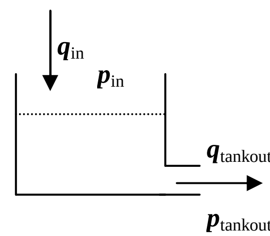
Mech Eng 4H03 Solution to Problem Set 3

Question 1

Five components can be identified from the given info: tank, valve, long pipe and two junctions.

1) Tank: Model as hydraulic capacitance



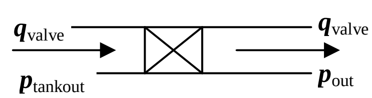
$$q_{in} - q_{tankout} = C_{tank} \frac{d(p_{tankout} - p_{in})}{dt}$$

where $C_{tank} = A/\rho g = A/Dg$ is the hydraulic capacitance of the tank.

We will use gauge pressure. Then p_{in}=0 and

$$q_{in} - q_{tankout} = C_{tankout} \frac{d(p_{tankout})}{dt} \qquad (1)$$

2) Valve: Model as hydraulic resistance

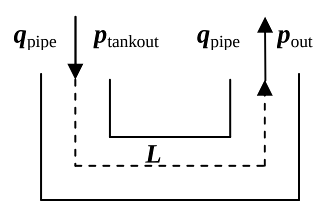


$$p_{tankout} - p_{out} = R_{valve} q_{valve}$$

Due to p_{out} = 0 gauge

$$p_{tankout} = R_{valve} q_{valve} \qquad (2)$$

where R_{valve} is hydraulic resistance of the valve.



3) Long pipe: Model as hydraulic inertance

$$p_{tankout} - p_{out} = I_{pipe} \frac{dq_{pipe}}{dt}$$

where $I_{pipe} = \frac{L\rho}{A_p} = \frac{LD}{A_p}$ is the inertance of the pipe.

Due to p_{out} = 0 gauge:

$$p_{tankout} = I_{pipe} \frac{dq_{pipe}}{dt} \qquad (3)$$

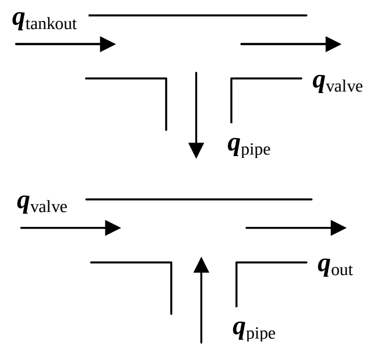
Taking Laplace transform of equations 1, 2 and 3 gives

$$Q_{in}(s) - Q_{tankout}(s) = C_{tankout}P_{tankout}(s)s \qquad (4)$$

$$P_{tankout}(s) = R_{valve}Q_{valve}(s) \qquad (5)$$

$$P_{tankout}(s) = I_{pipe}Q_{pipe}(s)s \qquad (6)$$

4) <u>Two junctions:</u>



For these two junctions we have:

$q_{tankout} = q_{valve} + q_{pipe}$ and

$q_{out} = q_{valve} + q_{pipe}$

Therefore:

$q_{tankout} = q_{out}$

Taking Laplace transform of the q_{out} and $q_{tankout}$ equations gives:

$$Q_{out}(s) = Q_{valve}(s) + Q_{pipe}(s) \qquad (7)$$

$$Q_{tankout}(s) = Q_{out}(s) \qquad (8)$$

From (2.5):

$$Q_{valve}(s) = \frac{P_{tankout}(s)}{R_{valve}} \qquad (9)$$

From (2.6):

$$Q_{pipe}(s) = \frac{P_{tankout}(s)}{I_{pipe}s} \qquad (10)$$

Sub. (9) and (10) into (7):

$$Q_{out}(s) = Q_{valve}(s) + Q_{pipe}(s) = \frac{P_{tankout}(s)}{R_1} + \frac{P_{tankout}(s)}{I_{pipe}s} = P_{tankout}(s)\frac{I_{pipe}s + R_{valve}}{R_{valve}I_{pipe}s}$$

Rearranging gives:

$$P_{tankout}(s) = Q_{out}(s)\frac{R_{valve}I_{pipe}s}{I_{pipe}s + R_{valve}} \qquad (11)$$

Sub. (8) and (11) into (4) and solve for the Laplace transfer function $Q_{out}(s)/Q_{in}(s)$:

$$Q_{in}(s) - Q_{tankout}(s) = C_{tankout} P_{tankout}(s) s$$

$$Q_{in}(s) - Q_{out}(s) = C_{tankout}\left(Q_{out}(s) \frac{R_{valve} I_{pipe} s}{I_{pipe} s + R_{valve}} \right) s$$

$$Q_{in}(s) = C_{tankout}\left(Q_{out}(s) \frac{R_{valve} I_{pipe} s}{I_{pipe} s + R_{valve}} \right) s + Q_{out}(s)$$

$$Q_{in}(s) = \left(\frac{C_{tankout} R_{valve} I_{pipe} s^2}{I_{pipe} s + R_{valve}} \right) Q_{out}(s) + Q_{out}(s)$$

$$\left(I_{pipe} s + R_{valve} \right) Q_{in}(s) = C_{tankout} R_{valve} I_{pipe} s^2 Q_{out}(s) + \left(I_{pipe} s + R_{valve} \right) Q_{out}(s)$$

$$\frac{Q_{out}(s)}{Q_{in}(s)} = \frac{I_{pipe} s + R_{valve}}{C_{tankout} R_{valve} I_{pipe} s^2 + I_{pipe} s + R_{valve}}$$

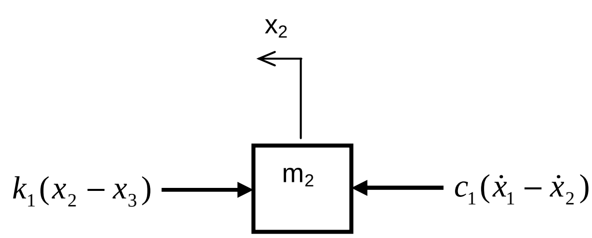
Question 2

<u>Mass 2</u>

The displacement is x_2. Assuming mass 1 has been displaced such that $\dot{x}_1 > \dot{x}_2$ and $x_2 > x_3$ the free body diagram is as shown. Taking forces to the left to be positive, from Newton's second law:

$$\sum F = ma$$

$$-k_1(x_2 - x_3) + c_1(\dot{x}_1 - \dot{x}_2) = m_2 \ddot{x}_2$$



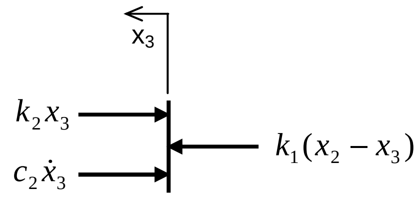
Taking the Laplace transform and rearranging gives:

$$m_2 X_2(s) s^2 + c_1 X_2(s) s + k_1 X_2(s) = k_1 X_3(s) + c_1 X_1(s) s$$

$$X_2(s) = \frac{k_1 X_3(s) + c_1 X_1(s) s}{m_2 s^2 + c_1 s + k_1} \qquad (1)$$

Massless connection at point 3

The displacement is x_3. The free body diagram is shown to the right. From Newton's second law:



$$\sum F = ma$$

$$-k_2 x_3 - c_2 \dot{x}_3 + k_1(x_2 - x_3) = m_3 \ddot{x}_3 = 0$$

Taking the Laplace transform and rearranging gives:

$$k_2 X_3(s) + c_2 X_3(s)s + k_1 X_3(s) = k_1 X_2(s) \text{ or}$$

$$X_2(s) = \frac{(c_2 s + k_1 + k_2)X_3(s)}{k_1} \qquad (2)$$

System Model

Substituting $X_2(s)$ from (1) into (2) and rearranging:

$$\frac{(c_2 s + k_1 + k_2)X_3(s)}{k_1} = \frac{k_1 X_3(s) + c_1 X_1(s)s}{m_2 s^2 + c_1 s + k_1} \quad \text{or}$$

$$(c_2 s + k_1 + k_2)(m_2 s^2 + c_1 s + k_1)X_3(s) = k_1^2 X_3(s) + c_1 k_1 X_1(s)s$$

Therefore, the desired Laplace transfer function is:

$$\frac{X_3(s)}{X_1(s)} = \frac{c_1 k_1 s}{c_2 m_2 s^3 + (c_1 c_2 + k_1 m_2 + k_2 m_2)s^2 + (k_1 c_2 + k_1 c_1 + k_2 c_1)s + k_1 k_2}$$

Question 3

Part a)

The local linear model at the point $(q_{1_0},\ \ q_{2_0}\ \ \ldots\ \ q_{n_0})$ for the mathematical model with n inputs $(q_1,\ \ q_2,\ \ \ldots\ \ q_n)$ is

$$\Delta y = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \bigg|_{q_i = q_{i_0}} \right) \Delta q_i \qquad (1)$$

where $\Delta y = y - y_0$ and $\Delta q_i = q_i - q_{i_0}$.

In this case, since only T, θ and v are changing, only three inputs are used in this model. Hence,

$$\Delta y = \left(\frac{\partial a}{\partial T} \bigg|_{T = T_0} \right) \Delta T + \left(\frac{\partial a}{\partial \theta} \bigg|_{\theta = \theta_0} \right) \Delta \theta + \left(\frac{\partial a}{\partial v} \bigg|_{v = v_0} \right) \Delta v \qquad (2)$$

Putting the acceleration model into (2) gives:

$$a - a_0 = \left(\frac{1}{m}\right)(T - T_0) + \left(-g\cos\theta\big|_{\theta=\theta_0}\right)(\theta - \theta_0) + \left(-\frac{2C_d v}{m}\Big|_{v=v_0}\right)(v - v_0)$$

Note $a_0 = \dfrac{T_0}{m} - g\sin\theta_0 - \dfrac{C_d v_0^2}{m}$. Organizing this equation:

$$a = \frac{T}{m} - g\theta\cos\theta_0 - \frac{2C_d v_0 v}{m} + g(\theta_0\cos\theta_0 - \sin\theta_0) + \frac{C_d v_0^2}{m} \qquad (3)$$

Equation (3) is the locally linear model of the rocket's acceleration.

Part b)

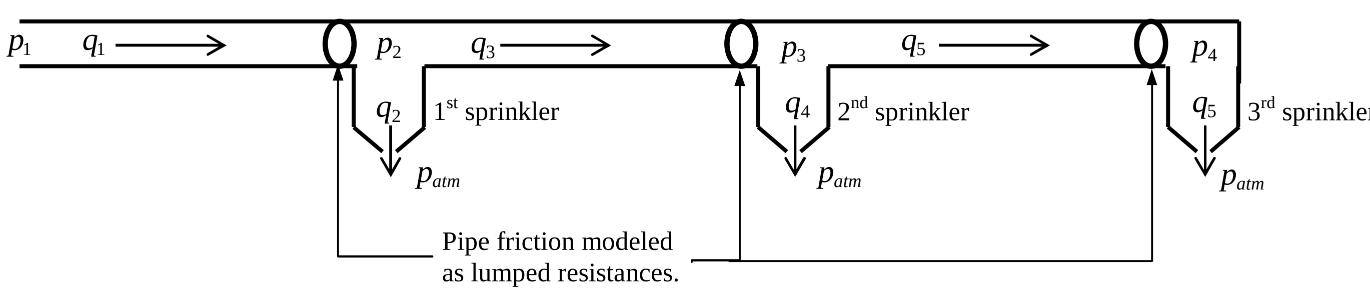
$$\frac{\partial a}{\partial \theta} = -g\cos\theta \qquad \frac{\partial}{\partial \theta}\left(\frac{\partial a}{\partial \theta}\right) = g\sin\theta$$

$$\frac{\partial a}{\partial v} = -\frac{2C_d v}{m} \qquad \frac{\partial}{\partial v}\left(\frac{\partial a}{\partial v}\right) = -\frac{2C_d}{m}$$

It is obvious that the second partial derivative with respect to v is a constant. Therefore, the error of the linearized model does not vary with v_0. The second derivative with θ is a positive sine wave. Therefore, the error will be largest when $\theta_0=\pm 90^0$, and the error will be smallest while $\theta_0=0^0$ or 180^0.

Question 4

Eight components can be identified from the given info: two junctions, three long pipes and three sprinklers. The pressures and flow rates are identified below:



1) Two junctions:

For these two junctions we have:

$q_1 = q_2 + q_3$ and

$q_3 = q_4 + q_5$

Taking the Laplace transfer of the governing equation for the 2nd junction gives:

$$Q_3(s) = Q_4(s) + Q_5(s) \qquad (1)$$

2) Sprinkler: Model each as a hydraulic resister as follows:

$$p_2 - p_{atm} = R_v q_2\,,$$

$$p_3 - p_{atm} = R_v q_4 \text{ and}$$

$$p_4 - p_{atm} = R_v q_5$$

Using gauge pressure, and taking the Laplace transform of each governing equation gives:

$$P_2(s) = R_v Q_2(s)\,, \qquad (2)$$

$$P_3(s) = R_v Q_4(s) \text{ and} \qquad (3)$$

$$P_4(s) = R_v Q_5(s) \qquad (4)$$

3) Long pipes: Model each as hydraulic inertor in series with hydraulic resistor

Total pressure drop = inertor pressure drop + resistor pressure drop

$$= I\frac{dq}{dt} + Rq$$

Applying this to the three long pipes gives:

$$p_1 - p_2 = I_1\frac{dq_1}{dt} + R_{L1}q_1\,,$$

$$p_2 - p_3 = I_2\frac{dq_3}{dt} + R_{L2}q_3 \text{ and}$$

$$p_3 - p_4 = I_3\frac{dq_5}{dt} + R_{L3}q_5$$

where $I_1 = \frac{L_1\rho}{A_{cs}}$, $I_2 = \frac{L_2\rho}{A_{cs}}$ and $I_3 = \frac{L_3\rho}{A_{cs}}$ are the inertances for pipes 1 to 3; R_{L1}, R_{L2} and R_{L3} are the resistances of the three pipes, ρ is the density of the fluid and A_{cs} is the cross-sectional area of the pipes.

Using gauge pressure, and taking the Laplace transform of each governing equation gives:

$$P_1(s) - P_2(s) = I_1 Q_1(s)s + R_{L1}Q_1(s)$$

$$P_2(s) - P_3(s) = I_2 Q_3(s)s + R_{L2}Q_3(s) \qquad (5)$$

$$P_3(s) - P_4(s) = I_3 Q_5(s)s + R_{L3}Q_5(s) \qquad (6)$$

Get desired transfer function

Sub. (3) and (4) into (6):

$$R_v Q_4(s) - R_v Q_5(s) = (I_3 s + R_{L3}) Q_5(s) \qquad (7)$$

Sub. (1), (2) and (3) into (5), and rearrange to get equation for $Q_4(s)$:

$$R_v Q_2(s) - R_v Q_4(s) = (I_2 s + R_{L2})(Q_4(s) + Q_5(s))$$

$$Q_4(s) = \frac{R_v Q_2(s) - (I_2 s + R_{L2}) Q_5(s)}{(I_2 s + R_{L2} + R_v)} \qquad (8)$$

Sub. (8) into (7):

$$R_v \left(\frac{R_v Q_2(s) - (I_2 s + R_{L2}) Q_5(s)}{(I_2 s + R_{L2} + R_v)} \right) - R_v Q_5(s) = (I_3 s + R_{L3}) Q_5(s)$$

Rearrange and solve for $Q_5(s)/Q_2(s)$:

$$R_v \left(R_v Q_2(s) - (I_2 s + R_{L2}) Q_5(s) \right) = (I_3 s + R_{L3} + R_v)(I_2 s + R_{L2} + R_v) Q_5(s)$$

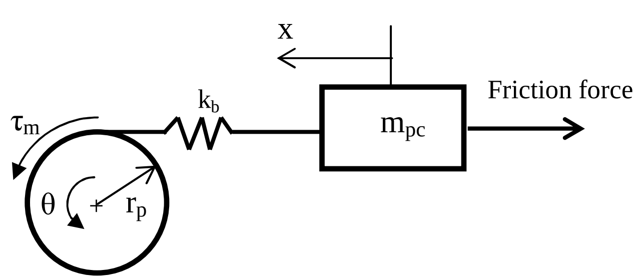
$$R_v^2 Q_2(s) = \left((I_3 s + R_{L3} + R_v)(I_2 s + R_{L2} + R_v) + R_v (I_2 s + R_{L2}) \right) Q_5(s)$$

$$\frac{Q_5(s)}{Q_2(s)} = \frac{R_v{}^2}{(I_3 s + R_{L3} + R_v)(I_2 s + R_{L2} + R_v) + R_v (I_2 s + R_{L2})}$$

$$= \frac{R_v{}^2}{I_2 I_3 s^2 + \left(I_2 R_{L3} + I_3 R_{L2} + 2 I_2 R_v + I_3 R_v \right) s + R_{L2} R_{L3} + 2 R_{L2} R_v + R_{L3} R_v + R_v^2}$$

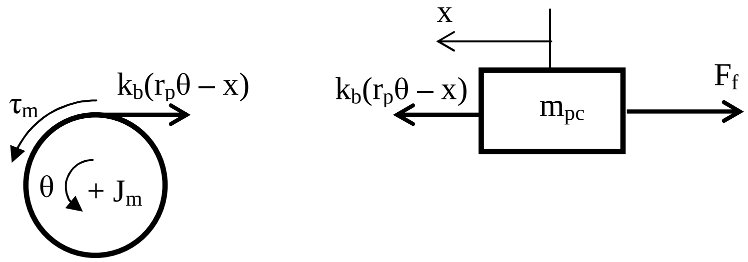
Question 5

a) Since the belt can only pull, and is flexible, the schematic diagram of the driven pulley, belt and carriage plus payload is:



k_b is the stiffness of the belt, and m_{pc} is the mass of the carriage + payload.

Assuming $r_p\theta > x$ the free body diagrams are:



J_m is the moment of inertia of the motor plus pulley.

b) Angular acceleration of the motor plus pulley:

$$\sum \tau = J\ddot{\theta}$$

$$\tau_m - \tau_{belt} = J_m\ddot{\theta}$$

$$\tau_m - r_p k_b\left(r_p\theta - x\right) = J_m\ddot{\theta}$$

$$\ddot{\theta} = \frac{1}{J_m}\left(\tau_m - k_b r_p^2\theta + k_b r_p x\right) \tag{1}$$

Linear acceleration of the payload plus carriage:

$$\sum F = ma$$

$$k_b\left(r_p\theta - x\right) - F_f = m_{pc}\ddot{x}$$

$$k_b\left(r_p\theta - x\right) - C_{viscous}\,\dot{x} - sign(\dot{x})F_{dynamic} - \left(1 - sign\left(\left|\dot{x}\right|\right)\right)F_{static} = m_{pc}\ddot{x}$$

$$\ddot{x} = \frac{1}{m_{pc}}\left(k_b r_p\theta - k_b x - C_{viscous}\,\dot{x} - sign(\dot{x})F_{dynamic} - \left(1 - sign\left(\left|\dot{x}\right|\right)\right)F_{static}\right) \tag{2}$$

c) Assuming $F_{dynamic}$ and F_{static} are negligible, (2) simplifies to:

$$\ddot{x} = \frac{1}{m_{pc}}\left(k_b r_p\theta - k_b x - C_{viscous}\,\dot{x}\right) \tag{3}$$

Taking the Laplace transform gives:

$$X(s)s^2 = \frac{1}{m_{pc}}\left(k_b r_p\Theta(s) - k_b X(s) - C_{viscous}\,X(s)s\right) \tag{4}$$

$$\left(s^2 + \frac{C_{viscous}}{m_{pc}}s + \frac{k_b}{m_{pc}}\right)X(s) = \frac{k_b r_p}{m_{pc}}\Theta(s) \tag{5}$$

Take Laplace transform of (1) and isolate $\Theta(s)$:

$$\Theta(s)s^2 = \frac{1}{J_m}\left(\tau_m(s) - k_b r_p^2 \Theta(s) + k_b r_p X(s)\right) \tag{6}$$

$$\left(s^2 + \frac{k_b r_p^2}{J_m}\right)\Theta(s) = \frac{1}{J_m}\left(\tau_m(s) + k_b r_p X(s)\right) \tag{7}$$

$$\Theta(s) = \frac{\tau_m(s) + k_b r_p X(s)}{J_m s^2 + k_b r_p^2} \tag{8}$$

Sub. (8) into (5) and solve for $X(s)/\tau_m(s)$:

$$\left(s^2 + \frac{C_{viscous}}{m_{pc}}s + \frac{k_b}{m_{pc}}\right)X(s) = \frac{k_b r_p}{m_{pc}}\left(\frac{\tau_m(s) + k_b r_p X(s)}{J_m s^2 + k_b r_p^2}\right)$$

$$\left(m_{pc}s^2 + C_{viscous}\, s + k_b\right)X(s) = k_b r_p\left(\frac{\tau_m(s) + k_b r_p X(s)}{J_m s^2 + k_b r_p^2}\right)$$

$$\left(\left(m_{pc}s^2 + C_{viscous}\, s + k_b\right)\left(J_m s^2 + k_b r_p^2\right) - k_b^2 r_p^2\right)X(s) = k_b r_p \tau_m(s)$$

$$\left(m_{pc}J_m s^4 + C_{viscous}\, J_m s^3 + \left(m_{pc}k_b r_p^2 \ + J_m k_b\right)s^2 + C_{viscous}\, k_b r_p^2 s\right)X(s) = k_b r_p \tau_m(s)$$

$$\frac{X(s)}{\tau_m(s)} = \frac{k_b r_p}{m_{pc}J_m s^4 + C_{viscous}\, J_m s^3 + \left(m_{pc}k_b r_p^2 \ + J_m k_b\right)s^2 + C_{viscous}\, k_b r_p^2 s}$$