert to $re^{i\phi}$: $-2 =$

compute the **argument** of $1-i=$

convert to $re^{i\phi}$: $-i=$

a) $-1-i = re^{i\theta}$

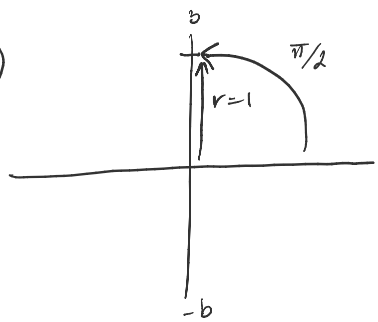


$\boxed{\theta = -\frac{3\pi}{4}}$

$r = \sqrt{2}$

$= \sqrt{2}e^{-i\frac{3\pi}{4}}$

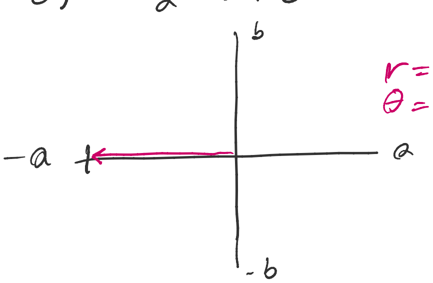
b)



$e^{i\pi/2} = i$

$|i| = 1 = r$

c) $-2 \rightarrow re^{i\theta}$

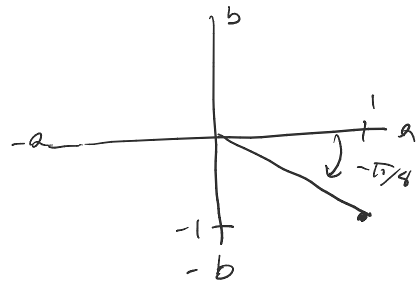


$r = 2$

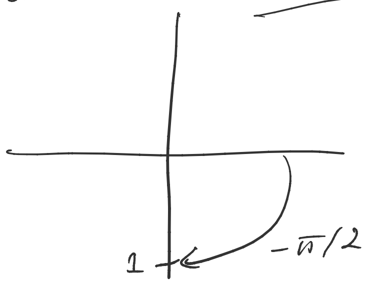
$\theta = \pi = -\pi$

$-2 = 2e^{i\pi} = 2e^{-i\pi}$

c) $\measuredangle(1-i) = \underline{\theta = -\pi/4}$



d) $-i \rightarrow re^{i\theta} = e^{-i\pi/2}$



A4

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1) LTI Systems

Given are the two impulse responses

$$h_1(n) = \begin{cases} 1 & n \text{ is even } \wedge n \geq 0 \\ 0 & n \text{ is odd } \vee n < 0 \end{cases}$$

and

$$h_2(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$$

Give difference equations and $[A, B, C, D]$ representations for both systems.

n	0	1	2	3	4	5
$h_1(n)$	1	0	1	0	1	0
$h_2(n)$	1	-1	1	-1	0	0

$$y_1(n) = x(n) + y(n-2)$$

$$S_1(n) = y(n-2) \rightarrow S_1(n+1) = y(n-1) = S_2$$

$$S_2(n) = y(n-1) \rightarrow S_2(n+1) = y(n) = x(n) + S_1$$

$$S(n+1) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} S(n) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} x(n)$$

$$y(n) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} S(n) + \underbrace{(1)}_{D} x(n)$$

$$y_2(n) = x(n) - x(n-1) + x(n-2) - x(n-3)$$

$\not{\#}$

$$y_2(n) = x(n) - \underbrace{x(n-1)}_{S_3} + \underbrace{x(n-2)}_{S_2} - \underbrace{x(n-3)}_{S_1}$$

$$S_1(n) = x(n-3) \longrightarrow S_1(n+1) = x(n-2) = S_2$$

$$S_2(n) = x(n-2) \longrightarrow S_2(n+1) = x(n-1) = S_3$$

$$S_3(n) = x(n-1) \longrightarrow S_3(n+1) = x(n)$$

$$S(n+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} S(n) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x(n)$$

$$y(n) = (-1 \ 1 \ -1) S(n) + (1) x(n)$$

2) Frequency Response

Compute the Frequency response of the following systems:

- $y(n) = x(n) - 2x(n-1) + x(n-2)$
- $y(n) = \frac{1}{2}y(n-1) + x(n)$
- $h(n) = \delta(n-2)$

$$x = e^{i\omega n}$$

$$1)\ H(\omega)\cancel{e^{i\omega n}} = \cancel{e^{i\omega n}} - 2\cancel{e^{i\omega n}} e^{-i\omega} + \cancel{e^{i\omega n}} e^{-2i\omega}$$

$$H(\omega) = 1 - 2e^{-i\omega} + e^{-2i\omega}$$

$$2)\ H(\omega)\cancel{e^{i\omega n}} = \frac{1}{2} H(\omega)\cancel{e^{i\omega n}} e^{-i\omega} + \cancel{e^{i\omega n}}$$

$$H(\omega) = \frac{1}{2} H(\omega) e^{-i\omega} + 1$$

$$H(\omega) = \frac{1}{2} H(\omega) e^{-i\omega} + 1$$

$$H(\omega) - \frac{1}{2} H(\omega) e^{-i\omega} = 1$$

$$H(\omega)\left(1 - \frac{1}{2} e^{-i\omega}\right) = 1$$

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-i\omega}}$$

3) $h(n) = \delta(n-2)$

$$H(\omega) = e^{-2i\omega}$$

3) Frequency, Periodicity

Given the two signals

$$x_1(t) = \sin(\frac{3}{2}t)$$
$$x_2(n) = \cos(\frac{\pi}{4}n)$$

1. Give period, frequency and normalized frequency of x_1 (with units).
2. Give period, frequency and normalized frequency of x_2 (with units). x_2 is played back on a computer with a sampling frequency of $f_s = 8000Hz$, what is the pitch (frequency) of the resulting audio signal?

$$x_1(t) = \sin\left(\frac{3}{2} t\right)$$

$$\frac{3}{2} p = k \cdot 2\pi$$

$$\boxed{p = 4\pi \ \frac{\text{seconds}}{\text{cycle}}}$$

$$\boxed{f = \frac{1}{4\pi} \ \text{cycles/seconds}}$$

$$\boxed{\omega = \frac{3}{2} \ \text{rad/s}}$$

$$x_2(n) = \cos\left(\frac{\pi}{4} n\right)$$

$x_2(n) = \cos\left(\frac{\pi}{4} n\right)$

$\boxed{\omega = \frac{\pi}{4} \text{ rad/sample}}$ $\quad \frac{\pi}{4} p = k \cdot 2\pi$

$\boxed{f = \frac{1}{8} \frac{\text{cycles}}{\text{sample}}}$ $\quad \boxed{p = 8 \frac{\text{samples}}{\text{cycle}}}$

$\text{pitch} = f \cdot f_s = \left(\frac{1}{8}\right) 8000 = 1000 \text{ Hz}$

4) Discrete Fouerier Series

Compute the discrete fourier series of the 4 periodig signal:

1, 1, 0, 0, 1, 1, 0, 0, 1, 1,

n	0	1	2	3	4
$y(n)$	1	1	0	0	1
	$y(n-4)$	$y(n-3)$			

$p = 4, \ \omega = \frac{\pi}{2}$

$p\omega = 2\pi$

$$X_k = \frac{1}{p} \sum_{k=0}^{p-1} x(n) e^{-i\omega n k}$$

$$X_k = \frac{1}{4} \sum_{k=0}^{3} x(n) e^{-\frac{i\pi}{2} n k}$$

$n \rightarrow$, $k \downarrow$

$$X_0 = \frac{1}{4}\left(\cancel{x(0)}^{1} e^{-\frac{i\pi}{2}(0)(0)} + \cancel{x(1)}^{1} e^{-\frac{i\pi}{2}(1)(0)} + \cancel{x(2)}^{0} + \cancel{x(3)}^{0}\right)$$

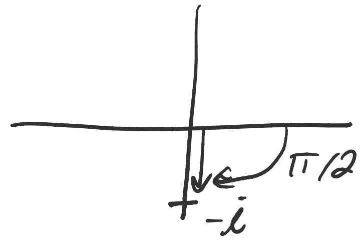
$$X_0 = \frac{1}{4}(2) = \frac{1}{2}$$

$\Delta\emptyset - 4 \quad 2$

$$X_1 = \frac{1}{4}\left(\cancel{x(\emptyset)}^{\,1} e^{-\frac{i\pi}{2}(\emptyset)(1)} + \cancel{x(1)}^{\,1} e^{-\frac{i\pi}{2}(1)(1)} + \cancel{x(2)}^{\,\emptyset} + \cancel{x(3)}^{\,\emptyset}\right)$$

$$X_1 = \frac{1}{4}\left(1 + e^{-i\pi/2}\right)$$

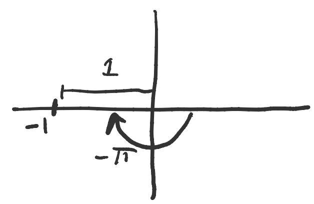
$$X_1 = \frac{1 + e^{-i\pi/2}}{4} \longrightarrow$$



$$X_1 = \frac{1 + (-i)}{4} = \frac{1 - i}{4}$$

$$X_2 = \frac{1}{4}\left(\cancel{x(\emptyset)}^{\,1} e^{-\frac{i\pi}{2}(\emptyset)(2)} + \cancel{x(1)}^{\,1} e^{-\frac{i\pi}{2}(1)(2)}\right)$$

$$X_2 = \frac{1}{4}\left(1 + e^{-i\pi}\right) \longrightarrow$$

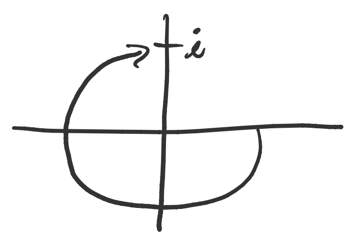


$= -1$

$$X_2 = \emptyset$$

$$X_3 = \frac{1}{4}\left(\cancel{x(\emptyset)}^{\,1} e^{-\frac{i\pi}{2}(\emptyset)(3)} + x(1) e^{-\frac{i\pi}{2}(1)(3)}\right)$$

$$X_3 = \frac{1}{4}\left(1 + e^{-i3\pi/2}\right)$$



$$X_3 = \frac{1 + i}{4}$$

MT1 prac

October 23, 2022 2:50 PM

1 Short Questions

1) Define the impulse response
2) What is the period and discrete frequency of $\cos(\frac{2\pi}{3}n)$
3) What is the frequency response of the system $y'' + y = x$.
4) What are the units for period and discrete frequency

1) when $x(n) = \delta(n)$, $h(n) = y(n)$

2) $\frac{2\pi}{3} p = k 2\pi$

$\boxed{p = 3}$ $\frac{\text{samples}}{\text{cycle}}$ $\longrightarrow$ $f = \frac{1}{3} \frac{\text{cycles}}{\text{sample}}$

3) $y'' + y = x$

$y = h(\omega) e^{i\omega n}$

$y'' = h(\omega)(i\omega)^2 e^{i\omega n}$

$h(\omega) \cancel{e^{i\omega n}} ((i\omega)^2 + 1) = \cancel{e^{i\omega n}}$

$$h(\omega) = \frac{1}{(i\omega)^2 + 1}$$

4) $p = \left[\frac{\text{samples}}{\text{cycle}}\right]$ $f = \left[\frac{\text{cycles}}{\text{sample}}\right]$

$\rightarrow)$ $\quad p = \left[\frac{\text{samples}}{\text{cycle}}\right] \qquad f = \left[\frac{\text{cycles}}{\text{sample}}\right]$

2 Frequency Response

Given the frequency response

$$H(\omega) = \cos(\omega)$$

compute the output to the signal $x(n) = 2 + \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\pi n + \frac{3\pi}{2})$

	$H(\omega)$	$\lvert H(\omega)\rvert$	$\measuredangle H(\omega)$
$\emptyset$	1	1	$\emptyset$
$\frac{\pi}{2}$	$\emptyset$	$\emptyset$	N/A
π	-1	1	$-\pi$

$$y(n) = (1)2 + (\emptyset)\cos(\tfrac{\pi}{2}n + \tfrac{\pi}{4}) + (1)\sin(\pi n + \tfrac{3\pi}{2} + (-\pi))$$
$$= 2 + \sin(\pi n + \pi/2)$$

3 State Space

Given the difference equation

$$y(n) = y(n-2) - x(n-2)$$

Determine the $[A, B, C, D]$ representation of the system.

$$y(n) = y(n-2) - x(n-2)$$

$$S_1(n) = x(n-2) \rightarrow S_1(n+1) = x(n-1) = S_2$$
$$S_2(n) = x(n-1) \rightarrow S_2(n+1) = x(n)$$
$$S_3(n) = y(n-2) \rightarrow S_3(n+1) = y(n-1) = S_4$$

$$S_3(n) = y(n-2) \longrightarrow S_3(n+1) = y(n-1) = S_4$$

$$S_4(n) = y(n-1) \longrightarrow S_4(n+1) = y(n) = S_3(n) - S_1(n)$$

$$S(n+1) = \overset{A}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}} S(n) + \overset{B}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} x(n)$$

$$y(n) = \overset{C}{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}} S(n) + \overset{D}{\begin{pmatrix} \emptyset \end{pmatrix}} x(n)$$

4 Convolution

Given is a LTI system by its impulse response

$$h(n) = \delta(n) + \delta(n-2)$$

Determine the output of the system to the input

$$x(n) = \delta(n) + \delta(n-2) + \delta(n-4)$$

To obtain full credit you have to give **all** non zero values of y.

※LTI

$$h(n) = \delta(n) + \delta(n-2)$$

$$x(n) = \delta(n) + \delta(n-2) + \delta(n-4)$$

	0	1	2	3	4	5	6	7
$\delta(n)$	1	0	0	...				
$h(n)$	1	0	1	0	...			
$\delta(n-2)$	0	0	1	0	0	...		
$h(n)$	0	0	1	0	1	...		
$\delta(n-4)$	0	0	0	0	1	0	0	0
$h(n)$	0	0	0	0	1	0	1	0

$$y(n) = (1 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 1)$$

$$y(n) = \delta(n) + 2\delta(n-2) + 2\delta(n-4) + \delta(n-6)$$

5 Frequency Response

- Compute the frequency response of the system

$$y(n) + y(n-1) = 2x(n) - 5x(n-2)$$

- Given the frequency response

$$H(\omega) = \frac{1 + e^{-i\omega}}{e^{-i\omega}}$$

compute the output to the signal $x(n) = \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\pi n + \frac{3\pi}{2})$

1) $y(n) = 2x(n) - 5x(n-2) - y(n-1)$

$$\cancel{e^{i\omega n}} H(\omega) + \cancel{e^{i\omega n}} H(\omega) e^{-i\omega} = 2\cancel{e^{i\omega n}} - 5\cancel{e^{i\omega n}} e^{-2i\omega}$$

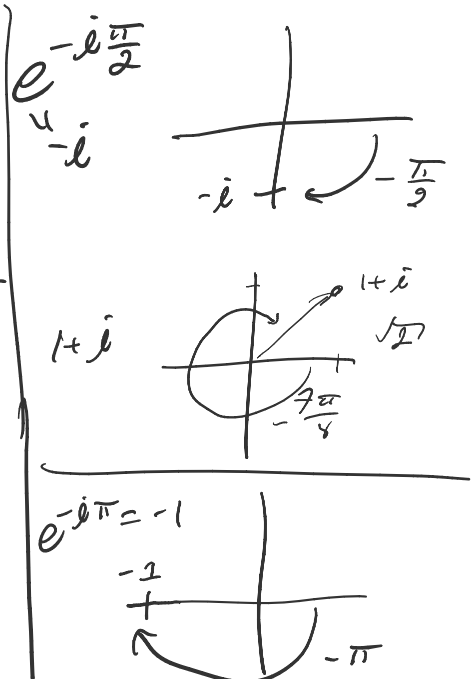
$$H(\omega)(1 + e^{-i\omega}) = 2 - 5e^{-2i\omega}$$

$$H(\omega) = \frac{2 - 5e^{-2i\omega}}{1 + e^{-i\omega}}$$

2) $H(\omega) = \dfrac{1 + e^{-i\omega}}{e^{-i\omega}}$, $x(n) = \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\pi n + \frac{3\pi}{2})$

2) $H(\omega) = \frac{1+e^{-i\omega}}{e^{-i\omega}}$, $x(n) = \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\pi n + \frac{3\pi}{2})$

	$H(\omega)$	$\lvert H(\omega)\rvert$	$\measuredangle H(\omega)$
$\frac{\pi}{2}$	$\frac{1+e^{-i\pi/2}}{e^{-i\pi/2}} = \frac{1-i}{-i} = 1+i$	$\sqrt{2}$	$-\frac{7\pi}{4}$
π	$\frac{1+e^{-i\pi}}{e^{-i\pi}} = \frac{1-1}{-1} = \varnothing$	$\varnothing$	N/A



$$y(n) = \sqrt{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} - \frac{7\pi}{4}\right) + \varnothing\sin(\ldots)$$
$$= \sqrt{2}\cos\left(\frac{\pi}{2}n - \frac{3\pi}{2}\right)$$

6 State Space

Given the following matrices:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \end{pmatrix}, D = 1$$

Compute the zero state output to the input $x(n) = \delta(n) + \delta(n-1)$. (I like to see $s(n)$ and $y(n)$ for $n = 0, 1, 2, 3$.)

$s(n+1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} s(n) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(n)$

$s(\varnothing) = \vec{\varnothing}$

$s(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cancel{s(\varnothing)}^{\vec{\varnothing}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cancel{x(\varnothing)}^{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$s(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cancel{x(1)}^{1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$s(3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cancel{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\vec{\varnothing}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y(n) = \begin{pmatrix} 1 & 1 \end{pmatrix} s(n) + (1)x(n)$

$y(\varnothing) = \begin{pmatrix} 1 & 1 \end{pmatrix} \cancel{s(\varnothing)}^{\vec{\varnothing}} + (1) = 1$

$y(1) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1)\cancel{x(1)}^{1} = 2$

$y(2) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cancel{x(2)}^{\vec{\varnothing}} = 2$

$y(3) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cancel{x(3)}^{\vec{\varnothing}} = 1$

$y(n) = \begin{pmatrix} 1 & 2 & 2 & 1 \end{pmatrix}$

7 Compute the Impulse Response

Given the system

$$y(n) = x(n) - 2x(n-1) + 3x(n-3)$$

Compute a close form expression (not just numbers) that represents the impulse response of this system.

$y(n) = x(n) - 2x(n-1) + 3x(n-3)$ FIR

$h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-3)$

8 Complex Numbers

Convert the following numbers into the complex exponential representation:

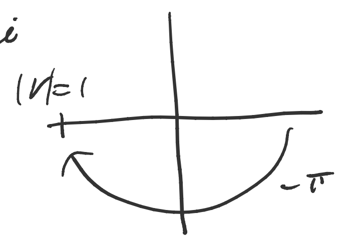
$$1 + i, -1, -i, -1 - i$$

Compute the argument of the following complex numbers:

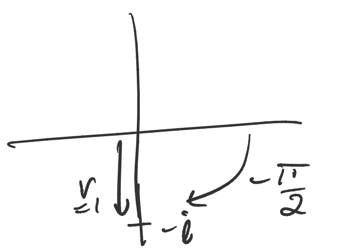
$$\frac{1}{1+i}, -2i, 2$$

1) $1 + i = \underline{\sqrt{2}\, e^{-\frac{7\pi}{4} i}}$

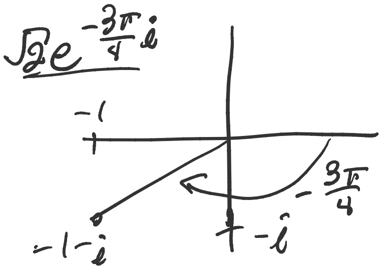
$-1 = \underline{e^{-\pi i}}$



$-i = \underline{e^{-\frac{\pi}{2} i}}$



$-1 - i = \underline{\sqrt{2}\, e^{-\frac{3\pi}{4} i}}$



2) $\frac{1}{1+i} = \frac{1}{\sqrt{2}\, e^{-\frac{7\pi}{4} i}} = \frac{1}{\sqrt{2}} e^{\frac{7\pi}{4} i}$ $\quad \measuredangle = \frac{7\pi}{4} = -\frac{\pi}{4}$

$-2i = 2e^{-\frac{\pi}{2} i}$ $\quad \measuredangle = -\frac{\pi}{2}$

$\measuredangle 2 = \varnothing$

9 State Space

Given the following state space equation:

$$s(n+1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} s(n) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(n)$$

$$y(n) = \begin{pmatrix} 1 & 1 \end{pmatrix} s(n) + x(n)$$

Compute the zero state output to the input $x(n) = \delta(n)$. (I like to see $s(n)$ and $y(n)$ for $n = 0, 1, 2, 3, 4$.)

$S(\emptyset) = \vec{\emptyset}$

$S(1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cancel{S(\emptyset)}^{\vec{\emptyset}} + \begin{pmatrix} 1 \\ \emptyset \end{pmatrix}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S(2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \emptyset = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$S(3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \emptyset = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$S(4) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$y(\emptyset) = 1$

$y(1) = (1 \ 1)\begin{pmatrix} 1 \\ \emptyset \end{pmatrix} = 1$

$y(2) = (1 \ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$

$y(3) = (1 \ 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3$

$y(4) = (1 \ 1)\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 5$

10 Discrete Fourier Series

Compute the Discrete dourier series with $N = 4$ of the signal $x(n) = 1 + \cos(\frac{1}{2}\pi n)$

$N = 4$ $\quad x(n) = 1 + \cos(\frac{1}{2}\pi n)$ $\quad \boxed{\omega = \frac{\pi}{2}}$

n

k

$X_k = \frac{1}{4}\sum_{n=0}^{N-1} x(n) e^{i\omega kn}$

$X_\emptyset = \frac{1}{4}(x(\emptyset)\cancel{e^{i\omega\emptyset\emptyset}}^{1} + x(1) + x(2))$

$X_\emptyset = \frac{1}{4}(2 + 1 + \overset{2}{\emptyset} + \overset{3}{1}) = 1$

$X_1 = \frac{1}{4}(2\cancel{e^{i\frac{\pi}{2}(1)(\emptyset)}}^{1} + \cancel{e^{-i\frac{\pi}{2}(1)(1)}}^{-i} + \emptyset + \cancel{e^{-i\frac{3\pi}{2}(1)}}^{+i}) = \frac{1}{2}$

$X_2 = \frac{1}{4}(2 + \cancel{e^{-i\pi}}^{-1} + \cancel{e^{-i3\pi}}^{-1}) = \emptyset$

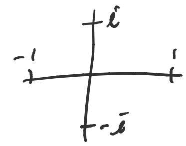
$X_3 = \frac{1}{4}(2 + \cancel{e^{-i\frac{3\pi}{2}}}^{i} + \cancel{e^{-i\frac{9\pi}{2}}}^{-i}) = \frac{1}{2}$

Synthesis $\quad X_n = \{1, \frac{1}{2}, \emptyset, \frac{1}{2}\}$

n

$N-1$

$$x(n) = \sum_{n=0}^{N-1} X_n e^{iwkn}$$



$$x(0) = (\overset{0}{1} + \overset{1}{\tfrac{1}{2}} + \overset{2}{0} + \overset{3}{\tfrac{1}{2}}) = 2$$

$$x(1) = (1 + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}}}_{i} + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}(3)(1)}}_{-i}) = 1$$

$$x(2) = (1 + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}(1)(2)}}_{-1} + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}(3)(2)}}_{-1}) = 1 - \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$x(3) = (1 + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}(1)(3)}}_{-i} + \tfrac{1}{2}\underbrace{e^{i\frac{\pi}{2}(3)(3)}}_{+i}) = 1$$

$$x(n) = \{2 \quad 1 \quad 0 \quad 1\}$$

4) Discrete Fouerier Series

Compute the discrete fourier series of the 4 periodig signal:

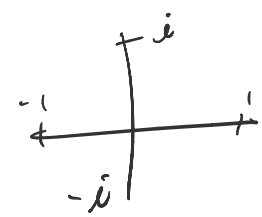
$$\boxed{1, 1, 0, 0,} 1, 1, 0, 0, 1, 1,$$

$$y(n) = \{\overset{y(1)}{1}, \overset{y(2)}{1}, \overset{y(3)}{0}, \overset{y(4)}{0}\}$$

$$f = \frac{1}{p} = \frac{1}{4}$$

$$w = 2\pi f = \frac{\pi}{2}$$

$$X_n = \frac{1}{p} \sum_{n=0}^{p-1} x(n) e^{iwkn}$$



$$X_0 = \frac{1}{4}(1 + 1 + 0 + 0) = \frac{1}{2}$$

$$X_1 = \frac{1}{4}(e^{-iw(1)(0)} + e^{-iw(1)(1)}) = \frac{1 + e^{-i\frac{\pi}{2}}}{4} = \frac{1 - i}{4}$$

$$X_2 = \frac{1}{4}(e^{-iw(2)(0)} + e^{-iw(2)(1)}) = \frac{1 + e^{-\pi i}}{4} = \frac{1 - 1}{4} = 0$$

$$X_3 = \frac{1}{4}(1 + e^{-iw(3)(1)}) = \frac{1 + e^{-\frac{3\pi}{2}i}}{4} = \frac{1 + i}{4}$$

✓

2 Frequency Response

Given the frequency response

$$H(\omega) = \cos(\omega)$$

compute the output to the signal $x(n) = 2 + \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\pi n + \frac{3\pi}{2})$

$H(\omega) = \cos(\omega)$

n	$H(\omega)$	$\lvert H(\omega)\rvert$	$\measuredangle H(\omega)$
$\emptyset$	1	1	$\emptyset$
$\frac{\pi}{2}$	$\emptyset$	$\emptyset$	N/A
π	-1	1	$-\pi$

$y(n) = 2(1) + (\emptyset)\cos(\ldots) + (1)\sin(\pi n + \frac{3\pi}{2} - \pi)$

$y(n) = 2 + \sin(\pi n - \frac{\pi}{2})$