Assignment 3

Thursday, May 26, 2022 10:43 PM

5b, 6b

Problem #1: Suppose that the random variable X has the following cumulative distribution function.

$$F(x) = \begin{cases} 0 & x < 4 \\ \frac{1}{48}(x^2 - 16) & 4 \le x < 8 \\ 1 & x > 8 \end{cases}$$

(a) Find $P(X > 5.33)$

(b) Find $P(3.38 < X < 4.20)$

(c) Find the mean of X.

(d) Find the variance of X.

$$F(x) = \begin{cases} 0 & x < 4 \\ \frac{1}{48}(x^2 - 16) & 4 \le x < 8 \\ 1 & x > 8 \end{cases} \Rightarrow f(x) = \begin{cases} 0 & x < 4 \\ \frac{1}{24}x & 4 \le x < 8 \\ 0 & x > 8 \end{cases}$$

a.) $P(X > 5.33) = 1 - P(X \le 5.33)$

$= 1 - F(5.33)$

$= 1 - \frac{1}{48}\left((5.33)^2 - 16\right)$

$= 0.741481$

b.) $P(3.38 < X < 4.20) = P(X \le 4.20) - P(X \le 3.38)$

$= \frac{1}{48}\left((4.20)^2 - 16\right) - 0$

$= 0.034167$

c.) $E(X) = \int_4^8 x \cdot \frac{1}{24}x \, dx = \frac{1}{72}x^3 \Big|_4^8 = \frac{56}{9}$

d.) $Var(x) = E(X^2) - [E(x)]^2$

$E(X^2) = \int_4^8 x^2 \cdot \frac{1}{24}x \, dx = \frac{1}{96} x^4 \Big|_4^8 = 40$

$Var(X) = 40 - \left(\frac{56}{9}\right)^2 = \frac{104}{81}$

Problem #2: Glaucoma is a disease of the eye that is manifested by high intraocular pressure. The distribution of intraocular pressure in the general population is approximately normal with mean 16 mm Hg and standard deviation 3 mm Hg.

(a) What percentage of people have an intraocular pressure lower than 13 mm Hg?

(b) Fill in the blank. Approximately 80% of adults in the general population have an intraocular pressure that is greater than ________ (how many?) mm Hg.

NOTE: Do **not** use the first half of the normal table (i.e., page 742 in the textbook, with negative z-values) because it will **not** be provided with the tests.

$\mu = 16 \qquad \sigma = 3$

a.) $z = \dfrac{x - \mu}{\sigma} = \dfrac{13 - 16}{3} = -1$

$P(z < -1) = 1 - 0.84134 = 0.15866$

$\therefore$ 15.866% of people have lower than 13 mm Hg

b.) $P = 80\%$ greater $\Rightarrow z = -0.85$

$-0.85 = \dfrac{x - 16}{3} \Rightarrow x = 13.45$

Problem #3: The weight of a sophisticated running shoe is normally distributed with a mean of 13 ounces.

(a) What must the standard deviation of weight be in order for the company to state that 97% of its shoes weight less than 14 ounces?

(b) Suppose that the standard deviation is actually 0.88. If we sample 7 such running shoes, find the probability that exactly 3 of those shoes weigh more than 14 ounces.

$\mu = 13$

a.) $P = 97\%$ greater $\Rightarrow z = 1.89$

$1.89 = \dfrac{14 - 13}{\sigma} \Rightarrow \sigma = 0.529101$

b.) $\sigma = 0.88$

$z = \dfrac{14 - 13}{\quad} = 1.136$

b.) $\sigma = 0.88$

$$z = \frac{14-13}{0.88} = 1.136$$

From table, $P(\text{weight} > 14) = 1 - 0.87286 = 0.12714$

Probability 3 out of 7 > 14 oz :

$$p = \binom{7}{3}(0.12714)^3(0.87286)^4 = 0.041754$$

Problem #4: Suppose that 25% of all steel shafts produced by a certain process are nonconforming but can be reworked (rather than having to be scrapped).

(a) In a random sample of 204 shafts, find the approximate probability that between 37 and 57 (inclusive) are nonconforming and can be reworked.

(b) In a random sample of 204 shafts, find the approximate probability that at least 54 are nonconforming and can be reworked.

Binomial Distribution, $n = 204$ $p = 0.25$

a.) X = nonconforming but reworkable

$P(37 \le X \le 57)$

$E(X) = n \cdot p = 204(0.25) = 51$

$\sigma = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{(204)(0.25)(1-0.25)} = 6.184658$

$$z_{37} = \frac{36.5 - 51}{6.1847} = -2.344$$

$$z_{57} = \frac{57.5 - 51}{6.1847} = 1.051$$

$$\begin{aligned} P(37 \le X \le 57) &= P(-2.344 < z < 1.051) \\ &= P(z < 1.051) - P(z < -2.344) \\ &= 0.85314 - 0.00964 \\ &= 0.8435 \end{aligned}$$

b.) $P(X \ge 54)$

$$z_{54} = \frac{53.5 - 51}{6.187} = 0.4041$$

$P(z \ge 0.4041) = 1 - P(z < 0.4041)$

$= 1 - 0.65542$
$= 0.34458$

Problem #5: A system consists of five components is connected in series as shown below.

As soon as one component fails, the entire system will fail. Assume that the components fail independently of one another.

(a) Suppose that each of the first two components have lifetimes that are exponentially distributed with mean 100 weeks, and that each of the last three components have lifetimes that are exponentially distributed with mean 133 weeks. Find the probability that the system lasts at least 43 weeks.

(b) Now suppose that each component has a lifetime that is exponentially distributed with the same mean. What must that mean be (in years) so that 86% of all such systems lasts at least one year?

$\mu_1 = \mu_2 = 100 \quad ; \quad \mu_3 = \mu_4 = \mu_5 = 133$

$P(X < x) = 1 - e^{-\lambda x} \quad ; \quad P(X \geq x) = e^{-\lambda x} \quad ; \quad \mu = \frac{1}{\lambda}$

a.) For components 1, 2 :

$\lambda = \frac{1}{\mu} = \frac{1}{100}$

$P(X < x) = 1 - e^{-\frac{1}{100}x}$

$P(X \geq 43) = 1 - P(X < 43)$

$= 1 - \left(1 - e^{-\frac{1}{100} \cdot 43}\right)$

$= 0.650509$

For components 3, 4, 5 :

$\lambda = \frac{1}{\mu} = \frac{1}{133}$

$P(X < x) = 1 - e^{-\frac{1}{133}x}$

$P(X \geq 43) = 1 - P(X < 43)$

$= 1 - \left(1 - e^{-\frac{1}{133} \cdot 43}\right)$

$= 0.723751$

$P(\text{system lasts} \geq 43) = (0.650509)^2 (0.723751)^3 = 0.160426$

b.) $P_{\text{overall}} = 86\% = 0.86$; $x = 1$

$$P(X \geq 1) = \left(e^{-\frac{1}{\mu} \cdot 1}\right)^5 = 0.86$$

$$\Rightarrow -\frac{1}{\mu} \cdot 5 = \ln(0.86)$$

$$\mu = \frac{-5}{\ln 0.86} = 33.15$$

Problem #6: Suppose that the random variables X and Y have the following joint probability density function.

$$f(x, y) = ce^{-4x-9y}, \quad 0 < y < x.$$

(a) Find the value of c.

(b) Find $P(X < \frac{1}{10}, Y < 2)$

a.) $$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy$$

$$1 = \int_0^{\infty} \int_0^{x} ce^{-4x-9y}\, dy\, dx$$

$$1 = \int_0^{\infty} \frac{1}{9} ce^{-13x} \cdot (e^{9x} - 1)\, dx$$

$$1 = \frac{c}{52} \quad \Rightarrow \quad c = 52$$

b.) $P(X < \frac{1}{10}, Y < 2) \Rightarrow (y < x < \frac{1}{10}) \cap (0 < y < 2)$

$$P(X < \tfrac{1}{10}, Y < 2) = \int_0^2 \int_y^{\frac{1}{10}} 52e^{-4x-9y}\, dx\, dy$$

$$= -\frac{-9e^{1.3} + 13e^{0.9} - 4}{9e^{1.3}}$$

$= 0.152885$

Problem #7: Suppose that the random variables X and Y have the following joint probability density function.

$$f(x, y) = ce^{-4x-9y}, \quad 0 < y < x.$$

(a) Find $P(X < 2, Y < \frac{1}{10})$.

(b) Find the marginal probability distribution of X.

$$f(x,y) = 52e^{-4x-9y} \quad , \quad 0<y<x$$

a.) $P(X<2, Y<\frac{1}{10}) \Rightarrow (y<x<2) \cap (0<y<\frac{1}{10})$

$$P(X<2, Y<\tfrac{1}{10}) = \int_0^{\frac{1}{10}} \int_y^2 52e^{-4x-9y}\, dx\, dy$$

wolfram alpha

$$= \int_0^{\frac{1}{10}} 13e^{-13y-8} \cdot (e^{4y} - e^{8})\, dy$$

$$= 0.727181$$

b.) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^x 52e^{-4x-9y}\, dy$$

$$= \frac{52}{9}\left(e^{-4x} - e^{-13x}\right)$$

Problem #8: Suppose that X and Y have the following joint probability density function.

$$f(x, y) = \frac{2}{477}x, \quad 0 < x < 9,\ y > 0,\ x-3 < y < x+3$$

(a) Find $E(XY)$.

(b) Find the covariance between X and Y.

a.) $x - 3 = 0 \Rightarrow x = 3$

$$E(XY) = \int_0^3 \int_0^{x+3} x \cdot y \cdot \frac{2x}{477}\, dy\, dx + \int_3^9 \int_{x-3}^{x+3} x \cdot y \cdot \frac{2x}{477}\, dy\, dx$$

$$= \frac{279}{530} + \frac{2160}{53}$$ wolfram - alpha

$$= \frac{21879}{530}$$

b.) $$E(X) = \int_0^3 \int_0^{x+3} x \cdot \frac{2x}{477}\, dy\, dx + \int_3^9 \int_{x-3}^{x+3} x \cdot \frac{2x}{477}\, dy\, dx$$

$$= \frac{21}{106} + \frac{312}{53} = \frac{645}{106}$$

$$E(Y) = \int_0^3 \int_0^{x+3} y \cdot \frac{2x}{477}\, dy\, dx + \int_3^9 \int_{x-3}^{x+3} y \cdot \frac{2x}{477}\, dy\, dx$$

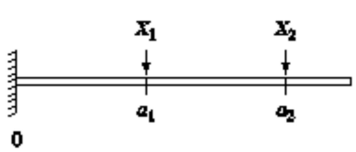
$$= \frac{51}{212} + \frac{312}{53} = \frac{1299}{212}$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{21879}{530} - \left(\frac{645}{106}\right)\left(\frac{1299}{212}\right)$$

$$= 3.996734$$

Problem #9: If two loads are applied to a cantilever beam as shown in the figure below, the bending moment at 0 due to the loads is $a_1X_1 + a_2X_2$.



Suppose that X_1 and X_2 are independent random variables with means 2 and 8 kips, respectively, and standard deviations 0.4 and 1.5 kips, respectively. Suppose that $a_1 = 4$ ft and $a_2 = 10$ ft.

(a) Find the expected value of the bending moment.

(b) Find the standard deviation of the bending moment.

(c) If X_1 and X_2 are normally distributed, what is the probability that the bending moment will exceed 108 kip-ft?

$\mu_1 = 2$; $\mu_2 = 8$
$\sigma_1 = 0.4$; $\sigma_2 = 1.5$
$a_1 = 4$; $a_2 = 10$

a.) $E(a_1X_1 + a_2X_2) = a_1E(X_1) + a_2E(X_2)$
$= 4 \cdot 2 + 10 \cdot 8$
$= 88$

b.) $Var(a_1X_1 + a_2X_2) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cancel{2a_1a_2 \cdot Cov(X_1, X_2)}$ 0
$= 4^2 \cdot 0.4^2 + 10^2 \cdot 1.5^2$
$= 2.56 + 225$
$= 227.56$

$\sigma = \sqrt{227.56} = 15.085092$

c.) $P(X > 108) = P\left(Z > \frac{108 - 88}{\sqrt{227.56}}\right)$
$= 1 - P(Z \leq 1.33)$
$= 1 - 0.90824$
$= 0.09176$