1. i) STABILITY:

First rule: Check that H(z) has no poles outside the unit circle. The poles are the roots of:

$$z^2-1.4z+0.49=0 \tag{6.20}$$

The quadratic formula gives: poles = $-(-1.4)\forall\text{sqrt}((-1.4)^2 - 4(1)(0.49)/(2(1))$ = 0.7 and 0.7. Since neither pole is outside the unit circle the first rule has not been breached.

Second rule: First check whether or not G(z) has any poles outside the unit circle. The poles are the roots of:

$$z^2-1.2z+0.27=0 \tag{6.21}$$

The quadratic formula gives: poles = $-(-1.2)\forall\text{sqrt}((-1.2)^2 - 4(1)(0.27)/(2(1))$ = 0.9 and 0.3. Since neither is outside the unit circle there is no need to check 1-H(z).

Third rule: First check the zeros of G(z). The zeros are roots of:

$$0.5z+0.6=0 \tag{6.22}$$

which gives a zero outside the unit circle at -1.2. For the rule not to be broken H(z) must contain this zero. The zeros of H(z) are the roots of:

$$0.0409z+0.0491=0 \tag{6.23}$$

Since this also gives a zero at -1.2 the rule has not been broken.

ii) CAUSALITY:

For G(z) the dead time is 2-1 = 1 sampling interval. For H(z) the dead time is 2-1 = 1 sampling interval. Since the dead time for H(z) is not less than for G(z), the causality rule has not been broken.

iii) STEADY STATE ERROR (STEP INPUT ONLY):

The rule requires that H(1)=1. Here we have:

$$H(1)= (0.0409(1)+0.0491) / (1^2-1.4(1)+0.49) = 0.09 / 0.09 = 1 \tag{6.24}$$

Therefore the steady state error rule has been passed.

iv) SENSITIVITY TO MODELLING ERRORS:

The greater the difference between the desired closed-loop dynamics and the open-loop dynamics the more sensitive the controller will be to modelling error. From part i) above, G(z) has poles at 0.9 and 0.3. Of these two, the pole at 0.9 is the dominant pole (since it is much slower than the one at 0.3). Again from part ii), H(z) has two poles at 0.7. Since 0.7 is not much closer to the origin than 0.9, the dynamics of the closed-loop will not be dramatically different than the open-loop, and therefore the controller will not be very sensitive to modelling errors.

2. Stability:

Setting z-0.9=0 we find G(z) has a pole at 0.9. Since |0.9|≤1, and G(z) has no zeros, the stability rules are not a concern. (We must also ensure that the pole of H(z) is inside the unit circle)

Transient Response:
The s-plane characteristic equation is: $\tau s+1=0$, or $1.4s+1=0$. Therefore $s=-1/1.4= -0.71$ is the pole.
The z-plane pole then equals: $z=e^{Ts} = e^{(0.5)(-0.71)} = 0.70$

Causality:
The dead time of G(z) is 3-0 = 3 sampling periods. Therefore we should make H(z) have a dead time of 3 sampling periods.

At this point we can put H(z) in the form:

$$H(z)=\frac{b_0 z^2 + b_1 z + b_2}{z^4(z-0.7)} \tag{6.25}$$

Steady State Error:
For no steady state error to a step:

$$H(1)=\frac{b_0+b_1+b_2}{1-0.7}=1 \tag{6.26}$$

$$b_0+b_1+b_2=0.3 \tag{6.27}$$

The simplest solution is: $b_0=0.3$, $b_1=b_2=0$.

Now we have:

$$H(z)=\frac{0.3}{z^2(z-0.7)} \tag{6.28}$$

The controller is given by:

$$D(z)=\frac{1}{G(z)}\frac{H(z)}{1-H(z)}=\frac{z^2(z-0.9)}{2}\frac{\dfrac{0.3}{z^2(z-0.7)}}{1-\dfrac{0.3}{z^2(z-0.7)}}$$

$$=\frac{z^2(z-0.9)0.3}{2(z^2(z-0.7)-0.3)}=\frac{0.15z^3-0.135z^2}{z^3-0.7z^2-0.3} \quad (6.29)$$

3. We have:

$$D(z)=\frac{U(z)}{E(z)}=\frac{0.15z^3-0.135z^2}{z^3-0.7z^2-0.3}$$

$$=\frac{0.15z^3-0.135z^2}{z^3-0.7z^2-0.3}\frac{z^{-3}}{z^{-3}}=\frac{0.15-0.135z^{-1}}{1-0.7z^{-1}-0.3z^{-2}} \quad (6.30)$$

To convert this transfer function to the discrete-time domain we start by cross multiplying to obtain:

$$0.15E(z)-0.135z^{-1}E(z)=U(z)-0.7z^{-1}U(z)-0.3z^{-3}U(z) \quad (6.31)$$

The inverse Z-transform of this equation equals:

$$0.15e(k)-0.135e(k-1)=u(k)-0.7u(k-1)-0.3u(k-3) \quad (6.32)$$

To implement this on a microprocessor we would just have to write it the form:

$$u(k)=0.15e(k)-0.135e(k-1)+0.7u(k-1)+0.3u(k-3) \quad (6.33)$$

where k is an integer representing the current sample number, u(k) is the controller output at time kT, and e(k) is the error at time kT.

4. Stability:

Setting the denominator of G(z) equal to zero we find that G(z) has poles at 0.8 and 0.4. The given desired characteristic equation has poles at 0.5 and 0.5. Since $|0.8|\leq 1$, $|0.4|\leq 1$, $|0.5|\leq 1$ and G(z) has no zeros, the stability rules are not a concern.

Causality:

The highest power of z in the numerator of G(z) is 0, and in the denominator it is 2, so its dead time is 2 sampling periods. Therefore we should make H(z) have a dead time of 2 sampling periods.

We have been given the desired closed-loop characteristic equation so at this point we can put H(z) in the form:

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 (z - 0.5)(z - 0.5)} \tag{6.34}$$

Steady State Error:

For no steady state error to a step:

$$H(1) = \frac{b_0 + b_1 + b_2}{(1 - 0.5)(1 - 0.5)} = 1 \tag{6.35}$$

$$b_0 + b_1 + b_2 = 0.25 \tag{6.36}$$

We could try the simplest solution, which is: b_0=0.25, b_1=b_2=0.
With this choice our H(z) is:

$$H(z) = \frac{0.25}{(z - 0.5)(z - 0.5)} \tag{6.37}$$

which has no zeros and two poles at 0.5. For the unit ramp the steady state error would then be:

$$e(\infty) = \sum_{i=1}^{n} \frac{T}{1 - p_i} - \sum_{j=1}^{m} \frac{T}{1 - q_j} = \frac{0.01}{1 - 0.5} + \frac{0.01}{1 - 0.5} - 0 = 0.04 \tag{6.38}$$

Since this is not equal to 0.005 it is not a valid solution, and we have to add a zero to H(z). Our new H(z) has the form:

$$H(z) = \frac{b_0 z + b_1}{z(z - 0.5)(z - 0.5)} \tag{6.39}$$

with an additional zero at $-b_1/b_0$ and an additional pole at 0. We can achieve an error of 0.005 by setting:

$$e(\infty) = \frac{0.01}{1 - 0} + \frac{0.01}{1 - 0.5} + \frac{0.01}{1 - 0.5} - \frac{0.01}{1 - (-b_1 / b_0)} = 0.005 \tag{6.40}$$

$$- \frac{0.01}{1 - (- b_1 / b_0)} = 0.005 \ - 0.05 \tag{6.41}$$

$$3.5 b_0 + 4.5 b_1 = 0 \tag{6.42}$$

Solving equations (6.36) and (6.42) for b_0 and b_1 gives: b_0= 1.125 and b_1= -0.875. The final answer is:

$$H(z) = \frac{1.125z - 0.875}{z(z - 0.5)(z - 0.5)} \tag{6.43}$$