McMaster University Final Exam

Name ____________________
Student Number ____________________

MECH ENG 4K03/6K03 ROBOTICS

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DURATION OF EXAMINATION: 2.5 HRS

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THIS EXAMINATION PAPER INCLUDES 5 PAGES (3 PAGES FOR QUESTIONS AND 2 PAGES FOR FORMULAS) AND 5 QUESTIONS.

Use of Casio FX-991 MS or MS Plus calculator.

Questions:

1. (8 points) Give the definitions of the following terms in robotics.

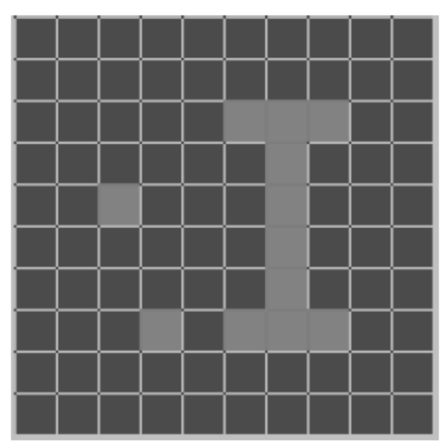
1) Hard automation
2) Flexible automation
3) Planar robot
4) Dextrous workspace

2. Short Answer Questions

1) (5 points) If the transformation matrices, ${}^{C}T_{D}$, ${}^{A}T_{B}$, ${}^{A}T_{E}$ and ${}^{E}T_{D}$, are known, derive the transformation equation for ${}^{B}T_{C}$ in terms of these matrices.

2) (5 points) Given the grayscale input image of the letter "I" from an optical character recognition application:

$$A = \begin{bmatrix} 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 130 & 130 & 130 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 130 & 75 & 75 & 75 \\ 75 & 75 & 130 & 75 & 75 & 75 & 130 & 75 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 130 & 75 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 130 & 75 & 75 & 75 \\ 75 & 75 & 75 & 130 & 75 & 130 & 130 & 130 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 \\ 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 \end{bmatrix}$$



Show a sample calculation for the row=8, col=5 pixel, if we apply a Laplacian 1 filter to the input image.

① 1) The use of specialized equipment to create a fixed process for assembly
2) Robots can be reprogrammed without shutdow when task changes
3) A robot whose end-effector motion covers a volume of 3D space
4) The volume of space the end-effector can reach with any desired orientation

② 1) ${}^{B}T_{C} = ({}^{A}T_{B})^{-1}({}^{A}T_{E})({}^{E}T_{D})({}^{C}T_{D})^{-1}$

2) Lap I $M = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$$S = \sum_{k=1}^{3} \sum_{p=1}^{3} m_{ij} = 1$$

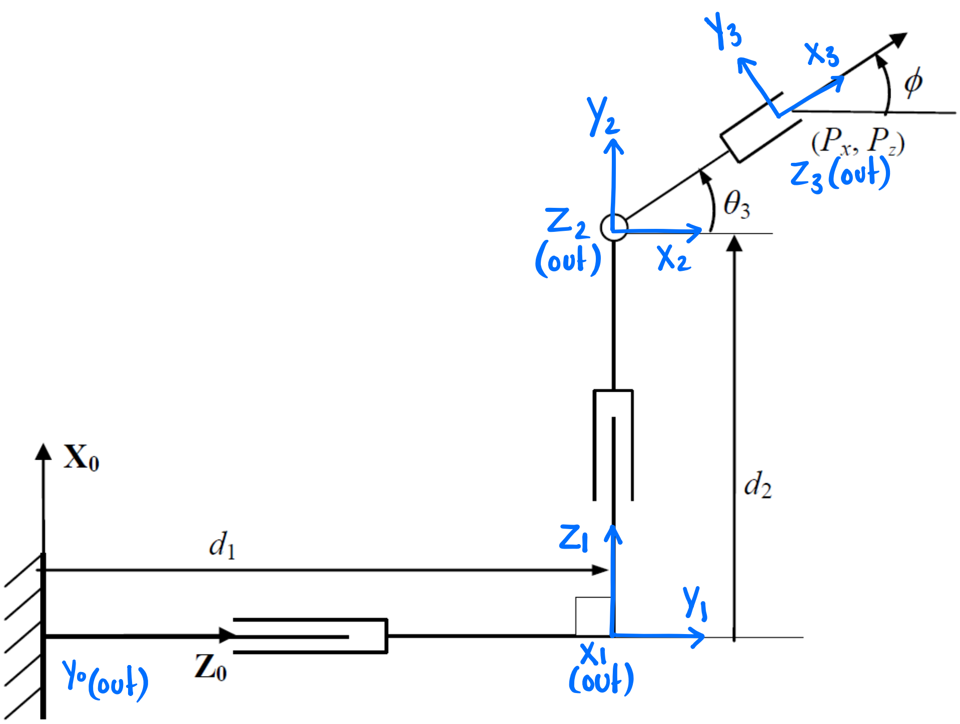
$$f_{85} = \frac{1}{S}\left(\sum_{k=1}^{3} \sum_{p=1}^{3} M_a\right)$$
$$= \frac{1}{1}(0\times 75 - 1\times 75 + 0\times 75 - 1\times 130 + 5\times 75 - 1\times 130 + 0\times 75 - 1\times 75 + 0\times 75)$$
$$= -35$$

3) $\bar{n}\cdot\bar{o} = 0 \longrightarrow n_x o_x + n_y o_y + n_z o_z = 0$ FRAME A: $0.77 \times -0.64 + 0.64 \times 0.77 + 0 \times 0 = 0$ ✓
$\bar{n}\cdot\bar{a} = 0$ $0.77 \times 0 + 0.64 \times 0 + 0 \times 1 = 0$ ✓
$\bar{a}\cdot\bar{o} = 0$ $0 \times -0.64 + 0 \times 0.77 + 1 \times 0 = 0$ ✓

3) (5 points) Which of the following matrices is a valid representation for a frame? Explain your answer.

$$A = \begin{bmatrix} 0.77 & -0.64 & 0 & 3 \\ 0.64 & 0.77 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & \sqrt{3}/2 & 0 & 3 \\ -1 & 0 & 0 & 0.5 \\ 0 & 1/2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 & 3 \\ -1 & 0 & 0 & 0.5 \\ 0 & 1/2 & -\sqrt{3}/2 & 15 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. (25 points) For the PPR planar robot shown in the following Figure:
a) Assign the frames using the D-H method.
b) Determine the D-H parameters and put them in the standard table form. Identify the joint variables.
c) Draw a diagram of the robot that properly shows the D-H frames, the joint variables, and any d or a parameters that are non-zero. (d_1 and d_2 are joint variables indicated in the figure; a_3 is the link length after the 3rd joint)
d) Calculate the A matrices and ${}^0T_3$
e) Its joint variables are d_1, d_2, and θ_3. Its end-effector position and orientation are given by P_x, P_z and $\emptyset$. Derive its inverse kinematics.



4. (25 points) For the planar PRP robot shown in the following Figure:
a) Derive the 3x3 manipulator Jacobian matrix. (The form used for calculating the linear velocity and angular velocity of the tool).
b) Determine the singular configuration(s) for this robot by examining the determinant of the Jacobian.
c) Draw the robot in a singular configuration and indicate which degree (s) of freedom have been lost.

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b)

$n+1$	θ_n	a_n	d_n	α_n
1	90	0	d_1	90
2	90	0	d_2	90
3	θ_3	a_3	0	0

d) $^0T_3 = A_1 A_2 A_3$

$$A_1 = \begin{bmatrix} \cancel{c(90)} & -s(90)\cancel{c(90)} & s(90)s(90) & 0 \\ s(90) & \cancel{c(90)c(90)} & \cancel{-c(90)s(90)} & 0 \\ 0 & s(90) & \cancel{c(90)} & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \cancel{c(90)} & -s(90)\cancel{c(90)} & s(90)s(90) & 0 \\ s(90) & \cancel{c(90)c(90)} & \cancel{-c(90)s(90)} & 0 \\ 0 & s(90) & \cancel{c(90)} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 c(0) & \cancel{s\theta_3 s(0)} & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 c(0) & \cancel{-c\theta_3 s(0)} & a_3 s\theta_3 \\ 0 & \cancel{s(0)} & c(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 + d_2 \\ 0 & 0 & 1 & 0 \\ c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e) $d_1 = P_x + a_3 \cos\varnothing$

$d_2 = P_y + a_3 \sin\varnothing$

$\theta_3 = \varnothing$

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$n+1$	θ	d	a	α
1	90°	d_1	0	90°
2	θ_2	0	0	90°
3	0	d_3	0	0

$$^nT_{n+1} = A_{n+1} = \begin{bmatrix} c\theta & -s\theta c\alpha & s\theta s\alpha & a c\theta \\ s\theta & c\theta c\alpha & -c\theta s\alpha & a s\theta \\ 0 & s\alpha & c\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c(90°) & -s(90°)c(90°) & s(90°)s(90°) & 0 \\ s(90°) & c(90°)c(90°) & -c(90°)s(90°) & 0 \\ 0 & s(90°) & c(90°) & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 c(90°) & s\theta_2 s(90°) & a c(90°) \\ s\theta_2 & c\theta_2 c(90°) & -c\theta_2 s(90°) & a s(90°) \\ 0 & s(90°) & c(90°) & d \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c(0) & -s(0)c(0) & s(0)s(0) & a c(0) \\ s(0) & c(0)c(0) & -c(0)s(0) & a s(0) \\ 0 & s(0) & c(0) & d \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = {}^0T_1 \times {}^0T_2 \times {}^0T_3 = A_1 \times A_2 \times A_3$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

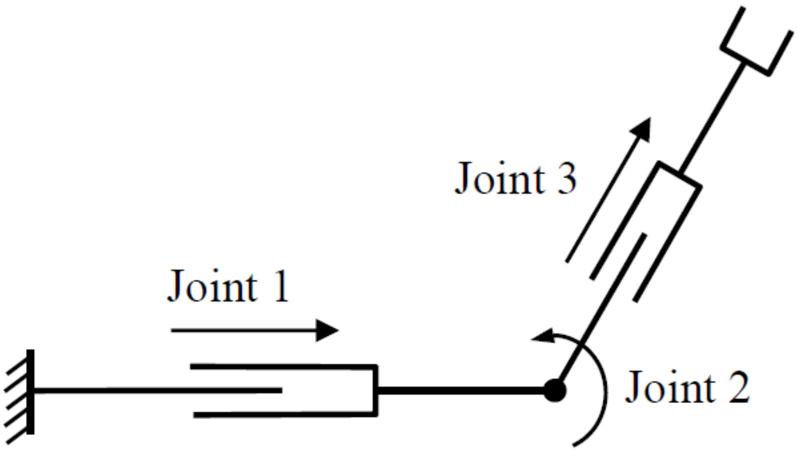
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ c\theta_2 & 0 & s\theta_2 & s\theta_2 \cdot d_3 \\ s\theta_2 & 0 & -c\theta_2 & -c\theta_2 \cdot d_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) $$J(\theta) = \begin{bmatrix} \frac{dP_y}{dd_1} & \frac{dP_y}{d\theta_2} & \frac{dP_y}{dd_3} \\ \frac{dP_z}{dd_1} & \frac{dP_z}{d\theta_2} & \frac{dP_z}{dd_3} \\ \varepsilon_1 z_0 & \varepsilon_2 z_1 & \varepsilon_3 z_2 \end{bmatrix} = \begin{bmatrix} 0 & c\theta_2 d_3 & s\theta_2 \\ 1 & s\theta_2 d_3 & -c\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

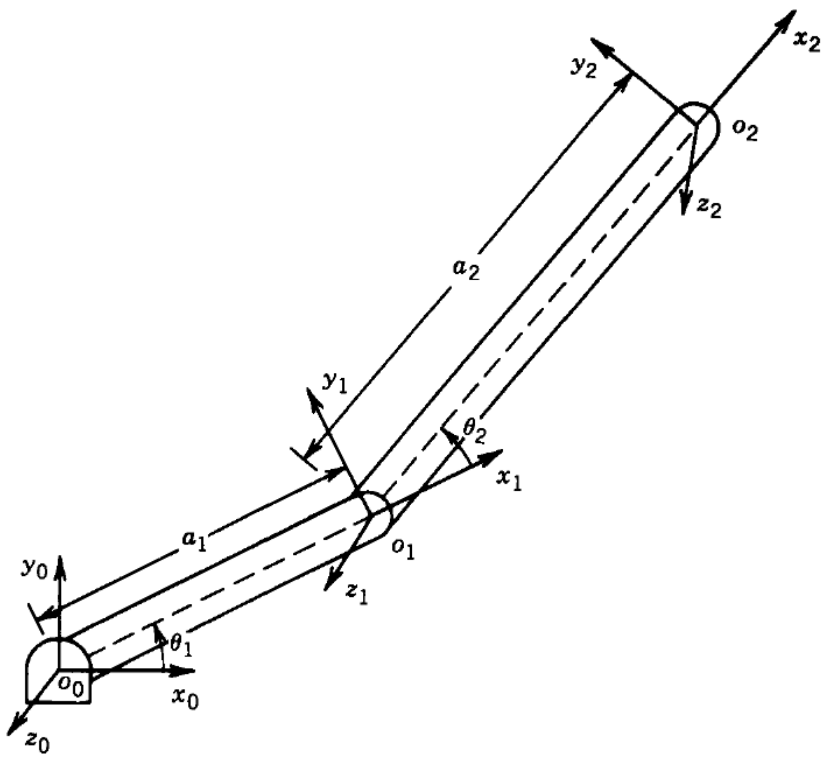
PRP → $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 0$

b) $\det(J) = (-1)(-s\theta_2) = s\theta_2$ → singularity when $s\theta_2 = 0$

↳ $\theta_2 = 0, 180°$



5. (27 points) For the RR planar robot in the following figure, if $a_1 = 0.4\text{m}$ and $a_2 = 0.3\text{m}$;
(a) Assuming the robot operates in the horizontal plane, calculate the joint torques such that the static force at the end-effector is $F_x = 20\text{N}$ and $F_y = -15\text{N}$ for the configuration $\theta_1 = 35^\circ$ and $\theta_2 = -75^\circ$
(b)Assuming the robot operates in the horizontal plane, calculate the static force applied by the end effector when $\tau_1 = 10\text{Nm}$, $\tau_2 = 5\text{Nm}$, $\theta_1 = 35^\circ$ and $\theta_2 = -75^\circ$.



(5) $O_{2y} = a_1 s\theta_1 + a_2 s\theta_{12} \longrightarrow \dot{O}_{2y} = \dot\theta_1 a_1 c\theta_1 + (\dot\theta_1+\dot\theta_2) a_2 c\theta_{12}$

$O_{2x} = a_1 c\theta_1 + a_2 c\theta_{12} \longrightarrow \dot{O}_{2x} = \dot\theta_1 a_1 s\theta_1 + (\dot\theta_1+\dot\theta_2) a_2 s\theta_{12}$

$$V_{o_1} = a_1\dot\theta_1$$
$$k_1 = \frac{1}{2} m_1 V_{o_1}^2 + \cancel{I_1}\omega_1^{2} \rightarrow 0$$
$$= \frac{1}{2} m_1 (a_1\dot\theta_1)^2$$

$$V_{o_2}^2 = \dot{x}_{o_2}^2 + \dot{y}_{o_2}^2$$
$$k_2 = \frac{1}{2} m_2 V_{o_2}^2 + \cancel{I_1}\omega_1^2 \rightarrow 0$$
$$= \frac{1}{2} m_2 (\dot{x}_{o_2}^2 + \dot{y}_{o_2}^2)$$

$\sin^2\theta_1 + \sin\theta_2$

$$= \frac{1}{2} m_2 \big[\dot\theta_1^2 a_1^2 \cancel{s^2\theta_1} + (\dot\theta_1+\dot\theta_2) a_2 s\theta_{12}\cdot\dot\theta_1 a_1 \cancel{s\theta_1} + (\dot\theta_1+\dot\theta_2)^2 a_2^2 \cancel{s\theta_{12}^2}$$
$$+ \dot\theta_1^2 a_1^2 \cancel{c^2\theta_1} + (\dot\theta_1+\dot\theta_2) a_2 c\theta_{12}\cdot\dot\theta_1 a_1 \cancel{c\theta_1} + (\dot\theta_1+\dot\theta_2)^2 a_2^2 \cancel{c^2\theta_{12}}\big]$$

$\cos^2\theta_1 + \cos\theta_2$

$$= \frac{1}{2} m_2 \big[\dot\theta_1^2 a_1^2 + (\dot\theta_1+\dot\theta_2) a_2 a_1 \dot\theta_1 (s\theta_2 + c\theta_2) + (\dot\theta_1+\dot\theta_2)^2 a_2^2\big]$$
$$= \frac{1}{2} m_2 \big[\dot\theta_1^2 a_1^2 + (\dot\theta_1^2+\dot\theta_2\dot\theta_1) a_2 a_1 (s\theta_2 + c\theta_2) + (\dot\theta_1^2 + 2\dot\theta_1\dot\theta_2 + \dot\theta_2^2) a_2^2\big]$$

$$P_1 = -m_1 G^T P_{o_1}$$
$$= -m_{o_1} \begin{bmatrix} 0 & -g \end{bmatrix} \begin{bmatrix} x_{o_1} \\ y_{o_1} \end{bmatrix}$$
$$= -m_{o1}(-g) y_{o1}$$
$$= -m_{o1}(-g) a_1 s\theta_1$$
$$= m_{o_1} g a_1 s\theta_1$$

$$P_2 = m_2 G^T P_{o_2}$$
$$= -m_{o_2}(-g) y_{o2}$$
$$= -m_{o2}(-g)(a_1 s\theta_1 + a_2 s\theta_{12})$$
$$= m_{o2} g (a_1 s\theta_1 + a_2 s\theta_{12})$$

$$L = k - P = k_1 + k_2 - P_1 - P_2$$
$$= \frac{1}{2} m_1 (a_1\dot\theta_1)^2 + \frac{1}{2} m_2 \big[\dot\theta_1^2 a_1^2 + (\dot\theta_1^2+\dot\theta_2\dot\theta_1) a_2 a_1 (s\theta_2 + c\theta_2) + (\dot\theta_1^2 + 2\dot\theta_1\dot\theta_2 + \dot\theta_2^2) a_2^2\big]$$
$$- m_{o_1} g a_1 s\theta_1 - m_{o2} g (a_1 s\theta_1 + a_2 s\theta_{12})$$
$$= \frac{1}{2} m_1 a_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \dot\theta_1^2 a_1^2 + \frac{1}{2} m_2 \dot\theta_1^2 + \dot\theta_2\dot\theta_1$$

(5) d) $a_1 = 0.4\text{m}$, $a_2 = 0.3\text{m}$, $F = \begin{bmatrix} 20 \\ -15 \end{bmatrix}\text{N}$, $\theta_1 = 35°$, $\theta_2 = -75°$

$$\tau = J(q)^T F_3$$

$$P_x = a_1 c\theta_1 + a_2 c\theta_{12}$$
$$P_y = a_1 s\theta_1 + a_2 s\theta_{12}$$

$$J(q) = \begin{bmatrix} \frac{dP_x(q)}{\theta_1} & \frac{dP_x(q)}{\theta_2} \\ \frac{dP_y(q)}{\theta_1} & \frac{dP_y(q)}{\theta_2} \end{bmatrix} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} & -a_2 s\theta_{12} \\ a_1 c\theta_1 + a_2 c\theta_{12} & a_2 c\theta_{12} \end{bmatrix}$$

$$J(q)^T = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} & a_1 c\theta_1 + a_2 c\theta_{12} \\ -a_2 s\theta_{12} & a_2 c\theta_{12} \end{bmatrix}$$

sub in values

$$= \begin{bmatrix} -0.4s(35°) - 0.3s(35-75°) & 0.4c(35°) + 0.3c(35-75°) \\ -0.3s(35-75) & 0.3c(35-75) \end{bmatrix}$$
$$= \begin{bmatrix} -0.0366 & 0.557 \\ 0.1928 & 0.2298 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -0.0366 & 0.557 \\ 0.1928 & 0.2298 \end{bmatrix} \begin{bmatrix} 20 \\ -15 \end{bmatrix}$$
$$= \begin{bmatrix} -9.087 \\ 0.409 \end{bmatrix} \text{Nm}$$

b) $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \text{Nm}$

$$F = (J(q)^T)^{-1}\tau$$

$$(J(q)^T)^{-1} = \frac{1}{(-0.0366\times 0.2298 - 0.557\times 0.1928)} \begin{bmatrix} -0.0366 & 0.557 \\ 0.1928 & 0.2298 \end{bmatrix}$$
$$= \frac{1}{-0.1158} \begin{bmatrix} 0.2298 & -0.557 \\ -0.1928 & -0.0366 \end{bmatrix}$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} -1.981 & 4.802 \\ 1.662 & 0.316 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 4.2 \\ 18.2 \end{bmatrix} \text{N}$$

Formulas

$$\mathrm{Rot}(X,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

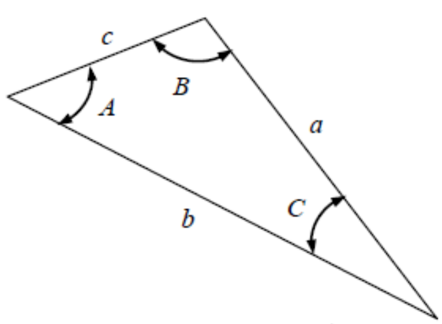
$$\mathrm{Rot}(Y,\theta)=\begin{bmatrix}C\theta & 0 & S\theta & 0\\ 0 & 1 & 0 & 0\\ -S\theta & 0 & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{Rot}(Z,\theta)=\begin{bmatrix}C\theta & -S\theta & 0 & 0\\ S\theta & C\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{Trans}(a,b,c)=\begin{bmatrix}1 & 0 & 0 & a\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & c\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$T^{-1}=\begin{bmatrix}n_x & n_y & n_z & -\vec{P}\bullet\vec{n}\\ o_x & o_y & o_z & -\vec{P}\bullet\vec{o}\\ a_x & a_y & a_z & -\vec{P}\bullet\vec{a}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$A_{n+1}={}^{n}T_{n+1}=\begin{bmatrix}C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1}\\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1}\\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1}\\ 0 & 0 & 0 & 1\end{bmatrix}$$



$$\cos A=\left(b^2+c^2-a^2\right)/\left(2bc\right)$$

$$S\theta_1C\theta_2+C\theta_1S\theta_2=S(\theta_1+\theta_2)=S\theta_{12}$$

$$C\theta_1C\theta_2-S\theta_1S\theta_2=C(\theta_1+\theta_2)=C\theta_{12}$$

if $a=\sin\theta$ and $b=\cos\theta$ then $\theta=\mathrm{atan2}(a,b)$

$$J(q)=\begin{bmatrix}\dfrac{\partial p_x(q)}{\partial q_1} & \dfrac{\partial p_x(q)}{\partial q_2}\\ \dfrac{\partial p_y(q)}{\partial q_1} & \dfrac{\partial p_y(q)}{\partial q_2}\end{bmatrix}$$

$$J(q)=\begin{bmatrix}\dfrac{\partial p_x(q)}{\partial q_1} & \dfrac{\partial p_x(q)}{\partial q_2} & \dfrac{\partial p_x(q)}{\partial q_3}\\ \dfrac{\partial p_y(q)}{\partial q_1} & \dfrac{\partial p_y(q)}{\partial q_2} & \dfrac{\partial p_y(q)}{\partial q_3}\\ \zeta_1 t_1 & \zeta_2 t_2 & \zeta_3 t_3\end{bmatrix}$$

$$J(q)=\begin{bmatrix}\dfrac{\partial p_x(q)}{\partial q_1} & \dfrac{\partial p_x(q)}{\partial q_2} & \dfrac{\partial p_x(q)}{\partial q_3}\\ \dfrac{\partial p_y(q)}{\partial q_1} & \dfrac{\partial p_y(q)}{\partial q_2} & \dfrac{\partial p_y(q)}{\partial q_3}\\ \dfrac{\partial p_z(q)}{\partial q_1} & \dfrac{\partial p_z(q)}{\partial q_2} & \dfrac{\partial p_z(q)}{\partial q_3}\end{bmatrix}$$

$$J(q)=\left[\begin{array}{cccc}\dfrac{\partial p_x(q)}{\partial q_1} & \dfrac{\partial p_x(q)}{\partial q_2} & \cdots & \dfrac{\partial p_x(q)}{\partial q_n}\\ \dfrac{\partial p_y(q)}{\partial q_1} & \dfrac{\partial p_y(q)}{\partial q_2} & \cdots & \dfrac{\partial p_y(q)}{\partial q_n}\\ \dfrac{\partial p_z(q)}{\partial q_1} & \dfrac{\partial p_z(q)}{\partial q_2} & \cdots & \dfrac{\partial p_z(q)}{\partial q_n}\\ \hdashline \zeta_1 \mathbf{z}_0(q) & \zeta_2 \mathbf{z}_1(q) & \cdots & \zeta_n \mathbf{z}_{n-1}(q)\end{array}\right]$$

$$z_i={}^{0}R_i\begin{bmatrix}0\\0\\1\end{bmatrix}\quad\text{where } {}^{0}R_i=\prod_{k=1}^{i}{}^{k-1}R_k$$

if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\det(J) = j_{11}(j_{33}j_{22} - j_{32}j_{23}) - j_{21}(j_{33}j_{12} - j_{32}j_{13}) + j_{31}(j_{23}j_{12} - j_{22}j_{13})$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J(q) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\tau = J(q)^T F$$

$$F_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

$$\tau_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

$$K_j = \tfrac{1}{2} m_j v_{cj}^{\ 2} + \tfrac{1}{2} I_j \omega_j^{\ 2}$$

$$P_j = -m_j G^T p_{cj}$$

$$\dot{\theta}_{\max} = \frac{\theta_h - \theta_b}{t_h - t_b} = \ddot{\theta}_d t_b$$

$$t_b = \frac{t_f}{2} - \frac{\sqrt{\ddot{\theta}_d^{\ 2} t_f^{\ 2} - 4\ddot{\theta}_d(\theta_f - \theta_i)}}{2\left|\ddot{\theta}_d\right|}$$

$\theta(t) = \theta_i + \tfrac{1}{2}\ddot{\theta}_d t^2,\ \dot{\theta}(t) = \ddot{\theta}_d t,$
and $\ddot{\theta}(t) = \ddot{\theta}_d$

$\theta(t) = \theta_i + \tfrac{1}{2}\ddot{\theta}_d t_b^{\ 2} + \ddot{\theta}_d t_b(t - t_b),\ \dot{\theta}(t) = \ddot{\theta}_d t_b,$
and $\ddot{\theta}(t) = 0$

$\theta(t) = \theta_f - \tfrac{1}{2}\ddot{\theta}_d(t_f - t)^2,\ \dot{\theta}(t) = \ddot{\theta}_d(t_f - t),$
and $\ddot{\theta}(t) = -\ddot{\theta}_d$

Gaussian $M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, Mean $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Lap1 $M = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$, Lap2 $M = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Sobel $M_h = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $M_v = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

Prewitt $M_h = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $M_v = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$$F = A + c\left(A - F_{smooth}\right)$$

The End