Q1

Alexander Bartella
400308868

March 6, 2022 6:42 PM

1. **Charge and Current Distributions**

Find expressions for the total charge on a circular disk defined by $r \le a$ and $z = 0$ if:

a. $\rho_s = \rho_{s_0} \sin^2\emptyset \left[\frac{C}{m^2}\right]$

b. $\rho_s = \rho_{s_0} e^{-r} \left[\frac{C}{m^2}\right]$

c. $\rho_s = \rho_{s_0} \cos\emptyset \left[\frac{C}{m^2}\right]$

d. $\rho_s = \rho_{s_0} e^{-r} \sin^2\emptyset \left[\frac{C}{m^2}\right]$

a) $\rho_s = \rho_{s\emptyset} \sin^2\phi \; [C/m^2]$ $\qquad ds = r\,dr\,d\phi$

$$Q = \int \rho_s \, ds \qquad \text{let } b = \rho_{s\emptyset}\sin^2\phi$$

$$= \int_\emptyset^{2\pi}\left(\int_\emptyset^a \underbrace{\rho_{s\emptyset}\sin^2\phi}_{b}\; r\,dr\right)d\phi$$

$$= \int_\emptyset^{2\pi}\left(\int_\emptyset^a b\, r\,dr\right)d\phi$$

$$= \int_0^{2\pi}\left[\frac{br^2}{2}\right]_\emptyset^a d\phi$$

$$= \int_0^{2\pi} b\left[\frac{r^2}{2}\right]_\emptyset^a d\phi$$

$$= \frac{a^2}{2}\int_0^{2\pi} \rho_{s\emptyset}\sin^2\phi \, d\phi$$

$$= \frac{a^2}{2}\int_0^{2\pi} \rho_{s\emptyset}\left(\frac{1-\cos 2\phi}{2}\right)d\phi$$

$$= \frac{a^2}{2}\int_0^{2\pi} \rho_{s0}\left(\frac{1-\cos 2\phi}{2}\right) d\phi$$

$$= \frac{\rho_{s0}\, a^2}{2}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{2\pi}$$

$$= \frac{\rho_{s0}\, a^2}{2}\left[\left(\frac{\cancel{2}\pi}{\cancel{2}} - \frac{\cancel{\sin 4\pi}}{4}\right) - \left(\frac{0}{2} - \frac{\sin 2(0)}{4}\right)\right]$$

$$= \frac{\rho_{s0}\, a^2 \pi}{2}$$

b) $\rho_s = \rho_{s0} e^{-r}$

$$Q = \int \rho_s \, ds \qquad ds = r\,dr\,d\phi$$

$$= \int_0^{2\pi}\left(\int_0^a r \rho_{s0} e^{-r}\, dr\right) d\phi$$

$$= \rho_{s0}\int_0^{2\pi}\int_0^a r e^{-r}\, dr\, d\phi$$

$$= \rho_{s0}\int_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^a d\phi$$

$$= \rho_{s0}\left[\left[(-ae^{-a}-e^{-a})-(0-\cancel{e^{0}}^{\,1})\right]\phi\right]_0^{2\pi}$$

$$= \rho_{s0}\left[\left[-ae^{-a}-e^{-a}+1\right]\phi\right]_0^{2\pi}$$

$$= \rho_{s0}\left[\left[-(1+a)e^{-a}+1\right]\phi\right]_0^{2\pi}$$

$$= 2\pi\rho_{s0}\left[1-(1+a)e^{-a}\right]$$

c) $\rho_s = \rho_{s0}\cos\phi$

$$Q = \int \rho_s \, ds$$

$$= \int_0^{2\pi}\left(\int_0^{a} \rho_{s0}\cos\phi \, r\,dr\right)d\phi$$

$$= \left(\frac{a^2}{2}\right)\int_0^{2\pi} \rho_{s0}\cos\phi \, d\phi$$

$$= \left(\frac{a^2}{2}\right)\rho_{s0}\sin\phi\Big|_0^{2\pi}$$

$$= 0$$

d) $\rho_s = \rho_{s0} e^{-r} \sin^2 \phi$

$$Q = \int \rho_s \, ds$$

$$= \rho_{s0} \int_0^{2\pi} \left(\int_0^a e^{-r} \sin^2 \phi \, r \, dr \right) d\phi$$

$$= \rho_{s0} \int_0^{2\pi} \left[-re^{-r} - e^{-r} \right]_0^a \sin^2 \phi \, d\phi$$

$$= \rho_{s0} \int_0^{2\pi} \left[1 - (1+a)e^{-a} \right] \sin^2 \phi \, d\phi$$

$$= \left[1 - (1+a) e^{-a} \right] \rho_{s0} \int_0^{2\pi} \sin^2 \phi \, d\phi$$

$$= \left[1 - (1+a) e^{-a} \right] \rho_{s0} \frac{1}{4} \left[2x - \sin(2x) \right]_0^{2\pi}$$

$$= \left[1 - (1+a) e^{-a} \right] \rho_{s0} \frac{1}{4} \left[(4\pi - 0) - (0 - 0) \right]$$

$$= \pi \rho_{s0} \left[1 - (1+a) e^{-a} \right]$$

Q2

March 6, 2022 6:43 PM

2. **Coulomb's Law**

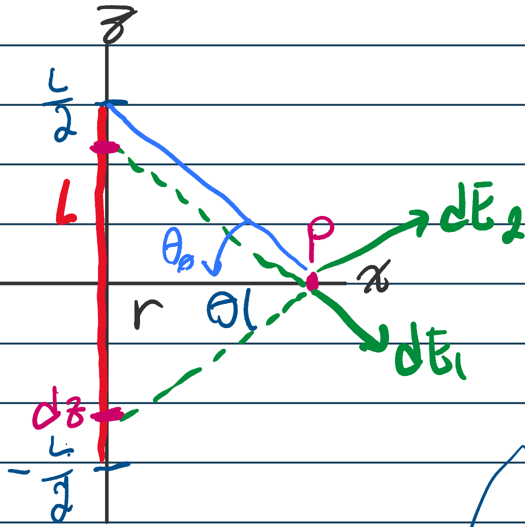
Between $z = -\frac{L}{2}$ and $z = \frac{L}{2}$ there is a line charge L with uniform density ρ_L along the z-axis.

a. Obtain an expression for the electric field at any point $P\ (r, \emptyset, 0)$ on the x-y plane using Coulomb's Law.

b. Show that your result from a) can simplify to the following expression for an infinite line of charge.

$$\mathbf{E} = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon_0 r}$$

a)



$$d\vec{E} = d\vec{E}_A + d\vec{E}_B$$

$$= \hat{r}\left[\frac{2\rho_L\cos\theta\, dz}{4\pi\varepsilon_0 R^2}\right]$$

$$\theta_\emptyset = \sin^{-1}\left(\frac{L/2}{\sqrt{r^2+(L/2)^2}}\right)$$

$$R = r/\cos\theta \ , \ z = r\tan\theta$$

$$\frac{dz}{d\theta} = \frac{d}{d\theta}(r\tan\theta)$$

$$dz = r\sec^2\theta\, d\theta$$

$$\vec{E} = \int_0^{\theta_\emptyset} d\vec{E}$$

$$= \int_\emptyset^{\theta_\emptyset} \hat{r}\left[\frac{\rho_L\cos\theta\, dz}{2\pi\varepsilon_0 R^2}\right]$$

θ

$$= \int_0^{\theta_0} \hat{r} \frac{\rho_L \cos^3\theta}{2\pi\varepsilon_0 r^2} \times \sec^2\theta \, d\theta$$

$$= \frac{\hat{r}\rho_L}{2\pi\varepsilon_0 r} \int_0^{\theta_0} \frac{\cos^3\theta}{\cos^2\theta} \, d\theta$$

$$= \frac{\hat{r}\rho_L}{2\pi\varepsilon_0 r} \int_0^{\theta_0} \cos\theta \, d\theta$$

$$= \frac{\hat{r}\rho_L}{2\pi\varepsilon_0 r} \sin(\theta_0) \qquad \theta_0 = \sin^{-1}\left(\frac{L/2}{\sqrt{r^2+(L/2)^2}}\right)$$

$$\boxed{\vec{E} = \frac{\hat{r}\rho_L}{2\pi\varepsilon_0 r} \cdot \frac{L/2}{\sqrt{r^2+(L/2)^2}}}$$

b)

as $L \to \infty$ ...

$$\sin(\theta_0) = \lim_{L \to \infty} \left(\frac{L/2}{\sqrt{r^2+(L/2)^2}}\right)$$

$$\sin(\theta_0) = \lim_{L \to \infty} \left(\sqrt{r^2 + (L/2)^2} \right)$$

* relative to L, $r \doteq \emptyset$

$$\sin(\theta_0) = \frac{L/2}{\sqrt{(L/2)^2}}$$

$$= \frac{L/2}{L/2}$$

$$\boxed{= 1}$$

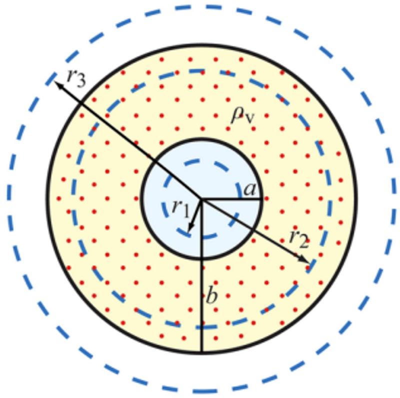
$\therefore$ as $L \to \infty$

$$\vec{E} = \frac{r P_L}{2\pi \epsilon_0 r} \cdot 1 \to \boxed{\vec{E} = \frac{r P_L}{2\pi \epsilon_0 r}}$$

Q3

March 6, 2022 6:43 PM

3. **Gauss's Law**
A spherical shell with outer radius b surrounds a charge-free cavity of radius $a < b$. If the shell contains a charge density given by $\rho_v = -\frac{\rho_{v_0}}{R^2}, a \leq R \leq b$, where ρ_{v_0} is a positive constant. Determine **D** in all regions.



per Gauss' law, $\vec{D} = \emptyset$ for $R < a$

for $b \geq R \geq a$ $\qquad dV' = 4\pi R^2 dR$

$$q = \int_{V'} \rho_v \, dV' = \int_a^R \left(-\frac{\rho_{v0}}{\cancel{R^2}}\right) 4\pi \cancel{R^2} \, dR$$

$$q = -4\rho_{v0}\pi (R)\Big|_a^R$$

$$q = -4\rho_{v0}\pi (R - a)$$

$$\vec{D} = \hat{R} \frac{q}{4\pi R^2}$$

$$= \hat{R} \frac{-\cancel{4}\rho_{v0}\cancel{\pi}(R-a)}{\cancel{4}\cancel{\pi}R^2}$$

$$\boxed{\vec{D} = -\frac{\hat{R}\,\rho_{v0}(R-a)}{R^2} \quad \text{for} \quad b \geq R \geq a}$$

for $R \geq b$

$$q = \int_{V'} \rho_v \, dV' = \int_a^b \left(-\frac{\rho_{v0}}{\cancel{R^2}}\right) 4\pi \cancel{R^2} \, dR$$

$$q = -4\rho_{v0}\pi (R)\Big|_a^b$$

$$q = -4\rho_{v0}\pi(b-a)$$

$$\boxed{\vec{D} = -\frac{\hat{R}\,\rho_{v0}(b-a)}{R^2} \quad \text{for} \quad R \geq b}$$