Problem 2.2 Calculate the line parameters R', L', G', and C' for a coaxial line with an inner conductor diameter of 0.5 cm and an outer conductor diameter of 1 cm, filled with an insulating material where $\mu = \mu_0$, $\varepsilon_r = 4.5$, and $\sigma = 10^{-3}$ S/m. The conductors are made of copper with $\mu_c = \mu_0$ and $\sigma_c = 5.8 \times 10^7$ S/m. The operating frequency is 1 GHz.

Solution: Given

$$a = (0.5/2) \text{ cm} = 0.25 \times 10^{-2} \text{ m},$$
$$b = (1.0/2) \text{ cm} = 0.50 \times 10^{-2} \text{ m},$$

combining Eqs. (2.5) and (2.6) gives

$$\begin{aligned} R' &= \frac{1}{2\pi}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}\left(\frac{1}{a}+\frac{1}{b}\right) \\ &= \frac{1}{2\pi}\sqrt{\frac{\pi(10^9 \text{ Hz})(4\pi\times 10^{-7} \text{ H/m})}{5.8\times 10^7 \text{ S/m}}}\left(\frac{1}{0.25\times 10^{-2} \text{ m}}+\frac{1}{0.50\times 10^{-2} \text{ m}}\right) \\ &= 0.788\ \Omega/\text{m}. \end{aligned}$$

From Eq. (2.7),

$$L' = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi\times 10^{-7} \text{ H/m}}{2\pi}\ln 2 = 139 \text{ nH/m}.$$

From Eq. (2.8),

$$G' = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi\times 10^{-3} \text{ S/m}}{\ln 2} = 9.1 \text{ mS/m}.$$

From Eq. (2.9),

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi\varepsilon_r\varepsilon_0}{\ln(b/a)} = \frac{2\pi\times 4.5\times(8.854\times 10^{-12} \text{ F/m})}{\ln 2} = 362 \text{ pF/m}.$$

Problem 2.3 A 1-GHz parallel-plate transmission line consists of 1.2-cm-wide copper strips separated by a 0.15-cm-thick layer of polystyrene. Appendix B gives $\mu_c = \mu_0 = 4\pi\times 10^{-7}$ (H/m) and $\sigma_c = 5.8\times 10^7$ (S/m) for copper, and $\varepsilon_r = 2.6$ for polystyrene. Use Table 2-1 to determine the line parameters of the transmission line. Assume $\mu = \mu_0$ and $\sigma \simeq 0$ for polystyrene.

Solution:

$$\begin{aligned} R' &= \frac{2R_s}{w} = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{2}{1.2\times 10^{-2}}\left(\frac{\pi\times 10^9\times 4\pi\times 10^{-7}}{5.8\times 10^7}\right)^{1/2} = 1.38 \quad (\Omega/\text{m}), \\ L' &= \frac{\mu d}{w} = \frac{4\pi\times 10^{-7}\times 1.5\times 10^{-3}}{1.2\times 10^{-2}} = 1.57\times 10^{-7} \quad (\text{H/m}), \\ G' &= 0 \qquad \text{because } \sigma = 0, \\ C' &= \frac{\varepsilon w}{d} = \varepsilon_0\varepsilon_r\frac{w}{d} = \frac{10^{-9}}{36\pi}\times 2.6\times\frac{1.2\times 10^{-2}}{1.5\times 10^{-3}} = 1.84\times 10^{-10} \quad (\text{F/m}). \end{aligned}$$

Section 2-5: The Lossless Line

Problem 2.6 In addition to not dissipating power, a lossless line has two important features: (1) it is dispertionless (μ_p is independent of frequency) and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties so as to satisfy the condition

$$R'C' = L'G' \quad \text{(distortionless line).}$$

Such a line is called a *distortionless* line because despite the fact that it is not lossless, it does nonetheless possess the previously mentioned features of the loss line. Show that for a distortionless line,

$$\alpha = R'\sqrt{\frac{C'}{L'}} = \sqrt{R'G'}, \qquad \beta = \omega\sqrt{L'C'}, \qquad Z_0 = \sqrt{\frac{L'}{C'}}.$$

Solution: Using the distortionless condition in Eq. (2.22) gives

$$\begin{aligned}
\gamma = \alpha + j\beta &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{G'}{C'} + j\omega\right)} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{R'}{L'} + j\omega\right)} \\
&= \sqrt{L'C'}\left(\frac{R'}{L'} + j\omega\right) = R'\sqrt{\frac{C'}{L'}} + j\omega\sqrt{L'C'}.
\end{aligned}$$

Hence,

$$\alpha = \mathfrak{Re}(\gamma) = R'\sqrt{\frac{C'}{L'}}, \qquad \beta = \mathfrak{Im}(\gamma) = \omega\sqrt{L'C'}, \qquad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}.$$

Similarly, using the distortionless condition in Eq. (2.29) gives

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\sqrt{\frac{R'/L' + j\omega}{G'/C' + j\omega}} = \sqrt{\frac{L'}{C'}}.$$

Problem 2.10 Using a slotted line, the voltage on a lossless transmission line was found to have a maximum magnitude of 1.5 V and a minimum magnitude of 0.6 V. Find the magnitude of the load's reflection coefficient.

Solution: From the definition of the Standing Wave Ratio given by Eq. (2.59),

$$S = \frac{|\widetilde{V}|_{\max}}{|\widetilde{V}|_{\min}} = \frac{1.5}{0.6} = 2.5.$$

Solving for the magnitude of the reflection coefficient in terms of S, as in Example 2-4,

$$|\Gamma| = \frac{S-1}{S+1} = \frac{2.5-1}{2.5+1} = 0.43.$$

Problem 2.15 A load with impedance $Z_{\mathrm{L}} = (25 - j50)\ \Omega$ is to be connected to a lossless transmission line with characteristic impedance Z_0, with Z_0 chosen such that the standing-wave ratio is the smallest possible. What should Z_0 be?

Solution: Since S is monotonic with $|\Gamma|$ (i.e., a plot of S *vs.* $|\Gamma|$ is always increasing), the value of Z_0 which gives the minimum possible S also gives the minimum possible $|\Gamma|$, and, for that matter, the minimum possible $|\Gamma|^2$. A necessary condition for a minimum is that its derivative be equal to zero:

$$\begin{aligned} 0 = \frac{\partial}{\partial Z_0}|\Gamma|^2 &= \frac{\partial}{\partial Z_0}\frac{|R_{\mathrm{L}} + jX_{\mathrm{L}} - Z_0|^2}{|R_{\mathrm{L}} + jX_{\mathrm{L}} + Z_0|^2} \\ &= \frac{\partial}{\partial Z_0}\frac{(R_{\mathrm{L}} - Z_0)^2 + X_{\mathrm{L}}^2}{(R_{\mathrm{L}} + Z_0)^2 + X_{\mathrm{L}}^2} = \frac{4R_{\mathrm{L}}(Z_0^2 - (R_{\mathrm{L}}^2 + X_{\mathrm{L}}^2))}{((R_{\mathrm{L}} + Z_0)^2 + X_{\mathrm{L}}^2)^2}. \end{aligned}$$

Therefore, $Z_0^2 = R_{\mathrm{L}}^2 + X_{\mathrm{L}}^2$ or

$$Z_0 = |Z_{\mathrm{L}}| = \sqrt{(25^2 + (-50)^2)} = 55.9\ \Omega.$$

A mathematically precise solution will also demonstrate that this point is a minimum (by calculating the second derivative, for example). Since the endpoints of the range may be local minima or maxima without the derivative being zero there, the endpoints (namely $Z_0 = 0\ \Omega$ and $Z_0 = \infty\ \Omega$) should be checked also.

Problem 2.19 Show that the input impedance of a quarter-wavelength long lossless line terminated in a short circuit appears as an open circuit.

Solution:

$$Z_{\text{in}} = Z_0\left(\frac{Z_{\mathrm{L}} + jZ_0 \tan\beta l}{Z_0 + jZ_{\mathrm{L}} \tan\beta l}\right).$$

For $l = \frac{\lambda}{4}$, $\beta l = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2}$. With $Z_{\mathrm{L}} = 0$, we have

$$Z_{\text{in}} = Z_0\left(\frac{jZ_0 \tan\pi/2}{Z_0}\right) = j\infty \quad \text{(open circuit)}.$$

Problem 2.23 Two half-wave dipole antennas, each with impedance of 75 Ω, are connected in parallel through a pair of transmission lines, and the combination is connected to a feed transmission line, as shown in Fig. 2.39 (P2.23(a)). All lines are 50 Ω and lossless.

(a) Calculate Z_{in_1}, the input impedance of the antenna-terminated line, at the parallel juncture.

(b) Combine Z_{in_1} and Z_{in_2} in parallel to obtain Z'_L, the effective load impedance of the feedline.

(c) Calculate Z_{in} of the feedline.

Solution:

(a)

$$Z_{\text{in}_1} = Z_0 \left[\frac{Z_{L_1} + jZ_0 \tan \beta l_1}{Z_0 + jZ_{L_1} \tan \beta l_1} \right]$$

$$= 50 \left\{ \frac{75 + j50 \tan[(2\pi/\lambda)(0.2\lambda)]}{50 + j75 \tan[(2\pi/\lambda)(0.2\lambda)]} \right\} = (35.20 - j8.62)\ \Omega.$$

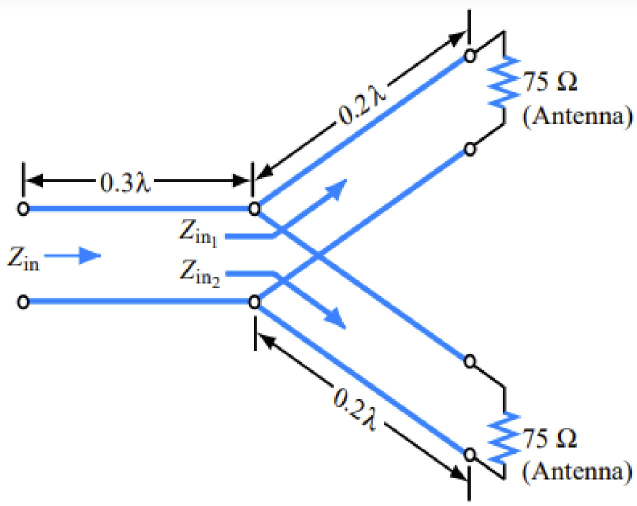


Figure P2.23: (a) Circuit for Problem 2.23.

(b)

$$Z'_L = \frac{Z_{\text{in}_1} Z_{\text{in}_2}}{Z_{\text{in}_1} + Z_{\text{in}_2}} = \frac{(35.20 - j8.62)^2}{2(35.20 - j8.62)} = (17.60 - j4.31)\ \Omega.$$

(c)

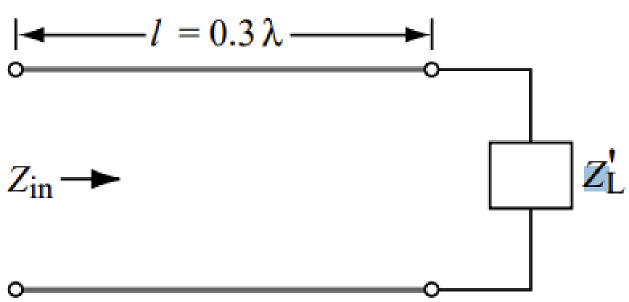


Figure P2.23: (b) Equivalent circuit.

$$Z_{\text{in}} = 50 \left\{ \frac{(17.60 - j4.31) + j50 \tan[(2\pi/\lambda)(0.3\lambda)]}{50 + j(17.60 - j4.31) \tan[(2\pi/\lambda)(0.3\lambda)]} \right\} = (107.57 - j56.7)\ \Omega.$$

Section 2-7: Special Cases

Problem 2.24 At an operating frequency of 300 MHz, it is desired to use a section of a lossless 50-Ω transmission line terminated in a short circuit to construct an equivalent load with reactance $X = 40\ \Omega$. If the phase velocity of the line is $0.75c$, what is the shortest possible line length that would exhibit the desired reactance at its input?

Solution:

$$\beta = \omega/u_{\mathrm{p}} = \frac{(2\pi \text{ rad/cycle}) \times (300 \times 10^6 \text{ cycle/s})}{0.75 \times (3 \times 10^8 \text{ m/s})} = 8.38 \text{ rad/m}.$$

On a lossless short-circuited transmission line, the input impedance is always purely imaginary; i.e., $Z_{\mathrm{in}}^{\mathrm{sc}} = jX_{\mathrm{in}}^{\mathrm{sc}}$. Solving Eq. (2.68) for the line length,

$$l = \frac{1}{\beta}\tan^{-1}\left(\frac{X_{\mathrm{in}}^{\mathrm{sc}}}{Z_0}\right) = \frac{1}{8.38 \text{ rad/m}}\tan^{-1}\left(\frac{40\ \Omega}{50\ \Omega}\right) = \frac{(0.675 + n\pi) \text{ rad}}{8.38 \text{ rad/m}},$$

for which the smallest positive solution is 8.05 cm (with $n = 0$).

Section 2-8: Power Flow on Lossless Line

Problem 2.31 A generator with $\widetilde{V}_{\mathrm{g}} = 300$ V and $Z_{\mathrm{g}} = 50\ \Omega$ is connected to a load $Z_{\mathrm{L}} = 75\ \Omega$ through a 50-Ω lossless line of length $l = 0.15\lambda$.

(a) Compute Z_{in}, the input impedance of the line at the generator end.

(b) Compute $\widetilde{I}_{\mathrm{i}}$ and $\widetilde{V}_{\mathrm{i}}$.

(c) Compute the time-average power delivered to the line, $P_{\mathrm{in}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{V}_{\mathrm{i}}\widetilde{I}_{\mathrm{i}}^*]$.

(d) Compute $\widetilde{V}_{\mathrm{L}}$, $\widetilde{I}_{\mathrm{L}}$, and the time-average power delivered to the load, $P_{\mathrm{L}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{V}_{\mathrm{L}}\widetilde{I}_{\mathrm{L}}^*]$. How does P_{in} compare to P_{L}? Explain.

(e) Compute the time average power delivered by the generator, P_{g}, and the time average power dissipated in Z_{g}. Is conservation of power satisfied?

Solution:

(a)

$$\beta l = \frac{2\pi}{\lambda} \times 0.15\lambda = 54°,$$

$$Z_{\mathrm{in}} = Z_0\left[\frac{Z_{\mathrm{L}} + jZ_0\tan\beta l}{Z_0 + jZ_{\mathrm{L}}\tan\beta l}\right] = 50\left[\frac{75 + j50\tan 54°}{50 + j75\tan 54°}\right] = (41.25 - j16.35)\ \Omega.$$

(b)

$$\widetilde{I}_{\mathrm{i}} = \frac{\widetilde{V}_{\mathrm{g}}}{Z_{\mathrm{g}} + Z_{\mathrm{in}}} = \frac{300}{50 + (41.25 - j16.35)} = 3.24e^{j10.16°} \quad \text{(A)},$$

$$\widetilde{V}_{\mathrm{i}} = \widetilde{I}_{\mathrm{i}} Z_{\mathrm{in}} = 3.24e^{j10.16°}(41.25 - j16.35) = 143.6e^{-j11.46°} \quad \text{(V)}.$$

(c)

$$P_{\mathrm{in}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{V}_{\mathrm{i}}\widetilde{I}_{\mathrm{i}}^{*}] = \frac{1}{2}\Re\mathfrak{e}[143.6\,e^{-j11.46^\circ} \times 3.24\,e^{-j10.16^\circ}]$$
$$= \frac{143.6 \times 3.24}{2}\cos(21.62^\circ) = 216 \quad \text{(W)}.$$

(d)

$$\Gamma = \frac{Z_{\mathrm{L}} - Z_0}{Z_{\mathrm{L}} + Z_0} = \frac{75-50}{75+50} = 0.2,$$
$$V_0^+ = \widetilde{V}_{\mathrm{i}}\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) = \frac{143.6\,e^{-j11.46^\circ}}{e^{j54^\circ} + 0.2\,e^{-j54^\circ}} = 150e^{-j54^\circ} \quad \text{(V)},$$
$$\widetilde{V}_{\mathrm{L}} = V_0^+(1+\Gamma) = 150e^{-j54^\circ}(1+0.2) = 180e^{-j54^\circ} \quad \text{(V)},$$
$$\widetilde{I}_{\mathrm{L}} = \frac{V_0^+}{Z_0}(1-\Gamma) = \frac{150e^{-j54^\circ}}{50}(1-0.2) = 2.4\,e^{-j54^\circ} \quad \text{(A)},$$
$$P_{\mathrm{L}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{V}_{\mathrm{L}}\widetilde{I}_{\mathrm{L}}^{*}] = \frac{1}{2}\Re\mathfrak{e}[180e^{-j54^\circ} \times 2.4\,e^{j54^\circ}] = 216 \quad \text{(W)}.$$

$P_{\mathrm{L}} = P_{\mathrm{in}}$, which is as expected because the line is lossless; power input to the line ends up in the load.

(e)

Power delivered by generator:

$$P_{\mathrm{g}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{V}_{\mathrm{g}}\widetilde{I}_{\mathrm{i}}] = \frac{1}{2}\Re\mathfrak{e}[300 \times 3.24\,e^{j10.16^\circ}] = 486\cos(10.16^\circ) = 478.4 \quad \text{(W)}.$$

Power dissipated in Z_{g}:

$$P_{Z_{\mathrm{g}}} = \frac{1}{2}\Re\mathfrak{e}[\widetilde{I}_{\mathrm{i}}\widetilde{V}_{Z_{\mathrm{g}}}] = \frac{1}{2}\Re\mathfrak{e}[\widetilde{I}_{\mathrm{i}}\widetilde{I}_{\mathrm{i}}^{*}Z_{\mathrm{g}}] = \frac{1}{2}|\widetilde{I}_{\mathrm{i}}|^2 Z_{\mathrm{g}} = \frac{1}{2}(3.24)^2 \times 50 = 262.4 \quad \text{(W)}.$$

Note 1: $P_{\mathrm{g}} = P_{Z_{\mathrm{g}}} + P_{\mathrm{in}} = 478.4$ W.

Problem 3.22 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2,$

(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0,$

(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2,$

(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi,$

(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outlines of each of the surfaces.

Solution:

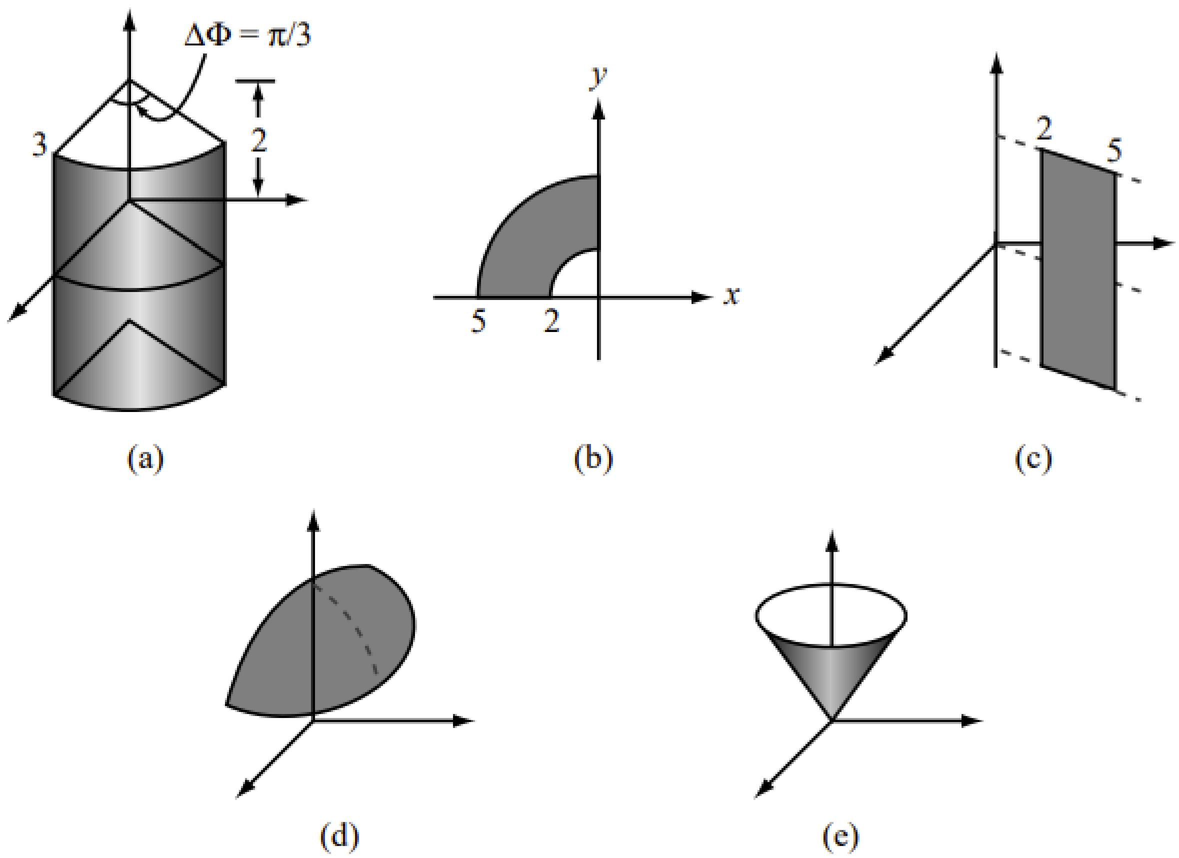
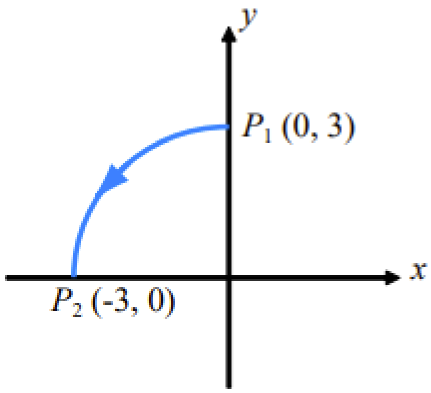


Figure P3.22: Surfaces described by Problem 3.22.

Problem 3.54 Evaluate the line integral of $\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$ along the segment P_1 to P_2 of the circular path shown in the figure.



Solution: We need to calculate:

$$\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

Since the path is along the perimeter of a circle, it is best to use cylindrical coordinates, which requires expressing both $\mathbf{E}$ and $d\boldsymbol{\ell}$ in cylindrical coordinates. Using Table 3-2,

$$\begin{aligned}\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)r\cos\phi - (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)r\sin\phi \\ &= \hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi\end{aligned}$$

The designated path is along the ϕ-direction at a constant $r = 3$. From Table 3-1, the applicable component of $d\boldsymbol{\ell}$ is:

$$d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}}\, r\, d\phi.$$

Hence,

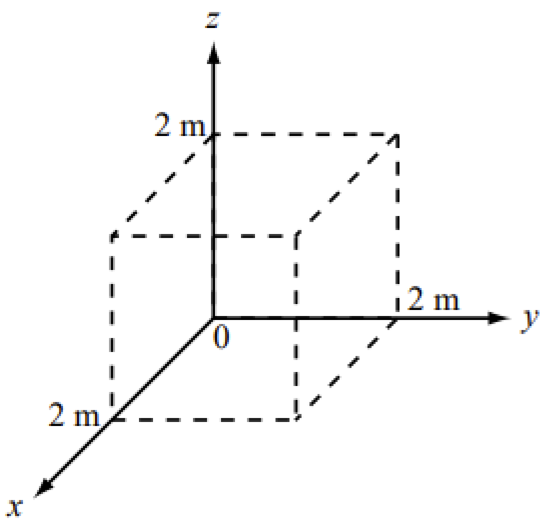
$$\begin{aligned}\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{\phi=180^\circ} \left[\hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}\, 2r\sin\phi\cos\phi\right] \cdot \hat{\boldsymbol{\phi}}\, r\, d\phi\Big|_{r=3} \\ &= \int_{90^\circ}^{180^\circ} -2r^2\sin\phi\cos\phi\, d\phi\Big|_{r=3} \\ &= -2r^2 \left.\left.\frac{\sin^2\phi}{2}\right|_{\phi=90^\circ}^{180^\circ}\right|_{r=3} = 9.\end{aligned}$$

Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

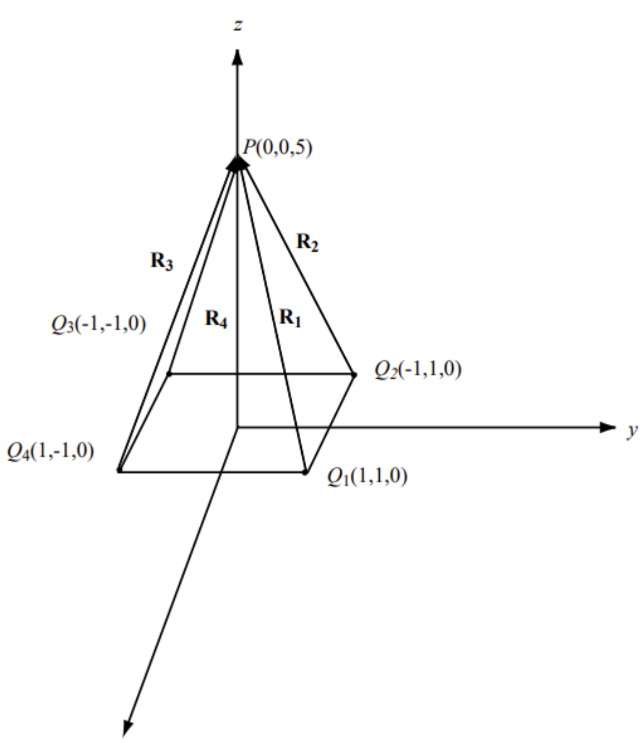
$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} xy^2 e^{-2z} \, dx \, dy \, dz$$

$$= \left(\frac{-1}{12} x^2 y^3 e^{-2z} \right) \Bigg|_{x=0}^{2} \Bigg|_{y=0}^{2} \Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$



Section 4-3: Coulomb's Law

Problem 4.9 A square with sides 2 m each has a charge of 40 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.



$$|R| = \sqrt{1^2+1^2+5^2} = \sqrt{27}.$$

$$\begin{aligned}
\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3}+\frac{\mathbf{R}_2}{|\mathbf{R}|^3}+\frac{\mathbf{R}_3}{|\mathbf{R}|^3}+\frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]\\
&= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{-\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}\right]\\
&= \hat{\mathbf{z}}\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\frac{5\times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0}\times 10^{-6}\ (\text{V/m}) = \hat{\mathbf{z}}51.2\ (\text{kV/m}).
\end{aligned}$$

Problem 4.10 Three point charges, each with $q = 3$ nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\text{ cm}, 0, 0)$, and the third at $(0, 2\text{ cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: Use Eq. (4.19) to determine the electric field at the origin due to the other two point charges [Fig. P4.10]:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\left[\frac{3\text{ nC }(-\hat{\mathbf{x}}0.02)}{(0.02)^3}\right] + \frac{3\text{ nC }(-\hat{\mathbf{y}}0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}}+\hat{\mathbf{y}})\ (\text{kV/m}) \text{ at } \mathbf{R} = 0.$$

Employ Eq. (4.14) to find the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ (μN).

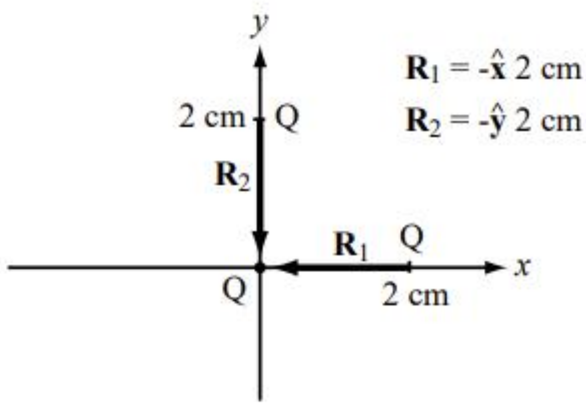
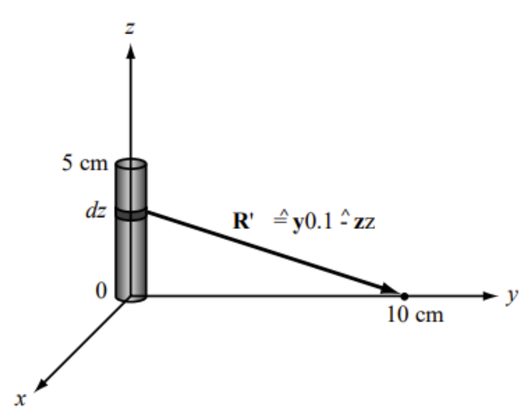


Figure P4.10: Locations of charges in Problem 4.10.

Problem 4.12 A line of charge with uniform density $\rho_l = 8$ (μC/m) exists in air along the z-axis between $z = 0$ and $z = 5$ cm. Find $\mathbf{E}$ at (0,10 cm,0).

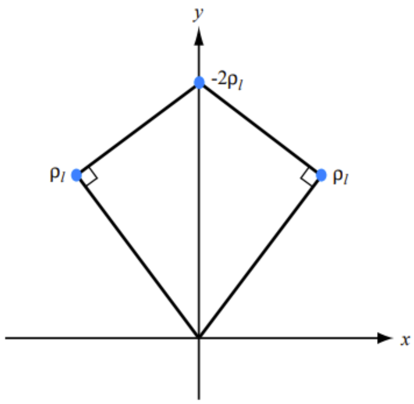
Solution: Use of Eq. (4.21c) for the line of charge shown in Fig. P4.12 gives

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'}\hat{\mathbf{R}}'\frac{\rho_l\,dl'}{R'^2},\\
R' &= \hat{\mathbf{y}}0.1 - \hat{\mathbf{z}}z\\
&= \frac{1}{4\pi\varepsilon_0}\int_{z=0}^{0.05}(8\times 10^{-6})\frac{(\hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z)}{[(0.1)^2+z^2]^{3/2}}\,dz\\
&= \frac{8\times 10^{-6}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{y}}10z+\hat{\mathbf{z}}}{\sqrt{(0.1)^2+z^2}}\right]\Bigg|_{z=0}^{0.05}\\
&= 71.86\times 10^3\,[\hat{\mathbf{y}}4.47-\hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}321.4\times 10^3 - \hat{\mathbf{z}}76.2\times 10^3 \quad (\text{V/m}).
\end{aligned}$$



Section 4-4: Gauss's Law

Problem 4.17 Three infinite lines of charge, all parallel to the z-axis, are located at the three corners of the kite-shaped arrangement shown in Fig. 4-29 (P4.17). If the two right triangles are symmetrical and of equal corresponding sides, show that the electric field is zero at the origin.



Solution: The field due to an infinite line of charge is given by Eq. (4.33). In the present case, the total **E** at the origin is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

The components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along $\hat{\mathbf{x}}$ cancel and their components along $-\hat{\mathbf{y}}$ add. Also, $\mathbf{E}_3$ is along $\hat{\mathbf{y}}$ because the line charge on the y-axis is negative. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{2\rho_l\cos\theta}{2\pi\varepsilon_0 R_1} + \hat{\mathbf{y}}\,\frac{2\rho_l}{2\pi\varepsilon_0 R_2}.$$

But $\cos\theta = R_1/R_2$. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_1}\,\frac{R_1}{R_2} + \hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_2} = 0.$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

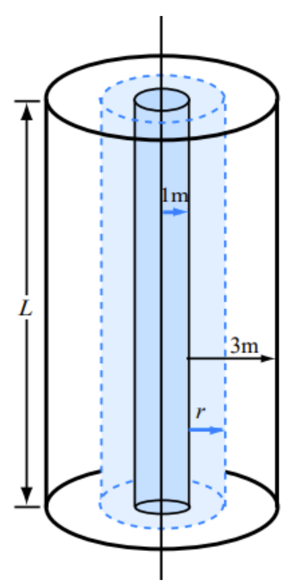
$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad (\text{C/m}^2),$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D}\cdot d\mathbf{s} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\Big|_{R=a}$$
$$= 2\pi\rho_0 a^3\int_0^{\pi}\sin\theta\,d\theta = -2\pi\rho_0 a^3\cos\theta\big|_0^{\pi} = 4\pi\rho_0 a^3 \quad (\text{C}).$$

Problem 4.25 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find **D** in all regions.



Solution: For $r < 1$ m, $\mathbf{D} = 0$.

For $1 \le r \le 3$ m,

$$\oint_S \hat{\mathbf{r}}D_r\cdot d\mathbf{s} = Q,$$
$$D_r\cdot 2\pi rL = \rho_{v0}\cdot\pi L(r^2-1^2),$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{\rho_{v0}\pi L(r^2-1)}{2\pi rL} = \hat{\mathbf{r}}\,\frac{\rho_{v0}(r^2-1)}{2r}, \qquad 1 \le r \le 3 \text{ m}.$$

For $r \ge 3$ m,

$$D_r\cdot 2\pi rL = \rho_{v0}\pi L(3^2-1^2) = 8\rho_{v0}\pi L,$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{4\rho_{v0}}{r}, \qquad r \ge 3 \text{ m}.$$

Section 4-5: Electric Potential

Problem 4.27 A square in the x–y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$ and the same at corner $(a/2, -a/2)$ and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x-axis.

(b) Evaluate V at $x = a/2$.

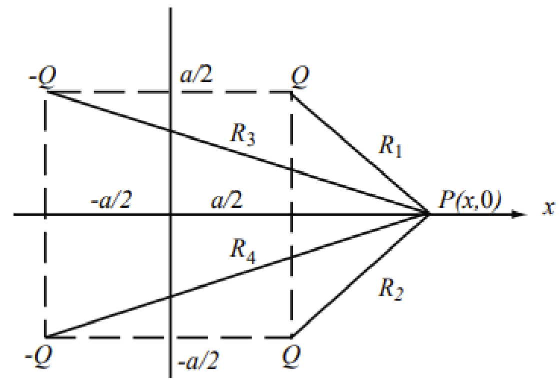
Solution: $R_1 = R_2$ and $R_3 = R_4$.

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} + \frac{-Q}{4\pi\varepsilon_0 R_3} + \frac{-Q}{4\pi\varepsilon_0 R_4} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

with

$$R_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

$$R_3 = \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}.$$



At $x = a/2$,

$$R_1 = \frac{a}{2},$$

$$R_3 = \frac{a\sqrt{5}}{2},$$

$$V = \frac{Q}{2\pi\varepsilon_0}\left(\frac{2}{a} - \frac{2}{\sqrt{5}\,a}\right) = \frac{0.55Q}{\pi\varepsilon_0 a}.$$

Problem 4.28 The circular disk of radius a shown in Fig. 4-7 (P4.28) has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P(0,0,z)$ on the z-axis.

(b) Use your result to find $\mathbf{E}$ and then evaluate it for $z = h$. Compare your final expression with Eq. (4.24), which was obtained on the basis of Coulomb's law.

Solution:

(a) Consider a ring of charge at a radial distance r. The charge contained in width dr is

$$dq = \rho_s(2\pi r\,dr) = 2\pi\rho_s r\,dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi\rho_s r\,dr}{4\pi\varepsilon_0 (r^2+z^2)^{1/2}}.$$

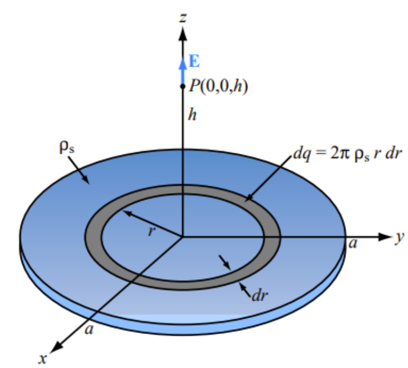


Figure P4.28: Circular disk of charge.

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0}(r^2+z^2)^{1/2}\Big|_0^a = \frac{\rho_s}{2\varepsilon_0}\left[(a^2+z^2)^{1/2} - z\right].$$

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}}\frac{\partial V}{\partial x} - \hat{\mathbf{y}}\frac{\partial V}{\partial y} - \hat{\mathbf{z}}\frac{\partial V}{\partial z} = \hat{\mathbf{z}}\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right].$$

Problem 4.39 A 100-m-long conductor of uniform cross section has a voltage drop of 4 V between its ends. If the density of the current flowing through it is 1.4×10^6 (A/m^2), identify the material of the conductor.

Solution: We know that conductivity characterizes a material:

$$\mathbf{J} = \sigma \mathbf{E}, \qquad 1.4 \times 10^6 \text{ (A/m}^2) = \sigma \left(\frac{4 \text{ (V)}}{100 \text{ (m)}} \right), \qquad \sigma = 3.5 \times 10^7 \quad \text{(S/m)}.$$

From Table B-2, we find that aluminum has $\sigma = 3.5 \times 10^7$ (S/m).

Problem 4.42 A 2×10^{-3}-mm-thick square sheet of aluminum has 5 cm $\times$ 5 cm faces. Find:

(a) the resistance between opposite edges on a square face, and

(b) the resistance between the two square faces. (See Appendix B for the electrical constants of materials).

Solution:

(a)

$$R = \frac{l}{\sigma A}.$$

For aluminum, $\sigma = 3.5 \times 10^7$ (S/m) [Appendix B].

$$l = 5 \text{ cm}, \qquad A = 5 \text{ cm} \times 2 \times 10^{-3} \text{ mm} = 10 \times 10^{-2} \times 10^{-6} = 1 \times 10^{-7} \text{ m}^2,$$

$$R = \frac{5 \times 10^{-2}}{3.5 \times 10^7 \times 1 \times 10^{-7}} = 14 \quad \text{(m}\Omega).$$

(b) Now, $l = 2 \times 10^{-3}$ mm and $A = 5 \text{ cm} \times 5 \text{ cm} = 2.5 \times 10^{-3} \text{ m}^2$.

$$R = \frac{2 \times 10^{-6}}{3.5 \times 10^7 \times 2.5 \times 10^{-3}} = 22.8 \text{ p}\Omega.$$

Problem 4.45 A 2-cm conducting sphere is embedded in a charge-free dielectric medium with $\varepsilon_{2r} = 9$. If $\mathbf{E}_2 = \hat{\mathbf{R}} 3 \cos\theta - \hat{\boldsymbol{\theta}} 3 \sin\theta$ (V/m) in the surrounding region, find the charge density on the sphere's surface.

Solution: According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s.$$

In the present case, $\hat{\mathbf{n}}_2 = \hat{\mathbf{R}}$ and $\mathbf{D}_1 = 0$. Hence,

$$\begin{aligned} \rho_s &= -\hat{\mathbf{R}} \cdot \mathbf{D}_2 |_{r=2 \text{ cm}} \\ &= -\hat{\mathbf{R}} \cdot \varepsilon_2 (\hat{\mathbf{R}} 3 \cos\theta - \hat{\boldsymbol{\theta}} 3 \sin\theta) \\ &= -27 \varepsilon_0 \cos\theta \quad \text{(C/m}^2). \end{aligned}$$

Problem 4.49 Dielectric breakdown occurs in a material whenever the magnitude of the field **E** exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

(a) At what value of r is $|E|$ maximum?

(b) What is the breakdown voltage if $a = 1$ cm, $b = 2$ cm, and the dielectric material is mica with $\varepsilon_r = 6$?

Solution:

(a) From Eq. (4.114), $\mathbf{E} = -\hat{\mathbf{r}}\rho_l/2\pi\varepsilon r$ for $a < r < b$. Thus, it is evident that $|\mathbf{E}|$ is maximum at $r = a$.

(b) The dielectric breaks down when $|\mathbf{E}| > 200$ (MV/m) (see Table 4-2), or

$$|\mathbf{E}| = \frac{\rho_l}{2\pi\varepsilon r} = \frac{\rho_l}{2\pi(6\varepsilon_0)(10^{-2})} = 200 \quad \text{(MV/m)},$$

which gives $\rho_l = (200 \text{ MV/m})(2\pi)6(8.854 \times 10^{-12})(0.01) = 667.6$ (μC/m).

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right) = \frac{(667.6\mu\text{C/m})}{12\pi(8.854 \times 10^{-12} \text{ F/m})}\ln(2) = 1.39 \quad \text{(MV)}.$$

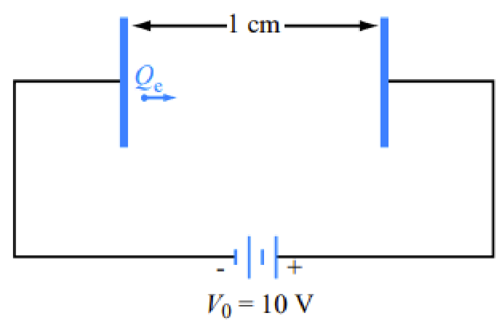
Thus, $V = 1.39$ (MV) is the breakdown voltage for this capacitor.

Problem 4.50 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

(a) the force acting on the electron,

(b) the acceleration of the electron, and

(c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.



Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e \frac{V}{d} = -1.6 \times 10^{-19} \frac{10}{0.01} = -1.6 \times 10^{-16} \quad \text{(N)}.$$

The force is directed from the negatively charged plate towards the positively charged plate.

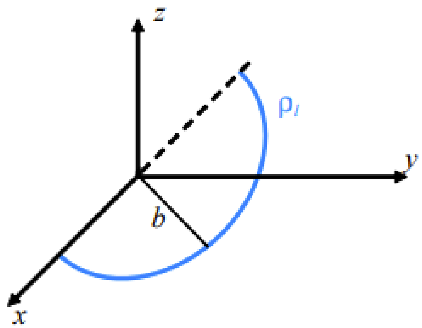
(b)

$$a = \frac{F}{m} = \frac{1.6 \times 10^{-16}}{9.1 \times 10^{-31}} = 1.76 \times 10^{14} \quad \text{(m/s}^2\text{)}.$$

(c) The electron does not get fast enough at the end of its short trip for relativity to manifest itself; classical mechanics is adequate to find the transit time. From classical mechanics, $d = d_0 + u_0 t + \frac{1}{2}at^2$, where in the present case the start position is $d_0 = 0$, the total distance traveled is $d = 1$ cm, the initial velocity $u_0 = 0$, and the acceleration is given by part (b). Solving for the time t,

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 0.01}{1.76 \times 10^{14}}} = 10.7 \times 10^{-9} \text{ s} = 10.7 \quad \text{(ns)}.$$

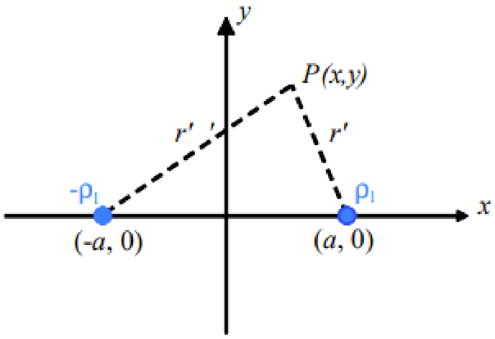
Problem 4.60 A line of charge of uniform density ρ_l occupies a semicircle of radius b as shown in the figure. Use the material presented in Example 4-4 to determine the electric field at the origin.



Solution: Since we have only half of a circle, we need to integrate the expression for $d\mathbf{E}_1$ given in Example 4-4 over ϕ from 0 to π. Before we do that, however, we need to set $h = 0$ (the problem asks for $\mathbf{E}$ at the origin). Hence,

$$\begin{aligned}
d\mathbf{E}_1 &= \frac{\rho_l b}{4\pi\varepsilon_0} \frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\, d\phi \bigg|_{h=0} \\
&= \frac{-\hat{\mathbf{r}}\rho_l}{4\pi\varepsilon_0 b}\, d\phi \\
\mathbf{E}_1 &= \int_{\phi=0}^{\pi} d\mathbf{E}_1 = -\frac{-\hat{\mathbf{r}}\rho_l}{4\varepsilon_0 b}\,.
\end{aligned}$$

Problem 4.62 Two infinite lines of charge, both parallel to the z-axis, lie in the x–z plane, one with density ρ_l and located at $x = a$ and the other with density $-\rho_l$ and located at $x = -a$. Obtain an expression for the electric potential $V(x,y)$ at a point $P(x,y)$ relative to the potential at the origin.



Solution: According to the result of Problem 4.30, the electric potential difference between a point at a distance r_1 and another at a distance r_2 from a line charge of density ρ_l is

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$

Applying this result to the line charge at $x = a$, which is at a distance a from the origin:

$$V' = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{r'}\right) \qquad (r_2 = a \text{ and } r_1 = r')$$
$$= \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{\sqrt{(x-a)^2+y^2}}\right).$$

Similarly, for the negative line charge at $x = -a$,

$$V'' = \frac{-\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{r''}\right) \qquad (r_2 = a \text{ and } r_1 = r')$$
$$= \frac{-\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{\sqrt{(x+a)^2+y^2}}\right).$$

The potential due to both lines is

$$V = V' + V'' = \frac{\rho_l}{2\pi\varepsilon_0}\left[\ln\left(\frac{a}{\sqrt{(x-a)^2+y^2}}\right) - \ln\left(\frac{a}{\sqrt{(x+a)^2+y^2}}\right)\right].$$

At the origin, $V = 0$, as it should be since the origin is the reference point. The potential is also zero along all points on the y-axis ($x = 0$).

Problem 5.4 The rectangular loop shown in Fig. 5-33 (P5.4) consists of 20 closely wrapped turns and is hinged along the z-axis. The plane of the loop makes an angle of 30° with the y-axis, and the current in the windings is 0.5 A. What is the magnitude of the torque exerted on the loop in the presence of a uniform field $\mathbf{B} = \hat{\mathbf{y}}\,2.4$ T? When viewed from above, is the expected direction of rotation clockwise or counterclockwise?

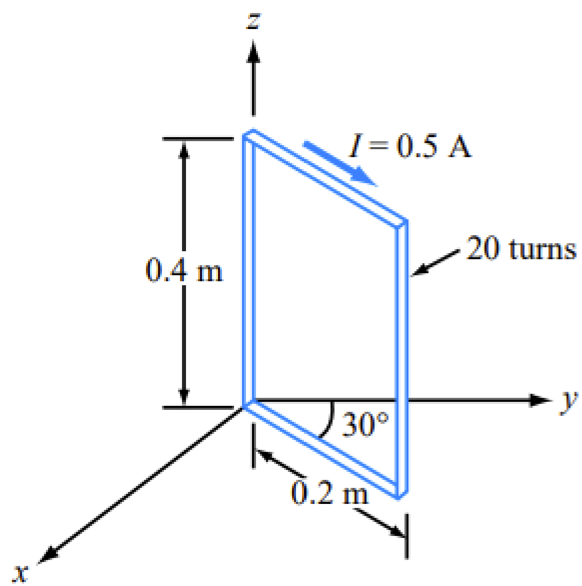


Figure P5.4: Hinged rectangular loop of Problem 5.4.

Solution: The magnetic torque on a loop is given by $\mathbf{T} = \mathbf{m} \times \mathbf{B}$ (Eq. (5.20)), where $\mathbf{m} = \hat{\mathbf{n}}NIA$ (Eq. (5.19)). For this problem, it is given that $I = 0.5$ A, $N = 20$ turns, and $A = 0.2\text{ m} \times 0.4\text{ m} = 0.08\text{ m}^2$. From the figure, $\hat{\mathbf{n}} = -\hat{\mathbf{x}}\cos 30° + \hat{\mathbf{y}}\sin 30°$. Therefore, $\mathbf{m} = \hat{\mathbf{n}}0.8\ (\text{A}\cdot\text{m}^2)$ and $\mathbf{T} = \hat{\mathbf{n}}0.8\ (\text{A}\cdot\text{m}^2) \times \hat{\mathbf{y}}2.4\text{ T} = -\hat{\mathbf{z}}1.66$ (N·m). As the torque is negative, the direction of rotation is clockwise, looking from above.

Problem 5.7 An 8 cm × 12 cm rectangular loop of wire is situated in the x–y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 50 A flowing with clockwise direction (when viewed from above). Determine the magnetic field at the center of the loop.

Solution: The total magnetic field is the vector sum of the individual fields of each of the four wire segments: $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4$. An expression for the magnetic field from a wire segment is given by Eq. (5.29).

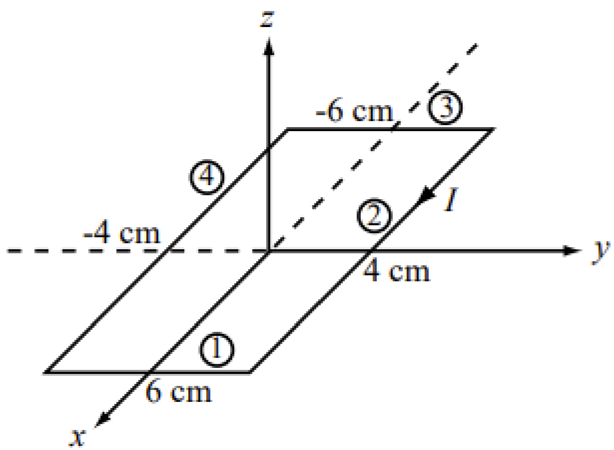


Figure P5.7: Problem 5.7.

For all segments shown in Fig. P5.7, the combination of the direction of the current and the right-hand rule gives the direction of the magnetic field as $-z$ direction at the origin. With $r = 6$ cm and $l = 8$ cm,

$$\mathbf{B}_1 = -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2+l^2}}$$
$$= -\hat{\mathbf{z}}\frac{4\pi \times 10^{-7} \times 50 \times 0.08}{2\pi \times 0.06 \times \sqrt{4 \times 0.06^2 + 0.08^2}} = -\hat{\mathbf{z}}9.24 \times 10^{-5} \quad \text{(T)}.$$

For segment 2, $r = 4$ cm and $l = 12$ cm,

$$\mathbf{B}_2 = -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2+l^2}}$$
$$= -\hat{\mathbf{z}}\frac{4\pi \times 10^{-7} \times 50 \times 0.12}{2\pi \times 0.04 \times \sqrt{4 \times 0.04^2 + 0.12^2}} = -\hat{\mathbf{z}}20.80 \times 10^{-5} \quad \text{(T)}.$$

Similarly,

$$\mathbf{B}_3 = -\hat{\mathbf{z}}9.24 \times 10^{-5} \quad \text{(T)}, \qquad \mathbf{B}_4 = -\hat{\mathbf{z}}20.80 \times 10^{-5} \quad \text{(T)}.$$

The total field is then $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 = -\hat{\mathbf{z}}0.60$ (mT).

Problem 5.9 The loop shown in Fig. 5-36 (P5.9) consists of radial lines and segments of circles whose centers are at point P. Determine the magnetic field $\mathbf{H}$ at P in terms of a, b, θ, and I.

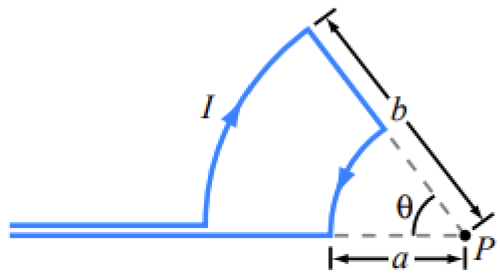


Figure P5.9: Configuration of Problem 5.9.

Solution: From the solution to Example 5-3, if we denote the z-axis as passing out of the page through point P, the magnetic field pointing out of the page at P due to the current flowing in the outer arc is $\mathbf{H}_{\text{outer}} = -\hat{\mathbf{z}} I\theta/4\pi b$ and the field pointing out of the page at P due to the current flowing in the inner arc is $\mathbf{H}_{\text{inner}} = \hat{\mathbf{z}} I\theta/4\pi a$. The other wire segments do not contribute to the magnetic field at P. Therefore, the total field flowing directly out of the page at P is

$$\mathbf{H} = \mathbf{H}_{\text{outer}} + \mathbf{H}_{\text{inner}} = \hat{\mathbf{z}}\frac{I\theta}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) = \hat{\mathbf{z}}\frac{I\theta(b-a)}{4\pi ab}.$$

Problem 5.12 Two infinitely long, parallel wires carry 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. 5-38 (P5.12).

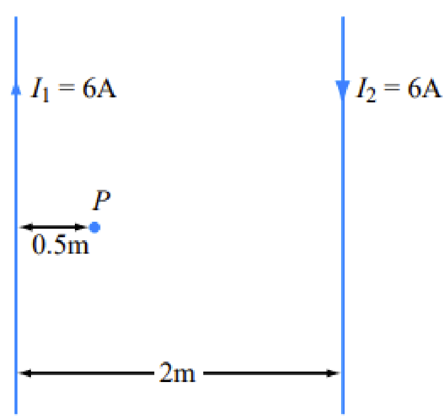


Figure P5.12: Arrangement for Problem 5.12.

Solution:

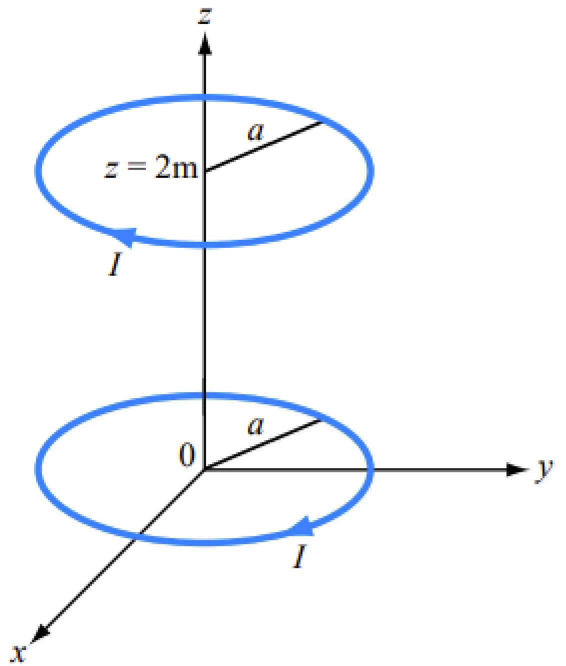
$$\mathbf{B} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\boldsymbol{\phi}}\frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\boldsymbol{\phi}}\frac{\mu_0}{\pi}(6+2) = \hat{\boldsymbol{\phi}}\frac{8\mu_0}{\pi} \quad \text{(T)}.$$

Problem 5.14 Two parallel, circular loops carrying a current of 40 A each are arranged as shown in Fig. 5-39 (P5.14). The first loop is situated in the x–y plane with its center at the origin and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

(a) $z = 0$,

(b) $z = 1$ m,

(c) $z = 2$ m.



Solution: The magnetic field due to a circular loop is given by (5.34) for a loop in the x–y plane carrying a current I in the $+\hat{\boldsymbol{\phi}}$-direction. Considering that the bottom loop in Fig. P5.14 is in the x–y plane, but the current direction is along $-\hat{\boldsymbol{\phi}}$,

$$\mathbf{H}_1 = -\hat{\mathbf{z}}\frac{Ia^2}{2(a^2+z^2)^{3/2}},$$

where z is the observation point along the z-axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z-2)$. Hence,

$$\mathbf{H}_2 = -\hat{\mathbf{z}}\frac{Ia^2}{2[a^2+(z-2)^2]^{3/2}}.$$

The total field is

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{\mathbf{z}}\frac{Ia^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{[a^2+(z-2)^2]^{3/2}}\right] \text{ A/m.}$$

(a) At $z = 0$, and with $a = 3$ m and $I = 40$ A,

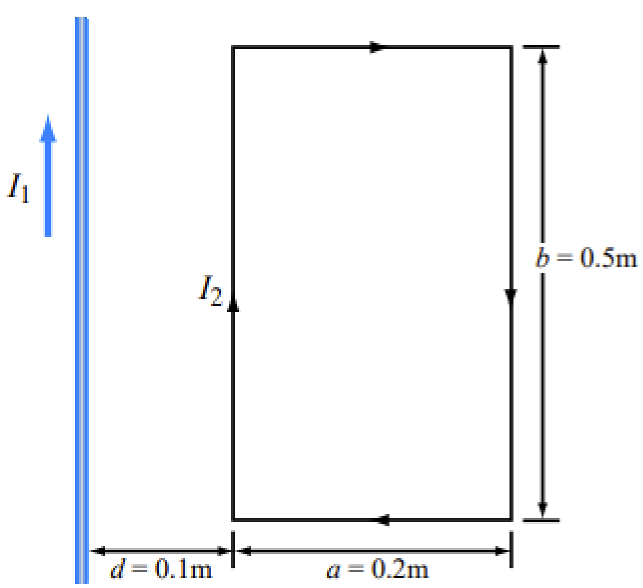
$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{3^3} + \frac{1}{(9+4)^{3/2}}\right] = -\hat{\mathbf{z}}\,10.5 \text{ A/m.}$$

(b) At $z = 1$ m (midway between the loops):

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{(9+1)^{3/2}} + \frac{1}{(9+1)^{3/2}}\right] = -\hat{\mathbf{z}}\,11.38 \text{ A/m.}$$

(c) At $z = 2$ m, $\mathbf{H}$ should be the same as at $z = 0$. Thus,

$$\mathbf{H} = -\hat{\mathbf{z}}\,10.5 \text{ A/m.}$$



$$\mathbf{B} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I_1}{2\pi r}.$$

In the plane of the loop, this magnetic field is

$$\mathbf{B} = \hat{\mathbf{y}}\frac{\mu_0 I_1}{2\pi x}.$$

Then, from Eq. (5.12), the force on the side of the loop nearest the wire is

$$\mathbf{F}_{m1} = I_2\boldsymbol{\ell}\times\mathbf{B} = I_2(\hat{\mathbf{z}}b)\times\left.\left(\hat{\mathbf{y}}\frac{\mu_0 I_1}{2\pi x}\right)\right|_{x=d} = -\hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi d}.$$

The force on the side of the loop farthest from the wire is

$$\mathbf{F}_{m2} = I_2\boldsymbol{\ell}\times\mathbf{B} = I_2(-\hat{\mathbf{z}}b)\times\left.\left(\hat{\mathbf{y}}\frac{\mu_0 I_1}{2\pi x}\right)\right|_{x=a+d} = \hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi(a+d)}.$$

Problem 5.21 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbf{H}$ for (a) $0 \le r \le a$ and (b) $r > a$.

Solution: This problem is very similar to Example 5-5.

(a) For $0 \le r_1 \le a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{r_1} \left(\frac{\hat{\mathbf{z}}J_0}{r}\right) \cdot (\hat{\mathbf{z}}r\,dr\,d\phi) = 2\pi \int_{r=0}^{r_1} J_0\,dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\boldsymbol{\phi}}J_0$.

(b) For $r \ge a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{\mathbf{z}}J_0}{r}\right) \cdot (\hat{\mathbf{z}}r\,dr\,d\phi) = 2\pi \int_{r=0}^{a} J_0\,dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $\mathbf{H}_2 = \hat{\boldsymbol{\phi}}J_0(a/r)$.

Problem 5.38 The rectangular loop shown in Fig. 5-48 (P5.38) is coplanar with the long, straight wire carrying the current $I = 20$ A. Determine the magnetic flux through the loop.

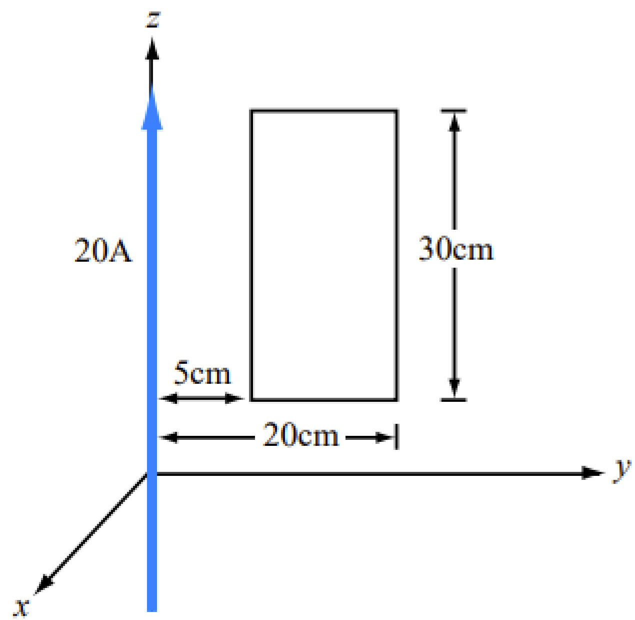


Figure P5.38: Loop and wire arrangement for Problem 5.38.

Solution: The field due to the long wire is, from Eq. (5.30),

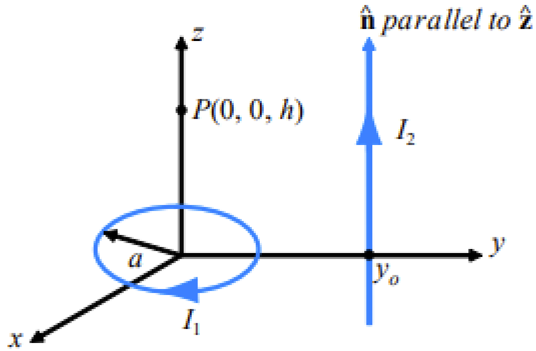
$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}}$ becomes $-\hat{\mathbf{x}}$ and r becomes y.

The flux through the loop is along $-\hat{\mathbf{x}}$, and the magnitude of the flux is

$$\begin{aligned}
\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} &= \frac{\mu_0 I}{2\pi}\int_{5\text{ cm}}^{20\text{ cm}} -\frac{\hat{\mathbf{x}}}{y}\cdot -\hat{\mathbf{x}}\,(30\text{ cm}\times dy) \\
&= \frac{\mu_0 I}{2\pi}\times 0.3\int_{0.05}^{0.2}\frac{dy}{y} \\
&= \frac{0.3\,\mu_0}{2\pi}\times 20\times \ln\left(\frac{0.2}{0.05}\right) = 1.66\times 10^{-6} \quad \text{(Wb)}.
\end{aligned}$$

Problem 5.39 A circular loop of radius a carrying current I_1 is located in the x–y plane as shown in the figure. In addition, an infinitely long wire carrying current I_2 in a direction parallel with the z-axis is located at $y = y_0$.



(a) Determine **H** at $P(0,0,h)$.

(b) Evaluate **H** for $a = 3$ cm, $y_0 = 10$ cm, $h = 4$ cm, $I_1 = 10$ A, and $I_2 = 20$ A.

Solution:

(a) The magnetic field at $P(0,0,h)$ is composed of $\mathbf{H}_1$ due to the loop and $\mathbf{H}_2$ due to the wire:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2.$$

From (5.34), with $z = h$,

$$\mathbf{H}_1 = \hat{\mathbf{z}} \frac{I_1 a^2}{2(a^2 + h^2)^{3/2}} \quad \text{(A/m)}.$$

From (5.30), the field due to the wire at a distance $r = y_0$ is

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} \frac{I_2}{2\pi y_0}$$

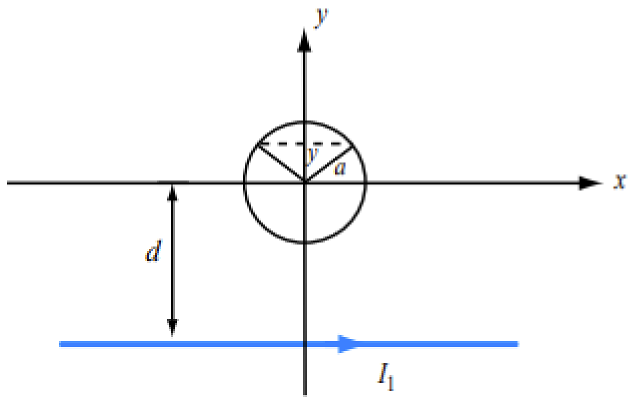
where $\hat{\boldsymbol{\phi}}$ is defined with respect to the coordinate system of the wire. Point P is located at an angel $\phi = -90°$ with respect to the wire coordinates. From Table 3-2,

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \\ &= \hat{\mathbf{x}} \qquad (\text{at } \phi = -90°). \end{aligned}$$

Hence,

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I_1 a^2}{2(a^2 + h^2)^{3/2}} + \hat{\mathbf{x}} \frac{I_2}{2\pi y_0} \quad \text{(A/m)}.$$

Problem 5.41 Determine the mutual inductance between the circular loop and the linear current shown in the figure.



Solution: To calculate the magnetic flux through the loop due to the current in the conductor, we consider a thin strip of thickness dy at location y, as shown. The magnetic field is the same at all points across the strip because they are all equidistant (at $r = d + y$) from the linear conductor. The magnetic flux through the strip is

$$\begin{aligned} d\Phi_{12} &= \mathbf{B}(y) \cdot d\mathbf{s} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi(d+y)} \cdot \hat{\mathbf{z}}2(a^2-y^2)^{1/2}\,dy \\ &= \frac{\mu_0 I(a^2-y^2)^{1/2}}{\pi(d+y)}\,dy \\ L_{12} &= \frac{1}{I}\int_S d\Phi_{12} \\ &= \frac{\mu_0}{\pi}\int_{y=-a}^{a}\frac{(a^2-y^2)^{1/2}\,dy}{(d+y)} \end{aligned}$$

Let $z = d + y \to dz = dy$. Hence,

$$\begin{aligned} L_{12} &= \frac{\mu_0}{\pi}\int_{z=d-a}^{d+a}\frac{\sqrt{a^2-(z-d)^2}}{z}\,dz \\ &= \frac{\mu_0}{\pi}\int_{d-a}^{d+a}\frac{\sqrt{(a^2-d^2)+2dz-z^2}}{z}\,dz \\ &= \frac{\mu_0}{\pi}\int\frac{\sqrt{R}}{z}\,dz \end{aligned}$$

where $R = a_0 + b_0 z + c_0 z^2$ and

$$\begin{aligned} a_0 &= a^2 - d^2 \\ b_0 &= 2d \\ c_0 &= -1 \\ \Delta &= 4a_0c_0 - b_0^2 = -4a^2 < 0 \end{aligned}$$

From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980, p. 84), we have

$$\int\frac{\sqrt{R}}{z}\,dz = \sqrt{R} + a_0\int\frac{dz}{z\sqrt{R}} + \frac{b_0}{z}\int\frac{dz}{\sqrt{R}}.$$

For

$$\sqrt{R}\Big|_{z=d-a}^{d+a} = \sqrt{a^2-d^2+2dz-z^2}\Big|_{z=d-a}^{d+a} = 0 - 0 = 0.$$

For $\int\frac{dz}{z\sqrt{R}}$, several solutions exist depending on the sign of a_0 and Δ.

For this problem, $\Delta < 0$, also let $a_0 < 0$ (i.e., $d > a$). Using the table of integrals,

$$\begin{aligned} a_0\int\frac{dz}{z\sqrt{R}} &= a_0\left[\frac{1}{\sqrt{-a_0}}\sin^{-1}\left(\frac{2a_0+b_0z}{z\sqrt{b_0^2-4a_0c_0}}\right)\right]_{z=d-a}^{d+a} \\ &= -\sqrt{d^2-a^2}\left[\sin^{-1}\left(\frac{a^2-d^2+dz}{az}\right)\right]_{z=d-a}^{d+a} \\ &= -\pi\sqrt{d^2-a^2}\,. \end{aligned}$$

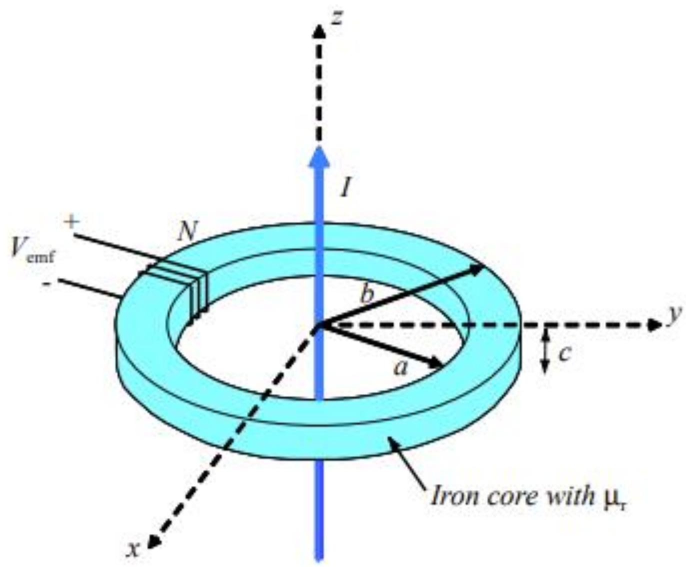
For $\int\frac{dz}{\sqrt{R}}$, different solutions exist depending on the sign of c_0 and Δ. In this problem, $\Delta < 0$ and $c_0 < 0$. From the table of integrals,

$$\begin{aligned} \frac{b_0}{z}\int\frac{dz}{\sqrt{R}} &= \frac{b_0}{2}\left[\frac{-1}{\sqrt{-c_0}}\sin^{-1}\frac{2c_0z+b_0}{\sqrt{-\Delta}}\right]_{z=d-a}^{d+a} \\ &= -d\left[\sin^{-1}\left(\frac{d-z}{a}\right)\right]_{z=d-a}^{d+a} = \pi d. \end{aligned}$$

Thus

$$\begin{aligned} L_{12} &= \frac{\mu_0}{\pi}\cdot\left[\pi d - \pi\sqrt{d^2-a^2}\right] \\ &= \mu_0\left[d - \sqrt{d^2-a^2}\right]. \end{aligned}$$

Problem 6.28 The transformer shown in the figure consists of a long wire coincident with the z-axis carrying a current $I = I_0 \cos \omega t$, coupling magnetic energy to a toroidal coil situated in the x–y plane and centered at the origin. The toroidal core uses iron material with relative permeability μ_r, around which 100 turns of a tightly wound coil serves to induce a voltage V_{emf}, as shown in the figure.



(a) Develop an expression for V_{emf}.

(b) Calculate V_{emf} for $f = 60$ Hz, $\mu_r = 4000$, $a = 5$ cm, $b = 6$ cm, $c = 2$ cm, and $I_0 = 50$ A.

Solution:

(a) We start by calculating the magnetic flux through the coil, noting that r, the distance from the wire varies from a to b

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_a^b \hat{\mathbf{x}} \frac{\mu I}{2\pi r} \cdot \hat{\mathbf{x}} c \, dr = \frac{\mu c I}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -\frac{\mu c N}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

$$= \frac{\mu c N \omega I_0}{2\pi} \ln\left(\frac{b}{a}\right) \sin \omega t \quad \text{(V)}.$$