Mechatronics 3DX4

Slides 5: Time Response

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Material based on lecture notes by P. Taylor and M. Lawford, and *Control Systems Engineering* by N. Nise.

Introduction

- ► Once we have obtained a mathematical representation of a system, our next step is to analyze its transient and steady-state response.

- ► In this section, we will focus on how to analyze a system's transient response.

- ► We already know how to determine the output response by solving differential equations, or taking inverse Laplace transforms.

- ► These methods are laborous and time consuming.

- ► We want to develop a technique where we can get the desired information about a system's transient and steady-state response, basically by inspection.

- ► Our first topic will be how to analyze poles and zeros to determine a system's response.

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Poles and Zeros

► Consider:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)}$$

► Poles of $G(s)$ are the roots of $D(s)$.

► Zeros of $G(s)$ are the roots of $N(s)$.

► Generally, at poles $G(s) = \infty$ unless the pole is cancelled by a matching zero.

► At zeros, $G(s) = 0$ unless the zero is cancelled by a matching pole.

Poles and Zeros of First Order System eg.

▶ A system's output response contains two parts:

1. **Forced** or **steady state response**: this is caused by the poles of the input function, $R(s)$.
2. **Natural** or **homogeneous response**: this is caused by the poles of the transfer function, $G(s)$.

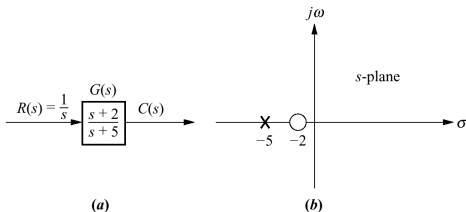▶ In example below, our transfer function is $G(s) = \dfrac{s+2}{s+5}$, and our input is $R(s) = \frac{1}{s}$.

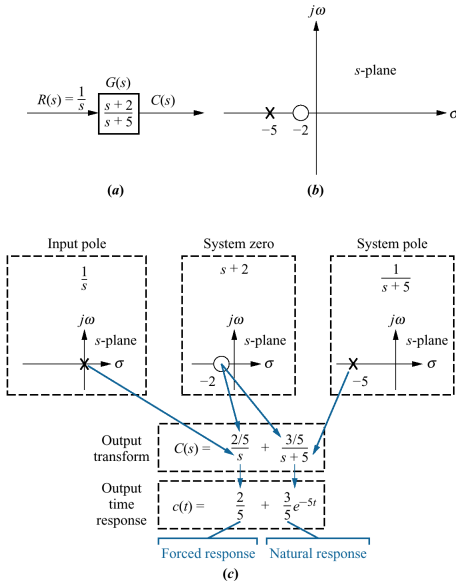Figure 4.1.

Poles and Zeros of First Order System eg. - II

Figure 4.1.

Poles and Zeros of First Order System eg. - III

1. Pole of input function generated forced response, $u(t)$.

2. Pole of transfer function generated natural response, e^{-5t}.

The above is not affected at all by the zero.

3. Pole on real axis, say at $-\alpha$, generates an exponential response, $e^{-\alpha t}$.

Farther to the left on negative axis, the faster the response decays.

4. Both the poles and zeros contribute to the amplitude of the response (ie. the $\frac{2}{5}$ and $\frac{3}{5}$ factors).

Pole at $-\alpha$ generates
response $Ke^{-\alpha t}$
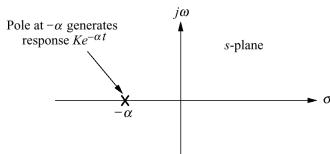
$j\omega$

s-plane

$-\alpha$

σ

Figure 4.2.

Evaluating Response using Poles eg.
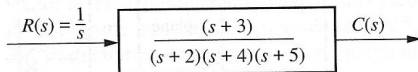
▶ Consider system shown below.

Figure 4.3.

▶ From inspection, we can immediately determine:

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural}}$$

▶ Using $\mathcal{L}\{\frac{1}{s+\alpha}\} = e^{-\alpha t}$ gives:

$$c(t) \equiv \underbrace{K_1}_{\text{Forced}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural}}$$

First Order Systems

▶ We will now examine first order systems without zeros so we can define performance specifications.

▶ We use systems of the form $G(s) = \frac{a}{s+a}$ as our base form for our definitions.

▶ If our input is the step function, $R(s) = \frac{1}{s}$, we get
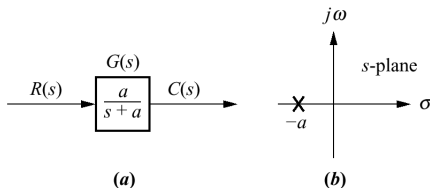
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

Figure 4.4.

Time Constant

▶ We will now examine first order systems without zeros so we can define performance specifications.

▶ Our first specification is the system's time constant, $\tau = \frac{1}{a}$.

▶ The time constant is the time required for step response to rise to 63% percent of its final value.

$$c(\tau) = 1 - e^{-a\tau} = 1 - e^{-a \cdot \frac{1}{a}} = 1 - e^{-1} \approx 1 - 0.37 = 0.63$$

▶ As $\frac{dc(t)}{dt} = ae^{-at}$, we thus have a equal to the slope at $t = 0$.
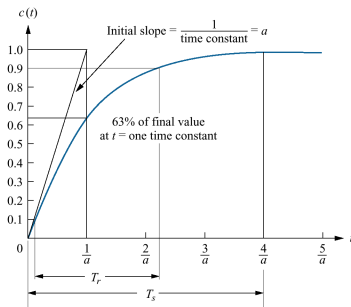
▶ We call a the exponential frequency.

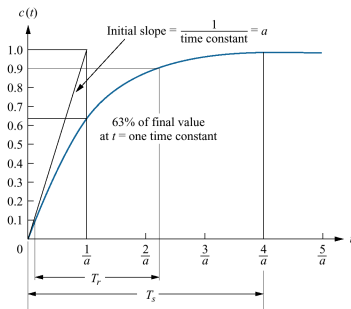**Figure 4.5**

Rise and Settling Time

- Rise time, T_r, is the time for the output to go from 10% to 90% of its final value.

- Can show that $T_r = \frac{2.2}{a}$.

- Settling time, T_s, is time required for the output to reach 98% of its final value.

- Setting $c(T_s) = 0.98$, we find that $T_s = \frac{4}{a}$.

Figure 4.5

Using Testing to Determine Transfer Function

▶ It is quite often not possible or practical to determine a system's transfer function by analytical means.

▶ In general, gain of system at $s = 0$ (D.C. gain) is not unity.

▶ A more general model would be $G(s) = \dfrac{K}{s + a}$.

▶ Step response is thus

$$C(s) = \frac{K}{s(s + a)} = \frac{\dfrac{K}{a}}{s} - \frac{\dfrac{K}{a}}{s + a}$$

▶ We thus have

$$c(t) = \frac{K}{a}(1 - e^{-at})$$

▶ How can we experimentally determine the values of K and a?

Using Testing to Determine Transfer Function - II

▶ For system to be a first-order system, its unit step response should have *no overshoot* and should have a *nonzero initial slope*, as in diagram below.

▶ From diagram, we can see the final value is about 0.72, thus 63% of that is $0.63 \times 0.72 = 0.45$.

▶ From diagram, the output reaches 0.45 at about $\tau = 0.13$ (time constant).

▶ We thus have $a = \dfrac{1}{\tau} = 7.7$.

▶ We next note that
$$c(\infty) = \frac{K}{a}(1 - e^{-at})|_{t \to \infty} = \frac{K}{a}$$

▶ Thus $K = a \cdot c(\infty) = (7.7)(0.72) = 5.54$
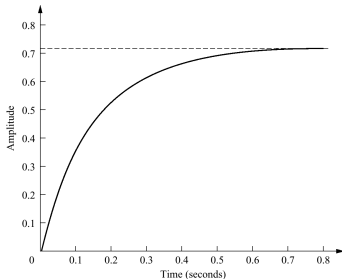
▶ Thus $G(s) = \dfrac{5.54}{s + 7.7}$

Figure 4.6

Second-Order Systems

▶ For first order systems, varying the systems parameters only changed the speed of the response.

▶ Form of a second order system we will analyze is
$G(s) = \frac{b}{s^2+as+b}$.

▶ Changes in these parameters can actually change the *form* of the system's response.

▶ May see responses similar to first-order system, damped oscillations, or undamped oscillations.

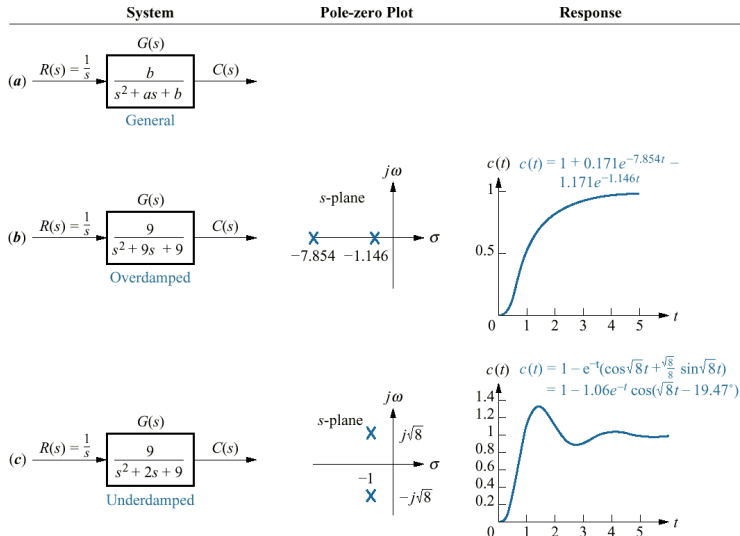# Second-Order System Examples

Figure 4.7.
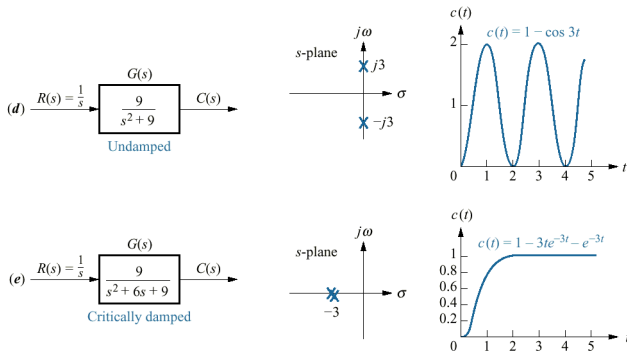
Second-Order System Examples -II

Figure 4.7.

▶ Like for first-order system, we want to determine information about system's steady state and transient response by examination.

Overdamped Response

▶ For overdamped response, we have a system with two non equal real poles.

▶ The unit step response to system below is

$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

▶ From inspection of poles, we know form of system's response will be:

$$c(t) = K_1 + K_2 e^{-\sigma_1 t} + K_3 e^{-\sigma_2 t}$$

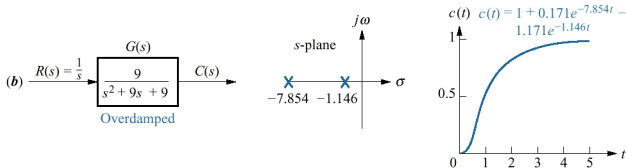where $-\sigma_1 = -7.854$ and $-\sigma_2 = -1.146$, are our two real poles.

Figure 4.7.

Critically Damped Response

▶ For critically damped response, we have a system with two equal real poles.

▶ The unit step response to system below is

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2} = \frac{K_1}{s} + \frac{K_2}{(s+3)} + \frac{K_3}{(s+3)^2}$$

▶ From inspection of poles, we know form of system's response will be:

$$c(t) = K_1 + K_2 e^{-\sigma_1 t} + K_3 t e^{-\sigma_1 t}$$
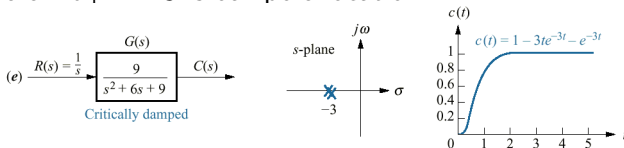
where $-\sigma_1 = -3$ is our pole location.

Figure 4.7.

Underdamped Response

▶ For underdamped response, we have a system with two complex conjugate poles (non zero real and imaginary parts).

▶ The unit step response to system below is

$$C(s) = \frac{9}{s(s^2 + 2s + 9)} = \frac{9}{s(s+1+j\sqrt{8})(s+1-j\sqrt{8})}$$

$$= \frac{K_1}{s} + \frac{\alpha + j\beta}{s+1+j\sqrt{8}} + \frac{\alpha - j\beta}{s+1-j\sqrt{8}}$$

▶ Thus the form of system's response will be:

$$c(t) = K_1 + e^{-\sigma_d t}[2\alpha \cos\omega_d t + 2\beta \sin\omega_d t]$$
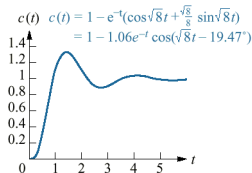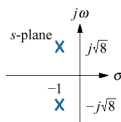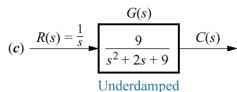
where $-\sigma_d \pm j\omega_d = -1 \pm j\sqrt{8}$.

Figure 4.7.

Underdamped Response - II

▶ For system with poles at $s = -\sigma_d \pm j\omega_d$, the real part ($\sigma_d$) determines the exponential frequency (decay rate) for the exponential envelope.

▶ The imaginary part, ω_d, determines the oscillation frequency of the sinusoids, and is called the damped frequency of oscillation.

▶ Can show that

$$e^{-\sigma_d t}[2\alpha \cos \omega_d t + 2\beta \sin \omega_d t]$$
$$= K_4 e^{-\sigma_d t} \cos(\omega_d t - \phi)$$

where $\phi = \tan^{-1}(\frac{\beta}{\alpha})$ and
$K_4 = \sqrt{(2\alpha)^2 + (2\beta)^2}$

$c(t)$

Exponential decay generated by real part of complex pole pair

Sinusoidal oscillation generated by imaginary part of complex pole pair

t
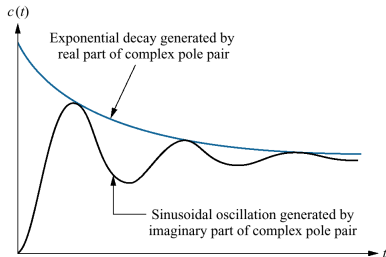
Figure 4.8

Undamped Response

- For undamped response, we have a system with two imaginary poles (zero real part).

- The unit step response to system below is

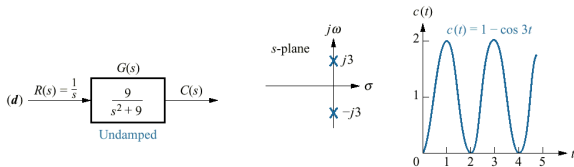$$C(s) = \frac{9}{s(s^2+9)} = \frac{9}{s(s+j3)(s-j3)} = \frac{K_1}{s} + \frac{\alpha+j\beta}{s+j3} + \frac{\alpha-j\beta}{s-j3}$$

- Thus the form of system's response will be:

$$c(t) = K_1 + e^{-(0)t}[2\alpha\,\cos\omega_d t + 2\beta\,\sin\omega_d t]$$
$$= K_1 + 2\alpha\,\cos\omega_d t + 2\beta\,\sin\omega_d t$$
$$= K_1 + K_4\cos(\omega_d t - \phi)$$

- where
  $\pm j\omega_d = \pm j3$.

Figure 4.7

Second-order System's Step Responses

▶ Critically damped case represents the transition between the underdamped and overdamped cases.

▶ Critically damped case is the fastest response without overshoot.
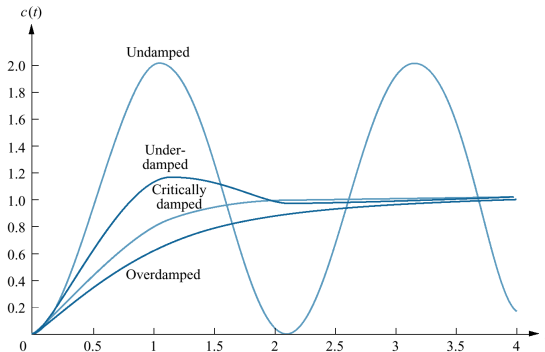
Figure 4.10.

General Second-Order Systems

► We now generalize our discussion of second-order systems and develop specifications to describe the response of the system.

1. The **natural frequency**, ω_n, of a second-order system is the frequency of oscillation of the system with damping removed.

2. The **damping ratio** of a second-order system is a way to describe a system's damped oscillation, independent of time scale.

   We define *damping ratio*, ζ, to be

   $$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency(rad/sec)}}$$

Deriving Parameters - ω_n

▶ We want to rewrite the second-order system shown below in terms of ω_n and ζ

$$G(s) = \frac{b}{s^2 + as + b} \tag{1}$$

▶ The quadratic equation tells us the poles are:

$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2} \tag{2}$$

▶ To determine ω_n, we need an undamped system; thus $a = 0$, $G(s) = \frac{b}{s^2 + b}$.

▶ Our poles are thus $s_{1,2} = \pm j\sqrt{b}$, giving $\omega_n = \sqrt{b}$ and $b = \omega_n^2$.

Deriving Parameters - ζ

$$G(s) = \frac{b}{s^2 + as + b} \qquad\qquad s_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}$$

► For an underdamped system, the poles must have a real part, $\sigma = \frac{-a}{2}$.

► The *exponential decay frequency* is equal to the absolute value of σ.

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency(rad/sec)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \qquad (3)$$

► We thus have:

$$a = 2\zeta\omega_n \qquad (4)$$

► We can now rewrite our system as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (5)$$

Relating Parameters to Poles

▶ Substituting into the poles equation gives:

$$s_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2} = \frac{-(2\zeta\omega_n)}{2} \pm \frac{\sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{6}$$

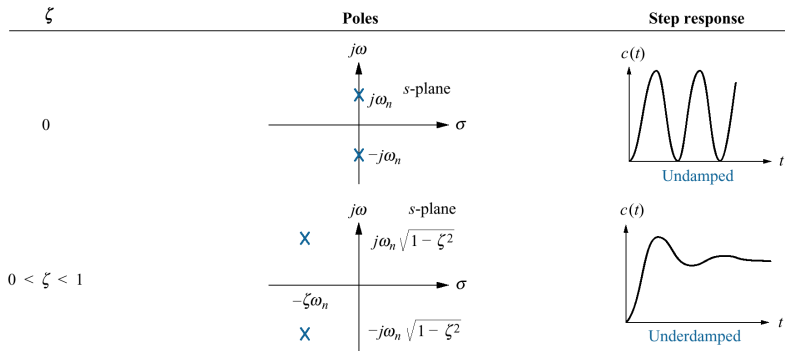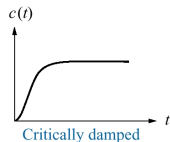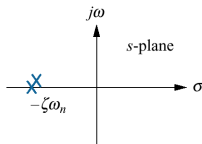| ζ	Poles	Step response
0	$j\omega_n$, $-j\omega_n$ on imaginary axis, s-plane	Undamped
$0 < \zeta < 1$	$-\zeta\omega_n$, $j\omega_n\sqrt{1-\zeta^2}$, $-j\omega_n\sqrt{1-\zeta^2}$, s-plane	Underdamped

Figure 4.11.

Relating Parameters to Poles - II

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
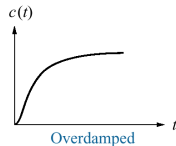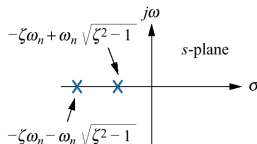

Figure 4.11.

Underdamped Second-order Systems

▶ We are mostly interested in underdamped systems as they give us the fastest response.

▶ Need to examine behavior more closely for analysis and design.

▶ Will now define transient specifications for underdamped responses.

▶ The step response is:

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7)$$

▶ Assuming $\zeta < 1$ (ie. underdamped case), partial fractions gives:

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \dfrac{\zeta}{\sqrt{1 - \zeta^2}}\omega_n\sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \quad (8)$$

Underdamped Second-order Systems - II

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

▶ We can now use the inverse Laplace transform below that we derived earlier

$$\mathcal{L}\{K_1 e^{-at}cos\omega t + K_2 e^{-at}sin\omega t\} = \frac{K_1(s + a) + K_2\omega}{(s + a)^2 + \omega^2}$$

▶ This gives:

$$c(t) = 1 - e^{-\zeta\omega_n t}(cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}sin\omega_n\sqrt{1-\zeta^2}\,t)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi) \qquad (9)$$

where $\phi = \tan^{-1}(\frac{\zeta}{\sqrt{1-\zeta^2}})$.

Underdamped Second-order Systems - III

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$$

where $\phi = \tan^{-1}(\frac{\zeta}{\sqrt{1-\zeta^2}})$.

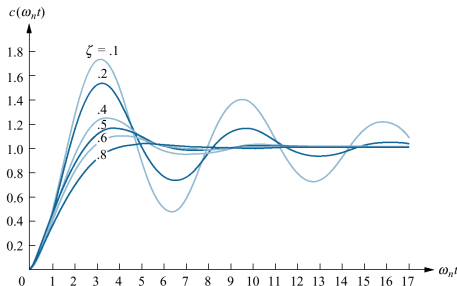▶ We can now plot the output with the time axis normalized to the natural frequency.

Figure 4.13.

Underdamped Response Specifications

Let $c_{final} = \lim_{t \to \infty} c(t)$.

1. **Rise time**, T_r, is the time for the output to go from 10% ($0.1c_{final}$) to 90% ($0.9c_{final}$) of its final value.
2. **Peak time**, T_p, is the time required to reach the first and largest peak, c_{max}.
3. **Percent overshoot**, **%OS**, is the percentage that the output overshoots the final value at $t = T_p$.

$$\text{\%OS} = \frac{c_{max} - c_{final}}{c_{final}} \times 100\%$$

4. **Settling time**, T_s, is time required for the output to reach and stay within $\pm 2\%$ of c_{final}.
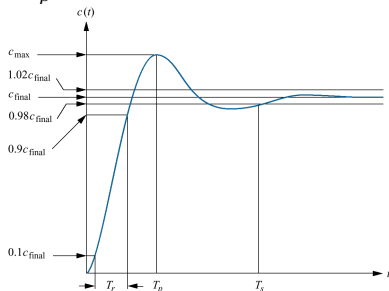

Figure 4.14

Calculating Peak time

▶ We can determine T_p by differentiating $c(t)$ in equation 9 and finding the first time it equals zero after $t = 0$.

$$\mathcal{L}\{\dot{c}(t)\} = sC(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{10}$$

▶ As we want in form of $\mathcal{L}\{sinwt\}$, we complete the square for the denominator giving

$$\mathcal{L}\{\dot{c}(t)\} = sC(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$= \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \tag{11}$$

▶ We thus have

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t) \tag{12}$$

Calculating Peak time - II

▶ Setting $\dot{c}(t) = 0$ gives

$$\omega_n \sqrt{1 - \zeta^2}\, t = n\pi$$

thus

$$t = \frac{n\pi}{\omega_n \sqrt{1 - \zeta^2}} \tag{13}$$

▶ At $n = 1$, we get first time derivative equals zero after $t = 0$. We thus have:

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \tag{14}$$

Calculating %OS

▶ Percent overshoot is: $\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100\%$

▶ To determine c_{max}, we need to evaluate $c(T_p)$ by substituting equation (14) into equation (9) reproduced below:

$$c(t) = 1 - e^{-\zeta \omega_n t}(cos\omega_n \sqrt{1 - \zeta^2}\,t + \frac{\zeta}{\sqrt{1 - \zeta^2}}sin\omega_n \sqrt{1 - \zeta^2}\,t)$$

$$c(T_p) = 1 - e^{-\zeta \pi/\sqrt{1-\zeta^2}}(cos\pi + \frac{\zeta}{\sqrt{1 - \zeta^2}}sin\pi)$$

$$= 1 + e^{-\zeta \pi/\sqrt{1-\zeta^2}} \qquad (15)$$

▶ From equation 9, it is easy to see that $c(\infty) = c_{final} = 1$.

▶ Substituting into the %OS formula gives:

$$\%OS = \frac{1 + e^{-\zeta \pi/\sqrt{1-\zeta^2}} - 1}{1} \times 100\%$$

$$\%OS = e^{-\zeta \pi/\sqrt{1-\zeta^2}} \times 100\% \qquad (16)$$

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Calculating %OS -II

$$\%\text{OS} = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

▶ However, what if we knew which %OS we wanted?

▶ We can use the equation above to solve for ζ in terms of %OS. This gives:

$$\zeta = \frac{-\ln(\%\text{OS}/100)}{\sqrt{\pi^2 + \ln^2(\%\text{OS}/100)}} \tag{17}$$
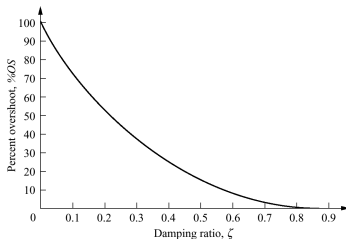

Figure 4.15.

Calculating 2% Settling Time

▶ Settling time is when the ouput reaches and stays within 2% of its final value.

▶ This occurs at latest when the exponential envelope of equation 9 reaches the value of 0.02. This gives:

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} = 0.02 \tag{18}$$

▶ Solving for t gives:

$$e^{-\zeta\omega_n t} = 0.02\sqrt{1-\zeta^2}$$

$$-\zeta\omega_n t = \ln(0.02\sqrt{1-\zeta^2})$$

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n} \tag{19}$$

$$T_s \approx \frac{4}{\zeta\omega_n} \tag{20}$$

Calculating Rise Time

▶ There does not exist a precise relationship between rise time and damping ratio.

▶ Instead, we use a computer and equation 9 to solve for $c(\omega_n t_1) = 0.1 c_{final}$ and $c(\omega_n t_2) = 0.9 c_{final}$, normalizing for the natural frequency.

▶ We then calculate the *normalized rize time*, $\omega_n T_r$, as
$\omega_n T_r = (\omega_n t_2) - (\omega_n t_1)$

▶ Then, we can use charts below to solve for T_r given a specific ζ and ω_n.

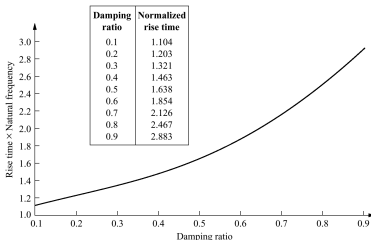| Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure 4.16.

System Response with Additional Poles

- ▶ In last section, we analyzed second-order systems.

- ▶ The formulas we have derived for percent overshoot, settling time, and peak time are only directly valid for systems with two complex poles and no zeros.

- ▶ However, sometimes we can approximate a higher-order system as a second-order system containing the dominant poles.

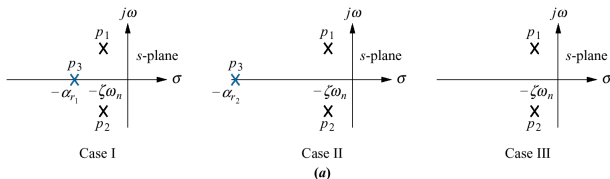- ▶ The dominant poles are the two poles farthest to the right.

Figure 4.23.

System Response with Additional Poles - II

▶ How far away do the additional poles have to be?.

▶ Depends on the accuracy you want.

▶ Text assumes that if a pole is five times more to the left than the dominant poles, then system is represented by the dominant poles.

▶ If above met, you would design using the second-order approximation, then simulate final system to make sure it satisfies the design specifications such as %OS, and T_s etc.

Example Three Pole Systems

▶ Compare step responses of systems below:
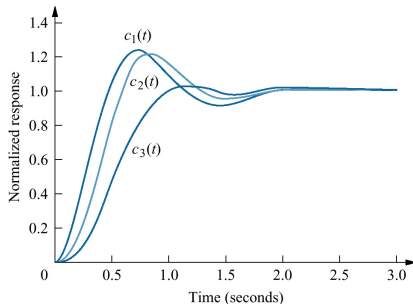
$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$

$$T_2(s) = \frac{10 \times 24.542}{(s+10)(s^2 + 4s + 24.542)}$$

$$T_3(s) = \frac{3 \times 24.542}{(s+3)(s^2 + 4s + 24.542)}$$

$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°)$

$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°)$

$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°)$

Figure 4.24.

Justification for Ignoring Nondominant Poles

▶ As long as the nondominant pole is far enough to the left, then its contribution to the output will be negligible.

▶ Easy to see that this will cause it to decay quickly, but what about its amplitude?

▶ Consider the third order system below

$$C(s) = \frac{bc}{s(s^2 + as + b)(s + c)} = \frac{A}{s} + \frac{Bs + C}{s^2 + as + b} + \frac{D}{s + c}$$

▶ If we assume steady state response is unity, and that the nondominant pole is at $s = -c$, we can then solve for the following constants using partial fractions:

$$A = 1; \qquad\qquad B = \frac{ca - c^2}{c^2 + b - ca}$$

$$C = \frac{ca^2 - c^2a - bc}{c^2 + b - ca}; \qquad D = \frac{-b}{c^2 + b - ca} \qquad (21)$$

Justification for Ignoring Nondominant Poles - II

$$A = 1; \qquad\qquad B = \frac{ca - c^2}{c^2 + b - ca}$$

$$C = \frac{ca^2 - c^2 a - bc}{c^2 + b - ca}; \qquad\qquad D = \frac{-b}{c^2 + b - ca}$$

► If we let $c \to \infty$, we find:

$$A = 1; \qquad\qquad B = -1$$

$$C = -a; \qquad\qquad D = 0. \qquad (22)$$

System Response with Zeros

▶ We now examine systems with Zeros.

▶ As we saw before, zeros don't change the type of system, but can affect the constants found during partial fraction expansion.

▶ Consider the system below

$$G(s) = \frac{(s+a)}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

where by partial fractions we have:

$$A = \frac{-b+a}{-b+c}; \qquad\qquad B = \frac{-c+a}{-c+b} \qquad (23)$$

System Response with Zeros -II

$$A = \frac{-b+a}{-b+c}; \qquad\qquad B = \frac{-c+a}{-c+b}$$

▶ When the zero is far to the left, it will be much larger than the poles, thus:

$$A \approx \frac{a}{-b+c}; \qquad\qquad B \approx \frac{a}{-c+b}$$

▶ Our system then becomes

$$G(s) \approx a \left[\frac{\frac{1}{-b+c}}{s+b} + \frac{\frac{1}{-c+b}}{s+c} \right] = \frac{a}{(s+b)(s+c)}$$

Nonminium-Phase System

▶ What if the zero is in the right half plane (ie. $a < 0$)?

▶ If $C(s)$ is the response of a system, then after adding zero at $-a$, we get:
$$(s + a)C(s) = sC(s) + aC(s)$$

▶ If the derivative term $sC(s)$ is larger than the scaled response $aC(s)$, the system will initially follow the derivative in the wrong direction!

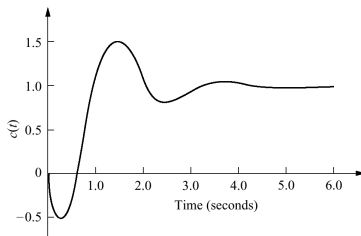▶ If we take $r(t) = -u(t)$ as our input, we could get a system like the one below.

Figure 4.26.

Pole Zero Cancellations

► Consider the system below:

$$G(s) = \frac{K(s + z_1)}{(s + p_1)(s^2 + as + b)} \quad (24)$$

► If z_1 and p_1 are close enough to each other, they can effectively cancel each other even though they are not exactly equal.

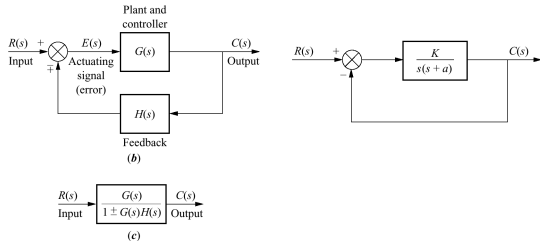$$C(s) = \frac{26.25(s + 4)}{s(s + 4.01)(s + 5)(s + 6)} \quad (25)$$

$$= \frac{0.87}{s} - \frac{5.3}{s + 5} + \frac{4.4}{s + 6} + \frac{-0.033}{s + 4.01} \quad (26)$$

$$\approx \frac{0.87}{s} - \frac{5.3}{s + 5} + \frac{4.4}{s + 6} \quad (27)$$

Analysis and Design of Feedback Systems

▶ Using the block diagram algebra we developed earlier, we can now apply our second-order system results to feedback systems.

▶ Applying feedback reduction, we find that the equivalent closed-loop transfer function of system on right is

$$T(s) = \frac{K}{s^2 + as + K} \tag{28}$$

Figures 5.6 and 4.14.

Analysis and Design of Feedback Systems - II

$$T(s) = \frac{K}{s^2 + as + K}$$

▶ As we increase K from zero, the poles of the system will go from overdamped $(0 \le K < \frac{a^2}{4})$, critically damped $(K = \frac{a^2}{4})$, to underdamped $(K > \frac{a^2}{4})$.

$$s_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2} \quad s_{1,2} = \frac{-a}{2} \quad s_{1,2} = \frac{-a}{2} \pm \frac{j\sqrt{4K - a^2}}{2}$$

Gain Design for Transient Response

▶ Design the value for system gain, K, such that the system response has 10% overshoot.

▶ Applying feedback reduction, our closed-loop transfer function becomes

$$T(s) = \frac{K}{s^2 + 5s + K} \qquad (29)$$

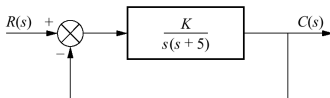$$\frac{R(s)}{+} \bigotimes_{-} \longrightarrow \boxed{\frac{K}{s(s+5)}} \xrightarrow{C(s)}$$

Figure 5.16.

Matlab solution to the problem

```
1  a=5
2  G=zpk([],[0 -a],1)
3  pos=10
4  zeta=-log(pos/100)/sqrt(pi^2+log(pos/100)^2)
5  K=a^2/(4*zeta^2)
6  Wn=sqrt(K)
7
8  rlocus(G)
9  sgrid(zeta,Wn)
10 pause
11
12 Gcl=K*G/(1+K*G)
13 step(Gcl)
14 [y,t]=step(Gcl);
15 max(y)
```