LAST (family) NAME: ______________________

FIRST (given) NAME: ______________________

ID # : ____________________

MATHEMATICS 2ZZ3: SAMPLE FINAL A: SOLUTIONS

Continued...

SOLUTIONS TO THE MUTIPLE-CHOICE PART OF THE EXAM

Part I: Multiple-choice questions: Enter your answers to Questions 1 to 22 on the scantron sheet provided, following the instructions given on page 2 and 3.

1. *(4 pts.)* The position of an object at time t is given by

$$\mathbf{r}(t) = (t\,\cos t)\,\mathbf{i} + (t\,\sin t)\,\mathbf{j} + t\,\mathbf{k}.$$

Its normal acceleration when $t = 0$ is given by

(A) 1

→ **(B)** 2

(C) -2

(D) 0

(E) $\sqrt{2}$

Solution. We have

$$\mathbf{r}'(t) = (\cos t - t\,\sin t)\,\mathbf{i} + (\sin t + t\,\cos t)\,\mathbf{j} + \mathbf{k},$$

and

$$\mathbf{r}''(t) = (-2\,\sin t - t\,\cos t)\,\mathbf{i} + (2\,\cos t - t\,\sin t)\,\mathbf{j}.$$

In particular, $\mathbf{r}'(0) = \langle 1, 0, 1\rangle$, $\mathbf{r}''(0) = \langle 0, 2, 0\rangle$, $\|\mathbf{r}'(0)\| = \sqrt{2}$,

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = \langle -2, 0, 2\rangle$$

which yields $\|\mathbf{r}'(0) \times \mathbf{r}''(0)\| = 2\sqrt{2}$. Thus, the normal acceleration at $t = 0$ is

$$a_N = \frac{\|\mathbf{r}'(0) \times \mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|} = \frac{2\sqrt{2}}{\sqrt{2}} = 2.$$

2. *(4 pts.)* The Fourier series expansion of the function $f(x) = x^2$ on $[-1, 1]$ is given by

(A) $\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x)$

(B) $\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$

(C) $\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin(n\pi x)$

→ **(D)** $\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$

(E) $\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x)$

Solution. Since $f(x)$ is even on $[-1, 1]$ and $L = 1$, its Fourier series on $[-1, 1]$ reduces to a Fourier cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,\cos(n\pi x),$$

where

$$a_n = 2\int_0^1 x^2\,\cos(n\pi x)\,dx, \quad n \ge 0.$$

If $n = 0$, we have

$$a_0 = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3},$$

and for $n \ge 1$,

$$a_n = 2\int_0^1 x^2\,\cos(n\pi x)\,dx = 2\int_0^1 x^2\left(\frac{\sin(n\pi x)}{n\,\pi}\right)'\,dx = 2\left[x^2\,\frac{\sin(n\pi x)}{n\,\pi}\right]_0^1 - 4\int_0^1 x\left(\frac{\sin(n\pi x)}{n\,\pi}\right)\,dx$$

$$= 4\int_0^1 x\left(\frac{\cos(n\pi x)}{(n\,\pi)^2}\right)'\,dx = 4\left[x\left(\frac{\cos(n\pi x)}{(n\,\pi)^2}\right)\right]_0^1 - 4\int_0^1 \frac{\cos(n\pi x)}{(n\,\pi)^2}\,dx$$

$$= \frac{4\,(-1)^n}{n^2\,\pi^2} - 4\left[\frac{\sin(n\pi x)}{(n\,\pi)^3}\right]_0^1 = \frac{4\,(-1)^n}{n^2\,\pi^2}.$$

The Fourier series expansion of $f(x) = x^2$ on $[-1,1]$ is thus

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,\cos(n\pi x)$$

3. *(4 pts.)* The (saddle-shaped) surface $z = x^2 - y^2$ within the cylindrical region $x^2 + y^2 \le 1$ has a surface area equal to

→ **(A)** $A = \frac{\pi}{6}\,(\sqrt{125} - 1)$

(B) $A = \frac{\pi}{12}\,(\sqrt{125} - 1)$

(C) $A = \frac{\pi}{6}\,(\sqrt{5} - 1)$

(D) $A = \frac{\pi^2}{6}\,(\sqrt{125} - 1)$

(E) $A = 2\,\pi$

Solution. The surface can be parametrized by

$$\mathbf{r}(x,y) = \langle x, y, x^2 - y^2\rangle, \quad (x,y) \in D,$$

where $D = \{(x,y),\ x^2 + y^2 \le 1$. We have

$$\mathbf{r}_x = \langle 1, 0, 2\,x\rangle, \quad \mathbf{r}_x = \langle 0, 1, 2\,y\rangle, \quad \mathbf{r}_x \times \mathbf{r}_y = \langle -2\,x, -2\,y, 1\rangle,$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{4\,x^2 + 4\,y^2 + 1}$. Hence,

$$A = \iint_D \|\mathbf{r}_x \times \mathbf{r}_y\|\,dx\,dy = \iint_D \sqrt{4\,x^2 + 4\,y^2 + 1}\,dx\,dy = \int_0^{2\,\pi}\int_0^1 \sqrt{4\,r^2 + 1}\,r\,dr\,d\theta$$

$$= 2\,\pi\left[\frac{(4\,r^2 + 1)^{3/2}}{12}\right]_0^1 = \frac{\pi}{6}\,(5^{3/2} - 1) = \frac{\pi}{6}\,(\sqrt{125} - 1)$$

4. *(4 pts.)* The flux of the vector field $\mathbf{F} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/(x^2+y^2+z^2)$ through the sphere $x^2+y^2+z^2 = 4$ (oriented using the outer pointing normal) is

(A) 4π

(B) $4\pi^2$

(C) -8π

$\rightarrow$ **(D)** 8π

(E) not defined because of the discontinuity at $(0,0,0)$.

Solution. We can parametrize the sphere S using spherical coordinates

$$\mathbf{r}(\phi,\theta) = \langle 2\cos\theta\,\sin\phi, 2\sin\theta\,\sin\phi, 2\cos\phi\rangle, \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

We have

$$\mathbf{r}_\phi = \langle 2\cos\theta\,\cos\phi, 2\sin\theta\,\cos\phi, -2\sin\phi\rangle,$$
$$\mathbf{r}_\theta = \langle -2\sin\theta\,\sin\phi, 2\cos\theta\,\sin\phi, 0\rangle,$$

and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin\phi\,\langle \cos\theta\,\sin\phi, \sin\theta\,\sin\phi, \cos\phi\rangle,$$

which gives an outward pointing normal. We have thus

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(\phi,\theta))\cdot \mathbf{r}_\phi \times \mathbf{r}_\theta\, d\phi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\langle 2\cos\theta\,\sin\phi, 2\sin\theta\,\sin\phi, 2\cos\phi\rangle \cdot 4\sin\phi\,\langle \cos\theta\,\sin\phi, \sin\theta\,\sin\phi, \cos\phi\rangle\, d\phi\, d\theta$$
$$= 2\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi\,[-\cos\phi]_0^{\pi} = 8\pi.$$

5. *(4 pts.)* Consider the vector field $\mathbf{F}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$. Using Stokes' Theorem, the flux of the vector field curl $\mathbf{F}$ through the part of the surface $z = (4 - x^2 - y^2)^3$ above the x, y plane with normal pointing upwards equals

(A) 2π

(B) 0

$\rightarrow$ **(C)** -4π

(D) π^2

(E) -2π

Solution. The boundary curve C is the circle $x^2+y^2 = 4$ on the x, y plane oriented counterclockwise as viewed from above and can be parametrized by

$$\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0\rangle, \quad 0 \le \theta \le 2\pi.$$

We have

$$\mathbf{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0\rangle.$$

By Stokes' theorem,

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} \langle 2\, \sin\theta, 0, 2\, \cos\theta\rangle \cdot \langle -2\, \sin\theta, 2\, \cos\theta, 0\rangle\, d\theta \\ &= -4 \int_0^{2\pi} \sin^2\theta\, d\theta = -2 \int_0^{2\pi} 1 - \cos(2\theta)\, d\theta \\ &= -2 \left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{2\pi} = -4\,\pi.\end{aligned}$$

6. *(4 pts.)* The equation of the plane tangent to the surface $z = \frac{1}{2\pi} \tan(\pi(x^2 + y^2))$ at the point $(1, -1, 0)$ is

(A) $x + y + z = 0$

(B) $x - y + z = 2$

→ **(C)** $x - y - z = 2$

(D) $2\,x + 2\,y - z = 0$

(E) $2\,x - 2\,y - z = 4$

Solution. The surface is the level surface $F(x, y, z) = 0$ where

$$F(x, y, z) = \frac{1}{2\pi} \tan(\pi(x^2 + y^2)) - z.$$

Since

$$F_x = x\, \sec^2(\pi(x^2 + y^2)), \quad F_y = y\, \sec^2(\pi(x^2 + y^2)), \quad F_z = -1,$$

we have

$$F_x(1, -1, 0) = \sec^2(2\pi) = 1, \quad F_y(1, -1, 0) = -\sec^2(2\pi) = -1, \quad F_z(1, -1, 0) = -1,$$

The equation of the tangent plane at $(1, -1, 0)$ is thus

$$(x - 1) - (y + 1) - z = 0, \quad \text{or} \quad x - y - z = 2.$$

7. *(4 pts.)* Compute the line integral

$$I = \oint_C 2\,x\,y\,dx - (x + y)^2\, dy$$

where C is the circle $x^2 + y^2 = 1$ oriented counterclockwise.

(A) $I = 4\,\pi$

→**(B)** $I = 0$

(C) $I = -2\,\pi$

(D) $I = -\pi$

(E) $I = 2\,\pi$

Solution. The circle can be parametrized by

$$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta\rangle, \quad 0 \le \theta \le 2\,\pi.$$

We have

$$\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta \rangle.$$

Hence,

$$\begin{aligned}
\oint_C 2\,x\,y\,dx - (x+y)^2\,dy &= \int_0^{2\pi} 2\,\cos\theta\,\sin\theta\,(-\sin\theta) - (\cos\theta + \sin\theta)^2\,(\cos\theta)\,d\theta \\
&= \int_0^{2\pi} -2\,\cos\theta\,\sin^2\theta - \cos\theta - 2\,\cos^2\theta\,\sin\theta \\
&= \left[-\frac{2}{3}\,\sin^3\theta - \sin\theta + \frac{2}{3}\,\cos^3\theta \right]_0^{2\pi} = 0.
\end{aligned}$$

8. *(4 pts.)* Find the value of a which makes the vector field

$$\mathbf{F}(x,y) = \frac{y + a\,(2\,x - y)}{x^2+y^2}\,\mathbf{i} + \frac{x + 2\,a\,y}{x^2+y^2}\,\mathbf{j}$$

conservative in the first quadrant $x, y > 0$.

(A) $a = 0$

(B) $a = -1$

(C) $a = 1$

$\rightarrow$ **(D)** $a = 2$

(E) No such a exists

Solution. Letting $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, we have that $\mathbf{F}$ is conservative on the simply connected region $x, y > 0$ iff $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. We have

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - 2\,x\,(x + 2\,a\,y)}{(x^2+y^2)^2} = \frac{-x^2 + y^2 - 4\,a\,x\,y}{(x^2+y^2)^2}$$

and

$$\frac{\partial P}{\partial y} = \frac{(1-a)(x^2+y^2) - 2\,y\,(y + a\,(2\,x - y))}{(x^2+y^2)^2} = \frac{(1-a)\,x^2 + (a-1)\,y^2 - 4\,a\,x\,y}{(x^2+y^2)^2}.$$

So $\mathbf{F}$ is conservative on the region iff $a = 2$.

9. *(4 pts.)* Evaluate the integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$

by changing to polar coordinates.

$\rightarrow$ **(A)** $I = \dfrac{\pi}{4}\,(e-1)$

(B) $I = \dfrac{\pi}{4}\,(e+1)$

(C) $I = -\dfrac{\pi}{4}\,(e-1)$

(D) $I = \dfrac{\pi}{2}\,(e-1)$

(E) $I = \dfrac{\pi}{2}\,(e+1)$

Solution. Letting $D = \{(x, y),\ 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\}$, we see that D is the region in the first quadrant inside the circle of radius 1 centered at the origin. D can thus be expressed in polar coordinates as

$$D^* = \{(r, \theta),\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1\}.$$

Therefore,

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 e^{r^2}\, r\, dr\, d\theta = \frac{\pi}{2} \left[\frac{e^{r^2}}{2} \right]_0^1 = \frac{\pi}{4}\,(e - 1).$$

10. *(4 pts.)* Find the area A of the region in the x, y plane bounded by the polar curve with equation $r = 2 + \cos(3\,\theta)$, $0 \le \theta \le 2\,\pi$.

(A) $A = \dfrac{5\,\pi}{2}$

(B) $A = \dfrac{15\,\pi}{4}$

(C) $A = \dfrac{7\,\pi}{4}$

(D) $A = \dfrac{11\,\pi}{3}$

$\rightarrow$ **(E)** $A = \dfrac{9\,\pi}{2}$

Solution.

$$A = \int_0^{2\pi} \int_0^{2+\cos(3\,\theta)} r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=2+\cos(3\,\theta)} d\theta = \frac{1}{2} \int_0^{2\pi} (2 + \cos(3\,\theta))^2\,d\theta$$
$$= \frac{1}{2} \int_0^{2\pi} 4 + 4\,\cos(3\,\theta) + \cos^2(3\,\theta)\,d\theta = \frac{1}{2} \int_0^{2\pi} 4 + 4\,\cos(3\,\theta) + \frac{1 + \cos(6\,\theta)}{2}\,d\theta$$
$$= \frac{1}{2} \int_0^{2\pi} \frac{9}{2} + 4\,\cos(3\,\theta) + \frac{\cos(6\,\theta)}{2}\,d\theta = \frac{1}{2} \left[\frac{9\,\theta}{2} + \frac{4}{3}\,\sin(3\,\theta) + \frac{\sin(6\,\theta)}{12}\right]_0^{2\pi} = \frac{9\,\pi}{2}$$

11. *(4 pts.)* Let D be the region in the first quadrant $x, y > 0$ below the graph of $y = x$ and inside the circle $(x-1)^2 + y^2 = 1$. Compute the integral

$$I = \iint_D 2\,(x-1)\,dA$$

$\rightarrow$ **(A)** $I = \frac{1}{3}$

(B) $I = \frac{1}{2}$

(C) $I = 0$

(D) $I = -\frac{1}{3}$

(E) $I = -\frac{1}{2}$

Solution. The region D can be expressed as the type II region

$$D = \{(x, y),\ 0 \le y \le 1,\ y \le x \le 1 + \sqrt{1 - y^2}\}.$$

Thus,

$$I = \int_0^1 \int_y^{1+\sqrt{1-y^2}} 2\,(x-1)\,dx\,dy = \int_0^1 \left[(x-1)^2\right]_{x=y}^{x=1+\sqrt{1-y^2}}\,dx$$
$$= \int_0^1 1 - y^2 - (y-1)^2\,dy = \int_0^1 -2\,y^2 + 2\,y\,dy = \left[-\frac{2}{3}\,y^3 + y^2\right]_0^1 = \frac{1}{3}.$$

12. *(4 pts.)* Let S_1 be the disk $x^2 + y^2 \leq 1$, $z = 0$ on the x, y plane, oriented using the upward pointing unit normal on S_1 and let S_2 be the part of the paraboloid $z = 1 - x^2 - y^2$ above the x, y plane, oriented so that the unit normal on S_2 has a positive z-component. For which of the following vector fields $\mathbf{F}$ is it true that

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}?$$

(**Hint:** You can avoid calculating the surface integrals by considering the solid region between the surfaces.)

(A) $\mathbf{F}(x, y, z) = (x^3)\,\mathbf{i} + (y^3)\,\mathbf{j} + (z^3)\,\mathbf{k}$ **(D)** $\mathbf{F}(x, y, z) = (-x^3)\,\mathbf{i} + (z^3)\,\mathbf{j} + (-z\,y^2)\,\mathbf{k}$

(B) $\mathbf{F}(x, y, z) = (x\,y^2)\,\mathbf{i} + (y\,z^2)\,\mathbf{j} + (z)\,\mathbf{k}$ $\rightarrow$ **(E)** $\mathbf{F}(x, y, z) = (z^3)\,\mathbf{i} + (x^3)\,\mathbf{j} + (y^3)\,\mathbf{k}$

(C) $\mathbf{F}(x, y, z) = (-y^3)\,\mathbf{i} + (-x^3)\,\mathbf{j} + (z)\,\mathbf{k}$

Solution. If V is the solid region under the graph of $z = 1 - x^2 - y^2$ and above the x, y plane, then V is the region enclosed by $S_1 \cup S_2$. Using the divergence theorem and taking the orientation of the surfaces into account, we have

$$-\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV,$$

for any smooth vector field $\mathbf{F}$. Note that for choice (E), we have $\nabla \cdot \mathbf{F} = 0$ and thus $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$. For the other choices, we have either $\nabla \cdot \mathbf{F} > 0$ or $\nabla \cdot \mathbf{F} < 0$ everywhere inside V, so the triple integral above cannot be zero.

13. *(4 pts.)* The volume of the solid region bounded below by the graph of $z = \sqrt{x^2 + y^2} + 1$ and above by the sphere $x^2 + y^2 + z^2 = 25$ can be computed, after passing to cylindrical coordinates, by the integral

$$\int_0^{2\pi} \int_0^a \int_{1+r}^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$$

where a is the number

(A) $a = 1$ **(D)** $a = 6$

$\rightarrow$ **(B)** $a = 3$ **(E)** $a = 7$

(C) $a = 5$

Solution. The two surfaces intersect when $z = r + 1 = \sqrt{25 - r^2}$. This implies $(r+1)^2 = 25 - r^2$ or $2\,r^2 + 2\,r - 24 = 2(r-3)\,(r+4) = 0$, i.e. $r = 3$. We have thus $0 \leq r \leq 3$.

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14. *(4 pts.)* Let V be the solid region inside the cylinder $x^2+y^2=4$ and between the planes $z=0$ and $z=4$. Let S be the closed surface bounding the region V oriented using the outward pointing normal. Consider the vector field

$$\mathbf{F}(x,y,z)=(x\,y^2\,z)\,\mathbf{i}+(x\,y)\,\mathbf{j}-(x\,z)\,\mathbf{k}.$$

Then, the flux of $\mathbf{F}$ across S, $I=\iint_S \mathbf{F}\cdot d\mathbf{S}$, is given by:
(**Hint:** Use the divergence theorem.)

→ **(A)** $I=32\pi$ **(D)** $I=4\pi$

(B) $I=16\pi$ **(E)** $I=2\pi$

(C) $I=8\pi$

Solution. We have $\nabla\cdot\mathbf{F}=y^2\,z$. Using the divergence theorem,

$$I=\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_V \nabla\cdot\mathbf{F}\,dV=\iiint_V y^2\,z\,dV.$$

Passing to cylindrical coordinates, we have thus

$$I=\int_0^{2\pi}\int_0^2\int_0^4 r^2\,\sin^2\theta\,z\,r\,dz\,dr\,d\theta=\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta\int_0^2 r^3\,dr\int_0^4 z\,dz$$
$$=\left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2\left[\frac{z^2}{2}\right]_0^4=\pi\,(4)\,(8)=32\,\pi$$

15. *(4 pts.)* Use spherical coordinates to evaluate the integral

$$I=\iiint_V z\,dV,$$

where V is the solid region inside the sphere $x^2+y^2+z^2=8$ and above the cone $z=\sqrt{x^2+y^2}$.

(A) $I=2\,\pi$ **(D)** $I=12\,\pi$

(B) $I=5\,\pi$ →**(E)** $I=8\,\pi$

(C) $I=\frac{7\pi}{3}$

Solution. The region V can be expressed in spherical coordinates as

$$V^*=\left\{(\rho,\theta,\phi)\ 0\le\rho\le 2\sqrt{2},\ 0\le\theta\le 2\,\pi,\ 0\le\phi\le\frac{\pi}{4}\right\}.$$

Thus,

$$I=\int_0^{2\,\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}}\rho\,\cos\phi\,\rho^2\,\sin\phi\,d\rho\,d\phi\,d\theta=2\,\pi\int_0^{\frac{\pi}{4}}\frac{\sin(2\phi)}{2}\,d\phi\int_0^{2\sqrt{2}}\rho^3\,d\rho$$
$$=\pi\left[-\frac{\cos(2\phi)}{2}\right]_0^{\frac{\pi}{4}}\left[\frac{\rho^4}{4}\right]_0^{2\sqrt{2}}=\pi\left(\frac{1}{2}\right)(16)=8\,\pi.$$

16. *(4 pts.)* The solution $u(x,t)$ of the heat equation $\dfrac{\partial u}{\partial t} = 5\,\dfrac{\partial^2 u}{\partial x^2}$ for $0 < x < \pi$, $t > 0$ satisfying the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$ and the initial condition $u(x,0) = 1$ for $0 < x < \pi$ is given by the series

(A) $\dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} \frac{\sin((1+2k)x)}{(1+2k)^2}\, e^{-25(1+2k)t}$

(B) $\dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{\cos((1+2k)x)}{1+2k}\, e^{-5(1+2k)^2 t}$

→**(C)** $\dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} \frac{\sin((1+2k)x)}{1+2k}\, e^{-5(1+2k)^2 t}$

(D) $\dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sin(2kx)}{2k}\, e^{-25(2k)^2 t}$

(E) $\dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} \frac{\cos((1+2k)x)}{1+2k}\, e^{-5(1+2k)t}$

Solution. The heat equation above has for solution the series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-5\,n^2 t}, \quad 0 < x < \pi \ t > 0,$$

where the coefficients b_n are chosen so that

$$u(x,0) = 1 = \sum_{n=1}^{\infty} b_n \sin(nx), \quad 0 < x < \pi.$$

Hence,

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi}\,\frac{(1-(-1)^n)}{n}.$$

Thus, $b_{2k} = 0$ and $b_{1+2k} = \frac{4}{\pi}\frac{1}{1+2k}$, for $k \geq 0$. Hence,

$$u(x,t) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{1+2k}\sin((1+2k)x)\, e^{-5\,(1+2k)^2 t}$$

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Part II: Provide all details and fully justify your answer in order to receive credit.

17. Let D be the region in the first quadrant ($x > 0$, $y > 0$) of the x, y plane where

$$1 \le x\,(x+y) \le 2 \quad \text{and} \quad 1 \le y\,(x+y) \le 2.$$

Consider the change of variables

$$x\,(x+y) = u, \quad y\,(x+y) = v.$$

(a) *(4 pts.)* Express the variables x, y in terms of the variables u, v. Find the region D^* in the u, v plane that corresponds to D when the change of variable above is used.

Solution. We have $\frac{v}{u} = \frac{y}{x}$ so $y = x\,\frac{v}{u}$. Substituting in the first equation yields

$$x^2 + x^2\,\frac{v}{u} = u \quad \text{or} \quad x^2 = \frac{u^2}{u+v}. \quad [1]$$

Thus

$$x = \frac{u}{\sqrt{u+v}} \quad [1] \quad \text{and} \quad y = x\,\frac{v}{u} = \frac{v}{\sqrt{u+v}} \quad [1]$$

and

$$D^* = \{(u,v),\ 1 \le u \le 2,\ 1 \le v \le 2\}. \quad [1]$$

(b) *(3 pts.)* Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ corresponding to the change of variables above.

Solution. We have

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad [1] \\
&= \begin{vmatrix} \frac{\sqrt{u+v} - \frac{u}{2}\,(u+v)^{-1/2}}{u+v} & -\frac{u}{2}\,(u+v)^{-3/2} \\ -\frac{v}{2}\,(u+v)^{-3/2} & \frac{\sqrt{u+v} - \frac{v}{2}\,(u+v)^{-1/2}}{u+v} \end{vmatrix} = \begin{vmatrix} \frac{\frac{u}{2}+v}{(u+v)^{3/2}} & \frac{-\frac{u}{2}}{(u+v)^{3/2}} \\ \frac{-\frac{v}{2}}{(u+v)^{3/2}} & \frac{\frac{v}{2}+u}{(u+v)^{3/2}} \end{vmatrix} \quad [1] \\
&= \frac{1}{2}\,\frac{u^2+v^2+2\,u\,v}{(u+v)^3} = \frac{1}{2}\,\frac{(u+v)^2}{(u+v)^3} = \frac{1}{2}\,\frac{1}{u+v}. \quad [1]
\end{aligned}$$

Alternatively,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2\,x+y & x \\ y & 2\,y+x \end{vmatrix} = 2\,(x^2+y^2+2\,x\,y) = 2\,(x+y)^2 = 2\,(\sqrt{u+v})^2 = 2\,(u+v).$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{2\,(u+v)}$$

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(c) *(5 pts.)* Use the change of variables above to compute the area of the region D.
(**Hint:** $\int \ln t\, dt = t\, \ln t - t + C$.)

Solution. Using the change of variable formula, the area of D is given by

$$\begin{aligned}
\iint_D 1\, dx\, dy &= \iint_{D^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \quad [1] \\
&= \int_1^2 \int_1^2 \frac{1}{2\,(u+v)}\, dv\, du \quad [1] \\
&= \frac{1}{2} \int_1^2 [\ln(u+v)]_{v=1}^{v=2}\, du = \frac{1}{2} \int_1^2 \ln(2+u) - \ln(1+u)\, du \quad [1] \\
&= \frac{1}{2} \left[(2+u)\, \ln(2+u) - (1+u)\, \ln(1+u)\right]_1^2 \quad [1] \\
&= \frac{1}{2} \left(4\, \ln 4 - 3\, \ln 3 - 3\, \ln 3 + 2\, \ln 2\right) \\
&= 5\, \ln 2 - 3\, \ln 3. \quad [1]
\end{aligned}$$

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18. Consider the surface S with equation $z = xy$ inside the cylinder $x^2 + y^2 = 4$, oriented so that its normal has a positive z-component and the vector field $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

(a) *(2 pts.)* Calculate the curl of $\mathbf{F}$ (i.e. $\nabla \times \mathbf{F}$).

Solution. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \quad [1]$$
$$= \langle -1, -1, -1 \rangle. \quad [1]$$

(b) *(3 pts.)* Calculate the flux of $\nabla \times \mathbf{F}$ through S.

Solution. We can parametrize S by

$$\mathbf{r}(x, y) = \langle x, y, x\,y \rangle, \quad (x, y) \in D, \quad \text{where} \quad D = \{(x, y),\ x^2 + y^2 \leq 4\}.$$

We have

$$\mathbf{r}_x = \langle 1, 0, y \rangle, \quad \mathbf{r}_y = \langle 0, 1, x \rangle,$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \langle -y, -x, 1 \rangle, \quad [1]$$

which gives the correct orientation on S. The flux of $\nabla \times \mathbf{F}$ through S is thus

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (\nabla \times \mathbf{F})\,(\mathbf{r}(x, y)) \cdot \mathbf{r}_x \times \mathbf{r}_y\,dx\,dy \quad [1]$$
$$= \iint_D \langle -1, -1, -1 \rangle \cdot \langle -y, -x, 1 \rangle\,dx\,dy$$
$$= -\iint_D 1\,dx\,dy = -\int_0^{2\pi} \int_0^2 r\,dr\,d\theta = -2\,\pi \left[\frac{r^2}{2} \right]_0^2 = -4\,\pi. \quad [1]$$

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(c) *(2 pts.)* Parametrize the boundary C of S, with an orientation consistent with the one on S. (Cylindrical coordinates are useful.)

Solution. We can parametrize C using the vector function

$$\mathbf{r}(\theta) = \langle 2\,\cos\theta, 2\,\sin\theta, 4\,\cos\theta\,\sin\theta\rangle, \quad 0 \le \theta \le 2\,\pi. \quad [2]$$

(d) *(4 pts.)* Calculate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ directly using the parametrization obtained in part (c).

(**Hint:** Some of the trigonometric terms have zero integrals because of symmetry.)

Solution. With the parametrization in (c), we have

$$\mathbf{r}'(\theta) = \langle -2\,\sin\theta, 2\,\cos\theta, 4\,(\cos^2\theta - \sin^2\theta)\rangle, \quad [1]$$

and

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\,d\theta \\
&= \int_0^{2\pi} \langle 2\,\sin\theta, 4\,\cos\theta\,\sin\theta, 2\,\cos\theta\rangle\cdot\langle -2\,\sin\theta, 2\,\cos\theta, 4\,(\cos^2\theta-\sin^2\theta)\rangle\,d\theta \quad [1] \\
&= \int_0^{2\pi} -4\,\sin^2\theta + 8\,\cos^2\theta\,\sin\theta + 8\,\cos\theta\,(1-2\,\sin^2\theta)\,d\theta \\
&= \int_0^{2\pi} -2\,(1-\cos(2\,\theta)) + 8\,\cos^2\theta\,\sin\theta + 8\,\cos\theta\,(1-2\,\sin^2\theta)\,d\theta \quad [1] \\
&= \left[-2\,\theta + \sin(2\,\theta) - 8\,\frac{\cos^3\theta}{3} - 8\,\sin\theta + \frac{16}{3}\,\sin^3\theta\right]_0^{2\pi} \\
&= -4\,\pi. \quad [1]
\end{aligned}$$

(e) *(1 pt.)* Explain how the results of (b) and (d) are related.

Solution. We must have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

by Stokes' theorem. [1]

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Some formulas you may use:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}, \qquad a_N = \kappa v^2 = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\,\mathbf{b}$$

$$\frac{d}{dt}\,[u(t)\,\mathbf{r}(t)] = u(t)\,\mathbf{r}'(t) + u'(t)\,\mathbf{r}(t),$$

$$\frac{d}{dt}\,[\mathbf{r}_1\cdot\mathbf{r}_2] = \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) + \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t), \quad \frac{d}{dt}\,[\mathbf{r}_1\times\mathbf{r}_2] = \mathbf{r}_1'(t)\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\mathbf{r}_2'(t),$$

$$\frac{d}{dt}\,(\cos t) = -\sin t, \qquad \frac{d}{dt}\,(\sin t) = \cos t.$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t), \qquad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t).$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta), \quad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad \cosh t = \frac{e^t + e^{-t}}{2} \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\rho = \sqrt{x^2+y^2+z^2}, \qquad x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi$$

$$r = \sqrt{x^2+y^2} = \rho\sin\phi$$

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_T \nabla\cdot\mathbf{F}\,dV$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \;\; 0 < x < L.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \;\; 0 < x < L.$$

*** **THE END** ***