LAST (family) NAME: ______________________________ **Test # 1**

FIRST (given) NAME: ______________________________ **Math 2ZZ3**

Test duration: 75 **min.**

SAMPLE TEST 1 A: SOLUTIONS

Instructions: You **must** use permanent ink. Tests submitted in pencil will not be considered later for remarking. This exam consists of 10 problems on 14 pages (make sure you have all 14 pages). The last two pages are for scratch or overflow work. The total number of points is 50. Do not add or remove pages from your test. No books, notes, or "cheat sheets" allowed. No calculator or other electronic devices are allowed. There is a formula that might be useful on the last page. **GOOD LUCK!**

SOLUTIONS

Continued...

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Part I: Enter your answer in the appropriate box. **Provide all details and fully justify your answer in order to receive credit.**

1. Consider the function

$$f(x) = \sin(x/2), \quad 0 < x < \pi.$$

(a) *(4 pts.)* Compute the half-range expansion of $f(x)$ as a cosine Fourier series of period 2π.

Solution. We have, for $n \geq 0$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \sin(x/2)\,\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin((1/2+n)x) + \sin((1/2-n)x)\,dx \\
&= \frac{1}{\pi}\left[-\frac{\cos((1/2+n)x)}{1/2+n} - \frac{\cos((1/2-n)x)}{1/2-n}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\frac{-\cos(\pi/2+n\pi)+1}{1/2+n} + \frac{-\cos(\pi/2-n\pi)+1}{1/2-n}\right\} \\
&= \frac{1}{\pi}\left\{\frac{1}{n+1/2} + \frac{1}{1/2-n}\right\} \\
&= \frac{1}{\pi}\,\frac{1}{1/4-n^2}.
\end{aligned}$$

Hence,

$$\boxed{f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,\cos(nx) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi}\,\frac{1}{1/4-n^2}\,\cos(nx).}$$

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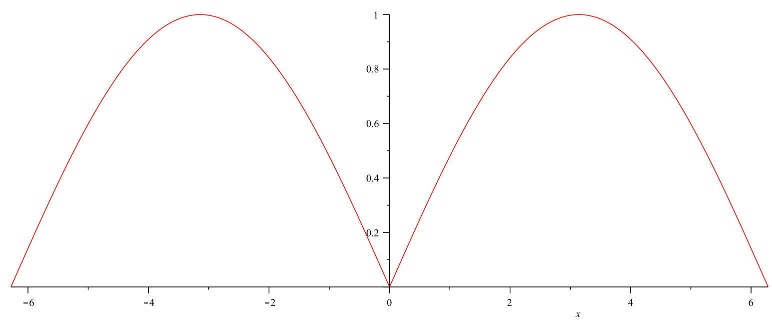
(b) *(2 pts.)* Draw a graph of the sum $S(x)$ of the cosine Fourier series obtained in part (a) for $-2\pi < x < 2\pi$. Indicate clearly the value of $S(x)$ on the graph at any point of discontinuity.

Solution.



(c) *(2 pts.)* What is the value of the sum of the cosine Fourier series computed in part (a) when $x = \frac{3\pi}{2}$, i.e. what is $S(\frac{3\pi}{2})$?

Solution. Since $S(x)$ is 2π-periodic,

$$S(3\pi/2) = S(-\pi/2).$$

Since $S(x)$ is even,

$$S(-\pi/2) = S(\pi/2) = \sin(\pi/4) = \frac{\sqrt{2}}{2}.$$

Continued. . .

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2. Let $f(x)$ be the function defined by the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(nx), \quad x \in \mathbb{R}.$$

(a) *(5 pts.)* Find a 2π-periodic solution of the differential equation

$$y'(x) + y(x) = f(x), \quad x \in \mathbb{R},$$

in the form of a Fourier series.

Solution. A particular solution of the DE

$$y'(x) + y(x) = \sin(nx)$$

has the form

$$y_n^p(x) = A_n \cos(nx) + B_n \sin(nx).$$

Since

$$\left(y_n^p\right)'(x) + y_n^p(x) = -A_n\, n \sin(nx) + B_n\, n \cos(nx) + A_n \cos(nx) + B_n \sin(nx) = \sin(nx),$$

This yields $B_n\, n + A_n = 0$ and $-A_n\, n + B_n = 1$ or $A_n = -n/(1+n^2)$ and $B_n = 1/(1+n^2)$. Thus,

$$y_n^p(x) = \frac{-n}{1+n^2} \cos(nx) + \frac{1}{1+n^2} \sin(nx).$$

A periodic solution of the original problem is thus given by

$$y(x) = \sum_{n=1}^{\infty} 2^{-n}\, y_n^p(x) = \sum_{n=1}^{\infty} \frac{-n}{2^n\,(1+n^2)} \cos(nx) + \frac{1}{2^n\,(1+n^2)} \sin(nx).$$

(b) *(1 pt.)* What is the value of the integral $\int_{-\pi}^{\pi} y(x)\, dx$. (Explain.)

Solution. There is no constant term in the cosine Fourier series representing $y(x)$, so

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x)\, dx = 0$$

showing that

$$\int_{-\pi}^{\pi} y(x)\, dx = 0$$

Continued. . .

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3. *(7 pts.)* Let $\mathbf{r}_1(t)$ denote the position vector of one end of a stick at time t and let $\mathbf{r}_2(t)$ denote the position of the other end. Prove that

$$\mathbf{r_1'} \cdot \mathbf{r_1} + \mathbf{r_2'} \cdot \mathbf{r_2} = \mathbf{r_1'} \cdot \mathbf{r_2} + \mathbf{r_2'} \cdot \mathbf{r_1}.$$

Hint: Notice that for any t

$$\|\mathbf{r}_2(t) - \mathbf{r}_1(t)\|^2 = (\mathbf{r}_2(t) - \mathbf{r}_1(t)) \cdot (\mathbf{r}_2(t) - \mathbf{r}_1(t)) = \ell^2,$$

where ℓ is the length of the stick.

Solution. We have

$$\ell^2 = (\mathbf{r}_2(t) - \mathbf{r}_1(t)) \cdot (\mathbf{r}_2(t) - \mathbf{r}_1(t)) = \mathbf{r}_2(t) \cdot \mathbf{r}_2(t) - 2\,\mathbf{r}_1(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_1(t)$$

and, differentiating both sides with respect to t yields

$$\begin{aligned} 0 &= \mathbf{r}_2'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_2'(t) \cdot \mathbf{r}_2(t) - 2\,\mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) - 2\,\mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_1(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_1'(t) \\ &= 2\,\mathbf{r}_2'(t) \cdot \mathbf{r}_2(t) - 2\,\mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) - 2\,\mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + 2\,\mathbf{r}_1'(t) \cdot \mathbf{r}_1(t) \end{aligned}$$

Thus

$$\mathbf{r}_2'(t) \cdot \mathbf{r}_2(t) - \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) - \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_1(t) = 0$$

or, equivalently,

$$\mathbf{r_1'} \cdot \mathbf{r_1} + \mathbf{r_2'} \cdot \mathbf{r_2} = \mathbf{r_1'} \cdot \mathbf{r_2} + \mathbf{r_2'} \cdot \mathbf{r_1}.$$

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4. Consider the curve parametrized by

$$\mathbf{r}(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j}, \quad -\infty < t < \infty.$$

(Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}$, $\sinh t = \dfrac{e^t - e^{-t}}{2}$.)

(a) *(5 pts)* Find the curvature $\kappa(t)$ of this curve.

Solution. Since $\mathbf{r}(t) = \left(\dfrac{e^t + e^{-t}}{2}\right)\mathbf{i} + \left(\dfrac{e^t - e^{-t}}{2}\right)\mathbf{j}$, we have

$$\mathbf{r}'(t) = \left(\frac{e^t - e^{-t}}{2}\right)\mathbf{i} + \left(\frac{e^t + e^{-t}}{2}\right)\mathbf{j} \quad \text{and} \quad \mathbf{r}''(t) = \left(\frac{e^t + e^{-t}}{2}\right)\mathbf{i} + \left(\frac{e^t - e^{-t}}{2}\right)\mathbf{j}.$$

Thus,

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{e^t - e^{-t}}{2} & \frac{e^t + e^{-t}}{2} & 0 \\ \frac{e^t + e^{-t}}{2} & \frac{e^t - e^{-t}}{2} & 0 \end{vmatrix} = \left(\frac{(e^t - e^{-t})^2}{4} - \frac{(e^t + e^{-t})^2}{4}\right)\mathbf{k} = -\mathbf{k}$$

Since

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2 + \left(\frac{e^t - e^{-t}}{2}\right)^2} = \sqrt{\frac{2\,e^{2t} + 2\,e^{-2t}}{4}} = \sqrt{\frac{e^{2t} + e^{-2t}}{2}}$$

and

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \| -\mathbf{k}\| = 1,$$

we have

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{2}}{(e^{2t} + e^{-2t})^{3/2}} = \frac{1}{\cosh(2t)^{3/2}}.$$

(b) *(2 pts.)* Find a point on the curve above where the curvature is maximized.

Solution. We need to find minimize $\frac{e^{2t}+e^{-2t}}{2} = \cosh(2t)$. Since $\cosh(2t) \geq 1$ for all t and $\cosh(0) = 1$, the curvature is maximized when $t = 0$ i.e. at the point $(1, 0)$.

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5. Consider the curve parametrized by

$$\mathbf{r}(t) = \frac{1}{t^2+1}\,\mathbf{i} + \frac{t}{t^2+1}\,\mathbf{j}, \quad 0 \le t \le 1.$$

(a) *(4 pts.)* Compute the arclength function $s(t)$.

Solution. We have

$$\mathbf{r}'(t) = \frac{-2\,t}{(t^2+1)^2}\,\mathbf{i} + \frac{1-t^2}{(t^2+1)^2}\,\mathbf{j}$$

and

$$\|\mathbf{r}'(t)\| = \sqrt{\frac{4\,t^2}{(t^2+1)^4} + \frac{(1-t^2)^2}{(t^2+1)^4}} = \frac{\sqrt{4\,t^2+1-2\,t^2+t^4}}{(t^2+1)^2} = \frac{\sqrt{1+2\,t^2+t^4}}{(t^2+1)^2} = \frac{\sqrt{(t^2+1)^2}}{(t^2+1)^2} = \frac{1}{t^2+1}.$$

Hence,

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t \frac{1}{u^2+1}\,du = \tan^{-1}(t).$$

(b) *(3 pts.)* Reparametrize the original curve using the arc-length parameter s. Specify the range of s. (**Hint.** Express the parameter t in terms of the arc-length parameter s)

Solution. Since $s = \tan^{-1} t$, we have

$$t = \tan s$$

and, since

$$t^2 + 1 = \tan^2 s + 1 = \frac{\sin^2 s + \cos^2 s}{\cos^2 s} = \frac{1}{\cos^2 s},$$

we have

$$x = \frac{1}{t^2+1} = \cos^2 s \;\left(= \frac{1+\cos(2s)}{2}\right) \quad \text{and} \quad y = \frac{t}{t^2+1} = \tan s\,\cos^2 s = \sin s\,\cos s \;\left(= \frac{\sin(2s)}{2}\right).$$

The curve can thus be reparametrized by

$$\mathbf{r}(t(s)) = (\cos^2 s)\,\mathbf{i} + (\sin s\,\cos s)\,\mathbf{j},$$

where

$$0 \le s \le \frac{\pi}{4}.$$

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PART II: Multiple choice. Indicate your choice very clearly. There is only one correct answer in each multiple-choice problem. Circle the letter (A,B,C,D or E) corresponding to your choice. Ambiguous answers will be marked as wrong. You don't need to justify your answers and no negative marks are given for a wrong answer.

6. *(3 pts.)* Using the fact that $\frac{1}{2}\int_{-1}^{1} xe^{-in\pi x}dx = \frac{(-1)^{n+1}}{in\pi}$ for any integer $n \geq 1$, the function $f(x) = x$, $-1 < x < 1$, can be represented by the Fourier series:

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in\pi}\cos(n\pi x)$

(B) $\displaystyle\sum_{-\infty}^{\infty} \frac{(-1)^{n+1}}{in\pi}e^{in\pi x}$

→ **(C)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{in\pi}(e^{-in\pi x} - e^{in\pi x})$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{in\pi}e^{in\pi x}$

(E) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi}\sin(\frac{n\pi x}{2})$

Solution. We have

$$c_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = 0.$$

Also

$$c_n = \frac{1}{2}\int_{-1}^{1} x\,e^{-in\pi x}dx = \frac{1}{2}\int_{-1}^{1} x\;(\cos(n\pi x) + i\,\sin(n\pi x))\;dx = \frac{(-1)^{n+1}}{in\pi}, \quad n \geq 1,$$

so that

$$c_{-n} = \frac{1}{2}\int_{-1}^{1} x\,e^{in\pi x}dx = \frac{1}{2}\int_{-1}^{1} x\;(\cos(n\pi x) - i\,\sin(n\pi x))\;dx = -\frac{(-1)^{n+1}}{in\pi}, \quad n \geq 1,$$

Thus, $f(x)$ has the (complex) Fourier series expansion

$$\begin{aligned} f(x) &\simeq \sum_{n=-\infty}^{\infty} c_n\,e^{in\pi x} = c_0 + \sum_{n=1}^{\infty} c_n\,e^{in\pi x} + c_{-n}\,e^{-in\pi x} \\ &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in\pi}\left(e^{in\pi x} - e^{-in\pi x}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n}}{in\pi}\left(e^{-in\pi x} - e^{in\pi x}\right) \end{aligned}$$

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7. *(3 pts.)* Consider the function $f(x)$ defined on the interval $(-\pi, \pi)$ by

$$f(x) = \begin{cases} 2 + x, & -\pi < x < 0, \\ 3\,x, & 0 < x < \pi. \end{cases}$$

What is sum of the (2π-periodic) Fourier series of $f(x)$ when it is evaluated at $x = 3\pi$?

(A) $2 - 3\,\pi$

(B) 2π

(C) $\dfrac{3\pi + 1}{2}$

(D) 2

$\rightarrow$ **(E)** $\pi + 1$

Solution. Since $\lim_{x\to-\pi^+} f(x) = 2-\pi$ and $\lim_{x\to\pi^-} f(x) = 3\,\pi$, the Fourier series converges to $\frac{2-\pi+3\,\pi}{2} = 1 + \pi$ at $x = 3\pi$.

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8. *(3 pts.)* Suppose that $z = \sqrt{2\,x^2 - y^2 + 5}$, $x = x(t)$, $y = y(t)$. Compute $\frac{dz}{dt}$ at $t = 0$ given that $x(0) = 2$, $y(0) = 3$, $x'(0) = 5$, $y'(0) = 2$.

$\rightarrow$ **(A)** 7

(B) $\dfrac{4}{3}$

(C) $\dfrac{5}{\sqrt{6}}$

(D) -3

(E) 4

Solution. We have, by the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2\,x}{\sqrt{2\,x^2 - y^2 + 5}}\,\frac{dx}{dt} - \frac{y}{\sqrt{2\,x^2 - y^2 + 5}}\,\frac{dy}{dt}$$

When $t = 0$, $\sqrt{2\,x^2 - y^2 + 5} = \sqrt{8 - 9 + 5} = \sqrt{4} = 2$. Hence,

$$\frac{dz}{dt} = \frac{4}{2}\,(5) - \frac{3}{2}\,(2) = \frac{14}{2} = 7$$

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9. *(3 pts.)* Consider the curve parametrized by

$$\mathbf{r}(t) = \langle t^3 - t, \frac{6t}{t+1}, (2t+1)^2 \rangle, \quad t > -1.$$

Let P be the point where the tangent line to the curve at the point $(0, 0, 1)$ intersect the plane $x + y + z$. Then,

(A) $P = \left(-\frac{7}{13}, \frac{4}{13}, \frac{3}{13}\right)$

(B) $P = (-1, 6, 5)$

(C) $P = \left(\frac{1}{5}, -\frac{4}{5}, \frac{3}{5}\right)$

$\rightarrow$ **(D)** $P = \left(\frac{1}{9}, -\frac{6}{9}, \frac{5}{9}\right)$

(E) $P = \left(\frac{1}{2}, -2, \frac{3}{2}\right)$

Solution. We have

$$\mathbf{r}'(t) = \langle 3\,t^2 - 1, \frac{6}{(t+1)^2}, 4\,(2t+1) \rangle.$$

Since $\langle 0, 0, 1 \rangle = \mathbf{r}(0)$ and $\mathbf{r}'(0) = \langle -1, 6, 4 \rangle$. The tangent to the curve at $(0, 0, 1)$ is parametrized by

$$x = -t, \quad y = 6\,t, \quad z = 1 + 4\,t.$$

That line intersects the plane $x + y + z$ when

$$-t + 6\,t + 1 + 4\,t = 0 \quad \text{or} \quad t = -\frac{1}{9}.$$

The point of intersection is thus $P = \left(\frac{1}{9}, -\frac{6}{9}, \frac{5}{9}\right)$

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SCRATCH

Some formulas you may use:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}, \qquad a_N = \kappa v^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{v} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta), \quad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta), \quad \cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\}, \quad -p < x < p,$$

where,

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \quad b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

THE END