Q1

February 27, 2022 12:03 PM

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1. **Vector Algebra**
   a. Determine whether the vector $\boldsymbol{C}$ is perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$ given:
   $$\boldsymbol{A} = 4\hat{x} + 5\hat{y}, \qquad \boldsymbol{B} = 7\hat{x} + 6\hat{y} + 8\,\hat{z}, \qquad \boldsymbol{C} = \hat{x} + 5\hat{z}$$
   Show your work and clearly state whether $\boldsymbol{C}$ is perpendicular to $\boldsymbol{A}$ and $\boldsymbol{B}$

   b. Find a vector $\boldsymbol{P}$ whose magnitude is 12 and whose direction is perpendicular to both vectors $\boldsymbol{Q}$ and $\boldsymbol{S}$, given:
   $$\boldsymbol{Q} = 5\hat{x} + 3\hat{y}, \qquad \boldsymbol{S} = 20\hat{y} - \hat{z}.$$

a) Dot product of 2 perpendicular vectors is $= 0$

if $\vec{C}$ is perp. to $\vec{A}$ & $\vec{B}$

$\vec{A} \cdot \vec{C} = 0$ & $A \cdot B = 0$

$\vec{A} \cdot \vec{C} = \langle 4, 5, 0 \rangle \cdot \langle 1, 0, 5 \rangle$

$= 4 \cdot 1 + 0 \cdot 5 + 5 \cdot 0$

$= 4 \neq 0$

$\therefore$ $\vec{C}$ is not perpendicular to $\vec{A}$

$B \cdot C = \langle 7, 6, 8 \rangle \cdot \langle 1, 0, 5 \rangle$

$= 7 \cdot 1 + 6 \cdot 0 + 8 \cdot 5$

$= 47$

$\therefore$ C is not perpendicular to A & B

b) $\vec{Q} = \langle 5, 3, 0 \rangle$ $\qquad$ $\vec{S} = \langle 0, 20, -1 \rangle$

5) $\overline{Q} = \langle 5, 3, 0 \rangle \qquad \overline{S} = \langle 0, 20, -1 \rangle$

$$\vec{Q} \times \vec{S} = \langle -3, 5, 100 \rangle = \vec{v}$$

$$\|\vec{v}\| = \sqrt{3^2 + 5^2 + 100^2} = \sqrt{10034}$$

$$\hat{v} = \vec{v} / \|\vec{v}\| = \left\langle \frac{-3}{\sqrt{10034}}, \frac{5}{\sqrt{10034}}, \frac{100}{\sqrt{10034}} \right\rangle$$

$$\vec{P} = 12\hat{v} = \left\langle \frac{-36}{\sqrt{10034}}, \frac{60}{\sqrt{10034}}, \frac{1200}{\sqrt{10034}} \right\rangle$$

Q2

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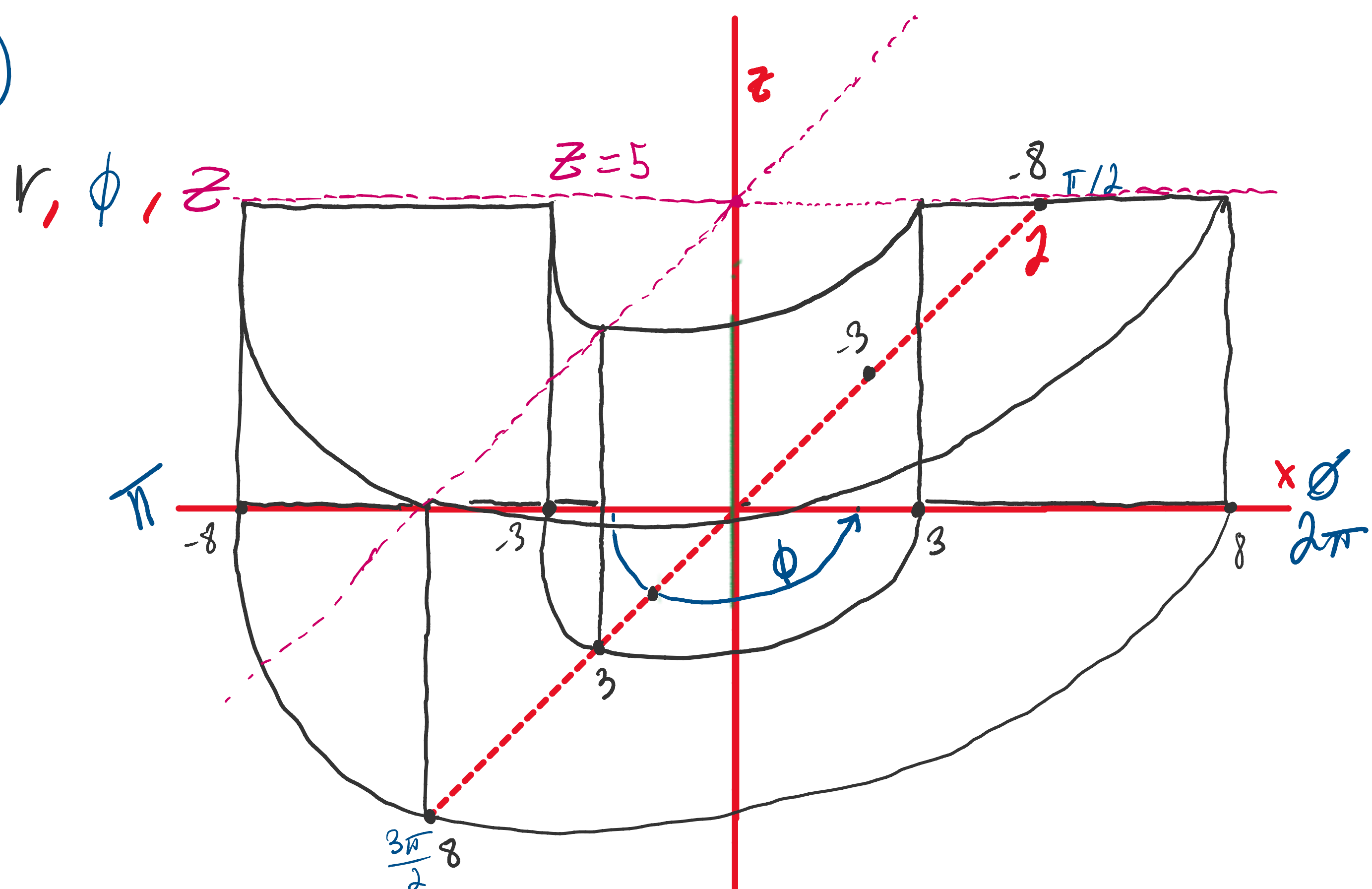
2. **Coordinate Systems**
   a. Provide a sketch (1) and [redacted]

$$3 \le r \le 8;\ \pi \le \varphi \le 2\pi;\ 0 \le z \le 5$$

   b. The surface area described by:

$$0 \le R \le 2;\ 90^\circ \le \theta \le 180^\circ;\ 45^\circ \le \varphi \le 90^\circ$$

a)



b) The surface area described by:

$$0 \le R \le 2;\ 90^\circ \le \theta \le 180^\circ;\ 45^\circ \le \varphi \le 90^\circ$$

$$dS_R = R^2 \sin\theta \, d\theta \, d\phi \, (\hat{R})$$

$$* \ R^2 = 2^2 = 4$$

$d S_R = R^2 \sin\theta \, d\theta \, d\varphi \, (R)$

$\star R^2 = 2^2 = 4$

$$S = \int_{90^\circ}^{180^\circ} \int_{45^\circ}^{90^\circ} R^2 \sin\theta \, d\phi \, d\theta$$

$$S = 4 \int_{90^\circ}^{180^\circ} \left[\sin\theta \cdot \phi \right]_{45^\circ}^{90^\circ} d\theta$$

$$S = 4 \left[-\cos\theta \right]_{90^\circ}^{180^\circ} \cdot \left[\phi \right]_{45^\circ}^{90^\circ}$$

$$S = 4(1 - \emptyset)(45)$$

$$S = 180$$

Q3

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3. **Gradient** Find the gradient of the following scalar functions
   a. $\boldsymbol{M} = 10/(x^2 + z^2)$
   b. $\boldsymbol{A} = xy^3z^2$
   c. $\boldsymbol{T} = e^R \sin\theta$
   d. $\boldsymbol{H} = R^3 \cos^2\theta$
   e. $\boldsymbol{S} = xy^2 - z^2$

a) $$\nabla M = -\frac{20x}{(x^2+z^2)^2} + \emptyset - \frac{20z}{(x^2+z^2)^2}$$

$$\nabla M = \frac{-20}{(x^2+z^2)^2}(x+z)$$

b) $$\nabla A = y^3z^2 + 3xy^2z^2 + 2xy^3z$$

c) $$\nabla T = e^R(\sin\theta + \cos\theta)$$

d) $$\nabla H = 3R^2\cos^2\theta - R^3 2\cos\theta\sin\theta$$

e) $$\nabla S = y^2 + 2xy - 2z$$

Q4

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4. **Divergence.** The Divergence theorem states the surface integral of a vector field over a closed surface, the flux through the surface, is equal to the volume integral of the divergence over the region within the surface. Given the Divergence theorem states:

$$\int_v \nabla \cdot \boldsymbol{E}\, dv = \oint_S \boldsymbol{E} \cdot d\boldsymbol{s}$$

The vector field V is given by:

$$\boldsymbol{V} = x^2\hat{\boldsymbol{x}} + y^3\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

Verify the divergence theorem by computing:

a. The total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each, parallel to the cartesian axes

b. The integral of $\nabla \cdot \boldsymbol{V}$ over the cubes volume.

c. Does the theorem hold true?

a) $x, y, z \in \{-1, 1\}$

$\Phi = \iint V \cdot ds$

surface 1

$ds = dy\,dz\,\hat{x}$ $\quad x = 1$

$V = x^2 = 1$

$\Phi_1 = \int_{-1}^{1}\int_{-1}^{1} (1)\, dy\, dz$

$\Phi_1 = \int_{-1}^{1} [y]_{-1}^{1}\, dz$

$\Phi_1 = \int_{-1}^{1} 2\, dz$

surface 2

$ds = dy\, dz(-\hat{x})$

$V = x^2 = 1$

$\Phi_2 = \int_{-1}^{1}\int_{-1}^{1} (-1)\, dy\, dz$

$\Phi_2 = \int_{-1}^{1} -2\, dz$

$\Phi_2 = -4$

$$\Phi_1 = \int_{-1}^{1} 2 \, dz$$

$$\Phi_1 = 4$$

<u>surface 3</u>

$$ds = dx\,dz\,(-\hat{y})$$

$$V = y^3 = (-1)^3 = -1$$

$$\Phi_2 = \int_{-1}^{1}\int_{-1}^{1} -(-1)\, dx\,dz$$

$$\Phi_2 = \int_{-1}^{1} 2 \, dz$$

$$\Phi_2 = 4$$

<u>surface 4</u>

$$ds = dx\,dz\,(\hat{y})$$

$$V = y^3 = 1^3 = 1$$

$$\Phi_3 = \int_{-1}^{1}\int_{-1}^{1} (1)\, dx\,dz$$

$$\Phi_3 = \int_{-1}^{1} 2 \, dz$$

$$\Phi_3 = 4$$

<u>surface 5</u>

$$ds = dx\,dy\,(-\hat{z})$$

$$V = z = -1$$

$$\Phi_5 = \int_{-1}^{1}\int_{-1}^{1} -(-1)\, dx\,dy$$

$$\Phi_5 = \int_{-1}^{1} 2 \, dz$$

$$\Phi_5 = 4$$

<u>surface 6</u>

$$ds = dx\,dy \cdot (\hat{z})$$

$$V = z = 1$$

$$\Phi_6 = \int_{-1}^{1}\int_{-1}^{1} (1)\, dx\,dz$$

$$\Phi_6 = \int_{-1}^{1} 2 \, dz$$

$$\Phi_6 = 4$$

$$\Phi_{out} = \sum \Phi = \overset{①}{4} - \overset{②}{4} + \overset{③}{4} + \overset{④}{4} + \overset{⑤}{4} + \overset{⑥}{4}$$

$$\Phi_{out} = 16$$

b) b. The integral of $\nabla \cdot \boldsymbol{V}$ over the cubes volume.

$$\nabla V = \frac{\partial}{\partial x} V + \frac{\partial}{\partial y} V + \frac{\partial}{\partial z} V$$

$$\nabla V = 2x + 3y^2 + 1$$

$$\Phi_{out} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x + 3y^2 + 1)\, dx\, dy\, dz$$

$$\Phi_{out} = \int_{-1}^{1}\int_{-1}^{1} \left[x^2 + 3y^2 x + x\right]_{-1}^{1} dy\, dz$$

$$\Phi_{out} = \int_{-1}^{1}\int_{-1}^{1} (6y^2 + 2)\, dy\, dz$$

$$\Phi_{out} = \int_{-1}^{1} \left[2y^3 + 2y\right]_{-1}^{1} dz$$

$$\Phi_{out} = \int_{-1}^{1} (8)\, dz$$

$$\Phi_{out} = \int_{-1}^{1} (8)\, dz$$

$$\Phi_{out} = \left[8z \right]_{-1}^{1}$$

$$\Phi_{out} = 8-(-8) = 16$$

c) $\because$ the answer from (a) & (b) match,

$\therefore$ the theorem holds true

Q5

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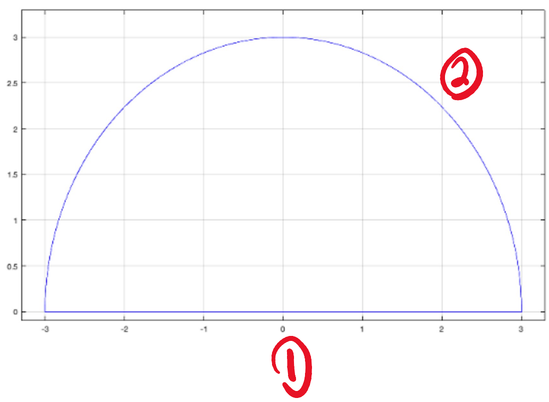
5. **Curl.** Stokes's theorem is a powerful equation that allows the conversion of a surface integral of the curl of a vector over an open surface S into a line integral, such as in the calculation of current through a closed magnetic field loop. Given that Stokes's theorem states:

$$\int_S (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$$

Verify Stokes's theorem for the vector field:

$$\boldsymbol{B} = r\cos\phi\,\hat{\boldsymbol{r}} + \sin\phi\,\hat{\boldsymbol{\phi}}$$

a. By evaluating $\oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$ over the semicircular contour shown below
b. By evaluating $\int_S (\nabla \times \mathbf{B}) \cdot d\boldsymbol{s}$ over the semicircular contour shown below



a) $\oint B\cdot dl = \int_{L1} B\cdot dl + \int_{L2} B\cdot dl$

$B\cdot dl = r\cos\phi\, dr + r\sin\phi\, d\phi$

$$\int_{L1} B\cdot dl = 2\int_{r=\emptyset}^{3} r\cos\phi\, dr \Big|_{\phi,z=\emptyset} + \int_{\phi=\emptyset}^{\emptyset} r\sin\phi\, d\phi \Big|_{z=\emptyset,\, r=3}$$

$$= 2\left(\frac{1}{2} r^2 \Big|_0^3\right)$$

$$= 9$$

$$\int_{L2} B\cdot dl = \int_{r=3}^{3} r\cos\phi\, dr \Big|_{\phi,z=\emptyset} + \int_{\phi=\emptyset}^{\pi} r\sin\phi\, d\phi \Big|_{z=\emptyset,\, r=-3}$$

$$= \emptyset + \left[-3\cos\phi\right]_{\emptyset}^{\pi}$$

$$= 3[\cos\pi - \cos(\emptyset)]$$

$$= 3[(-1) - 1]$$

$$= 6$$

$$\therefore \oint B \cdot dl = 9 + 6 = 15$$

b) $\int_S (\nabla \times \vec{B}) \, ds$

$$\nabla \times \vec{B} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ r\cos\phi & r\sin\phi & \emptyset \end{vmatrix}$$

$$= \frac{1}{r}[\sin\phi + r\sin\phi] + \emptyset\hat{\phi} + \phi\hat{r}$$

$$= \frac{\sin\phi}{r} + \sin\phi$$

$$ds = r\,dr\,d\theta(\hat{z})$$

$$\int_S (\nabla \times \vec{B}) ds = \int_\emptyset^\pi \int_0^3 \left(\frac{\sin\phi}{r} + \sin\phi\right) r\,dr\,d\theta$$

$$= \int_0^\pi \int_0^3 (\sin\phi + r\sin\phi)\,dr\,d\theta$$

$$\int^\pi 15 \sin\phi \, d\theta$$

$$= \int_0^{\pi} \frac{15}{2} \sin\phi \; d\theta$$

$$= \underline{15}$$

Q6

March 1, 2022 1:50 AM

6. **Bonus Question**: Answer one of the following questions. Clearly state whether a, b, or c is being answered.
   a. Find the values for $\boldsymbol{V} = ax^2\hat{\boldsymbol{x}} + by^3\hat{\boldsymbol{y}} + c\hat{\boldsymbol{z}}$ where the divergence at $P = (6,4,7)$ is equal to $\nabla \cdot \mathbf{V} = 10$

   b. Describe $\boldsymbol{A}$ in cylindrical coordinates and evaluate it at
   $$P = (2, \pi, \pi/4)$$
   $$\boldsymbol{A} = \sin^2\theta\, cos\varphi \widehat{\boldsymbol{R}} + \cos\theta\, \widehat{\boldsymbol{\theta}} - sin\varphi\, \widehat{\boldsymbol{\varphi}}$$

   c. Convert the following coordinates
      i. From cartesian to cylindrical and spherical coordinates: $P_1 = (5,10,15)$
      ii. From cylindrical to spherical and cartesian: $P_2 = \left(1, \frac{\pi}{2}, -1\right)$
      iii. From spherical to cylindrical: $P_3 = (4, \pi, \pi)$

c) i) $P_1 = (5, 10, 15)$
(x, y, z)

cylindrical

$R = \sqrt{5^2 + 10^2} = \sqrt{125}$

$\phi = \tan^{-1}(10/5) = \tan^{-1}(2) = 1.11$ rad

$z = 15$

$P_1 = (\sqrt{125}, 1.11 \text{ rad}, 15)$

spherical

$R = \sqrt{5^2 + 10^2 + 15^2} = \sqrt{350}$

$$R = \sqrt{5^2 + 10^2 + 15^2} = \sqrt{350}$$

$$\theta = \tan^{-1}\left(\sqrt{5^2+10^2}/15\right) = 0.75 \text{ rad}$$

$$\phi = \tan^{-1}(10/5) = 1.11 \text{ rad}$$

$$P_1 = (\sqrt{350}, 0.75 \text{ rad}, 1.11 \text{ rad})$$

ii. From cylindrical to spherical and cartesian: $P_2 = \left(1, \frac{\pi}{2}, -1\right)$

ii) $P_2 = (1, \pi/2, -1)$
(r, ϕ, z)

spherical

$$R = \sqrt{1 + 1} = \sqrt{2}$$

$$\theta = \tan^{-1}(1/-1) = -0.785 \text{ rad}$$

$$\phi = \pi/2$$

$$P_1 = (\sqrt{2}, -0.785 \text{ rad}, \pi/2 \text{ rad})$$

cartesian

$$x = 1 \cos(\pi/2) = 0$$
$$y = 1 \sin(\pi/2) = 1$$
$$z = -1$$

$$P_1 = (0, 1, -1)$$

iii. From spherical to cylindrical: $P_3 = (4, \pi, \pi)$

iii) $P_3 = (4, \pi, \pi)$
(R, θ, ϕ)

$r = 4\sin(\pi) = \emptyset$
$\phi = \pi$
$z = 4\cos(\pi) = -4$

$P_3 = (\emptyset, \pi, -4)$