ENGPHYS 2A04 Winter 2022 – Assignment 4 Solutions

DUE MONDAY FEBRUARY 14th, 8AM

1. Consider a coaxial air line with inner diameter 13 mm, and outer diameter 15 mm. Both conductors are made of copper (refer to your textbook for material constants).
   a. Compute the transmission line parameters for a signal at 10 kHz.
   Solution:
   Copper: $\mu_{Cu} = 1\mu_0 = 4\pi \times 10^{-7}\frac{H}{m}, \sigma_{Cu} = 5.8 \times 10^7 \frac{S}{m}$

$$a = \frac{13mm}{2} = 6.5mm, b = \frac{15mm}{2} = 7.5mm$$

$$R_S = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \sqrt{\frac{\pi(10000)(4\pi \times 10^{-7})}{(5.8 \times 10^7)}} = 2.609 \times 10^{-5}$$

$$\boldsymbol{R' = \frac{R_S}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{2.609\times10^{-5}}{2\pi}\left(\frac{1}{6.5\times10^{-3}}+\frac{1}{7.5\times10^{-3}}\right) = 1.19 \times 10^{-3}\ \Omega/m}$$

$$\boldsymbol{L' = \frac{\mu}{2\pi}ln\left(\frac{b}{a}\right) = \frac{\mu_0}{2\pi}ln\left(\frac{b}{a}\right) = \frac{4\pi\times10^{-7}}{2\pi}ln\left(\frac{7.5}{6.5}\right) = 2.86 \times 10^{-8}\ H/m}$$

$\boldsymbol{G' = \frac{2\pi\sigma}{ln\left(b/a\right)} = \frac{2\pi(0)}{ln\left(7.5/6.5\right)} = 0\ S/m}$, *because the insulator is air which has no conductance.*

$$\boldsymbol{C' = \frac{2\pi\epsilon}{ln\left(b/a\right)} = \frac{2\pi(8.854\times10^{-12})}{ln\left(7.5/6.5\right)} = 3.89 \times 10^{-10}\ F/m}$$

   b. Compute the attenuation constant and the phase constant of the transmission line at the specified operating frequency.

   Solution:

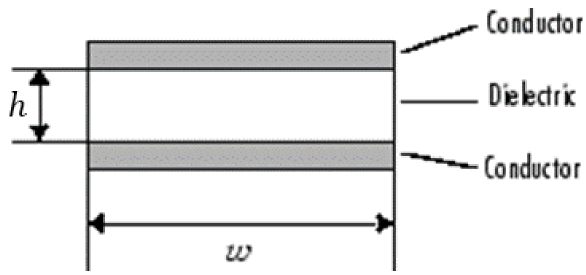
$$\omega = 2\pi f = 2\pi(10000) = 20000\pi$$

$$\begin{aligned}
\gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= \sqrt{\left(1.19 \times 10^{-3} + j(20000\pi)(2.86 \times 10^{-8})\right)\left(0 + j(20000\pi)(3.89 \times 10^{-10})\right)} \\
&= \sqrt{(1.19 \times 10^{-3} + j1.80 \times 10^{-3})(j2.44 \times 10^{-5})} \\
&= \sqrt{-4.392 \times 10^{-8} + j2.904 \times 10^{-8}} = \sqrt{5.265 \times 10^{-8}e^{j2.557}} \\
&= (5.265 \times 10^{-8})^{0.5}e^{j2.557^{0.5}} = 2.295 \times 10^{-4}e^{j1.279} = 6.6 \times 10^{-5} + j2.2 \times 10^{-4}
\end{aligned}$$

$$\boldsymbol{\alpha = \Re e\{\gamma\} = 6.6 \times 10^{-5}\ Np/m}$$

$$\boldsymbol{\beta = \mathfrak{Im}\{\gamma\} = 2.2 \times 10^{-4}\ rad/m}$$

2. Calculate the phase velocity of a $3kHz$ signal travelling on a parallel-plate transmission line. Assume the conductors are gold, and assume the dielectric is air. The conductors are $5mm$ wide, separated by a $2mm$ dielectric. Does this value make sense?



Solution:

Because of the resistances of the conductors and the dielectric, $G' = 0$. This results in:

$$\beta = \Im m\left\{\sqrt{(R' + j\omega L')(j\omega C')}\right\} = \Im m\left\{\sqrt{(j\omega R' - \omega^2 L')(C')}\right\} = \Im m\left\{\sqrt{(j\omega R'C' - \omega^2 L'C')}\right\}$$

From $L'C' = \mu\epsilon$:

$$\beta = \Im m\left\{\sqrt{j\omega R'C' - \omega^2\mu\epsilon}\right\}$$

Substituting in expressions for R' and C':

$$\beta = \Im m\left\{\sqrt{j\omega\frac{2R_S}{w}\frac{\epsilon w}{h} - \omega^2\mu\epsilon}\right\} = \Im m\left\{\sqrt{j\omega\frac{2R_S\epsilon}{h} - \omega^2\mu\epsilon}\right\} = \Im m\left\{\sqrt{j\omega\frac{2\sqrt{\pi f\mu_c/\sigma_c}\epsilon}{h} - \omega^2\mu\epsilon}\right\}$$

Inputting the known values:

$$\beta =$$

$$= \Im m\left\{\sqrt{j\left(2\pi(3\times10^3)\right)\frac{2\sqrt{\pi(3\times10^3)(4\pi\times10^{-7})/(4.1\times10^7)}8.854\times10^{-12}}{2\times10^{-3}} - \left(2\pi(3\times10^3)\right)^2\mu_0\epsilon_0}\right\}$$

$$= \Im m\left\{\sqrt{j(18849.6)\frac{2(1.6996\times10^{-5})8.854\times10^{-12}}{2\times10^{-3}} - (3.553\times10^8)(4\pi\times10^{-7})(8.854\times10^{-12})}\right\}$$

$$= \Im m\left\{\sqrt{j2.837\times10^{-9} - 3.953\times10^{-9}}\right\} = \Im m\left\{\sqrt{10^{-9}(-3.953 + j2.837)}\right\}$$

$$= \Im m\left\{\sqrt{(4.8657\times10^{-9})e^{j2.5191}}\right\} = \Im m\{6.975\times10^{-5}e^{j1.2596}\}$$

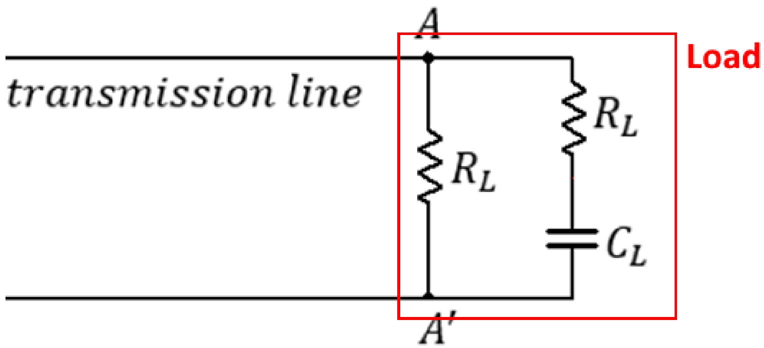
$$= \Im m\{2.136\times10^{-5} + j6.64\times10^{-5}\} = 6.64\times10^{-5}\ rad/s$$

Calculating phase velocity based on this:

$$\boldsymbol{u_P = \frac{\omega}{\beta} = \frac{2\pi f}{6.64\times10^{-5}} = \frac{2\pi(3000)}{6.64\times10^{-5}} = 2.84\times10^8\ m/s}$$

Yes, it makes sense because it is slightly less than the speed of light in a vacuum.

3. The following load is placed at the end of a 120Ω transmission line carrying a 2MHz signal, with $R_L = 80\Omega$ and $C_L = 11nF$:



Calculate the voltage reflection coefficient at the load, and report it in polar form. Explain the meaning of this coefficient.

Solution:

Calculate the impedance of the load.

$$Z_{load} = \left(\frac{1}{R_L} + \frac{1}{R_L + Z_C}\right)^{-1} = \left(\frac{1}{R_L} + \frac{1}{R_L - \frac{j}{\omega C}}\right)^{-1} = \left(\frac{1}{R_L} + \frac{1}{R_L - \frac{j}{2\pi f C}}\right)^{-1}$$

$$= \left(\frac{1}{80} + \frac{1}{80 - \frac{j}{2\pi(2\times 10^6)(11\times 10^{-9})}}\right)^{-1} = \left(0.0125 + \frac{1}{80 - j7.234}\right)^{-1}$$

$$= \left(0.0125 + (0.01240 + j1.121\times 10^{-3})\right)^{-1} = (0.0249 + j1.121\times 10^{-3})^{-1}$$

$$= 40.1 - j1.805 = 40.1e^{-j0.0450}$$

Next, compute the normalized load impedance.

$$z_L = \frac{Z_{Load}}{Z_0} = \frac{40.1e^{-j0.0450}}{120} = 0.334e^{-j0.0450} = 0.334 - j0.0150\ \Omega$$

From here, the voltage reflection coefficient can be computed.

$$\boldsymbol{\Gamma = \frac{z_L - 1}{z_L + 1} = \frac{0.334 - j0.0150 - 1}{0.334 - j0.0150 + 1} = \frac{-0.666 - j0.0150}{1.334 - j0.0150} = 0.5e^{-j3.1}}$$

This coefficient gives the ratio of amplitudes of the reflected and incident voltage waves.

4. A 10-metre section of a 100Ω lossless transmission line is driven by a source with $v_g(t) = 12\cos\left(2\pi\times 10^6 t - \frac{\pi}{3}\right)\ (V)$, and $Z_g = 150\Omega$. The line has relative permittivity $\epsilon_r = 2.1$, and is terminated by a load with impedance $Z_L = (120 - j40)\Omega$. Express complex values in polar form. Determine:
   a. λ on the line.
   b. The reflection coefficient at the load.
   c. The input impedance.
   d. The input voltage $\widetilde{V}_i$.
   e. The time-domain input voltage $v_i(t)$.

Solution:

a) *Lossless, so* $u_P = \frac{c}{\sqrt{\epsilon_r}} = \frac{3\times10^8}{\sqrt{2.1}} = 2.07 \times 10^8\ m/s$*. Then,*

$$\boldsymbol{\lambda = \frac{2\pi u_P}{\omega} = \frac{2\pi\left(2.07 \times 10^8\right)}{2\pi \times 10^6} = 207\ m}$$

b) *The reflection coefficient is given by:*

$$\boldsymbol{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(120 - j40) - 100}{(120 - j40) + 100} = \frac{20 - j40}{220 - j40} = 0.12 - j0.16 = 0.2e^{-j0.927}\ (\Omega)}$$

c) *First need to find* βl*:*

$$\beta = \frac{\omega}{u_P} = \frac{2\pi \times 10^6}{2.07 \times 10^8} = 0.00966\pi\ (\frac{rad}{m})$$

$$\beta l = 0.03035(10) = 0.0966\pi\ (rad)$$

$$\boldsymbol{Z_i = Z_0\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} = 100\frac{(120-j40)+j100\tan 0.3035}{100+j(120-j40)\tan 0.3035} = 93.62 - j38.98 = 101.4e^{-j0.395}\ \Omega}$$

d) *Now that input impedance is found:*

$$\boldsymbol{\widetilde{V_i} = \frac{\widetilde{V_g}Z_i}{Z_g + Z_i} = \frac{12e^{-j\frac{\pi}{3}}101.4e^{-j0.395}}{150 + 101.4e^{-j0.395}} = \frac{1216.8e^{-j1.442}}{246.7e^{-j0.1588}} = 4.9e^{-j1.28}}$$

e) *Converting the above to time domain:*

$$\boldsymbol{v_i(t) = \Re e\{\widetilde{V_i}e^{j\omega t}\} = \Re e\{4.9e^{-j1.28}e^{j\omega t}\} = \Re e\{4.9e^{j(\omega t - 1.28)}\}}$$
$$\boldsymbol{= 4.9\cos\left(2\pi \times 10^6 t - 1.28\right)}$$

5. Using McMaster's library catalogue, Google Scholar, or some other resource, find an academic paper that discusses an application of transmission lines. In fewer than 5 sentences, summarize the paper's topic, and **explain how it is related to the transmission line theory covered in class.** Cite your source using IEEE, APA, or some comparable format.

6. BONUS: In addition to not dissipating power, a lossless line has two important features: (1) it is dispersionless (u_P independent of frequency), and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties to satisfy the condition

$$R'C' = L'G' \qquad (distortionless\ line)$$

Such a line is called a *distortionless* line, because despite the fact that it is not lossless, it nonetheless possesses the previously mentioned features of the lossless line. Show that for a distortionless line:

$$\alpha = R'\sqrt{\frac{C'}{L'}} = \sqrt{R'G'}$$
$$\beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

Solution: Using the distortionless condition in Eq. (2.22) gives

$$\begin{aligned}
\gamma = \alpha + j\beta &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{G'}{C'} + j\omega\right)} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{R'}{L'} + j\omega\right)} \\
&= \sqrt{L'C'}\left(\frac{R'}{L'} + j\omega\right) = R'\sqrt{\frac{C'}{L'}} + j\omega\sqrt{L'C'}.
\end{aligned}$$

Hence,

$$\alpha = \Re\mathfrak{e}(\gamma) = R'\sqrt{\frac{C'}{L'}}, \qquad \beta = \Im\mathfrak{m}(\gamma) = \omega\sqrt{L'C'}, \qquad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}.$$

Similarly, using the distortionless condition in Eq. (2.29) gives

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\sqrt{\frac{R'/L' + j\omega}{G'/C' + j\omega}} = \sqrt{\frac{L'}{C'}}.$$

ASSIGNMENT SUBMISSION INSTRUCTIONS

- Each question is worth equal points, except for bonus questions.
- Show all your work for full marks.
- Clearly label your name and student number at the top of the first page of your assignment.
- All assignments should be submitted in pdf format to the assignments drop box on Avenue to Learn.
- No late assignments will be accepted. A grade of 0% will be given for late assignments. If you have completed part of the assignment, submit the portion you have completed before the deadline for partial marks.