3. VELOCITIES, JACOBIANS AND SINGULARITIES

3.1 Introduction

In this chapter we will study the issues involved in controlling the robot's velocity. For example, if the robot is being used for arc welding the velocity of the welding gun (end-effector) must be precisely controlled to produce a high quality weld. In this chapter we will study the relationship between the velocities of the joints and the velocities of the end-effector, and related topics.

3.2 The Manipulator Jacobian

If variables x and y are related by a nonlinear function we can write

$$y = f(x) \tag{3.1}$$

The relationship between the differential changes (i.e. very small changes) in x and y is described by

$$\delta y = \frac{\partial f}{\partial x}\delta x \tag{3.2}$$

where δy is the differential change in y and δx is the differential change in x. Dividing both sides of equation (3.2) by a differential change in time gives the velocity relationship

$$\dot{y} = \frac{\partial f}{\partial x}\dot{x} \tag{3.3}$$

where $\dot{y}$ is the instantaneous velocity of y and $\dot{x}$ is the instantaneous velocity of x. If y is a function of several independent variables then, after applying the chain rule, equation (3.2) can be written in the matrix form:

$$\begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_1} & \ddots & & \vdots \\ \vdots & & & \\ \frac{\partial f_i}{\partial x_1} & \cdots & & \frac{\partial f_i}{\partial x_j} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_j \end{bmatrix} \tag{3.4}$$

The $i \times j$ matrix of partial derivatives is known as a "Jacobian matrix". The form of equation (3.4) can be used to represent the relationship between the differential changes in end-effector's position and orientation in base coordinates (frame {0}), and the differential changes in the joint variables for an n-DOF manipulator, as follows

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = J(q) \begin{bmatrix} dq_1 \\ dq_2 \\ \vdots \\ dq_n \end{bmatrix} \tag{3.5}$$

where q_i are the joint variables; $J(q)$ is the "manipulator Jacobian"; dq_i are differential changes in the joint variables; dx, dy and dz are the differential changes of the tool along the x_0, y_0 and z_0 axes; and δx, δy and δx are the differential changes of the tool about the x_0, y_0 and z_0 axes.

This can also be written in terms of velocities:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J(q) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{3.6}$$

where $\dot{q}_i$ are the joint velocities; v_x, v_y and v_z are the linear velocities of the tool along the x_0, y_0 and z_0 axes; and ω_x, ω_y and ω_z are the angular velocities of the tool about the x_0, y_0 and z_0 axes.

Note that J is a function of the joint variables, or in other words it depends on the configuration of the robot arm. There are several ways to derive the manipulator Jacobian. We will study Schilling's approach [1] because it is relatively simple, and builds upon the forward kinematic equations we've already studied.

The approach begins by partitioning $J(q)$ into two $3 \times$ n blocks as follows:

$$J(q) = \begin{bmatrix} \frac{\partial p_x(q)}{\partial q_1} & \frac{\partial p_x(q)}{\partial q_2} & \cdots & \frac{\partial p_x(q)}{\partial q_n} \\ \frac{\partial p_y(q)}{\partial q_1} & \frac{\partial p_y(q)}{\partial q_2} & \cdots & \frac{\partial p_y(q)}{\partial q_n} \\ \frac{\partial p_z(q)}{\partial q_1} & \frac{\partial p_z(q)}{\partial q_2} & \cdots & \frac{\partial p_z(q)}{\partial q_n} \\ \hline \zeta_1 z_0(q) & \zeta_2 z_1(q) & \cdots & \zeta_n z_{n-1}(q) \end{bmatrix} = \begin{bmatrix} J_A \\ \hline J_B \end{bmatrix} \tag{3.7}$$

where p_x, p_y and p_z are the expressions from the forward kinematics solution for ${}^0T_6$. ζ_i is a parameter indicating the joint type where $\zeta_i = 0$ if joint i is prismatic and $\zeta_i = 1$ if joint i is revolute. The z_i are coordinates of the z-axes for each frame relative to the base frame. These may be computed using

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } z_i = {}^0R_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } i = 1 \text{ to } n-1 \tag{3.8}$$

where ${}^0R_i = \prod_{k=1}^{i} {}^{k-1}R_k$ and ${}^{k-1}R_k$ is the 3 x 3 rotation portion of A_k.

Example 3.1

We are interested in deriving the manipulator Jacobian for the planar 3R manipulator shown in Figure 3.1. Note that x_3 in the direction of tool approach. For this robot:

$$ {}^{0}T_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 & a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} \\ S\theta_{123} & C\theta_{123} & 0 & a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.9} $$

$$ {}^{0}R_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.10} $$

and

$$ \begin{aligned} {}^{0}R_2 &= {}^{0}R_1\,{}^{1}R_2 \\ &= \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ S\theta_2 & C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C\theta_1C\theta_2 - S\theta_1S\theta_2 & -C\theta_1S\theta_2 - S\theta_1C\theta_2 & 0 \\ S\theta_1C\theta_2 + C\theta_1S\theta_2 & -S\theta_1S\theta_2 + C\theta_1C\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{3.11} $$

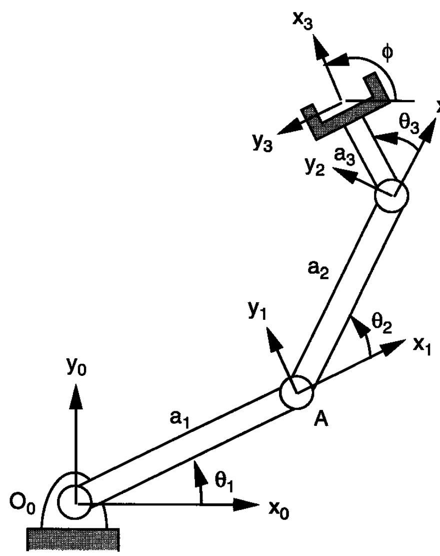

Figure 3.1 Planar 3R manipulator [2].

From equation (3.9):

$$ \begin{aligned} p_x &= a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123}, \\ p_y &= a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\ p_z &= 0 \end{aligned} \tag{3.12} $$

Applying equation (3.8) gives:

$$ z_1 = {}^{0}R_1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.13} $$

Similarly, $z_2 = {}^{0}R_2 [0 \quad 0 \quad 1]^T = [0 \quad 0 \quad 1]^T$.

We also know that $q_1 = \theta_1$, $q_2 = \theta_2$ and $q_3 = \theta_3$; and $\zeta_1 = \zeta_2 = \zeta_3 = 1$ since all joints are revolute. Applying equation (3.7), and recalling that $\frac{d(\cos\theta)}{d\theta} = -\sin\theta$ and $\frac{d(\sin\theta)}{d\theta} = \cos\theta$, now gives

$$ J(q) = \begin{bmatrix} -a_1S\theta_1 - a_2S\theta_{12} - a_3S\theta_{123} & -a_2S\theta_{12} - a_3S\theta_{123} & -a_3S\theta_{123} \\ a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} & a_2C\theta_{12} + a_3C\theta_{123} & a_3C\theta_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{3.14} $$

Examining equation (3.14) the zero rows could have been predicted since v_z, ω_x and ω_y will always be zero for this planar robot. For the planar case we can reduce (3.6) to

$$\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -a_1S\theta_1 - a_2S\theta_{12} - a_3S\theta_{123} & -a_2S\theta_{12} - a_3S\theta_{123} & -a_3S\theta_{123} \\ a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} & a_2C\theta_{12} + a_3C\theta_{123} & a_3C\theta_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{3.15}$$

End of example 3.1.

In-Class Exercise 3.1

Derive the manipulator Jacobian for an RRP spherical manipulator with the A matrices and the forward kinematics solution:

$$A_1 = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$${}^0T_3 = \begin{bmatrix} C\theta_1C\theta_2 & -S\theta_1 & C\theta_1S\theta_2 & d_3C\theta_1S\theta_2 \\ S\theta_1C\theta_2 & C\theta_1 & S\theta_1S\theta_2 & d_3S\theta_1S\theta_2 \\ S\theta_2 & 0 & C\theta_2 & d_3C\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3 Using the Manipulator Jacobian to Predict a Robot's Repeatability

Clearly from equation (3.6), the manipulator Jacobian may be used to calculate the end-effector velocities given the joint velocities. The manipulator Jacobian has other important uses. For example it may be used to predict a robot's repeatability as a function of the errors of its joint variables and its configuration. This can form a useful guide for programming the robot for high precision applications such as assembly.

The errors at the joints are usually due to a combination of sources such as gear backlash, friction, and sensor errors. Assuming the joint variable errors have been measured, equation (3.5) can be used to predict the resulting error in the end-effector's position and orientation as follows

$$\begin{bmatrix} e_x \\ e_y \\ e_z \\ e_{rx} \\ e_{ry} \\ e_{rz} \end{bmatrix} = J(q) \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \tag{3.16}$$

where e_i are the joint variable errors; e_x, e_y and e_z are the error of the tool position along the x_0, y_0 and z_0 axes; and e_{rx}, e_{ry} and e_{rz} are the orientation errors of the tool about the x_0, y_0 and z_0 axes. Unfortunately, since the joint errors can be positive or negative, applying equation (3.16) once is not sufficient. For an n-DOF robot we must apply this equation 2^n times and save the maximum predicted values for the end-effector errors.

Example 3.2

We will assume we have the same 3R planar robot as example 3.1. We are given: $e_1 = e_2 = e_3 = \pm 0.1°$ and a_1=0.5 m, a_2=0.5 m, and a_3=0.1 m.

(a) What is the expected error at the end-effector when $\theta_1 = 0, \theta_2 = 116.6°$ and $\theta_3 = 0$?

Since n=3 equation (3.16) must be calculated 2^3=8 times. First, we need to substitute the values of the joint variables into equation (3.15) to calculate J(q) for this arm configuration, giving

$$J(q) = \begin{bmatrix} -a_1 S\theta_1 - a_2 S\theta_{12} - a_3 S\theta_{123} & -a_2 S\theta_{12} - a_3 S\theta_{123} & -a_3 S\theta_{123} \\ a_1 C\theta_1 + a_2 C\theta_{12} + a_3 C\theta_{123} & a_2 C\theta_{12} + a_3 C\theta_{123} & a_3 C\theta_{123} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -0.5364 & -0.5364 & -0.0894 \\ 0.2312 & -0.2688 & -0.0448 \\ 1 & 1 & 1 \end{bmatrix} \qquad (3.17)$$

For the 8 combinations $[e_1 \; e_2 \; e_3]^T$, $[e_1 \; e_2 \; -e_3]^T$, $[e_1 \; -e_2 \; e_3]^T$, $[e_1 \; -e_2 \; -e_3]^T$, $[-e_1 \; e_2 \; e_3]^T$, $[-e_1 \; e_2 \; -e_3]^T$, $[-e_1 \; -e_2 \; e_3]^T$, and $[-e_1 \; -e_2 \; -e_3]^T$ the predicted tool errors calculated using equation (3.16) are:

$$\begin{array}{l} [-0.0020 \quad -0.0001 \quad 0.0052]^T \\ [-0.0017 \quad 0.0000 \quad 0.0017]^T \\ [-0.0002 \quad 0.0008 \quad 0.0017]^T \\ [\ 0.0002 \quad 0.0010 \quad -0.0017]^T \\ [-0.0002 \quad -0.0010 \quad 0.0017]^T \\ [\ 0.0002 \quad -0.0008 \quad -0.0017]^T \\ [\ 0.0017 \quad -0.0000 \quad -0.0017]^T \\ [\ 0.0020 \quad 0.0001 \quad -0.0052]^T \end{array} \qquad (3.18)$$

Assuming the joint errors reflect the repeatability of the joints, from (3.18) the predicted repeatability of the tool's position and orientation is: e_x=±0.0020 m, e_y=±0.0010 m and e_{rz}=±0.0052 radians = ±0.3°.

(b) How does the predicted repeatability vary as a function of the tool's position?

This can be solved by first using the inverse kinematics equations (to be studied in Chapter 4) to solve for the joint variables at many locations within the workspace. Next the manipulator Jacobian and the tool's predicted repeatability must be calculated at each location.

Since we are only interested in the effect of the tool's position, the tool's orientation angle ϕ (see Figure 3.1) will be kept constant at 90°. Plots of e_x and e_y versus tool position are shown in Figures 3.2 and 3.3, respectively.

From these plots we can observe that the variation of the error is complex. The general trends that the error in x increases as the arm becomes outstretched in the y direction, and vice-versa, are also apparent. To further complicate matters, the results also depend on the arm configuration. The plots shown are for the θ_2>0 ("elbow down") configuration.

End of example 3.2.

Note: Normally this approach is only useful for predicting the random portion of the error (*i.e.* the repeatability) since the accuracy is caused by other factors in addition to errors in the joint variables.

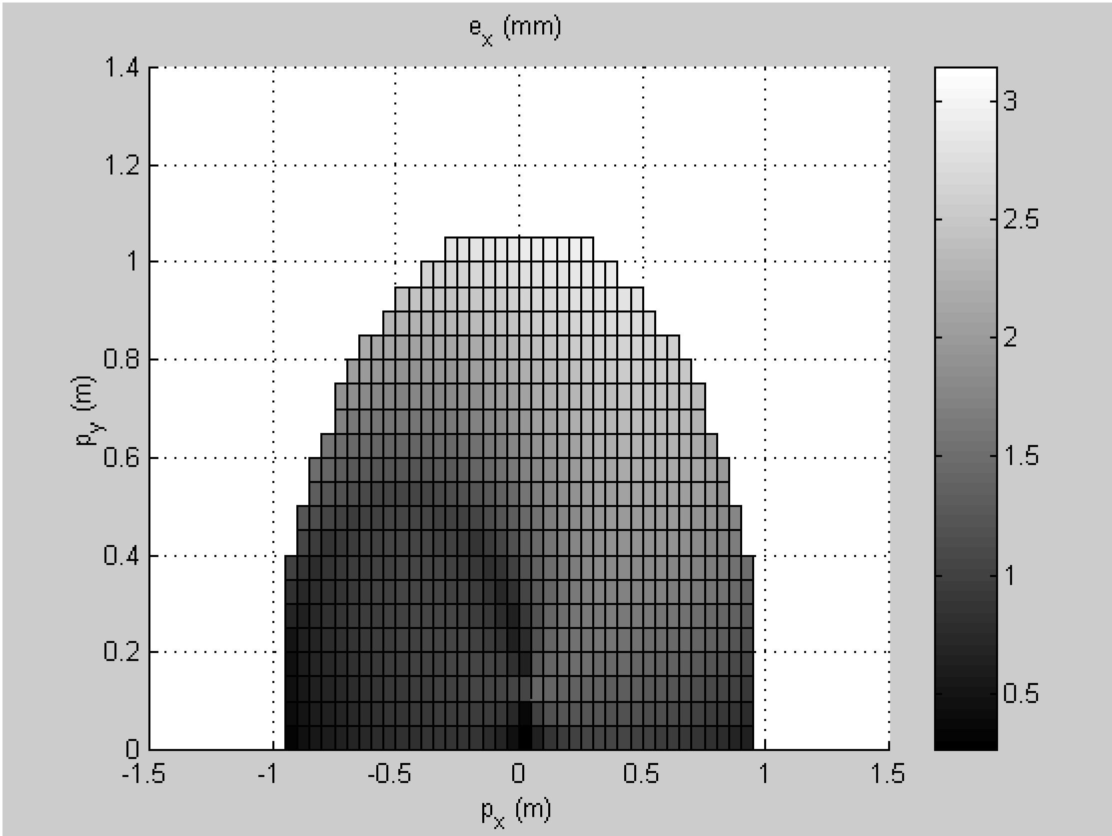

Figure 3.2

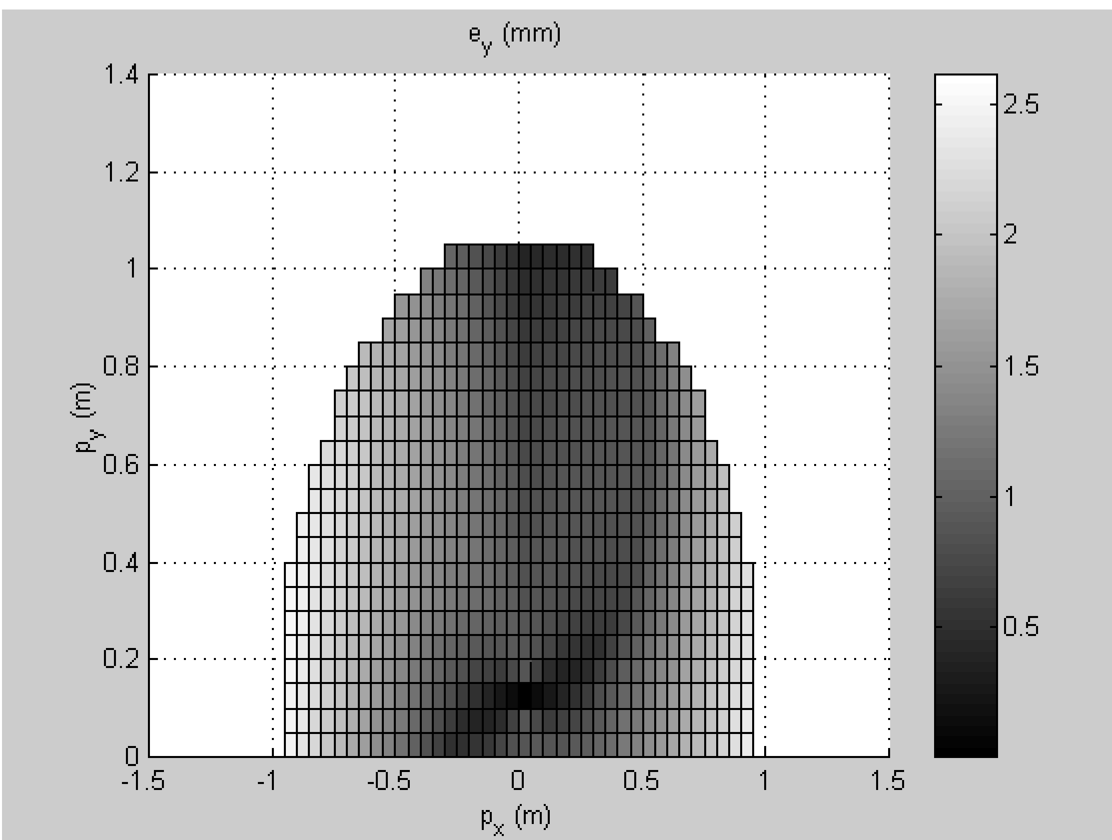

Figure 3.3

3.4 Applying Tranformations to Jacobian Matrices

The manipulator Jacobian is in term of the base frame. Sometimes we may be interested in the Jacobian relative to another coordinate frame. For a planar robot, this transformation is accomplished using:

$$ {}^{A}J(q) = {}^{A}R_{B}\,{}^{B}J(q) \tag{3.19}$$

where ${}^{A}J(q)$ is relative to frame {A}, ${}^{B}J(q)$ is relative to frame {B} and ${}^{A}R_{B}$ is the rotation matrix from ${}^{A}T_{B}$. Note that the translation component is not involved since a velocity vector is not altered by translation.

For a spatial robot, the same transformation is accomplished using:

$$ {}^{A}J(q) = \left[\begin{array}{c|c} {}^{A}R_{B} & 0 \\ \hline 0 & {}^{A}R_{B} \end{array}\right] {}^{B}J(q) \tag{3.20}$$

Example 3.3

(a) Given the manipulator Jacobian from example 3.1 derive the Jacobian for the tool frame.

The answer is obtained as follows:

$$\begin{aligned}
{}^{3}J(q) &= {}^{3}R_{0}\,{}^{0}J(q) \\
&= {}^{0}R_{3}^{-1}\,{}^{0}J(q) \\
&= {}^{0}R_{3}^{T}\,{}^{0}J(q) \\
&= \begin{bmatrix} C\theta_{123} & S\theta_{123} & 0 \\ -S\theta_{123} & C\theta_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} -a_1S\theta_1 - a_2S\theta_{12} - a_3S\theta_{123} & -a_2S\theta_{12} - a_3S\theta_{123} & -a_3S\theta_{123} \\ a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} & a_2C\theta_{12} + a_3C\theta_{123} & a_3C\theta_{123} \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -C\theta_{123}(a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123}) + S\theta_{123}(a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123}) & -C\theta_{123}(a_2S\theta_{12} + a_3S\theta_{123}) + S\theta_{123}(a_2C\theta_{12} + a_3C\theta_{123}) & 0 \\ S\theta_{123}(a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123}) + C\theta_{123}(a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123}) & S\theta_{123}(a_2S\theta_{12} + a_3S\theta_{123}) + C\theta_{123}(a_2C\theta_{12} + a_3C\theta_{123}) & a_3 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a_2S\theta_3 + a_1S\theta_{23} & a_2S\theta_3 & 0 \\ a_3 + a_2C\theta_3 + a_1C\theta_{23} & a_3 + a_2C\theta_3 & a_3 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Calculate the velocity of the tool in tool frame coordinates when
$\theta_1 = 90°, \theta_2 = 0, \theta_3 = 0, \dot{\theta}_1 = -90°/s, \dot{\theta}_2 = 180°/s,$ and $\dot{\theta}_3 = 90°/s$

After the values are converted to radians and radians/s we substitute them into:

$$\begin{bmatrix} v_{x_3} \\ v_{y_3} \\ \omega_{z_3} \end{bmatrix} = {}^3J(q)\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{3.21}$$

to find

$$\begin{aligned} \begin{bmatrix} v_{x_3} \\ v_{y_3} \\ \omega_{z_3} \end{bmatrix} &= \begin{bmatrix} a_2 S\theta_3 + a_1 S\theta_{23} & a_2 S\theta_3 & 0 \\ a_3 + a_2 C\theta_3 + a_1 C\theta_{23} & a_3 + a_2 C\theta_3 & a_3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1.1 & 0.6 & 0.1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{-\pi}{2} \\ \pi \\ \frac{\pi}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0.3142 \\ 3.142 \end{bmatrix} \end{aligned} \tag{3.22}$$

Note that v_{x3}=0 due to the arm's current configuration.

End of example 3.3

Note that we could have derived ${}^3J(q)$ directly using the method described in section 3.8 of Niku's textbook. We could also have obtained the manipulator Jacobian by transforming this ${}^3J(q)$ to the base frame using equation (3.19).

3.5 The Inverse Jacobian and Singularities

For the purposes of controlling the velocity of the end-effector, we require an equation to calculate the required joint velocities from the specified end-effector velocities. We will term this the "inverse velocity problem". The equation may be obtained from equation (3.6) in the following way

$$J^{-1}(q)\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J^{-1}(q)J(q)\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{3.23}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = J^{-1}(q)\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \tag{3.24}$$

The critical issue with this result (equation (3.24)) is whether *J(q)* is invertible for all values of *q*. For the vast majority of robots there will be some values of *q* (*i.e.* some arm configurations) where *J(q)* is singular (*i.e.* it is non-invertible). These configurations are known as "singular configurations" or "singularities". Three things take place at a singularity. First, the robot looses one or more degrees of freedom in Cartesian space. Second, one or more of the required joint velocities tends towards infinity. Third, the numerical solutions for the inverse kinematics problem and the inverse velocity problem will "blow up" unless special precautions are taken. Obviously, knowing where these singular configurations are is crucial when developing a robot and when operating it.

Singularities may be grouped into two classes:

1. Workspace boundary singularities: These occur when the manipulator is fully stretched out or folded back on itself such that the end-effector is near or at the boundary of the workspace;
2. Workspace interior singularities: These occur in the interior of the workspace are typically caused by two joints becoming collinear.

A matrix is singular when its determinant equals zero. Therefore by studying when the determinant of J(q) equals zero we can predict when singularities will occur. For a simple robot det(J(q)) is relatively easy to obtain. For a general 6 DOF arm determining the equation for det(J(q)) is very difficult. We will discuss a solution to this case after the example.

Example 3.4

An RR planar robot is shown in Figure 3.4. With this simple robot only the velocities in x and y can be controlled, and the manipulator Jacobian takes the form:

$$J(q) = \begin{bmatrix} -a_1 S\theta_1 - a_2 S\theta_{12} & -a_2 S\theta_{12} \\ a_1 C\theta_1 + a_2 C\theta_{12} & a_2 C\theta_{12} \end{bmatrix} \tag{3.25}$$

The determinant is then

$$\begin{aligned} \det(J(q)) &= \begin{vmatrix} -a_1 S\theta_1 - a_2 S\theta_{12} & -a_2 S\theta_{12} \\ a_1 C\theta_1 + a_2 C\theta_{12} & a_2 C\theta_{12} \end{vmatrix} \\ &= (-a_1 S\theta_1 - a_2 S\theta_{12})(a_2 C\theta_{12}) - \\ &\quad (a_1 C\theta_1 + a_2 C\theta_{12})(-a_2 S\theta_{12}) \\ &= -a_1 a_2 S\theta_1 C\theta_{12} + a_1 a_2 C\theta_1 S\theta_{12} \\ &= a_1 a_2 S\theta_2 \end{aligned} \tag{3.26}$$

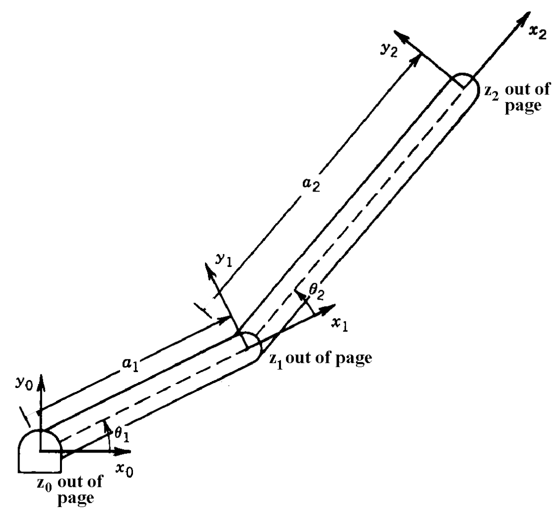

Figure 3.4 A RR planar robot [3].

since $\sin(\theta_a - \theta_b) = \sin\theta_a \cos\theta_b - \cos\theta_a \sin\theta_b$. From equation (3.26) the robot is at a singularity whenever $S\theta_2 = 0$ or whenever $\theta_2 = 0$ or $\pm 180°$. These correspond to the configurations when the arm is completely stretched out straight, and when it is completely folded back on itself. Since these occur at the boundary of the workspace they are both workspace boundary singularities. Note that these may occur for any value of θ_1.

It is also interesting to see what happens to the joint velocities near to a singular configuration. To do this we need the equation for the inverse of *J(q)*, which is:

$$J^{-1}(q) = \frac{1}{\det(J(q))}\begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{a_1 a_2 S\theta_2}\begin{bmatrix} a_2 C\theta_{12} & a_2 S\theta_{12} \\ -a_1 C\theta_1 - a_2 C\theta_{12} & -a_1 S\theta_1 - a_2 S\theta_{12} \end{bmatrix} \tag{3.27}$$

The required joint velocities may be calculated using

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J^{-1}(q) \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{3.28}$$

Let's say that $a_1 = a_2 = 0.5\,\text{m}$. Then if we start at the configuration $\theta_1 = 45°$ and $\theta_2 = -90°$ with the end-effector lying on the x_0-axis and move at v_x=1 m/s and v_y=0, we obtain the following results:

$\theta_1 = 45°$, $\theta_2 =$ -90°: $\dot{\theta}_1 =$ -1.4 rad/s, $\dot{\theta}_2 =$ 2.8 rad/s
$\theta_1 = 30°$, $\theta_2 =$ -60°: $\dot{\theta}_1 =$ -2.0 rad/s, $\dot{\theta}_2 =$ 4.0 rad/s
$\theta_1 = 15°$, $\theta_2 =$ -30°: $\dot{\theta}_1 =$ -3.9 rad/s, $\dot{\theta}_2 =$ 7.7 rad/s
$\theta_1 = 10°$, $\theta_2 =$ -20°: $\dot{\theta}_1 =$ -5.8 rad/s, $\dot{\theta}_2 =$ 11.5 rad/s
$\theta_1 = 5°$, $\theta_2 =$ -10°: $\dot{\theta}_1 =$ -11.5 rad/s, $\dot{\theta}_2 =$ 22.9 rad/s
$\theta_1 = 1°$, $\theta_2 =$ -2°: $\dot{\theta}_1 =$ -57.3 rad/s, $\dot{\theta}_2 =$ 114.6 rad/s

Clearly, as the robot approaches the singularity the controller will demand very large joint velocities (unless it has been programmed to avoid this situation) that will not be good for the motors or the robot's mechanical structure.

End of example 3.4

For the case of a 6 DOF robot with a spherical wrist, the approach of Spong and Vidyasagar [3] can be used to decouple the problem of analysing the singularities into two simpler problems. The first problem is to determine the singularities due to the configuration of the arm or "arm singularities". The second problem is to determine the singularities due to the configuration of the wrist or "wrist singularities". This decoupling is done by first partitioning the manipulator Jacobian into four 3×3 matrices:

$$\mathrm{J(q)} = \left[\begin{array}{c|c} \mathrm{J}_{11} & \mathrm{J}_{12} \\ \hline \mathrm{J}_{21} & \mathrm{J}_{22} \end{array}\right] \tag{3.29}$$

Next, it is assumed that the origins of frames 3-6 are coincident. Due to this assumption and the spherical wrist [3]:

$$\det(\mathrm{J(q)}) = \det(J_{11}) * \det(J_{22}) \tag{3.30}$$

and the arm singularities may be obtained from a study of $\det(J_{11})$, while the wrist singularities can be obtained from a study of $\det(J_{22})$. A study of $\det(J_{22})$ concludes that a spherical wrist has a singularity whenever $\theta_5 = 0$ or $\pm 180°$. This singularity is a workspace interior singularity. Next we must analyse $\det(J_{11})$. Since the origins of the frames 3-6 are coincident, $\det(J_{11})$ may be obtained using

$$J_{11}(q) = \begin{bmatrix} \frac{\partial p_x(q)}{\partial q_1} & \frac{\partial p_x(q)}{\partial q_2} & \frac{\partial p_x(q)}{\partial q_3} \\ \frac{\partial p_y(q)}{\partial q_1} & \frac{\partial p_y(q)}{\partial q_2} & \frac{\partial p_y(q)}{\partial q_3} \\ \frac{\partial p_z(q)}{\partial q_1} & \frac{\partial p_z(q)}{\partial q_2} & \frac{\partial p_z(q)}{\partial q_3} \end{bmatrix} \tag{3.31}$$

where p_x, p_y and p_z are the expressions from the forward kinematics solution for ${}^0T_3$ (*i.e.* position vector for the wrist centre).

Example 3.5

Let's derive $J_{11}(q)$ and analyse the arm singularities for the "elbow manipulator" shown in Figure 3.5. The projection of this arm onto the plane perpendicular to joints 2 and 3 is shown in Figure 3.6.

For this robot:

$$ {}^0T_3 = \begin{bmatrix} & & & p_x \\ & {}^0R_3 & & p_y \\ & & & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & a_2C\theta_1C\theta_2 + a_3(C\theta_1C\theta_2C\theta_3 - C\theta_1S\theta_2S\theta_3) \\ & {}^0R_3 & & a_2S\theta_1C\theta_2 + a_3(S\theta_1C\theta_2C\theta_3 - S\theta_1S\theta_2S\theta_3) \\ & & & a_2S\theta_2 + a_3(S\theta_2C\theta_3 + C\theta_2S\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.32} $$

Applying equation (3.31) gives

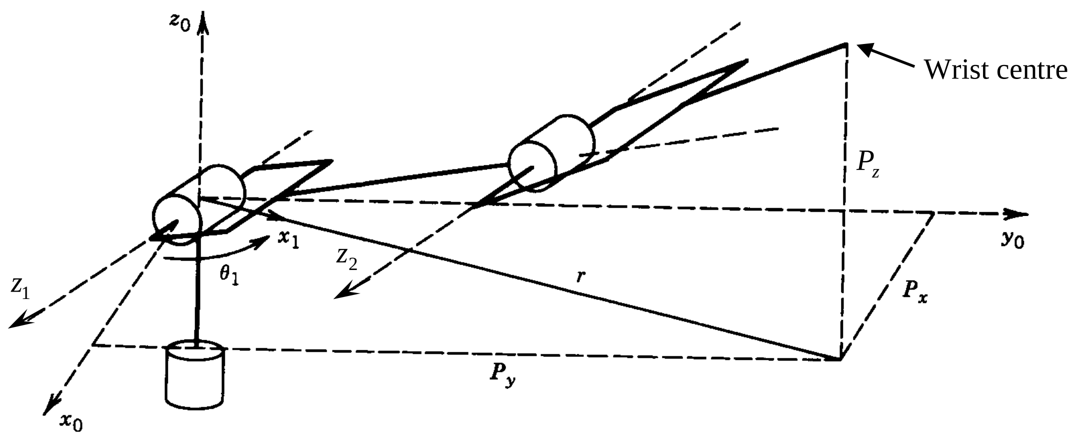

Figure 3.5 Articulated robot that is also known as an "elbow manipulator" [3].

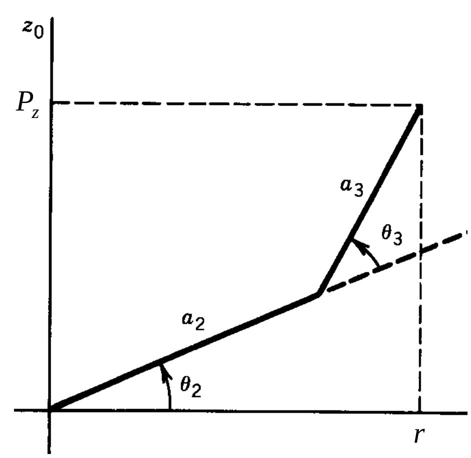

Figure 3.6 Projection onto the plane formed by links 2 and 3 [3].

$$J_{11} = \begin{bmatrix} -a_2 S\theta_1 C\theta_2 - a_3 S\theta_1 C\theta_{23} & -a_2 C\theta_1 S\theta_2 - a_3 C\theta_1 S\theta_{23} & -a_3 C\theta_1 S\theta_{23} \\ a_2 C\theta_1 C\theta_2 + a_3 C\theta_1 C\theta_{23} & -a_2 S\theta_1 S\theta_2 - a_3 S\theta_1 S\theta_{23} & -a_3 S\theta_1 S\theta_{23} \\ 0 & a_2 C\theta_2 + a_3 C\theta_{23} & a_3 C\theta_{23} \end{bmatrix} \quad (3.33)$$

This has the determinant

$$\det(J_{11}) = a_2 a_3 S\theta_3 (a_2 C\theta_2 + a_3 C\theta_{23}) \quad (3.34)$$

This is zero and the arm is at a singularity whenever

$$S\theta_3 = 0 \quad (3.35)$$

and whenever

$$a_2 C\theta_2 + a_3 C\theta_{23} = 0 \quad (3.36)$$

Equation (3.35) has the solutions $\theta_3 = 0$ or $\pm 180°$. These correspond to the configurations when the arm is completely stretched out straight, and when it is completely folded back on itself that we saw with the 2R planar manipulator. As before these are classified as workspace boundary singularities. Note that these may occur for any value of θ_1 or θ_2. From the arm geometry shown in Figure 3.6 it is apparent that $a_2 C\theta_2 + a_3 C\theta_{23} = r$ so the solutions to equation (3.36) place the wrist centre somewhere along the z_0 axis. There are an infinite number of these configurations. One example is shown in Figure 3.7. These are classified as workspace interior singularities.

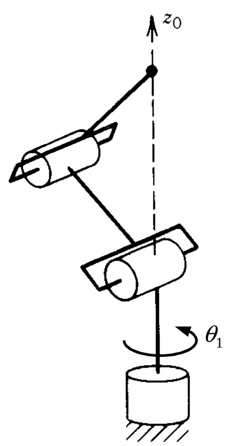

Figure 3.7 A singular configuration that is a solution to equation (3.36) [3].

3.6 Other Approaches to the Inverse Velocity Problem

Obtaining the required joint velocities using equation (3.24) may not be feasible for the actual motion control of a robot. For proper motion control $J^{-1}(q)$ must be calculated every few milliseconds. Computing the inverse numerically or by using the equations of the symbolic inverse may be too slow. There are other potential approaches.

The first alternate approach is to numerically integrate the desired instantaneous end-effector velocities to get a series of desired positions and orientations for the end-effector. Then obtain the required joint angles using the inverse kinematics equations. This must be done every few milliseconds as before.

The second alternate approach is described in section 3.10 of Niku's textbook. This involves reworking the inverse kinematics equations and then differentiating them. The advantages of this approach are not obvious.

References

1. R.J. Schilling, "Fundamentals of Robotics", Prentice-Hall, 1990.
2. L.-W. Tsai, "Robot Analysis", John Wiley & Sons, 1999.
3. M.W. Spong and M. Vidyasagar, "Robot Dynamics and Control", John Wiley & Sons, 1989.

Addendum to Chapter 3:

Jacobians for Robots with Two or Three DOF

Planar Robot with Two DOF

In this case we can only control the linear velocities of the tool in the X and Y directions so equations (3.6) and (3.7) are simplified to:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = J(q)\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \text{where} \quad J(q) = \begin{bmatrix} \frac{\partial p_x(q)}{\partial q_1} & \frac{\partial p_x(q)}{\partial q_2} \\ \frac{\partial p_y(q)}{\partial q_1} & \frac{\partial p_y(q)}{\partial q_2} \end{bmatrix}$$

Planar Robot with Three DOF

With this robot the planar angular velocity and linear velocities can be controlled. Equations (3.6) and (3.7) now become:

$$\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} = J(q)\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad \text{where} \quad J(q) = \begin{bmatrix} \frac{\partial p_x(q)}{\partial q_1} & \frac{\partial p_x(q)}{\partial q_2} & \frac{\partial p_x(q)}{\partial q_3} \\ \frac{\partial p_y(q)}{\partial q_1} & \frac{\partial p_y(q)}{\partial q_2} & \frac{\partial p_y(q)}{\partial q_3} \\ \zeta_1 t_1 & \zeta_2 t_2 & \zeta_3 t_3 \end{bmatrix}$$

where $\zeta_i = 0$ if joint *i* is prismatic and $\zeta_i = 1$ if joint *i* is revolute; and
$t_i = 1$ if joint *i* is revolute and its Z axis points out of the page,
$t_i = -1$ if joint *i* is revolute and its Z axis points into the page and
$t_i = 0$ if joint *i* is prismatic.

Spatial Robot with Three DOF

In this case we can only control the linear velocities of the tool in the X, Y and Z directions so equations (3.6) and (3.7) are simplified to:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = J(q)\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad \text{where} \quad J(q) = \begin{bmatrix} \frac{\partial p_x(q)}{\partial q_1} & \frac{\partial p_x(q)}{\partial q_2} & \frac{\partial p_x(q)}{\partial q_3} \\ \frac{\partial p_y(q)}{\partial q_1} & \frac{\partial p_y(q)}{\partial q_2} & \frac{\partial p_y(q)}{\partial q_3} \\ \frac{\partial p_z(q)}{\partial q_1} & \frac{\partial p_z(q)}{\partial q_2} & \frac{\partial p_z(q)}{\partial q_3} \end{bmatrix}$$