ENG PHYS 2A04 Tutorial 8

Electricity and Magnetism

Your TAs today

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Chapter 4

Problem 4.48 - Question

Find $\mathbf{E}_1$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2\ (\frac{V}{m})$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}(\frac{C}{m^2})$. What angle does E_2 make with the z-axis?

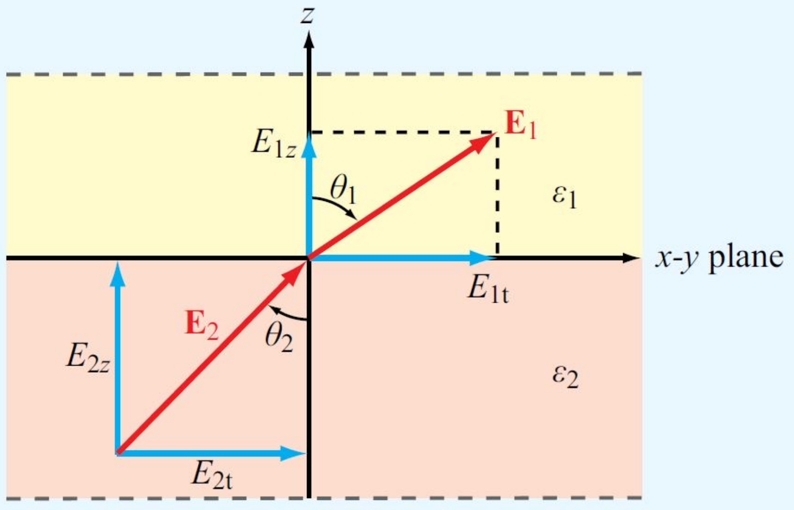

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

Problem 4.48 - Question

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a) **Find $\boldsymbol{E}_1$**

b) Find θ_2

E_1 is comprised of 3 component vectors: $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$

E_{1t} constitutes the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ components

E_{1z} constitutes the $\hat{\boldsymbol{z}}$ component only

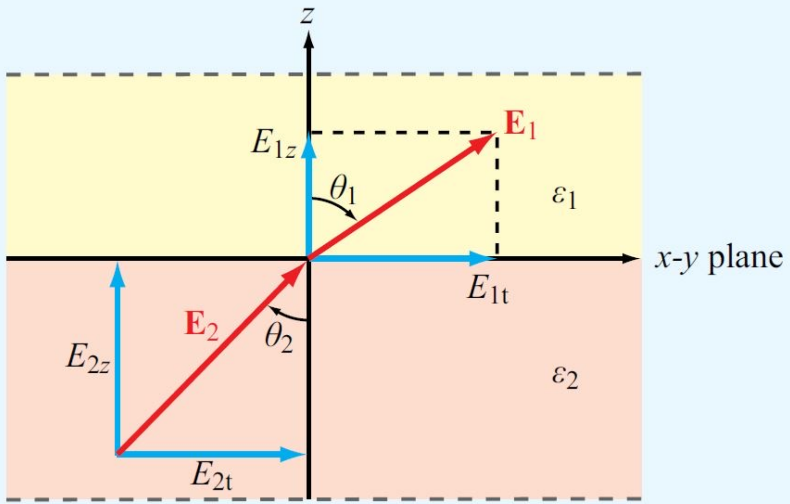

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

Problem 4.48 – Work (a)

$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 \left(\frac{V}{m}\right)$$

Table 4-3 Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ϵ_1	Medium 2 Conductor
Tangential E	$\boxed{\mathbf{E}_{1t} = \mathbf{E}_{2t}}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\epsilon_1 = \mathbf{D}_{2t}/\epsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
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Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Problem 4.48 – Work (a)

$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2\left(\frac{V}{m}\right)$$

$$\varepsilon_1(E_1 \cdot \hat{n}) - \varepsilon_2(E_2 \cdot \hat{n}) = \rho_s$$
$$\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s$$

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1}$$
$$= \frac{3.54\times10^{-11} + 18\varepsilon_0 \cdot 2}{2\varepsilon_0}$$

$$E_{1z} = 20\left(\frac{V}{m}\right)$$

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$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2\left(\frac{V}{m}\right)$$

$$E_{1z} = 20\left(\frac{V}{m}\right)$$

Therefore, this is the answer for $\boldsymbol{E_1}$

$$E_1 = E_{1t} + E_{1z}$$

$$E_1 = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 + \hat{\boldsymbol{z}}20\left(\frac{V}{m}\right)$$

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a) Find E_1 $\quad E_1 = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 + \hat{\boldsymbol{z}}20(\frac{V}{m})$

b) Find $\boldsymbol{\theta_2}$

Use trig!

$E_{2z} = |E_2| \cos\theta_2$

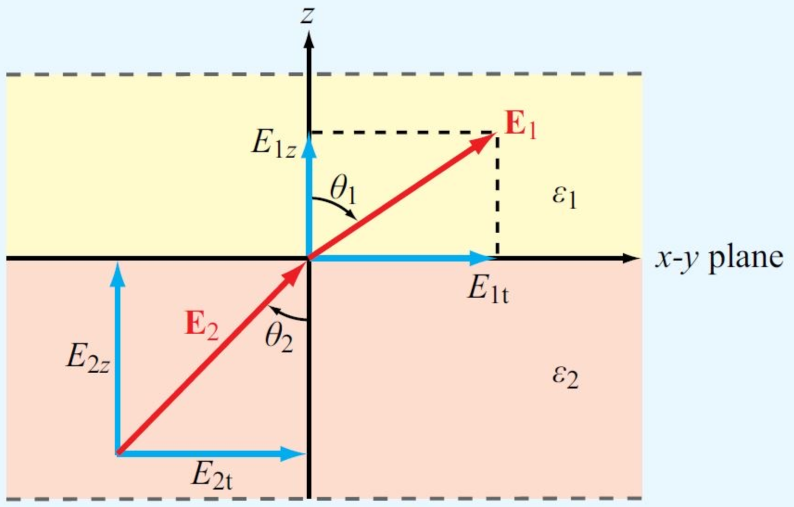

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

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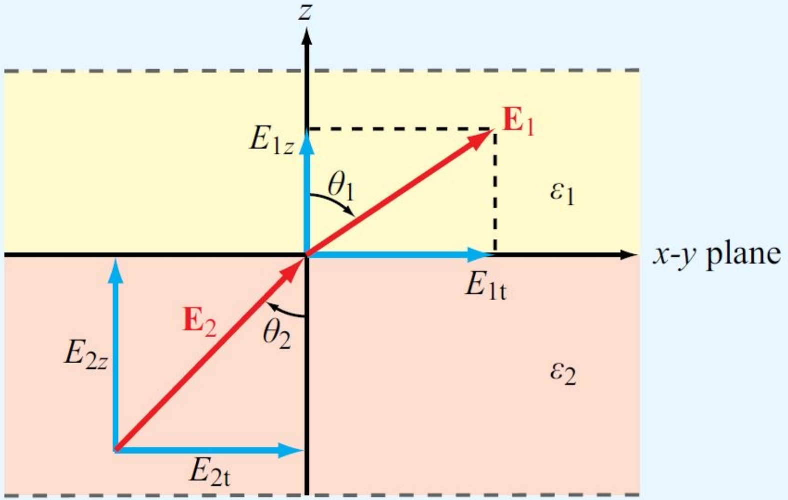
Use trig!

$$E_{2z} = |E_2| \cos\theta_2$$

$$\theta_2 = \cos^{-1}\left(\frac{E_{2z}}{|E_2|}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{3^2 + 2^2 + 2^2}}\right)$$

$$= 61°$$



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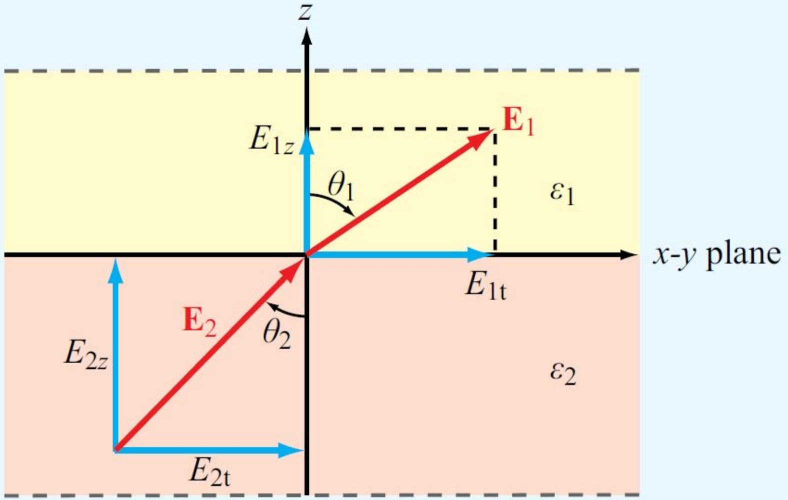

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

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Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

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$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_R S\varepsilon_0 = (150)4\pi(0.05)^2\varepsilon_0 = \frac{3\pi\varepsilon_0}{2} \quad \text{(C)}.$$

Problem 4.54 - Question

4.54 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area (**Fig. P4.54**). If the voltage across the capacitor is 10 V, find the following:

(a) The force acting on the electron.

(b) The acceleration of the electron.

(c) The time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

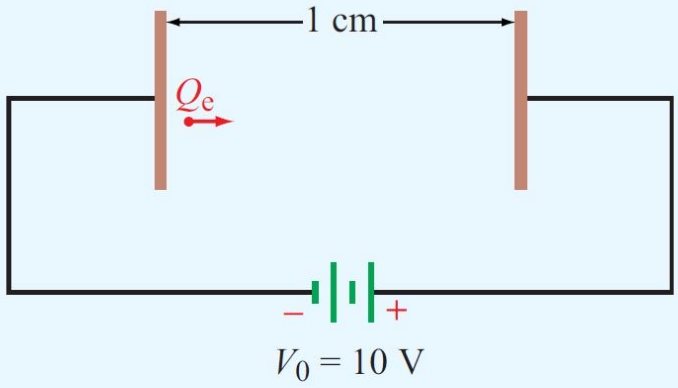

Figure P4.54 Electron between charged plates of Problem 4.54.

Problem 4.54 - Solution

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Part = a

$$F = Q_e E = Qe\frac{V}{d}$$

Putting Values in above equation we get

$$F = -1.6\times\frac{10}{0.01} = -1.6\times10^{-16}\ \text{N}$$

Part = b

$$a = \frac{F}{m} = \frac{1.6\times10^{-16}}{9.1\times10^{-31}} = 1.76\times10^{14}\ \ m/s^2$$

Part = c

$$t = \sqrt{\frac{2d}{a}} = \left(\frac{2\times0.01}{1.76\times10^{14}}\right)^{1/2}$$

$$t = 10.7\times10^{-9}\text{sec} = 10\cdot7(ns)$$

Problem 4.58 - Question

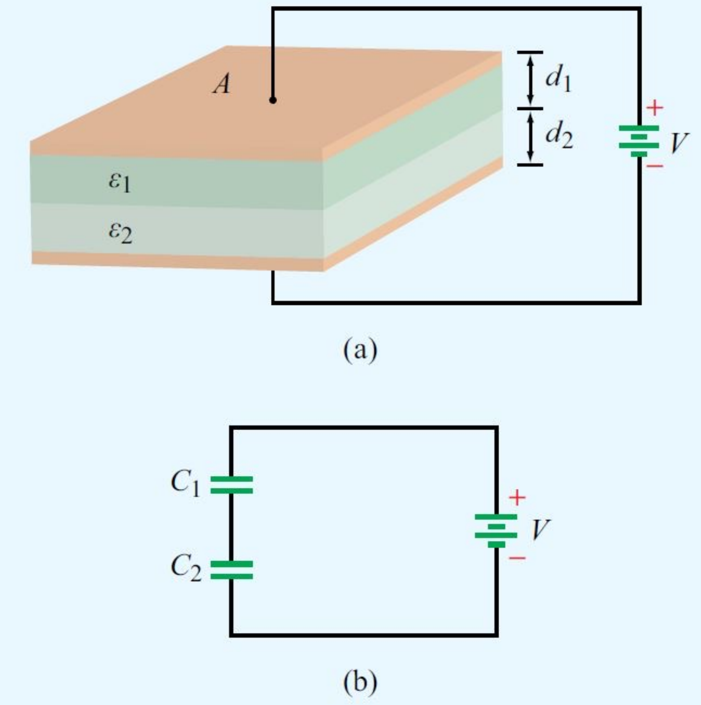

Figure P4.58 (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).

4.58 The capacitor shown in **Fig. P4.58** consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{4.136}$$

where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \qquad C_2 = \epsilon_2 \frac{A}{d_2}.$$

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1, ϵ_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (4.136).

Part = a (Solution)

If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2$$

and,

$$V_1 = E_1 d_1 \quad \text{and} \quad V_2 = E_2 d_2$$

So,

$$V = E_1 d_1 + E_2 d_2 \longrightarrow \text{(A)}$$

According to boundary conditions, the normal component of **D** is continuous across the boundary

$$D_{1n} = D_{2n}$$

From Equation 4.15

$$D = \varepsilon E$$

So,

$$\varepsilon_1 E_1 = \varepsilon_2 E_2 \Longrightarrow E_2 = \frac{\varepsilon_1 E_1}{\varepsilon_2}$$

$$\text{(A)} \longrightarrow V = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2 \Longrightarrow E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \quad \text{and} \quad E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}$$

Part = b (Solution)

From Eq= 4.122, stored potential energy is given by,

$$W_e = \frac{1}{2}\varepsilon E^2 (Ad)$$

So,

$$W_{e1} = \frac{1}{2}\varepsilon_1 {E_1}^2 (Ad_1)$$

Replacing E_1 by

$$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \Rightarrow W_{e1} = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2}\right)^2 (Ad_1) \Rightarrow W_{e_1} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\, \mathrm{d}_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right)$$

$$W_{e_1} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\, \mathrm{d}_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right)$$ And $$W_{e_2} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1^2 \varepsilon_2\ A\, \mathrm{d}_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right)$$

$$W_e = W_1 + W_2 = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\, \mathrm{d}_1 + \varepsilon_1^2 \varepsilon_2\ A\, \mathrm{d}_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right)$$

But,

$$w_e = \frac{1}{2}CV^2$$

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A\, \mathrm{d}_1 + \varepsilon_1^2 \varepsilon_2\ A\, \mathrm{d}_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \Rightarrow \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2\, \mathrm{d} + \varepsilon_1\, \mathrm{d}_2)}{\left(\varepsilon_{2d_1} + \varepsilon_1 d_2\right)^2} \Rightarrow \frac{\varepsilon_1 \varepsilon_2 A}{\left(\varepsilon_{2d_1} + \varepsilon_1 d_2\right)}$$

Part = c (Solution)

From solution of part b we have,

$$C = \frac{\varepsilon_1 \varepsilon_2 A}{(\varepsilon_{2d_1} + \varepsilon_1 d_2)}$$

Multiply numerator and denominator by A/d_1d_2

$$C = \frac{\varepsilon_1 \varepsilon_2 A \, (A/d_1d_2)}{(\varepsilon_{2d_1} + \varepsilon_1 d_2)(A/d_1d_2)}$$

$$C = \frac{\dfrac{\varepsilon_1 A}{d_1} \cdot \dfrac{\varepsilon_2 A}{d_2}}{\dfrac{A\varepsilon_2\, \mathrm{d}_1}{d_1 d_2} + \dfrac{A\varepsilon\, \mathrm{d}_2}{\mathrm{d}_1 d_2}} = \frac{\dfrac{\varepsilon_1 A}{d_1} \cdot \dfrac{\varepsilon_2 A}{d_2}}{\dfrac{A\varepsilon_2}{d_2} + \dfrac{A\varepsilon_1}{d_1}} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_2 = \frac{\varepsilon_2 A}{d_2} \qquad C_1 = \frac{\varepsilon_1 A}{d_1}$$