MATH 2ZZ3: SAMPLE TEST #2

Note that problems # 5,6 and 11 in the file M2ZZ3-t1-sample-b.pdf are also from sections covered in Test 2

SOLUTIONS

1. Let $f(x, y, z) = x^2 + y^3 + z$ and let C be the path parametrized by the vector-function

$$\mathbf{r}(t) = \left\langle 2+t, t^2, \frac{6}{2+t}\right\rangle, \quad 0 \le t \le 1.$$

The value of the path integral

$$I = \int_C \nabla f \cdot d\mathbf{r}$$

is

(A) $I = 2$

(B) $I = -2$

→**(C)** $I = 5$

(D) $I = 0$

(E) $I = 3$

Solution. Since $\mathbf{r}(0) = \langle 2,\, 0,\, 3\rangle$ and $\mathbf{r}(1) = \langle 3,\, 1,\, 2\rangle$, the fundamental theorem of line integrals shows that

$$\int_C \nabla f \cdot d\mathbf{r} = f(3,1,2) - f(2,0,3) = 12 - 7 = 5.$$

2. Let D be the region in the first quadrant $(x, y \ge 0)$ bounded by the line $y = 1 - x$ and the circle $x^2 + y^2 = 1$. Use polar coordinates to compute the double integral

$$I = \iint_D \frac{x+y}{x^2+y^2}\, dA.$$

(A) $I = 3 - \dfrac{2\pi}{3}$

(B) $I = 4 + \dfrac{\pi}{6}$

(C) $I = 1 - \dfrac{3\pi}{2}$

→**(D)** $I = 2 - \dfrac{\pi}{2}$

(E) $I = 3 - \dfrac{2\pi}{3}$

Solution. The line $y = 1 - x$ has equation $r\,\cos\theta + r\,\sin\theta = 1$ or $r = \frac{1}{\cos\theta + \sin\theta}$ in polar coordinates while the circle $x^2 + y^2 = 1$ has equation $r = 1$. The region D is expressed as the region

$$D^* = \{(r, \theta),\ 0 \le \theta \le \frac{\pi}{2},\ \frac{1}{\cos\theta + \sin\theta} \le r \le 1\}.$$

We have thus

$$
\begin{aligned}
I &= \iint_{D^*} \frac{r\,\cos\theta + r\,\sin\theta}{r^2}\, r\,dr\,d\theta = \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \cos\theta + \sin\theta\,dr\,d\theta \\
&= \int_0^{\pi/2} (\cos\theta + \sin\theta)\left(1 - \frac{1}{\cos\theta + \sin\theta}\right) d\theta \\
&= \int_0^{\pi/2} \cos\theta + \sin\theta - 1\,d\theta = \big[\sin\theta - \cos\theta - \theta\big]_0^{\pi/2} = 2 - \frac{\pi}{2}.
\end{aligned}
$$

3. Let D be the region in the x, y plane bounded by the triangle with vertices at $(0,0)$, $(0,3)$ and $(2,3)$. The integral

$$I = \iint_D e^{y^2}\,dA$$

is given by:

(A) $I = 0$

$\rightarrow$**(B)** $I = \dfrac{e^9 - 1}{3}$

(C) $I = \dfrac{e^4 - 1}{2}$

(D) $I = e^3 + 2$

(E) $I = 3\,e^2 + 2\,e^3$

Solution. The region D can be expressed as the type II region

$$D = \left\{(x,y),\ 0 \le y \le 3,\ 0 \le x \le \frac{2\,y}{3}\right\}.$$

Thus,

$$I = \int_0^3 \int_0^{\frac{2\,y}{3}} e^{y^2}\,dx\,dy = \int_0^3 \frac{2\,y\,e^{y^2}}{3}\,dy = \frac{1}{3}\left[e^{y^2}\right]_0^3 = \frac{e^9 - 1}{3}.$$

4. Let L be the lamina bounded by the curves $y = e^{-x}$, $x = 0$, $x = 1$ and $y = 0$ with mass density per unit of area given by $\rho(x,y) = 1$. The x-coordinate of its center of mass is:

(A) $\overline{x} = \dfrac{e^{-1} - 2}{e^{-1} - 1}$

(B) $\overline{x} = \dfrac{e - 2}{e^{-1} - 1}$

(C) $\overline{x} = \dfrac{e - 1}{e - 2}$

(D) $\overline{x} = \dfrac{e^{-1} - 2}{e - 1}$

$\rightarrow$**(E)** $\overline{x} = \dfrac{e - 2}{e - 1}$

Solution. The mass of the lamina is

$$m = \int_0^1 \int_0^{e^{-x}} 1\,dy\,dx = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = 1 - e^{-1}.$$

We have also

$$M_y = \int_0^1 \int_0^{e^{-x}} x\, dy\, dx = \int_0^1 x\, e^{-x}\, dx = \int_0^1 x\,(-e^{-x})'\, dx$$
$$= \big[x\,(-e^{-x})\big]_0^1 + \int_0^1 e^{-x}\, dx = -e^{-1} + 1 - e^{-1} = 1 - 2\,e^{-1}.$$

Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{1 - 2\,e^{-1}}{1 - \ e^{-1}} = \frac{e-2}{e-1}.$$

5. Compute the volume V of the solid region below the graph of the function $z = \dfrac{y}{(1+x^3)^2}$ and above the triangle with vertices $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$ in the x, y plane.

→**(A)** $V = \dfrac{1}{12}$

(B) $V = \dfrac{1}{3}$

(C) $V = \dfrac{2}{7}$

(D) $V = \dfrac{3}{8}$

(E) $V = \dfrac{1}{8}$

Solution. If D denotes the region bounded by the triangle in the x, y plane, we have

$$V = \iint_D \frac{y}{(1+x^3)^2}\, dA.$$

The region D can be expressed as the region of type 1

$$D = \{(x,y),\ 0 \le x \le 1,\ 0 \le y \le x\}.$$

Therefore,

$$V = \int_0^1 \int_0^x \frac{y}{(1+x^3)^2}\, dy\, dx = \int_0^1 \frac{1}{(1+x^3)^2} \left[\frac{y^2}{2}\right]_0^x dx = \frac{1}{2} \int_0^1 \frac{x^2}{(1+x^3)^2}\, dx$$
$$= \frac{1}{6}\left[-\frac{1}{1+x^3}\right]_0^1 = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}.$$

6. When written in polar coordinates, the integral

$$I = \int_{-1}^1 \int_{|x|}^{\sqrt{2-x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$$

becomes

(A) $I = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \frac{r}{1+r^2}\, dr\, d\theta$

(B) $I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \int_0^{2} \frac{r}{1+r^2}\, dr\, d\theta$

→**(C)** $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{2}} \frac{r}{1+r^2}\, dr\, d\theta$

(D) $I = \int_0^{\frac{3\pi}{2}} \int_0^{\sqrt{2}} \frac{r}{1+r^2}\, dr\, d\theta$

(E) $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{1} \frac{r}{1+r^2}\, dr\, d\theta$

Solution. Note that the curve $y = \sqrt{2-x^2}$ is the part of the circle of radius $\sqrt{2}$ centered at the origin above the x-axis. Thus,

$$I = \iint_D \frac{1}{1+x^2+y^2}\, dA,$$

where D is the region in the half-plane $y \geq 0$, bounded by the lines $y = \pm x$ and the circle of radius $\sqrt{2}$ centered at the origin, which is expressed in polar coordinates as the region

$$D^* = \left\{(r,\theta),\ \frac{\pi}{4} \leq \theta \leq \frac{3\,\pi}{4},\ 0 \leq r \leq \sqrt{2}\right\}.$$

Hence,

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{2}} \frac{r}{1+r^2}\, dr\, d\theta.$$

7. Find the area A of the region in the x, y plane bounded by the curves $y = x^2 - 4$ and $y = 14 - x^2$.

(A) $A = 36$

(B) $A = 25$

(C) $A = 4$

→**(D)** $A = 72$

(E) $A = 96$

Solution. The curves $y = x^2 - 4$ and $y = 14 - x^2$ intersect when $y = x^2 - 4 = 14 - x^2$. This yields $x^2 = 9$ or $x = \pm 3$, so the points of intersection are $(\pm 3, 5)$. The region bounded by the two curves can thus be written as

$$D = \{(x,y),\ -3 \leq x \leq 3,\ x^2 - 4 \leq y \leq 14 - x^2\}.$$

Hence,

$$A = \iint_D 1\, dA = \int_{-3}^{3} \int_{x^2-4}^{14-x^2} 1\, dy\, dx = \int_{-3}^{3} 18 - 2\,x^2\, dx = 2\int_0^3 18 - 2\,x^2\, dx$$
$$= 2\left[18\,x - \frac{2\,x^3}{3}\right]_0^3 = 2\,(36) = 72.$$

8. Let C be the semicircle $x^2 + y^2 = 4$, $x \geq 0$, oriented anticlockwise. Compute the path integral $I = \int_C (x+y)^2\, ds$.

(**Hint:** You can parametrize C using cosine and sine functions.)

(A) $I = 4\,\pi^2$

(B) $I = 4$

(C) $I = 2$

→**(D)** $I = 8\,\pi$

(E) $I = 2\,\pi$

Solution. Letting $x = 2\,\cos t$, $y = 2\,\sin t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, we have $x'(t) = -2\,\sin t$, $y'(t) = 2\,\cos t$, $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \sqrt{4\,(\sin^2 t + \cos^2 t)}\,dt = 2\,dt$. Thus,

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\,\cos t + 2\,\sin t)^2\, 2\,dt = 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t + \sin^2 t + 2\,\sin t\,\cos t\,dt \\ &= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \sin(2\,t)\,dt = 8\left[t - \frac{\cos(2\,t)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 8\,\pi \end{aligned}$$

9. The closed path C consists of three line segments:

from $(1,0,0)$ to $(0,0,2)$, from $(0,0,2)$ to $(0,1,0)$, and from $(0,1,0)$ back to $(1,0,0)$.

The work W done by the force $F(x,y,z) = \langle xy,\, xz,\, yz\rangle$ along C is

→**(A)** $W = -\frac{1}{2}$

(B) $W = \frac{1}{2}$

(C) $W = 0$

(D) $W = \frac{2}{3}$

(E) $W = -\frac{2}{3}$

Solution. If we denote the line segments above by C_1, C_2 and C_3, respectively and we let $P = x\,y$, $Q = x\,z$ and $R = y\,z$, we note that $P = R = 0$ on C_1, $P = Q = 0$ on C_2 and $Q = R = 0$ on C_3. Thus,

$$W = \int_{C_1} Q\,dy + \int_{C_2} R\,dz + \int_{C_3} P\,dx.$$

Since $y = 0$ on C_1, $\int_{C_1} Q\,dy = 0$. On C_2, we can let $y = t$ and $z = 2(1-t)$, $dz = -2\,dt$, for $0 \leq t \leq 1$. So

$$\int_{C_2} y\,z\,dz = -\int_0^1 t\,(1-t)\,4\,dt = -4\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = -\frac{2}{3}.$$

On C_2, we can let $x = t$ and $y = 1-t$, $dx = dt$, for $0 \leq t \leq 1$. So

$$\int_{C_3} x\,y\,dz = \int_0^1 t\,(1-t)\,dt = \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1}{6}.$$

Thus,

$$W = 0 - \frac{2}{3} + \frac{1}{6} = -\frac{1}{2}.$$

10. Let $a > 1$. The path integral $I = \displaystyle\int_C \frac{x\,dy + y\,dx}{x^2 + y^2}$, where C denotes the line segment from $(1, 1)$ to (a, a), is

(A) $I = a\,\dfrac{\pi}{4}$

$\rightarrow$**(B)** $I = \ln a$

(C) $I = \dfrac{1}{2}\,\ln a$

(D) $I = (a - 1)\,\dfrac{\pi}{4}$

(E) undefined because the path passes through $(0, 0)$.

Solution. We can parametrize C by letting $x = y = t$, for $1 \le t \le a$ so that $dx = dy = dt$. Hence,

$$I = \int_0^a \frac{2\,t}{2\,t^2}\,dt = \int_1^a \frac{1}{t}\,dt = \ln a.$$

11. For what value of c is the vector field $\mathbf{F}(x, y, z) = \langle 2\,y + 2\,z,\, 2\,x - 3\,z,\, -3\,y + c\,x\rangle$ a gradient (or conservative) vector field?

$\rightarrow$**(A)** $c = 2$

(B) $c = -2$

(C) $c = -3$

(D) $c = 3$

(E) no such c exists

Solution. $\mathbf{F}$ is conservative iff $\nabla \times \mathbf{F} = \mathbf{0}$. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2\,y + 2\,z & 2\,x - 3\,z & -3\,y + c\,x \end{vmatrix} = 0\,\mathbf{i} + (2 - c)\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

if and only if $c = 2$. Note that for $c = 2$, a potential function is $g(x, y, z) = 2\,x\,y + 2\,x\,z - 3\,y\,z$.

12. Let C be the path from $(1, 0, 1)$ to $(2, 3, 2)$ parametrized by

$$x = \frac{1}{2}\,(3 - \cos(\pi t)), \quad y = 3\,t^3, \quad z = 1 + t, \quad 0 \le t \le 1,$$

and consider the functions $P = 2\,x\,y$, $Q = x^2 + z^2$ and $R = 2\,y\,z$. Compute the path integral $I = \int_C P\,dx + Q\,dy + R\,dz$.

(**Hint:** Does the field $\langle P, Q, R\rangle$ have a potential function?)

(A) $I = -24$

(B) $I = 0$

$\rightarrow$**(C)** $I = 24$

(D) $I = 12$

(E) $I = -12$

Solution. By inspection, a potential function is $g(x, y, z) = x^2\,y + y\,z^2$. Thus,

$$I = g(2, 3, 2) - g(1, 0, 1) = 24 - 0 = 24.$$

13. The vector field $\mathbf{F}(x, y) = \langle -1/y, x/y^2 \rangle$ is conservative in the region

(A) $x > 0$

$\rightarrow$ **(B)** $y > 0$

(C) $x^2 + y^2 < 1$

(D) $x + y > 0$

(E) $x - y > 0$

Solution. Since the vector field $\mathbf{F}$ is undefined when $y = 0$, the region cannot intersect the line $y = 0$. The only possible region where a potential function can exist is thus the half-plane $y > 0$. In fact, a potential function is $g(x, y) = -\frac{x}{y}$ which is well-defined on that region.

THE END