In-Class Test (#3)

Name ___________________

Student Number ___________________

ROBOTICS 4K03

INSTRUCTOR NAME: Fengjun Yan

DURATION OF EXAMINATION: 50 MINS

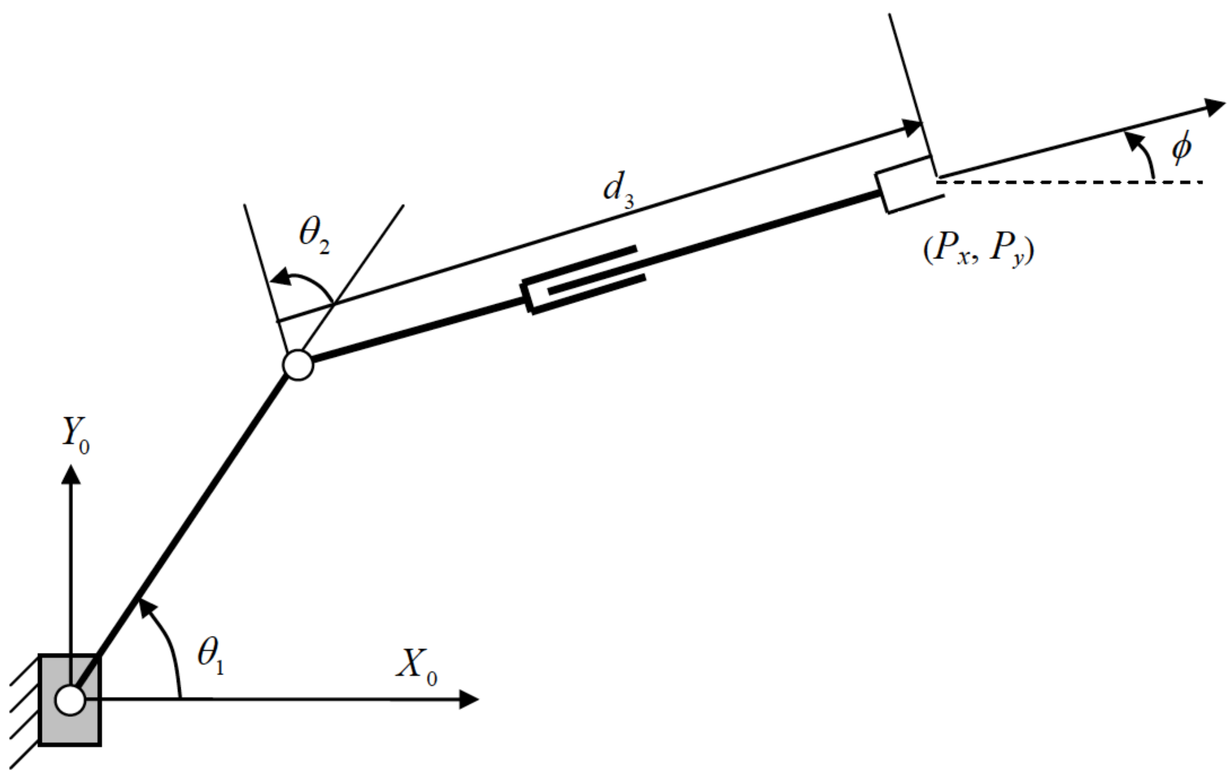
Nov. 15th, 2021

THIS EXAMINATION PAPER INCLUDES 2 PAGES AND 2 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Use of Casio FX-991 calculator. This paper must be returned with your answers.

Questions:

1. (45 points) A RRP planar robot is shown in the following figure. Its joint variables are θ_1 and θ_2, and d_3. Its end-effector position and orientation are given by P_x and P_y, and ϕ. Derive the inverse kinematics equations for this robot.



Solutions:

$\Theta_1 + \Theta_2 - 90^\circ = \phi$

$P_x = a_1 . C\Theta_1 + d_3 \, C\phi$ -----> $C\Theta_1 = \frac{P_x - d_3 \, C\,\phi}{a_1}$

$P_y = a_1 . S\Theta_1 + d_3 S\phi$ -----> $S\Theta_1 = \frac{P_y - d_3 S\phi}{a_1}$

$C^2\Theta_1 + S^2\,\Theta_1 = \left(\frac{P_x - d_3\,C\,\phi}{a_1}\right)^2 + \left(\frac{P_y - d_3\,S\,\phi}{a_1}\right)^2 = 1$

$d_3{}^2 - 2(P_x\,C\phi + P_y\,S\phi)]\,d_3 + P_x{}^2 + P_y{}^2 - a_1{}^2 = 0$

$$d_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where:

a= 1

$\mathrm{b} = -2(P_x C\emptyset + P_y S\emptyset)$

$\mathrm{c} = P_x^2 + P_y^2 - a_1^2$

$$d_3 = P_x C\emptyset + P_y S\emptyset \pm \frac{\sqrt{[2(P_x C\emptyset + P_y S\emptyset)]^2 - 4(P_x^2 + P_y^2 - a_1^2)}}{2}$$

Assume $\Delta = \left[2\left(P_x\,C\,\phi\; + \;P_y\,S\,\phi\right)\right]^2 - 4\,(P_x^2 + P_y^2 - a_1^2)$

If $\Delta < 0$, there will be No solution for d_3.

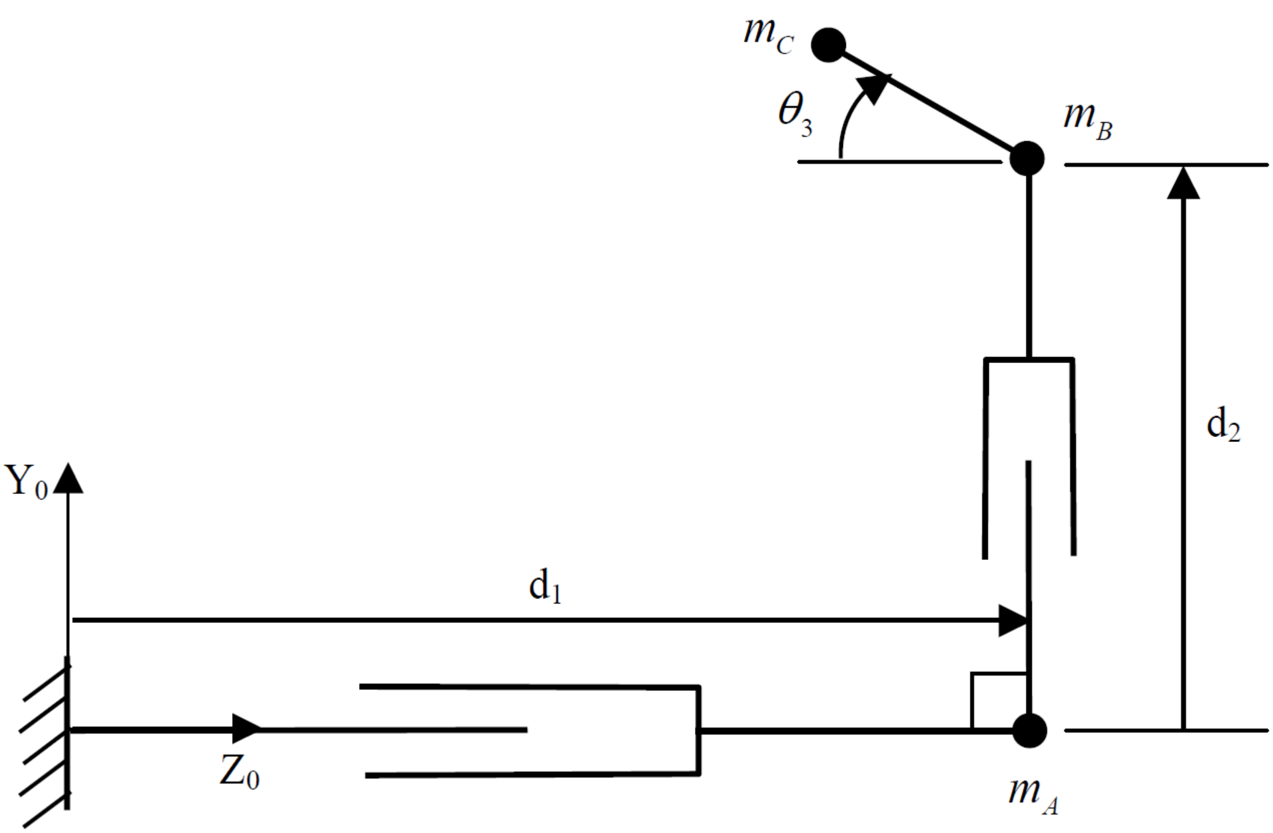
If $\Delta = 0$, there will be only ONE solution for d_3.

If $\Delta > 0$, there will be TWO solutions for d_3.

$$\theta_1 = \mathrm{atan2}\left(\frac{p_y - d_3 \sin\phi}{a_1}, \frac{p_x - d_3 \cos\phi}{a_1}\right)$$

$$\theta_2 = \frac{\pi}{2} - (\theta_1 - \phi)$$

2. (55 points)
The planar PPR robot shown in the figure operates in the vertical plane (*i.e.* gravity acts in the $-Y_0$ direction). The masses of the links are concentrated at points A, B and C as shown.
Derive the Lagrangian function L and calculate the force/torque for the *i*th joint only. (*i depends on the first letter of your Last Name: i=1, when the first letter of your Last Name is from A-I; i=2, when the first letter of your Last Name is from J-R; i=3, when the first letter of your Last Name is from S-Z*)



Solutions: (**Calculate the force for 1st joint is sufficient**)

$$v_A = \dot{d}_1 \Rightarrow K_1 = \tfrac{1}{2} m_A v_A^2 = \tfrac{1}{2} m_A \dot{d}_1^2$$

$$\begin{cases} \dot{z}_B = \dot{d}_1 \\ \dot{y}_B = \dot{d}_2 \end{cases} \Rightarrow K_2 = \tfrac{1}{2} m_B v_B^2 = \tfrac{1}{2} m_B (\dot{d}_1^2 + \dot{d}_2^2)$$

$$\begin{cases} z_C = d_1 - a_3 C\theta_3 \Rightarrow \dot{z}_C = \dot{d}_1 + a_3 \dot{\theta}_3 S\theta_3 \\ y_C = d_2 + a_3 S\theta_3 \Rightarrow \dot{y}_C = \dot{d}_2 + a_3 \dot{\theta}_3 C\theta_3 \end{cases} \Rightarrow \begin{cases} K_3 = \tfrac{1}{2} m_C v_C^2 = \tfrac{1}{2} m_C \left\{ (\dot{d}_1 + a_3 \dot{\theta}_3 S\theta_3)^2 + (\dot{d}_2 + a_3 \dot{\theta}_3 C\theta_3)^2 \right\} \\ K_3 = \tfrac{1}{2} m_C (\dot{d}_1^2 + \dot{d}_2^2 + a_3^2 \dot{\theta}_3^2 + 2 a_3 \dot{d}_1 \dot{\theta}_3 S\theta_3 + 2 a_3 \dot{d}_2 \dot{\theta}_3 C\theta_3) \end{cases}$$

$$P_1 = -m_A G^T p_{cA} = -m_A \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} y_A \\ z_A \end{bmatrix} = -m_A \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} 0 \\ d_1 \end{bmatrix} = 0$$

$$P_2 = -m_B G^T p_{cB} = -m_B \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} y_B \\ z_B \end{bmatrix} = -m_B \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} = m_B g d_2$$

$$P_3 = -m_C G^T p_{cC} = -m_C \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} y_C \\ z_C \end{bmatrix} = -m_C \begin{bmatrix} -g & 0 \end{bmatrix} \begin{bmatrix} d_2 + a_3 S\theta_3 \\ d_1 - a_3 C\theta_3 \end{bmatrix} = m_C g (d_2 + a_3 S\theta_3)$$

$$L = K - P = K_1 + K_2 + K_3 - P_1 - P_2 - P_3$$

$$L = \tfrac{1}{2} m_A \dot{d}_1^2 + \tfrac{1}{2} m_B (\dot{d}_1^2 + \dot{d}_2^2) + \tfrac{1}{2} m_C (\dot{d}_1^2 + \dot{d}_2^2 + a_3^2 \dot{\theta}_3^2 + 2a_3 \dot{d}_1 \dot{\theta}_3 S\theta_3 + 2a_3 \dot{d}_2 \dot{\theta}_3 C\theta_3) - m_B g d_2 - m_C g (d_2 + a_3 S\theta_3)$$

$$L = \tfrac{1}{2}(m_A + m_B + m_C)\dot{d}_1^2 + \tfrac{1}{2}(m_B + m_C)\dot{d}_2^2 + \tfrac{1}{2} m_C a_3^2 \dot{\theta}_3^2 + m_C a_3 (\dot{d}_1 \dot{\theta}_3 S\theta_3 + \dot{d}_2 \dot{\theta}_3 C\theta_3) - (m_B + m_C) g d_2 - m_C g a_3 S\theta_3$$

For joint 1:

$$\left. \begin{aligned} F_1 &= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{d}_1}\right) - \frac{\partial L}{\partial d_1} \\ \frac{\partial L}{\partial \dot{d}_1} &= (m_A + m_B + m_C)\dot{d}_1 + m_C a_3 \dot{\theta}_3 S\theta_3 \\ \frac{\partial L}{\partial d_1} &= 0 \end{aligned} \right\} \Rightarrow F_1 = (m_A + m_B + m_C)\ddot{d}_1 + m_C a_3 \ddot{\theta}_3 S\theta_3 + m_C a_3 \dot{\theta}_3^2 C\theta_3$$

For joint 2:

$$\left. \begin{aligned} F_2 &= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{d}_2}\right) - \frac{\partial L}{\partial d_2} \\ \frac{\partial L}{\partial \dot{d}_2} &= (m_B + m_C)\dot{d}_2 + m_C a_3 \dot{\theta}_3 C\theta_3 \\ \frac{\partial L}{\partial d_2} &= -(m_B + m_C) g \end{aligned} \right\} \Rightarrow F_2 = (m_B + m_C)\ddot{d}_2 + m_C a_3 \ddot{\theta}_3 C\theta_3 - m_C a_3 \dot{\theta}_3^2 S\theta_3 + (m_B + m_C) g$$

For joint 3:

$$\left. \begin{aligned} \tau_3 &= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} \\ \frac{\partial L}{\partial \dot{\theta}_3} &= m_C a_3^2 \dot{\theta}_3 + m_C a_3 (\dot{d}_1 S\theta_3 + \dot{d}_2 C\theta_3) \\ \frac{\partial L}{\partial \theta_3} &= m_C a_3 (\dot{d}_1 \dot{\theta}_3 C\theta_3 - \dot{d}_2 \dot{\theta}_3 S\theta_3) - m_C g a_3 C\theta_3 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \tau_3 &= m_C a_3^2 \ddot{\theta}_3 + m_C a_3 (\ddot{d}_1 S\theta_3 + \dot{d}_1 \dot{\theta}_3 C\theta_3 + \ddot{d}_2 C\theta_3 - \dot{d}_2 \dot{\theta}_3 S\theta_3) \\ &\quad - m_C a_3 (\dot{d}_1 \dot{\theta}_3 C\theta_3 - \dot{d}_2 \dot{\theta}_3 S\theta_3) + m_C g a_3 C\theta_3 \\ \tau_3 &= m_C a_3^2 \ddot{\theta}_3 + m_C a_3 \ddot{d}_1 S\theta_3 + m_C a_3 \ddot{d}_2 C\theta_3 + m_C g a_3 C\theta_3 \end{aligned} \right.$$

We can write the final answer in the matrix form (optional for this question):

$$\begin{bmatrix} F_1 \\ F_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} m_A + m_B + m_C & 0 & m_C a_3 S\theta_3 \\ 0 & m_B + m_C & m_C a_3 C\theta_3 \\ m_C a_3 S\theta_3 & m_C a_3 C\theta_3 & m_C a_3^2 \end{bmatrix} \begin{bmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & m_C a_3 C\theta_3 \\ 0 & 0 & -m_C a_3 S\theta_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1^2 \\ \dot{d}_2^2 \\ \dot{\theta}_3^2 \end{bmatrix} + \begin{bmatrix} 0 \\ (m_B + m_C) g \\ m_C g a_3 C\theta_3 \end{bmatrix}$$