Section 4 Mathematical Modeling of Dynamic Systems

4.1 Introduction

Mathematical models are the basis of:

1) Simulation of mechatronic designs for "virtual prototyping".
   $\Rightarrow$ Try out designs quickly and inexpensively.

2) Optimization of a mechatronic design and its operating conditions.
   $\Rightarrow$ Improve performance and/or reduce cost

3) Model-based control algorithm design.
   $\Rightarrow$ Better predictions lead to faster and precise control.

All three of these applications of mathematical models are common in industry.

****
* **Examples will be presented after we cover the building blocks.** *
****

4.2.1 Mechanical Systems Building Blocks

Building blocks: (i) mass, (ii) spring and (iii) damper.

Spring: $$F = kx \tag{4.1}$$

Damper: $$F = cv \tag{4.2}$$

Combine using free-body diagrams and $\sum F = ma$.

4.2.2 Electrical Systems Building Blocks

Building blocks: (i) inductor, (ii) capacitor and (iii) resistor.

Inductor: $$V = L\frac{di}{dt} \tag{4.3}$$

Capacitor: $$i = C\frac{dV}{dt} \tag{4.4}$$

Resistor: $$V = iR \tag{4.5}$$

Combine using Kirchhoff's laws:

1. The sum of the currents at a junction must equal zero.
2. The sum of voltage drops in a loop must equal the voltage supply.

4.2.3 Hydraulic Systems Building Blocks

Building blocks: (i) hydraulic resistance, (ii) hydraulic capacitance and (iii) hydraulic inertance.

Hydraulic resistance: $P_1 - P_2 = RQ$ (4.6)

Hydraulic capacitance:

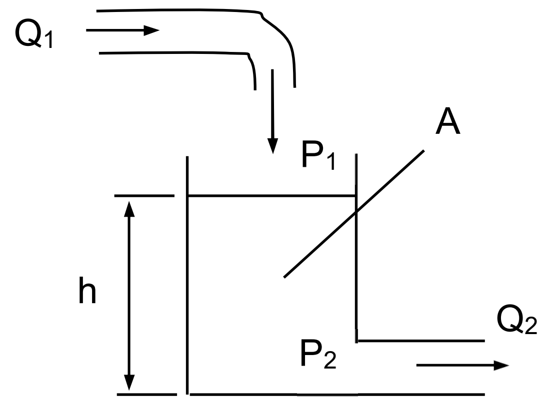
$$Q_1 - Q_2 = C\frac{d(P_2 - P_1)}{dt} \tag{4.12}$$

or

$$P_2 - P_1 = \frac{1}{C}\int (Q_1 - Q_2)dt \tag{4.13}$$

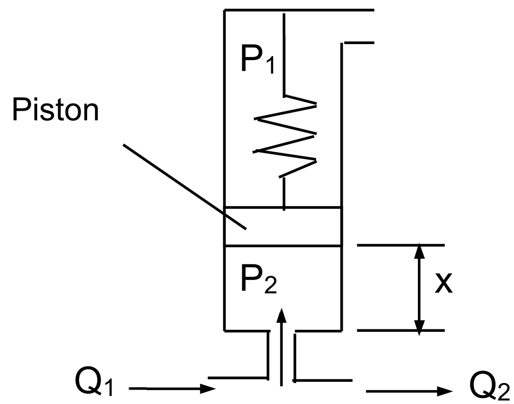
For a pressure head:

$$C = \frac{A}{\rho g}$$

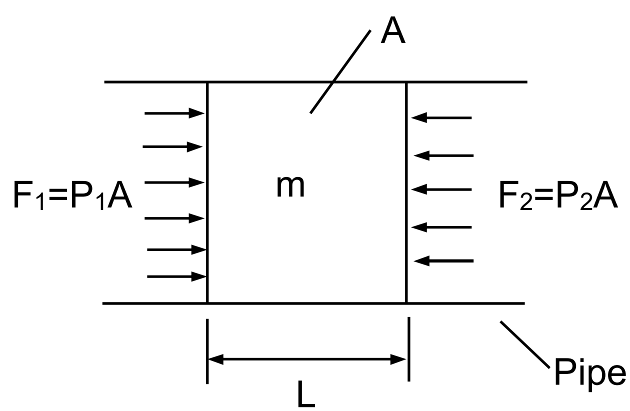


For a spring and piston accumulator:

$$C = \frac{A^2}{k} \qquad (4.18)$$



Hydraulic inertance:



$$P_1 - P_2 = I\frac{dQ}{dt} \qquad (4.22)$$

where $I = \frac{L\rho}{A}$ is the hydraulic inertance for the block of fluid.

Combine hydraulic elements using equivalents to Kirchhoff's laws:

1. Sum of flow rates at a junction = 0
2. Sum of pressure drops around loop = supply pressure

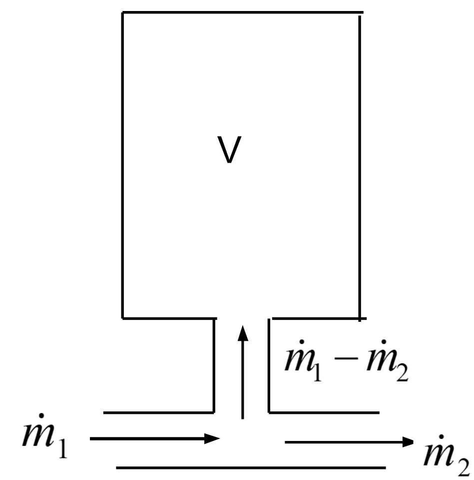
4.2.4 Pneumatic Systems Building Blocks

Similar to hydraulic except for changes due to the compressibility of air.

Pneumatic resistance: $P_1 - P_2 = R\dot{m}$ (4.23)

Pneumatic capacitance: $\dot{m}_1 - \dot{m}_2 = C\dfrac{dP}{dt}$ (4.27)

For a pneumatic accumulator (as shown): $C = \dfrac{V}{R_g T}$



Pneumatic inertance:

$$P_1 - P_2 = I\frac{d\dot{m}}{dt} \qquad (4.28)$$

For a block of air in a long pipe: $I = \dfrac{L}{A}$

Pneumatic building blocks are combined in the same manner as hydraulic ones.

4.2.5 Thermal Systems Building Blocks

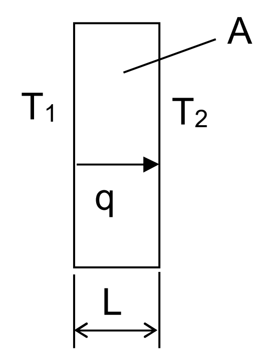
Building blocks: (i) thermal resistance and (ii) thermal capacitance.

Thermal resistance:

$$T_1 - T_2 = Rq \qquad (4.29)$$

where q is the rate of heat flow, $T_1 - T_2$ is the temperature drop across the resistance element and R is the thermal resistance.

For unidirectional conduction through a solid: $R = \dfrac{L}{Ak}$ (4.30)



where A is the cross-sectional area, L is the length over which the heat must flow and k is the thermal conductivity of the material.

For convection with liquids and gases the thermal resistance is:

$$R = \frac{1}{Ah} \qquad (4.31)$$

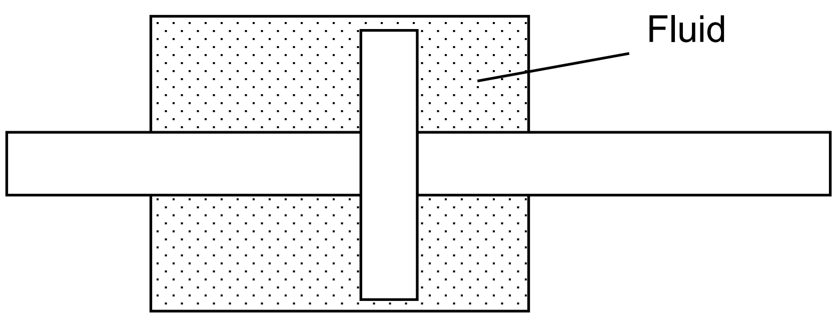
where A is the surface area across which there is a temperature difference, and h is the coefficient of heat transfer.

Thermal capacitance:

$$q_1 - q_2 = mc\frac{dT}{dt} = C\frac{dT}{dt} \tag{4.32}$$

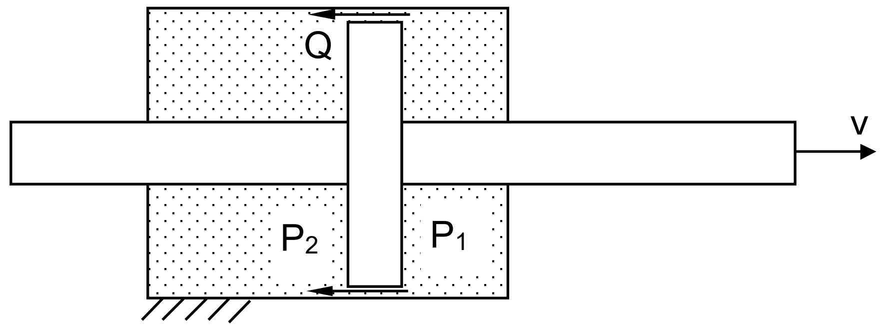
where m is the mass of the element, c is the specific heat capacity for the material and $C = mc$ is the thermal capacitance.

Example 4.1: Fluid damper



How can we verify that this design behaves correctly? We will assume that the mass of the piston and rods may be neglected.

Solution



Gap around the piston may be modeled as a hydraulic resistor:

$$P_1 - P_2 = R_{gap} Q \tag{4.33}$$

where Q is the volume flow rate through the gap, R_{gap} is the hydraulic resistance of the gap, P_1 is the pressure before the resistor, and P_2 is the pressure after the resistor.

Since $Q = vA$ this may be rewritten as:

$$P_1 - P_2 = R_{gap} v A_{piston} \tag{4.34}$$

where A_{piston} is the cross-sectional area of the piston. Multiplying both sides of this equation by the area of the piston gives:

$$A_{piston}P_1 - A_{piston}P_2 = A_{piston}R_{gap}vA_{piston} \tag{4.35}$$

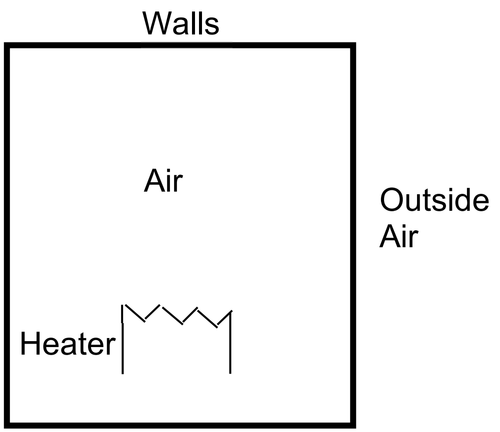
$$F_1 - F_2 = A_{piston}^2 R_{gap} v \tag{4.36}$$

$$F = cv \tag{4.37}$$

where F is the net force (opposing the velocity), and $c = A_{piston}^2 R_{gap}$. Since (4.37) matches (4.2) the design behaves correctly.

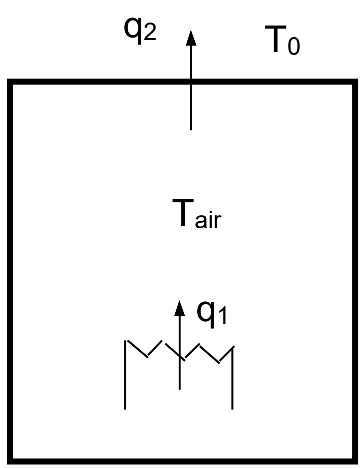
Example 4.2: Room heating

Determine the Laplace transfer function for a mechatronic system consisting of an electric heater inside a room. The room is surrounded by air at a temperature of T_0. We will assume that the air in the room has a uniform temperature, the walls do not store any heat, and there is no heat transfer through the floor.



Solution

We must first determine the thermal building blocks involved and their equations. We will assume that the heater is hotter than the air in the room and the air in the room is hotter than the air outside.



Since the walls do not store any heat they may be modeled as a thermal resistor. So we have:

$$T_{air} - T_0 = R_{walls} q_2 \tag{4.38}$$

Since the temperature of the air inside is uniform its resistance is not an issue so we can model it as a thermal capacitor:

$$q_1 - q_2 = C_{air}\frac{dT}{dt} \tag{4.39}$$

where q_2 is the heat lost through the walls, q_1 is the heat produced by the heater and C_{air} is the thermal capacitance of the air in the room.

To obtain the system model, substitute q_2 from (4.38) into (4.39):

$$q_1 - \left(\frac{T_{air} - T_0}{R_{walls}}\right) = C_{air}\frac{dT_{air}}{dt} \tag{4.40}$$

$$R_{walls}C_{air}\frac{dT_{air}}{dt} + T_{air} = R_{walls}q_1 + T_0 \tag{4.41}$$

Taking the Laplace transform of both sides (assuming zero initial conditions) gives:

$$R_{walls}C_{air}T_{air}(s)s + T_{air}(s) = R_{walls}q_1(s) + T_0(s) \tag{4.42}$$

The desired transfer function is:

$$T_{air}(s) = \frac{R_{walls} q_1(s)}{R_{walls} C_{air} s + 1} + \frac{T_0(s)}{R_{walls} C_{air} s + 1} \tag{4.43}$$

This is a first order lag transfer function with two inputs (q_1 and T_0) and one output (T_{air}). The time constant is $R_{walls} C_{air}$.

Example 4.3: An application of our room heating model

Given: 3 × 3 × 3 m room, 0.05 m thick brick walls, and the outside air temperature is –10 °C. Determine:

(a) The smallest (lowest power) heater needed for the room's air temperature to reach 22 °C.

(b) The time constant.

Solution

(a) If the q_1 and T_0 are held constant the final value for T_{air} is obtained by finding the limit of the transfer function (4.43) as s approaches zero. The results in:

$$T_{air\ steadystate} = R_{walls} q_1 + T_0 \tag{4.48}$$

So the heater output required is:

$$q_1 = \frac{T_{air\ steadystate} - T_0}{R_{walls}} \tag{4.49}$$

The total cross-sectional area of the walls (including the ceiling) is:

$$A = 5(3\ m)(3\ m) = 45\ m^2 \tag{4.50}$$

The thermal conductivity of brick is 0.073 W/m°K. If we assume the heat loss is only due to conduction, the wall resistance is:

$$R_{walls} = \frac{L}{Ak} = \frac{0.05\ m}{(45\ m^2)(0.073\ W/mK)} = 1.52 \times 10^{-2}\ K/W \tag{4.51}$$

So the answer is:

$$q_1 = \frac{T_{air\,steadystate} - T_0}{R_{walls}} = \frac{295\ °K - 263\ K}{1.52 \times 10^{-2}\ K/W} = 2.10 \times 10^3\ W \tag{4.52}$$

(b) For this we will need to find the capacitance of the air in the room. Assuming the air is at −10 °C and 1.01×10^5 Pa at the beginning, the mass of the air is:

$$m = \frac{PV}{R_g T} = \frac{(1.01 \times 10^5\ Pa)(3\ m)(3\ m)(3\ m)}{(287\ J/kg°K)(-10 + 273)°K} = 36.1\ kg \tag{4.53}$$

The specific heat capacity of air in this temperature range is 1006 J/kg°K so we have:

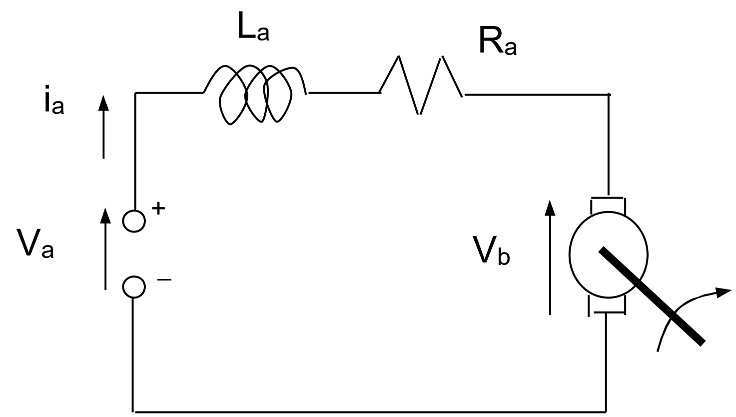
$$C_{air} = mc = (36.1\ kg)(1006\ J/kgK) = 3.63 \times 10^4\ J/K \tag{4.54}$$

The answer is (recalling that 1 W = 1 J/s):

$$\tau = R_{walls} C_{air} = (1.52 \times 10^{-2}\ K/W)(3.63 \times 10^4\ J/K) = 552\ s \tag{4.55}$$

Example 4.4: PM DC Motor Revisited

The electrical elements (armature coils, commutation, and PMs) may be modeled by the circuit shown below. There is a single loop and the voltage supply is V_a. So Kirchhoff's second law (sum of voltage drops in a loop = voltage supply) gives:



$$V_a = V_b + L_a \frac{di_a}{dt} + R_a i_a \tag{4.56}$$

Since the back EMF equals $V_b = K_b\omega$, this becomes:

$$V_a = K_b\omega + L_a \frac{di_a}{dt} + R_a i_a \tag{4.57}$$

The motor internally produces a torque proportional to the armature current. The viscous friction of the motor bearings creates an opposing torque proportional to the angular velocity, $K_d\omega$. Therefore the motor's output torque is:

$$T_{motor} = K_t i_a - K_d \omega \tag{4.58}$$

The motor must also overcome the torque due to an external load, T_{load} . Therefore Newton's second law in rotational form gives:

$$J\frac{d\omega}{dt} = \sum T \tag{4.59}$$

$$J\frac{d\omega}{dt} = T_{motor} - T_{load} \tag{4.60}$$

$$J\frac{d\omega}{dt} = K_t i_a - K_d \omega - T_{load} \tag{4.61}$$

If the amplifier we are using adjusts the armature current the transfer function between the angular velocity, the current and the load torque is obtained by taking the Laplace transform of (4.61) as follows:

$$J\omega(s)s = K_t i_a(s) - K_d \omega(s) - T_{load} \tag{4.62}$$

$$\omega(s) = \left(\frac{K_t}{Js + K_d}\right) i_a(s) - \left(\frac{1}{Js + K_d}\right) T_{load} \tag{4.63}$$

Similarly, for the angular position $\theta(s) = \omega(s)/s$ and:

$$\theta(s) = \left(\frac{K_t}{s(Js + K_d)}\right) i_a(s) - \left(\frac{1}{s(Js + K_d)}\right) T_{load} \tag{4.64}$$

If the amplifier adjusts the armature voltage then we must first take the Laplace transform of (4.57) and solve for i_a(s):

$$V_a(s) = K_b\omega(s) + L_a s\, i_a(s) + R_a i_a(s) \tag{4.65}$$

$$i_a(s) = \frac{V_a(s) - K_b\omega(s)}{L_a s + R_a} \tag{4.66}$$

Substituting this into (4.63) gives the angular velocity transfer function:

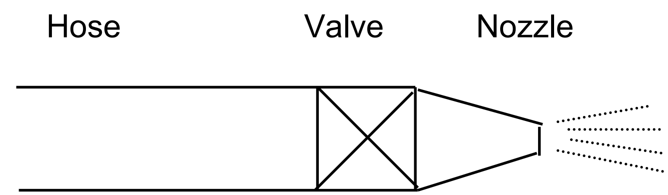
$$\omega(s)=\left(\frac{K_t}{(L_a s+R_a)(Js+K_d)+K_b K_t}\right)V_a(s)-\left(\frac{L_a s+R_a}{(L_a s+R_a)(Js+K_d)+K_b K_t}\right)T_{load} \quad (4.67)$$

The corresponding angular position transfer function is:

$$\theta(s)=\left(\frac{K_t}{s\left[(L_a s+R_a)(Js+K_d)+K_b K_t\right]}\right)V_a(s)-\left(\frac{L_a s+R_a}{s\left[(L_a s+R_a)(Js+K_d)+K_b K_t\right]}\right)T_{load} \quad (4.68)$$

Although not technically correct, the ratio L_a/R_a is often called the "electrical time constant" and J/K_d is termed the "mechanical time constant". Typically the electrical time constant is much smaller than the mechanical one. This means that the mechanical portion of the dynamics will dominate the response. (Recall the concept of a dominant pole from your control systems course).

Example 4.5: Fuel Injector

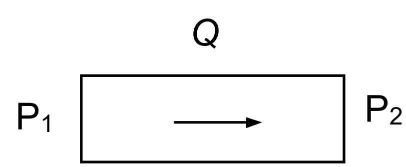


Fuel injector for an internal combustion engine consisting of a long hose followed by a valve and nozzle. We will assume the valve and nozzle can be modelled by a single linear element and that the fuel is incompressible. We wish to determine the transfer function between the volume flow rate and the supply pressure when the valve is open.

Solution

The hydraulic elements and there governing equations are:

(i) Fuel in the hose:

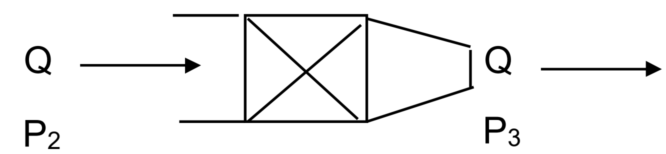


This may be modeled as an inertor:

$$P_1 - P_2 = I\frac{dQ}{dt} = \frac{L\rho}{A}\frac{dQ}{dt} \tag{4.69}$$

where P_1 is the input pressure (here the supply pressure), P_2 is the output pressure, L is the length of the hose, A is its cross-sectional area, ρ is the density of the fuel and Q is the volume flow rate.

(ii) Valve and nozzle:



The valve and nozzle may be modeled as a combined hydraulic resistance R, with the governing equation:

$$P_2 - P_3 = RQ \tag{4.70}$$

Note that Q is the same as in the hose since the fluid is incompressible (and there is nowhere else for it to go).

To obtain the system model we first choose to measure all pressures relative to the engine pressure. In this case: $P_3 = 0$ and

$$P_2 = RQ \tag{4.71}$$

Substituting (4.71) into (4.70) gives:

$$P_1 - RQ = \frac{L\rho}{A}\frac{dQ}{dt} \tag{4.72}$$

$$P_1 = \frac{L\rho}{A}\frac{dQ}{dt} + RQ \tag{4.73}$$

Taking the Laplace transform gives:

$$P_1(s) = \frac{L\rho}{A} sQ(s) + RQ(s) \tag{4.74}$$

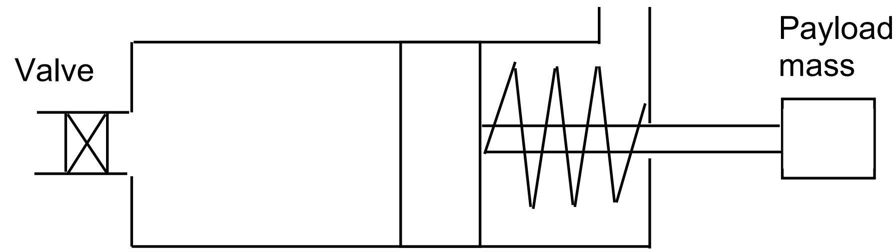
The desired transfer function is then:

$$\frac{Q(s)}{P_1(s)} = \frac{1}{(L\rho / A)s + R} \tag{4.75}$$

Again this is a first order lag transfer function.

Questions: What is the steady state gain of this transfer function?
Does this gain make physical sense?

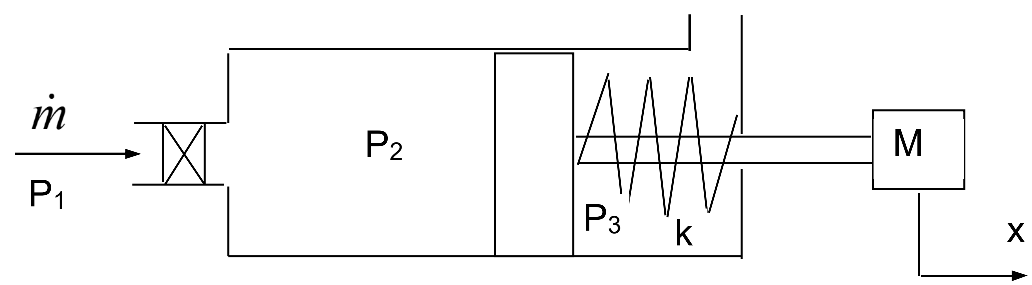
Example 4.6: Single Acting Pneumatic Cylinder



A single acting pneumatic cylinder is being used to move a payload mass horizontally. Assume the mass of the piston and rod can be neglected. A valve is used to control the flow into the extend side of the piston. The retract side of the piston is open to the atmosphere.

Determine the transfer function between the displacement of the payload and the pressure input to the valve (when the valve is open).

Solution



Payload mass=M,
area of the extend side of the piston=A,
input pressure for valve=P_1 ,
output pressure for valve=P_2
and pressure on retract side of piston=P_3.

Assuming the open valve can be modeled as pneumatic resistor we have:

$$P_1 - P_2 = R_{valve}\dot{m} \tag{4.76}$$

where $\dot{m}$ is the mass flow rate into the extend side of the cylinder. By definition of the mass flow rate:

$$\dot{m} = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho\frac{dV}{dt} + V\frac{d\rho}{dt} \tag{4.77}$$

Note that the volume is not constant because the piston moves. If we assume that the spring force is small then the volume will change more rapidly than the density of the air. We define the displacement of the payload, x, to be zero when the piston is fully retracted. Neglecting the density change we then find:

$$\dot{m} = \rho\frac{dV}{dt} = \rho\frac{d(Ax)}{dt} = \rho A\dot{x} \tag{4.78}$$

Substituting this into (4.76) gives:

$$P_1 - P_2 = R_{valve}\rho A\dot{x}$$

$$P_1 = R_{valve}\rho A\dot{x} + P_2 \qquad (4.79)$$

Now we need to find the relationship between P_2 and x. From Newton's second law:

$$\sum F = M\ddot{x}$$

$$P_2 A - P_3 A - kx = M\ddot{x} \qquad (4.80)$$

If we use gauge pressures then P_3=0 since it is open to the atmosphere. This simplifies (4.80) and allows us to solve for P_2 in terms of x:

$$P_2 A - kx = M\ddot{x}$$

$$P_2 = \frac{M}{A}\ddot{x} + \frac{k}{A}x \qquad (4.81)$$

Substituting this into (4.79) gives:

$$P_1 = R_{valve}\rho A\dot{x} + \frac{M}{A}\ddot{x} + \frac{k}{A}x \tag{4.82}$$

We may then obtain the desired transfer function:

$$P_1(s) = R_{valve}\rho As\,x(s) + \frac{M}{A}s^2 x(s) + \frac{k}{A}x(s) \tag{4.83}$$

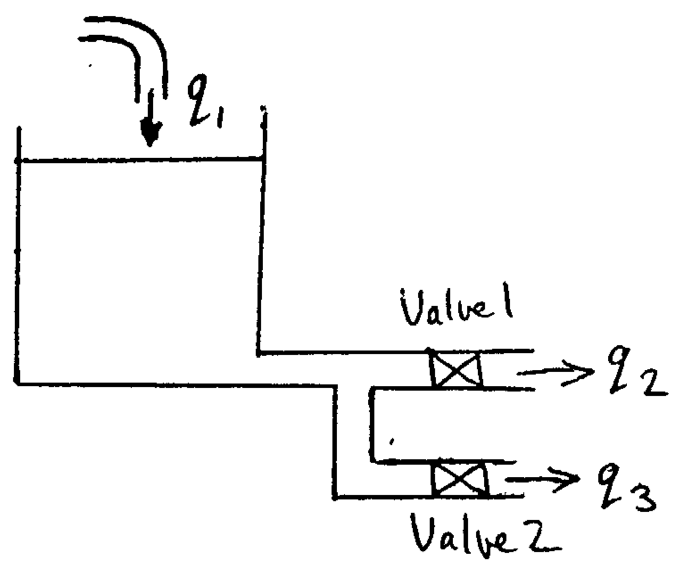
$$\frac{x(s)}{P_1(s)} = \frac{A}{Ms^2 + R_{valve}\rho A^2 s + k} \tag{4.84}$$

Questions: What kind of transfer function is this?
What is the steady state gain of this transfer function?
Does this gain make physical sense?

Example 4.7: Hydraulic System

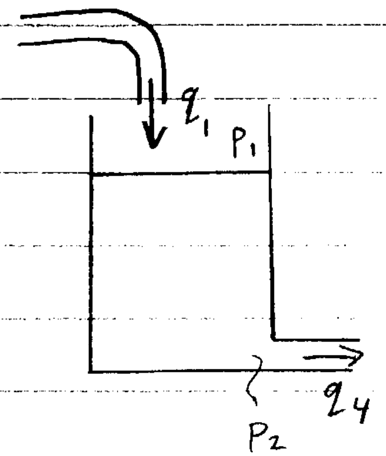
The hydraulic system shown in the figure consists of a tank, some short pipes and two valves. The valves are not identical. The pressure at the top of the tank and after the valves is atmospheric.

Determine the transfer function between the input volume flow rate and the output volume flow rate for valve 1.



Solution

Tank

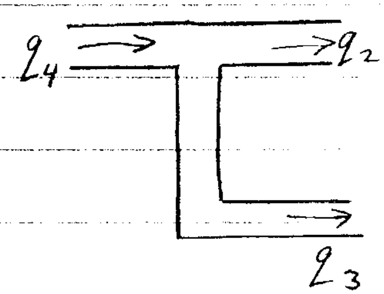


$$q_1 - q_4 = C\frac{dp}{dt} = C\frac{d(p_2 - p_1)}{dt}$$

If we use gauge pressures then:

$$q_1 - q_4 = C\frac{dp_2}{dt} \quad \ldots (1)$$

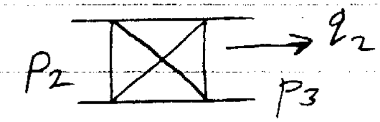
Junction



$$q_4 = q_2 + q_3 \quad \ldots (2)$$

Valve 1

$$p_2 - p_3 = R_1 q_2$$

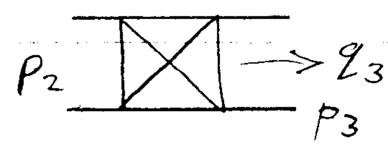


$p_3 = 0$ gauge

$$\therefore \quad p_2 = R_1 q_2 \quad \ldots (3)$$

Valve 2

$$p_2 - p_3 = R_2 q_3$$



$p_3 = 0$ gauge.

$$\therefore \quad p_2 = R_2 q_3 \quad \ldots (4)$$

System Model

Laplace transform of (1) gives:

$$q_1(s) - q_4(s) = C p_2(s) s \qquad \ldots (5)$$

Laplace transform of (3) gives:

$$p_2(s) = R_1 q_2(s) \qquad \ldots (6)$$

Subst. $p_2(s)$ from (6) into (5):

$$q_1(s) - q_4(s) = C R_1 q_2(s) s \qquad \ldots (7)$$

Equating the Laplace transforms of (3) & (4) gives:

$$R_1 q_2(s) = R_2 q_3(s)$$

$$\therefore q_3(s) = \frac{R_1}{R_2} q_2(s) \qquad \ldots (8)$$

Subst. q_4 from (2) and q_3 from (8) into (7):

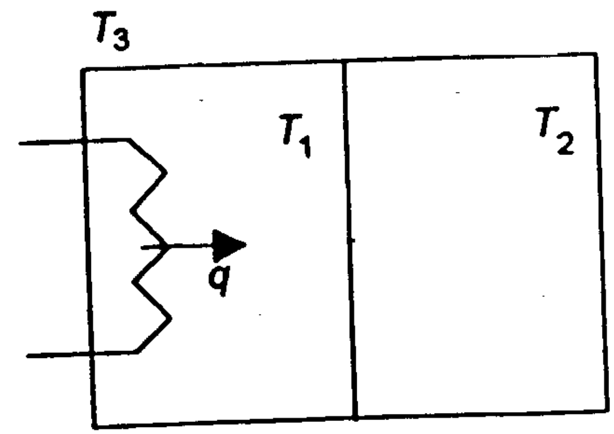
$$q_1(s) - q_2(s) - \frac{R_1}{R_2} q_2(s) = CR_1 q_2(s)\, s$$

$$q_1(s) = \left(CR_1 s + \frac{R_1}{R_2} + 1 \right) q_2(s)$$

$$\frac{q_2(s)}{q_1(s)} = \frac{1}{CR_1 s + R_1/R_2 + 1} \quad \text{where } C = \frac{A}{\rho g}$$

Example 4.8: Thermal System

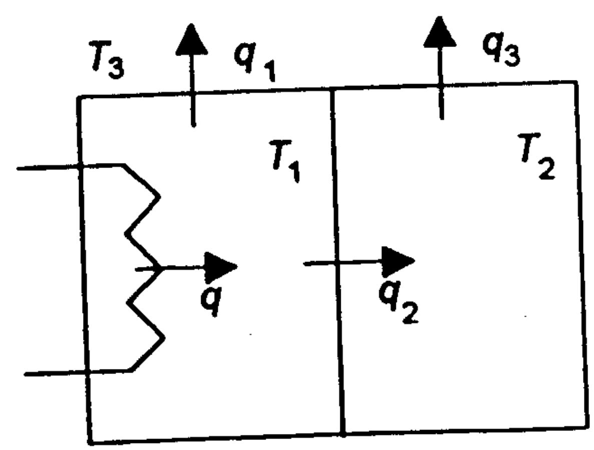
A thermal system consisting of two sealed compartments is shown in the schematic to the right. The first compartment contains a heater whose heat flow output is *q*. The two compartments are otherwise identical in design.



Determine the transfer function for relating the output T_1 to the inputs q and T_3.

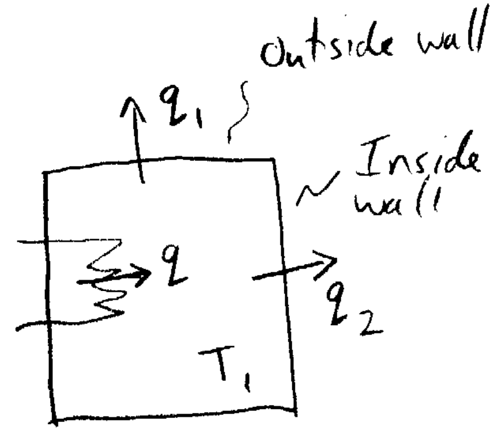
Solution

Assuming $T_1>T_2$ and $T_2>T_3$ the directions of heat flow are as shown to the right.



First Compartment:

Temp. is uniform ∴ Just capacitance



$$q - q_1 - q_2 = C \frac{dT_1}{dt}$$

Taking Laplace transform gives:

$$q - q_1 - q_2 = C T_1 s \qquad (1)$$

Outside Wall

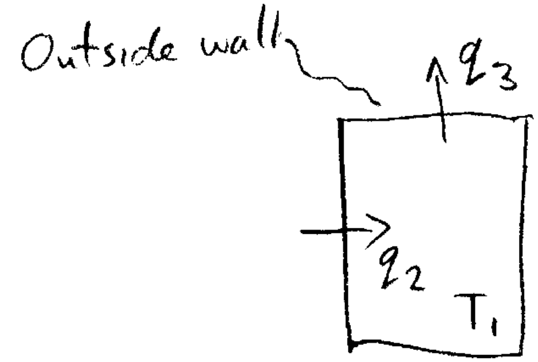
All walls do not store heat ∴ Just resistance

$$T_1 - T_3 = R q_1 \qquad (2)$$

Inside Wall

$$T_1 - T_2 = R q_2 \qquad (3)$$

Second Compartment:



As before just cap.

$$q_2 - q_3 = C\frac{dT_2}{dt}$$

Taking Laplace transform gives:

$$q_2 - q_3 = C T_2 s \qquad (4)$$

Note: From the diagram, the compartments have the same capacitance and we can reasonably assume all walls have the same resistance.

System Model $\Rightarrow$ We must eliminate T_2, q_1, q_2 and q_3

Combining (1), (2) and (3) gives:

$$q - (T_1 - T_3)/R - (T_1 - T_2)/R = C T_1 s \qquad (6)$$

Combining (3), (4) and (5) gives:

$$(T_1 - T_2)/R - (T_2 - T_3)/R = C T_2 s$$

$$T_1 + T_3 = R C T_2 s + 2 T_2$$

$$T_2 = (T_1 + T_3)/(RCs + 2) \quad \ldots (7)$$

Substitute (7) into (6):

$$q - (T_1 - T_3)/R - T_1/R + \left[(T_1 + T_3)/(RCs+2)\right]/R = CT_1 s$$

$$Rq - 2T_1 + T_3 + T_1/(RCs+2) + T_3/(RCs+2) = RCT_1 s$$

$$(R^2Cs + 2R)q - (2RCs+4)T_1 + (RCs+2)T_3 + T_1 + T_3$$
$$= RCs(RCs+2)T_1$$

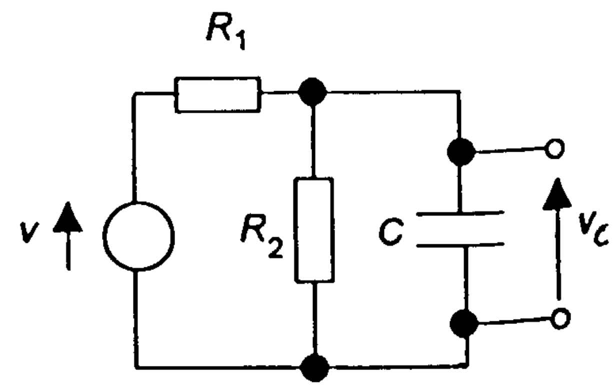
$$(R^2Cs + 2R)q + (RCs+3)T_3 = (R^2C^2s^2 + 4RCs + 3)T_1$$

$$\therefore\ T_1(s) = \frac{R^2Cs + 2R}{R^2C^2s^2 + 4RCs + 3}\, q(s) + \frac{RCs + 3}{R^2C^2s^2 + 4RCs + 3}\, T_3(s)$$

(Note if $q(s) = 0$ in the steady state $T_1 = T_3$ which makes sense)

Example 4.9: Electrical system

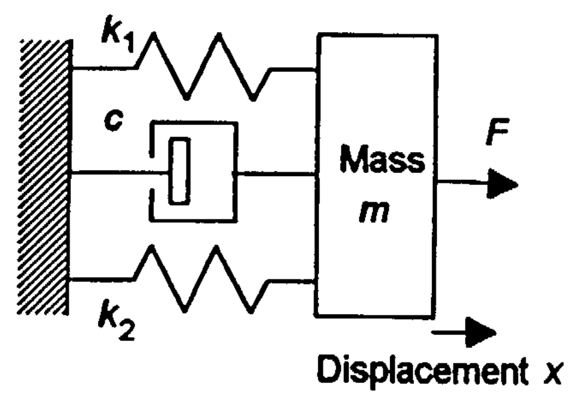
Determine the transfer function between the output, v_c, and the input, v, for the circuit shown below.



Any questions about this example?

Example 4.10: Mechanical system

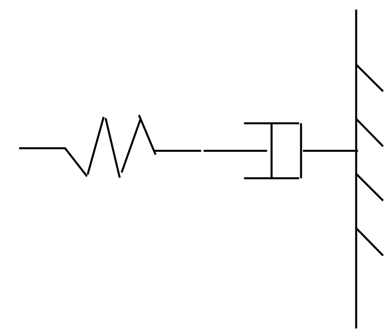
For the two spring, mass and damper system shown to the right, determine the transfer function between the displacement output, x, and the force input, F.



Any questions about this example?

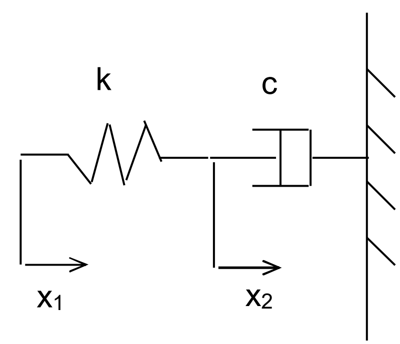
Example 4.11: Mechanical elements with a massless connection

A spring and damper are connected in series as shown to the right. Determine the transfer function between the input displacement of the spring and the output displacement of the damper.

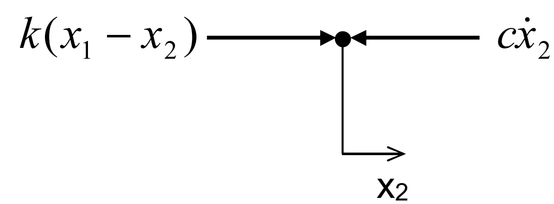


Solution

Denoting the spring constant as k, the damping coefficient as c and the displacements as shown in the figure to the right.



Assuming $x_1 > x_2$ and $\dot{x}_2 > 0$ the free-body diagram for the massless connection between the spring and damper is shown below. From Newton's second law we find:



$$\sum F = ma = 0 \qquad (4.85)$$

$$k(x_1 - x_2) - c\dot{x}_2 = 0 \tag{4.86}$$

$$kx_1 = kx_2 + c\dot{x}_2 \tag{4.87}$$

Taking the Laplace transform of both sides and rearranging gives the final answer:

$$kx_1(s) = kx_2(s) + csx_2(s) \tag{4.88}$$

$$\frac{x_2(s)}{x_1(s)} = \frac{k}{cs+k} \tag{4.89}$$

Questions: What is the steady state gain of this transfer function?
Does this make physical sense?
What is the significance of a larger *c* value in terms of the time and frequency responses?

4.2.7 Non-linear Elements

- Any mathematical model is an approximation of reality.
- All real elements are non-linear to some extent.
- Only the control theory for linear systems is fully developed

- Two classes of real elements:

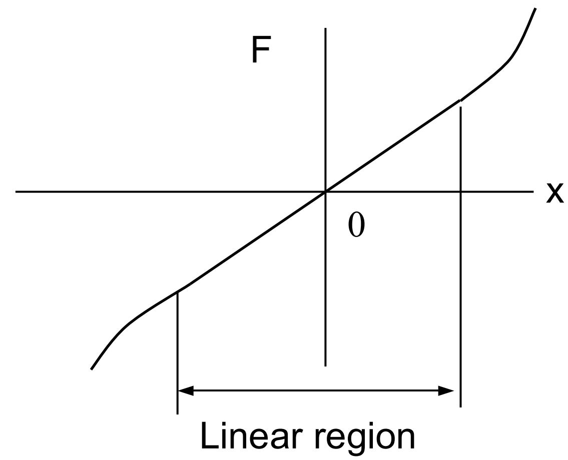
1) Those that have an operating region over which linearity may be assumed.
2) Those that may be approximated by a linear system at a particular operating condition.

Element from the first class: We must determine the approximately linear region and ensure that we do not apply the model outside of this region.

Element from the second class: We can approximate its behavior at a particular operating point by a linear model. This is known as "local linearization".

Example 4.12

The response curve for a typical spring is shown in the figure to the right. This curve has an approximately linear region so it belongs to the first class of real elements.

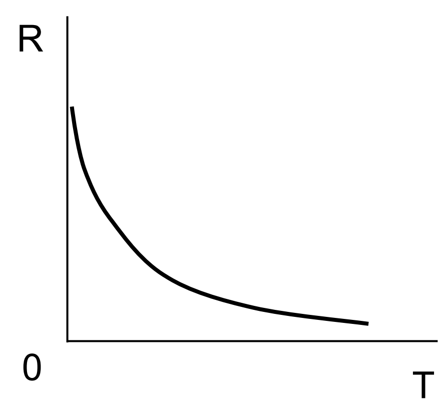


Example 4.13

The input-output response of a thermistor is given by:

$$R = C_1 e^{-C_2 T} \tag{4.95}$$

where R is the resistance, T is the temperature, and C_1 and C_2 are constant parameters. An example is shown in the figure below. The objective is to determine a linear model for this element.



Solution

The response curve contains no approximately linear region so the element belongs to the second class described above. Applying equation (4.91) gives:

$$R - R_0 = \left(\left. \frac{df}{dT} \right|_{T=T_0} \right)(T - T_0) \tag{4.96}$$

$$R - R_0 = \left(\left. -C_2 C_1 e^{-C_2 T} \right|_{T=T_0} \right)(T - T_0) \tag{4.97}$$

$$R - R_0 = \left(-C_2 C_1 e^{-C_2 T_0} \right)(T - T_0) \tag{4.98}$$

$$R = \left(-C_2 C_1 e^{-C_2 T_0} \right)(T - T_0) + R_0 \tag{4.99}$$

Equation (4.99) is the local linear model for the operating point (R_0, T_0).

Example 4.14

We have modelled valves as a linear hydraulic resistance but the valve sizing equation (equation 3.10) shows that the relationship between the pressure drop and the volume flow rate is actually non-linear.

Assuming the fluid density is constant while the flow rate, flow coefficient, and pressure drop are all variable (as is the case with a proportional valve or a servo valve) we can now derive a locally linearized equation.

Solution

First we'll put the equation in a form similar to equation (4.6) for the hydraulic resistor, as follows:

$$C_V = 4.22 \times 10^4 Q \sqrt{\frac{\rho}{P_{drop}}} \tag{4.100}$$

$$C_V{}^2 = (4.22 \times 10^4)^2 Q^2 \frac{\rho}{P_{drop}} \tag{4.101}$$

$$P_{drop} = 1.78\times10^9 Q^2 \frac{\rho}{C_V{}^2} \tag{4.102}$$

where $P_{drop} = P_1 - P_2$ is the pressure drop across the valve. Next, if ρ is constant and the our operating point is (Q_0, C_{V0}, P_{drop0}) then the local linear model is given by:

$$\Delta P_{drop} = \left(\left.\frac{\partial f}{\partial Q}\right|_{Q=Q_0}\right)\Delta Q + \left(\left.\frac{\partial f}{\partial C_{\mathrm{V}}}\right|_{C_V=C_{V0}}\right)\Delta C_V \tag{4.103}$$

$$\Delta P_{drop} = \left(2(1.78\times10^9)\frac{\rho Q_0}{C_{V0}^2}\right)\Delta Q + \left(-2(1.78\times10^9)\frac{\rho Q_0{}^2}{C_{V0}^3}\right)\Delta C_V \tag{4.104}$$

$$\Delta P_{drop} = \left(3.56\times10^9\frac{\rho Q_0}{C_{V0}^2}\right)\Delta Q - \left(3.56\times10^9\frac{\rho Q_0{}^2}{C_{V0}^3}\right)\Delta C_V \tag{4.105}$$

So the effective resistance of the valve will increase when C_{V0} is decreased and when Q_0 is increased.

Actuator Friction

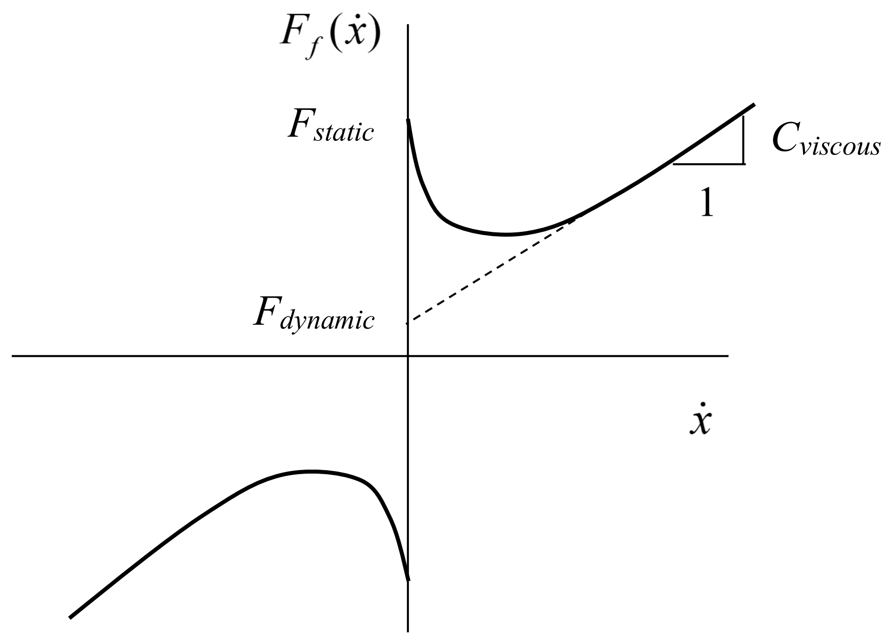
Actuators are also subject to several non-linearities. The first one we'll discuss is friction. Friction consists of a combination of static, dynamic and viscous components. One model of the friction force is:

$$F_f(\dot{x}) = C_{viscous}\dot{x} + sign(\dot{x})F_{dynamic} + (1 - sign(|\dot{x}|))F_{static} \quad (4.106)$$

where $\dot{x}$ is the velocity, F_{static} is the static friction force, $F_{dynamic}$ is the dynamic friction force, $C_{viscous}$ is the coefficient of viscous friction

and $sign(\dot{x}) = \begin{cases} 1, \dot{x} > 0 \\ 0, \dot{x} = 0 \\ -1, \dot{x} < 0 \end{cases}$.

More realistically there will be transition region at low velocities as shown in the figure.



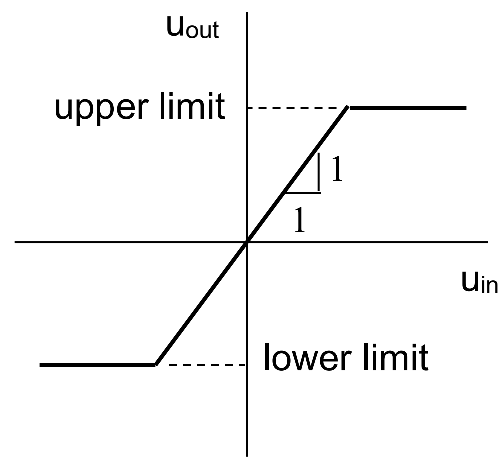
We often want to operate our control system near to the point $\dot{x} = 0$ but the response curve is discontinuous and not differentiable at this point.

For this operating point no linear model may be obtained (unless both the static and dynamic components are zero).

Actuator Saturation

All actuators have a limited range, for example a valve can be 100% open but not 150% open.

If our control system commands a larger value then the actuator saturates.



Actuator saturation is not differentiable and cannot be linearized.

It is also not a good solution to stay in the linear region (avoiding the upper and lower limits) because this means we have oversized our actuator for the given application (making the product larger and more expensive).

4.2.8 Dead Time (or Time Delay) Elements

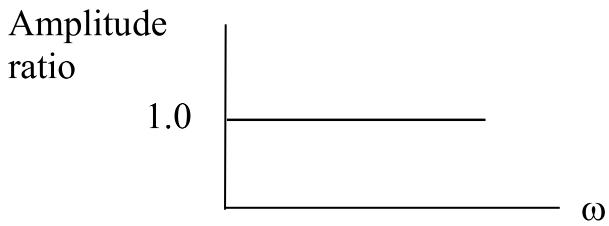
In the time domain the response of a dead time element is simply:

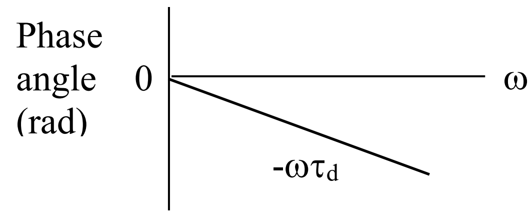
$$out(t) = \begin{cases} 0 & t < \tau_d \\ in(t-\tau_d) & t \geq \tau_d \end{cases} \tag{4.107}$$

where τ_d is the period of dead time. The transfer function for a dead time element is:

$$\frac{Out(s)}{In(s)} = e^{-\tau_d s} \tag{4.107}$$

The non-algebraic form of this transfer function makes it difficult to work with dead time elements in the Laplace domain. The frequency response is shown below.





Dead-time elements add an unbounded phase lag to a control system that tends to destabilize the system (reducing the phase margin).

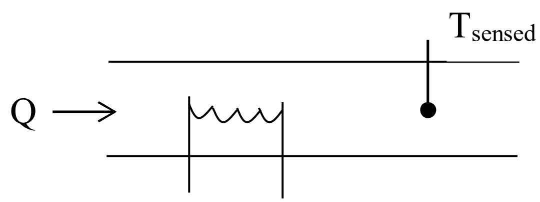
So we would like to minimize the dead time wherever possible.

One approach is to design the mechatronic system so that the actuator and sensor are close together (known as “collocated”).

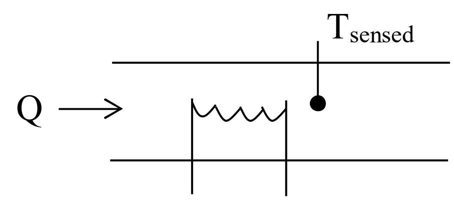
This moves the dead time is moved outside of the control system so the phase margin is unaffected by it.

For example if we are heating a fluid within a pipe and controlling its temperature then it is best if the temperature sensor is just downstream of the heater. This type of dead time is known as "transport delay"..

Design with large transport delay:



Design with smaller transport delay:



Another example is a long-distance teleoperated robot.

A person controls the remotely located robot over a communication link. The transmission delay resulting from the long-distance communications makes stable control very difficult.

One solution to this problem is send high-level commands to the robot and then let it perform the control locally (operating autonomously).

This is the approach that was used in the Mars Pathfinder mission.

Computation Delays in Digital Control Systems

Another source of dead time inside a control loop is computation time. Processing the information from the sensor and determining the appropriate command for the actuator takes time and adds delay to the control system.

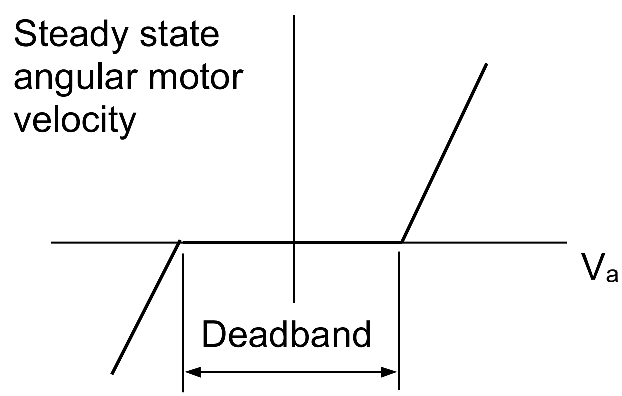
A faster microcontroller will reduce this delay while adding cost to the product, so there is a cost/performance trade-off.

Actuator Dead Time caused by Deadband

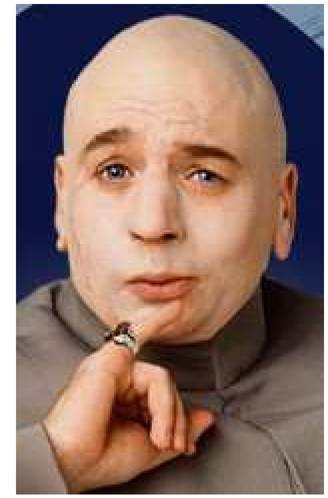
The deadband we covered in the sensors chapter can also apply to actuators. This is a non-linearity that is typically caused by the static friction component.

When a linear controller is used with an actuator with a dead band the dead band will produce a time delay.

If the deadband is known and constant then a non-linear controller may be used to eliminate this problem.



Example 4:15: The Evil Housemate



You're taking a shower the morning after a late night out and your housemate takes revenge on you for eating the last KD by flushing the toilet. After recovering from this predicament you wonder, why is it so hard to get the water back to the right temperature?

Solution

Actuator = you turning the taps

Sensor = your skin!

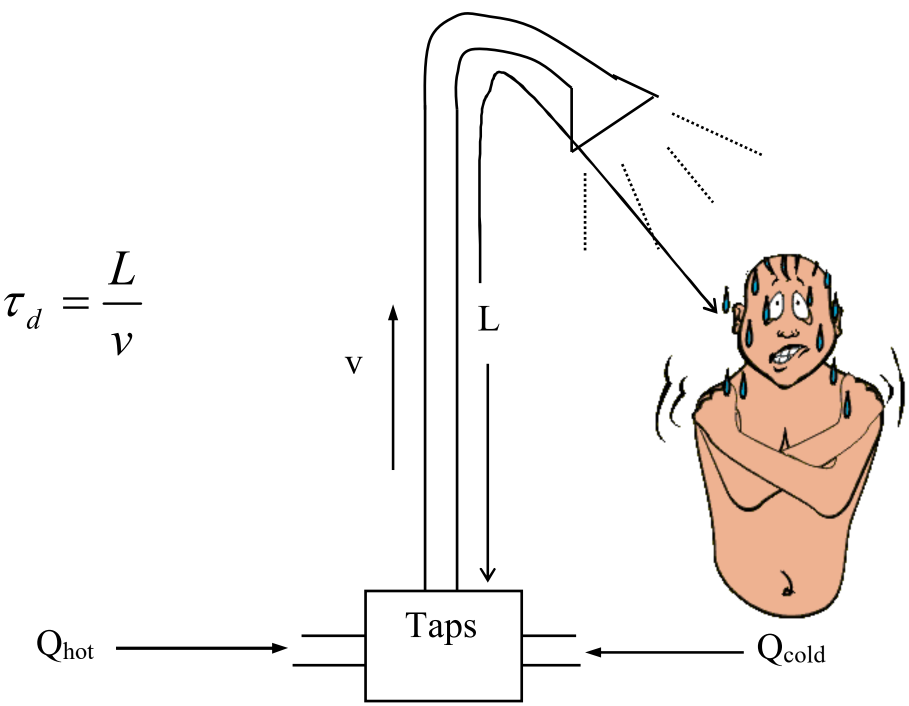
The transport delay is

$$\tau_d = \frac{L}{v}$$

inside the loop and is not constant.

This can lead to some

nasty temperature oscillations ☹.



4.3 Introduction to the State Space Method

The system dynamics are modeled by a set of first order differential equations.

Each first order equation defines the dynamics of a "state variable".

The set of equations is then written in the matrix form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.108}$$

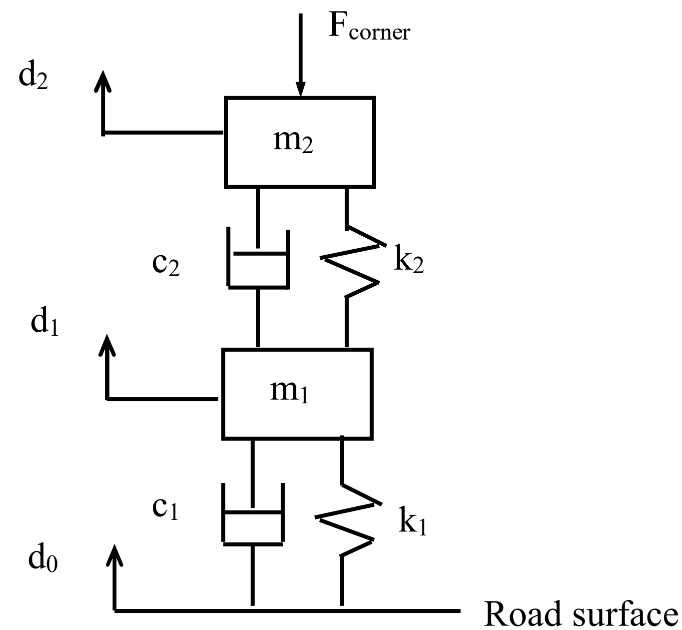
where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $\dot{\mathbf{x}}(t) = [\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]^T$, $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T$, $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix. The $x_i(t)$ are the states. An *n*th order model will have *n* states. The $u_i(t)$ are the inputs. The system outputs are given by:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.109}$$

where $\mathbf{y}(t) = [y_1(t), y_2(t), \ldots, y_p(t)]^T$, $\mathbf{C}$ is a $p \times n$ matrix and $\mathbf{D}$ is usually the zero matrix. The $y_i(t)$ are the outputs. The **A, B** and **C** matrices define the dynamic behavior of the model.

Example 4.16: 1/4 car suspension model

m_1 is the combined mass of the wheel, tire and suspension components; k_2 and c_2 are the spring constant and damping coefficient for the spring and shock absorber; k_1 and c_1 are the spring constant and damping coefficient for the tire; and F_{corner} is the force due to cornering. The position of the road surface is also variable so that bumpy roads may be simulated.



Outputs: positions of the tire and car

Inputs: cornering force, position and velocity of the road surface.

Let's derive the state space model.

Solution

From free-body diagrams (not shown) of the two masses and Newton's 2nd law we may obtain:

$$m_2\ddot{d}_2 = c_2\dot{d}_1 - c_2\dot{d}_2 + k_2 d_1 - k_2 d_2 - F_{corner} \quad (4.110)$$

$$m_1\ddot{d}_1 = c_2\dot{d}_2 - c_2\dot{d}_1 + k_2 d_2 - k_2 d_1 + c_1\dot{d}_0 - c_1\dot{d}_1 + k_1 d_0 - k_1 d_1 \quad (4.111)$$

Question: How many state variables will we need for this system?
Answer: We have two 2nd order differentiation equations so we need 4 state variables.

Question: What should we define as the state variables?
Answer: For each differential equation of order q, we'll define the states as the zero'th to the (q–1)th derivatives

Here we should set: $x_1 = d_1$, $x_2 = \dot{d}_1$, $x_3 = d_2$, $x_4 = \dot{d}_2$ and $u_1 = F_{corner}$, $u_2 = d_0$ and $u_3 = \dot{d}_0$.

Substituting these state variable definitions into equations (4.110) and (4.111) gives:

$$m_2\dot{x}_4 = c_2x_2 - c_2x_4 + k_2x_1 - k_2x_3 - u_1 \tag{4.112}$$

$$m_1\dot{x}_2 = c_2x_4 - c_2x_2 + k_2x_3 - k_2x_1 + c_1u_3 - c_1x_2 + k_1u_2 - k_1x_1 \tag{4.113}$$

Rearranging these equations and noting that $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$ (from our state variable definitions) the first part of the state space model is.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & \frac{-c_1-c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 \\ -\frac{1}{m_2} & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} \tag{4.114}$$

For the outputs to be the positions of the tire and the car we require: $y_1 = d_1 = x_1$, and $y_2 = d_2 = x_3$. So we have:

$$\underset{\uparrow \atop \mathbf{y}}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \underset{\uparrow \atop \mathbf{C}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{\uparrow \atop \mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} \tag{4.115}$$

This completes the state space model.

- Advantages of State Space Models vs. Transfer Function Models

Simulation of State Space Models

State space models are the basis of many computer aided design tools for control systems design. Recalling the first matrix equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

We can approximate $\dot{\mathbf{x}}(t)$ using the forward difference:

$$\dot{\mathbf{x}}(t) = \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} \tag{4.117}$$

$$\therefore\ \mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t\dot{\mathbf{x}}(t) \tag{4.118}$$

where t is the current time, and Δt is a small time change or "timestep". This is also known as Euler integration.

Calculating $\mathbf{y}$ using (4.109) is straightforward.

In Matlab the code would be:

```
x(1)=x0;  % Initialize the vector of states
for k=1:N  % Loop N times
    xDot(k)=a*x(k)+B*u(k);  % Equation (4.108)
    x(k+1)=x(k)+deltaT*xDot(k);  % Equation (4.118)
    y(k)=C*x(k);  % Equation (4.109)
    u(k)=……  % Either an open-loop input or your control algorithm goes
    here
end
t=deltaT*[1:N];  % Time values
plot(y,t); % Graphs the result
```

4.4 Discrete Time Models

Discrete time transfer functions involve the use of Z transforms and exist in the Z domain. An example of a discrete time Z transfer function is:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2z+3}{z^2 - 0.7z - 0.18} \tag{4.119}$$

Note that like the variable s, the variable z may be complex number. This transfer function is second order and has the zero: z = -1.5, and the poles: z = 0.9 and z = -0.2.

The Z transfer function for a plant may be derived from its Laplace transform or may be obtained from experimental data as discussed further in the next section.

There will be more coverage of Z transfer functions in Chapter 6 of the course.

4.5 Introduction to the System Identification Method

With the system identification method the mathematical model is obtained from experimental data.

Advantages:

Disadvantages:

Combined approach: