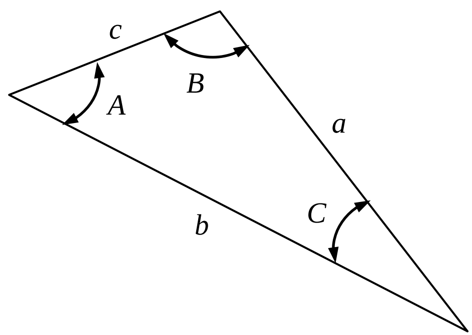
4. INVERSE KINEMATICS

4.0 Introduction

The next kinematics problem we need to solve in order to control the motion of a robot arm is known as the "inverse kinematics problem". With this problem the desired position and orientation of the end-effector is known and we wish to determine the corresponding values for the joint variables. Solving the inverse kinematics problem is more difficult than the forward kinematics problem since it involves a set of nonlinear trigonometric equations.

4.1 Mathematical Preliminaries

The solution of the inverse kinematics equations involves a combination of geometry and trigonometry. One useful mathematical tool is the cosine law. This is illustrated in Figure 4.0 below.



$$\cos A = \left(b^2 + c^2 - a^2\right)/\left(2bc\right)$$

$$\cos B = \left(a^2 + c^2 - b^2\right)/\left(2ac\right)$$

$$\cos C = \left(a^2 + b^2 - c^2\right)/\left(2ab\right)$$

Figure 4.0 Illustration of the cosine law.

A second useful mathematical tool is the function *atan2*, the four-quadrant arctangent. If we try to solve for an angle using *acos*, *asin* or *atan* we may obtain the incorrect answer since $\cos(-\theta) = \cos(\theta)$, $\sin(\pi - \theta) = \sin(\theta)$ and $\tan(\pi + \theta) = \tan(\theta)$. The function atan2 avoids this problem by using both the *cos* and *sin* terms. If we know that $a = \sin(\theta)$ and $b = \cos(\theta)$, then $\theta = \mathrm{atan2}(a, b)$.

4.2 Too Few or Too Many Answers

Several possible outcomes exist when solving the inverse kinematics equations of a robot. Assuming the robot has six DOF the desired position and orientation of the end-effector can be written as ${}^0T_6$. No solution for the joint variables will exist if ${}^0T_6$ is outside the robot's dextrous workspace (*i.e.* due to the length of the links and the motion ranges of the joints). The second possibility is that a single solution to ${}^0T_6$ exists. The third possibility is that multiple solutions exist as a result of the robot's particular kinematics. An example is shown in Figure 4.1. In this figure four solutions to the same ${}^0T_6$ for a PUMA 560 robot are illustrated. The existence of a finite number of multiple solutions can be beneficial if obstacles in the workspace must be avoided, as shown in Figure 4.2, but also complicates the control and programming of the robot. The fourth possibility is the robot's kinematics together with ${}^0T_6$ result in a singular configuration (or singularity). An infinite number of solutions exist for one or more of the joint variables when a robot is at a singularity. For the example shown in Figure 4.3 there are infinitely many solutions for θ_1 since the end-effector intersects the Z_0 axis. In fact, the inverse kinematics solution may become numerically unstable when the robot is close to a singular configuration. This can cause erratic robot motions, and such locations should be avoided.

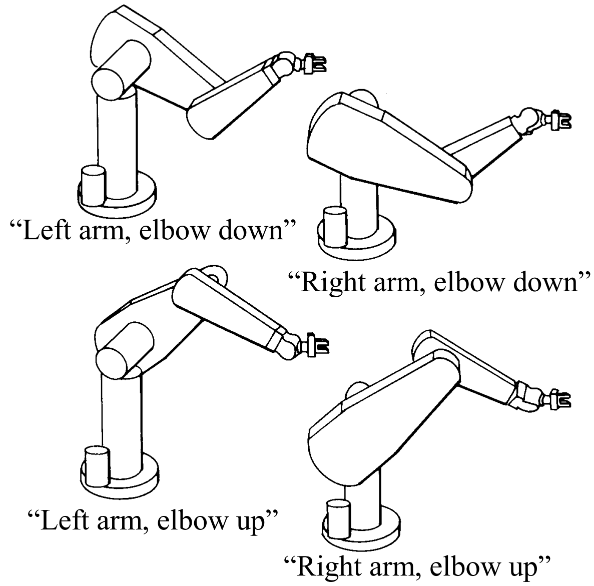


Figure 4.1 Four solutions to the same ${}^0T_6$ for a PUMA 560 robot [1].

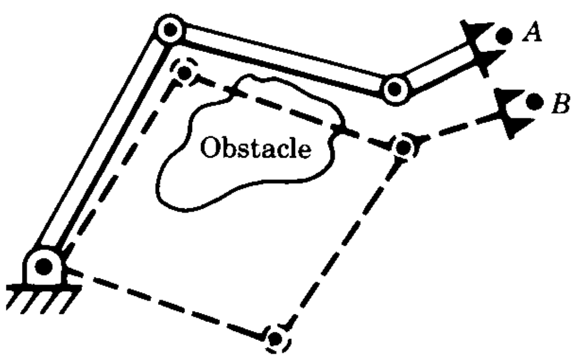


Figure 4.2 An example where using the elbow down configuration avoids a collision between the robot arm and an obstacle [1].

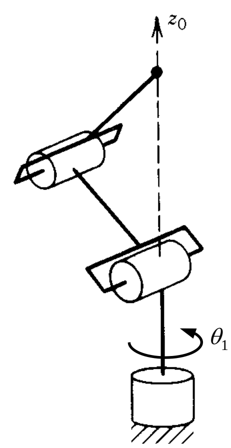


Figure 4.3 An example of a singular configuration [2].

4.3 Inverse Kinematics for Planar Two DOF Robots

We will start with the same planar two DOF robots we studied in Chapter 2 (see Figures 2.14-2.17). The desired planar position of the tool (or end-effector) frame is given and we are interested in finding the equations for the joint variables.

Example 4.1

For the RR robot, the tool moves in the XY plane, the joint variables are θ_1 and θ_2 and we'll denote the given position vector of the tool frame as $[p_x \;\; p_y]^T$. The link lengths a_1 and a_2 are also known. Please see Figure 4.4. For the triangle with the side lengths a_1, a_2 and $\sqrt{p_x^2 + p_y^2}$, from the cosine law we have:

$$\cos(\pi - \theta_2) = \frac{a_1^2 + a_2^2 - p_x^2 - p_y^2}{2a_1 a_2} \tag{4.1}$$

But $\cos(\pi - \theta_2) = -\cos\theta_2$ so this may be rewritten as:

$$\cos\theta_2 = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1 a_2} = U \tag{4.2}$$

Since $(\sin\theta)^2 + (\cos\theta)^2 = 1, \sin\theta_2 = \pm\sqrt{1-U^2}$ and the equation for θ_2 is:

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-U^2}, U\right) \tag{4.3}$$

Note that this equation gives two solutions for θ_2. These take the form of the elbow up and elbow down solutions we saw previously. Note that the term elbow up is typically used even if the arm is operating in the horizontal plane.

After examining the figure again, the angles α, β and θ_1 are given by:

$$\alpha = \operatorname{atan2}\left(\frac{p_y}{\sqrt{p_x^2 + p_y^2}}, \frac{p_x}{\sqrt{p_x^2 + p_y^2}}\right), \tag{4.4}$$

$$\beta = \operatorname{atan2}\left(\frac{a_2 \sin\theta_2}{\sqrt{p_x^2 + p_y^2}}, \frac{a_1 + a_2\cos\theta_2}{\sqrt{p_x^2 + p_y^2}}\right), \tag{4.5}$$

and

$$\theta_1 = \alpha - \beta \tag{4.6}$$

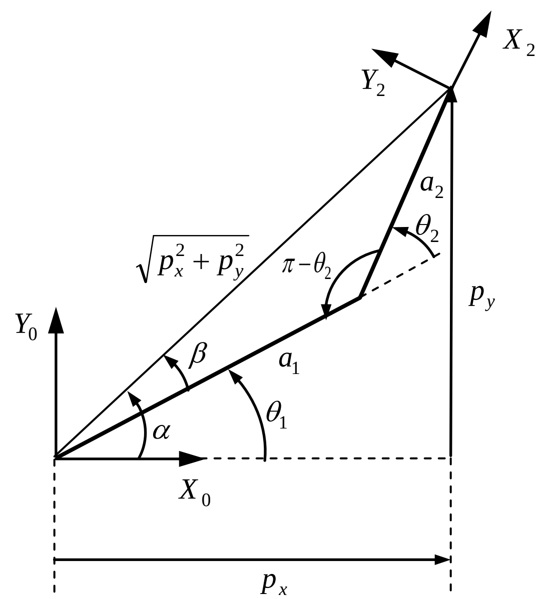


Figure 4.4 Geometry of the planar RR robot.

Note that substituting the two values of θ_2 into equations (4.4)-(4.6) gives two corresponding values for θ_1. This completes the inverse kinematics equations.

End of example 4.1.

Note: If a planar two DOF robot has one or two revolute joints then its inverse kinematics equations will have two solutions.

Example 4.2

For the planar PR robot, the tool moves in the XZ plane, the joint variables are d_1 and θ_2 and we'll denote the given position vector of the tool as $[p_x \; p_z]^T$. Please see Figure 4.5 for the robot geometry. From trigonometry we find:

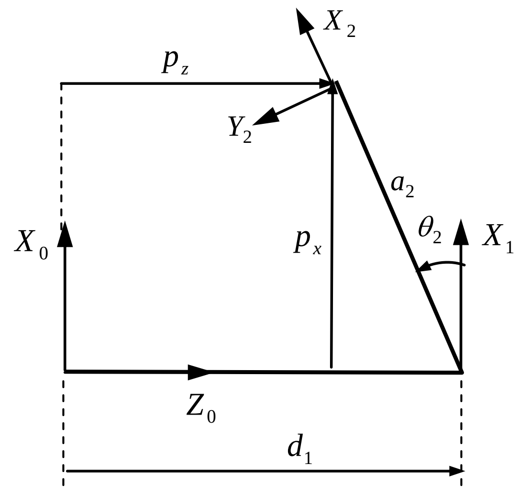


Figure 4.5 Geometry of the planar PR robot.

$$\cos\theta_2 = \frac{p_x}{a_2} = U \tag{4.7}$$

As before, $\sin\theta_2 = \pm\sqrt{1-U^2}$ so we have:

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-U^2}, U\right) \tag{4.8}$$

From the geometry:

$$d_1 - a_2\sin\theta_2 = p_z \tag{4.9}$$

$$\therefore d_1 = p_z + a_2\sin\theta_2 \tag{4.10}$$

Again, there are two valid solutions. This completes the inverse kinematics equations.

End of example 4.2.

In-class Exercises 1 & 2

We will derive the inverse kinematics equations for the RP and PP robots as in-class exercises.

4.4 Inverse Kinematics for Planar Three DOF Robots

With the planar three DOF robots (see Figures 2.19-2.25), the desired planar position and orientation of the tool is given and we are interested in finding the equations for the joint variables. These robots fall into two groups. If the last joint is revolute, the solution is similar to the associated two DOF robot; if the last joint is prismatic, the similarity does not exist. This will illustrated by two examples.

Example 4.3

The geometry of a planar RRR robot is shown in Figure 4.6. As before the link lengths are known. The position of the tool frame is given by $[p_x \; p_y]^T$ and its orientation is specified by the angle ϕ.

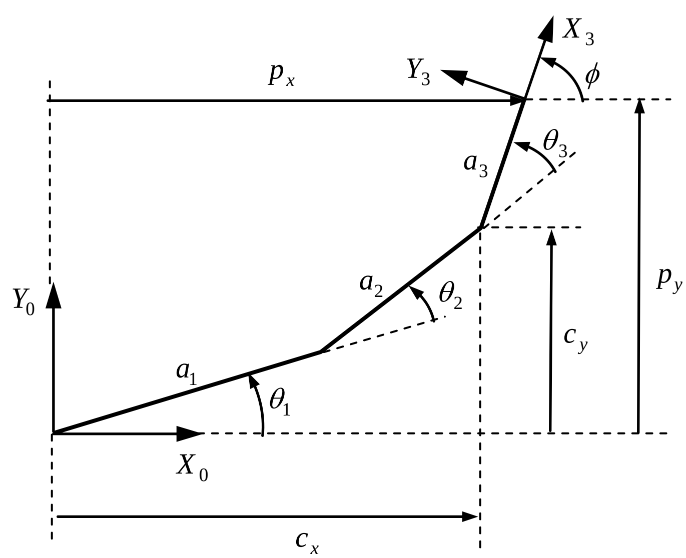


Figure 4.6 Geometry of the planar RRR robot.

Note that the last joint is revolute. Since a_3 is known the position of joint 3 may be calculated using:

$$c_x = p_x - a_3 \cos\phi \tag{4.11}$$

and

$$c_y = p_y - a_3 \sin\phi \tag{4.12}$$

From the geometry:

$$\theta_3 = \phi - \theta_1 - \theta_2 \tag{4.13}$$

The rest of the solution follows the RR planar robot we already solved in example 4.1. The remaining angles are given by:

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-U^2}, U\right) \text{ where } U = \frac{c_x^2 + c_y^2 - a_1^2 - a_2^2}{2a_1a_2} \text{ and} \tag{4.14}$$

$$\theta_1 = \operatorname{atan2}\left(\frac{c_y}{\sqrt{c_x^2+c_y^2}}, \frac{c_x}{\sqrt{c_x^2+c_y^2}}\right) - \operatorname{atan2}\left(\frac{a_2\sin\theta_2}{\sqrt{c_x^2+c_y^2}}, \frac{a_1 + a_2\cos\theta_2}{\sqrt{c_x^2+c_y^2}}\right) \tag{4.15}$$

Note that there are two valid solutions for the joint variables.

End of example 4.3.

Example 4.4

The geometry of a planar PRP robot is shown in Figure 4.7. The position of the tool frame is given by $[p_x \;\; p_z]^T$ and its orientation is specified by the angle ϕ. The joint variables are d_1, θ_2 and d_3.

Note that the last joint is not revolute. From the geometry we find:

$$\theta_2 = \phi \tag{4.16}$$

From trigonometry in the X_0 direction:

$$d_3 \cos\beta = p_x \text{ where } \beta = \tfrac{\pi}{2} - \theta_2 \tag{4.17}$$

$$\therefore d_3 = \frac{p_x}{\cos\left(\frac{\pi}{2} - \theta_2\right)} = \frac{p_x}{\sin\theta_2} \tag{4.18}$$

In the Z_0 direction:

$$d_1 + d_3 \sin\beta = p_z \tag{4.19}$$

$$\begin{aligned} d_1 &= p_z - d_3 \sin\left(\tfrac{\pi}{2} - \theta_2\right) \\ &= p_z - d_3 \cos\theta_2 \end{aligned} \tag{4.20}$$

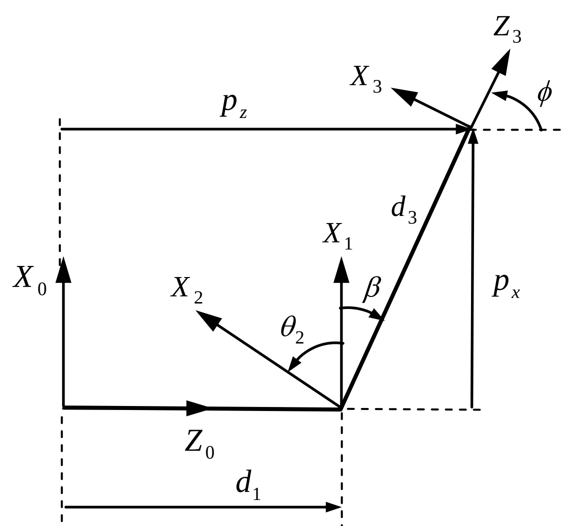


Figure 4.7 Geometry of the planar PRP robot.

This completes the inverse kinematics equations. Note that this robot has only one valid solution for the joint variables.

End of example 4.4.

Note: If a planar three DOF robot has two or three revolute joints then it's inverse kinematics equations will have two solutions.

In-class Exercises 3 & 4

The figures for in-class exercises 3 & 4 are included at the end of this chapter.

4.5 Checking the Forward and Inverse Kinematics Equations

It is possible to check your derivations of the forward and inverse kinematics equations using the equations themselves. The recommended approach is as follows:

1. Start with the forward kinematics equations. Use random values for the joint variables and calculate the resulting position and orientation of the tool frame. Note that in the planar case you will have to calculate the angle ϕ from the appropriate directional vector;
2. Input the position and orientation values into your inverse kinematics equations and solve for the joint variables. If there is more than one solution to the joint variables then continue with step 4;
3. If the joint variables you calculated in step 2 match the values used in step 1 then your equations are correct*;
4. If one of the resulting solutions for the joint variables matches the values you used in step 1 then your equations are partly correct*. Enter the other solutions back into the forward kinematics equations. They should all produce the same position and orientation values as in step 1. If they do then your equations are correct*.

***Warnings:** It is recommended that this testing approach is performed using several random sets of joint variables. It is possible that your equations pass the test but are still incorrect due to an error in common, although this is unlikely.

4.6 Kinematic Decoupling Method for Spatial Robots

If a manipulator uses a wrist whose joints intersect at a point (as with most industrial robots) the task of deriving the equations that solve the inverse kinematics problem can be simplified. The problem can be decoupled into two simpler problems, known as "inverse position kinematics" and "inverse orientation kinematics". We will assume the robot consists of three major axes followed by a Euler wrist but the method can also be applied to robots with fewer DOF. In fact we already applied a simplified version of this method with the three DOF planar robots whose last joint is revolute.

4.6.1 Inverse Position Kinematics

In this section the derivation of the equations for the first three joint variables (*i.e.* the major axes) will be described. We begin by partitioning the given ${}^{0}T_{6}$ into a 3 × 3 rotation matrix R and a 3 × 1 position vector D as follows:

$$
{}^{0}T_{6} = \begin{bmatrix} & R & & D \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.21}
$$

Since this matrix is given, and assuming the D-H parameters are known, the position vector of the wrist centre may be calculated using:

$$C = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} d_x - d_6 r_{13} \\ d_y - d_6 r_{23} \\ d_z - d_6 r_{33} \end{bmatrix} \tag{4.22}$$

This equation may be understood by recalling that the origin of frame {5} is at the wrist centre, the origin of frame {6} was translated by d_6 along the Z_5-axis, and Z_5 and Z_6 are collinear. With the components of C known the equations for the first three joint variables may be found using conventional geometry.

Example 4.5

We will first consider an articulated robot with a "shoulder offset" similar to the one shown in Figure 4.8. We will denote the offset between frames {1} and {2}, *b*, as seen in Figure 4.9. Joint 2 is usually called the "shoulder" and this is known as a "shoulder offset". This is also known as an "elbow manipulator with shoulder offset". The projection of links 2 and 3 onto the X_0Y_0 plane for the "left arm" and "right arm" configurations are shown in Figures 4.10 and 4.11, respectively.

Figure 4.8 Elbow manipulator with a shoulder offset.

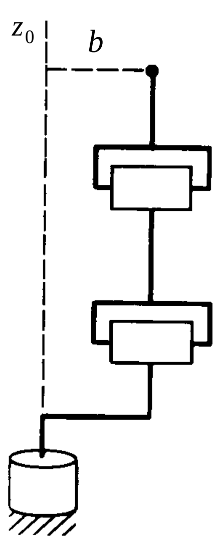


Figure 4.9 Elbow manipulator with offset shoulder [2].

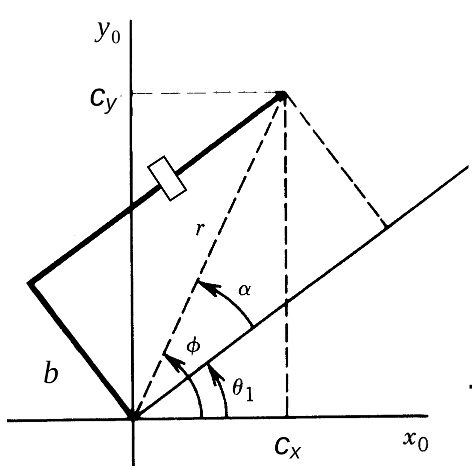


Figure 4.10 Projection of left arm configuration [2].

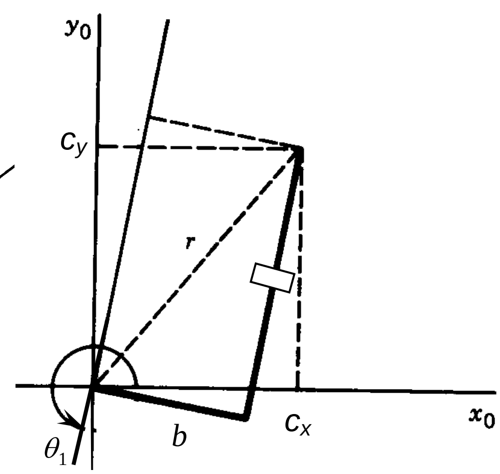


Figure 4.11 Projection of right arm configuration [2].

An examination of Figure 4.10 reveals that

$$\theta_1 = \phi - \alpha = \text{atan2}\left(\frac{c_y}{\sqrt{{c_x}^2 + {c_y}^2}}, \frac{c_x}{\sqrt{{c_x}^2 + {c_y}^2}}\right) - \text{atan2}\left(\frac{b}{r}, \frac{\sqrt{r^2 - b^2}}{r}\right) \tag{4.23}$$

for the left arm configuration. Similarly, for the right arm configuration (Figure 4.11):

$$\theta_1 = \phi + \alpha + \pi = \text{atan2}\left(\frac{c_y}{\sqrt{{c_x}^2 + {c_y}^2}}, \frac{c_x}{\sqrt{{c_x}^2 + {c_y}^2}}\right) + \text{atan2}\left(\frac{b}{r}, \frac{\sqrt{r^2 - b^2}}{r}\right) + \pi \tag{4.24}$$

As shown in Figure 4.12, the shoulder offset also alters the projection onto the plane formed by links 2 and 3. Now we find:

$$\cos\theta_3 = \frac{r^2 - b^2 + {c_z}^2 - {a_2}^2 - {a_3}^2}{2a_2a_3} = V \tag{4.25}$$

and

$$\theta_3 = \text{atan2}\left(\pm\sqrt{1 - V^2}, V\right). \tag{4.26}$$

For the left arm configuration, the positive square root gives the elbow down solution and the negative square root gives the elbow up solution. From the geometry of Figure 4.12, the corresponding solutions for angle θ_2 are given by:

$$\theta_2 = \text{atan2}\left(\frac{c_z}{\sqrt{{c_z}^2 + r^2 - b^2}}, \frac{\sqrt{r^2 - b^2}}{\sqrt{{c_z}^2 + r^2 - b^2}}\right) - \text{atan2}\left(\frac{a_3 \sin\theta_3}{\sqrt{{c_z}^2 + r^2 - b^2}}, \frac{a_2 + a_3\cos\theta_3}{\sqrt{{c_z}^2 + r^2 - b^2}}\right) \tag{4.27}$$

This completes the solution for the left arm configuration. For the right arm configuration, equation (4.26) applies except that the positive square root gives the elbow up solution and vice-versa. Finally, for the right arm configuration:

$$\theta_2 = \pi - \text{atan2}\left(\frac{c_z}{\sqrt{{c_z}^2 + r^2 - b^2}}, \frac{\sqrt{r^2 - b^2}}{\sqrt{{c_z}^2 + r^2 - b^2}}\right) - \text{atan2}\left(\frac{a_3 \sin\theta_3}{\sqrt{{c_z}^2 + r^2 - b^2}}, \frac{a_2 + a_3\cos\theta_3}{\sqrt{{c_z}^2 + r^2 - b^2}}\right) \tag{4.28}$$

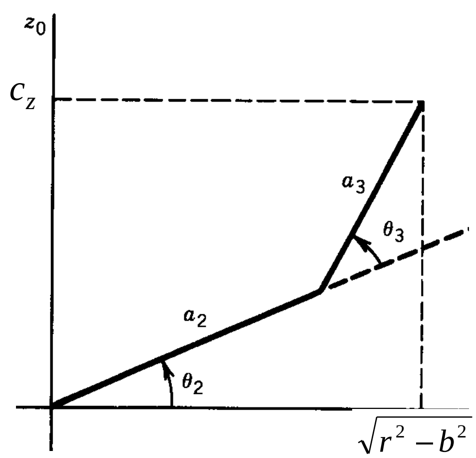


Figure 4.12 Projection onto the plane formed by links 2 and 3 for the left arm configuration of the elbow manipulator with shoulder offset [2].

There are four solutions to the joint variables (assuming the joints have sufficient range), namely "left arm, elbow up", "left arm, elbow down", "right arm, elbow up" and "right arm, elbow down".

End of example 4.5.

Example 4.6

Next, we will consider an articulated robot similar to the one in Figure 4.8, but with no shoulder offset. This robot can be thought of as the robot from the previous example, but with the offset *b* set to zero. It is known as an "elbow manipulator". Several companies make robots of this type. In this figure *C* is projected onto the x_0 - y_0 plane. We'll start with the case known as "not flipped", as seen in the left hand side (top and bottom) of Figure 4.13.

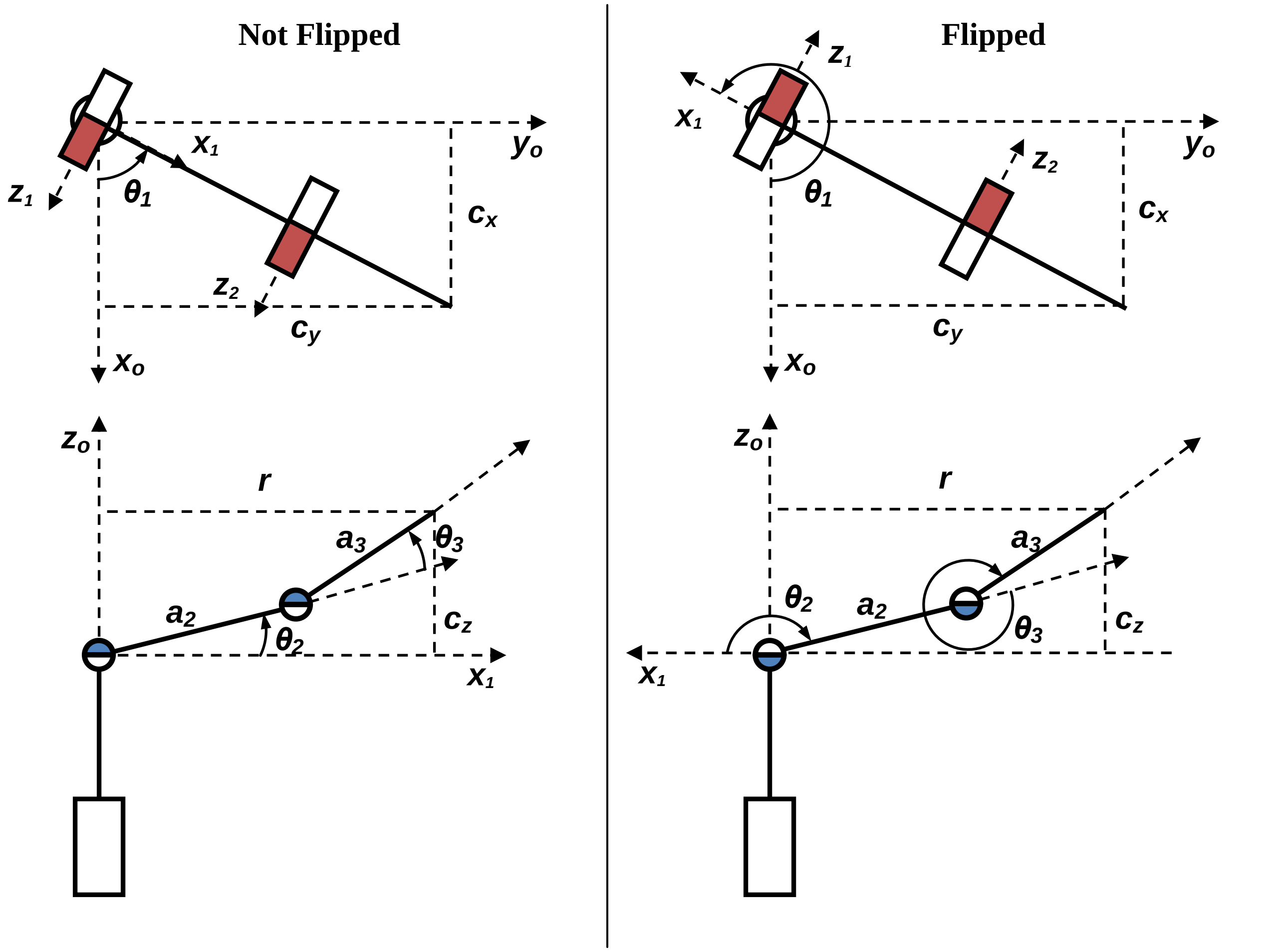


Figure 4.13 Left side: Arm in "not flipped" configuration. Right side: Arm in "flipped" configuration.
Top (L&R): projection onto the plane formed by x_0 and y_0.
Bottom (L&R): projection onto the plane formed by links 2 and 3.

In this case we have:

$$\theta_1 = \text{atan2}\left(\frac{c_y}{\sqrt{c_x^{\,2}+c_y^{\,2}}}, \frac{c_x}{\sqrt{c_x^{\,2}+c_y^{\,2}}}\right) \tag{4.29}$$

Note that atan2(u,v) does not work when u=v=0 so this equation numerically breaks down in the vicinity of the singular position c_x=c_y=0 that we saw in Figure 4.3. The angles for joints 2 and 3 can be found by projecting onto the plane formed by links 2 and 3. This projection is shown in Figure 4.13 (bottom). Using the Law of Cosines and Figure 4.13 we find:

$$\cos\theta_3 = \frac{r^2 + C_z^{\,2} - a_2^{\,2} - a_3^{\,2}}{2a_2a_3} = U \tag{4.30}$$

Since $(\sin\theta)^2 + (\cos\theta)^2 = 1$, angle θ_3 is given by:

$$\theta_3 = \text{atan2}\left(\pm\sqrt{1-U^2}, U\right) \tag{4.31}$$

where the positive square root gives the elbow down solution and the negative square root gives the elbow up solution. From the geometry of Figure 4.13, the corresponding solutions for angle θ_2 are given by:

$$\theta_2 = \text{atan2}\left(\frac{c_z}{\sqrt{c_z^{\,2}+r^2}}, \frac{r}{\sqrt{c_z^{\,2}+r^2}}\right) - \text{atan2}\left(\frac{a_3\sin\theta_3}{\sqrt{c_z^{\,2}+r^2}}, \frac{a_2 + a_3\cos\theta_3}{\sqrt{c_z^{\,2}+r^2}}\right) \tag{4.32}$$

If joints 1 and 2 have enough range another pair of valid solutions is given by "flipping" the arm, as seen in Figure 4.13(right hand side, top and bottom). It can be seen that in order to "flip" the arm and end up in the same configuration one must manipulate all the joints. First by adding π to θ_1 the arm is rotated around and is now facing in the opposite direction. Then by replacing the θ_2 values with $\pi - \theta_2$ to flip the arm over itself, and finally changing the sign of θ_3 to correct the elbow. This gives a total of four solutions for the joint variables, namely "arm flipped, elbow up", "arm flipped, elbow down", "arm not flipped, elbow up" and "arm not flipped, elbow down".

Note that this derivation of the inverse kinematics for a spatial robot was not very different than what we've seen with planar robots. We projected the arm onto a plane and solved for one joint variable, and then we projected the arm onto a second plane to find the equations for the remaining joint variables. This procedure of projecting the arm onto two planes works for most spatial robots.

End of example 4.6.

Regarding Multiple Solutions for the Inverse Position Kinematics Equations of Spatial Robots

If none of the major axes of a spatial robot are revolute joints then its inverse position kinematics equations will have one solution.

If one of the major axes of a spatial robot is a revolute joint then its inverse position kinematics equations will have two solutions.

If two or more of the major axes of a spatial robot are revolute joints then its inverse position kinematics equations will have four solutions.

4.6.2 Inverse Orientation Kinematics

Covering the method for solving the inverse orientation kinematics in detail is beyond the scope of this course. In this section an overview of the method will be presented.

The equations for the inverse orientation kinematics may be used to calculate the values of the wrist's joint angles (*i.e.* the minor axes). The method begins by first calculating the orientation of the end-effector relative to frame {3} as follows:

$${}^3R_6 = {}^0R_3^{-1} * R = {}^0R_3^T * R = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \tag{4.33}$$

where ${}^{0}R_3$ can be calculated using forward kinematics and the joint variables we found using the inverse position kinematics equations. We have assumed that the robot is equipped with an Euler wrist as is common with industrial robots. For an Euler wrist the rotation matrix is

$$ {}^{3}R_6 = \begin{bmatrix} C\theta_4 C\theta_5 C\theta_6 - S\theta_4 S\theta_6 & -C\theta_4 C\theta_5 S\theta_6 - S\theta_4 C\theta_6 & C\theta_4 S\theta_5 \\ S\theta_4 C\theta_5 C\theta_6 + C\theta_4 S\theta_6 & -S\theta_4 C\theta_5 S\theta_6 + C\theta_4 C\theta_6 & S\theta_4 S\theta_5 \\ -S\theta_5 C\theta_6 & S\theta_5 S\theta_6 & C\theta_5 \end{bmatrix} \tag{4.34} $$

Equating (4.33) and (4.34) gives the matrix equation

$$ \begin{bmatrix} C\theta_4 C\theta_5 C\theta_6 - S\theta_4 S\theta_6 & -C\theta_4 C\theta_5 S\theta_6 - S\theta_4 C\theta_6 & C\theta_4 S\theta_5 \\ S\theta_4 C\theta_5 C\theta_6 + C\theta_4 S\theta_6 & -S\theta_4 C\theta_5 S\theta_6 + C\theta_4 C\theta_6 & S\theta_4 S\theta_5 \\ -S\theta_5 C\theta_6 & S\theta_5 S\theta_6 & C\theta_5 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \tag{4.35} $$

The remainder of the method involves solving for angles θ_4, θ_5 and θ_6 using the known *u* values. It should be noted that an Euler wrist has two inverse kinematic solutions known as "flip" and "no flip" that will be explained further in class. This means that the full inverse kinematics equations (position and orientation) for an articulated robot with an Euler wrist will have eight solutions altogether.

4.7 Specifying End-Effector Orientation

We know that the end-effector's orientation can be represented using the r_{ij} values in equation (4.21). In practice, the orientation is often specified in terms of three rotation angles. There are a few different standard representations.

The first method is in terms of "Euler angles". The end-effector's orientation is obtained by a rotation of ϕ (phi) about the z-axis, then a rotation of θ about the current y-axis, and finally a rotation of φ (psi) about the current z-axis. ϕ, θ and φ are known as the Euler angles. Since the rotations are about the current axes we post-multiply the matrices. So we have:

$$ \text{Euler}(\phi, \theta, \varphi) = \text{Rot}(z, \phi) * \text{Rot}(y, \theta) * \text{Rot}(z, \varphi) \tag{4.36} $$

Note that ϕ, θ and φ correspond to θ_4, θ_5 and θ_6 for the Euler wrist.

The second method is in terms of "Roll, Pitch and Yaw" angles. The end-effector's orientation is obtained by a rotation of *roll* ("rolling") about the z-axis, then a rotation of *pitch* ("pitching") about the current y-axis, and finally a rotation of *yaw* about the current x-axis. These are analogous to the nautical terms if we imagine the boat is pointing initially along the z-axis. Since the rotations are about the current axes we post-multiply the matrices. So we have:

$$ \text{RPY}(roll, pitch, yaw) = \text{Rot}(z, roll) * \text{Rot}(y, pitch) * \text{Rot}(x, yaw) \tag{4.37} $$

Note that this can also be interpreted as yaw-pitch-roll relative to the initial coordinate frame.

4.8 World vs. Tool Coordinates

Depending on the application we may want to specify the orientation (and also the position) of the end-effector relative to the fixed base (or world) coordinate frame or relative to the current tool frame. For example, let's say we wish to use the robot to screw in a bolt. Assume the robot's end-effector is a screwdriver. To bring it to the bolt, we would then specify the position and orientation of the screwdriver in world coordinates. To screw in the bolt the screwdriver will have to rotated and translated. Although this rotation and translation could be specified in world coordinates it would be much easier to specify this in tool coordinates. Assuming the screwdriver is aligned with tool z-axis this would be simply a rotation about the tool z-axis combined with a translation along the tool z-axis. So far we have assumed that we are using base or world coordinates. To make the desired transformation (rotations and translations) relative to the current tool frame we must post-multiply as follows:

$$(\text{specified } {}^0T_6) = (\text{current } {}^0T_6) * T \tag{4.38}$$

where T is the desired transformation.

Example 4.7

A robot is holding a peg and the peg is aligned with the tool z-axis. We wish to push the peg into a hole that is at the position $[7\ \ 8\ \ 9]^T$ in world coordinates. The hole is angled at a 45° rotation relative to the world x-axis. This would normally be done as two movements. To move the peg to the top of the hole with the proper alignment we require:

$${}^0T_6 = \mathrm{Trans}(7,8,9) * \mathrm{Rot}(x,45°) = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 0.707 & -0.707 & 8 \\ 0 & 0.707 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.39}$$

Next we want to push the peg into the hole. If the hole is 0.5 deep this will require translating 0.5 in the tool z direction. So to complete the task we require:

$${}^0T_6 = (\text{current } {}^0T_6) * \mathrm{Trans}(0,0,0.5) = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 0.707 & -0.707 & 8 \\ 0 & 0.707 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 0.707 & -0.707 & 7.646 \\ 0 & 0.707 & 0.707 & 9.354 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.40}$$

End of example 4.7.

Note: Additional inverse kinematics examples are presented in chapter 4 of Craig's textbook, and in section 2.7 of Niku's textbook.

References

1. J.J. Craig, "Introduction to Robotics", Addison Wesley Longman, 1989.
2. M.W. Spong and M. Vidyasagar, "Robot Dynamics and Control", John Wiley & Sons, 1989.

In-class Exercises 3 and 4:

In-class Exercise 3: Inverse Kinematics Equations for a Planar PRR Robot.

The robot's geometry is shown in the Fig. 4.14. Derive its inverse kinematics equations.

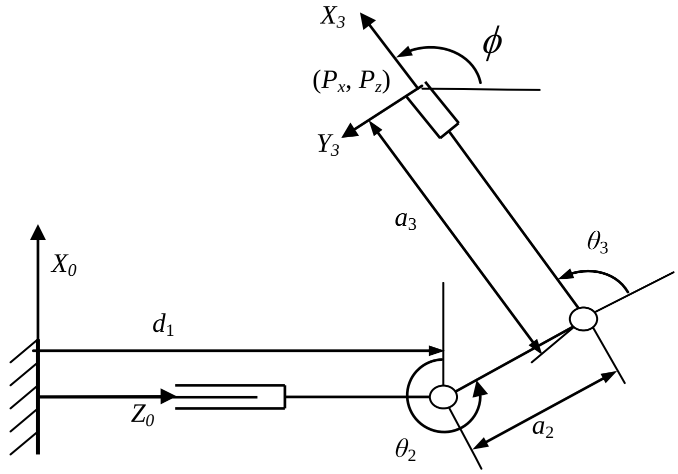


Figure 4.14 Geometry of the planar PRR robot

In-class Exercise 4: Inverse Kinematics Equations for a Planar RPP Robot.

The robot's geometry is shown in the Fig. 4.15. Derive its inverse kinematics equations.

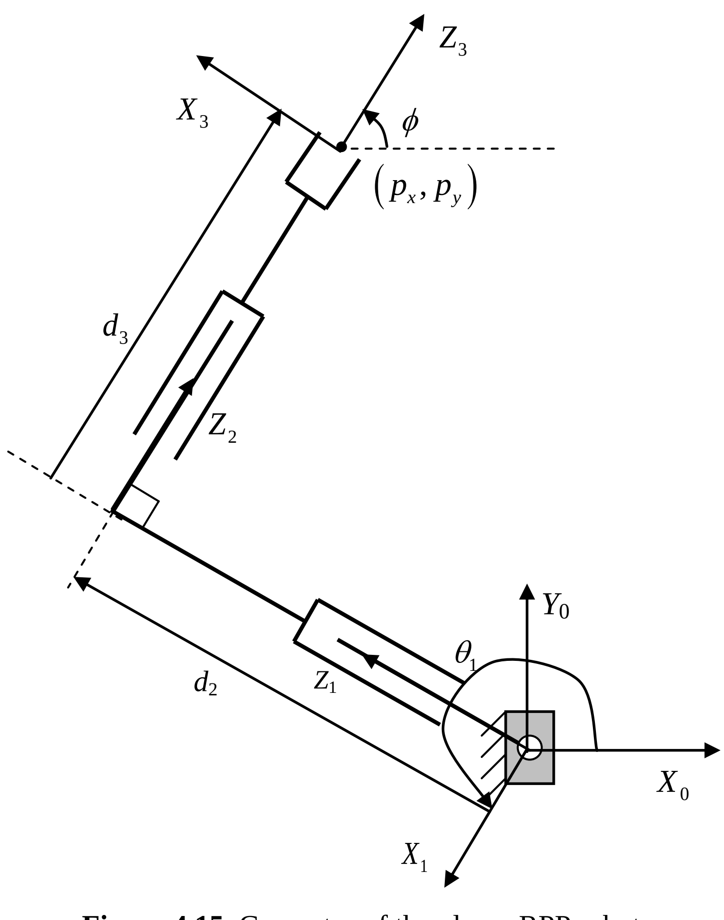


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Figure 4.15 Geometry of the planar RPP robot