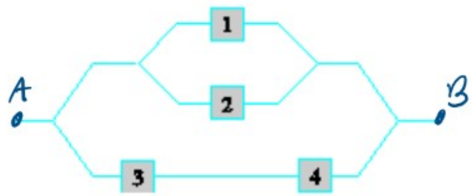
CM2

September 24, 2022 10:40 PM

Problem #1: Consider the system of components connected as in the figure below. Components 1 and 2 are connected in parallel, so that subsystem works if and only if either 1 or 2 works. Since 3 and 4 are connected in series, that subsystem works if and only if both 3 and 4 work. Suppose that $P(1 \text{ works}) = 0.7$, $P(2 \text{ works}) = 0.75$, $P(3 \text{ works}) = 0.86$, and $P(4 \text{ works}) = 0.83$. Find the probability that the system works.



$P(1) = 0.7$ $\quad P(3) = 0.86$

$P(2) = 0.75$ $\quad P(4) = 0.83$

$$P(A \to B) = P(UP \cup DOWN)$$

$$P(UP) = P(1 \cup 2) = P(1) + P(2) - P(1 \cap 2)$$
$$= P(1) + P(2) - P(1)P(2) \leftarrow \text{independence}$$
$$= 0.7 + 0.75 - 0.7 \cdot 0.75$$
$$= 0.925$$

$$P(DOWN) = P(3 \cap 4) = P(3)P(4)$$
$$= 0.86 \cdot 0.83$$
$$= 0.7138$$

$$P(A \to B) = P(UP \cup DOWN)$$
$$= P(UP) + P(DOWN) - P(UP)P(DOWN)$$
$$= 0.925 + 0.7138 - 0.925 \cdot 0.7138$$
$$= 0.978535$$

$= 0.978535$

Problem #2: Consider purchasing a system of audio components consisting of a receiver, a pair of speakers, and a CD player. Let A_1 be the event that the receiver functions properly throughout the warranty period. Let A_2 be the event that the speakers function properly throughout the warranty period. Let A_3 be the event that the CD player functions properly throughout the warranty period. Suppose that these events are (mutually) independent with $P(A_1) = 0.93$, $P(A_2) = 0.84$, and $P(A_3) = 0.71$.

(a) What is the probability that at least one component needs service during the warranty period?

(b) What is the probability that exactly one of the components needs service during the warranty period?

$P(A_1) = 0.93 \quad P(A_2) = 0.84 \quad P(A_3) = 0.71$

$P(\bar{A}_1) = 0.07 \quad P(\bar{A}_2) = 0.16 \quad P(\bar{A}_3) = 0.29$

a) what is the probability of one or more failure?

$$P(1 \text{ or more fail}) = 1 - P(\text{all working})$$
$$= 1 - P(A_1)P(A_2)P(A_3)$$
$$= 1 - 0.554652$$
$$= 0.445348$$

b) Probability exactly one needs service

$$P(\text{exactly 1 fail}) = P(\bar{A}_1 \text{ only fails}) \cup P(\bar{A}_2 \text{ only}) \cup P(\bar{A}_3 \text{ only})$$
$$= P(\bar{A}_1 \text{ only}) + P(\bar{A}_2 \text{ only}) + P(\bar{A}_3 \text{ only})$$
$$= P(\bar{A}_1)P(A_2)P(A_3)$$
$$+ P(A_1)P(\bar{A}_2)P(A_3)$$
$$+ P(A_1)P(A_2)P(\bar{A}_3)$$

$$= 0.07 \cdot 0.84 \cdot 0.71$$
$$+ 0.93 \cdot 0.16 \cdot 0.71$$
$$+ 0.93 \cdot 0.84 \cdot 0.29$$

$$= 0.373944$$

Problem #3: A box consists of 9 components, 6 of which are defective.

(a) Components are selected and tested one at a time, without replacement, until a non-defective component is found. Let X be the number of tests required. Find $P(X = 3)$.

(b) Components are selected and tested, one at a time without replacement, until two *consecutive* non defective components are obtained. Let X be the number of tests required. Find $P(X = 5)$.

9 components, 6 defective, 3 working

a) ✗ ✗ ✓ ... ...

$6/9$ $5/8$ $3/7$

what is the chance of 2 failures & then one sucess?

$$P(X=3) = \frac{6}{9} \times \frac{5}{8} \times \frac{3}{7} = 0.178571$$

b)

1) ✗ ✗ ✗ ✓ ✓

$6/9$ $5/8$ $4/7$ $3/6$ $2/5$

2) ✗ ✓ ✗ ✓ ✓ ...

$6/9$ $3/8$ $5/7$ $2/6$ $1/5$

3) ✓ ✗ ✗ ✓ ✓ ...

$3/9$ $6/8$ $5/7$ $2/6$ $1/5$

$$P(X=5) = P(1 \cup 2 \cup 3)$$

$P(✓ \cap ✗ \cap ✗ \cap ✓ \cap ✓)$

,

$P(\checkmark \cap \times \cap \times \cap \checkmark \cap \checkmark)$

$$P(X=5) = P(1 \cup 2 \cup 3)$$
$$= P(1) + P(2) + P(3)$$
$$= \left(\tfrac{6}{9}\right)\left(\tfrac{5}{8}\right)\left(\tfrac{4}{7}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{5}\right) + \left(\tfrac{6}{9}\right)\left(\tfrac{3}{8}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{5}\right)$$
$$+ \left(\tfrac{3}{9}\right)\left(\tfrac{6}{8}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{5}\right)$$
$$= 0.071428$$

Problem #4: An assembly consists of two mechanical components. Suppose that the probabilities that the first and second components meet specifications are 0.99 and 0.85, respectively. Let X be the number of components in the assembly that meet specifications. A B

(a) Find the mean of X.

(b) Find the variance of X.

$P(A) = 0.99 \qquad P(B) = 0.85$

$P(\bar{A}) = 0.01 \qquad P(\bar{B}) = 0.15$

$\text{range}(X) = \{0, 1, 2\} \leftarrow$ posses

$$P(X=\varnothing) = P(\bar{A}) \cap P(\bar{B}) = 0.01 \times 0.15 = 0.0015$$

$$P(X=1) = P(\bar{A} \cap B) + P(A \cap \bar{B})$$
$$= P(\bar{A})P(B) + P(A)P(\bar{B})$$
$$= (0.01)(0.85) + (0.99)(0.15)$$
$$= 0.157$$

$$P(X=2) = P(A)P(B) = 0.99 \cdot 0.85 = 0.8415$$

a) $$E[X] = \sum x P(x)$$
$$= (\varnothing)(0.0015) + (1)(0.157) + (2)(0.8415)$$
$$= 1.84$$

$$b)\ var(X) = \left(\sum_x x^2 P(X=x)\right) - \mu^2$$

$$= \left((\emptyset)^2 P(X=\emptyset) + (1)^2 P(X=1) + (2)^2 P(X=2)\right) - (1.84)^2$$

$$= (0.157 + (4)(0.8415)) - (3.3856)$$

$$= 0.1374$$

Problem #5: A manufacturing process has 45 customer orders to fill. Each order requires one component part that is purchased from a supplier. However, typically 4% of the components are identified as defective, and the components can be assumed to be independent.

(a) If the manufacturer stocks 47 components, what is the probability that the 45 orders can be filled without reordering components?

(b) Let *X* be the number of good (i.e., non-defective) components among the 47 in stock. Find the mean of *X*.

(c) Find the variance of *X* [from part (b)].

45 orders, 4% are defective

a) 47 componens

what are the odds you choose 45 components without choosing a defective one?

at least the first 45 of the total 47 must be working

$$P(x=45) = \binom{47}{45}(0.96)^{45}(0.04)^2$$

$$P(x=45) = 0.275519$$

$$P(x=\underline{46}) = \binom{47}{46}(0.96)^{46}(0.04) = 0.287498$$

$$P(x=\underline{47}) = \binom{47}{47}(0.96)^{47}(1) = 0.146807$$

$$P(\text{at least 45 working}) = P(x=45) \cup P(x=46) \cup P(x=47)$$

$$= 0.275519 + 0.287498 + 0.146807$$

$= 0.275519 + 0.287498 + 0.146807$

$= 0.709824$

b) $E[X] = \underset{n}{(47)}\underset{p}{(0.96)}$ Binomial(n, p)

$= 45.12$

c) $Var(X) = nP(1-p)$

$= (47)(0.96)(0.04)$

$= 1.8048$

Problem #6: The probability that a randomly selected box of a certain type of cereal has a particular prize is 0.11. Suppose that you purchase box after box until you have obtained 2 of these prizes.

(a) What is the probability that you purchase exactly 7 boxes?

(b) What is the probability that you purchase at least 10 boxes?

(c) How many boxes would you expect to purchase, on average?

$P(\text{prize}) = 0.11$

a)

1) ✓ ✗ ✗ ✗ ✗ ✗ ✓

2) ✗ ✓ ✗ ✗ ✗ ✗ ✓

3) ✗ ✗ ✓ ✗ ✗ ✗ ✓

4) ✗ ✗ ✗ ✓ ✗ ✗ ✓

5) $\times$ $\times$ $\times$ $\times$ $\checkmark$ $\times$ $\checkmark$

6) $\times$ $\times$ $\times$ $\times$ $\times$ $\checkmark$ $\checkmark$

negative binomial distribution

$$P(X=7) = \binom{7-1}{2-1}(0.11)^2(0.89)^5$$

$$= 0.040540$$

b) $P(X \geq 10) = 1 - P(X<10)$

$$= 1 - P(X=9) - P(X=8) \ldots$$

$$= 1 - \binom{9-1}{2-1}(0.11)^2(0.89)^7 - \binom{8-1}{2-1}(0.11)^2(0.89)^6 \ldots$$

$$= 0.740079$$

c) $E[X] = r/p = 2/0.11 = 18.1818\ldots$

Problem #7: A geologist has collected 21 specimens of basaltic rock and and 17 specimens of granite. The geologist instructs a laboratory assistant to randomly select 8 of the specimens for analysis.

(a) What is the probability that at least 4 of the selected specimens are granite?

(b) What is the expected number of granite specimens in the sample?

(c) If this same process is repeated every day, how many days (on average) will it take before getting a sample consisting entirely of granite?

21 of b & 19 of g

choose 8 for analysis

a) 4 or more are granite?

__ __ __ __ __ __ __ __ → 4 granite, 4 - b Rock

hypergeometric

hypergeometric

$$\text{range}(X) = \{0,1,2,3,\overbrace{4,5,6,7,8}\}$$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$$

$$P(x=4) = \binom{17}{4}\binom{21}{4} / \binom{17+21}{8}$$

$$= 0.291273698$$

$$P(x=5) = \binom{17}{5}\binom{21}{3} / \binom{17+21}{8}$$

$$= 0.168291458$$

$$P(x=6) = \binom{17}{6}\binom{21}{2} / \binom{17+21}{8}$$

$$= 0.053144671$$

$$P(x=7) = \binom{17}{7}\binom{21}{1} / \binom{17+21}{8}$$

$$= 0.008351305$$

$$P(x=8) = \binom{17}{8}\binom{21}{0} / \binom{17+21}{8}$$

$$= 0.000497101$$

$$P(x \geq 4) = 0.52158213$$

(b) What is the expected number of granite specimens in the sample?

$$E[X] = nk/N$$

$$= 8 \cdot 17/(17+21)$$

$$= 68/19$$

$= 68/19$

(c) If this same process is repeated every day, how many days (on average) will it take before getting a sample consisting entirely of granite?

$P(X=8) = 0.000497101$

$1/P(X=8) \doteq 2012$ days

Problem 7

Poisson's dist.

in 225g = 14.4

in 43g = 2.75

$\lambda = 2.75$, unit length = 43g

$$P(X=x) = \frac{(\lambda T)^x}{x!} e^{-\lambda T}$$

$T = 1$

$$= \frac{(2.75)^x e^{-2.75}}{x!}$$

a) $P(X \geq 3) = ?$

no maximum $\therefore$ do $1-P$

$$P(X \geq 3) = 1 - P(X < 3)$$

$$P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$P(X=0) = \frac{(2.75)^{0} e^{-2.75}}{0!}$$

$$= 0.063927861$$

$$P(X=1) = \frac{2.57 e^{-2.75}}{1!}$$

$$= 0.175801618$$

$$P(X=2) = \frac{(2.57)^{2} e^{-2.75}}{2!}$$

$$= 0.241727225$$

$$P(X<3) = 0.481456704$$

$$P(X \geq 3) = 1 - P(X<3)$$

$$= 0.518543295$$

b) 13 weeks, 0 [illegible] for EXACTLY 6 weeks.

binomial RV

$$P(X=k) = \binom{n}{k} p^{k} (1-p)^{n-k}$$

$$P(\text{exactly } 6) = \binom{13}{\ } P(X=0)^{6}\, P(X>0)^{9-6}$$

$$P(\text{exactly } 6) = \binom{13}{6} P(X=\emptyset)^6 P(X>\emptyset)^{9-6}$$

$$P(X>\emptyset) = 1 - P(X=\emptyset)$$

$$P(X=\emptyset) = 0.063927861$$

$$P(X>\emptyset) = 0.936072139$$

$$P(X=6) = \binom{13}{6}(0.063927861)^6(0.936072139)^3$$

$$= 0.000009606\,9$$

Problem #9: Let X denote the vibratory stress (psi) on a wind turbine blade at a particular wind speed in a wind tunnel. Suppose that X has the following *Rayleigh* pdf.

$$f(x) = \begin{cases} (x/\theta^2)\, e^{-x^2/(2\theta^2)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) If $\theta = 91$, find the probability that the vibratory stress is between 81 and 314.

(b) If $\theta = 91$, then 78% of the time the vibratory stress is greater than what value?

a) $$P(81 \le X \le 314) = \int_{81}^{314} f(x)\,dx$$

$$= \int_{81}^{314} \frac{x}{\theta^2} e^{-x^2/2\theta^2}\,dx$$

$$= \left[\frac{1}{2\theta^2} x^2 \left(\frac{2\theta^2}{-x^2}\right) e^{-x^2/2\theta^2}\right]_{81}^{314}$$

$$= \left(-e^{-x^2/2\theta^2}\right)_{81}^{314}, \theta = 91$$

$$= \left(-e^{-x^2/2\theta^2}\right)_{81} \quad , \theta = 91$$

$$= 0.670309571$$

b) $P(X \geq x) = 0.78 = \int_x^{\infty} f(x)\,dx \leftarrow$ no

$P(X \leq x) = 1 - 0.78$

$$1 - 0.78 = \int_0^x f(x)\,dx$$

$$\cancel{1} - 0.78 = -e^{-\frac{x^2}{2\theta^2}} \cancel{+1}$$

$$-0.78 = -e^{-x^2/2\theta^2}$$

$$0.78 = e^{-x^2/2\theta^2}$$

$$\ln(0.78) = \frac{-x^2}{2\theta^2}$$

$$\sqrt{-2\theta^2 \ln(0.78)} = x$$

$$x = 64.14839852$$