Problem 1:

(a) Calculate the transformation matrix representing a rotation of -45° about the Z-axis of reference frame, followed by a translation of [2,-4,5] along the X, Y and Z axes of the reference frame, followed by a rotation of 90° about Y-axis of current frame.

(b) Compute the inverse of your T matrix from (a) using the method from Chapter 2 and confirm that $T^{-1}T = I$.

Solutions:

(a)

$$T = Trans(2,-4,5) * Rot(Z,-45) * I * Rot(Y,90)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C(-45) & -S(-45) & 0 & 0 \\ S(-45) & C(-45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C(90) & 0 & S(90) & 0 \\ 0 & 1 & 0 & 0 \\ -S(90) & 0 & C(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.707 & 0.707 & 0 & 2 \\ -0.707 & 0.707 & 0 & -4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.707 & 0.707 & 2 \\ 0 & 0.707 & -0.707 & -4 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$T^{-1} = \begin{bmatrix} & R^T & & -p \bullet n \\ & & & -p \bullet o \\ & & & -p \bullet a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-p \bullet n = -[(2*0) + ((-4)*0) + (5*(-1))] = 5$$

$$-p \bullet o = -[(2*0.707) + ((-4)*0.707) + (5*0)] = 1.414$$

$$-p \bullet a = -[(2*0.707) + ((-4)*(-0.707)) + (5*0) = -4.242]$$

$$\Rightarrow T^{-1} = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 0.707 & 0.707 & 0 & 1.414 \\ 0.707 & -0.707 & 0 & -4.242 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1}T = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 0.707 & 0.707 & 0 & 1.414 \\ 0.707 & -0.707 & 0 & -4.242 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.707 & 0.707 & 2 \\ 0 & 0.707 & -0.707 & -4 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Problem 2:

a) Based on Figure 4, calculate the matrices ${}^{A}T_{B}$, ${}^{B}T_{C}$ and ${}^{A}T_{C}$ by multiplying the appropriate pure translation and pure rotation matrices.

(b) Show that ${}^{A}T_{C} = {}^{A}T_{B} * {}^{B}T_{C}$.

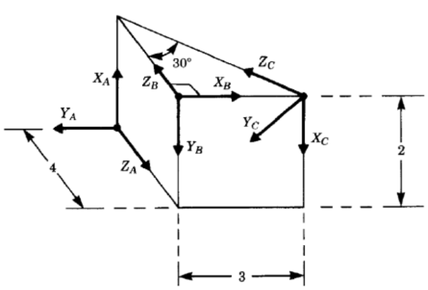

Figure 4. Frames at the corners of a wedge.

Solutions:

(a)

For transformation matrix ${}^{A}T_{B}$, the following series of transformation is listed:

1) Translation [2, 0, 4] along X, Y, and Z axes of current frame (frame {A}).
2) Rotation of 180° about the current X axis.
3) Rotation of 90° about the current Z axis.

$$
\begin{aligned}
{}^{A}T_{B} &= Trans(2,0,4) * Rot(X,180) * Rot(Z,90) \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C(180) & -S(180) & 0 \\ 0 & S(180) & C(180) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C(90) & -S(90) & 0 & 0 \\ S(90) & C(90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

For transformation matrix ${}^{B}T_{C}$, the following series of transformation is listed:

1) Translation of 3 units along the current X axis (frame {B}).
2) Rotation of 90° about the current Z axis.
3) Rotation of -30° about the current X axis.

$$
\begin{aligned}
{}^{B}T_{C} &= Trans(3,0,0) * Rot(Z,90) * Rot(X,-30) \\
&= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C(90) & -S(90) & 0 & 0 \\ S(90) & C(90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C(-30) & -S(-30) & 0 \\ 0 & S(-30) & C(-30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -0.866 & -0.5 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

For transformation matrix ${}^{A}T_{C}$, the following series of transformation is listed:

1) Translation [2, -3, 4] along the X, Y, Z axes of current frame (frame {A}).
2) Rotation of 180° about the current Z axis.
3) Rotation of 150° about the current X axis.

$$
\begin{aligned}
{}^{A}T_{C} &= Trans(2,-3,4) * Rot(Z,180) * Rot(X,150) \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C(180) & -S(180) & 0 & 0 \\ S(180) & C(180) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C(150) & -S(150) & 0 \\ 0 & S(150) & C(150) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.866 & -0.5 & 0 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 0.866 & 0.5 & -3 \\ 0 & 0.5 & -0.866 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

(b)

$$
{}^{A}T_{B} * {}^{B}T_{C} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -0.866 & -0.5 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 0.866 & 0.5 & -3 \\ 0 & 0.5 & -0.866 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{A}T_{C}
$$

Problem 3:

If ${}^{C}T_{D}$, ${}^{A}T_{B}$, ${}^{A}T_{E}$, and ${}^{E}T_{D}$ are known, derive the transformation equation for ${}^{B}T_{C}$ in terms of these matrices.

Solutions:

$$
\begin{aligned}
{}^{A}T_{E} * {}^{E}T_{D} &= {}^{A}T_{B} * {}^{B}T_{C} * {}^{C}T_{D} \\
{}^{A}T_{B}^{-1} * {}^{A}T_{E} * {}^{E}T_{D} &= \left({}^{A}T_{B}^{-1} * {}^{A}T_{B}\right) * {}^{B}T_{C} * {}^{C}T_{D} \\
{}^{A}T_{B}^{-1} * {}^{A}T_{E} * {}^{E}T_{D} &= {}^{B}T_{C} * {}^{C}T_{D} \\
{}^{A}T_{B}^{-1} * {}^{A}T_{E} * {}^{E}T_{D} * {}^{C}T_{D}^{-1} &= {}^{B}T_{C} * \left({}^{C}T_{D} * {}^{C}T_{D}^{-1}\right) \\
{}^{A}T_{B}^{-1} * {}^{A}T_{E} * {}^{E}T_{D} * {}^{C}T_{D}^{-1} &= {}^{B}T_{C} \\
\text{or } {}^{B}T_{C} &= {}^{A}T_{B}^{-1} * {}^{A}T_{E} * {}^{E}T_{D} * {}^{C}T_{D}^{-1}
\end{aligned}
$$