EP2A04 Tutorial 1

TUTORIAL TAS:

FRASER MCCAULEY | MCCAULJ@MCMASTER.CA

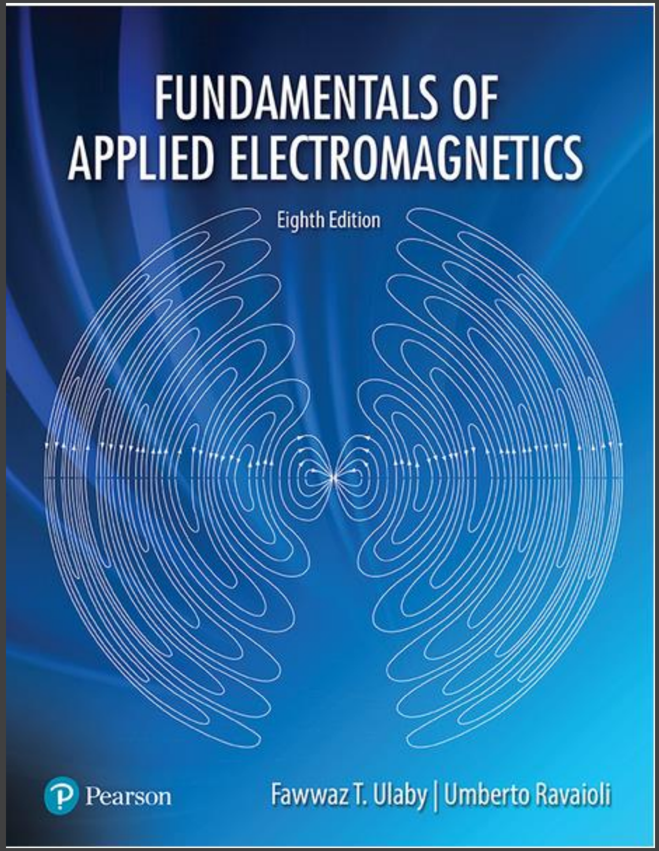
FATEMEH BAKHSHANDEH | BAKHSHAF@MCMASTER.CA

Your Textbook

Fundamentals of Applied Electromagnetics, Eighth Edition

Ulaby & Ravaioli

Seventh edition also acceptable, with some inconsistencies



Tutorial Problem 1 (1.1, or 1.3 in 7th ed.)

A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

Goal: **Wavelength**

Givens: **Frequency, Phase Velocity**

$$u_P = f\lambda \rightarrow \lambda = \frac{u_P}{f}$$

Convert to SI: $f = 180\frac{1}{60s} = 3\frac{1}{s} = 3Hz$ $\quad u_P = \frac{300cm}{10s} = \frac{3m}{10s} = 0.3\frac{m}{s}$

Solution: $\lambda = \frac{u_P}{f} = \frac{0.3}{3} = 0.1m$

Tutorial Problem 2 (1.2)

For the pressure wave described in Example 1-1, plot the following:

A) $p(x,t)$ vs x at $t = 0$.

B) $p(x,t)$ vs t at $x = 0$.

Be sure to use appropriate scales for and so that each of your plots covers at least two cycles.

The equation: $$\boldsymbol{p(x,t) = 10\cos\left(2\pi \times 10^3 t - \frac{4\pi}{3}x + \frac{\pi}{3}\right)}$$

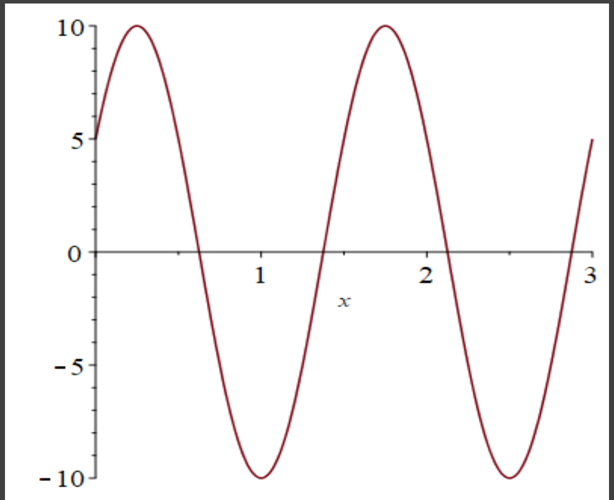
Tutorial Problem 2 (1.2) *continued…*

The equation: $$\boldsymbol{p(x,t) = 10\cos\left(2\pi \times 10^3 t - \frac{4\pi}{3}x + \frac{\pi}{3}\right)}$$

A) Substituting $t = 0$:

$$p(x,0) = 10\cos\left(-\frac{4\pi}{3}x + \frac{\pi}{3}\right)$$

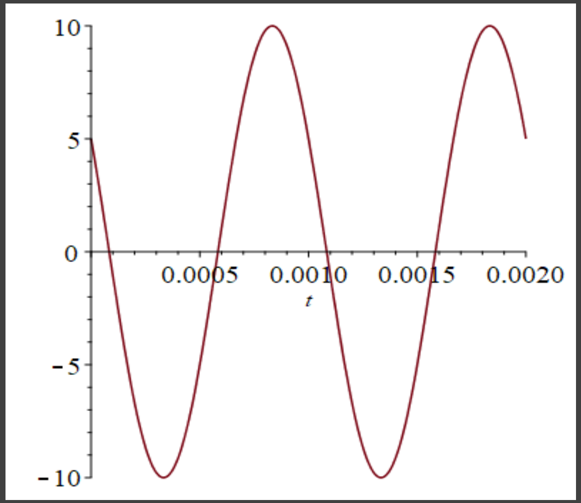
Plotting this for 2 wavelengths ($\lambda = \frac{3}{2}$):



B) Substituting $x = 0$:

$$p(0,t) = 10\cos\left(2\pi \times 10^3 t + \frac{\pi}{3}\right)$$

Plotting this for 2 periods ($T = \frac{1}{f} = 0.001s$):



Tutorial Problem 3 (1.8)

Two waves on a string are given by the functions:

$y_1(x,t) = 4\cos(20t - 30x) \quad (cm)$

$y_2(x,t) = -4\cos(20t + 30x) \quad (cm)$

Where x is in centimeters. The waves are said to interfere constructively when their superposition $|y_S| = |y_1 + y_2|$ is a maximum, and they interfere destructively when $|y_S|$ is a minimum.

A) What are the directions of propagation of waves $y_1(x,t)$ and $y_2(x,t)$?

B) At $t = (\pi/50)\ s$, what location x do the two waves interfere constructively, and what is the corresponding value of $|y_S|$?

C) At $t = (\pi/50)\ s$, what location x do the two waves interfere destructively, and what is the corresponding value of $|y_S|$?

Tutorial Problem 3 (1.8) *continued...*

A) What are the directions of propagation of waves $y_1(x,t)$ and $y_2(x,t)$?

$$y_1(x,t) = 4\cos(20t - 30x) \qquad (cm)$$

The t and x terms have opposite signs, so the wave is propagating in the positive x-direction.

$$y_2(x,t) = -4\cos(20t + 30x) \qquad (cm)$$

The t and x terms have the same sign, so the wave is propagating in the negative x-direction.

Tutorial Problem 3 (1.8) *continued…*

B) At $t = (\pi/50)\ s$, what location x do the two waves interfere constructively, and what is the corresponding value of $|y_S|$?

First, substitute in $t = (\pi/50)\ s$:

$$y_1(x, \pi/50) = 4\cos\left(20\left(\frac{\pi}{50}\right) - 30x\right) = 4\cos\left(\frac{2\pi}{5} - 30x\right)$$

$$y_2(x, \pi/50) = -4\cos\left(20\left(\frac{\pi}{50}\right) + 30x\right) = -4\cos\left(\frac{2\pi}{5} + 30x\right)$$

$$\left|y_S(x, \frac{\pi}{50})\right| = \left|4\cos\left(\frac{2\pi}{5} - 30x\right) - 4\cos\left(\frac{2\pi}{5} + 30x\right)\right| \quad \rightarrow \quad 2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

$$\left|y_S\left(x, \frac{\pi}{50}\right)\right| = \left|4\left(2\sin\left(\frac{2\pi}{5}\right)\sin(30x)\right)\right| = \left|8\sin\left(\frac{2\pi}{5}\right)\sin(30x)\right|$$

Tutorial Problem 3 (1.8) *continued…*

B) At $t = (\pi/50)\ s$, what location x do the two waves interfere constructively, and what is the corresponding value of $|y_S|$?

$$\left|y_S\left(x, \frac{\pi}{50}\right)\right| = |7.61\sin(30x)|$$

To find where they constructively interfere, we find the peaks of the superposition by differentiating and finding the zeros:

$$\frac{dy_S}{dx} = \frac{d}{dx}(7.61\sin(30x)) = 30 * 7.61\cos(30x) = 228.3\cos(30x)$$

Find where $\frac{dy_S}{dx} = 0$. Cosine is 0 at $\theta = \left(n + \frac{1}{2}\right)\pi$, so:

$$x = \frac{\left(n + \frac{1}{2}\right)\pi}{30} = \frac{\pi}{60}, \frac{\pi}{20}, \frac{\pi}{12} \ldots (cm)$$
$$x = 0.052cm, 0.157cm, 0.262cm\ \ldots$$

The amplitude of the superposition at these values is:

$$\left|y_S\left(\frac{\pi}{60}, \frac{\pi}{50}\right)\right| = \left|7.61\sin\left(30\frac{\pi}{60}\right)\right| = 7.61\text{ cm}$$

Tutorial Problem 3 (1.8) *continued…*

C) At $t = (\pi/50)\ s$, what location x do the two waves interfere destructively, and what is the corresponding value of $|y_S|$?

$$\left|y_S\left(x, \frac{\pi}{50}\right)\right| = |7.61\sin(30x)|$$

By a similar logic, we can find the points of destructive interference by finding the zeros of the superposition. sin is zero at $\theta = n\pi$, so they interfere destructively at:

$$x = \frac{n\pi}{30} = 0, \frac{\pi}{30}, \frac{\pi}{15}\ \ldots (cm)$$
$$x = 0cm, 0.105cm, 0.209cm\ \ldots$$

The amplitude of the superposition at these values is:

$$\left|y_S\left(0, \frac{\pi}{50}\right)\right| = |7.61\sin(30(0))| = 0\text{ cm}$$

Tutorial Problem 4 (1.12)

Given two waves characterized by

$y_1(t) = 3\cos(\omega t)$

$y_2(t) = 3\sin(\omega t + 60°)$,

Does $y_2(t)$ lead or lag $y_1(t)$ and by what phase angle?

Sin(x) lag cos(x) by 90°. Adding in a phase shift of +60° will create a lead of 60°. The total effect is $y_2(t)$ lagging $y_1(t)$ by $30°$.

Good luck!

Assignment 1 is due 8AM on January 24.

No late submissions will be accepted – if your work is incomplete, submit whatever you have before the deadline for part marks!

Question 5 is bonus, to help us know where the class stands with circuits content.

Show all of your work for full marks.

ENG PHYS 2A04 Tutorial 2

Electricity and Magnetism

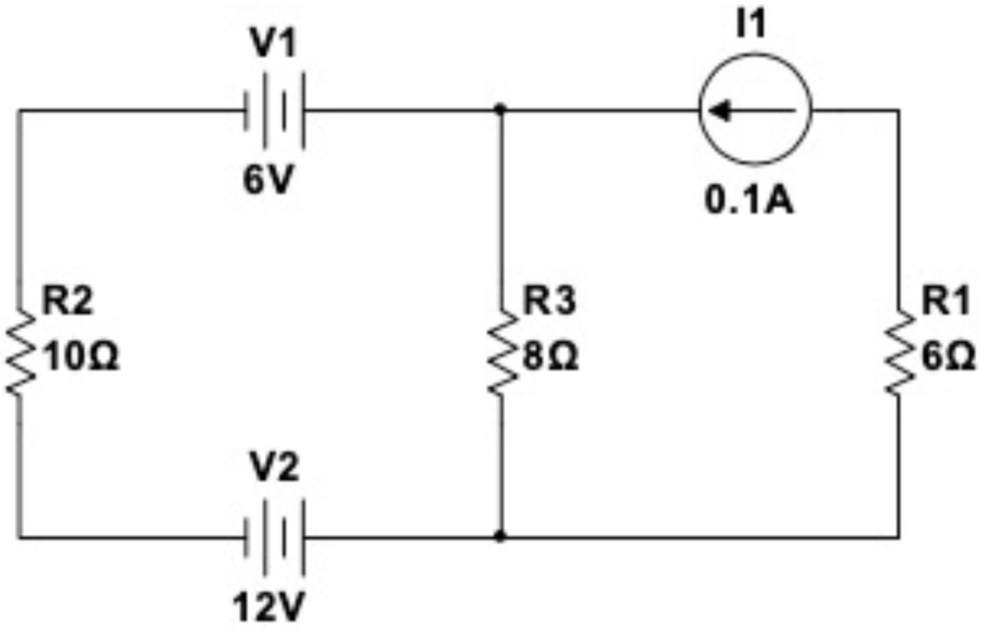
Your TAs today

- Tommy Kean
  - keant1@mcmaster.ca
- Muhammad Munir
  - munirm6@mcmaster.ca

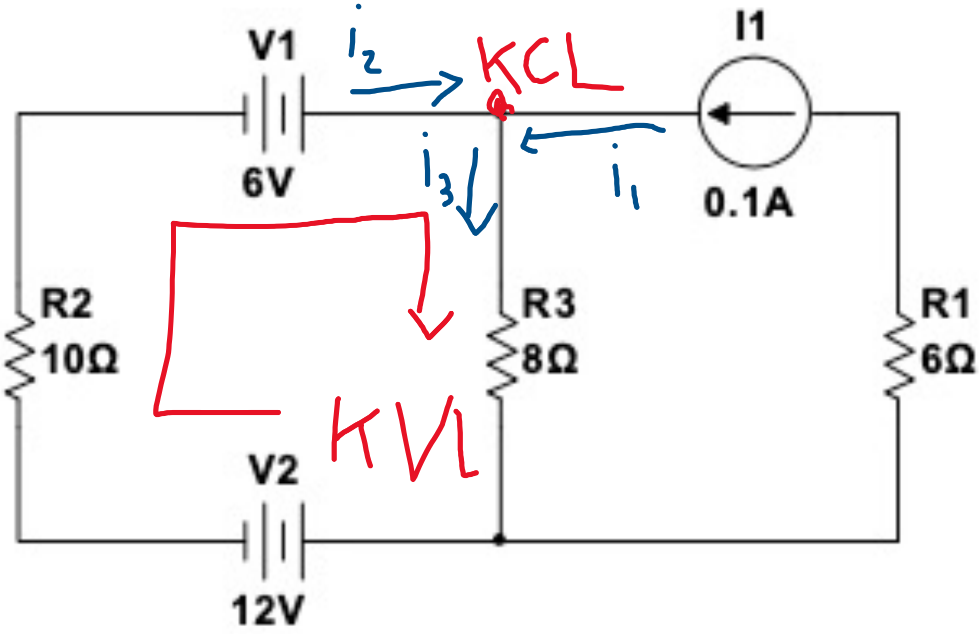
Chapter 28: DC Circuits (Serway, 9th edition)

Problem 1 Example in Lecture

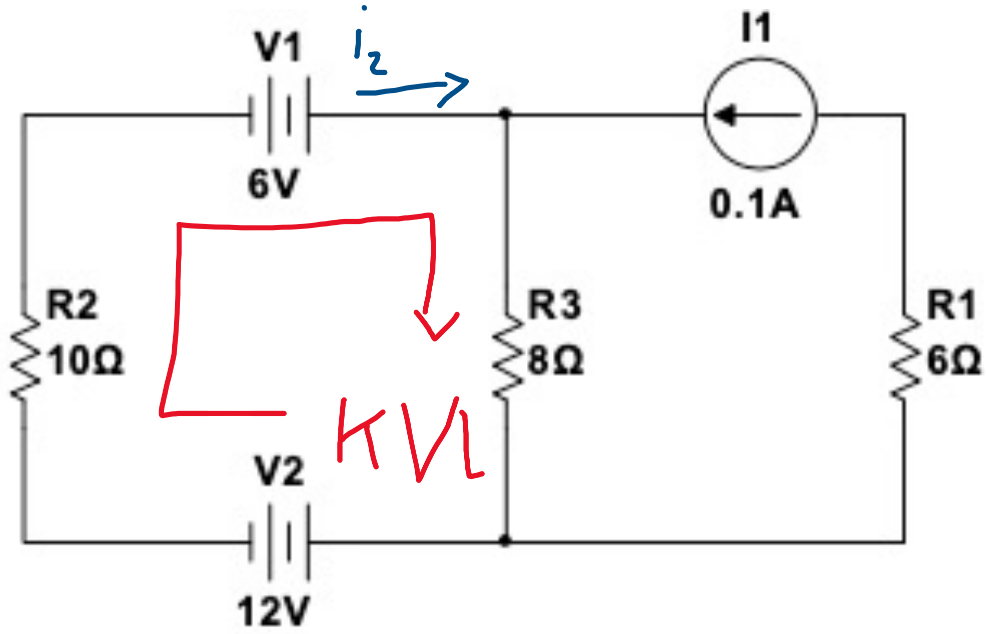
Find the current across R_1, R_2 and R_3



Example in Lecture



Example in Lecture



Step 1 - Loop Rule (KCL) :

$$\sum_{closed\ loop} \Delta V = 0$$

$$V_1 + V_{R_3} + V_2 + V_{R_2} = 0$$

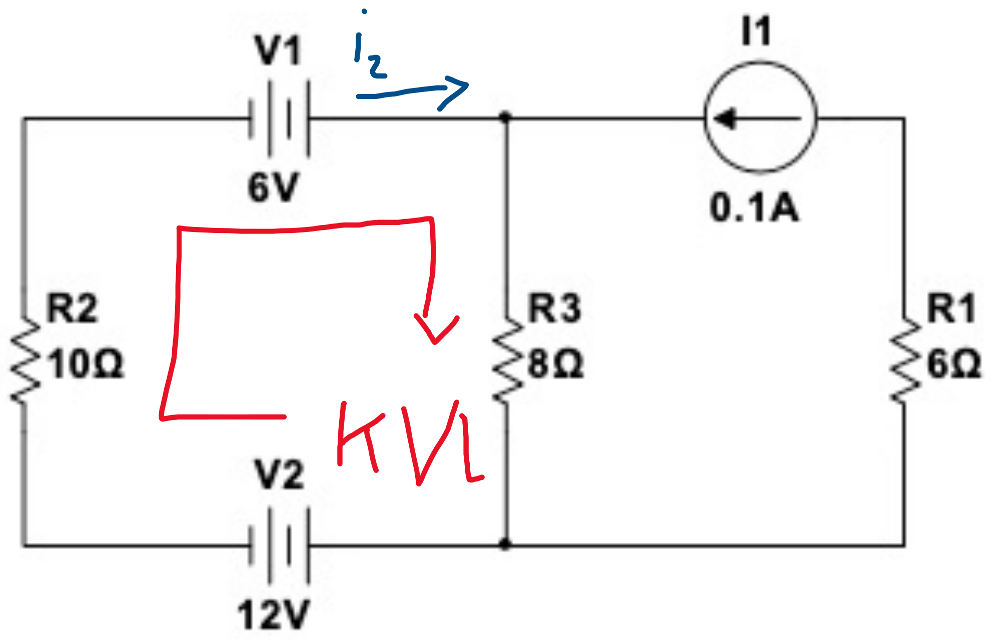
$$V_1 + (i_3 R_3) + V_2 + (i_2 R_2) = 0$$

$$V_1 + i_3 R_3 + V_2 + i_2 R_2 = 0$$

$$(-6) + (12) + \ i_3(8) + i_2(10) = 0$$

$$6 + \ 8i_3 + 10i_2 = 0$$

Example in Lecture



Step 2 - Junction Rule :

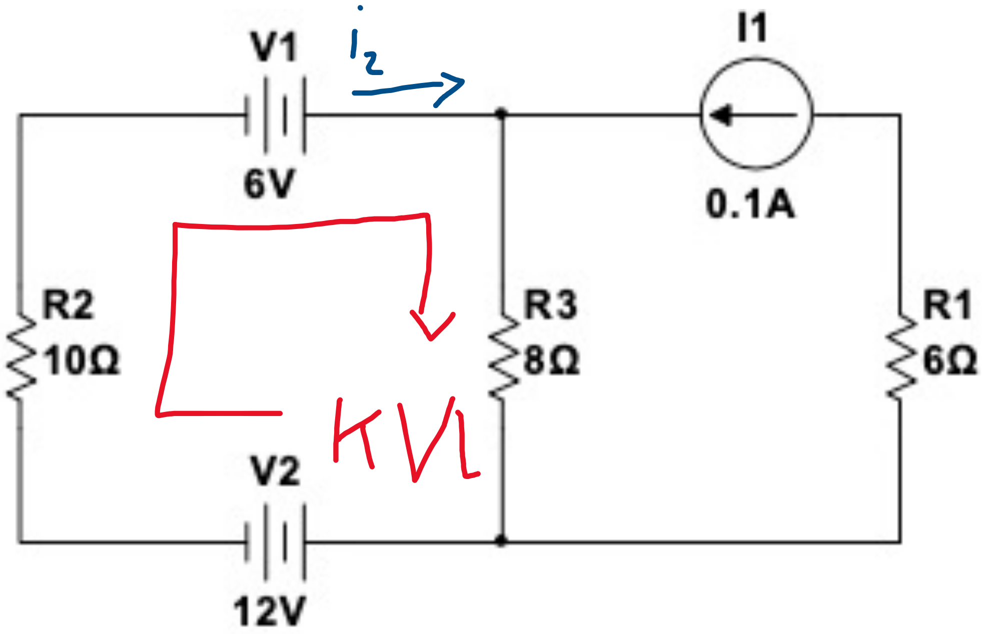
$$\sum_{junction} \mathrm{i} = 0$$

$$i_1 + i_2 - i_3 = 0$$

$$0.1 + i_2 - i_3 = 0$$

$$i_3 = 0.1 + i_2$$

Example in Lecture



Step 3 - Substitute:

$$6 + \ 8i_3 + 10i_2 = 0$$
$$6 + \ 8(0.1 + i_2) + 10i_2 = 0$$
$$6 + \ 0.8 + 8i_2 + 10i_2 = 0$$
$$6.8 + 18\, i_2 = 0$$
$$i_2 = \frac{-6.8}{18}$$
$$i_2 = -0.377$$

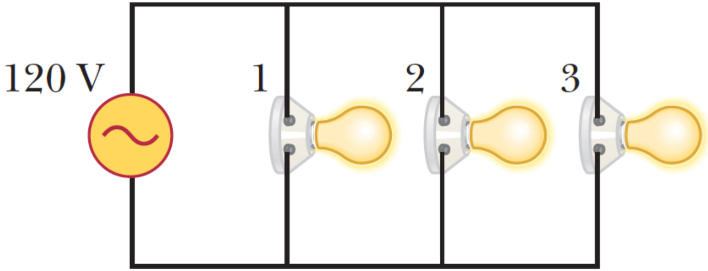
$$i_3 = \ 0.1 + i_2$$
$$i_3 = \ 0.1 + (-0.377)$$
$$i_3 = \ -0.277$$

$$i_1 = 0.1\ A,\ i_2 = \ -0.377\ A,\ i_3 = \ -0.277\ A$$

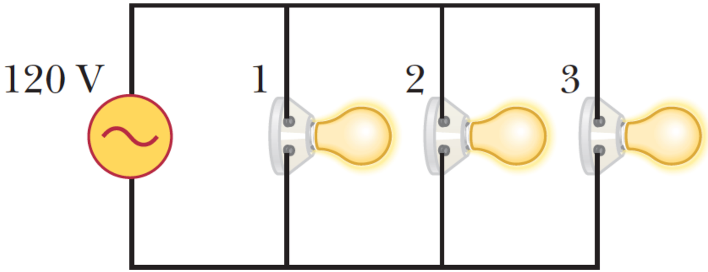
Chapter 33: AC Circuits (Serway, 9th edition)

Problem 2

The figure below shows three lightbulbs connected to a 120-V AC (rms) household supply voltage. Bulb 1 has a power rating of 40 W, bulb 2 has a 75 W rating and bulb 3 has a 60 W rating. Find (a) the rms current in each bulb and (b) the resistance of each bulb. (c) What is the total resistance of the combination of the three lightbulbs:



Problem 2



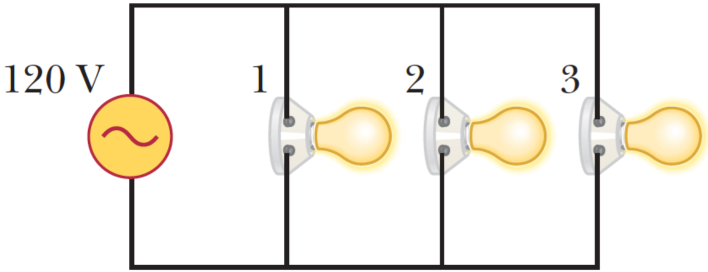
Find the RMS Current:

$$I_{1,rms} = \frac{P_1}{V} = \frac{40W}{120V} = 0.33A$$

$$I_{2,rms} = \frac{P_2}{V} = \frac{75W}{120V} = 0.625A$$

$$I_{3,rms} = \frac{P_3}{V} = \frac{60W}{120V} = 0.5A$$

Problem 2



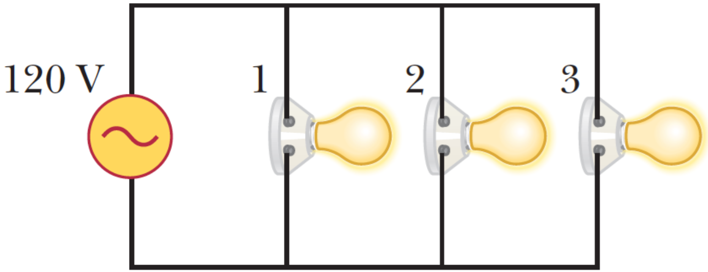
Find the resistance of each bulb :

$$R_1 = \frac{V^2}{P_1} = \frac{(120V)^2}{40W} = 360\ \Omega$$

$$R_2 = \frac{V^2}{P_2} = \frac{(120V)^2}{75W} = 192\ \Omega$$

$$R_3 = \frac{V^2}{P_3} = \frac{(120V)^2}{60W} = 240\ \Omega$$

Problem 2



the total resistance of the combination of the three lightbulbs :

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$R_{total} = \left(\frac{1}{360} + \frac{1}{192} + \frac{1}{240}\right)^{-1} = 82.3\ \Omega$$

Problem 3

What maximum current is delivered by an AC source M with ΔV_{max} = 48.0 V and f=90.0 Hz when connected across a 3.70-μF capacitor?

Problem 3

What maximum current is delivered by an AC source with ΔV_{max} = 48.0 V and f=90.0 Hz when connected across a 3.70-μF capacitor?

1. Find the V_{RMS}

$$\Delta V_{RMS} = \frac{\Delta V_{max}}{\sqrt{(2)}}$$

$$\Delta V_{RMS} = 33.94\, V$$

Problem 3

What maximum current is delivered by an AC source M with ΔV_{max} = 48.0 V and f=90.0 Hz when connected across a 3.70-μF capacitor?

2. Find the reactance

$$x_c = \frac{1}{2\pi f C}$$

$$x_c = \frac{1}{2\pi(90\ Hz)(3.7\times10^{-6})}$$

$$x_c = 477.94\Omega$$

Problem 3

What maximum current is delivered by an AC source M with ΔV_{max} = 48.0 V and f=90.0 Hz when connected across a 3.70-µF capacitor?

3. Find the I_{rms}

$$I_{rms} = \frac{\Delta V_{RMS}}{x_c}$$

$$I_{rms} = \frac{33.94\ V}{477.94\Omega}$$

$$I_{rms} = 0.071\ A$$

Problem 3

What maximum current is delivered by an AC source M with ΔV_{max} = 48.0 V and f=90.0 Hz when connected across a 3.70-μF capacitor?

3. Find the I_{rms}

$$I_{peak} = I_{rms}\sqrt{2}$$

$$I_{peak} = (0.071\ A)\sqrt{2}$$

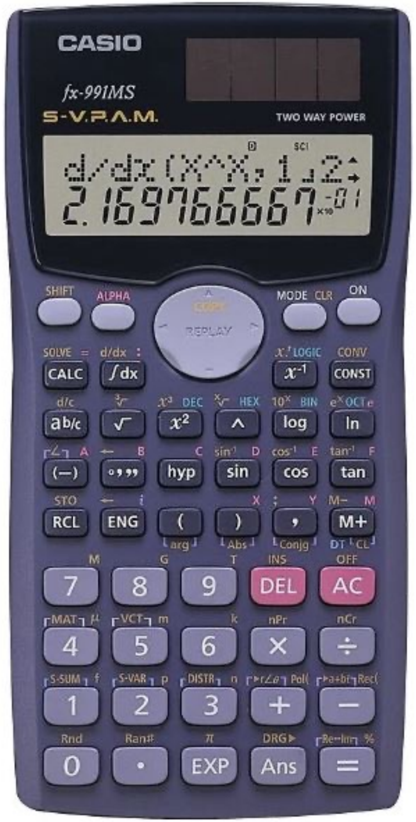
$$I_{peak} = 0.100\ A$$

Section 1-7: Review of Phasors (Ulaby, 8th ed.)

McMaster’s Standard Calculator

- Casio FX-991MS
- 2nd Edition (left) available at McMaster Bookstore: $21.99 (link below)
- Will need to solve phasor equations

https://campusstore.mcmaster.ca/cgi-mcm/ws/gmdetail.pl?pwsPRODIDG1=2207018&sType=gm&proddesc=Casio%20fx%2D991%20Calculator&pwsGROUP=



Phasor Conversion – Calculator Setup

Enable complex numbers:

- “CMPLX” appears at top of display

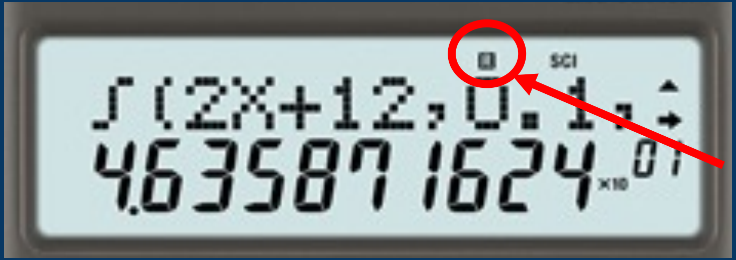
MODE CLR → S-VAR p 2

Ensure calculator is in degree mode

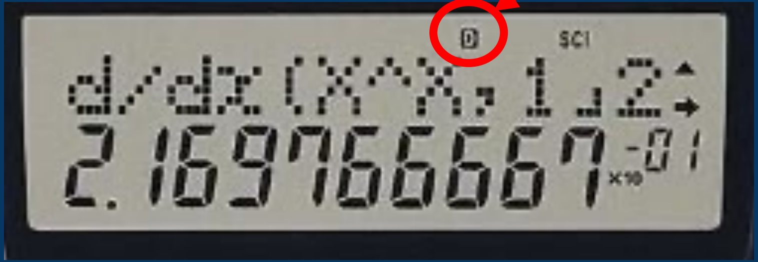
MODE CLR x4 → S-SUM f 1

- For consistent units in polar form

GOOD!

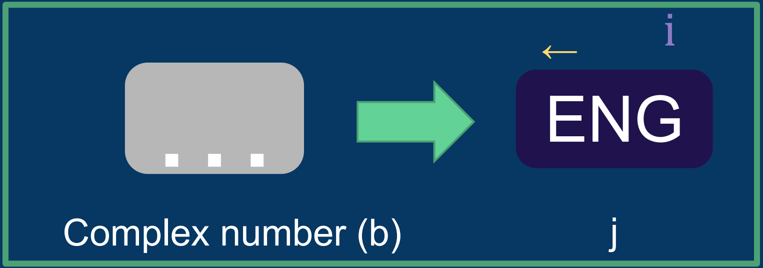


BAD!

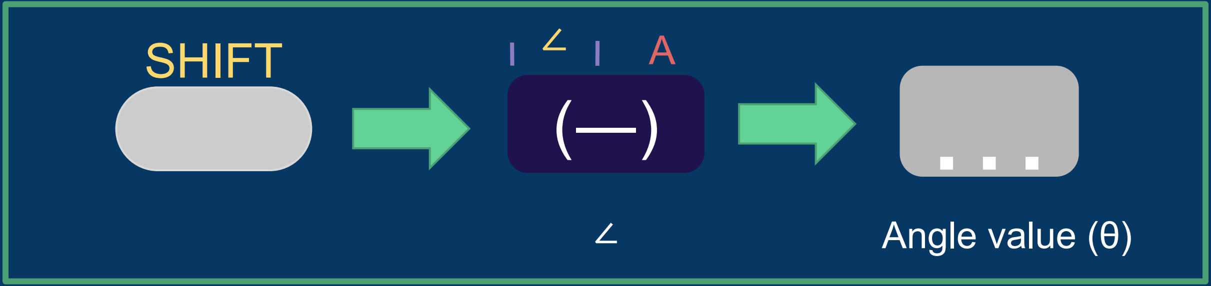


Phasor Conversion – Inputting Complex Values

To enter complex number in rectangular form:

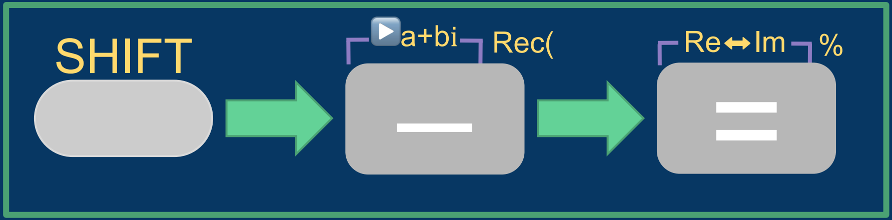


To enter angle value in polar form:

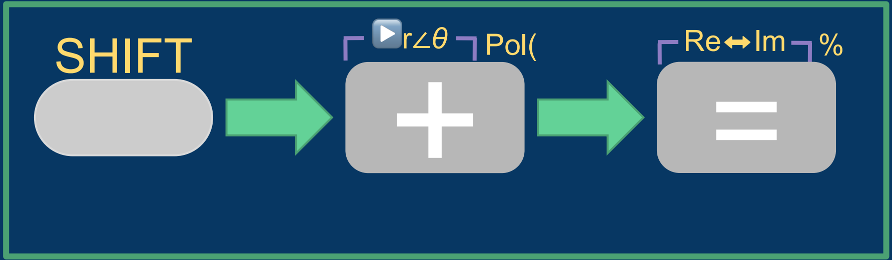


Phasor Conversion – Output Viewing

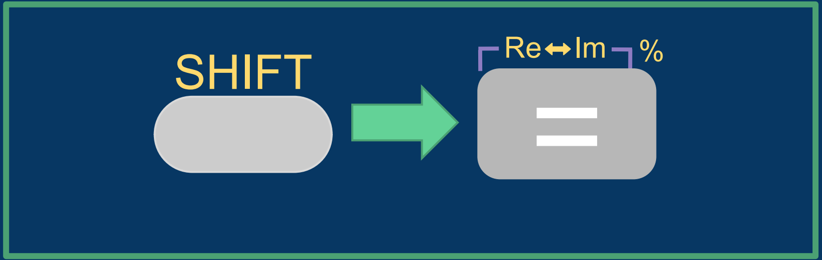
To view answer in rectangular form:



To view answer in polar form:



To change between real and imaginary (rectangular) or radius and angle (polar):



Phasor Conversion - Question

Convert the following phasors to rectangular form (a + bj):

a) $9 \angle -60°$

b) $12 \angle -45°$

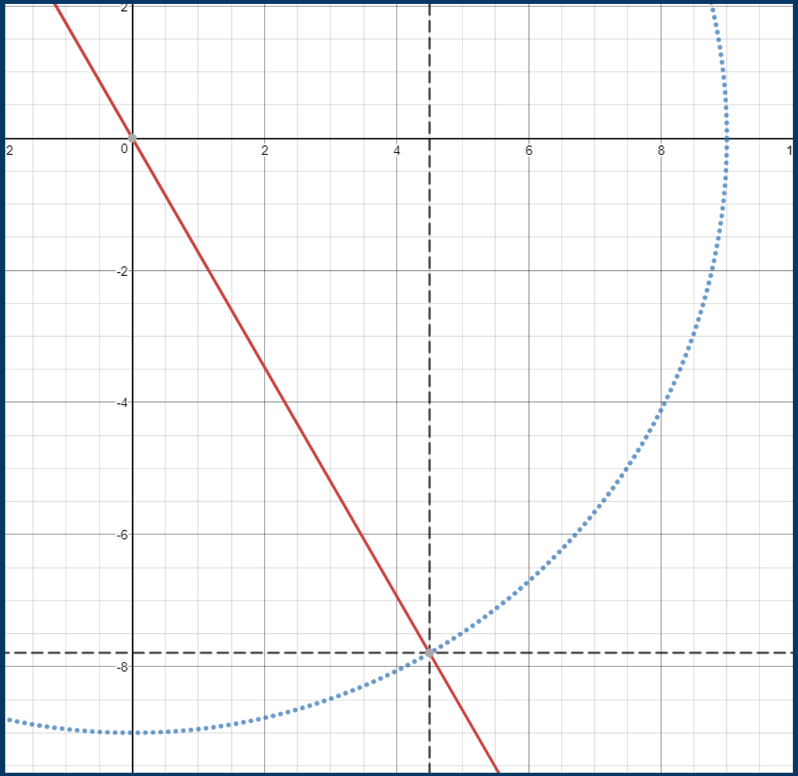
Convert the following phasors to polar form ($r \angle \theta$):

a) $3 + 9j$

b) $-0.4 + 0.3j$

Phasor Conversion - Solution

a) Convert the following to rectangular form (a + bj):



$9 \angle -60°$

9 SHIFT (-) (-) 6 0

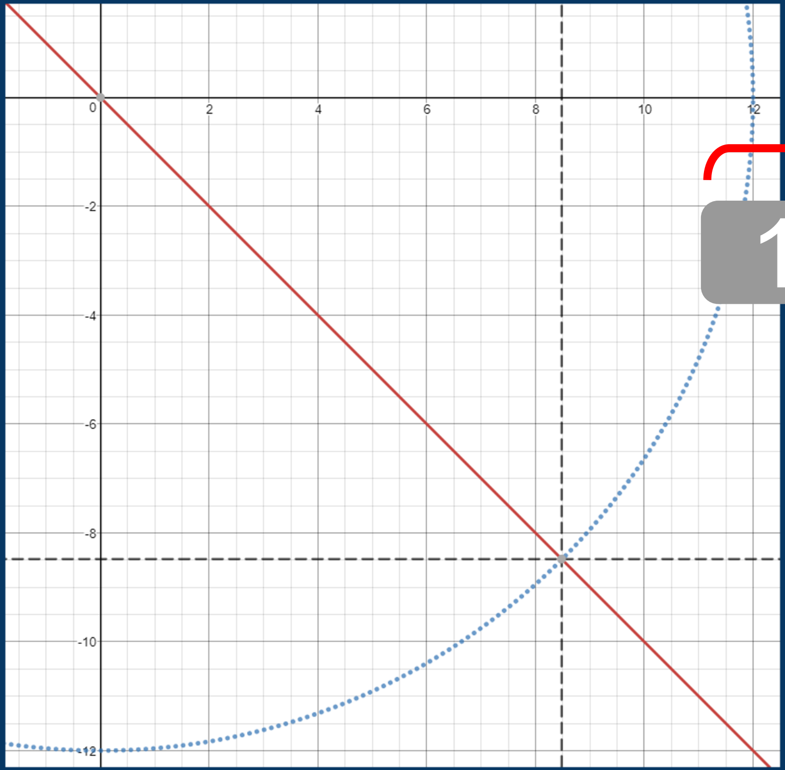
= a + bj

= SHIFT =

Answer: 4.5 - 7.79j

Phasor Conversion - Solution

b) Convert the following to rectangular form (a + bj):



12 ∠ -45°

1 | 2 | SHIFT | (-) | (-) | 4 | 5

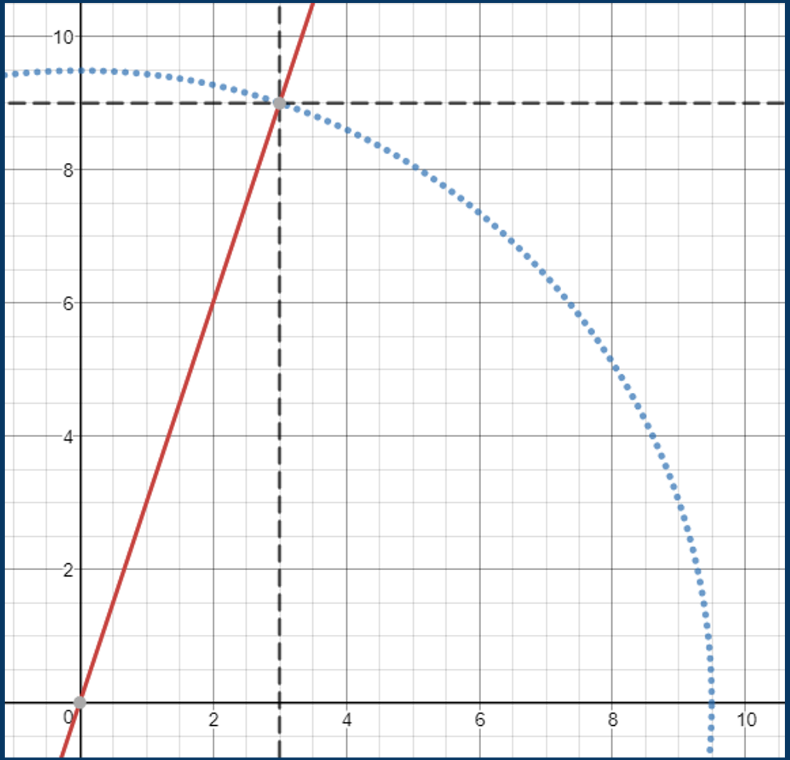
= a + bj

= | SHIFT | =

Answer: 8.48 – 8.48j

Phasor Conversion - Solution

a) Convert the following to polar form (r ∠ θ):



3 + 9 j (r ∠ θ)

i

3 + 9 ENG SHIFT +

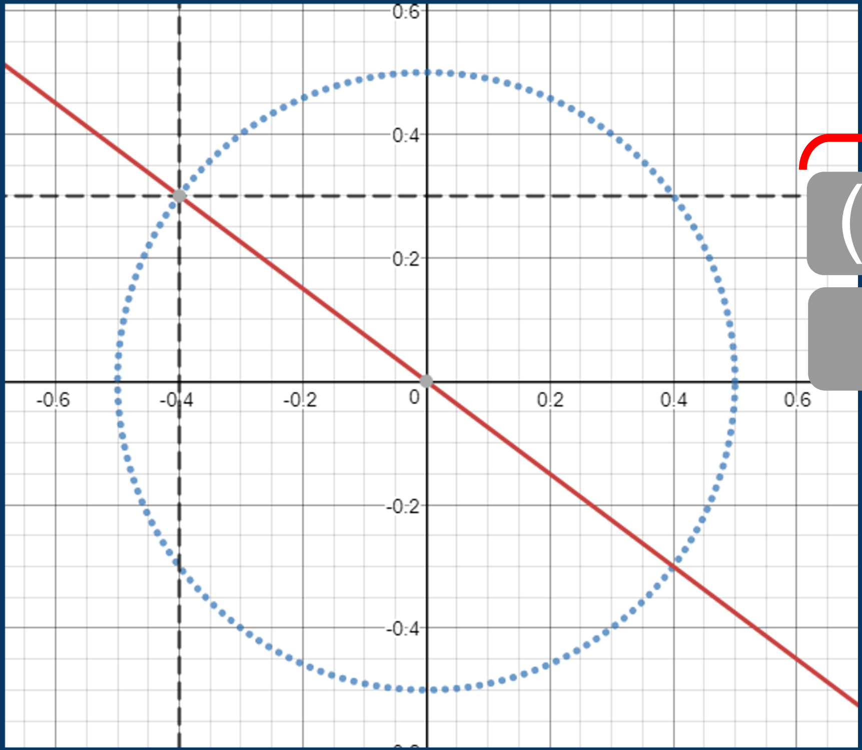
= r ∠ θ

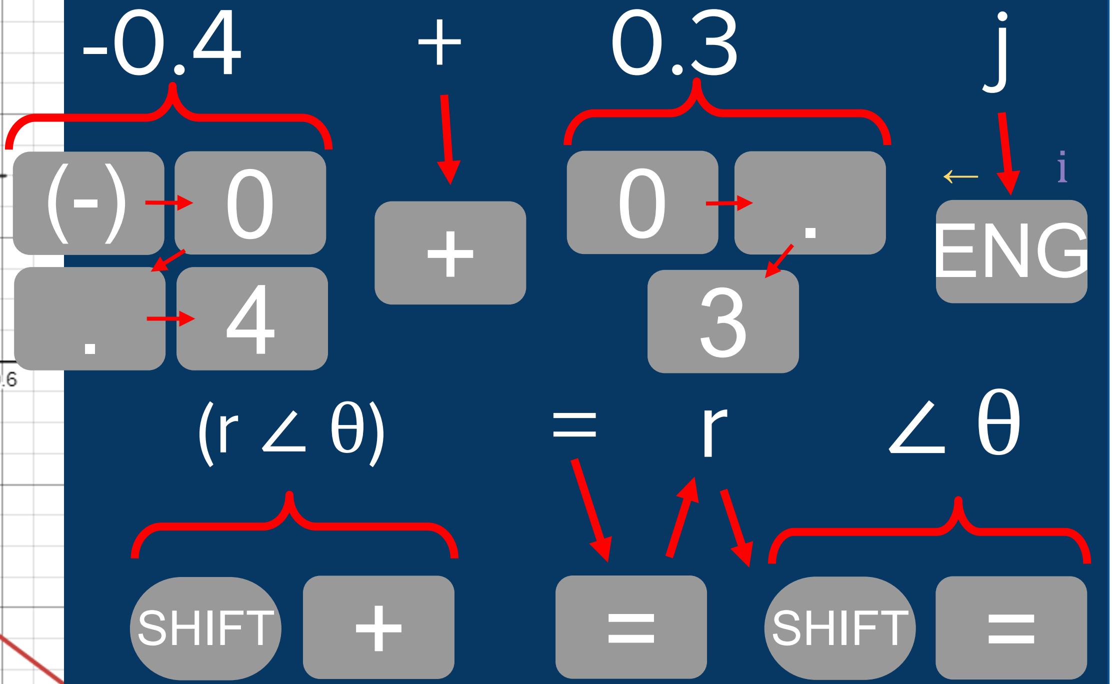
= SHIFT =

Answer: 9.49 ∠ 71.56°

Phasor Conversion - Solution

a) Convert the following to polar form ($r \angle \theta$):





Answer: $0.5 \angle 143.13°$

Problem 1.28 – Question

Find the phasors of the following time functions:

a) $v(t) = 9\cos\left(\omega t - \frac{\pi}{3}\right)$ (V)

b) $v(t) = 12\sin\left(\omega t + \frac{\pi}{4}\right)$ (V)

c) $i(x,t) = 5e^{-3x}\sin\left(\omega t + \frac{\pi}{6}\right)$ (A)

d) $i(t) = -2\cos\left(\omega t + \frac{3\pi}{4}\right)$ (A)

e) $i(t) = 4\sin\left(\omega t + \frac{\pi}{3}\right) + 3\cos\left(\omega t - \frac{\pi}{6}\right)$ (A)

Problem 1.28 – Solution (a)

a) $v(t) = 9\cos\left(\omega t - \frac{\pi}{3}\right)$ (V)

Adopt cosine reference ($v(t) = V_0 \cos(\omega t + \phi)$). Is this already done?

- Yes! $V_0 = 9$, $\phi = -\frac{\pi}{3}$

From sinusoidal time domain to phasor domain:

- $v(t) = V_0 \cos(\omega t + \phi) \;\rightarrow \mathbf{V} = \mathrm{V}_0 e^{j\phi}$

$$\therefore v(t) = 9\cos\left(\omega t - \frac{\pi}{3}\right) \text{ (V)} \rightarrow \boldsymbol{\tilde{V} = 9e^{-j\frac{\pi}{3}}\ (V)}$$

Problem 1.28 – Solution (b)

b) $v(t) = 12\sin\left(\omega t + \frac{\pi}{4}\right)$ (V)

Adopt cosine reference ($v(t) = V_0\cos(\omega t + \phi)$). Is this already done?

- No! → $v(t) = 12\sin\left(\omega t + \frac{\pi}{4}\right) = 12\cos(\omega t + \frac{\pi}{4} - \frac{\pi}{2}) = 12\cos(\omega t - \frac{\pi}{4})$
- $V_0 = 12, \phi = -\frac{\pi}{4}$

Sinusoidal time domain → phasor domain: $v(t) = V_0\cos(\omega t + \phi) \rightarrow \mathbf{V} = \mathrm{V}_0 e^{j\phi}$

$$\therefore v(t) = 12\sin\left(\omega t + \frac{\pi}{4}\right)(\mathrm{V}) = 12\cos\left(\omega t - \frac{\pi}{4}\right)(V) \rightarrow \boldsymbol{V}^{\sim} = \mathbf{12}\boldsymbol{e}^{-j\frac{\pi}{4}}\,(\boldsymbol{V})$$

Problem 1.28 – Solution (c)

c) $i(x,t) = 5e^{-3x}\sin\left(\omega t + \frac{\pi}{6}\right)$ (A)

Adopt cosine reference ($i(t) = V_0\cos(\omega t + \phi)$). Is this already done?

- No! →$i(x,t) = 5e^{-3x}\sin\left(\omega t + \frac{\pi}{6}\right) = 5e^{-3x}\cos(\omega t + \frac{\pi}{6} - \frac{\pi}{2}) = 5e^{-3x}\cos(\omega t - \frac{\pi}{3})$
- $V_0 = 5e^{-3x}, \phi = -\frac{\pi}{3}$

Sinusoidal time domain → phasor domain: $v(t) = V_0\cos(\omega t + \phi) \rightarrow \mathbf{V} = \mathrm{V}_0 e^{j\phi}$

$\therefore i(x,t) = 5e^{-3x}\sin\left(\omega t + \frac{\pi}{6}\right)$ (A) $= 5e^{-3x}\cos(\omega t - \frac{\pi}{3})\,(A) \rightarrow \boldsymbol{V^{\sim} = 5e^{-3x}e^{-j\frac{\pi}{3}}\,(V)}$

Problem 1.28 – Solution (d)

d) $i(t) = -2\cos\left(\omega t + \frac{3\pi}{4}\right)$ (A)

Adopt cosine reference ($i(t) = I_0 \cos(\omega t + \phi)$). Is this already done?

- Yes! $I_0 = -2,\ \phi = \frac{3\pi}{4}$
- One missing step: $-1 = e^{-j\pi}$ → $I_0 = -2 = 2(-1) = 2e^{-j\pi}$

Sinusoidal time domain → phasor domain $i(t) = I_0 \cos(\omega t + \phi) \ \rightarrow \mathbf{I} = \mathrm{I}_0 e^{j\phi}$

$$\therefore i(t) = -2\cos\left(\omega t + \frac{3\pi}{4}\right)\ (\mathrm{A}) \rightarrow I = 2e^{-j\pi}e^{j\frac{3\pi}{4}}\ (A) \rightarrow \boldsymbol{I = 2e^{-j\frac{\pi}{4}}\ (A)}$$

Problem 1.28 – Solution (e)

e) $i(t) = 4\sin\left(\omega t + \frac{\pi}{3}\right) + 3\cos\left(\omega t - \frac{\pi}{6}\right)\ (\mathrm{A})$

Adopt cosine reference ($i(t) = I_0\cos(\omega t + \phi)$). Is this already done?

- Kinda… →$4\sin\left(\omega t + \frac{\pi}{3}\right) = 4\cos\left(\omega t + \frac{\pi}{3} - \frac{\pi}{2}\right) = 4\cos\left(\omega t - \frac{\pi}{6}\right)$
- $I_0 = 4,\ \phi = -\frac{\pi}{6}$ and $I_0 = 3,\ \phi = -\frac{\pi}{6}$

Sinusoidal time domain → phasor domain $i(t) = I_0\cos(\omega t + \phi)\ \rightarrow \mathbf{I} = \mathrm{I}_0 e^{j\phi}$

$\therefore i(t) = 4\sin\left(\omega t + \frac{\pi}{3}\right) + 3\cos\left(\omega t - \frac{\pi}{6}\right)(\mathrm{A}) \rightarrow \boldsymbol{I} = 4e^{-j\frac{\pi}{6}} + 3e^{-j\frac{\pi}{6}} \rightarrow \boldsymbol{I = 7e^{-j\frac{\pi}{6}}(A)}$

ENGPHYS 2A04 Tutorial 3

Electricity and Magnetism

Your TAs Today

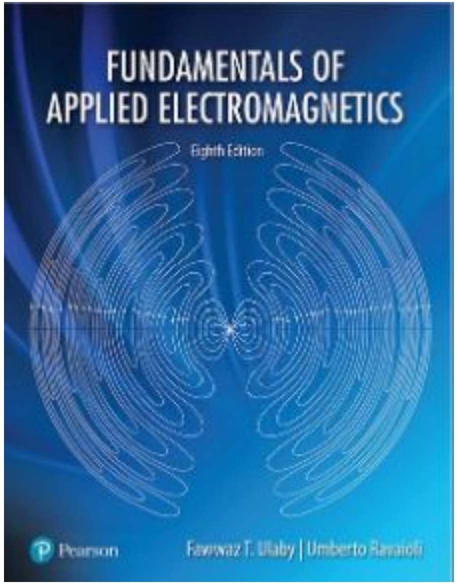
- Joanne Lee
  leej298@mcmaster.ca
- Fatemeh Bakhshandeh
  bakhshaf@mcmaster.ca

Your Textbook

Fundamentals of Applied Electromagnetics Eighth Edition.

Ulaby & Ravaioli

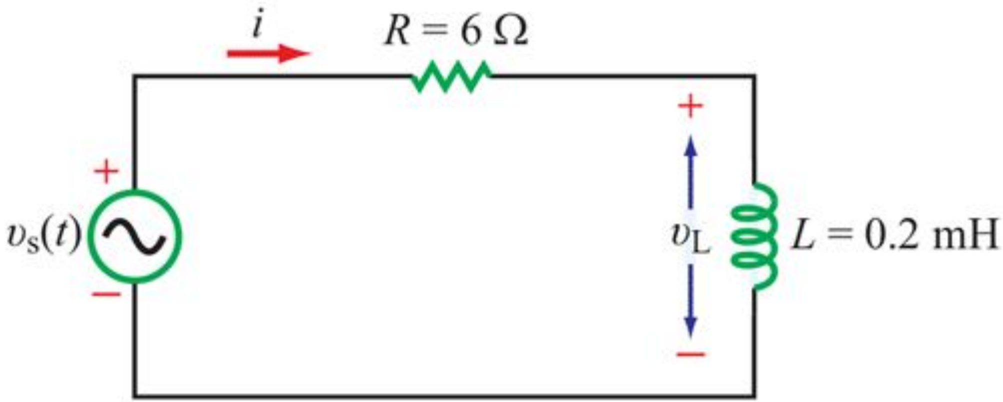
Seventh edition also acceptable, with some inconsistencies



Lecture Problem (Example 1-4 Page 40)

The voltage source of the RL circuit shown below is given by:

V_s (t) =5 sin($4*10^4$t-30 °). Find an expression for the voltage across the inductor.



Lecture Problem Solution

Before converting to the phasor domain, we express the initial equation in terms of cosine.

$$\begin{aligned} v_s(t) &= 5\sin(4 \times 10^4 t - 30^\circ) \\ &= 5\cos(4 \times 10^4 t - 120^\circ) \qquad \text{(V)}. \end{aligned}$$

The corresponding voltage phasor equation is

$$\widetilde{V}_s = 5e^{-j120^\circ} \qquad \text{(V)},$$

Lecture Problem Solution

The voltage loop equation of the RL circuit is $$Ri + L\frac{di}{dt} = v_s(t).$$

The corresponding phasor equation is $$R\tilde{I} + j\omega L\tilde{I} = \tilde{V}_s.$$

Solving for current phasor ***I*** the equation becomes:

$$I = \frac{V_s}{R + j\omega L}$$

$$I = \frac{5e^{-j120°}}{6 + j4 * 10^4 * 2 * 10^{-4}}$$

$$I = \frac{5e^{-j120°}}{6 + j8}$$

$$I = \frac{5e^{-j120°}}{10e^{j53.1°}}$$

$$I = 0.5e^{-j173.1°}$$

Lecture Problem Solution

The voltage phasor across the inductor is related to the current *I* by

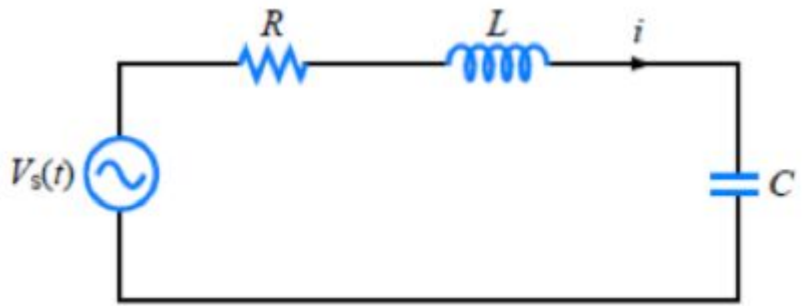
$$\begin{aligned}
\widetilde{V}_{\mathrm{L}} &= j\omega L\widetilde{I} \\
&= j4\times 10^{4}\times 2\times 10^{-4}\times 0.5e^{-j173.1^{\circ}} \\
&= 4e^{j(90^{\circ}-173.1^{\circ})} = 4e^{-j83.1^{\circ}} \qquad \text{(V)},
\end{aligned}$$

Corresponding Instantaneous Voltage $V_L(t)$ is:

$$\begin{aligned}
v_{\mathrm{L}}(t) &= \mathfrak{Re}\left[\widetilde{V}_{\mathrm{L}}e^{j\omega t}\right] \\
&= \mathfrak{Re}\left[4e^{-j83.1^{\circ}}e^{j4\times 10^{4}t}\right] \\
&= 4\cos(4\times 10^{4}t-83.1^{\circ})
\end{aligned}$$

Problem 33.4 - Phasor Method

A series RLC AC circuit has: $R = 425\ \Omega$, $L = 1.25$ H, $C = 3.50\ \mu F$, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V.



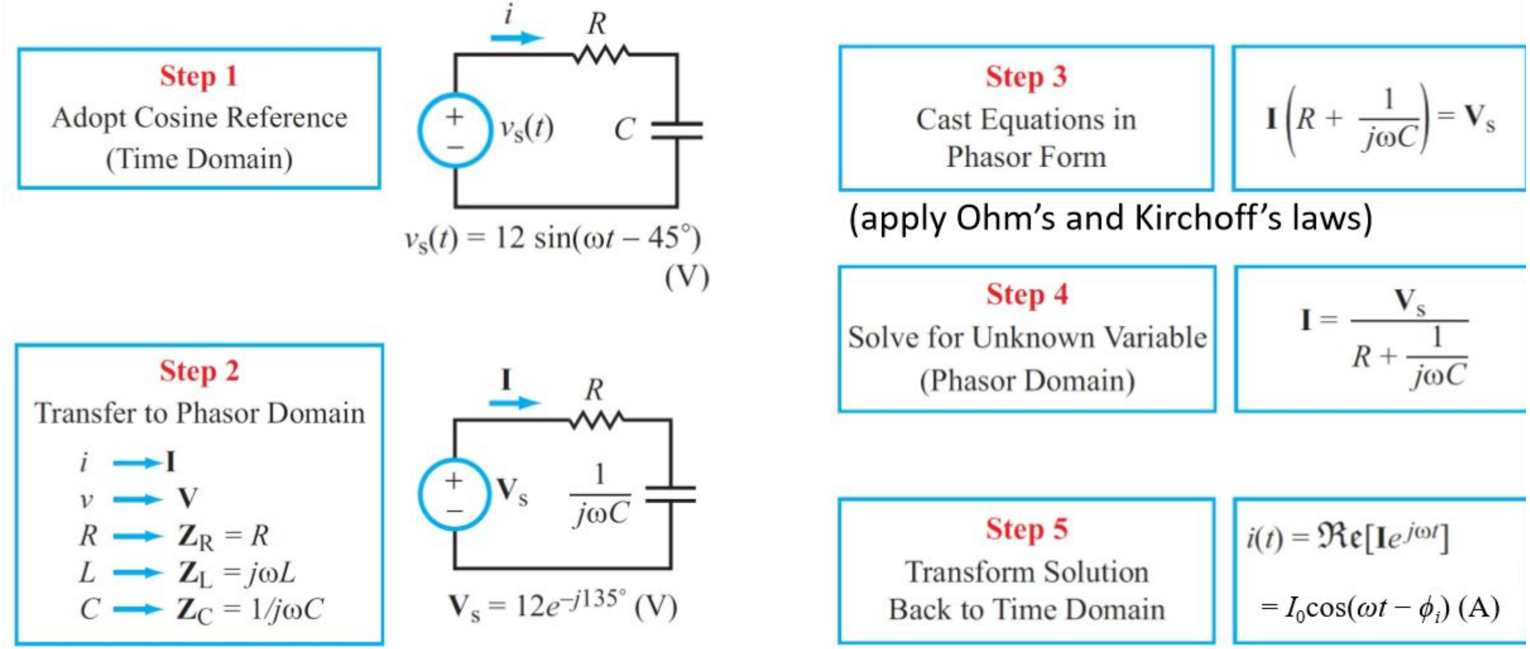
a) Determine the inductive reactance and the capacitive reactance of the circuit.

b) Find the impedance and phase angle between the current and voltage.

c) Find the maximum current in the circuit.

d) Find both the maximum voltage and the instantaneous voltage across each element.

AC Phasor Analysis



(apply Ohm's and Kirchoff's laws)

Example 33.4 Initial Steps

First create the time domain cosine equation for the given circuit using the information given.

$$V_s(t) = \Delta V_{max} \cos(\omega t + \phi)$$

$$V_s(t) = 150 \cos(377t + \phi)$$

Then transfer the equation into the phasor domain.

$$i \rightarrow \tilde{I}$$

$$V_s(t) = \tilde{V} = \Delta V_{max} e^{j\phi}$$

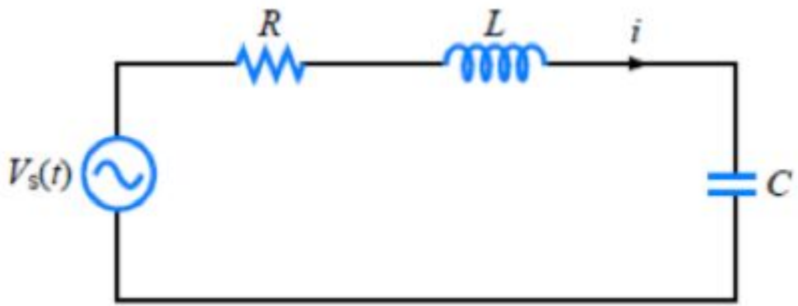
$$R \rightarrow Z_R = R = 425\ \Omega$$

$$L \rightarrow Z_L = j\omega L = jX_L$$

$$C \rightarrow Z_C = \frac{1}{j\omega C} = -jX_C$$

Example 33.4 Solution to A)

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V.



a) Determine the inductive reactance and capacitive reactance of circuit.

$$X_L = \omega L = (377 s^{-1})(1.25\ H)$$

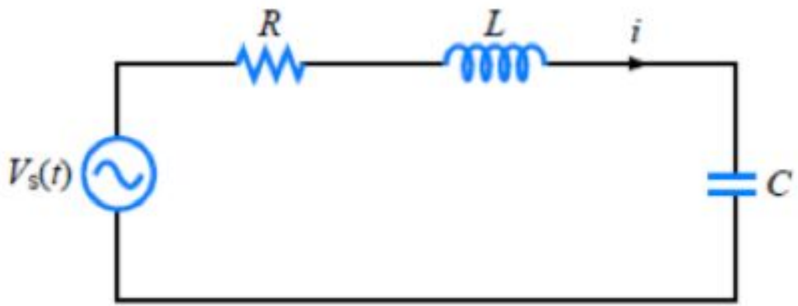
$$X_L = 471\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(377 s^{-1})(3.50 * 10^{-6}\ F)}$$

$$X_C = 758\ \Omega$$

Example 33.4 Solution to B)

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V.



b) Find the impedance and phase angle between current and voltage

$$Z = Z_R + Z_L + Z_C = R + jX_L - jX_C$$

$$Z = 425 + 471j - 758j$$

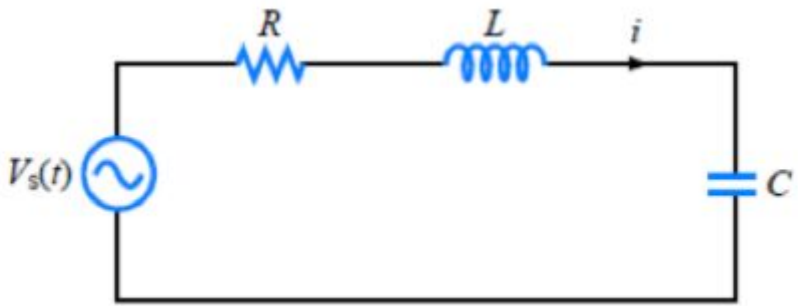
$$Z = 425 - 287j\ \Omega$$

Convert to phasor using method from before.

$$Z \cong 513\ \Omega\angle -34.0°$$

Example 33.4 Solution to C)

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V.



c) Find the maximum current by applying Kirchoff's Rules in phasor form and solve for unknowns.

$$\tilde{V} = V_R + V_L + V_C$$

$$\tilde{I} = (Z_R + Z_L + Z_C) = \tilde{I}Z$$

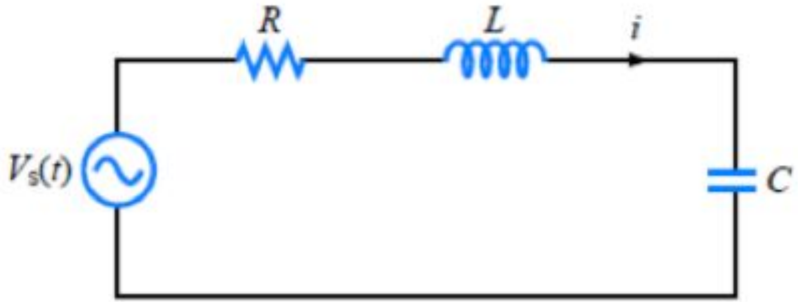
$$\tilde{I} = \frac{\tilde{V}}{Z}$$

The max current occurs when: $\tilde{V} = \Delta V_{max}$

$$\tilde{I}_{max} = \frac{\Delta V_{max}}{|Z|} = \frac{150\ V}{513\ \Omega} = 0.292\ A$$

Example 33.4 Solution to D) Max Voltage

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, ω = 377 s^{-1}, and ΔV_{max} = 150 V.



c) Find both the maximum voltage and the instantaneous voltage across each element.

Max voltage occurs when: $\tilde{I} = \tilde{I}_{max}$

$$\Delta V_R = \tilde{I}_{max} Z_R = (0.292\ A)(425\ \Omega)$$

$$\Delta V_R = 124\ V$$

$$\Delta V_L = \tilde{I}_{max} Z_L = (0.292\ A)(471j\ \Omega)$$

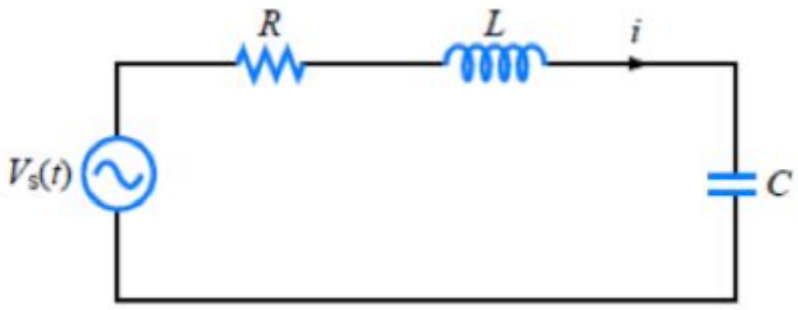
$$\Delta V_L = 138j\ V$$

$$\Delta V_C = \tilde{I}_{max} Z_C = (0.292\ A)(-758j\ \Omega)$$

$$\Delta V_C = -222j\ V$$

Example 33.4 Solution to D) Instantaneous Voltage

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, ω = 377 s^{-1}, and ΔV_{max} = 150 V.



c) Find both the maximum voltage and the instantaneous voltage across each element.

Transform all solutions back to time domain.

$$v(t) = \Re\mathrm{e}\left[V_0 e^{j\phi} e^{j\omega t}\right] = V_0 \cos(\omega t + \phi)$$

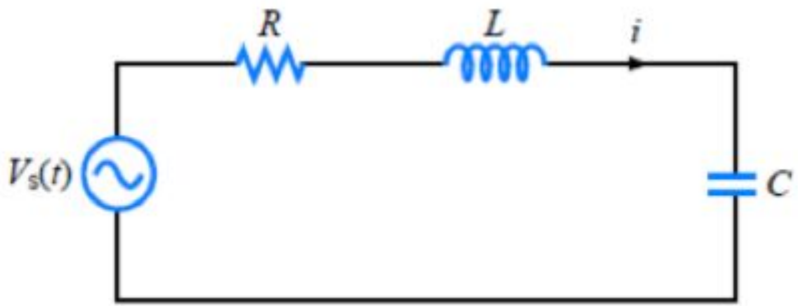
$$\Delta V_R = V_0 e^{j\phi} = 124\ V$$

$$\Delta V_L = V_0 e^{j\phi} = 138j = 138 e^{\frac{j\pi}{2}}\ V$$

$$\Delta V_C = V_0 e^{j\phi} = -222j = 222 e^{-\frac{j\pi}{2}}\ V$$

Example 33.4 Solution to D) Instantaneous Voltage

A series RLC AC circuit has: R = 425 Ω, L = 1.25 H, C = 3.50 μF, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V.



c) Find both the maximum voltage and the instantaneous voltage across each element.

$$v(t) = \Re e\left[V_0 e^{j\phi} e^{j\omega t}\right] = V_0 \cos(\omega t + \phi)$$

$$v_R(t) = \Re e\left[\Delta V_R e^{j\omega t}\right] = \Re e\left[124 e^{j377t}\right]$$

$$v_R(t) = 124 \cos 377t$$

$$v_L(t) = \Re e\left[\Delta V_L e^{j\omega t}\right] = \Re e\left[138 e^{\frac{j\pi}{2}} e^{j377t}\right]$$

$$v_L(t) = 138 \sin 377t$$

$$v_C(t) = \Re e\left[\Delta V_C e^{j\omega t}\right] = \Re e\left[222 e^{\frac{-j\pi}{2}} e^{j377t}\right]$$

$$v_C(t) = -222 \sin 377t$$

Example 2.1

A transmission line of length l connects to a sinusoidal voltage source with an oscillation frequency f. Assuming that the velocity of wave propagation on the line is c, for which of the following situations is it reasonable to ignore the presence of the transmission line in the solution of the circuit.

a) $l = 20$ cm, $f = 20$ kHz,
b) $l = 50$ km, $f = 60$ Hz
c) $l = 20$ cm, $f = 600$ MHz,
d) $l = 1$ mm, $f = 100$ GHz,

Example 2.1 Solution

Transmission line effects are negligible when $\frac{l}{\lambda} < 0.01$

It may be necessary to account for both the phase shift due to time delay and the presence of reflected signals that may have bounced back.

Line length

Frequency of the oscillating wave

$$\boxed{\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{lf}{c} = \frac{lf}{3\times10^8\ m/s}}$$

Wavelength

Velocity of wave propagation

Speed of light

$$\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{lf}{c} = \frac{lf}{3 * 10^8\ m/s}$$

Example 2.1 Solution

a) $l = 20\ cm,\ f = 20\ kHz$ $= \frac{lf}{3 * 10^8\ m/s} = \frac{20 * 10^{-3}\ m * 20 * 10^3\ Hz}{3 * 10^8\ m/s} = 1.33 * 10^{-5}$ ← negligible

b) $l = 50\ km,\ f = 60\ Hz$ $= \frac{lf}{3 * 10^8\ m/s} = \frac{50 * 10^3\ m * 60\ Hz}{3 * 10^8\ m/s} = 0.01$ ← included

c) $l = 20\ cm,\ f = 600\ MHz,$ $= \frac{lf}{3 * 10^8\ m/s} = \frac{20 * 10^{-2}\ m * 600 * 10^6\ Hz}{3 * 10^8\ m/s} = 0.4$ ← included

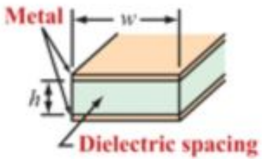
d) $l = 1\ mm,\ f = 100\ GHz,$ $= \frac{lf}{3 * 10^8\ m/s} = \frac{1 * 10^{-3}\ m * 20 * 10^9\ Hz}{3 * 10^8\ m/s} = 0.33$ ← included

Example 2.4

A 1 GHz parallel-plate transmission line consists of 1.2 cm wide copper strips separated by a 0.15 cm thick layer of polystyrene.

Appendix B gives $\mu_c = \mu_0 = 4\pi * 10^{-7}\ H/m\ and\ \sigma_c = 5.8 * 10^7 S/m\ for\ copper, and\ \varepsilon_r = 2.6$

for polystyrene. Use Table 2-1 to determine the line parameters of the transmission line. Assume that $\mu_c = \mu_0$ and $\sigma \approx 0$ for polystyrene.



Example 2.4 Solution

From table 2-1, the following parameter formulas can be determined for a parallel-plate:

$$R' = \frac{2R_S}{w}$$

$$L' = \frac{\mu h}{w} \qquad R_S = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

$$G' = \frac{\sigma w}{h}$$

$$C' = \frac{\epsilon w}{h}$$

For copper:

$$w = 1.2\ cm = 0.012\ m$$
$$\mu_c = \mu_0 = 4\pi * 10^{-7}\ H/m$$
$$\sigma_c = 5.8 * 10^{8}\ S/m$$

Other values:

$$f = 1\ GHz = 1 * 10^{9}\ Hz$$
$$\varepsilon = 8.854 * 10^{-12}\ F/m$$

For Polystyrene:

$$h = 0.15\ cm = 0.0015\ m$$
$$\mu = \mu_0 = 4\pi * 10^{-7}\ H/m$$
$$\sigma = 0$$
$$\varepsilon_r = 2.6$$

Example 2.4 Solution

From table 2-1, the following parameter formulas can be determined for a parallel-plate:

$$R' = \frac{2R_S}{w} \qquad R' = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{2}{1.2 * 10^{-2}}\sqrt{\frac{\pi * 1 * 10^9 * 4\pi * 10^{-7}}{5.8 * 10^7}} = 1.38\ \Omega/m$$

$$L' = \frac{\mu h}{w} \qquad L' = \frac{\mu h}{w} = \frac{4\pi * 10^{-7} * 0.0015}{0.012} = 1.57 * 10^{-7}\ H/m$$

$$G' = \frac{\sigma w}{h} \qquad G' = \frac{\sigma w}{h} = 0\ S/m$$

$$C' = \frac{\epsilon w}{h} \qquad C' = \frac{\epsilon w}{h} = \frac{\varepsilon_0 \varepsilon_r w}{h} = \frac{8.854 * 10^{-12} * 2.6 * 0.012}{0.0015} = 1.84 * 10^{-10}\ F/m$$

EP2A04 TUTORIAL 4

FEBRUARY 7TH, 2022

TEACHING ASSISTANTS

- Fraser McCauley, mccaulj@mcmaster.ca
- Muhammad Munir, munirm6@mcmaster.ca

QUESTION 2.5

For the parallel-plate transmission line of Problem 2.4, the line parameters are given by $R' = 1\ \Omega/m, L' = 167\ nH/m, G' = 0$ and $C' = 172\ pF/m$. Find α, β, u_P and Z_0 at $1\ GHz$.

Givens:

$$R' = 1\ \Omega/m$$
$$L' = 167\ nH/m$$
$$G' = 0$$
$$C' = 172\ pF/m$$
$$f = 1\ GHz$$

Plan:

- Find $\gamma = \alpha + j\beta$
- Isolate real and imaginary components
- Use β and ω to obtain u_P
- Find Z_0 using the transmission line parameters

Unknowns: α, β, u_P, Z_0

Find $\gamma = \alpha + j\beta$

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$$\gamma = \sqrt{\left(1 + j(2\pi \times 10^9)(167 \times 10^{-9})\right)\left(j(2\pi \times 10^9)(172 \times 10^{-12})\right)}$$

$$\gamma = 0.016 + j33.7$$

QUESTION 2.5

For the parallel-plate transmission line of Problem 2.4, the line parameters are given by $R' = 1\ \Omega/m, L' = 167\ nH/m, G' = 0$ and $C' = 172\ pF/m$. Find α, β, u_P and Z_0 at $1\ GHz$.

Isolate real and imaginary components

$$\alpha = \Re e\{\gamma\} = \Re e\{0.016 + j33.7\} = 0.016\,\frac{Np}{m}$$

$$\beta = \Im m\{\gamma\} = \Im m\{0.016 + j33.7\} = 33.7\,\frac{rad}{m}$$

Use β and ω to obtain u_P

$$u_P = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi(1\times 10^9)}{33.7} = 1.86\times 10^8\ m/s$$

Find Z_0 using the transmission line parameters

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

$$= \sqrt{\frac{1 + j(2\pi\times 10^9)(167\times 10^{-9})}{j(2\pi\times 10^9)(172\times 10^{-12})}}$$

$$= \sqrt{\frac{1 + j1049.3}{j1.081}} = \sqrt{970.7 - j0.9251}$$

$$= \sqrt{970.7e^{-j9.53\times 10^{-4}}} = 31.2e^{-j4.765\times 10^{-4}}$$

$$= (31.2 - j0.015)\ \Omega$$

QUESTION 2.6

A coaxial line with inner and outer conductor diameters of 0.5 cm and 1 cm, respectively, is filled with an insulating material with $\epsilon_r = 4.5$ and $\sigma = 10^{-3}\ S/m$. The conductors are made of copper. Calculate the line parameters at 1 GHz.

Givens:

$d_{inner} = 0.5\ cm$

$d_{outer} = 1\ cm$

$\epsilon_r = 4.5$

$\sigma = 10^{-3}\ \frac{S}{m}$

$f = 1\ GHz$

Hidden givens:

Copper parameters!

$\sigma_{Cu} = 5.8 \times 10^7\ S/m$

$\mu_{r-Cu} = 1$

Unknowns: R', L', G', C'

Plan:

- Compute R_S
- Use the coaxial line formulae to compute the line parameters

Compute R_S

$$R_S = \sqrt{\frac{\pi f \mu_C}{\sigma_C}} = \sqrt{\pi(1 \times 10^9)(4\pi \times 10^{-7})/(5.8 \times 10^7)}$$

$$= 8.2502 \times 10^{-3}$$

QUESTION 2.6

A coaxial line with inner and outer conductor diameters of 0.5 cm and 1 cm, respectively, is filled with an insulating material with $\epsilon_r = 4.5$ and $\sigma = 10^{-3}\ S/m$. The conductors are made of copper. Calculate the line parameters at 1 GHz.

Use the coaxial line formulae to compute the line parameters

$$R' = \frac{R_S}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{R_S}{2\pi}\left(\frac{2}{d_{inner}} + \frac{2}{d_{outer}}\right) = \frac{8.2502\times10^{-3}}{2\pi}\left(\frac{2}{0.5\times10^{-2}} + \frac{2}{1\times10^{-2}}\right) = 0.788\ \Omega/m$$

$$L' = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{\mu}{2\pi}\ln\left(\frac{d_{outer}}{d_{inner}}\right) = \frac{(4\pi\times10^{-7})}{2\pi}\ln\left(\frac{1}{0.5}\right) = 1.39\times10^{-7}\ H/m$$

$$G' = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi\times10^{-3}}{\ln(1/0.5)} = 9.06\times10^{-3}\ S/m$$

$$C' = \frac{2\pi\epsilon}{\ln(b/a)} = \frac{2\pi\epsilon_r\epsilon_0}{\ln(b/a)} = \frac{2\pi(4.5)(8.854\times10^{-12})}{\ln(1/0.5)} = 3.61\times10^{-10}\ F/m$$

QUESTION 2.18

Polyethylene with $\epsilon_r = 2.25$ is used as the insulating material in a lossless coaxial line with a characteristic impedance of 50Ω. The radius of the inner conductor is $1.2\ mm$. (A) What is the radius of the outer conductor? (B) What is the phase velocity of the line?

Plan:

- Write characteristic impedance equation using line parameters
- Solve for b
- Compute the phase velocity

Givens:

$$\epsilon_r = 2.25$$
$$Z_0 = 50\ \Omega$$
$$a = 1.2\ mm$$

Hidden givens:

Lossless! So…

$$R' = G' = 0$$

Write characteristic impedance equation using line parameters

Since R' and G' are 0, the equation simplifies to:

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)}{\left(\frac{2\pi\epsilon}{\ln(b/a)}\right)}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}\left(\ln\left(\frac{b}{a}\right)\right)^2} = \sqrt{\frac{\mu}{4\pi^2\epsilon}}\ln\left(\frac{b}{a}\right)$$

QUESTION 2.18

Polyethylene with $\epsilon_r = 2.25$ is used as the insulating material in a lossless coaxial line with a characteristic impedance of 50Ω. The radius of the inner conductor is $1.2\ mm$. (A) What is the radius of the outer conductor? (B) What is the phase velocity of the line?

Solve for b

$$Z_0 = \sqrt{\frac{\mu}{4\pi^2\epsilon}}\ln\left(\frac{b}{a}\right)$$

$$\frac{Z_0}{\sqrt{\frac{\mu}{4\pi^2\epsilon}}} = \ln\left(\frac{b}{a}\right)$$

$$b = a * \exp\left(\frac{Z_0}{\sqrt{\frac{\mu}{4\pi^2\epsilon}}}\right)$$

$$b = (1.2 \times 10^{-3}) * \exp\left(\frac{50}{\sqrt{\frac{4\pi \times 10^{-7}}{4\pi^2(2.25)(8.854 \times 10^{-12})}}}\right)$$

$$b = (1.2 \times 10^{-3}) * \exp\left(\frac{50}{\sqrt{\frac{10^5}{\pi(2.25)(8.854)}}}\right)$$

$$b = 4.2\ mm$$

Compute the phase velocity

Lossless line, so…

$$u_P = \frac{c}{\sqrt{\epsilon_r}}$$

$$u_P = \frac{3 \times 10^8}{\sqrt{2.25}}$$

$$u_P = 2 \times 10^8\ m/s$$

QUESTION 2.27

At an operating frequency of $300\ MHz$, a lossless $50\ \Omega$ air-spaced transmission line $2.5\ m$ in length is terminated with an impedance $Z_L = (40 + j20)\ \Omega$. Find the input impedance.

Givens
$f = 300\ MHz$
$Z_0 = 50\ \Omega$
$l = 2.5\ m$
$Z_L = (40 + j20)\ \Omega$

Unknowns: Z_i

Plan:
- Calculate the phase constant of the line
- Plug into the input impedance formula

Calculate the phase constant of the line

Since the line is air filled, $u_P = c$. Therefore…

$$\beta = \frac{\omega}{c} = \frac{2\pi(300 \times 10^6)}{3 \times 10^8} = 2\pi\ rad/s$$

Plug into the input impedance formula

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

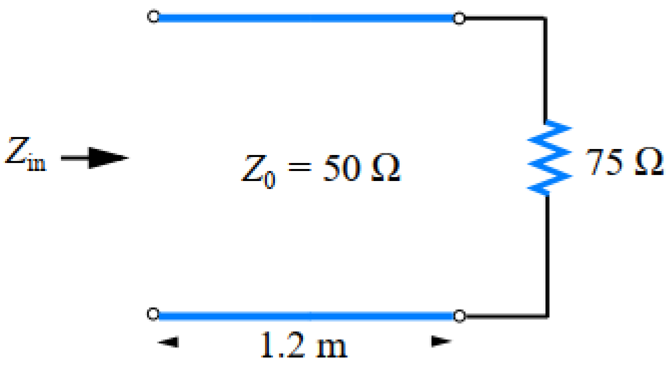
$$Z_i = 50 \frac{40 + j20 + j(50)\tan\big(2\pi(2.5)\big)}{50 + j(40 + j20)\tan\big(2\pi(2.5)\big)}$$

$$Z_i = 50 \frac{40 + j20}{50}$$

$$Z_i = (40 + j20)\ \Omega$$

QUESTION 2.35

For the lossless transmission line circuit shown in the figure, determine the equivalent series lumped-element circuit at $400\ MHz$ at the input to the line. The line has a characteristic impedance of $50\ \Omega$ and the insulating layer has $\epsilon_r = 2.25$.



Givens
$f = 400\ MHz$
$Z_0 = 50\ \Omega$
$l = 1.2\ m$
$Z_L = 75\ \Omega$
$\epsilon_r = 2.25$

Unknowns: Z_i

Plan:

- Compute phase constant of line
- Compute equivalent input impedance
- Design equivalent circuit based on impedance

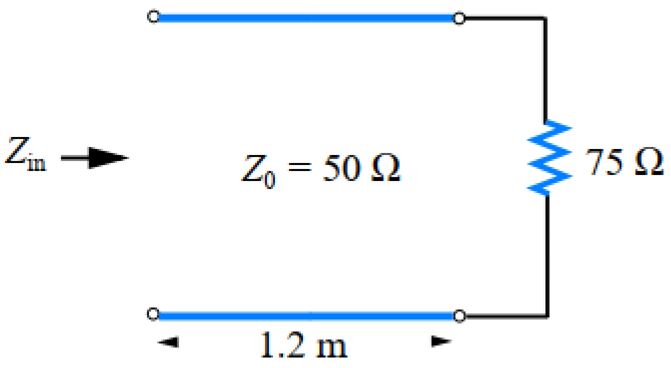
Compute phase constant of line

Lossless line, so $u_P = \frac{c}{\sqrt{\epsilon_r}} = \frac{3\times10^8}{\sqrt{2.25}} = 2 \times 10^8\ m/s$

$$\beta = \frac{\omega}{u_P} = \frac{2\pi(400 \times 10^6)}{2 \times 10^8} = 4\pi\ rad/m$$

QUESTION 2.35

For the lossless transmission line circuit shown in the figure, determine the equivalent series lumped-element circuit at $400\ MHz$ at the input to the line. The line has a characteristic impedance of $50\ \Omega$ and the insulating layer has $\epsilon_r = 2.25$.



Compute equivalent input impedance

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

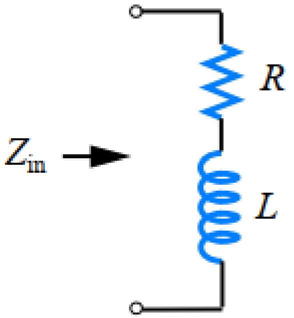
$$Z_i = 50 \frac{75 + j(50) \tan\big((4\pi)(1.2)\big)}{50 + j(75) \tan\big((4\pi)(1.2)\big)}$$

$$Z_i = 50 \frac{75 - j36.327}{50 - j54.491}$$

$$Z_i = (52.38 + j20.76)\ \Omega$$

Design equivalent circuit based on impedance

Imaginary part is positive, so if this is a series circuit, it must be a resistor and an inductor!



$$R = 52.38\ \Omega$$

$$L = \frac{20.76}{\omega} = 8.3\ nH$$

REMINDER

- Assignment 4 is out now, and is due at 8AM on February 14
- 4 calculation questions, 1 research question, 1 bonus derivation question
- Submit PDFs please!
- Good luck!

ENG PHYS 2A04 Tutorial 5

Electricity and Magnetism

Your TAs today

- Yang Liu

  liuy514@mcmaster.ca

- Muhammad Munir Khan

  munirm6@mcmaster.ca

Chapter 3

Problem 3.17 – Question

Find a vector **G** whose magnitude is 4 and whose direction is perpendicaular to both vectors **E** and **F**, where:

$\mathbf{E} = \hat{x} + \hat{y}2 - \hat{z}2$ and $\mathbf{F} = \hat{y}3 - \hat{z}6$

Problem 3.17 – Question

Find a vector **G** whose magnitude is 4 and whose direction is perpendicular to both vectors **E** and **F**, where:

$\mathbf{E} = \hat{x} + \hat{y}2 - \hat{z}2$ and $\mathbf{F} = \hat{y}3 - \hat{z}6$

Eq. 3.1: $\boldsymbol{G} = G\hat{\boldsymbol{g}}$

Eq. 3.22: vector normal to the plane containing **E** and **F**: **E**×**F**

Eq. 3.22: unit vector: $\dfrac{\mathbf{E}\times\mathbf{F}}{|\mathbf{E}\times\mathbf{F}|}$

Problem 3.17 – Question

Find a vector **G** whose magnitude is 4 and whose direction is perpendicaular to both vectors **E** and **F**, where:

$\mathbf{E} = \hat{x} + \hat{y}2 - \hat{z}2$ and $\mathbf{F} = \hat{y}3 - \hat{z}6$

$$\mathbf{G} = \pm 4 \frac{\mathbf{E}\times\mathbf{F}}{|\mathbf{E}\times\mathbf{F}|}$$

Problem 3.17 – Question

Find a vector **G** whose magnetude is 4 and whose direction is perpendicular to both vectors **E** and **F**, where:

$\mathbf{E} = \hat{x} + \hat{y}2 - \hat{z}2$ and $\mathbf{F} = \hat{y}3 - \hat{z}6$

Eq.3.28: $$\boldsymbol{E}\times\boldsymbol{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \boldsymbol{E_x} & \boldsymbol{E_y} & \boldsymbol{E_z} \\ \boldsymbol{F_x} & \boldsymbol{F_y} & \boldsymbol{F_z} \end{vmatrix}$$

$$= \left(\boldsymbol{E_y}\,\boldsymbol{F_z} - \boldsymbol{E_z}\,\boldsymbol{F_y}\right)\hat{x} + \left(\boldsymbol{E_z F_x} - \boldsymbol{E_x F_z}\right)\hat{y} + \left(\boldsymbol{E_x F_y} - \boldsymbol{E_y F_x}\right)\hat{z}$$

$$= -\hat{x}6 + \hat{y}6 + \hat{z}3$$

Problem 3.17 – Question

Find a vector **G** whose magnetude is 4 and whose direction is perpendicaular to both vectors **E** and **F**, where:

$\mathbf{E} = \hat{x} + \hat{y}2 - \hat{z}2$ and $\mathbf{F} = \hat{y}3 - \hat{z}6$

Eq. 3.4: $|\mathbf{E}\times\mathbf{F}| = +\sqrt{(-6)^2 + (6)^2 + (3)^2} = 9$

$$\mathbf{G} = \pm 4\frac{\mathbf{E}\times\mathbf{F}}{|\mathbf{E}\times\mathbf{F}|} = \pm 4\frac{(-\hat{x}6 + \hat{y}6 + \hat{z}3)}{9} = \pm(-\hat{x}\frac{8}{3} + \hat{y}\frac{8}{3} + \hat{z}\frac{4}{3})$$

Problem 3.26 – Question

Find the volume described by the following, Also sketch the outline of the volume:

$$2 \le r \le 5; \frac{\pi}{2} \le \varphi \le \pi; 0 \le z \le 2$$

Recongnize that this is a cylindrical integration

Problem 3.26 – Question

Find the volume described by the following, Also sketch the outline of the volume:

$$2 \leq r \leq 5; \frac{\pi}{2} \leq \varphi \leq \pi; 0 \leq z \leq 2$$

Table 3–1: $V = \int_{z=0}^{2} \int_{\varphi=\pi/2}^{\pi} \int_{r=2}^{5} r dr d\varphi dz$

Problem 3.26 – Question

Find the volume described by the following, Also sketch the outline of the volume:

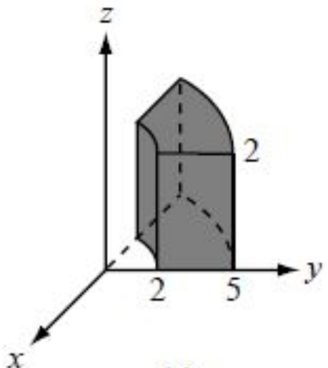
$$2 \leq r \leq 5; \frac{\pi}{2} \leq \varphi \leq \pi; 0 \leq z \leq 2$$

$$V = \left(\frac{1}{2} r^2 \Big|_{r=2}^{5}\right)\left(\varphi \Big|_{\varphi=\pi/2}^{\pi}\right)\left(z \Big|_{z=0}^{2}\right) = \frac{21\pi}{2}$$

Problem 3.26 – Question

Find the volume described by the following, Also sketch the outline of the volume:

$$2 \leq r \leq 5; \frac{\pi}{2} \leq \varphi \leq \pi; 0 \leq z \leq 2$$



(a)

Problem 3.34 – Question

Transform the following vectors into cylindrical coordinates and then evaluate them at the indicated points:

a) $\mathbf{A} = \hat{\boldsymbol{x}}(x + y)$ at $P_1 = (1,2,3)$

b) $\mathbf{C} = \hat{\boldsymbol{x}}\frac{y^2}{(x^2+y^2)} - \hat{\boldsymbol{y}}\frac{x^2}{(x^2+y^2)} + \hat{\boldsymbol{z}}4$ at $P_2 = (1,-1,2)$

Problem 3.34 – Question

$\mathbf{A} = \hat{\boldsymbol{x}}(x + y)$ at $P_1 = (1,2,3)$

Convert to cylindrical coordinates

Table 3-2: $r = +\sqrt{x^2 + y^2} = +\sqrt{1^2 + 2^2} = \sqrt{5},$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 63.4°$$

$$z = z = 3$$

$$P_1 = \left(\sqrt{5}, 63.4°, 3\right)$$

Problem 3.34 – Question

$\mathbf{A} = \hat{\boldsymbol{x}}(x + y)$ at $P_1 = (1,2,3)$

Convert to cylindrical coordinates

Table 3-2: $\boldsymbol{A} = \hat{\boldsymbol{r}}\mathrm{A}_r + \hat{\boldsymbol{\varphi}}\mathrm{A}_\varphi + \hat{\mathbf{z}}\mathrm{A}_z$

$$\boldsymbol{x} = \boldsymbol{r}\cos\varphi;\ \boldsymbol{y} = \boldsymbol{r}\sin\varphi$$

$$\mathrm{A}_r = \mathrm{A}_x\cos\varphi + \mathrm{A}_y\sin\varphi = (\boldsymbol{x} + \boldsymbol{y})\cos\varphi = (\boldsymbol{r}\cos\varphi + \boldsymbol{r}\sin\varphi)\cos\varphi$$

$$\mathrm{A}_\varphi = -\mathrm{A}_x\sin\varphi + \mathrm{A}_y\cos\varphi = -(\boldsymbol{x} + \boldsymbol{y})\sin\varphi = -(\boldsymbol{r}\cos\varphi + \boldsymbol{r}\sin\varphi)\sin\varphi$$

$$\mathrm{A}_z = 0$$

$$\mathbf{A} = \hat{\boldsymbol{r}}r\cos\varphi\,(\cos\varphi + \sin\varphi) - \hat{\boldsymbol{\varphi}}r\sin\varphi\,(\cos\varphi + \sin\varphi)$$

Problem 3.34 – Question

$\mathbf{A} = \hat{\boldsymbol{x}}(x+y)$ at $P_1 = (1,2,3)$

$$\mathbf{A} = \hat{\boldsymbol{r}}r\cos\varphi\,(\cos\varphi + \sin\varphi) - \hat{\boldsymbol{\varphi}}r\sin\varphi\,(\cos\varphi + \sin\varphi)$$

$$\boldsymbol{P_1} = \left(\sqrt{5}, 63.4°, 3\right)$$

$$\mathbf{A}(\boldsymbol{P_1}) = \hat{\boldsymbol{r}}1.34 - \hat{\boldsymbol{\varphi}}2.68$$

Problem 3.34 – Question

$\mathbf{C} = \hat{\boldsymbol{x}}\frac{y^2}{(x^2+y^2)} - \hat{\boldsymbol{y}}\frac{x^2}{(x^2+y^2)} + \hat{\mathbf{z}}4$ at $P_2 = (1,-1,2)$

Convert to cylindrical coordinates

Table 3-2: $r = +\sqrt{x^2 + y^2} = +\sqrt{1^2 + (-1)^2} = \sqrt{2}$,

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{1}\right) = -45°$$

$$z = z = 2$$

$$P_2 = \left(\sqrt{2}, -45°, 2\right)$$

Problem 3.34 – Question

$\mathbf{C} = \hat{\boldsymbol{x}}\frac{y^2}{(x^2+y^2)} - \hat{\boldsymbol{y}}\frac{x^2}{(x^2+y^2)} + \hat{\mathbf{z}}4$ at $P_2 = (1,-1,2)$

Table 3-2: $\boldsymbol{C} = \hat{\boldsymbol{r}}C_r + \hat{\boldsymbol{\varphi}}C_\varphi + \hat{\mathbf{z}}C_z$

$$\boldsymbol{x} = \boldsymbol{r}\cos\varphi;\ \boldsymbol{y} = \boldsymbol{r}\sin\varphi$$

$$C_r = \sin\varphi\cos\varphi\,(\sin\varphi - \cos\varphi)$$

$$C_\varphi = -(\sin^3\varphi + \cos^3\varphi)$$

$$C_z = 4$$

$$C = \hat{\boldsymbol{r}}\sin\varphi\cos\varphi\,(\sin\varphi - \cos\varphi) - \hat{\boldsymbol{\varphi}}(\sin^3\varphi + \cos^3\varphi) + \hat{\mathbf{z}}4$$

Problem 3.34 – Question

$\mathbf{C} = \hat{\boldsymbol{x}}\frac{y^2}{(x^2+y^2)} - \hat{\boldsymbol{y}}\frac{x^2}{(x^2+y^2)} + \hat{\boldsymbol{z}}4$ at $P_2 = (1,-1,2)$

$$\mathrm{C} = \hat{\boldsymbol{r}}\sin\varphi\cos\varphi\,(\sin\varphi - \cos\varphi) - \hat{\boldsymbol{\varphi}}(\sin^3\varphi + \cos^3\varphi) + \hat{\boldsymbol{z}}4$$

$$P_2 = \left(\sqrt{2}, -45°, 2\right)$$

$$\mathbf{C}(\boldsymbol{P_2}) = \hat{\boldsymbol{r}}0.707 - \hat{\boldsymbol{\varphi}}4$$

Problem 3.36-e – Question

Find the gradiant of the following scalr function:

$$\boldsymbol{S} = 4x^2e^{-z} + y^3$$

Problem 3.36-e – Solution

Find the gradiant of the following scalr function:

$$\boldsymbol{S} = 4x^2e^{-z} + y^3$$

$$\textbf{AS, } \nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

Problem 3.36-e – Solution

So

$$\nabla \boldsymbol{S} = \hat{\boldsymbol{x}}\frac{\partial S}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial S}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial S}{\partial z}$$

Putting values of function S

$$\nabla \boldsymbol{S} = \hat{\boldsymbol{x}}\frac{\partial(4x^2e^{-z} + y^3)}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial(4x^2e^{-z} + y^3)}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial(4x^2e^{-z} + y^3)}{\partial z}$$

$$\boxed{\nabla \boldsymbol{S} = \hat{\boldsymbol{x}}8xe^{-z} + \hat{\boldsymbol{y}}3y^2 - \hat{\boldsymbol{z}}4x^2e^{-z}}$$

Problem 3.48 – Question

For the vector field $\boldsymbol{E} = \hat{\boldsymbol{x}}xz - \hat{\boldsymbol{y}}yz^2 - \hat{\boldsymbol{z}}xy$, verify the divergence theorem by computing

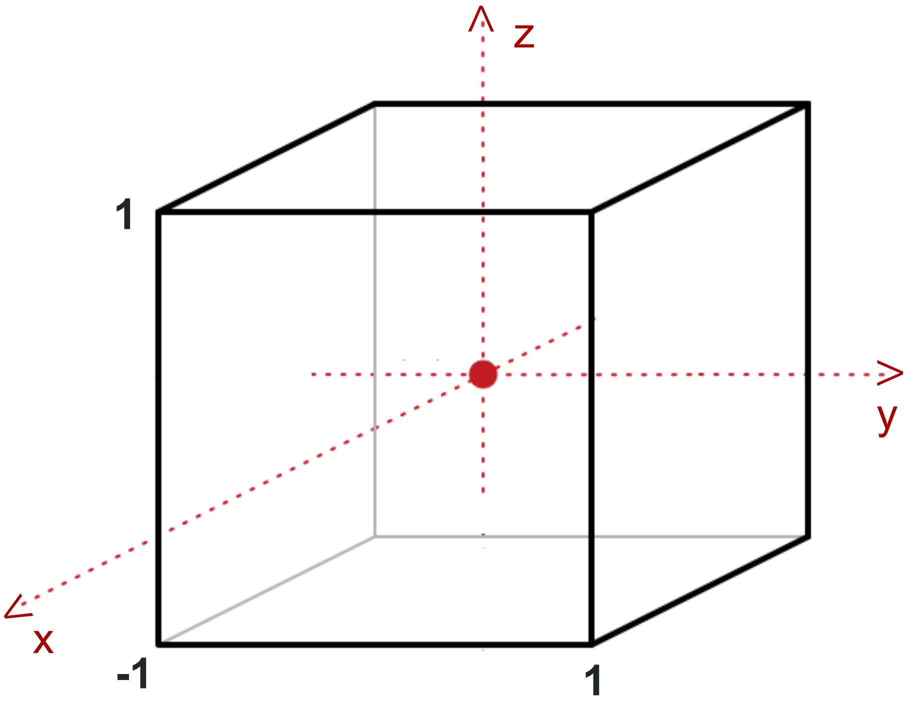
a) The total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes.

b) The integral of $\nabla \cdot \boldsymbol{E}$ over the cube's volume.

Divergence Theorem!

$$\oint_S \vec{E} \cdot d\vec{s} = \int_{\boldsymbol{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) dV$$

Problem 3.48 – Solution



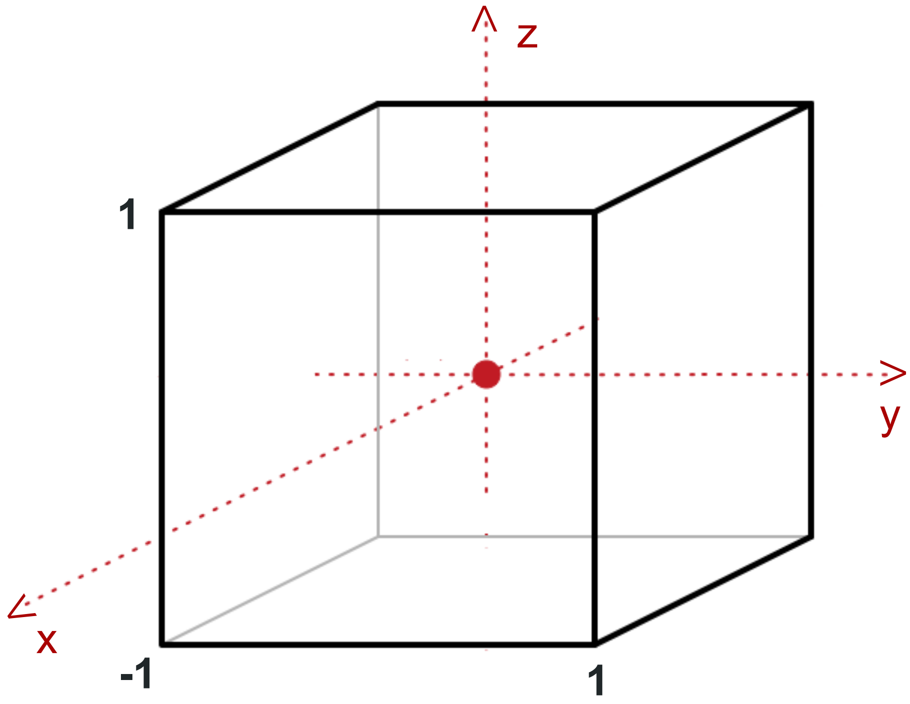
Part (a):

Goal → Calculate total outward flux

Given → $\boldsymbol{E} = \hat{\boldsymbol{x}}xz - \hat{\boldsymbol{y}}yz^2 - \hat{\boldsymbol{z}}xy$

Problem 3.48 – Solution

Part (a):



1) How to calculate the total flux?

Flux through each face → Add them all

2) Flux: $F = \oint_S \vec{E} \cdot d\vec{s}$

$$F_{Total} = F_{top} + F_{bottom} + F_{right} + F_{left} + F_{front} + F_{back}$$

Problem 3.48 – Solution

Part (a):

$$F_{top} = \int \left(\vec{E}\big|_{z=1} \cdot (\hat{z}\, dx\, dy) \right)$$

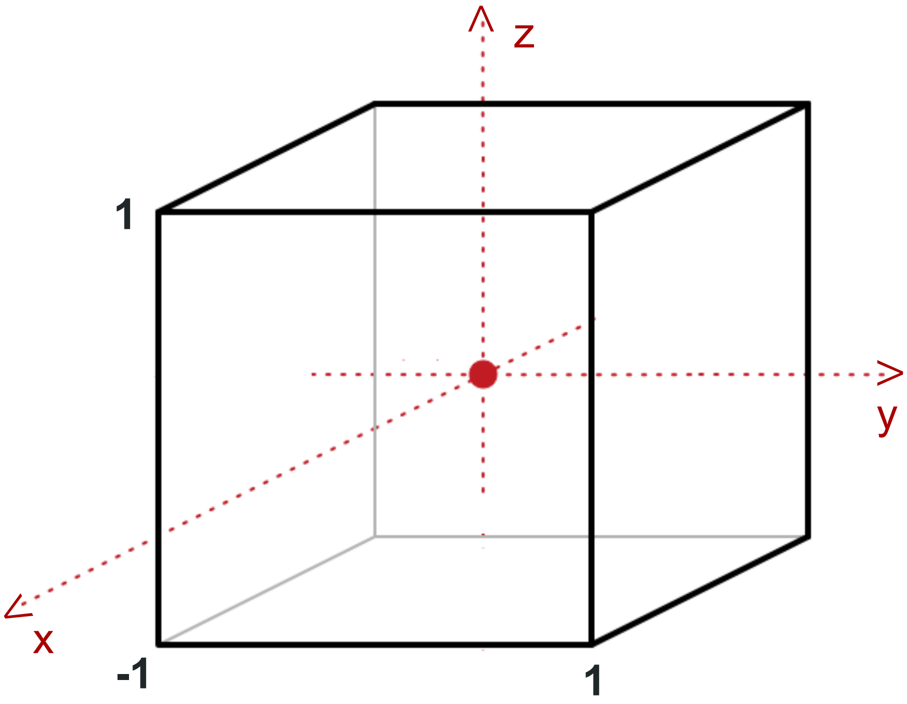
$$F_{bottom} = \int \left(\vec{E}\big|_{z=-1} \cdot (-\hat{z}\, dx\, dy) \right)$$

$$F_{right} = \int \left(\vec{E}\,\Big|_{y=1} \cdot (\hat{y}\, dx\, dz) \right)$$

$$F_{left} = \int \left(\vec{E}\,\Big|_{y=-1} \cdot (-\hat{y}\, dx\, dz) \right)$$

$$F_{front} = \int \left(\vec{E}\big|_{x=1} \cdot (\hat{x}\, dy\, dz) \right)$$

$$F_{back} = \int \left(\vec{E}\big|_{x=-1} \cdot (-\hat{x}\, dy\, dz) \right)$$



Problem 3.48 – Solution

Part (a):

$$F_{top} = 0$$

$$F_{bottom} = 0$$

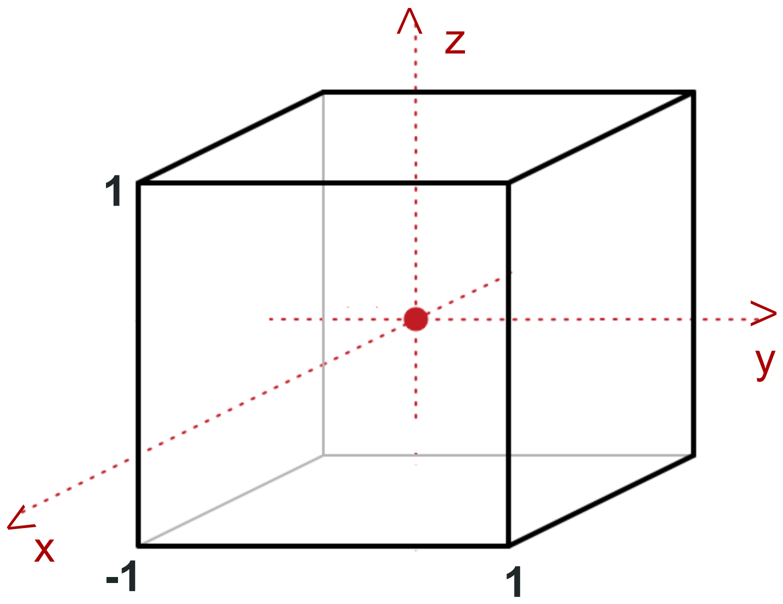
$$F_{right} = -\frac{4}{3}$$

$$F_{left} = -\frac{4}{3}$$

$$F_{front} = 0$$

$$F_{back} = 0$$

$$\boxed{F_{Total} = -\frac{8}{3}}$$



Problem 3.48 – Solution

Part (a):

An Idea how to calculate F_{right} and Fleft

$$\boldsymbol{F_{right}} = \int_{x=1}^{1}\int_{z=-1}^{1} \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy \Big|_{y=1} \cdot (\hat{y}\, dx\, dz)$$

$$\boldsymbol{F_{right}} = -\int_{x=1}^{1}\int_{z=-1}^{1} z^2 dzdx$$

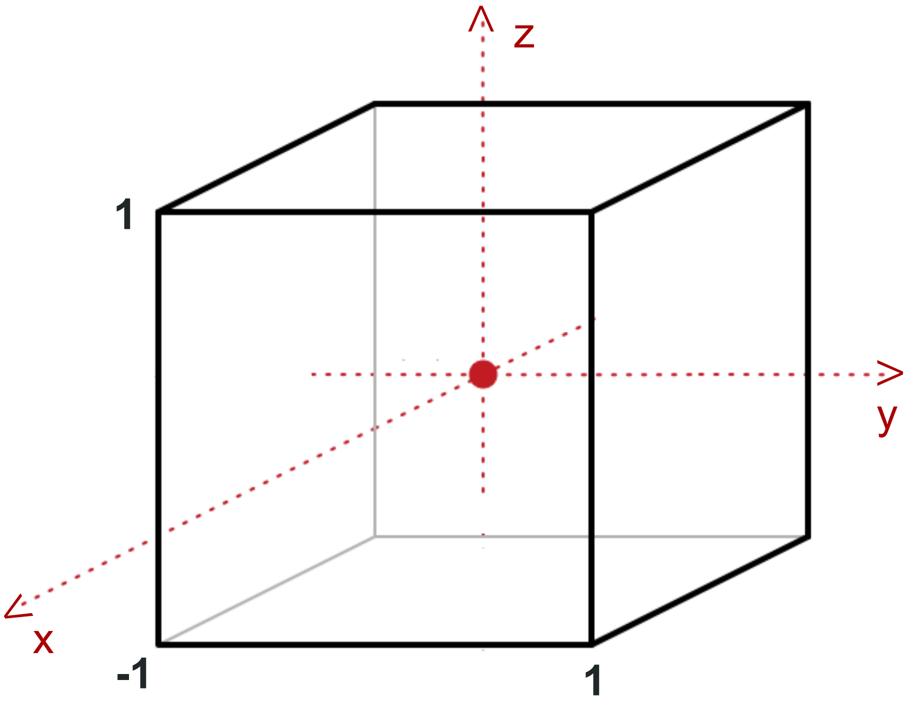
$$\boldsymbol{F_{right}} = -\left(\left(\frac{xz^3}{3}\right)\Big|_{z=-1}^{1}\right)\Big|_{x=-1}^{1}\right) = -\frac{4}{3}$$

Problem 3.48 – Solution

Part=b

The integral of $\nabla \cdot \boldsymbol{E}$ over the cube's volume.

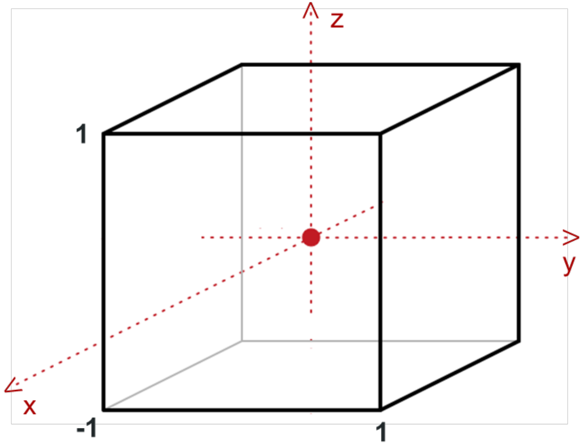
$$\nabla \cdot \boldsymbol{E} = \frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z}$$



Problem 3.48 – Solution

Part=b

$$\nabla \cdot \boldsymbol{E} = \frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z}$$



Put

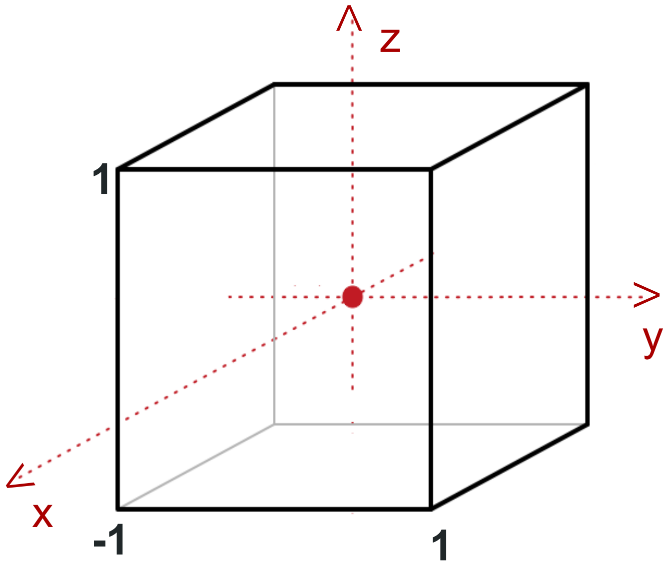
$$\boldsymbol{E_x} = (\hat{\boldsymbol{x}}xz\ )$$
$$\boldsymbol{E_y} = (-\hat{\boldsymbol{y}}yz^2$$
$$\boldsymbol{E_z} = (-\hat{\mathbf{z}}xy)$$

$$\nabla \cdot \boldsymbol{E} = \frac{\partial(\hat{\boldsymbol{x}}xz)}{\partial x} + \frac{\partial(-\hat{\boldsymbol{y}}yz^2)}{\partial y} + \cancel{\frac{\partial(-\hat{\mathbf{z}}xy)}{\partial z}}$$

Zero

Problem 3.48 – Solution

Part=b



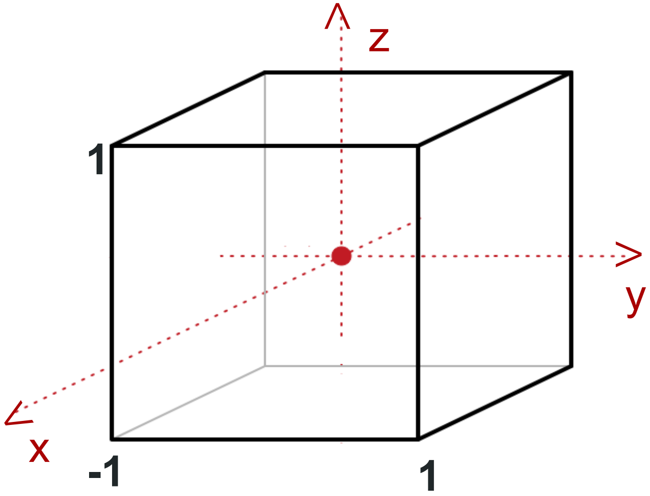
The integral of $\nabla \cdot \boldsymbol{E}$ over the cube's volume.

We get,

$$\nabla \cdot \boldsymbol{E} = -z^2 + z$$

Problem 3.48 – Solution

Part=b



$$\iiint (\nabla \cdot \boldsymbol{E})dV = \int_{x-1}^{1}\int_{y-1}^{1}\int_{z-1}^{1} \nabla.\left(\hat{\boldsymbol{x}}xz - \hat{\boldsymbol{y}}yz^2 - \hat{\boldsymbol{z}}xy\right) dxdydz$$

$$\iiint (\nabla \cdot \boldsymbol{E})dV = \int_{x-1}^{1}\int_{y-1}^{1}\int_{z-1}^{1} \left(z - z^2\right) dxdydz$$

Problem 3.48 – Solution

Part=b

$$\int\int\int(\nabla \cdot \boldsymbol{E})dV = \left(\left(\left(\boldsymbol{xy}\left(\frac{\boldsymbol{z}^2}{2} - \frac{\boldsymbol{z}^3}{\boldsymbol{3}}\right)\right)\Big|_{\boldsymbol{z}=-\boldsymbol{1}}^{\boldsymbol{1}}\right)\Big|_{\boldsymbol{y}=-\boldsymbol{1}}^{\boldsymbol{1}}\right)\Big|_{\boldsymbol{x}=-\boldsymbol{1}}^{\boldsymbol{1}}\Bigg) = \boxed{-\frac{\boldsymbol{8}}{\boldsymbol{3}}}$$

Divergence Theorem!

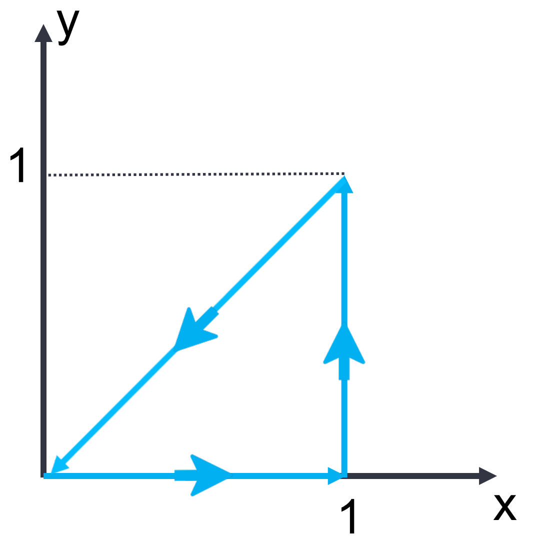
$$\oint_S \vec{E} \cdot d\vec{s} = \int_{\boldsymbol{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) dV$$

Problem 3.52-Question

For the vector field $\boldsymbol{E} = \hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2)$, calculate

a) $\oint_C \vec{E} \cdot d\vec{l}$ around the triangular contour shown in the Figure

b) $\int_S (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot d\boldsymbol{s}$ over the area of the triangle

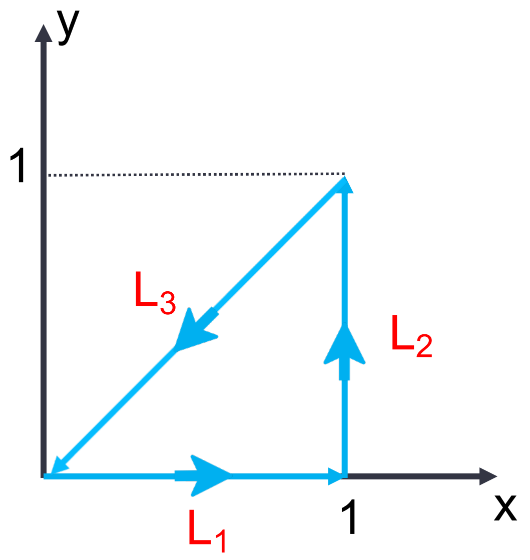


Stokes' Theorem!

$$\oint_C \vec{E} \cdot d\vec{l} = \int_{\boldsymbol{S}} (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot d\boldsymbol{s}$$

Problem 3.52

Part (a)



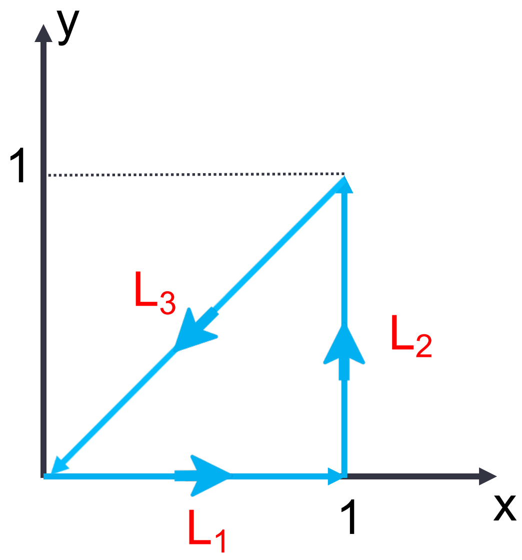
Split the countor C in 3 sections:

$$\oint_C \vec{E} \cdot d\vec{l} = \int_{L_1} \vec{E} \cdot d\overrightarrow{l_1} + \int_{L_2} \vec{E} \cdot d\overrightarrow{l_2} + \int_{L_3} \vec{E} \cdot d\overrightarrow{l_3}$$

Problem 3.52

Part (a)

$$\boldsymbol{E} = \hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2)$$



$$\int_{L_1} \vec{E} \cdot d\vec{l_1} = \int_0^1 [\hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2)]\Big|_{y=0} \cdot \hat{\boldsymbol{x}}dx$$

$$\int_{L_2} \vec{E} \cdot d\vec{l_2} = \int_1^1 [\hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2)]\Big|_{x=1} \cdot \hat{\boldsymbol{y}}dy$$

$$\int_{L_3} \vec{E} \cdot d\vec{l_3} = \int [\hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2)]\Big|_{y=x} \cdot (\hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy)$$

Problem 3.52
Part (a)

$Practice\ for\ solving\ L_2$

$$L_2 = \int_{L_2} \vec{E} \cdot \overrightarrow{dl_2} = \int [\hat{\boldsymbol{x}}xy - \hat{\boldsymbol{y}}(x^2 + 2y^2).(\hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}\,dy + \hat{\boldsymbol{z}}dz)]$$

$$L_2 = \int_{x=1}^{1} [xy)|_{\boldsymbol{z=0}} dx - \int_{y=0}^{1} (x^2 + 2y^2 \Big|_{\boldsymbol{x=1,z=o}}^{\boldsymbol{1}} dy \int_{z=0}^{1} (0)\Big|_{\boldsymbol{x=1}}^{\boldsymbol{1}} \boldsymbol{dz}$$

$$L_2 = 0 - (y + \frac{2y^2}{3}\Big|_{\boldsymbol{y=1}}^{\boldsymbol{0}} + \boldsymbol{0} = -\frac{\boldsymbol{5}}{\boldsymbol{3}}$$

$L_1\ and\ L_3\ can\ be\ solved\ by\ same\ process$

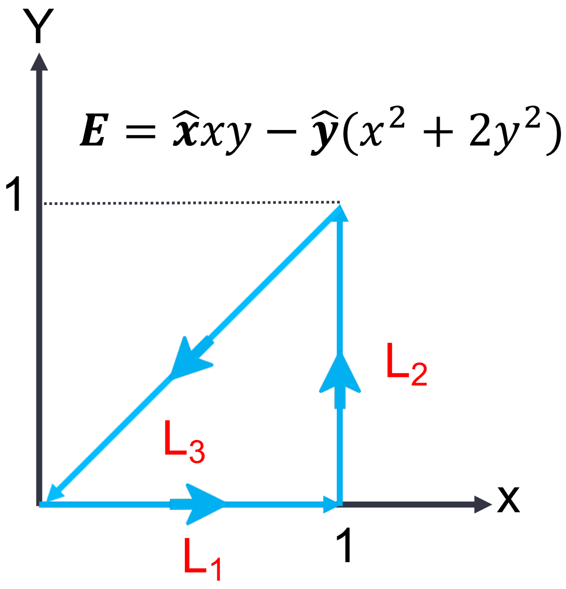
Problem 3.52

Part (a)

$$\int_{L_1} \vec{E} \cdot d\vec{l_1} = 0$$

$$\int_{L_2} \vec{E} \cdot d\vec{l_2} = -\frac{5}{3}$$

$$\int_{L_3} \vec{E} \cdot d\vec{l_3} = \frac{2}{3}$$



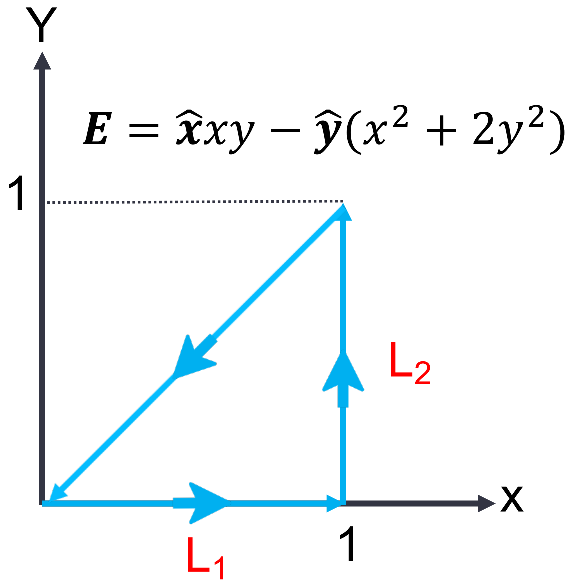
$$\oint_C \vec{E} \cdot d\vec{l} = \int_{L_1} \vec{E} \cdot d\vec{l_1} + \int_{L_2} \vec{E} \cdot d\vec{l_2} + \int_{L_3} \vec{E} \cdot d\vec{l_3}$$

$$\boxed{\oint_C \vec{E} \cdot d\vec{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1}$$

Problem 3.52

Part (b)

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s}$$

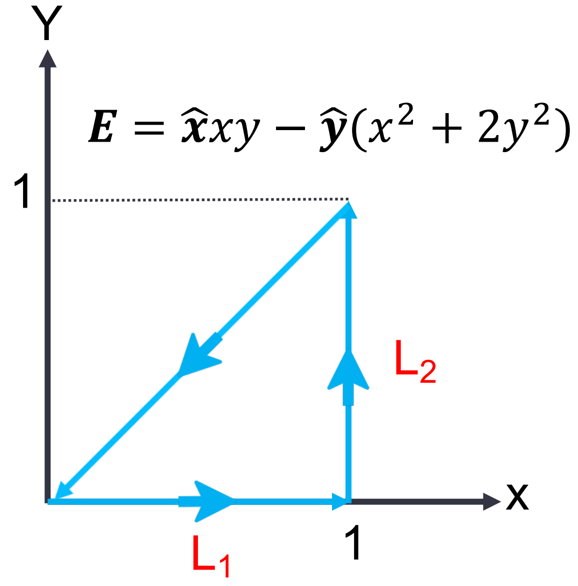


Calculate the curl:

$$(\nabla \times \boldsymbol{E}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -3x\hat{\boldsymbol{z}}$$

Problem 3.52

Part (b)



$$d\boldsymbol{s} = dxdy\hat{\mathbf{z}} \quad \text{and} \quad (\nabla\times\boldsymbol{E}) = -3x\hat{\mathbf{z}}$$

$$\iint (\nabla\times\boldsymbol{E})\cdot d\boldsymbol{s} = \int_{x-0}^{x=1}\int_{y=0}^{y=x} \left((-3x\hat{\mathbf{z}}).(\hat{\mathbf{z}}dydx)\right)\Big|_{z=\mathbf{0}}$$

$$\iint (\nabla\times\boldsymbol{E})\cdot d\boldsymbol{s} = -\int_{x=0}^{x=1}\int_{y=0}^{y=x} 3x\,dydx\,) = -\int_{x=0}^{1} 3x(x-0)dx = -(x^3)\Big|_{\mathbf{0}}^{\mathbf{1}} = \boxed{-\mathbf{1}}$$

Stokes' Theorem!

$$\oint_C \vec{E} \cdot d\vec{l} = \int_{\boldsymbol{S}} (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot d\boldsymbol{S}$$

ENGPHYS 2A04 Tutorial 6

ELECTRICITY AND MAGNETISM

Your TAs Today

- Joanne Lee

leej298@mcmaster.ca

- Seung Il Lee (Alex)

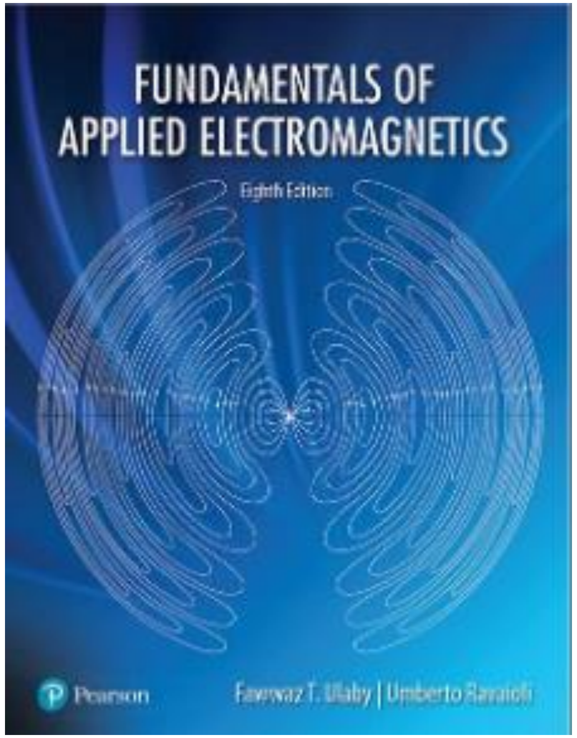
lee449@mcmaster.ca

Your Textbook

Fundamentals of Applied Electromagnetics Eighth Edition

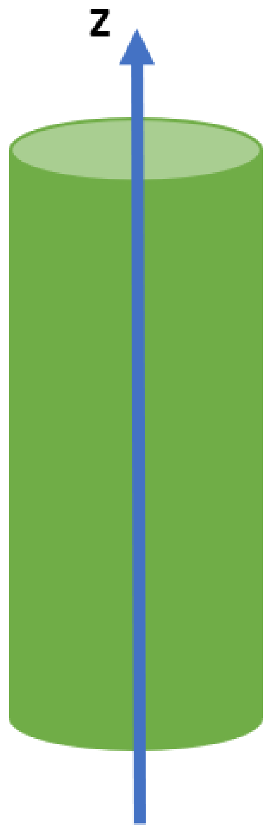
Ulaby & Ravaioli

Seventh Edition also acceptable, with some inconsistencies



Problem 4.9

A circular beam of charge of radius *a* consists of electrons moving with a constant speed *u* along the +z direction. The beam's axis is coincident with the z-axis and the electron charge density by,



$$\rho_v = -cr^2 \;\; [c/m^3]$$

Where c is constant and *r* is the radial distance from the axis of the beam.

A) Determine the charge density per unit length.

B) Determine the current crossing the z-plane.

Problem 4.9

Solution to A)

$$\rho_v = \frac{dq}{dv}$$

$$dv = dl\, ds$$

$$\frac{dq}{dl} = \rho_l$$

$$\rho_v = \frac{dq}{dv} = \frac{dq}{dl\, ds} = \frac{\rho_l}{ds}$$

$$\rho_l = \rho_v ds \rightarrow \rho_l = \iint \rho_v ds$$

$$ds = r dr d\emptyset \quad \text{(polar coordinates)}$$

$$\rho_l = \int_{r=0}^{a} \int_{\emptyset=0}^{2\pi} -cr^2 * r\, dr\, d\emptyset$$

Problem 4.9

Solution to A)

$$\rho_l = \int_{r=0}^{a} \int_{\emptyset=0}^{2\pi} -cr^2 * r \, dr \, d\emptyset = -2\pi c \frac{r^4}{4}$$

$$\rho_l = \frac{-\pi c a^4}{2} \left[\frac{C}{m}\right]$$

Problem 4.9

Solution to B)

$$\boldsymbol{I = \frac{charge}{time} = \left[\frac{Coulomb}{Second}\right] = [A]}$$

$$I = \int \boldsymbol{J} \cdot d\boldsymbol{s} = \int_{r=0}^{a} \int_{\emptyset=0}^{2\pi} (-cur^2\hat{\mathbf{z}}) \cdot (\hat{\mathbf{z}}) r \, dr \, d\emptyset$$

as $\boldsymbol{J} = \rho_v \boldsymbol{u} = -cr^2 \cdot u \, \hat{\mathbf{z}}$

$$I = -2\pi cu \int_{r=0}^{a} r^3 \, dr = -\frac{\pi c a^4 u}{2} = \rho_l u$$

$$Units\ for\ \rho_l u \rightarrow \left[\frac{C}{m}\right]\left[\frac{m}{s}\right] = \left[\frac{C}{s}\right] = [A]$$

Problem 4.15

Electric charge is distributed along an arc located in the x-y plane and defined by $r = 2\ cm$ and $0 \le \emptyset \le \frac{\pi}{4}$

If $\rho_l = 5\frac{\mu C}{m}$, find $\boldsymbol{E}$ at $(0,0,z)$ and then evaluate it at:

A) The origin

B) z = 5 cm

C) z = -5 cm

$$\widehat{\boldsymbol{R}}' = -0.02\cos\emptyset\hat{\boldsymbol{x}} - 0.02\sin\emptyset\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

$$\boldsymbol{R}' = -0.02\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}}$$

$$dl' = r\,d\emptyset = 0.02\,d\emptyset$$

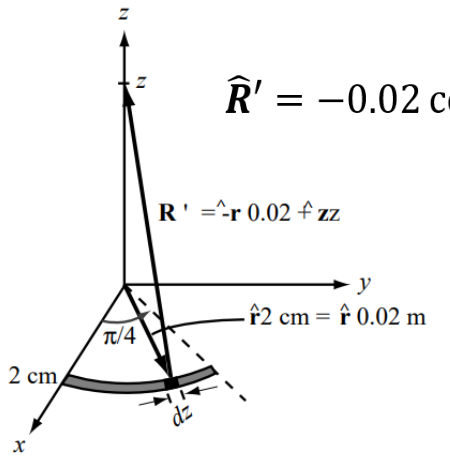


Figure P4.13: Line charge along an arc.

Problem 4.15

Solution:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'} \widehat{\boldsymbol{R}}' \frac{\rho_l dl'}{R'^2} = \frac{1}{4\pi\varepsilon_0}\int_{\emptyset=0}^{\frac{\pi}{4}} \rho_l \frac{-0.02\cos\emptyset\hat{\boldsymbol{x}} - 0.02\sin\emptyset\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{((0.02)^2 + (z^2)^{3/2}} 0.02 d\phi$$

$$\Rightarrow \quad \boldsymbol{E} = \frac{898.8}{((0.02)^2 + (z^2)^{3/2}}[-0.014\hat{\boldsymbol{x}} - 0.006\hat{\boldsymbol{y}} + 0.78z\hat{\boldsymbol{z}}]\left[\frac{V}{m}\right]$$

Problem 4.15

Solution

A) At origin (z = 0)

$$\boldsymbol{E} = [-1.6\hat{\boldsymbol{x}} - 0.66\hat{\boldsymbol{y}}] \left[\frac{MV}{m}\right]$$

B) z = 5 cm

$$\boldsymbol{E} = [-81.4\hat{\boldsymbol{x}} - 33.7\hat{\boldsymbol{y}} + 226\hat{\boldsymbol{z}}] \left[\frac{kV}{m}\right]$$

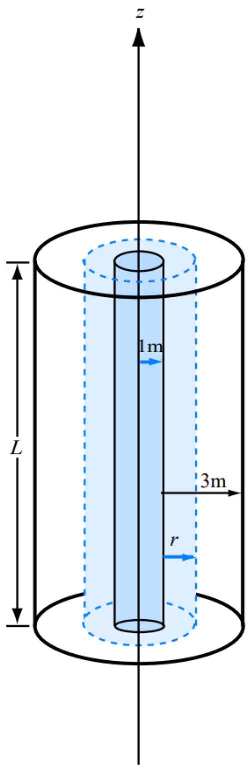
C) z = -5 cm

$$\boldsymbol{E} = [-81.4\hat{\boldsymbol{x}} - 33.7\hat{\boldsymbol{y}} - 226\hat{\boldsymbol{z}}] \left[\frac{kV}{m}\right]$$

Problem 4.27

An infinitely long cylindrical shell extending between r = 1 m and r = 3 m contains a uniform charge density ρ_{v_0}

Apply Gauss' law to find ***D*** in all regions.



Gauss' Law:

$$\oint_S (D_r \hat{\boldsymbol{r}}) \cdot d\boldsymbol{s} = Q$$

Ignore axial direction

Out of symmetry

$$D_r \cdot 2\pi r = Q$$

Problem 4.27

Solution:

$D_r \cdot 2\pi r = Q$

For r $< 1\, m, Q = 0, \boldsymbol{D} = 0$

For $1 \leq$ r ≤ 3 m

$Q = \rho_{v_0} \cdot \pi L(r^2 - 1)$

$D_r \cdot 2\pi r L = \rho_{v_0} \cdot \pi L(r^2 - 1)$

$$\boldsymbol{D} = D_r \hat{\boldsymbol{r}} = \frac{\rho_{v_0} \cdot \pi(r^2 - 1)}{2\pi r}\hat{\boldsymbol{r}} \quad \left[\frac{C}{m^2}\right]$$

Problem 4.27

Solution:

For r ≥ 3 m,

$$Q = \rho_{v_0} \cdot \pi(3^2 - 1^2) = 8\rho_{v_0} \cdot \pi$$

$$D_r \cdot 2\pi r L = 8\rho_{v_0} \cdot \pi L$$

$$\boldsymbol{D} = D_r \hat{\boldsymbol{r}} = \frac{4\rho_{v_0}}{r} \hat{\boldsymbol{r}} \quad \left[\frac{C}{m^2}\right]$$

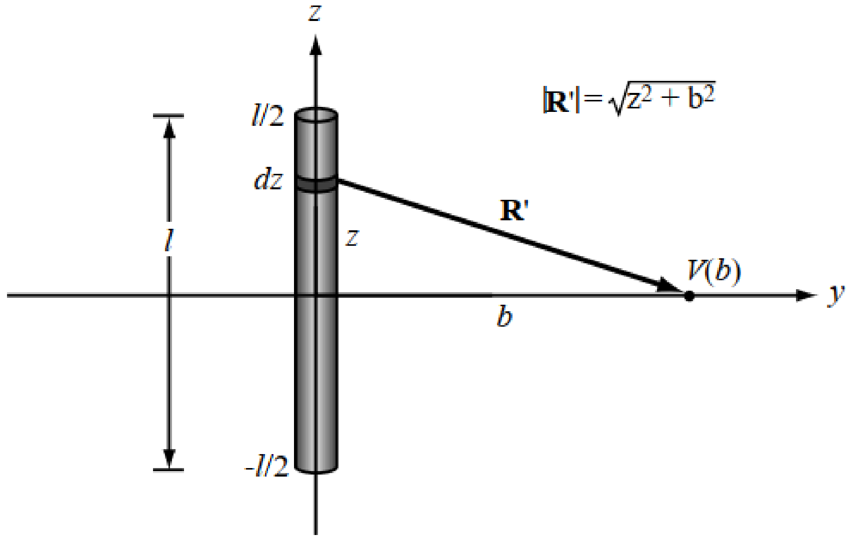
2A04 Tutorial 7

March 7th, 2022

Fraser McCauley & Jacob Saunders

1. Problem 4.34

Find the electric potential V at a location a distance b from the origin in the $x - y$ plane due to a line charge with charge density ρ_l and of length l. The line charge is coincident with the z axis and extends from $z = -l/2$ to $z = l/2$.



For a line charge, we have the formula:

$$V = \frac{1}{4\pi\varepsilon}\int_{l'} \frac{\rho_l}{R'}\,dl'$$

In this specific case, we can integrate along z with the following limits:

$$V = \frac{\rho_l}{4\pi\varepsilon}\int_{-l/2}^{l/2} \frac{dz}{R'}$$

$$V = \frac{\rho_l}{4\pi\varepsilon}\int_{-l/2}^{l/2} \frac{dz}{\sqrt{z^2 + b^2}}$$

$$\boldsymbol{V = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{\sqrt{l^2 + 4b^2} + l}{\sqrt{l^2 + 4b^2} - l}\right)}$$

$$V = \frac{\rho_l}{4\pi\varepsilon}\int_{-l/2}^{l/2}\frac{dz}{\sqrt{z^2+b^2}}$$

$$\int \frac{1}{\sqrt{u^2+a^2}}\,du = \left|\ln\left(u+\sqrt{u^2+a^2}\right)\right| + C$$

Let $C_1 = \frac{\rho_l}{4\pi\varepsilon}$

$$V = C_1 \ln\left(z+\sqrt{z^2+b^2}\right)\Big|_{-l/2}^{l/2} = C_1\left(\ln\left(\frac{l}{2}+\sqrt{\frac{l^2}{4}+b^2}\right) - \ln\left(-\frac{l}{2}+\sqrt{\frac{l^2}{4}+b^2}\right)\right)$$

$$V = C_1 \ln\left(\frac{l/2+\sqrt{l^2/4+b^2}}{-l/2+\sqrt{l^2/4+b^2}}\right)$$

$$V = C_1 \ln\left(\frac{l+\sqrt{4(l^2/4+b^2)}}{-l+\sqrt{4(l^2/4+b^2)}}\right)$$

$$V = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{\sqrt{l^2+4b^2}+l}{\sqrt{l^2+4b^2}-l}\right)$$

2. Problem 4.41

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\left(\frac{m^2}{Vs}\right), \mu_h = 0.05\left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

a. The conductivity of silicon,
b. The current I flowing in the bar,
c. The drift velocities $\overrightarrow{u_e}$ and $\overrightarrow{u_h}$,
d. The resistance of the bar, and
e. The power dissipated in the bar.

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13\left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05\left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

What does each parameter mean???

2. Problem 4.41a

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right), \mu_h = 0.05\ \left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

The conductivity of silicon

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05\ \left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

How can we calculate conductivity?

$$\sigma = (N_e\mu_e + N_h\mu_h)e \quad (\frac{S}{m})$$

$$\sigma = \left(1.5 \times 10^{16}(0.13 + 0.05)\right)(1.602 \times 10^{-19})$$

$$\boldsymbol{\sigma = 4.33 \times 10^{-4}\ (\frac{S}{m})}$$

2. Problem 4.41a

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right), \mu_h = 0.05\ \left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

The conductivity of silicon

$$\boldsymbol{\sigma = 4.33 \times 10^{-4}\ (\frac{S}{m})}$$

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05\ \left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

Table B-2 Conductivity $\boldsymbol{\sigma}$ of Some Common Materials[a]

Material	Conductivity σ (S/m)	Material	Conductivity σ (S/m)
Conductors		**Semiconductors**	
Silver	6.2×10^7	Pure germanium	2.2
Copper	5.8×10^7	Pure silicon	4.4×10^{-4}
Gold	4.1×10^7	**Insulators**	
Aluminum	3.5×10^7	Wet soil	$\sim 10^{-2}$
Tungsten	1.8×10^7	Fresh water	$\sim 10^{-3}$
Zinc	1.7×10^7	Distilled water	$\sim 10^{-4}$
Brass	1.5×10^7	Dry soil	$\sim 10^{-4}$
Iron	10^7	Glass	10^{-12}
Bronze	10^7	Hard rubber	10^{-15}
Tin	9×10^6	Paraffin	10^{-15}
Lead	5×10^6	Mica	10^{-15}
Mercury	10^6	Fused quartz	10^{-17}
Carbon	3×10^4	Wax	10^{-17}
Seawater	4		
Animal body (average)	0.3 (poor cond.)		

[a]These are low-frequency values at room temperature (20 °C).

2. Problem 4.41b

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right), \mu_h = 0.05\ \left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

The current I flowing in the bar

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05\ \left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

How can we compute the current?

$$I = JA$$

$$I = \sigma EA$$

$$I = \sigma \frac{V}{l} \pi r^2$$

$$I = (4.33 \times 10^{-4}) \frac{5}{0.08} \pi 0.004^2$$

$$\boldsymbol{I = 1.36\mu A}$$

2. Problem 4.41c

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right), \mu_h = 0.05\ \left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

The drift velocities $\overrightarrow{u_e}$ and $\overrightarrow{u_h}$

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13\ \left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05\ \left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

From the definition of the mobility:

$$\overrightarrow{u_e} = -\mu_e \vec{E}, \qquad \overrightarrow{u_h} = \mu_h \vec{E}$$

$$\overrightarrow{u_e} = -\mu_e E \frac{\vec{E}}{E}, \qquad \overrightarrow{u_h} = \mu_h E \frac{\vec{E}}{E}$$

What is the fraction at the end there? Why do we need it?

$$\overrightarrow{u_e} = -\mu_e \frac{V}{l} \widehat{\boldsymbol{E}}, \qquad \overrightarrow{u_h} = \mu_h \frac{V}{l} \widehat{\boldsymbol{E}}$$

$$\overrightarrow{u_e} = -0.13 \frac{5}{0.08} \widehat{\boldsymbol{E}}, \qquad \overrightarrow{u_h} = 0.05 \frac{5}{0.08} \widehat{\boldsymbol{E}}$$

$$\boldsymbol{\overrightarrow{u_e} = -8.125 \widehat{E}}, \qquad \boldsymbol{\overrightarrow{u_h} = 3.125 \widehat{E}}$$

2. Problem 4.41d,e

A cylindrical bar of silicon has a radius of 4mm and a length of 8cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13 \left(\frac{m^2}{Vs}\right), \mu_h = 0.05 \left(\frac{m^2}{Vs}\right), N_e = 1.5 \times 10^{16} electrons/m^3$, and $N_h = N_e$, find the following:

Resistance of, and power dissipated across, the bar.

Givens

$r = 4mm$

$l = 8cm$

$V = 5V$

$\mu_e = 0.13 \left(\frac{m^2}{Vs}\right)$

$\mu_h = 0.05 \left(\frac{m^2}{Vs}\right)$

$N_e = N_h = 1.5 \times 10^{16} electrons/m^3$

We already found $I = 1.36\mu A$. How can we find the resistance?

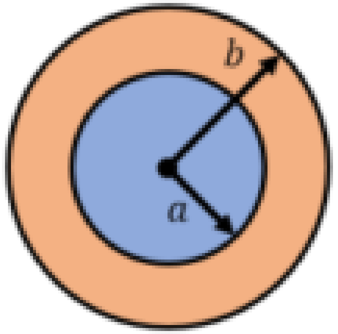
Ohm's Law: $\boldsymbol{R = \frac{V}{I} = \frac{5}{1.36\times10^{-6}} = 3.68\ M\Omega}$

With the same parameters, we can find the power dissipated:

$$\boldsymbol{P = VI = 5\left(1.36 \times 10^{-6}\right) = 6.8\ \mu W}$$

3. Problem 4.44

A coaxial resistor of length l consists of two concentric cylinders. The inner cylinder has radius a and is made of a material with conductivity σ_1, and the outer cylinder, extending between $r = a$ and $r = b$, is made of a material with conductivity σ_2. If the two ends of the resistor are capped with conducting plates, show that the resistance between the two ends is



$$R = \frac{l}{\pi\left(\sigma_1 a^2 + \sigma_2 (b^2 - a^2)\right)}.$$

So how can we find R_1 and R_2?

These are basically parallel resistors!

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

Each R will follow $R = \frac{l}{\sigma A}$ where A is the surface area.

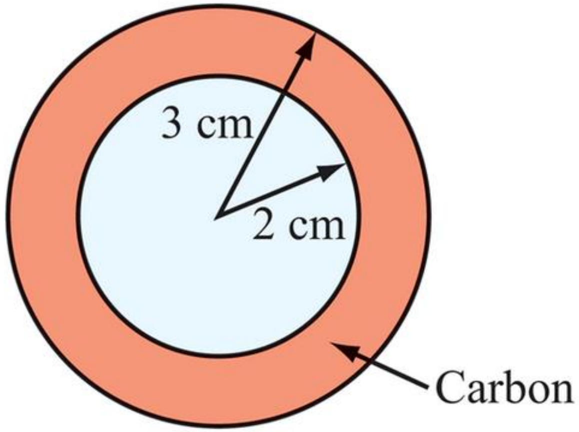
R= $\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \left(\frac{\sigma_1 A_1}{l} + \frac{\sigma_2 A_2}{l}\right)^{-1}$… what are the areas?

$$R = \left(\frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi (b^2 - a^2)}{l}\right)^{-1} = \left(\frac{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)}{l}\right)^{-1}$$

$$\boldsymbol{R = \frac{l}{\sigma_1 \pi a^2 + \sigma_2 \pi (b^2 - a^2)} = \frac{l}{\pi\left(\sigma_1 a^2 + \sigma_2 (b^2 - a^2)\right)}}$$

4. Problem 4.45

Apply the result of Problem 4.44 (*that's the one we just did!*) to find the resistance of a 20 cm long hollow cylinder made of carbon with $\sigma = 3 \times 10^4\ S/m.$



$$R = \frac{l}{\pi\left(\sigma_1 a^2 + \sigma_2 (b^2 - a^2)\right)}.$$

Givens

$a = 2cm$

$b = 3cm$

$\sigma = 3 \times 10^4\ S/m$

$l = 20cm$

Which σ is the given value? And what's the other value?

Cylinder is <u>hollow</u>… so take $\sigma_1 = 0$ and $\sigma_2 = 3 \times 10^4\ S/m$

$$\boldsymbol{R = \frac{0.2}{\pi\left((0)(0.02)^2 + (3 \times 10^4)(0.03^2 - 0.02^2)\right)}}$$

$$= \boldsymbol{\frac{0.2}{\pi\left((3 \times 10^4)(5 \times 10^{-4})\right)} = 4.2\ m\Omega}$$

Reminders

- Assignment 7 is out, and is due at 8AM on March 14.
- Complete the mid-semester survey for a 5% bonus on Assignment 7! Survey must be completed by Sunday March 13th.
- Have a great week!

ENG PHYS 2A04 Tutorial 8

Electricity and Magnetism

Your TAs today

- Thomas Kean
  Keant1@mcmaster.ca
- Batoul Hashemi
  hashemb@mcmaster.ca

Chapter 4

Problem 4.48 - Question

Find $\mathbf{E}_1$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2\ (\frac{V}{m})$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}(\frac{C}{m^2})$. What angle does E_2 make with the z-axis?

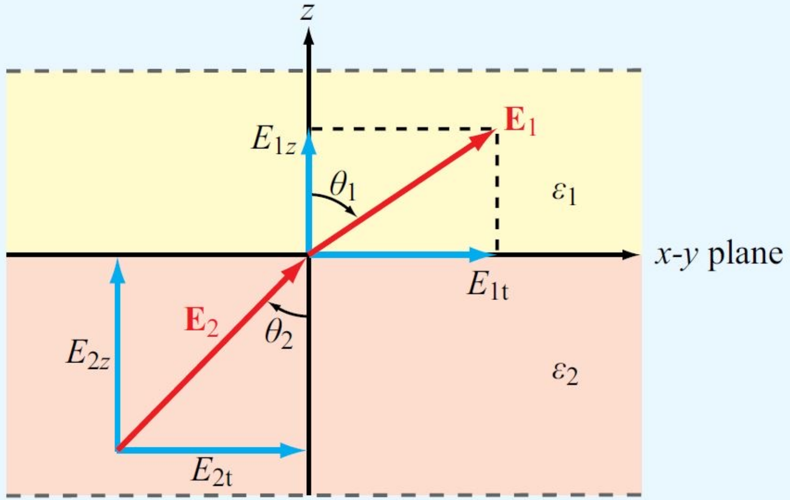


Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

Problem 4.48 - Question

Find $\mathbf{E}_1$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2\ (\frac{V}{m})$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}(\frac{C}{m^2})$. What angle does E_2 make with the z-axis?

a) Find $\boldsymbol{E}_1$

b) Find θ_2

E_1 is comprised of 3 component vectors: $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$

E_{1t} constitutes the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ components

E_{1z} constitutes the $\hat{\boldsymbol{z}}$ component only

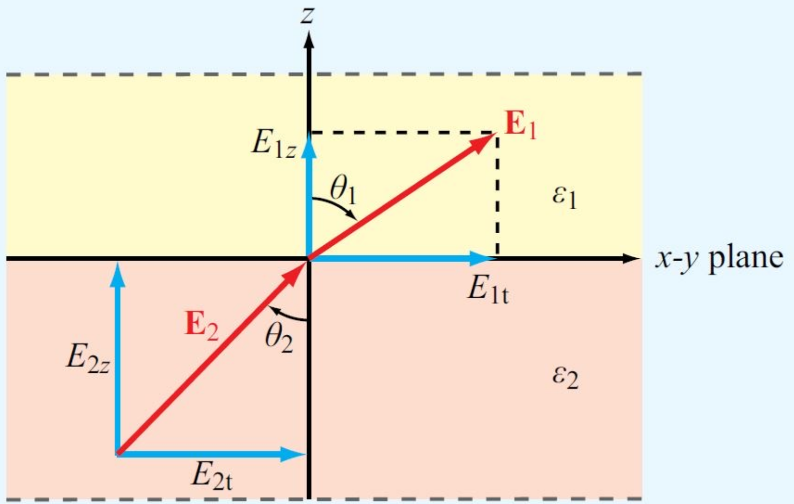


Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

Problem 4.48 – Work (a)

$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 \left(\frac{V}{m}\right)$$

Table 4-3 Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ϵ_1	Medium 2 Conductor
Tangential E	$\boxed{\mathbf{E}_{1t} = \mathbf{E}_{2t}}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\epsilon_1 = \mathbf{D}_{2t}/\epsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\epsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Problem 4.48 – Work (a)

$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2\left(\frac{V}{m}\right)$$

$$\varepsilon_1(E_1 \cdot \hat{n}) - \varepsilon_2(E_2 \cdot \hat{n}) = \rho_s$$

$$\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s$$

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1}$$

$$= \frac{3.54\times10^{-11} + 18\varepsilon_0 \cdot 2}{2\varepsilon_0}$$

$$E_{1z} = 20\left(\frac{V}{m}\right)$$

Table 4-3 Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ϵ_1	Medium 2 Conductor
Tangential E	$\boxed{\mathbf{E}_{1t} = \mathbf{E}_{2t}}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\epsilon_1 = \mathbf{D}_{2t}/\epsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$	$E_{1n} = \rho_s/\epsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Problem 4.48 – Work (a)

$$E_1 = E_{1t} + E_{1z}$$

$$E_{1t} = E_{2t} = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2\left(\frac{V}{m}\right)$$

$$E_{1z} = 20\left(\frac{V}{m}\right)$$

Therefore, this is the answer for $\boldsymbol{E_1}$

$$E_1 = E_{1t} + E_{1z}$$

$$E_1 = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 + \hat{\boldsymbol{z}}20\left(\frac{V}{m}\right)$$

Table 4-3 Boundary conditions for the electric fields.

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Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Problem 4.48 - Question

$\boxed{\text{Find } \mathbf{E}_1}$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2\ (\frac{V}{m})$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}(\frac{C}{m^2})$. What $\boxed{\text{angle does } E_2 \text{ make}}$ with the z-axis?

a) Find E_1 $\quad E_1 = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 + \hat{\boldsymbol{z}}20(\frac{V}{m})$

b) **Find $\boldsymbol{\theta_2}$**

Use trig!

$E_{2z} = |E_2| \cos\theta_2$

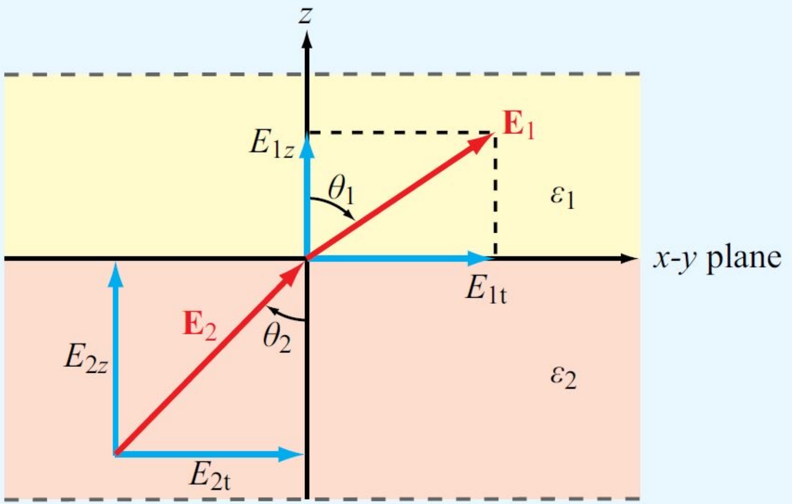

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

Problem 4.48 - Question

Find $\boldsymbol{E}_1$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2\ (\frac{V}{m})$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}(\frac{C}{m^2})$. What angle does E_2 make with the z-axis?

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b) **Find $\boldsymbol{\theta_2}$**

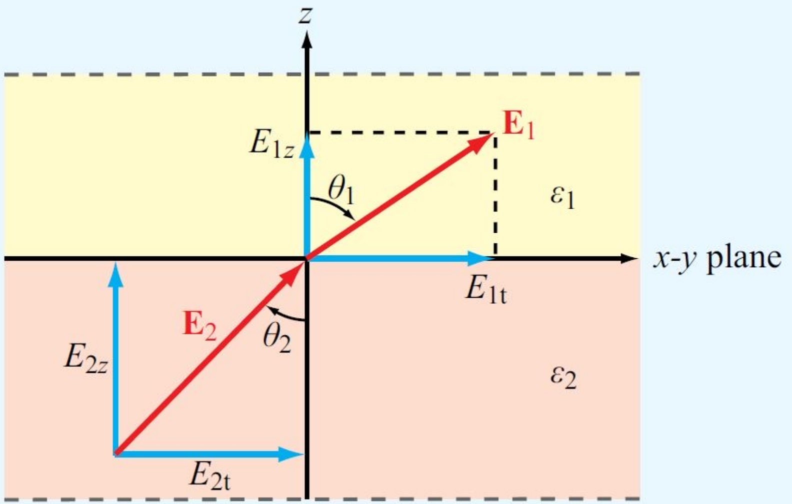
Use trig!

$$E_{2z} = |E_2| \cos\theta_2$$

$$\theta_2 = \cos^{-1}\left(\frac{E_{2z}}{|E_2|}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{3^2+2^2+2^2}}\right)$$

$$= 61°$$



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Problem 4.48 - Question

Find $\mathbf{E}_1$ if $\boldsymbol{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2 \left(\frac{V}{m}\right)$, $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$ and the boundary has a surface charge density $\rho_s = 3.54\times10^{-11}\left(\frac{C}{m^2}\right)$. What angle does E_2 make with the z-axis?

a) Find E_1 $\quad E_1 = \hat{\boldsymbol{x}}3 - \hat{\boldsymbol{y}}2 + \hat{\boldsymbol{z}}20\left(\frac{V}{m}\right)$

b) Find θ_2 $\quad \theta_2 = 61°$

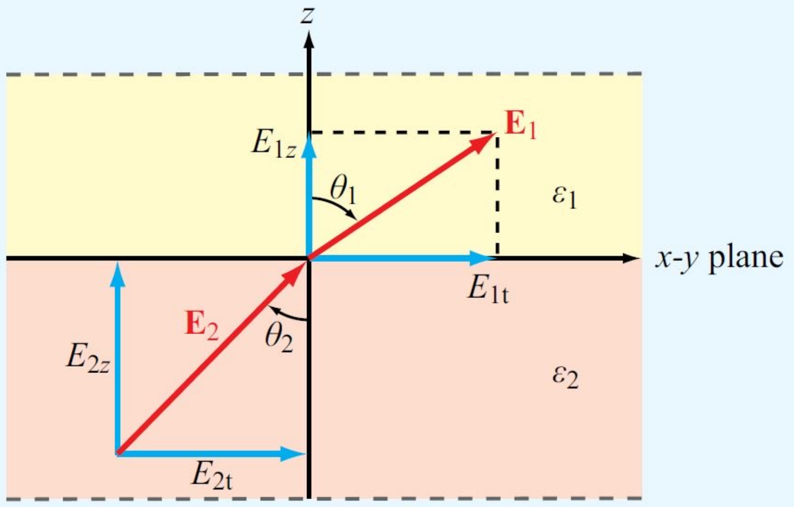

Figure 4-19 Application of boundary conditions at the interface between two dielectric media (Example 4-10).

*4.50 If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

***4.50** If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Table 4-3 **Boundary conditions for the electric fields.**

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Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

*4.50 If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Table 4-3 Boundary conditions for the electric fields.

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Normal E	$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\epsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

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$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_R S \varepsilon_0 = (150)4\pi(0.05)^2\varepsilon_0 = \frac{3\pi\varepsilon_0}{2} \quad \text{(C)}.$$

Problem 4.54 - Question

4.54 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area (**Fig. P4.54**). If the voltage across the capacitor is 10 V, find the following:

(a) The force acting on the electron.

(b) The acceleration of the electron.

(c) The time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

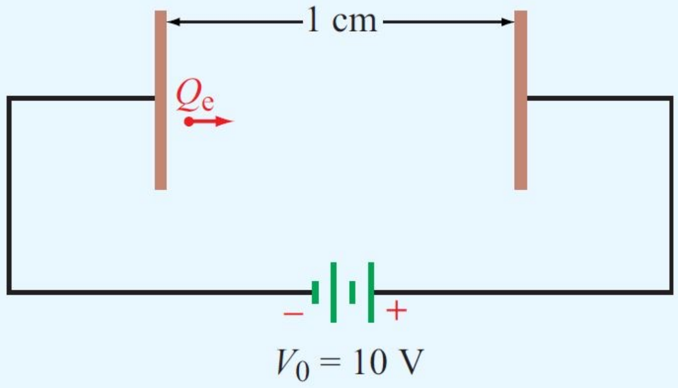


Figure P4.54 Electron between charged plates of Problem 4.54.

Problem 4.54 - Solution

4.54 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area (**Fig. P4.54**). If the voltage across the capacitor is 10 V, find the following:

(a) The force acting on the electron.

(b) The acceleration of the electron.

(c) The time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

Part = a

$$F = Q_e E = Qe\frac{V}{d}$$

Putting Values in above equation we get

$$F = -1.6\times\frac{10}{0.01} = -1.6\times10^{-16}\text{ N}$$

Part = b

$$a = \frac{F}{m} = \frac{1.6\times10^{-16}}{9.1\times10^{-31}} = 1.76\times10^{14}\ \ m/s^2$$

Part = c

$$t = \sqrt{\frac{2d}{a}} = \left(\frac{2\times0.01}{1.76\times10^{14}}\right)^{1/2}$$

$$t = 10.7\times10^{-9}\text{sec} \ = 10\cdot7(ns)$$

Problem 4.58 - Question

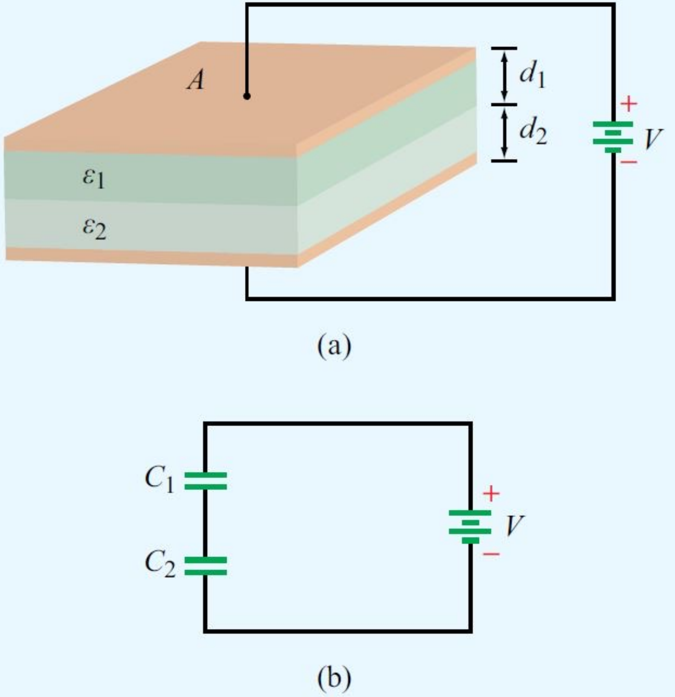


Figure P4.58 (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).

4.58 The capacitor shown in **Fig. P4.58** consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{4.136}$$

where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \qquad C_2 = \epsilon_2 \frac{A}{d_2}.$$

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1, ϵ_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (4.136).

Part = a (Solution)

If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2$$

and,

$$V_1 = E_1 d_1 \quad \text{and} \quad V_2 = E_2 d_2$$

So,

$$V = E_1 d_1 + E_2 d_2 \longrightarrow \text{(A)}$$

According to boundary conditions, the normal component of **D** is continuous across the boundary

$$D_{1n} = D_{2n}$$

From Equation 4.15

$$D = \varepsilon E$$

So,

$$\varepsilon_1 E_1 = \varepsilon_2 E_2 \Longrightarrow E_2 = \frac{\varepsilon_1 E_1}{\varepsilon_2}$$

$$\text{(A)} \longrightarrow V = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2 \Longrightarrow E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \quad \text{and} \quad E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}$$

Part = b (Solution)

From Eq= 4.122, stored potential energy is given by,

$$W_e = \frac{1}{2}\varepsilon E^2 (Ad)$$

So,

$$W_{e1} = \frac{1}{2}\varepsilon_1 {E_1}^2 (Ad_1)$$

Replacing E_1 by

$$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \Rightarrow W_{e1} = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2}\right)^2 (Ad_1) \Rightarrow W_{e_1} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\,\mathrm{d}_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right)$$

$$W_{e_1} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\,\mathrm{d}_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right)$$ And $$W_{e_2} = \frac{1}{2}V^2 \left(\frac{\varepsilon_1^2 \varepsilon_2\ A\,\mathrm{d}_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right)$$

$$W_e = W_1 + W_2 = \frac{1}{2}V^2 \left(\frac{\varepsilon_1\ \varepsilon_2^2 A\,\mathrm{d}_1 + \varepsilon_1^2 \varepsilon_2\ A\,\mathrm{d}_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right)$$

But,

$$w_e = \frac{1}{2}CV^2$$

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A\,\mathrm{d}_1 + \varepsilon_1^2 \varepsilon_2\ A\,\mathrm{d}_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \Rightarrow \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2\,\mathrm{d} + \varepsilon_1\,\mathrm{d}_2)}{\left(\varepsilon_{2d_1} + \varepsilon_1 d_2\right)^2} \Rightarrow \frac{\varepsilon_1 \varepsilon_2 A}{\left(\varepsilon_{2d_1} + \varepsilon_1 d_2\right)}$$

Part = c (Solution)

From solution of part b we have,

$$C = \frac{\varepsilon_1 \varepsilon_2 A}{(\varepsilon_{2d_1} + \varepsilon_1 d_2)}$$

Multiply numerator and denominator by A/d_1d_2

$$C = \frac{\varepsilon_1 \varepsilon_2 A\, (A/d_1d_2)}{(\varepsilon_{2d_1} + \varepsilon_1 d_2)(A/d_1d_2)}$$

$$C = \frac{\frac{\varepsilon_1 A}{d_1} \cdot \frac{\varepsilon_2 A}{d_2}}{\frac{A\varepsilon_2\, \mathrm{d}_1}{d_1 d_2} + \frac{A\varepsilon\, \mathrm{d}_2}{\mathrm{d}_1 d_2}} = \frac{\frac{\varepsilon_1 A}{d_1} \cdot \frac{\varepsilon_2 A}{d_2}}{\frac{A\varepsilon_2}{d_2} + \frac{A\varepsilon_1}{d_1}} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_2 = \frac{\varepsilon_2 A}{d_2} \qquad C_1 = \frac{\varepsilon_1 A}{d_1}$$

ENGPHYS 2A04 TUTORIAL 9

ELECTRICITY & MAGNETISM

Your TAs Today

- Joanne Lee

leej298@mcmaster.ca

- Muhammad Munir

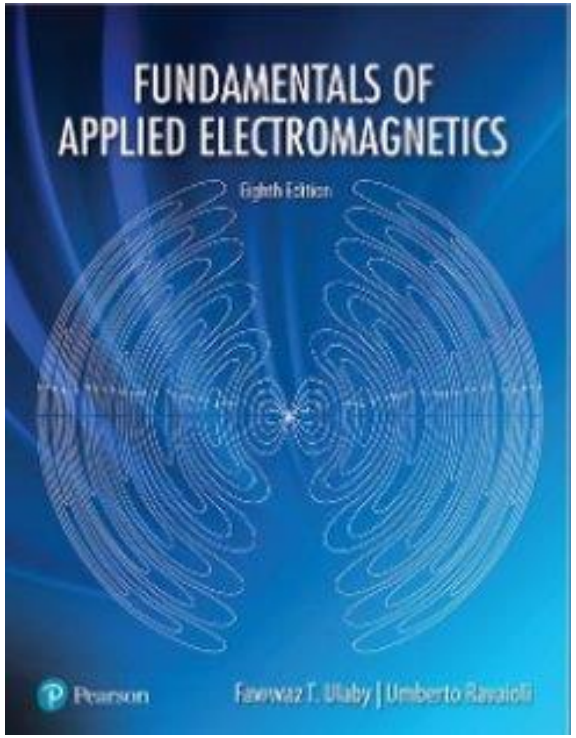
munirm6@mcmaster.ca

Your Textbook

Fundamentals of Applied Electromagnetics Eighth Edition

Ulaby & Ravaioli

Seventh Edition also acceptable, with some inconsistencies



Problem 5.1

An electron with a speed of 8 x 10^6 m/s is projected along the positive x direction into a medium containing a uniform magnetic flux density $\mathbf{B} = (\hat{\boldsymbol{x}}4 - \hat{\boldsymbol{z}}3)T$

Given that e = 1.6*10^{19} C and the mass of an electron is me = 9.1*10^{-31} kg, determine the initial acceleration vector of the electron (at the moment it is projected into the medium).

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Given that e = 1.6*10^{19} C and the mass of an electron is me = 9.1*10^{-31} kg, determine the initial acceleration vector of the electron (at the moment it is projected into the medium).

Known Values:

$$Electron\ speed: u = 8 * 10^6 \frac{m}{s}$$

$$Magnetic\ Flux\ Density: \mathbf{B} = (\hat{\boldsymbol{x}}4 - \hat{\mathbf{z}}3)T$$

$$Elementary\ Charge: e = 1.6 * 10^{19}\ C$$

$$Electron\ Mass: m_e = 9.1 * 10^{-31}\ kg$$

Problem 5.1

Particle of a charge *q* moving with velocity **u** in a magnetic field experiences magnetic force $\mathbf{F}_m$ given by:

$$Electron\ speed: u = 8 * 10^6 \frac{m}{s}$$

$$Magnetic\ Flux\ Density: \mathbf{B} = (\hat{\boldsymbol{x}}4 - \hat{\boldsymbol{z}}3)T$$

$$Elementary\ Charge: e = 1.6 * 10^{19}\ C$$

$$Electron\ Mass: m_e = 9.1 * 10^{-31}\ kg$$

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} \quad (\mathrm{N})$$

Use Newton's Second Law: F = m * a

Rearrange equation and substitute **F** for equation above.

Problem 5.1

$$a = \frac{\mathbf{F}_m}{m_e} = \frac{q\mathbf{u} \times \mathbf{B}}{m_e}$$

Assuming q = -e

$$Electron\ speed: u = 8 * 10^6 \frac{m}{s}$$

$$Magnetic\ Flux\ Density: \mathbf{B} = (\hat{\boldsymbol{x}}4 - \hat{\mathbf{z}}3)T$$

$$Elementary\ Charge: e = 1.6 * 10^{19}\ C$$

$$Electron\ Mass: m_e = 9.1 * 10^{-31}\ kg$$

$$= \frac{-1.6 * 10^{-19}}{9.1 * 10^{-31}} (\hat{\boldsymbol{x}}8 * 10^6) \times (\hat{\boldsymbol{x}}4 - \hat{\mathbf{z}}3)$$

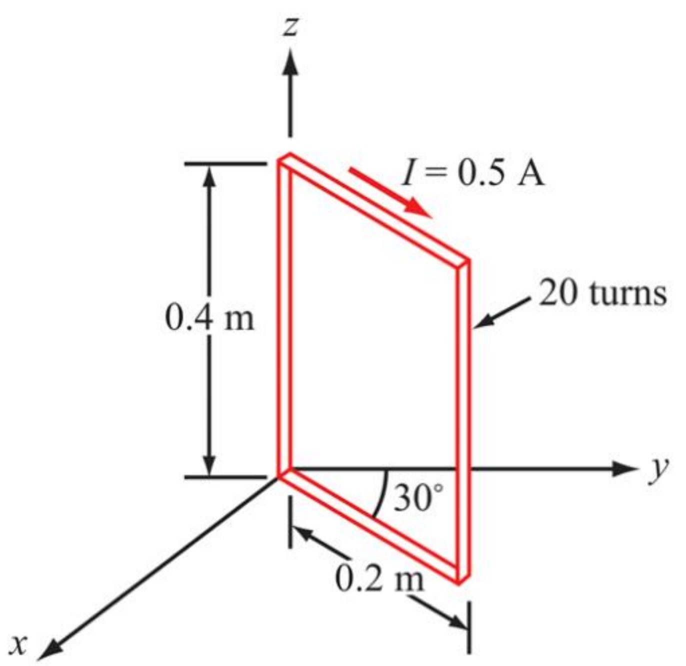
$$= -\hat{\boldsymbol{y}}4.22 * 10^{18} \quad (m/s^2)$$

$$\overline{a} \times \overline{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 8000000 & 0 & 0 \\ 4 & 0 & -3 \end{vmatrix} = \boldsymbol{i}(0\cdot(-3) - 0\cdot 0) - \boldsymbol{j}(8000000\cdot(-3) - 0\cdot 4) + \boldsymbol{k}(8000000\cdot 0 - 0\cdot 4) =$$

$$= \boldsymbol{i}(0 - 0) - \boldsymbol{j}(-24000000 - 0) + \boldsymbol{k}(0 - 0) = \{0; 24000000; 0\}$$

Problem 5.4

The rectangular loop shown in Fig. P5.4 consists of 20 closely wrapped turns and is hinged along the z axis. The plane of the loop makes an angle of 30° with the y axis, and the current in the windings is 0.5 A. What is the magnitude of the torque exerted on the loop in the presence of a uniform field $\mathbf{B} = \hat{\mathbf{y}}2.4$ T? When viewed from above, is the expected direction of rotation clockwise or counterclockwise?



(5.19)

$$\mathbf{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}m \quad (\mathrm{A}\cdot\mathrm{m}^2),$$

(5.20)

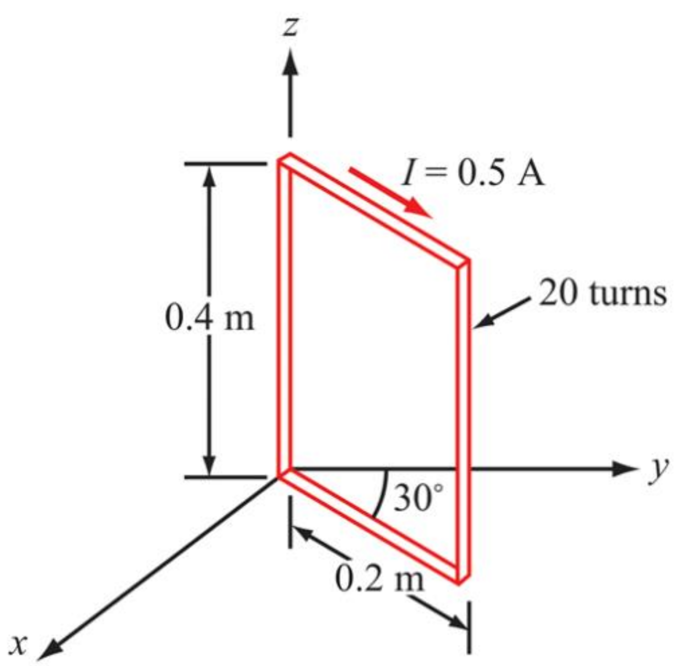
$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad (\mathrm{N}\cdot\mathrm{m}).$$

Problem 5.4

Magnetic Moment m:

$$\boldsymbol{m} = \widehat{\boldsymbol{n}}NIA = \widehat{\boldsymbol{n}}m\ (A \cdot m^2)$$

where $\widehat{\boldsymbol{n}}$ is the surface normal of the loop and governed by the following *right-hand rule: When the four fingers of the right-hand advance in the direction of th current* ***I****, the direction of the thumb specifies the direction of* $\widehat{\boldsymbol{n}}$.



$$N = 20$$
$$I = 0.5\ A$$
$$A = l \times w = 0.2\ m \times 0.4\ m = 0.08\ m^2$$
$$\therefore m = NIA = (20)(0.5\ A)(0.08\ m^2) = 0.8\ A \cdot m^2$$

(5.19)

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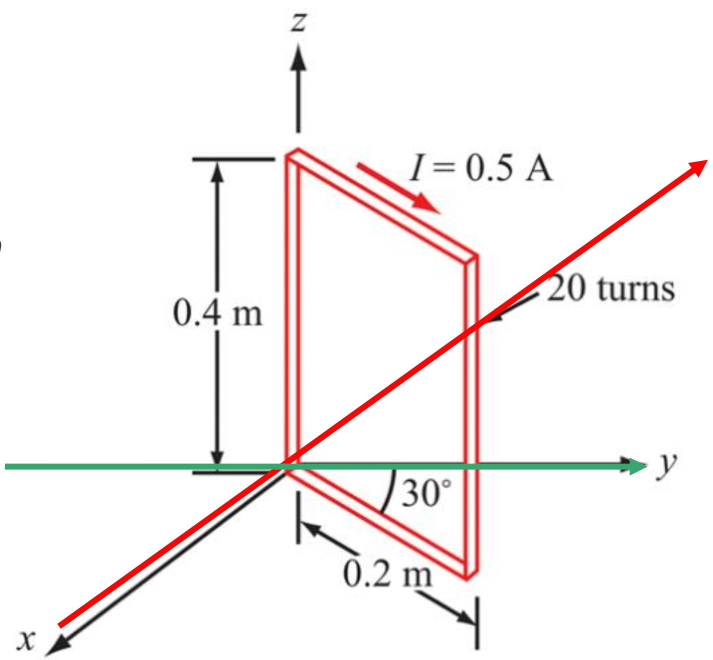

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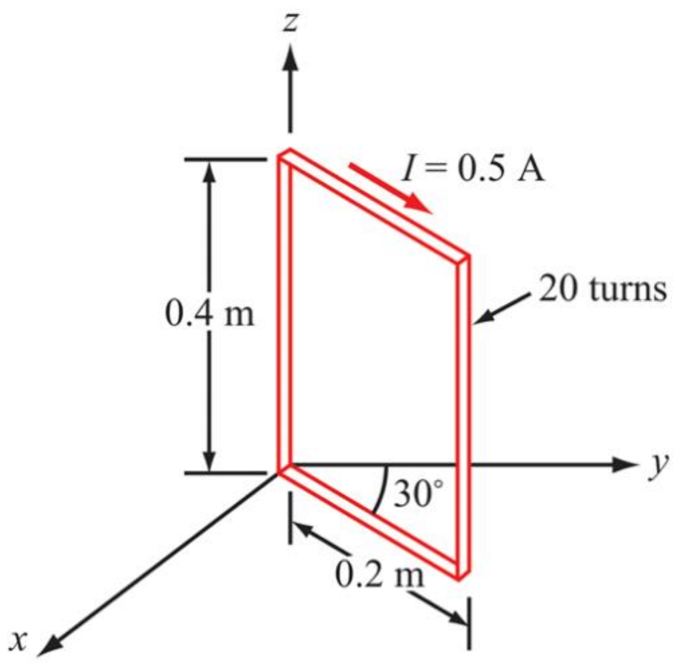

$$\hat{\mathbf{n}} = -\hat{\mathbf{x}}\cos 30° + \hat{\boldsymbol{y}}\sin 30°$$

(5.19)

$$\mathbf{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}m \quad (\mathrm{A} \cdot \mathrm{m}^2),$$

Problem 5.4

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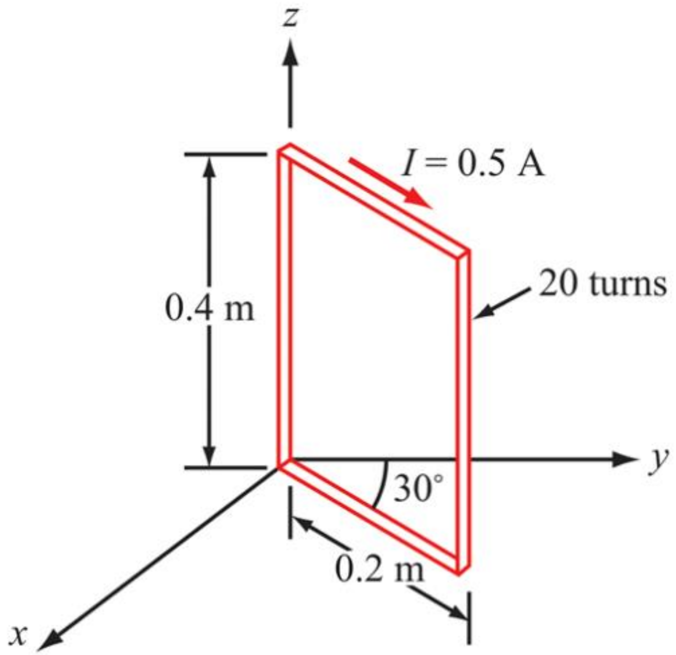
$N = 20$
$I = 0.5\ A$
$A = l \times w = 0.2\ m \times 0.4\ m = 0.08\ m^2$
$\therefore m = NIA = (20)(0.5\ A)(0.08\ m^2) = 0.8\ A \cdot m^2$

$$\hat{\mathbf{n}} = -\hat{\mathbf{x}}\cos 30° + \hat{\boldsymbol{y}}\sin 30°$$



Torque is defined as:

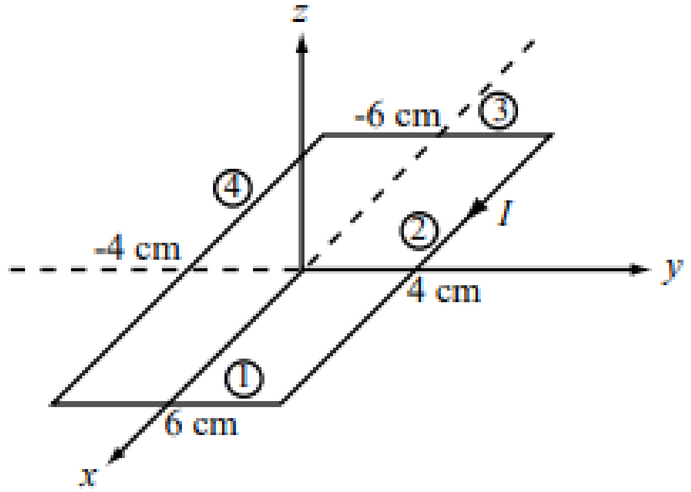
$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = \hat{\mathbf{n}}m \times \mathbf{B}$$

$$= (-\hat{\mathbf{x}}\cos 30° + \hat{\boldsymbol{y}}\sin 30°)0.8 \times \hat{\boldsymbol{y}}2.4 \cong -1.66\ \hat{\boldsymbol{z}}\ N \cdot m$$

Negative torque indicates clockwise.

Problem 5.7

An 8 cm x 12 cm rectangular loop of wire is situation in the x-y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 50 flowing with clockwise direction (when viewed from above). Determine the magnetic field at the center of the loop.



$Biot\ Savart\ Law$: $\quad d\boldsymbol{H} = \frac{1}{4\pi}\frac{d\boldsymbol{l}\times\widehat{\boldsymbol{R}}}{R^2}\ \left[\frac{A}{m}\right]$

Where $d\boldsymbol{H} = differential\ magnetic\ field\ intensity$

$d\boldsymbol{l} = differential\ length\ vector$

$\widehat{\boldsymbol{R}} = distance\ vector\ between\ d\boldsymbol{l}\ \ and\ the\ observation\ point$

Problem 5.7

$Biot\ Savart\ Law$: $d\boldsymbol{H} = \frac{1}{4\pi}\frac{d\boldsymbol{l}\times\hat{\boldsymbol{R}}}{R^2}\left[\frac{A}{m}\right] \rightarrow \boldsymbol{H} = \int_{\boldsymbol{l}} d\boldsymbol{H}$

$\boldsymbol{B} = \mu\boldsymbol{H}$

Break conductor into 4 segments and calculate each segment's contribution to total magnetic field.

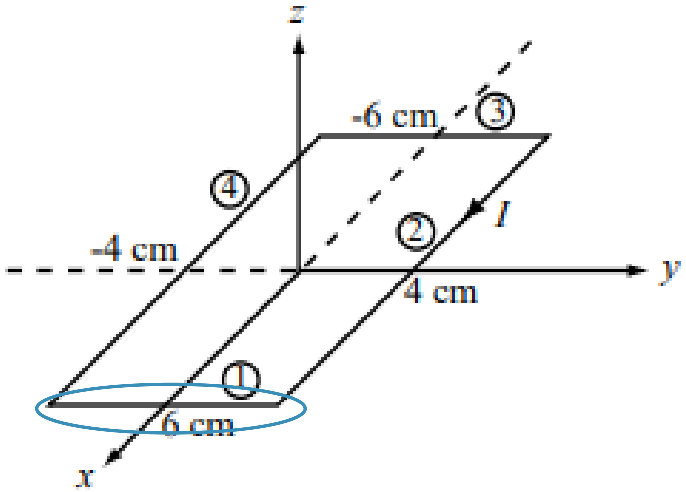
Segment 1 (blue circle):

Right hand Rule gives direction of magnetic field, $\therefore$ $\boldsymbol{B_1}$is along –z direction.

Using equation for wire of finite length.

$|\boldsymbol{B}| = \mu|\boldsymbol{H}| = \mu_0\frac{Il}{2\pi r\sqrt{4r^2+l^2}}\ \ [T]$

Using:



$$\begin{aligned}\mu_0 &= \mu = 4\pi\times10^{-7}\ N/A^2\\ I &= 50\ A\\ r &= 6\ cm\\ l &= 8cm\end{aligned}$$

Problem 5.7

$$|\boldsymbol{B}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}} \quad [T]$$

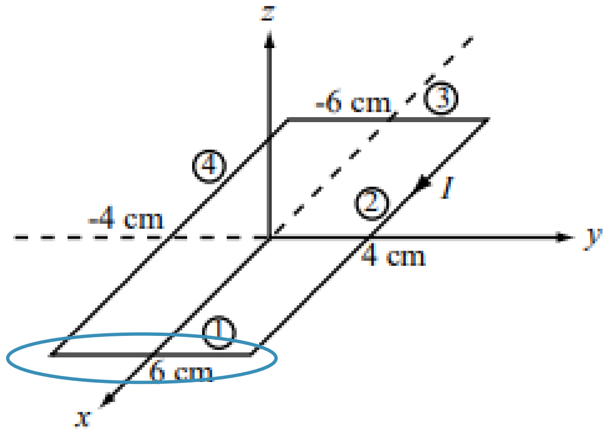
Segment 1 (blue circle):

$$|\boldsymbol{B_1}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}}(-\mathbf{z})$$

$$= 4\pi * 10^{-7}\ NA^{-2} \frac{50\ A\ * 0.08\ m}{2\pi(0.06\ m)\sqrt{4(0.06\ m)^2 + (0.08\ m)^2}}(-\mathbf{z})$$

$$= -9.24 * 10^{-5}\boldsymbol{z}\ \ [T]$$

$$\boldsymbol{B_1} = \ \ -9.24 * 10^{-5}\boldsymbol{z}\ [T]$$



Problem 5.7

$$|\boldsymbol{B}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}} \quad [T]$$

Segment 2 (blue circle):

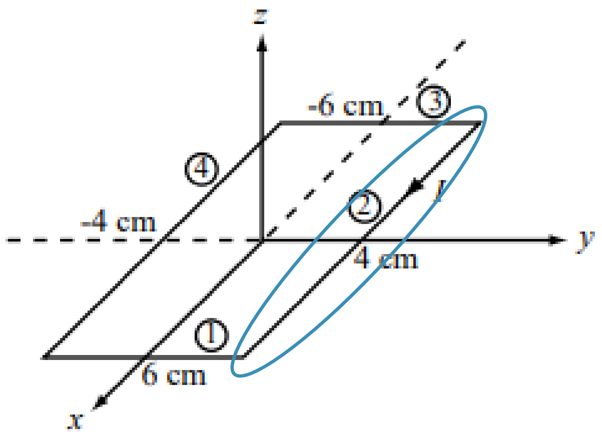
$\boldsymbol{B_2}$ *is along* $-z$ *direction.*

$$|\boldsymbol{B_2}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}}(-\mathbf{z})$$

$$= 4\pi * 10^{-7}\ NA^{-2} \frac{50\ A\ * 0.12\ m}{2\pi(0.04\ m)\sqrt{4(0.04\ m)^2 + (0.12\ m)^2}}(-\boldsymbol{z})$$

$$= -20.80 * 10^{-5}\boldsymbol{z}\ \ [T]$$

$$\boldsymbol{B_2} = -20.80 * 10^{-5}\boldsymbol{z}\ [T]$$

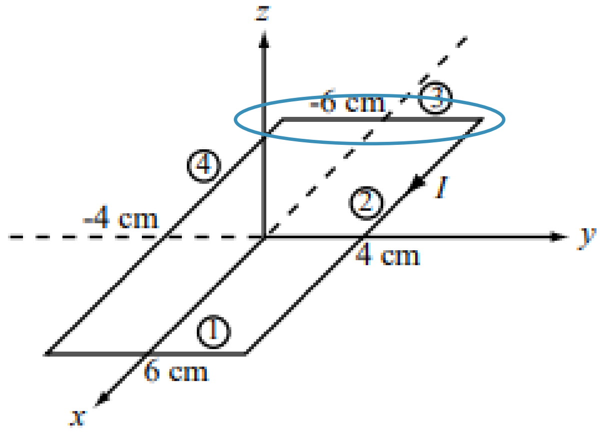


Problem 5.7

$$|\boldsymbol{B}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}} \quad [T]$$

Segment 3 (blue circle):

$$|\boldsymbol{B_3}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}}(-\mathbf{z})$$



$$= 4\pi * 10^{-7}\ NA^{-2}\frac{50\ A\ * 0.08\ m}{2\pi(0.06\ m)\sqrt{4(0.06\ m)^2 + (0.08\ m)^2}}(-\boldsymbol{z})$$

$$= -9.24 * 10^{-5}\boldsymbol{z}\ \ [T]$$

$$\boldsymbol{B_3} = \ -9.24 * 10^{-5}\boldsymbol{z}\ [T]$$

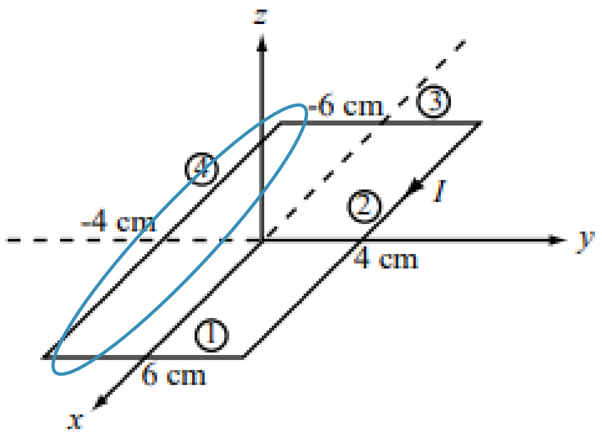
Problem 5.7

$$|\boldsymbol{B}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}} \quad [T]$$

Segment 4 (blue circle):

$\boldsymbol{B_4}$ *is along* $-z$ *direction.*

$$|\boldsymbol{B_4}| = \mu|\boldsymbol{H}| = \mu_0 \frac{Il}{2\pi r\sqrt{4r^2+l^2}} (-\mathbf{z})$$



$$= 4\pi * 10^{-7}\ NA^{-2} \frac{50\ A\ * 0.12\ m}{2\pi(0.04\ m)\sqrt{4(0.04\ m)^2 + (0.12\ m)^2}} (-\boldsymbol{z})$$

$$= -20.80 * 10^{-5}\boldsymbol{z}\ \ [T]$$

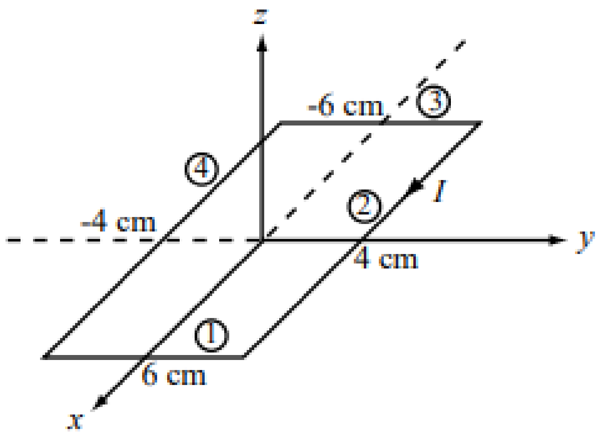
$$\boldsymbol{B_4} = -20.80 * 10^{-5}\boldsymbol{z}\ [T]$$

Problem 5.7

$\boldsymbol{B} = \boldsymbol{B_1} + \boldsymbol{B_2} + \boldsymbol{B_3} + \boldsymbol{B_4}$

$= -9.24 * 10^{-5}\boldsymbol{z} + -20.80 * 10^{-5}\boldsymbol{z} + -9.24 * 10^{-5}\boldsymbol{z} + -20.80 * 10^{-5}\boldsymbol{z}$

$\boldsymbol{B} = -0.60 * 10^{-3}\boldsymbol{z}\ [T]$



2A04 Tutorial 10

March 28th, 2022

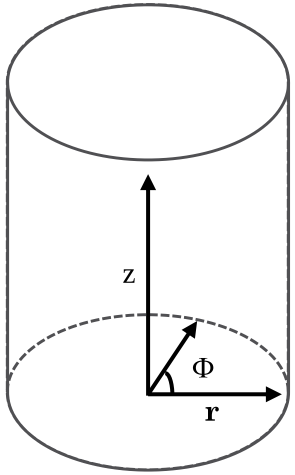
Fraser McCauley & Alex Lee

1. Problem 5.24

In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\boldsymbol{H} = \hat{\boldsymbol{\Phi}}\frac{4}{r}[1-(1+3r)e^{-3r}]$$

Find the current density **J**.



What equation relates **J** to **H**?

$$\boldsymbol{J} = \nabla \times \boldsymbol{H}$$

Curl of a system in cylindrical coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

H_r and H_z are zero, so the curl simplifies to:

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(r\boldsymbol{H}_{\boldsymbol{\varphi}})\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(4[1-(1+3r)e^{-3r}])\right]$$

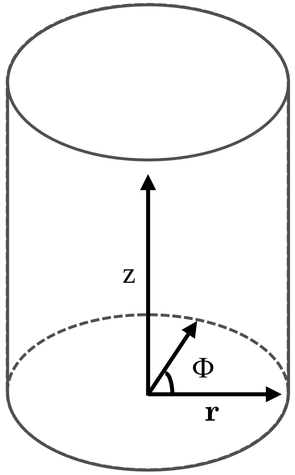
$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}[4-4(1+3r)e^{-3r}]\right]$$

1. Problem 5.24

In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\boldsymbol{H} = \hat{\boldsymbol{\Phi}}\frac{4}{r}[1-(1+3r)e^{-3r}]$$

Find the current density **J**.



H_r and H_z are zero, so the curl simplifies to:

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\boldsymbol{H}_{\boldsymbol{\varphi}}\right)\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(4[1-(1+3r)e^{-3r}])\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}[4-4(1+3r)e^{-3r}]\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}\left[\frac{\partial}{\partial r}[(1+3r)e^{-3r}]\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}[3e^{-3r} - 3e^{-3r}(1+3r)]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}\left[3e^{-3r}\left(1-(1+3r)\right)\right]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}\frac{-4}{r}[3e^{-3r}(-3r)]$$

$$\boldsymbol{J} = \hat{\boldsymbol{z}}36e^{-3r}\ \ A/m^2$$

2. Problem 5.27

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

a) Determine **B**.

b) Use Eq 5.66 to calculate the magnetic flux passing through a square loop with 0.25-m long edges if the loop is in the x-y plane, its center is at the origin, and its edges are parallel to the x and y axes.

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

2. Problem 5.27a

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

a) Determine **B**.

What equation relates **A** to **B**?

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

- There is no component in y ($A_y = 0, \frac{\partial A_y}{\partial anything} = 0$)
- The x component only depends on y ($\frac{\partial A_x}{\partial z} = 0$)
- The z component only depends on x ($\frac{\partial A_z}{\partial y} = 0$)

$$\boldsymbol{B} = \hat{\boldsymbol{x}}(0 - 0) + \hat{\boldsymbol{y}}\left(0 - \frac{\partial A_z}{\partial x}\right) + \hat{\boldsymbol{z}}\left(0 - \frac{\partial A_x}{\partial y}\right)$$

$$\boldsymbol{B} = -\hat{\boldsymbol{y}}\frac{\partial A_z}{\partial x} - \hat{\boldsymbol{z}}\frac{\partial A_x}{\partial y} = -\hat{\boldsymbol{y}}\frac{\partial}{\partial x}(2 + \sin\pi x) - \hat{\boldsymbol{z}}\frac{\partial}{\partial y}5\cos\pi y$$

$$\boldsymbol{B} = -\hat{\boldsymbol{y}}\pi\cos\pi x + \hat{\boldsymbol{z}}5\pi\sin\pi y$$

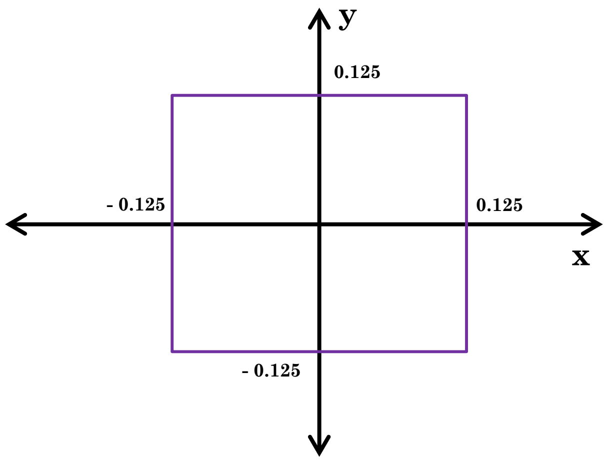
2. Problem 5.27b

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

b) Use Eq 5.66 to calculate the magnetic flux passing through a square loop with 0.25-m long edges if the loop is in the x-y plane, its center is at the origin, and its edges are parallel to the x and y axes.

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{(Wb)}$$



$$\Phi = \int_{-0.125}^{0.125}\int_{-0.125}^{0.125} (-\hat{\boldsymbol{y}}\pi\cos\pi x + \hat{\boldsymbol{z}}5\pi\sin\pi y)\hat{\boldsymbol{z}}dxdy$$

$$\Phi = \int_{-0.125}^{0.125}\int_{-0.125}^{0.125} (5\pi\sin\pi y)\hat{\boldsymbol{z}}dxdy = \hat{\boldsymbol{z}}\int_{-0.125}^{0.125} x5\pi\sin\pi y\,\Big|_{x=-0.125}^{x=0.125} dy$$

$$\Phi = -\hat{\boldsymbol{z}}x5\pi\frac{\cos\pi y}{\pi}\Big|_{x=-0.125}^{x=0.125}\Big|_{y=-0.125}^{y=0.125} = -\hat{\boldsymbol{z}}5(0.25)\left(\cos\frac{\pi}{8} - \cos\frac{-\pi}{8}\right)$$

$$\Phi = 0$$

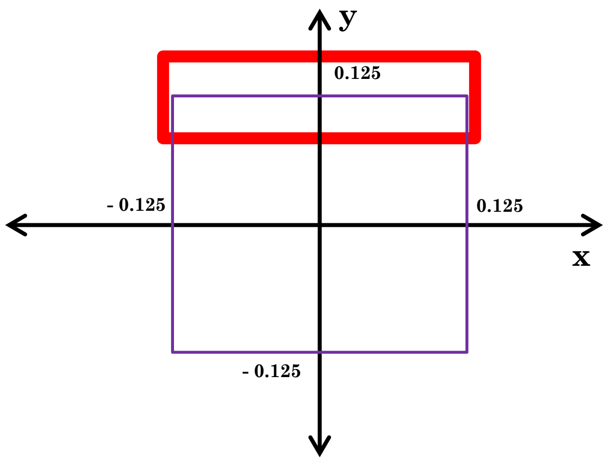
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along y=0.125:

$$S_{top} = \int_{-0.125}^{0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \cdot \hat{\boldsymbol{x}}dx$$

$$S_{top} = \int_{-0.125}^{0.125} 5\cos(0.125\pi)\, dx = 4.6194x\Big|_{x=-0.125}^{x=0.125}$$

S_{top} =1.15485

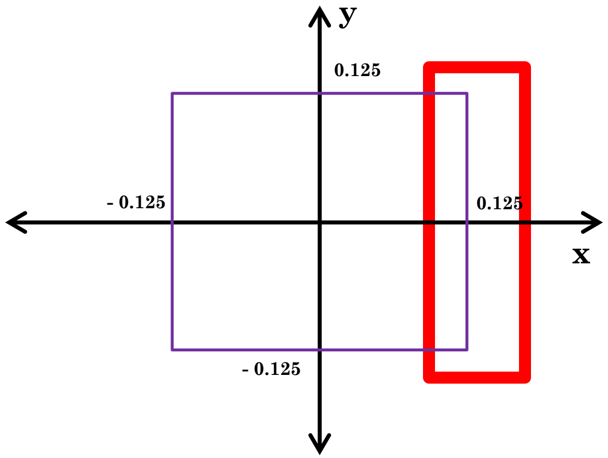
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along x=0.125:

$$S_{right} = \int_{0.125}^{-0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \cdot \hat{\boldsymbol{y}}dy$$

$$S_{right} = \int_{0.125}^{-0.125} 0dx = 0$$

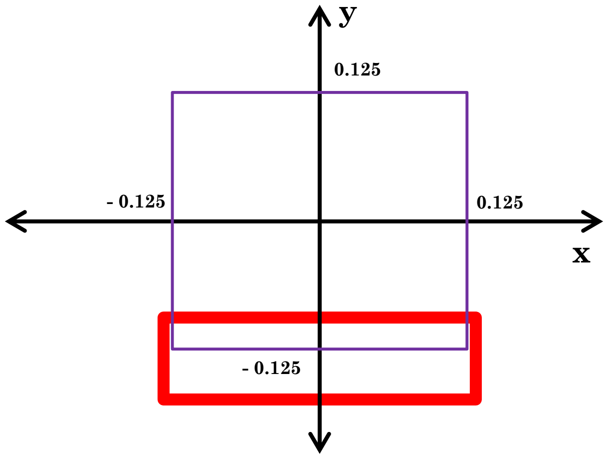
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along y=-0.125:

$$S_{bottom} = \int_{0.125}^{-0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2 + \sin\pi x) \cdot \hat{\boldsymbol{x}}dx$$

$$S_{bottom} = \int_{0.125}^{-0.125} 5\cos(0.125\pi)\,dx = 4.6194x\Big|_{x=0.125}^{x=-0.125}$$

$$S_{bottom} = -1.15485$$

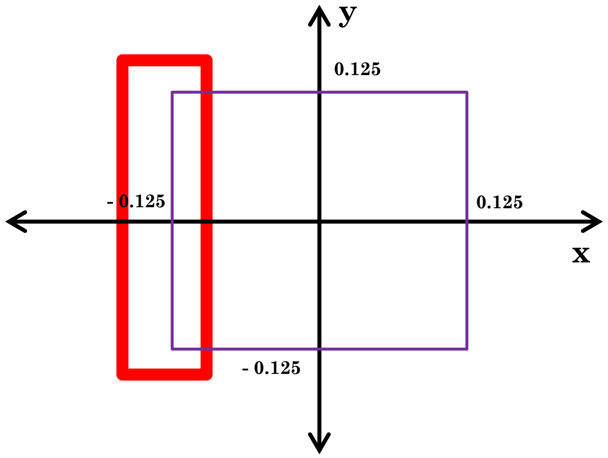
2. Problem 5.27c

In a given region of space, the vector magnetic potential is given by

$$\boldsymbol{A} = \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x) \quad (Wb/m)$$

c) Calculate $\boldsymbol{\Phi}$ again using Eq 5.67

$$\boldsymbol{\Phi} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$$



$$\Phi = S_{top} + S_{right} + S_{bottom} + S_{left}$$

Along x=-0.125:

$$S_{left} = \int_{-0.125}^{0.125} \hat{\boldsymbol{x}}5\cos\pi y + \hat{\boldsymbol{z}}(2+\sin\pi x)\cdot\hat{\boldsymbol{y}}dy$$

$$S_{left} = \int_{0.125}^{-0.125} 0dx = 0$$

$$\Phi = 1.15485 + 0 - 1.15485 + 0$$
$$\Phi = 0$$

3. Problem 5.31

Iron contains $8.5 \times 10^{28} atoms/m^3$. At saturation, the alignment of the electrons' spin magnetic moments in iron can contribute 1.5T to the total magnetic flux density **B**. If the spin magnetic moment of a single electron is 9.27×10^{-24} $(A \cdot m^2)$, how many electrons per atom contribute to the saturated field?

The number of electrons contributing to the bulk magnetization *per atom* is n_e, the total number of contributing electrons is N_e, and the number of atoms is N_{atoms}.

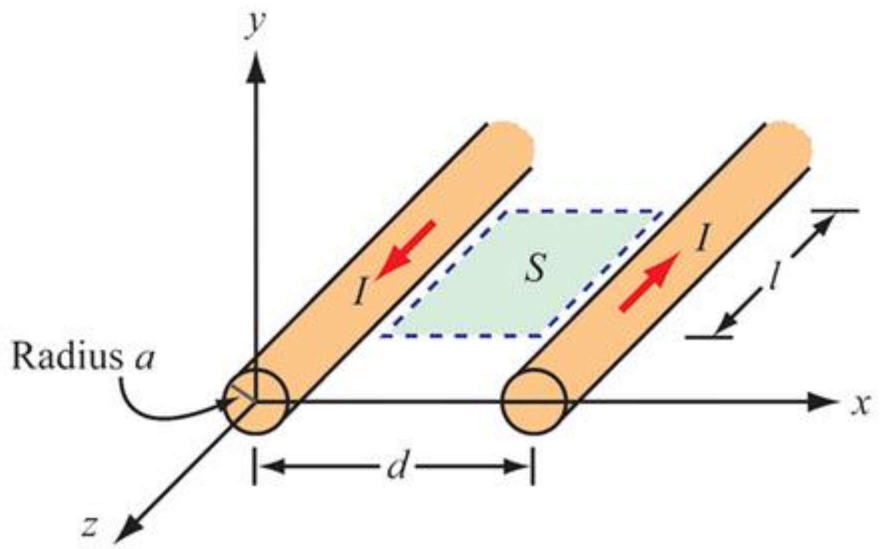
So, $n_e = \frac{N_e}{N_{atoms}}$

From $B_m = \mu_0 M = \mu_0 N_e m_s \rightarrow N_e = \frac{B_m}{\mu_0 m_s}$

$$n_e = \frac{N_e}{N_{atoms}} = \frac{B_m}{\mu_0 m_s N_{atoms}} = \frac{1.5}{\mu_0 (9.27 \times 10^{-24})(8.5 \times 10^{28})} = 1.5\ electrons/atom$$

4. Problem 5.37

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Notes:

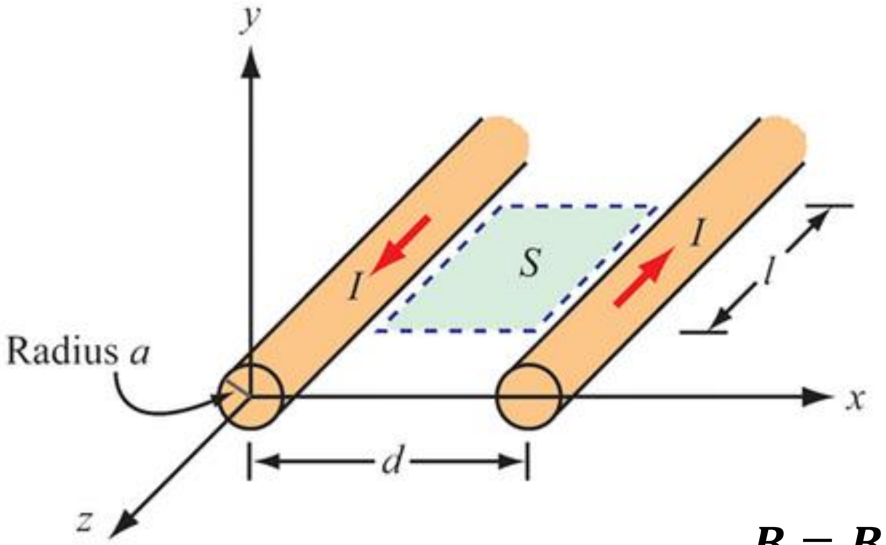
- Self-inductance is based on magnetic flux
- Magnetic flux is based on magnetic field strength
- ASSUMPTION: radii are much smaller than the separation distance

Plan:

- Calculate the magnetic field caused by each wire
- Calculate the magnetic flux
- Calculate the inductance

4. Problem 5.37 – step 1

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic field caused by each wire

Currents are both I, the magnetic field at a position x for long wires is:

$$\boldsymbol{B_1} = \hat{y}\frac{\mu I}{2\pi x} \qquad \boldsymbol{B_2} = \hat{y}\frac{\mu I}{2\pi(d-x)}$$

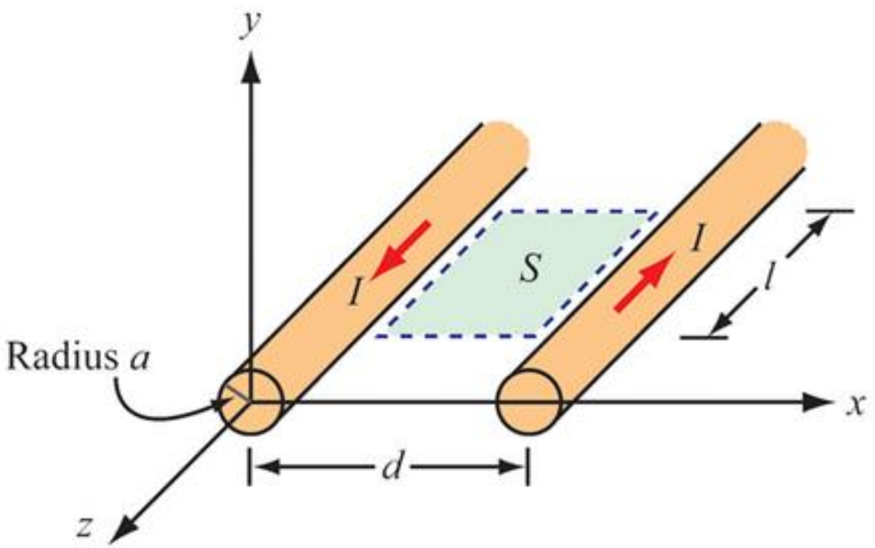
They're both positive, because the currents are in opposite directions.

$$\boldsymbol{B} = \boldsymbol{B_1} + \boldsymbol{B_2} = \hat{\boldsymbol{y}}\frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) = \hat{\boldsymbol{y}}\frac{\mu I}{2\pi}\left(\frac{d-x}{x(d-x)} + \frac{x}{x(d-x)}\right)$$

$$\boldsymbol{B} = \hat{\boldsymbol{y}}\frac{\mu I d}{2\pi x(d-x)}$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

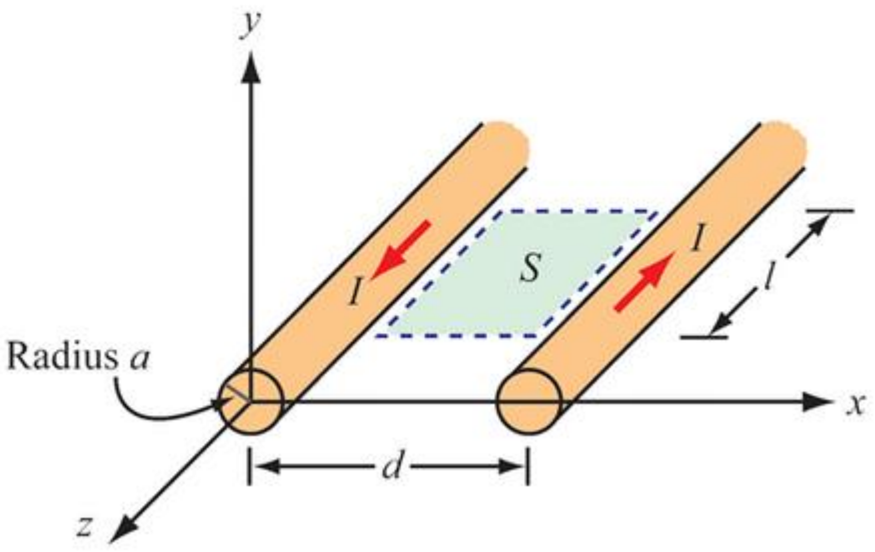
$$\boldsymbol{\Phi} = \int\int \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$\boldsymbol{\Phi} = \int_{z=0}^{l}\int_{x=a}^{d-a} \hat{\boldsymbol{y}}\frac{\mu I d}{2\pi x(d-x)}\cdot\hat{\boldsymbol{y}}dxdz$$

$$\boldsymbol{\Phi} = \frac{\mu I d}{2\pi}\int_{z=0}^{l}\int_{x=a}^{d-a}\frac{1}{x(d-x)}dxdz$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

$$\boldsymbol{\Phi} = \frac{-\mu I d}{2\pi} \int_{z=0}^{l} \int_{x=a}^{d-a} \frac{1}{x(x-d)} dx dz$$

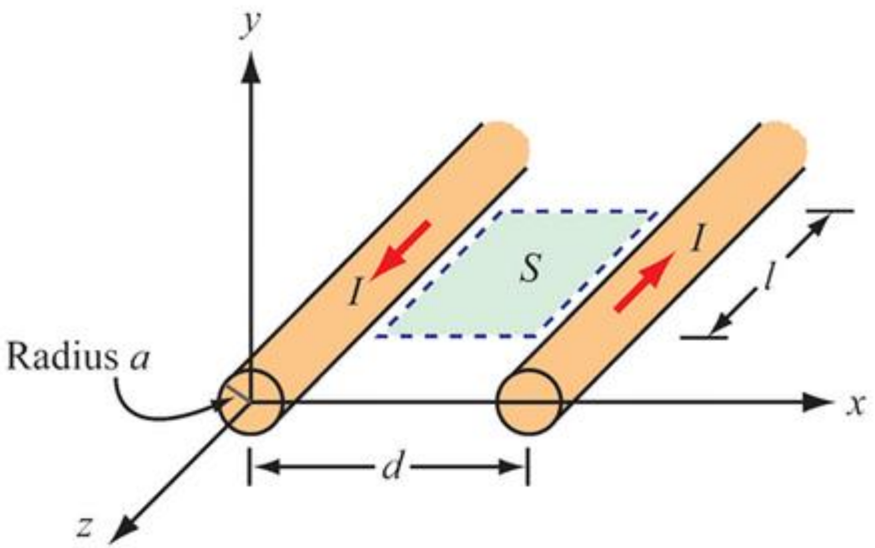
Use common integral:

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \; a \neq b$$

$$\boldsymbol{\Phi} = \frac{-\mu I d}{2\pi} \int_{z=0}^{l} \left(\frac{1}{-d-0} \ln \frac{x}{x-d} \Big|_{x=a}^{d-a} \right) dz = \frac{\mu I l}{2\pi} \left(\ln \frac{d-a}{-a} - \ln \frac{a}{a-d} \right)$$

4. Problem 5.37 – step 2

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the magnetic flux

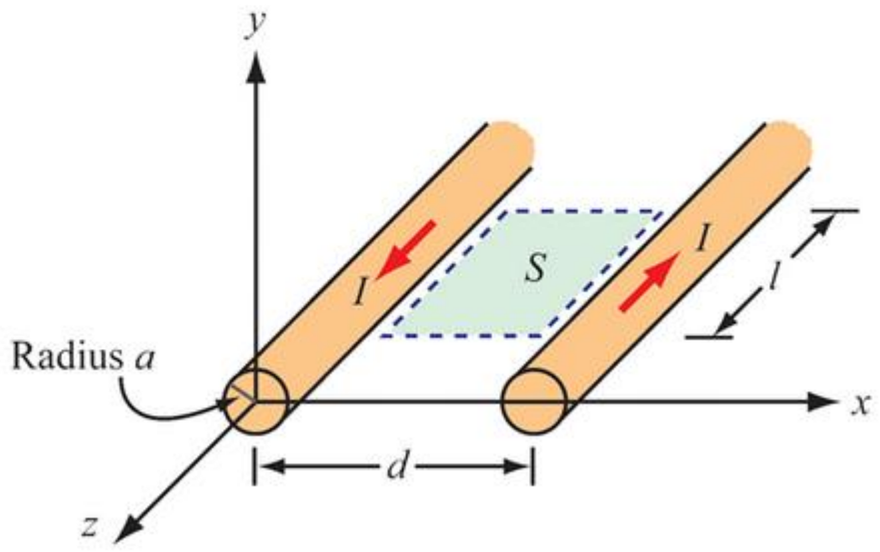
$$\boldsymbol{\Phi} = \frac{\mu I l}{2\pi}\left(\ln\frac{d-a}{-a} - \ln\frac{a}{a-d}\right)$$

$$\boldsymbol{\Phi} = \frac{\mu I l}{2\pi}\left(\ln\frac{(d-a)(a-d)}{-a^2}\right) = \frac{\mu I l}{2\pi}\left(\ln\frac{(d-a)^2}{a^2}\right)$$

$$\boldsymbol{\Phi} = \frac{\mu I l}{\pi}\ln\frac{d-a}{a}$$

4. Problem 5.37

Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-28a in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.



Calculate the inductance

There is only one ‘loop’, so inductance is:

$$L = \frac{\Phi}{I} = \frac{\mu I l}{\pi} \ln\frac{d-a}{a} \Big/ I = \frac{\mu l}{\pi} \ln\frac{d-a}{a}$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\mu l}{\pi} \ln\frac{d-a}{a}$$

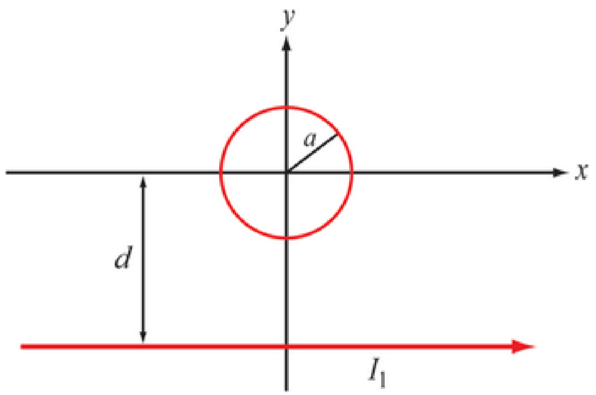
Assuming a is very small relative to d:

$$L' \approx \frac{\mu l}{\pi} \ln\frac{d}{a}$$

5. Problem 5.40

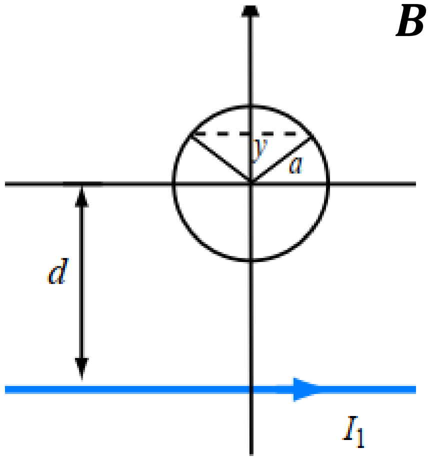
Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).



Find the magnetic field induced by the current on the linear conductor, for any line of constant y:

$$\boldsymbol{B} = \hat{\boldsymbol{z}} \frac{\mu_0 I}{2\pi(d+y)}$$



Find the flux through each infinitesimal strip of constant y inside the loop.

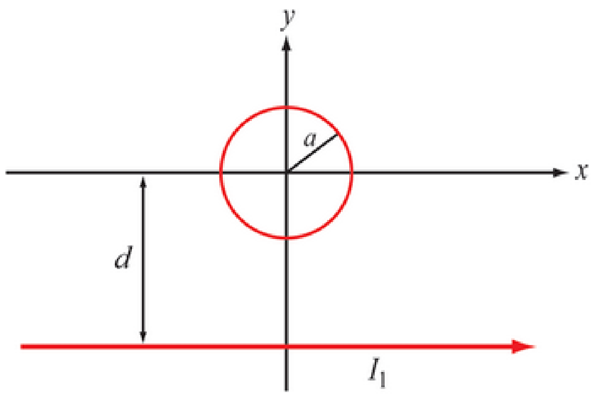
$$d\Phi = \boldsymbol{B}(y) \cdot d\boldsymbol{s} = \hat{\boldsymbol{z}} \frac{\mu_0 I}{2\pi(d+y)} \cdot \hat{\boldsymbol{z}} 2\sqrt{(a^2 - y^2)}dy$$

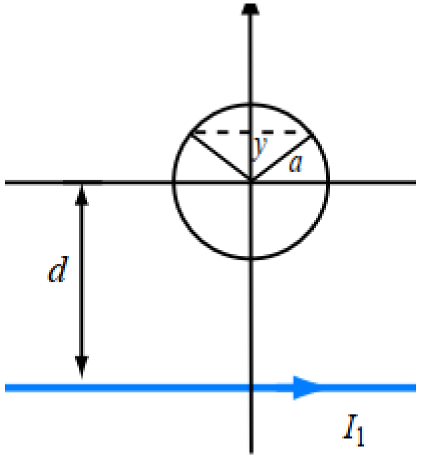
5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$d\Phi = \mathbf{B}(y)\cdot d\mathbf{s} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi(d+y)}\cdot\hat{\mathbf{z}}2\sqrt{(a^2-y^2)}dy$$

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).





$$d\Phi = \frac{\mu_0 I\sqrt{(a^2-y^2)}}{\pi(d+y)}dy$$

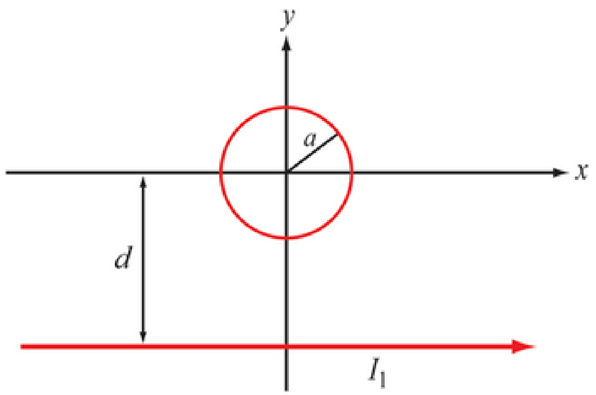
$$L = \frac{\Phi}{I} = \frac{1}{I}\int_S d\Phi = \frac{1}{I}\int_{y=-a}^{a}\frac{\mu_0 I\sqrt{(a^2-y^2)}}{\pi(d+y)}dy$$

$$L = \frac{\mu_0}{\pi}\int_{y=-a}^{a}\frac{\sqrt{(a^2-y^2)}}{(d+y)}dy$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Figure P5.40 Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 40).



Solving the following integral:

$$L = \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{\sqrt{(a^2 - y^2)}}{(d + y)} dy$$

Let $z = d + y$: $L = \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{(a^2 - (z - d)^2)}}{z} dz$

$$L = \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{(a^2 - d^2) + 2dz - z^2}}{z} dz$$

Let $R = a_0 + b_0 z + c_0 z^2$ be the polynomial $(a^2 - d^2) + 2dz - z^2$:

From a table of integrals: $\int \frac{\sqrt{R}}{z} dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

So now we're solving *these* integrals:

Let $R = a_0 + b_0 z + c_0 z^2$ be the polynomial $(a^2 - d^2) + 2dz - z^2$:

$$\int \frac{\sqrt{R}}{z} dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

Evaluating the first term over $d - a$ to $d + a$:

$$\sqrt{R}\Big|_{d-a}^{d+a} = \sqrt{(a^2 - d^2) + 2d(d+a) - (d+a)^2} - \sqrt{(a^2 - d^2) + 2d(d-a) - (d-a)^2} = 0$$

So our integral is now $a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

From a table of integrals:

$$a_0 \int \frac{dz}{z\sqrt{R}} = a_0 \left[\frac{1}{\sqrt{-a_0}} \sin^{-1} \left(\frac{2a_0 + b_0 z}{z\sqrt{{b_0}^2 - 4a_0 c_0}} \right) \right]_{z=d-a}^{d+a}$$

$$= -\sqrt{d^2 - a^2} \left[\sin^{-1} \left(\frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a}$$

$$= -\pi \sqrt{d^2 - a^2}$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

$$a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}$$

From a table of integrals:

$$\frac{b_0}{z} \int \frac{dz}{\sqrt{R}} = \frac{b_0}{2} \left[\frac{-1}{\sqrt{-c_0}} \sin^{-1} \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right]_{z=d-a}^{d+a}$$

$$= -d \left[\sin^{-1} \left(\frac{d-z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d$$

5. Problem 5.40

Determine the mutual inductance between the circular loop and the linear current shown in Fig P5.40.

Last step!

$$L = \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{\sqrt{(a^2 - y^2)}}{(d + y)} dy$$

$$L = \frac{\mu_0}{\pi}\left(\pi d - \pi\sqrt{d^2 - a^2}\right)$$

$$L = \mu_0\left(d - \sqrt{d^2 - a^2}\right)$$

Reminders

- Assignment 10 is out, and is due at 8AM on April 4th.
- Good luck!

ENG PHYS 2A04 Tutorial 11

Electricity and Magnetism

Your TAs today

- Thomas Kean

  keant1@mcmaster.ca

- Joanne Lee

  leej298@mcmaster.ca

Chapter 6

Problems

Problem 6.3

Problem 6.4

Problem 6.5

Problem 6.7

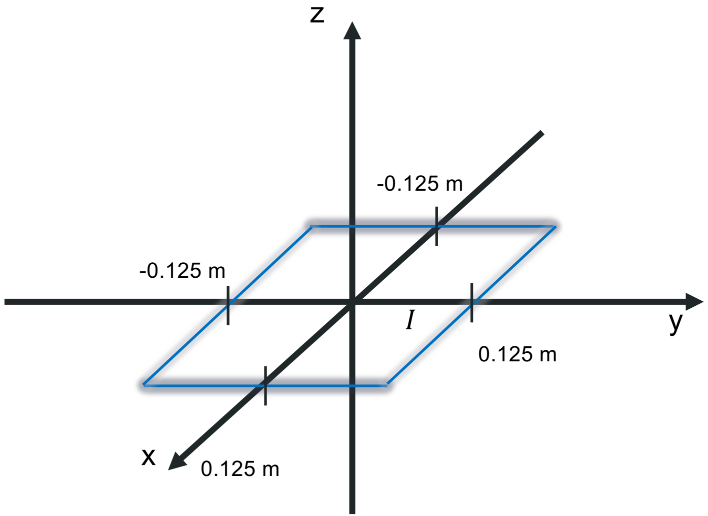
Problem 6.3 – Question

A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the x- or y-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) $\mathbf{B} = \hat{\mathbf{z}}\, 20 e^{-3t}$ (T)

(b) $\mathbf{B} = \hat{\mathbf{z}}\, 20 \cos x \cos 10^3 t$ (T)

(c) $\mathbf{B} = \hat{\mathbf{z}}\, 20 \cos x \sin 2y \cos 10^3 t$ (T)



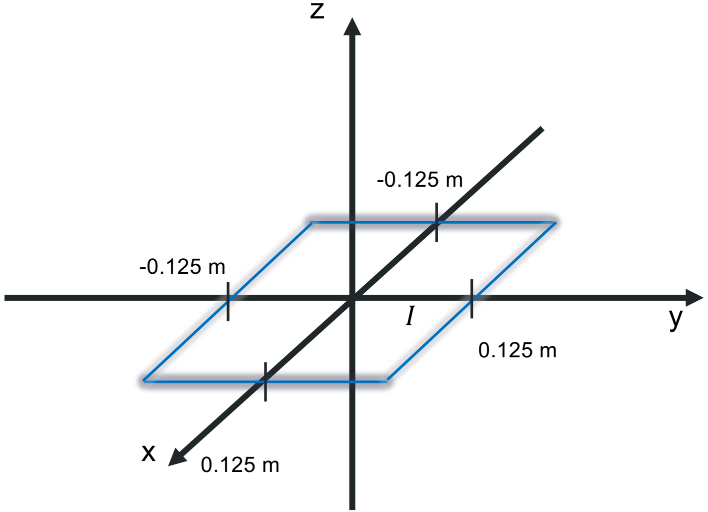
Problem 6.3 – Details

A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the x- or y-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) $\mathbf{B} = \hat{\mathbf{z}}\, 20e^{-3t}$ (T)

(b) $\mathbf{B} = \hat{\mathbf{z}}\, 20 \cos x \cos 10^3 t$ (T)

(c) $\mathbf{B} = \hat{\mathbf{z}}\, 20 \cos x \sin 2y \cos 10^3 t$ (T)



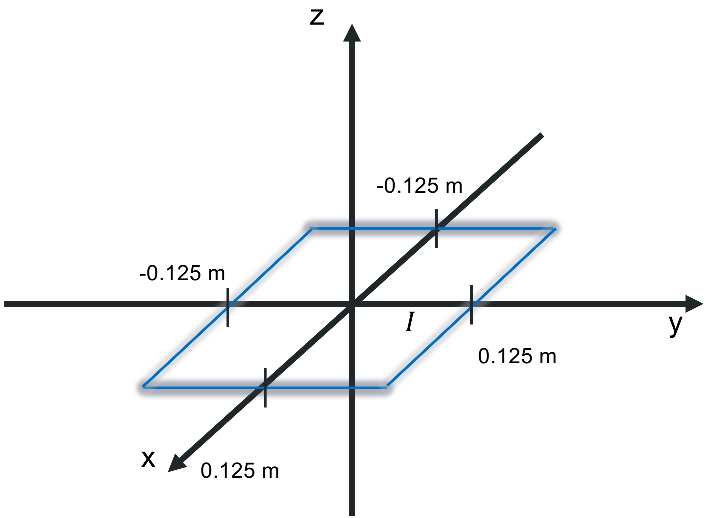
Solution? → Apply Faraday's Law!

Problem 6.3 – Solution (a)

Faraday's Law states: $V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$

$N = 100$ turns, $d\mathbf{s} = \hat{\mathbf{z}}\, dxdy$

$\mathbf{B} = \hat{\mathbf{z}}\, 20e^{-3t}$ (T)

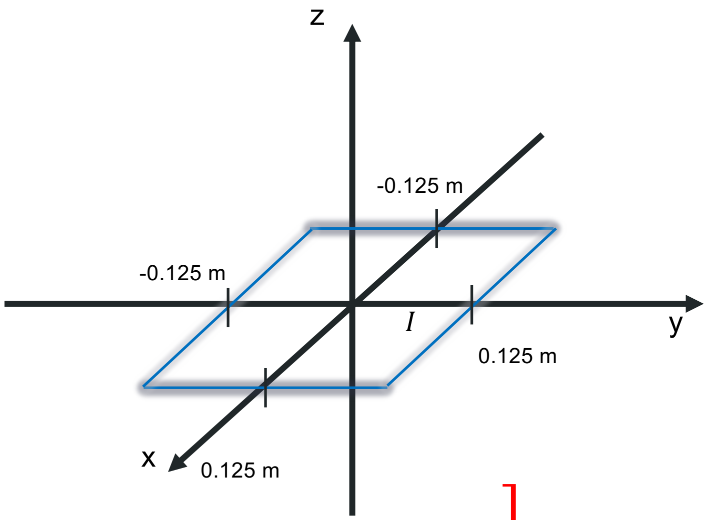


$$\rightarrow V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\left[\int_{-0.125}^{0.125}\int_{-0.125}^{0.125} \hat{\mathbf{z}}\, 20e^{-3t}\cdot\hat{\mathbf{z}}\, dxdy\right]$$

$$= -100\frac{d}{dt}[(20e^{-3t})(0.25)^2] = -100(0.0625)(20)\frac{d}{dt}[e^{-3t}] = -125(-3)e^{-3t}$$

$$\rightarrow\therefore \boldsymbol{V_{emf} = 375e^{-3t}}\ \textbf{(V)}$$

Problem 6.3 – Solution (b)



$$V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s},\ N = 100 \text{ turns},\ d\mathbf{s} = \hat{\mathbf{z}}\,dxdy$$

$$\mathbf{B} = \hat{\mathbf{z}}\,20\cos x\cos 10^3 t\ (\mathrm{T})$$

$$\rightarrow V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\left[\int_{-0.125}^{0.125}\int_{-0.125}^{0.125}\hat{\mathbf{z}}\,20\cos x\cos 10^3 t\cdot\hat{\mathbf{z}}\,dxdy\right]$$

$$= -100\frac{d}{dt}[20\cos 10^3 t\,(\sin 0.125 - \sin -0.125)(0.25)]$$

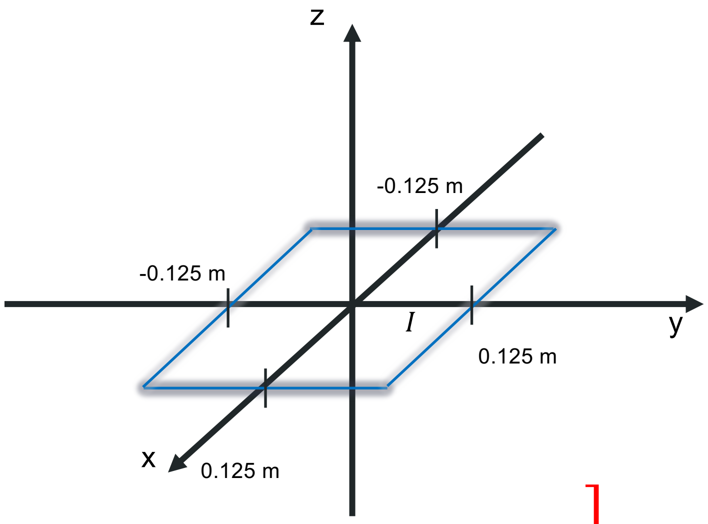
$$= -100(20)(0.25)(0.125 - (-0.125))\frac{d}{dt}[\cos 10^3 t] = -125(-1000\sin 10^3 t)$$

$$\rightarrow\therefore \boldsymbol{V_{emf} = 125\sin 10^3 t\ \ (\mathrm{kV})}$$

Problem 6.3 – Solution (c)

$V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$, $N = 100$ turns, $d\mathbf{s} = \hat{\mathbf{z}}\,dxdy$

$\mathbf{B} = \hat{\mathbf{z}}\,20\cos x\sin 2y\cos 10^3 t$ (T)



$$\rightarrow V_{emf} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\left[\int_{-0.125}^{0.125}\int_{-0.125}^{0.125}\hat{\mathbf{z}}\,20\cos x\sin 2y\cos 10^3 t\cdot\hat{\mathbf{z}}\,dxdy\right]$$

$$\boxed{=0} \qquad \boxed{\cos -2y = \cos 2y}$$

$$= -100\frac{d}{dt}\left[20\cos 10^3 t\left(\frac{-\cos 2y + \cos(-2y)}{2}\right)(\sin 0.125 - \sin -0.125)\right]$$

$$\rightarrow\therefore \boldsymbol{V_{emf} = 0\ (\mathrm{V})}$$

Problem 6.4 – Question

A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2 Ω resistor is connected across its open ends?

Problem 6.4 – Details

A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2 Ω resistor is connected across its open ends?

Goal → find current in closed loop

Let:
- V_{emf} = induced emf
- $I = 5\text{ A}$ = current of stationary conducting loop
- $R = 0.5\ \Omega$ = internal resistance of stationary conducting loop
- $I' = ?$ = current of new loop with $2\ \Omega$ resistor inserted
- $R' = 2\ \Omega$ = equivalent resistance

Problem 6.4 – Solution

A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2 Ω resistor is connected across its open ends?

Does V_{emf} change with the resistance? → NO ⟶ $V_{emf} = I \cdot R = I' \cdot R'$

$$\rightarrow I' = \frac{I \cdot R}{R'} = \frac{5\ \mathrm{A} \cdot 0.5\ \Omega}{2\ \Omega + 0.5\ \Omega} \rightarrow \therefore \boldsymbol{I' = 1\ \mathrm{A}}$$

Let:

- V_{emf} = induced emf
- $I = 5\ \mathrm{A}$ = current of stationary conducting loop
- $R = 0.5\ \Omega$ = internal resistance of stationary conducting loop
- $I' = ?$ = current of new loop with $2\ \Omega$ resistor inserted
- $R' = 2\ \Omega$ = equivalent resistance

Problem 6.5

A circular-loop TV antenna with 0.02 m^2 area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 30 (mV). What is the peak magnitude of B of the incident wave?

Problem 6.5

Solution:

TV loop antennas have one turn.

At maximum orientation with the loop area of A and Uniform magnetic field of B=$|B|$,

$\Phi = \int B.ds = \pm BA$

F=300 MHz, $\omega = 2\pi \times 300 \times 10^6 = 6\pi \times 10^8 rad/s$

$$V_{emf} = -N\frac{d\Phi}{dt} = -A\frac{d}{dt}[B_0 \cos(\omega t + \alpha_0)] = AB_0\omega \sin(\omega t + \alpha_0)$$

$$-1 \leq \sin(\omega t + \alpha_0) \leq 1$$

$$V_{emf} = AB_0\omega$$

$$B_0 = \frac{V_{emf}}{A\omega} = \frac{0.03V}{0.02m^2(6\pi \times 10^8 rad/s)}$$

$$B_0 = 0.8nA/m$$

Problem 6. 7

The rectangular conducting loop shown in Fig. P6.7 rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\boldsymbol{y}}50 \text{ (mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

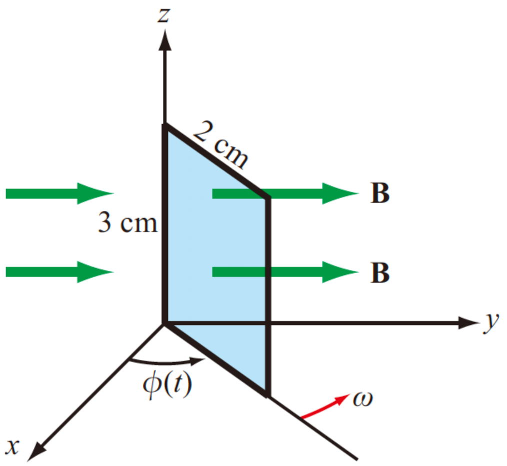


Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

Problem 6. 7

The rectangular conducting loop shown in Fig. P6.7 rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}50 \text{ (mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

Analysis:

A moving loop with a time-varying in a static field B

$$I_{ind} = \frac{V_{emf}}{R_i} \Rightarrow \boxed{V_{emf} = V_{emf}^{m}} = \boxed{-}\frac{d\phi}{dt}$$

Lenz's Law

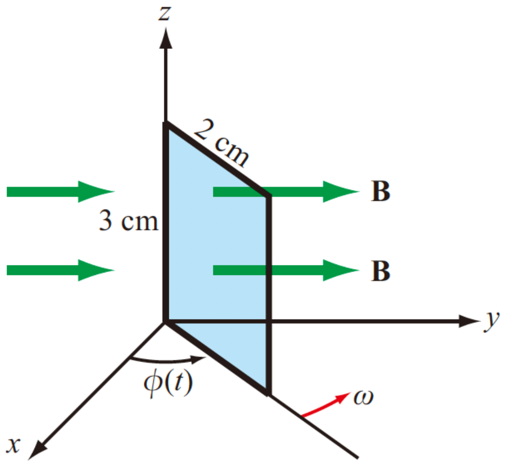


Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

Solution:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \hat{\mathbf{y}}\, 50 \times 10^{-3} \cdot \hat{\mathbf{y}}(2 \times 3 \times 10^{-4}) \cos\phi(t) = 3 \times 10^{-5} \cos\phi(t),$$

$$\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad \text{(rad/s)},$$

$$\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad \text{(Wb)},$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad \text{(V)},$$

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad \text{(mA)}.$$

Don’t forget the direction of current, because current is a vector.

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \le \phi \le \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.