# A2

October 20, 2022 4:36 PM

**Problem 1** [4 points] Write the body of the MATLAB function

```
function  B = inverse(A)
```

For an $n \times n$ nonsingular matrix $A$, it should compute the inverse $A^{-1}$ of $A$. You can use only the `lu` function of MATLAB. Your `inverse` function must compute $A^{-1}$ in $O(n^3)$ operations.

```
function B = inverse(A)
[L, U, P]=lu(A);

len = length(A);
B = zeros(size(A));
I = eye(size(A));

    for i=1:len
        b = I(:,i);
        B(:,i) = bSub(U, fSub(L, P*b));
    end

end

function x = bSub(U, b)
    len = length(b);
    x = zeros(len, 1);
    x(len) = b(len)/U(len, len);
    for i=(length(b)-1):-1:1
        sum = 0;
        for j=len:-1:i
            sum = sum + U(i, j)*x(j);
        end
        x(i) = (b(i)-sum)/U(i, i);
    end
end

function x = fSub(L, b)
    x = zeros(length(b), 1);
    for i=1:length(b)
        sum = 0;
        for j=i-1:-1:1
            sum = sum + L(i, j)*x(j);
        end
        x(i)=(b(i)-sum)/L(i,i);
    end
end
```

```
Command Window
>> question_1_test
n=   700 time= 1.1e+00 error=2.04e-14
n= 1400 time= 9.9e+00 ratio=   8.8 error=5.66e-14
n= 2800 time= 9.6e+01 ratio=   9.6 error=9.12e-14
n= 5600 time= 1.2e+03 ratio= 12.7 error=2.31e-13
fx >>
```

**Problem 2** [6 points] Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.3 & 0.6 & 0.9 \\ 0.6 & 1.5 & 3 \end{bmatrix}$$

and $b = [1.4, 1.8, 6]^T$.

a. [2 points] Show that $A$ is singular.

a) $A$ is singular iff $|A| = \emptyset$

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$|A| = 0.1 \begin{vmatrix} 0.6 & 0.9 \\ 1.5 & 3 \end{vmatrix} - 0.3 \begin{vmatrix} 0.3 & 0.7 \\ 1.5 & 3 \end{vmatrix} + 0.6 \begin{vmatrix} 0.3 & 0.7 \\ 0.6 & 0.9 \end{vmatrix}$$

$$|A| = 0.1(0.6\times3 - 1.5\times0.9) - 0.3(0.3\times3 - 1.5\times0.7) + 0.6(0.3\times0.9 - 0.6\times0.7)$$

$$|A| = \emptyset$$

b. [2 points] If we were to use Gaussian elimination with partial pivoting to solve this system using exact arithmetic, show where the process fails.

$$\begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.3 & 0.6 & 0.9 \\ 0.6 & 1.5 & 3 \end{bmatrix} x = \begin{bmatrix} 1.4 \\ 1.8 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.1 & 0.3 & 0.7 & 1.4 \\ 0.3 & 0.6 & 0.9 & 1.8 \\ 0.6 & 1.5 & 3 & 6 \end{array}\right]$$

$r_2 = r_2 - r_1 \times 0.3$
$r_3 = r_3 - r_1 \times 0.1$

$$\left[\begin{array}{ccc|c} 1 & 2.5 & 5 & 10 \\ 0 & -0.15 & -0.6 & -1.2 \\ \emptyset & 0.05 & 0.2 & 0.4 \end{array}\right]$$

$r_2 = r_2 / (-0.15)$

$$\begin{bmatrix} 0.6 & 1.5 & 3 & \vert & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.6 & 1.5 & 3 & 6 \\ 0.3 & 0.6 & 0.9 & 1.8 \\ 0.1 & 0.3 & 0.7 & 1.4 \end{array}\right]$$

$r_1 = r_1 / 0.6$

$$\left[\begin{array}{ccc|c} 1 & 2.5 & 5 & 10 \\ 0.3 & 0.6 & 0.9 & 1.8 \\ 0.1 & 0.3 & 0.7 & 1.4 \end{array}\right]$$

$r_2 = r_2 / (-0.15)$

$$\left[\begin{array}{ccc|c} 1 & 2.5 & 5 & 10 \\ \emptyset & 1 & 4 & 8 \\ \emptyset & 0.05 & 0.2 & 0.4 \end{array}\right]$$

$r_3 = r_3 - r_2 \times 0.05$

$$\left[\begin{array}{ccc|c} 1 & 2.5 & 5 & 10 \\ \emptyset & 1 & 4 & 8 \\ \emptyset & \emptyset & \emptyset & \emptyset \end{array}\right]$$

infinite solutions

$$x_1 + 2.5x_2 + 5x_3 = 10$$

$$x_2 + 4x_3 = 8$$

$$x_2 = 8 - 4x_3$$

$$x_1 + 2.5(8 - 4x_3) = 10$$

$$x_1 - 10x_3 = -20$$

During gaussian elimination, the process fails when performing backward substitution on the matrix. During backward substitution, when a row composed of entirely zeroes is encountered, division by zero would be performed during pivoting and therefore the process would fail.

c. [1 point] Although $A$ is singular, cond(A) does not return Inf. Why?

```
a =

    0.1000    0.3000    0.7000
    0.3000    0.6000    0.9000
    0.6000    1.5000    3.0000

>> cond(a)

ans =

   3.6114e+16
```

```
>> inv(a)
Warning: Matrix is close to singular

ans =

   1.0e+16 *
```

Matlab represents all numbers as binary in double precision. Cond(A) returns the ratio of the largest singular value of A to the smallest. Since A is singular, in theory cond(A) should return inf. However, keeping in mind that Matlab represents all numbers as binary in double precision, it is likely that roundoff error is causing the calculation of the determinant of the matrix to not equal zero (but a number very close to zero).

As observed when taking the inverse of A using inv(A), the coefficient of the matrix is very large, which means that the determinant of the matrix is very close to zero. As we know, det(A) == 0 means that A is singular, and through exact arithmetic, we know that the actual determinant is zero. However, as previously discussed, the roundoff error present in matlab due to the

```
ans =

   1.0e+16 *

    5.8743    1.9581   -1.9581
   -4.6994   -1.5665    1.5665
    1.1749    0.3916   -0.3916
```

== 0 means that A is singular, and through exact arithmetic, we know that the actual determinant is zero. However, as previously discussed, the roundoff error present in matlab due to the representation of very small numbers. This error is present in the calculation of the determinant, since the result in matlab is a very small number not equal to zero, thereby making the matrix **close to singular** but not actually singular in matlab.

Since the matrix is close to singular (and det(A) != 0) then the calculation of cond(A) would return a number close to infinity but not infinity itself.

Inverse of a Matrix

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

d. [1 point] If we solve $Ax = b$ in Matlab using `A\b`, how accurate is the computed $x$?

```
>> x = a\[1.4;1.8;6]
Warning: Matrix is cl
RCOND =   1.850372e-18

x =

     0
     0
     2
```

The computed x is completely accurate. Substituting the acquired x3 into the previously derived formulae for x1 and x2 should result in x1 = 0 and x2 = 0 as shown by the A\b operation.

$x_1 - 10(2) = -20 \rightarrow x_1 = 0$

$x_2 = 8 - 4(2) \rightarrow x_2 = 0$

Therefore since the exact arithmetic results match the computed results then the computed results are in fact accurate, and the system has infinite solutions.

**Problem 3** [4 points] The gamma function has the following values: $\Gamma(0.5) = \sqrt{\pi}$, $\Gamma(0.75) = \sqrt{\pi/2}$, $\Gamma(1) = 1$.

a. [2 points] Derive the quadratic interpolation polynomial interpolating these values.

b. [2 points] From this polynomial, determine the value of $x$ for which $\Gamma(x) = 1.5$.

$(0.5, \sqrt{\pi})$ $(0.75, \sqrt{\pi/2})$ $(1, 1)$

$(0.5, \sqrt{\pi})$ $(0.75, \sqrt{\pi/2})$ $(1,1)$

a) $$L_1(x) = \frac{(x-0.75)(x-1)}{(0.5-0.75)(0.5-1)} = \frac{x^2 - 1.75x + 0.75}{0.125}$$

$$= \frac{x^2 - 7x/4 + 3/4}{1/8} = 8x^2 - 14x + 6$$

$$L_2(x) = \frac{(x-0.5)(x-1)}{(0.75-0.5)(0.75-1)} = \frac{x^2 - 1.5x + 0.5}{-0.0625}$$

$$= \frac{x^2 - 3x/2 + 1/2}{1/16} = -16x^2 + 24x - 8$$

$$L_3(x) = \frac{(x-0.5)(x-0.75)}{(1-0.5)(1-0.75)} = \frac{x^2 - 1.25x + 0.375}{0.125}$$

$$= \frac{x^2 - 10x/8 + 3/8}{1/8} = 8x^2 - 10x + 3$$

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$= \sqrt{\pi}(8x^2 - 14x + 6) + \sqrt{\tfrac{\pi}{2}}(-16x^2 + 24x - 8) + (1)(8x^2 - 10x + 3)$$

$$= 8\sqrt{\pi}x^2 - 14\sqrt{\pi}x + 6\sqrt{\pi} - 16\sqrt{\tfrac{\pi}{2}}x^2 + 24\sqrt{\tfrac{\pi}{2}}x - 8\sqrt{\tfrac{\pi}{2}} + 8x^2 - 10x + 3$$

$$= x^2\left(8\sqrt{\pi} - 16\sqrt{\tfrac{\pi}{2}} + 8\right) + x\left(-14\sqrt{\pi} + 24\sqrt{\tfrac{\pi}{2}} - 10\right) + \left(6\sqrt{\pi} - 8\sqrt{\tfrac{\pi}{2}} + 3\right)$$

$$= \left(8+\sqrt{\pi}\left(8-8\sqrt{2}\right)\right)x^2 + \left(\sqrt{\pi}\left(12\sqrt{2}-14\right)-10\right)x + \left(3+\sqrt{\pi}\left(6-4\sqrt{2}\right)\right)$$

$$\left(\approx 2.126604608x^2 - 4.734814632x + 3.60821001\right)$$

b) $p(x) = \frac{3}{2} \rightarrow \varnothing = p(x) - \frac{3}{2}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-\left(\sqrt{\pi}\left(12\sqrt{2}-14\right)-10\right) \pm \left(484\pi - 336\pi\sqrt{2} + 280\sqrt{\pi} - 240\sqrt{2}\sqrt{\pi} + 100 - 448\pi + 320\pi\sqrt{2} - 240\sqrt{\pi} + 176\sqrt{2}\sqrt{\pi} - 48\right)^{1/2}}{16 - 16\sqrt{2}\sqrt{\pi} + 16\sqrt{\pi}}$$

$$x = \frac{10 - \sqrt{\pi}\left(12\sqrt{2}-14\right) \pm \sqrt{40\sqrt{\pi} - 16\sqrt{\pi}\sqrt{2} + 36\pi - 64\sqrt{2}\sqrt{\pi} + 52}}{16 - 16\sqrt{2}\sqrt{\pi} + 16\sqrt{\pi}}$$

$x_1 \approx 1.611167523$ $\qquad$ $x_2 \approx 0.6152992926$

**Problem 4** [4 points] Given the three data points $(-1, 1)$, $(0, 0)$, $(1, 1)$, determine the interpolating polynomial of degree two using :

a. [1 point] monomial basis

b. [1 point] Lagrange basis

c. [1 point] Newton basis

[1 point] Show that the three representations give the same polynomial.

a)

| | | | |
|---|---|---|---|
| $x_i$ | $-1$ | $\varnothing$ | $1$ |
| $y_i$ | $1$ | $0$ | $1$ |

$$P_2(x) = c_\varnothing + c_1 x + c_2 x^2$$

$$P_2(-1) = c_\varnothing + c_1(-1) + c_2(-1)^2 = 1$$

$$P_2(\varnothing) = c_\varnothing + \varnothing c_1 + \varnothing^2 c_2 = \varnothing$$

$$P_2(1) = c_\varnothing + (1)c_1 + (1)^2 c_2 = 1$$

$P_2(1) = C_\varnothing + (1)C_1 + (1)^2 C_2 = 1$

$C_\varnothing = \varnothing$

$C_\varnothing - C_1 + C_2 = 1 \qquad C_\varnothing + C_1 + C_2 = 1$

$C_2 - C_1 = 1 \qquad C_1 + C_2 = 1$

$C_2 = 1 - C_1$

$1 - C_1 - C_1 = 1$

$-2C_1 = \varnothing$

$C_1 = \varnothing$

$\boxed{C_2 = 1}$

$P(x) = x^2$

b) $(-1, 1)\ (\varnothing, \varnothing)\ (1, 1)$

$$L_1(x) = \frac{(x-\varnothing)(x-1)}{(-1-\varnothing)(-1-1)} = \frac{x^2 - x}{2}$$

$$L_2(x) = \frac{(x+1)(x-1)}{(\varnothing+1)(\varnothing-1)} = -x^2 + 1$$

$$L_3(x) = \frac{(x+1)(x+\varnothing)}{(1-\varnothing)(1+1)} = \frac{x^2 + x}{2}$$

$P(x) = \cancel{y_1}^{1} L_1 + \cancel{y_2}^{\varnothing} L_2 + \cancel{y_3}^{1} L_3$

$$P(x) = y_1 L_1 + y_2 L_2 + y_3 L_3$$

$$= L_1 + L_3$$

$$= \frac{x^2}{2} - \frac{x}{2} + \frac{x^2}{2} + \frac{x}{2}$$

$$P(x) = x^2$$

c) $(-1, 1)$ $(0, 0)$ $(1, 1)$

$$b_1 = y_1 = 1$$

$$b_2 = \frac{0 - 1}{0 + 1} = -1$$

$$b_3 = \frac{\frac{1-0}{1-0} - \frac{0-1}{0+1}}{1-(-1)} = \frac{1-(-1)}{2} = 1$$

$$b_1 = 1, \quad b_2 = -1, \quad b_3 = 1$$

$$P(x) = 1 - 1(x-(-1)) + 1(x-(-1))(x-0)$$

$$P(x) = 1 - x - 1 + x(x+1)$$

$$P(x) = -x + x^2 + x$$

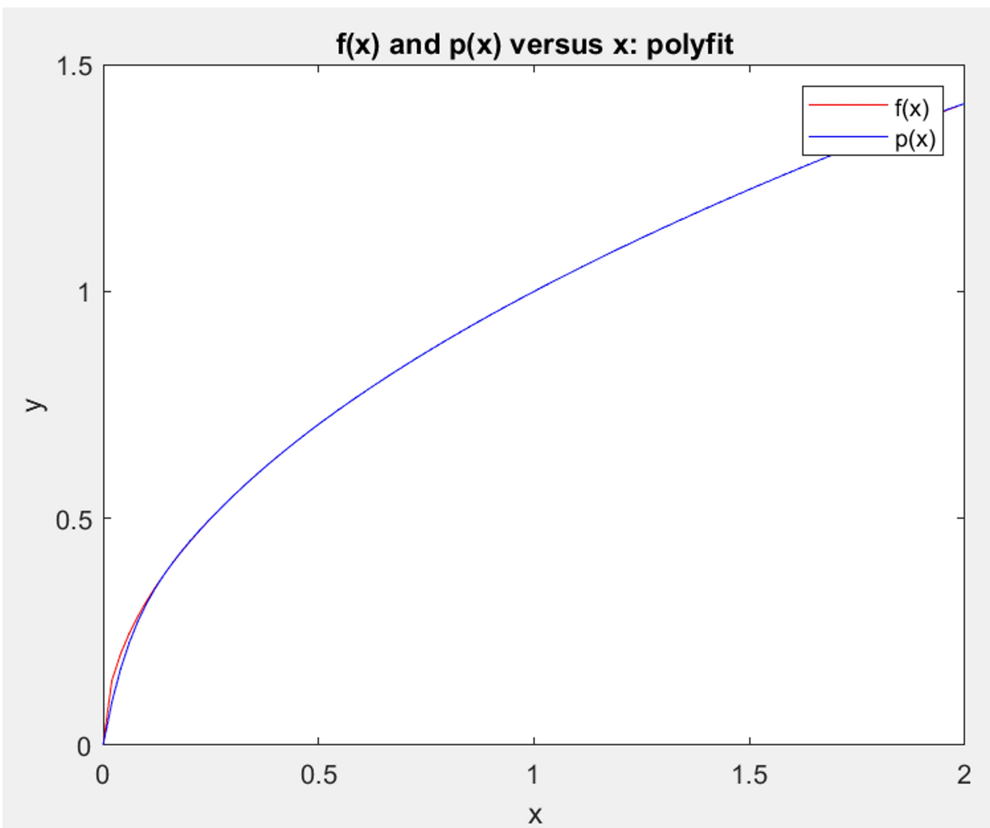
$$P(x) = x^2$$

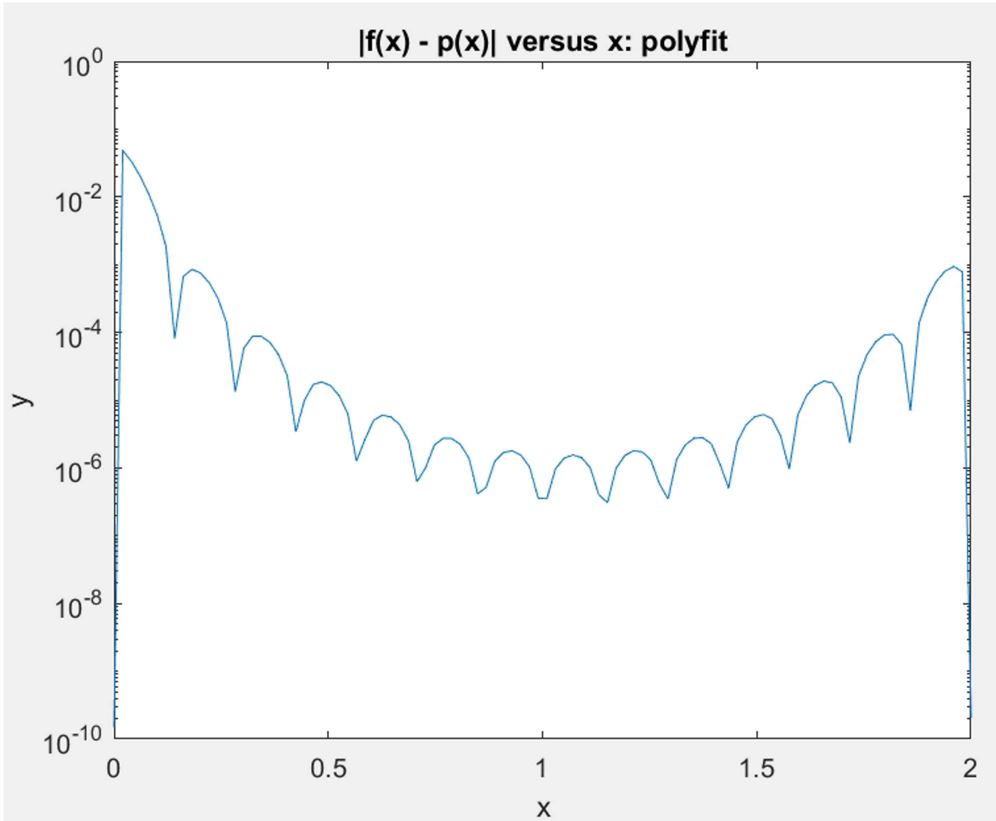
$P(x) = x^2$

**Problem 5** [8 points] Consider $f(x) = \sqrt{x}$ on $[0, 2]$.

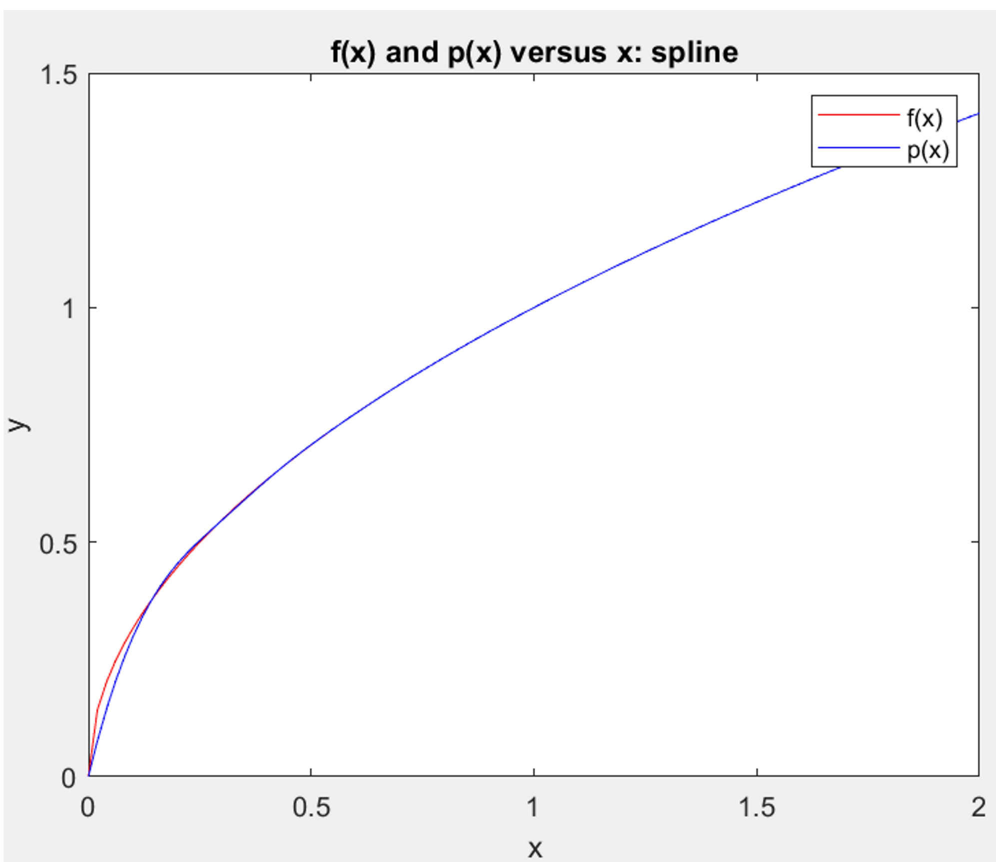
a. [3 points] Interpolate $f(x)$ at 15 evenly spaced points $x_0 = 0 < x_1 < \cdots < 2 = x_{14}$ in $[0, 2]$. You can use the `polyfit` function; see also `linspace`. Denote the resulting interpolation polynomial by $p(x)$.

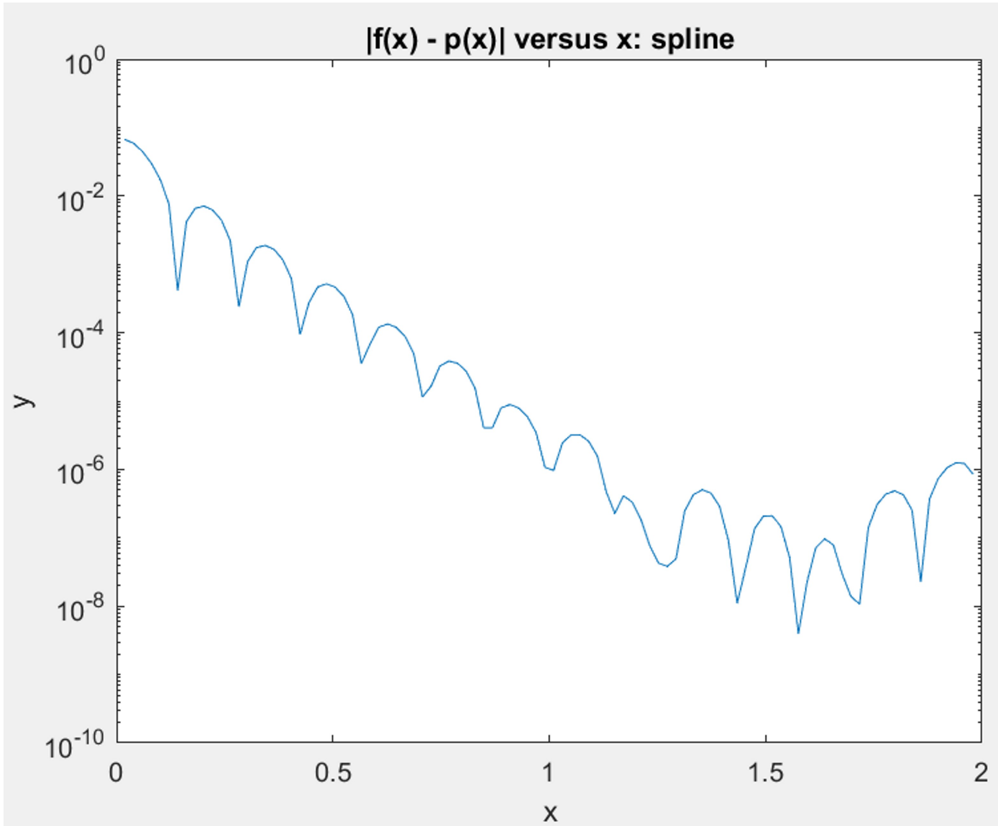
- Plot on the same plot $f(x)$ and $p(x)$ versus $x$ at 100 evenly spaced points in $[0, 2]$.
- Plot $|f(x) - p(x)|$ versus $x$ at these points; use `semilogy`.



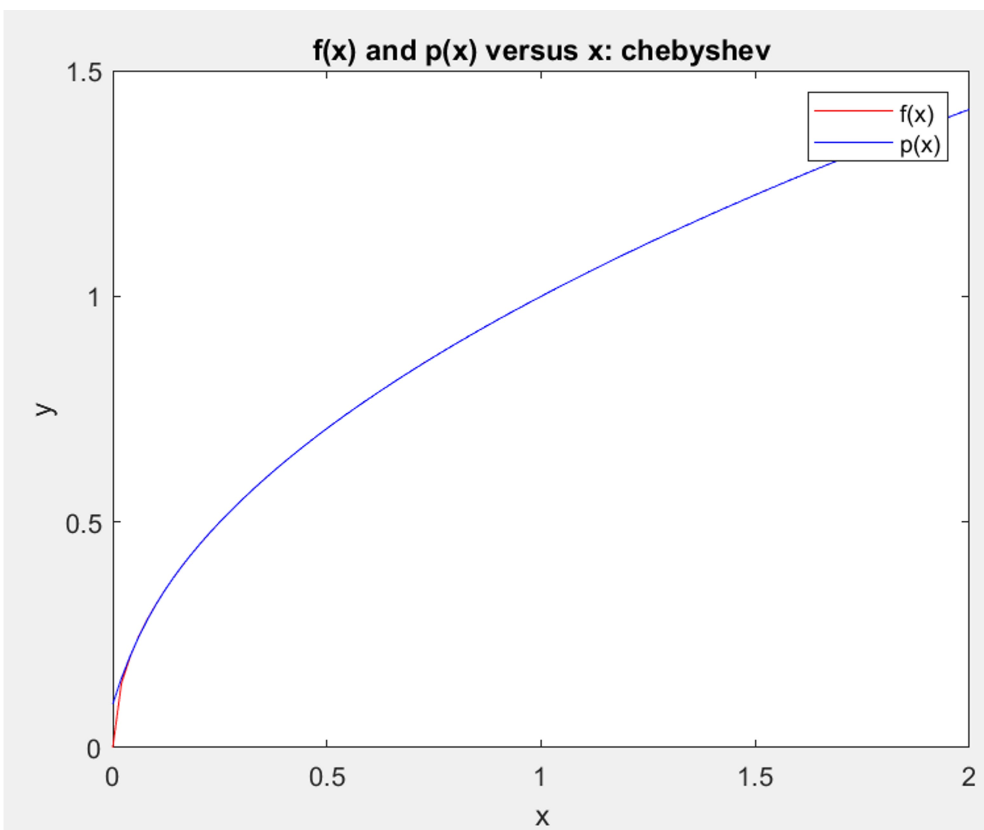


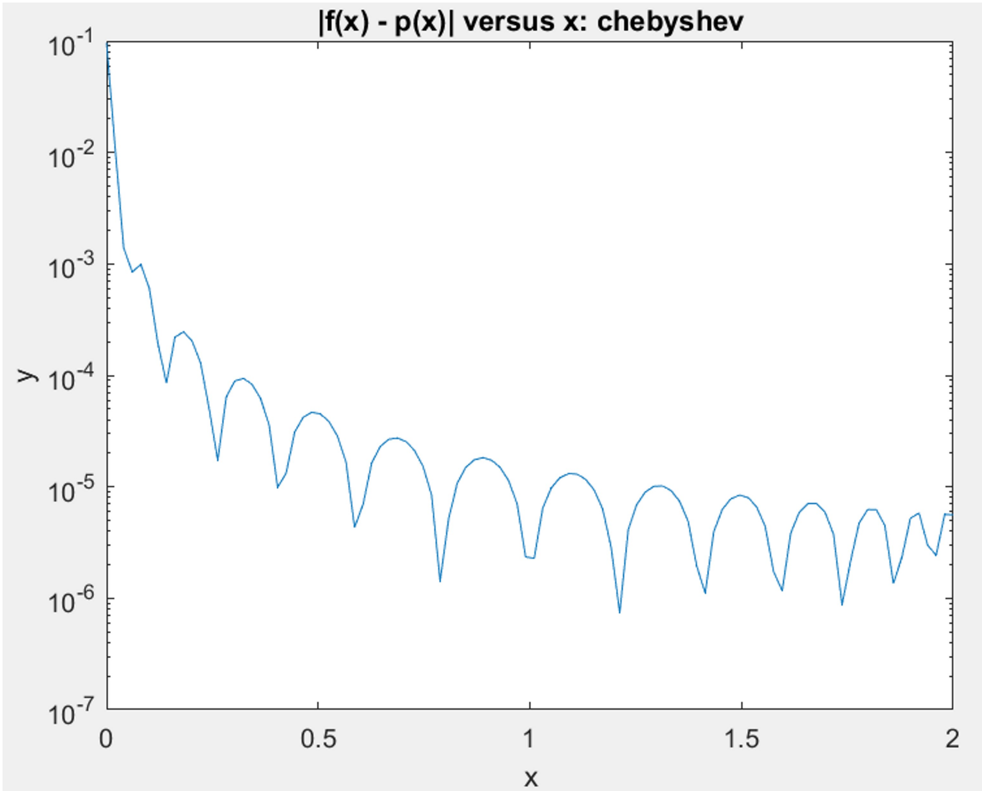

b. [2 points] Instead of `polyfit` use `spline` and produce the plots in a.






c. [2 points] Repeat a. with 15 Chebyshev points.






d. [1 point] Explain the differences in the error plots in a. and c.

The error plot from c demonstrates much smaller overall error than the plot from a. This is due to the use of chebyshev points. Chebyshev nodes allow us to choose points such that the "h" factor (or product (x-xi) ) is minimized inside of the polynomial interpretation error function. The minimization of this term allows for less overall error caused by the points themselves, and this is shown inside of the graphs by the overall reduction in error in graph c in comparison to graph a.

**Problem 6** [2 points] Suppose you interpolate $e^{x^2}$ at $x = 0, 0.5, 1$ by a polynomial of degree 2. Derive a bound for the error of this interpolation.

| $x$ | $y$ |
|---|---|
| $\varnothing$ | $1$ |
| $0.5$ | $e^{0.25}$ |
| $1$ | $e$ |

$n = 2$

$$h = \frac{b-a}{n} = \frac{1-\varnothing}{2} = 0.5$$

$$|f(x) - P_2(x)| \le \frac{M}{4(3)} h^3$$

$$f'(x) = 2x e^{x^2}$$

$$f''(x) = e^{x^2}(4x^2+1)$$

$$f'''(x) = 4xe^{x^2}(2x^2+3)$$

$$M = \max_{0 \le t \le 1} |f'''(t)|$$

$$= \max_{0 \le t \le 1} |4xe^{x^2}(2x^2+3)|$$

max occurs @ $x = 1$

$$= |4(1)e^{(1)^2}(2(1)^2+3)|$$

$$= 20e$$

$$|f(x) - P_2(x)| \le \frac{M}{4(3)}(0.5)^3$$

$$|f(x) - P_2(x)| \leq \frac{20e}{12}(0.125)$$

$$|f(x) - P_2(x)| \leq \frac{5e}{24} \approx 0.566308714$$