

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

11 Linked List Structures

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

February 7, 2022

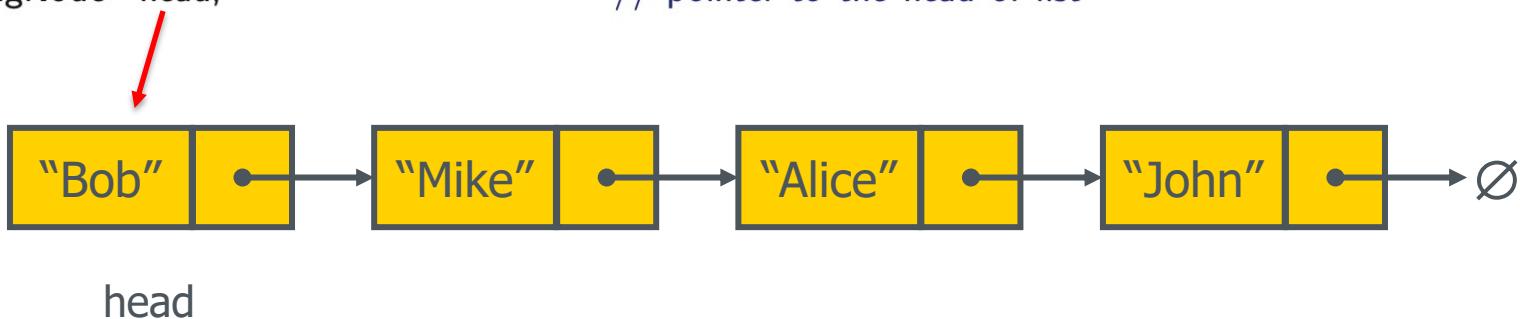
Administration

- The midterm will be in-person
 - Location: PGCLL B138
 - Date: Monday, February 14, 2022
 - Time: 1:30 PM - 3:30 PM
- The university policy is that we are back fully in person as of Feb 7th. If someone feels unsafe coming to campus to take a test, they can apply for an exemption for in person learning through student accessibility services, though these will only be approved if appropriate medical documentation is provided.
- I will make an announcement about the sources that you can study for the mid-term.

Review Singly Linked List - AddFront

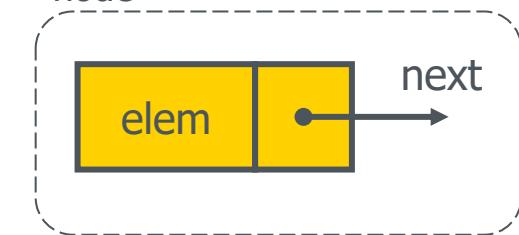
- For storing strings only!

```
class StringLinkedList {  
public:  
    StringLinkedList();  
    ~StringLinkedList();  
    bool empty() const;  
    const string& front() const;  
    void addFront(const string& e);  
    void removeFront();  
private:  
    StringNode* head;  
};
```



```
class StringNode {  
private:  
    string elem;  
    StringNode* next;  
  
friend class StringLinkedList;  
};  
  
// a linked list of strings  
  
// empty list constructor  
// destructor  
// is list empty?  
// get front element  
// add to front of list  
// remove front item list  
  
// pointer to the head of list
```

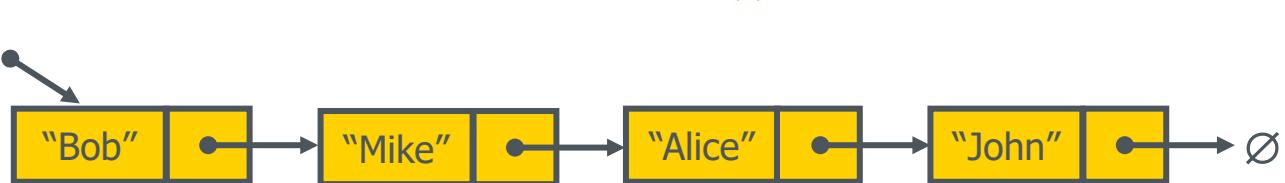
```
// a node in a list of strings  
// element value  
// next item in the list  
  
// provide StringLinkedList access  
node
```



Singly Linked List - addFront

- Insert element at the head of the singly linked list

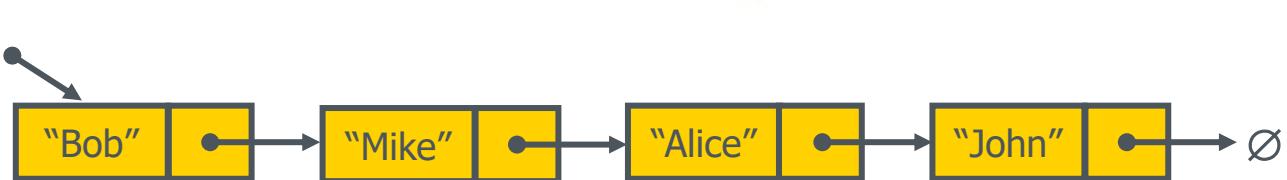
```
void StringLinkedList::addFront(const string& e) { // add to front of list
    StringNode* v = new StringNode; // create new node
    v->elem = e; // store data
    v->next = head; // head now follows v
    head = v; // v is now the head
}
```



Singly Linked List - addFront

- Insert element at the head of the singly linked list

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}
```

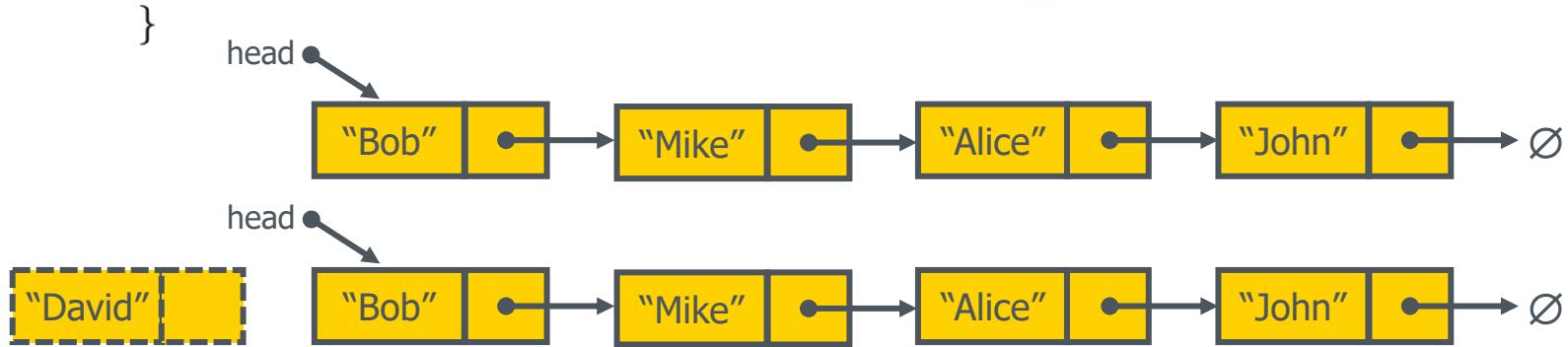


- Notice friendship!

Singly Linked List - addFront

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void StringLinkedList::addFront(const string& e) { // add to front of list
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}
```

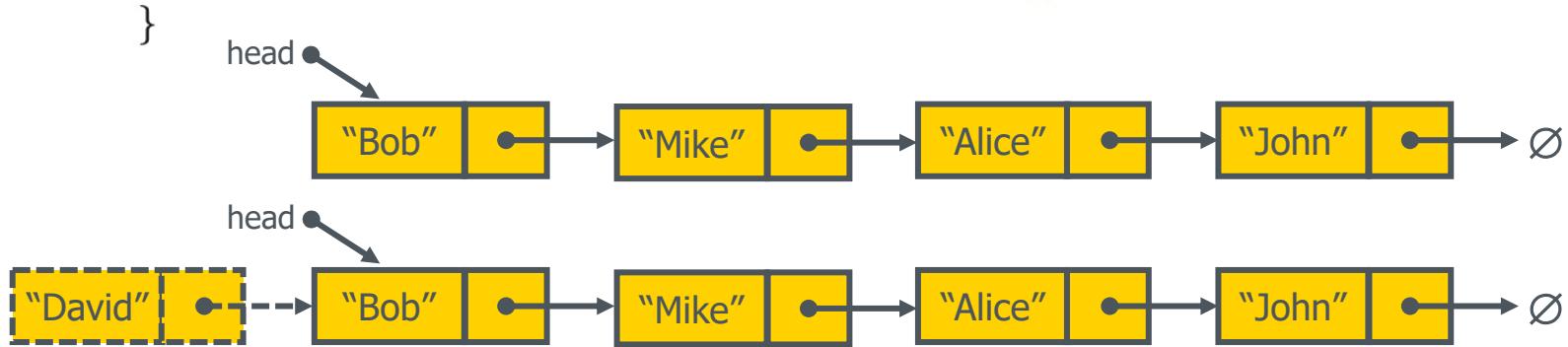


- Notice friendship!

Singly Linked List - addFront

- Insert element at the head of the singly linked list

```
void StringLinkedList::addFront(const string& e) { // add to front of list
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    v->elem = e; // store data
    v->next = head; // head now follows v
    head = v; // v is now the head
}
```

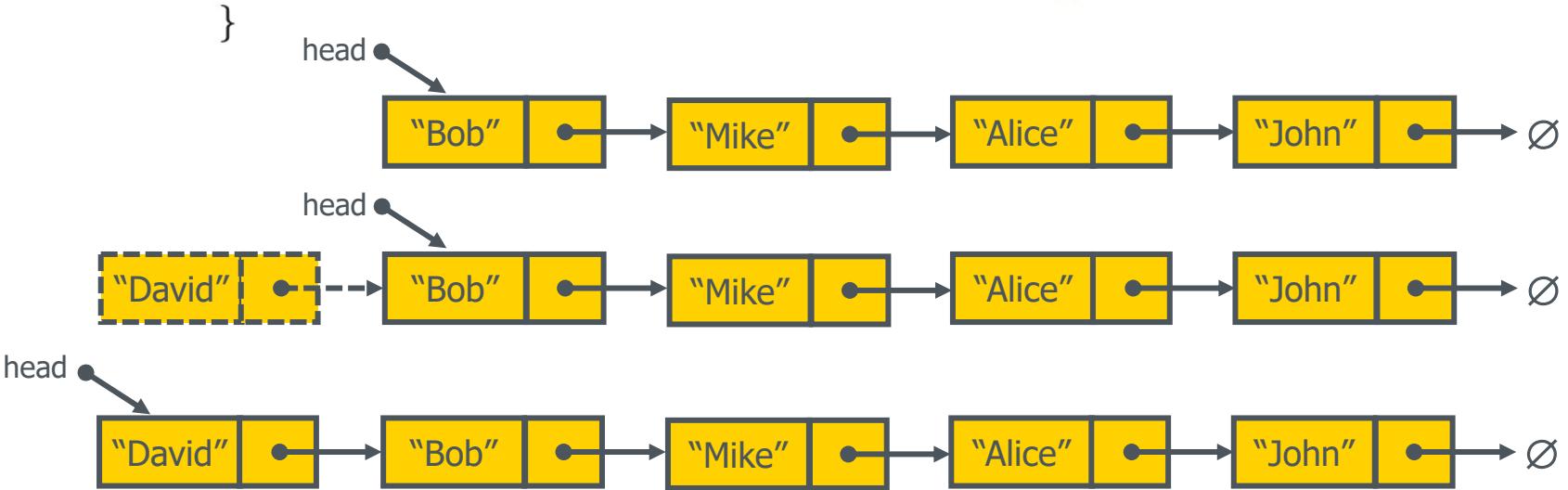


- Notice friendship!

Singly Linked List - addFront

- Insert element at the head of the singly linked list

```
void StringLinkedList::addFront(const string& e) { // add to front of list
    StringNode* v = new StringNode; // create new node
    v->elem = e; // store data
    v->next = head; // head now follows v
    head = v; // v is now the head
}
```

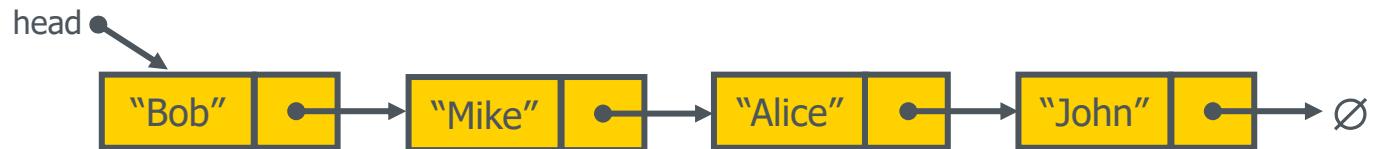


- Notice friendship!

Singly Linked List - removeFront

- Remove an element from the head of the singly linked list

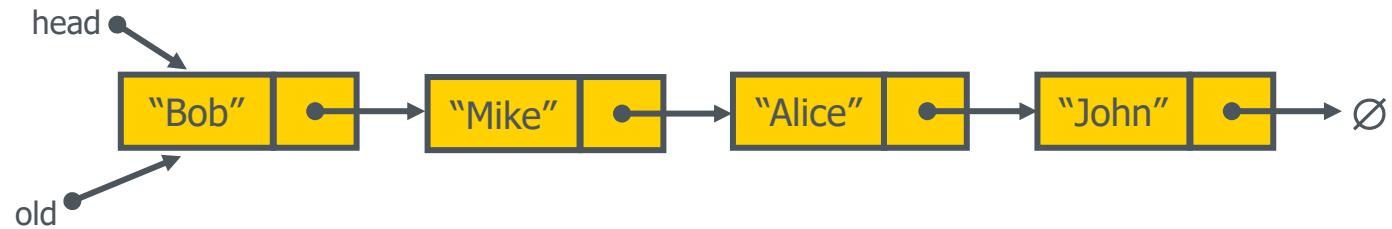
```
void StringLinkedList::removeFront() {           // remove front item
    StringNode* old = head;                      // save current head
    head = old->next;                           // skip over old head
    delete old;                                  // delete the old head
}
```



Singly Linked List - removeFront

- Remove an element from the head of the singly linked list

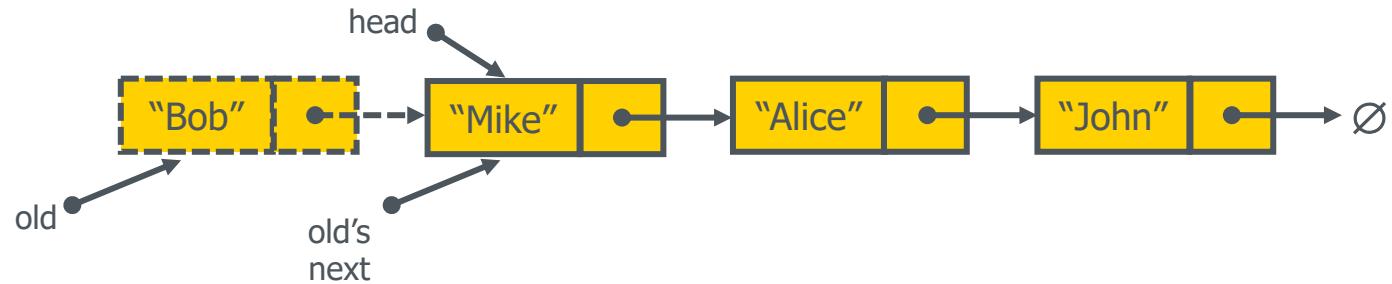
```
void StringLinkedList::removeFront() {  
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    head = old->next; // skip over old head  
    delete old; // delete the old head  
}
```



Singly Linked List - removeFront

- Remove an element from the head of the singly linked list

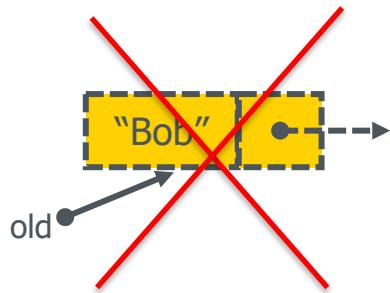
```
void StringLinkedList::removeFront() {  
    StringNode* old = head; // save current head  
    head = old->next; // skip over old head  
    delete old; // delete the old head  
}
```



Singly Linked List - removeFront

- Remove an element from the head of the singly linked list

```
void StringLinkedList::removeFront() {           // remove front item
    StringNode* old = head;                     // save current head
    head = old->next;                          // skip over old head
    delete old;                                // delete the old head
}
```

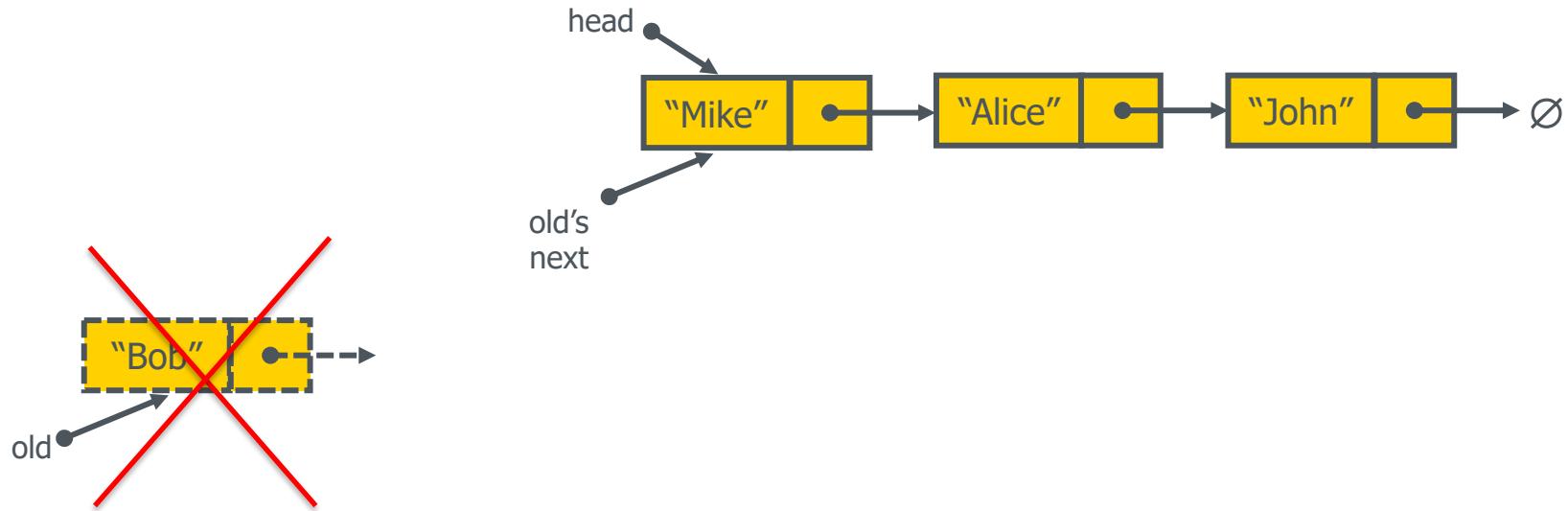


- Avoid memory leak!

Singly Linked List - removeFront

- Remove an element from the head of the singly linked list

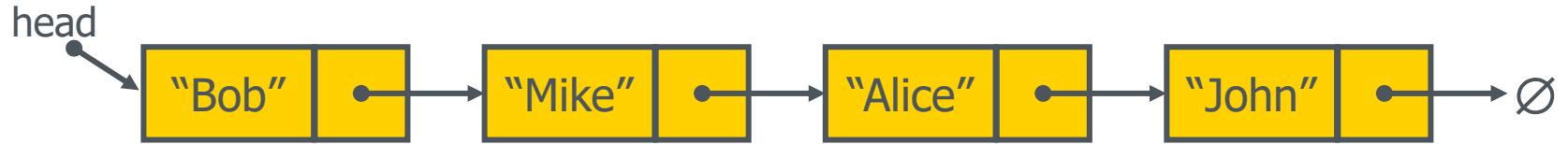
```
void StringLinkedList::removeFront() {  
    StringNode* old = head; // save current head  
    head = old->next; // skip over old head  
    delete old; // delete the old head  
}
```



- Avoid memory leak!

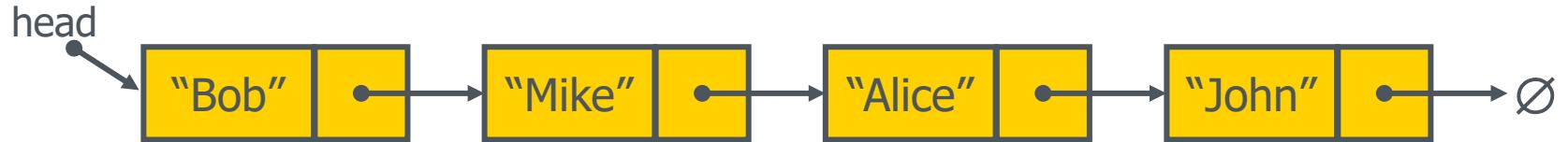
Other Operations on Singly Linked Lists

- How to insert at the tail?

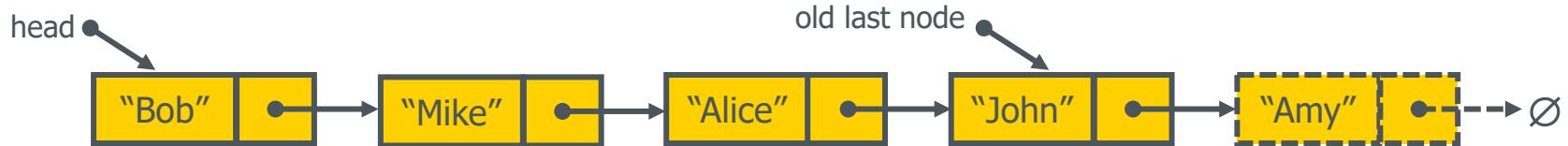


Other Operations on Singly Linked Lists

- How to insert at the tail?



- Allocate a new node
- Insert new element
- Have new node point to Null
- Traverse to find the old last node (how?) and have it to point to new node



Generic Singly Linked List

- We assumed elements were strings, now we want arbitrary element types
- It is easy using C++'s **Template** mechanism

```
class StringNode {           // a node in a list of strings
private:
    string elem;           // element value
    StringNode* next;      // next item in the list

    friend class StringLinkedList; // provide StringLinkedList access
};
```

Before

```
template <typename E>
class SNode {               // singly linked list node
private:
    E elem;                // linked list element value
    SNode<E>* next;        // next item in the list
    friend class SLinkedList<E>; // provide SLinkedList access
};
```

After

Generic Singly Linked List

```
class StringLinkedList {                                // a linked list of strings
public:
    StringLinkedList();                            // empty list constructor
    ~StringLinkedList();                          // destructor
    bool empty() const;                         // is list empty?
    const string& front() const;                // get front element
    void addFront(const string& e);             // add to front of list
    void removeFront();                         // remove front item list
private:
    StringNode* head;                           // pointer to the head of list
};
```

Before

```
template <typename E>
class SLinkedList {                                // a singly linked list
public:
    SLinkedList();                            // empty list constructor
    ~SLinkedList();                          // destructor
    bool empty() const;                     // is list empty?
    const E& front() const;                 // return front element
    void addFront(const E& e);              // add to front of list
    void removeFront();                    // remove front item list
private:
    SNode<E>* head;                      // head of the list
};
```

After

Generic Singly Linked List

Before

```
StringLinkedList::StringLinkedList()  
: head(NULL) { }
```

```
bool StringLinkedList::empty() const  
{ return head == NULL; }
```

```
const string& StringLinkedList::front() const  
{ return head->elem; }
```

```
StringLinkedList::~StringLinkedList()  
{ while (!empty()) removeFront(); }
```

```
void StringLinkedList::addFront(const string& e) {  
    SNode* v = new SNode;  
    v->elem = e;  
    v->next = head;  
    head = v;  
}
```

```
void StringLinkedList::removeFront() {  
    SNode* old = head;  
    head = old->next;  
    delete old;  
}
```

After

```
template <typename E>  
SLinkedList<E>::SLinkedList()  
: head(NULL) { }
```

```
template <typename E>  
bool SLinkedList<E>::empty() const  
{ return head == NULL; }
```

```
template <typename E>  
const E& SLinkedList<E>::front() const  
{ return head->elem; }
```

```
template <typename E>  
SLinkedList<E>::~SLinkedList()  
{ while (!empty()) removeFront(); }
```

```
template <typename E>  
void SLinkedList<E>::addFront(const E& e) {  
    SNode<E>* v = new SNode<E>;  
    v->elem = e;  
    v->next = head;  
    head = v;  
}
```

```
template <typename E>  
void SLinkedList<E>::removeFront() {  
    SNode<E>* old = head;  
    head = old->next;  
    delete old;  
}
```

Usage

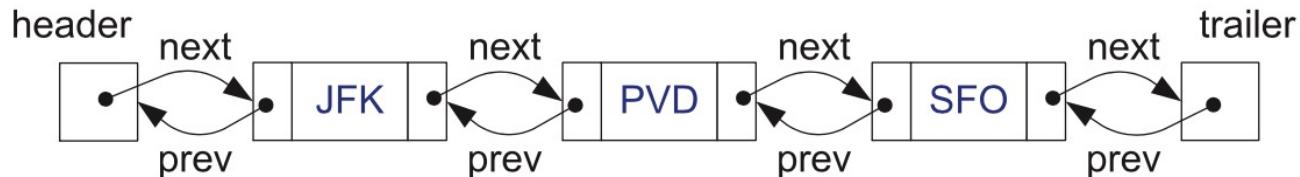
```
SLinkedList<string> a; // list of strings  
a.addFront("MSP");  
// ...  
SLinkedList<int> b; // list of integers  
b.addFront(13);
```

Limitations of Singly Linked Lists

- Not easy to remove an element at the tail (or any other node)
 - We don't have a quick way to access to the node immediately preceding the one we want to delete!
 - Some operations need the tree traversal from beginning to the end



- A better Data Structure
 - Doubly Linked List
 - Offers traversing in both directions
 - Quick update operations (insert or remove at any given position)

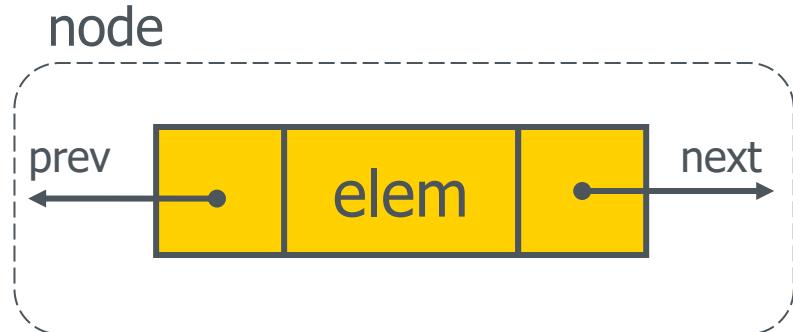


Administration

- The first mid-term's material will be up to this slide!

Doubly Linked Lists

- Allows traversing in both directions (forward and reverse)
- Each node stores
 - element
 - link to the next node
 - link to the previous node
- Sentinel nodes
 - Dummy **header** node
 - Dummy **trailer** node



\emptyset



Doubly Linked List C++ Classes Declaration

- Notice `typedef`!
 - Generic, like templates

```
class DLinkedList {  
public:  
    DLinkedList();  
    ~DLinkedList();  
    bool empty() const;  
    const Elem& front() const;  
    const Elem& back() const;  
    void addFront(const Elem& e);  
    void addBack(const Elem& e);  
    void removeFront();  
    void removeBack();  
private:  
    DNode* header;  
    DNode* trailer;  
protected:  
    void add(DNode* v, const Elem& e);  
    void remove(DNode* v);  
};
```

```
typedef string Elem;  
class DNode {  
private:  
    Elem elem;  
    DNode* prev;  
    DNode* next;  
    friend class DLinkedList;  
};  
// doubly linked list  
  
// constructor  
// destructor  
// is list empty?  
// get front element  
// get back element  
// add to front of list  
// add to back of list  
// remove from front  
// remove from back  
// local type definitions  
// list sentinels  
  
// local utilities  
// insert new node before v  
// remove node v
```

// list element type
// doubly linked list node

// node element value
// previous node in list
// next node in list
// allow DLinkedList access



Doubly Linked List Definitions

- Constructor
- isEmpty?
 - header and trailer pointing each other
- Return front and back elements
- Dynamic memory allocation
 - We need destructor
- Destructor
 - remove nodes until list is empty

```
DLinkedList::DLinkedList() {  
    header = new DNode;  
    trailer = new DNode;  
    header->next = trailer;  
    trailer->prev = header;  
}  
  
// constructor  
// create sentinels  
  
// have them point to each other  
  
bool DLinkedList::empty() const // is list empty?  
{ return (header->next == trailer); }  
  
const Elem& DLinkedList::front() const // get front element  
{ return header->next->elem; }  
  
const Elem& DLinkedList::back() const // get back element  
{ return trailer->prev->elem; }  
  
DLinkedList::~DLinkedList() {  
    while (!empty()) removeFront();  
    delete header;  
    delete trailer;  
}  
  
// destructor  
// remove all but sentinels  
// remove the sentinels
```



Doubly Linked List - Add Element

- add() is protected
 - Utility function, why?

```
// insert new node before v
void DLinkedList::add(DNode* v, const Elem& e) {
    DNode* u = new DNode; u->elem = e; // create a new node for e
    u->next = v; // link u in between v
    u->prev = v->prev; // ...and v->prev
    v->prev->next = v->prev = u;
}

void DLinkedList::addFront(const Elem& e) // add to front of list
{ add(header->next, e); }

void DLinkedList::addBack(const Elem& e) // add to back of list
{ add(trailer, e); }
```



Doubly Linked List - Remove Element

- `remove()` is protected
 - Utility function

```
void DLinkedList::remove(DNode* v) {           // remove node v
    DNode* u = v->prev;                      // predecessor
    DNode* w = v->next;                      // successor
    u->next = w;
    w->prev = u;
    delete v;                                // unlink v from list
}

void DLinkedList::removeFront()                // remove from front
{ remove(header->next); }

void DLinkedList::removeBack()                 // remove from back
{ remove(trailer->prev); }
```



Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

12 Linked Lists - Continued

Department of Computing and Software

Instructor:

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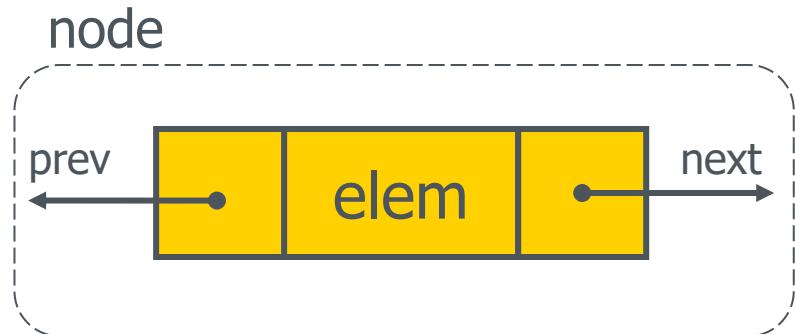
February 9, 2022

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- Read the announcement about resources that you can study for the mid-term.
- My Office Hour:
 - Today at 15:00 in **ITB-159** in-person (or virtually using teams as usual)

Doubly Linked Lists

- Allows traversing in both directions (forward and reverse)
- Each node stores
 - element
 - link to the next node
 - link to the previous node
- Sentinel nodes
 - Dummy **header** node
 - Dummy **trailer** node



\emptyset



Doubly Linked List C++ Classes Declaration

- Notice `typedef`!
 - Generic, like templates

```
class DLinkedList {  
public:  
    DLinkedList();  
    ~DLinkedList();  
    bool empty() const;  
    const Elem& front() const;  
    const Elem& back() const;  
    void addFront(const Elem& e);  
    void addBack(const Elem& e);  
    void removeFront();  
    void removeBack();  
private:  
    DNode* header;  
    DNode* trailer;  
protected:  
    void add(DNode* v, const Elem& e);  
    void remove(DNode* v);  
};
```

```
typedef string Elem;  
class DNode {  
private:  
    Elem elem;  
    DNode* prev;  
    DNode* next;  
    friend class DLinkedList;  
};  
// doubly linked list  
  
// constructor  
// destructor  
// is list empty?  
// get front element  
// get back element  
// add to front of list  
// add to back of list  
// remove from front  
// remove from back  
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// local utilities  
// insert new node before v  
// remove node v
```

// list element type
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Doubly Linked List Definitions

- Constructor
- isEmpty?
 - header and trailer pointing each other
- Return front and back elements
- Dynamic memory allocation
 - We need destructor
- Destructor
 - remove nodes until list is empty

```
DLinkedList::DLinkedList() {  
    header = new DNode;  
    trailer = new DNode;  
    header->next = trailer;  
    trailer->prev = header;  
}  
  
// constructor  
// create sentinels  
  
// have them point to each other  
  
bool DLinkedList::empty() const // is list empty?  
{ return (header->next == trailer); }  
  
const Elem& DLinkedList::front() const // get front element  
{ return header->next->elem; }  
  
const Elem& DLinkedList::back() const // get back element  
{ return trailer->prev->elem; }  
  
DLinkedList::~DLinkedList() {  
    while (!empty()) removeFront();  
    delete header;  
    delete trailer;  
}  
  
// destructor  
// remove all but sentinels  
// remove the sentinels
```



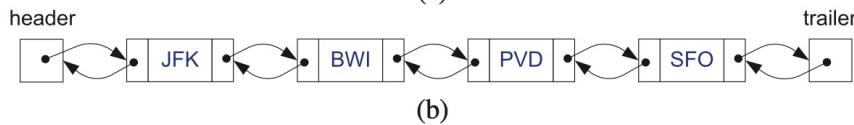
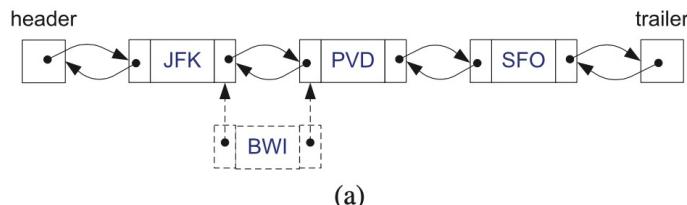
Doubly Linked List - Add Element

- add() is protected
 - Utility function, why?

```
// insert new node before v
void DLinkedList::add(DNode* v, const Elem& e) {
    DNode* u = new DNode; u->elem = e; // create a new node for e
    u->next = v; // link u in between v
    u->prev = v->prev; // ...and v->prev
    v->prev->next = v->prev = u;
}
```

```
void DLinkedList::addFront(const Elem& e) // add to front of list
{ add(header->next, e); }
```

```
void DLinkedList::addBack(const Elem& e) // add to back of list
{ add(trailer, e); }
```



Doubly Linked List - Remove Element

- remove() is protected
 - Utility function

```
void DLinkedList::remove(DNode* v) {  
    DNode* u = v->prev;  
    DNode* w = v->next;  
    u->next = w;  
    w->prev = u;  
    delete v;  
}
```

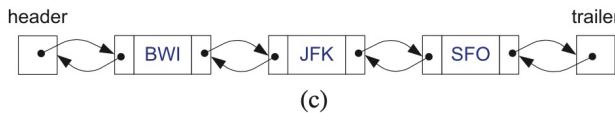
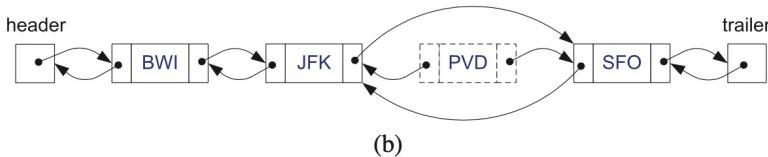
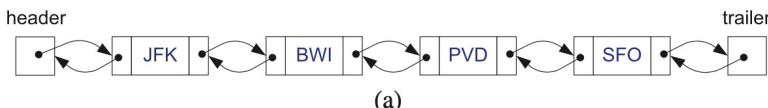
// remove node v
// predecessor
// successor
// unlink v from list

```
void DLinkedList::removeFront()  
{ remove(header->next); }
```

// remove from front

```
void DLinkedList::removeBack()  
{ remove(trailer->prev); }
```

// remove from back



Doubly Linked List - Reverse

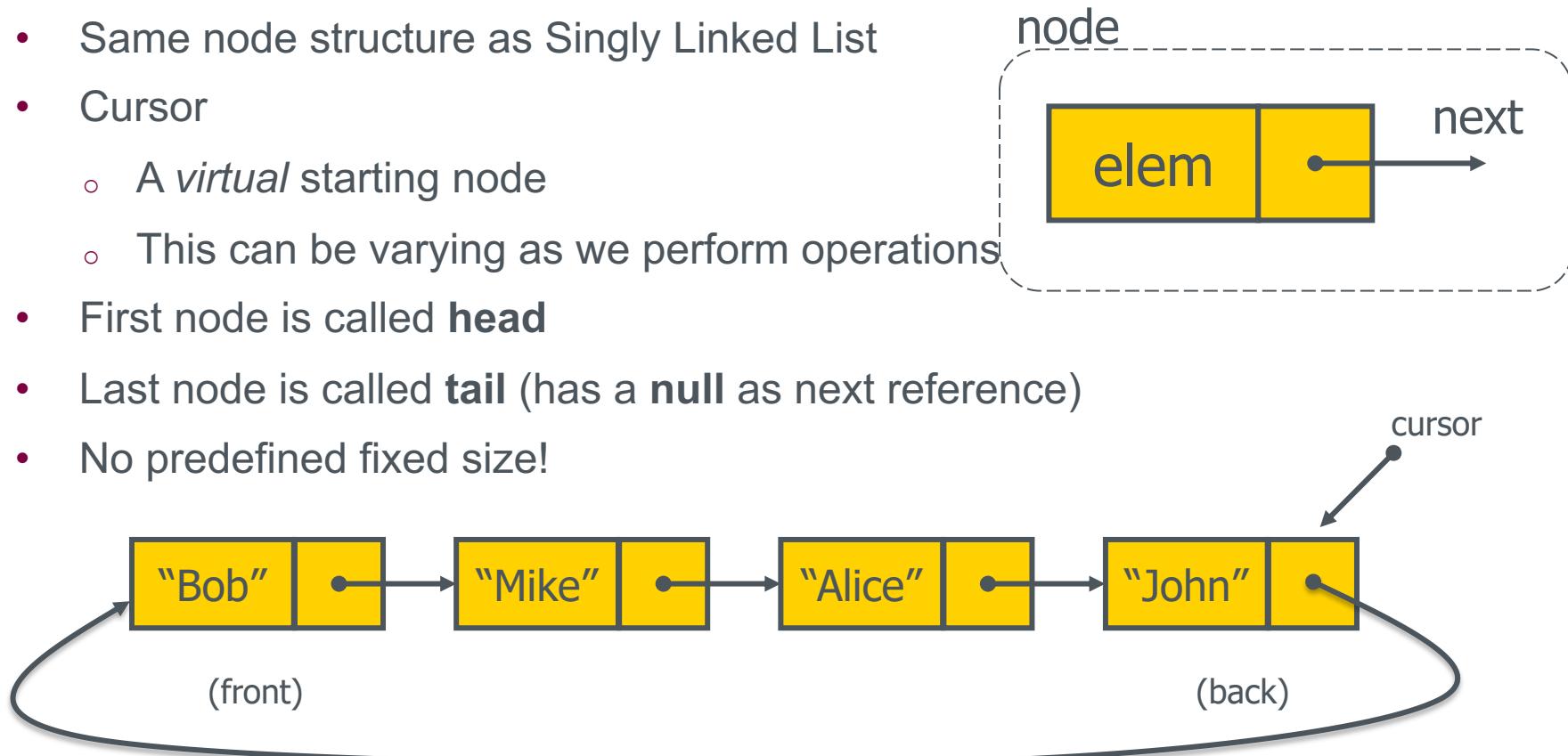
- This is not a function declared in the class
- See the tutorial of week 4 for a possible implementation in the class

```
void listReverse(DLinkedList& L) {           // reverse a list
    DLinkedList T;                          // temporary list
    while (!L.empty()) {                   // reverse L into T
        string s = L.front();  L.removeFront();
        T.addFront(s);
    }
    while (!T.empty()) {                   // copy T back to L
        string s = T.front();  T.removeFront();
        L.addBack(s);
    }
}
```



Circularly Linked List

- A variant of Singly Linked List
 - Rather than having a **head** or a **tail**, it forms a cycle
- Same node structure as Singly Linked List
- Cursor
 - A *virtual* starting node
 - This can be varying as we perform operations
- First node is called **head**
- Last node is called **tail** (has a **null** as next reference)
- No predefined fixed size!



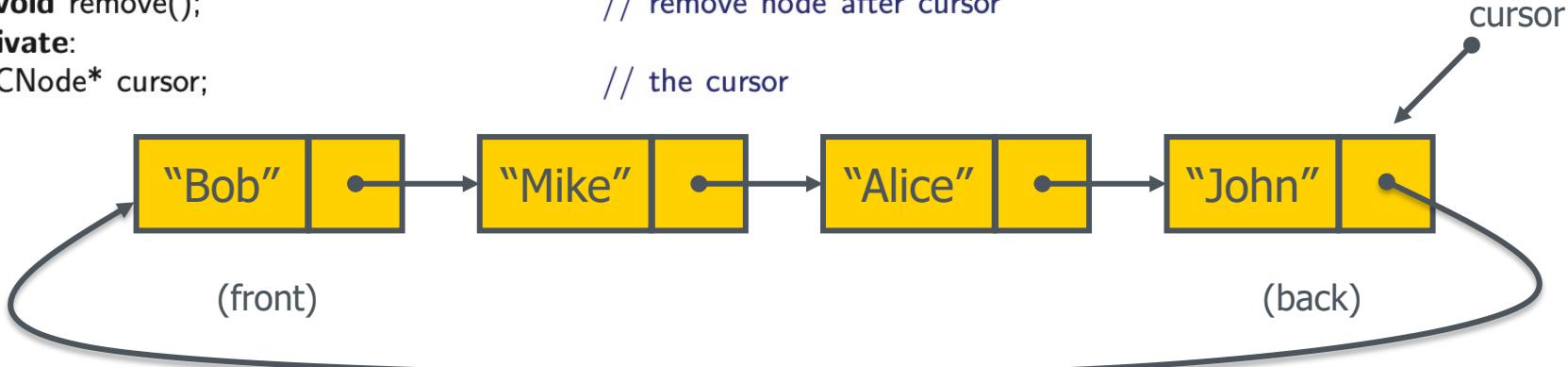
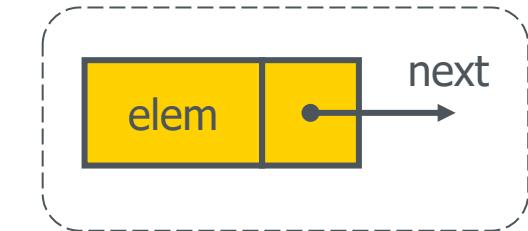
Circularly Linked List C++ Classes Declaration

- Notice `typedef`!
 - Generic, like templates

```
class CircleList {  
public:  
    CircleList();  
    ~CircleList();  
    bool empty() const;  
    const Elem& front() const;  
    const Elem& back() const;  
    void advance();  
    void add(const Elem& e);  
    void remove();  
private:  
    CNode* cursor;  
};
```

```
typedef string Elem;  
class CNode {  
private:  
    Elem elem;  
    CNode* next;  
  
    friend class CircleList;  
};  
// a circularly linked list  
  
// constructor  
// destructor  
// is list empty?  
// element at cursor  
// element following cursor  
// advance cursor  
// add after cursor  
// remove node after cursor  
  
// the cursor
```

```
// element type  
// circularly linked list node  
  
// linked list element value  
// next item in the list  
  
// provide CircleList access  
node
```



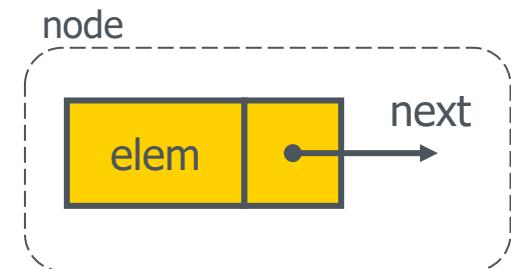
Circular Linked List Definitions

- Constructor
 - Set cursor to Null
- Destructor
 - remove nodes until list is empty



```
CircleList::CircleList()  
: cursor(NULL) { }  
CircleList::~CircleList()  
{ while (!empty()) remove(); }
```

// constructor
// destructor



Circular Linked List Definitions

- Constructor
 - Set cursor to Null
- Destructor
 - remove nodes until list is empty
- is Empty?
 - check if cursor is Null
- Return **front** element
 - return element immediately after cursor
- Return **back** element
 - return element referenced by cursor
- **advance():**
 - advances the cursor to the next node

```
CircleList::CircleList()
    : cursor(NULL) { }

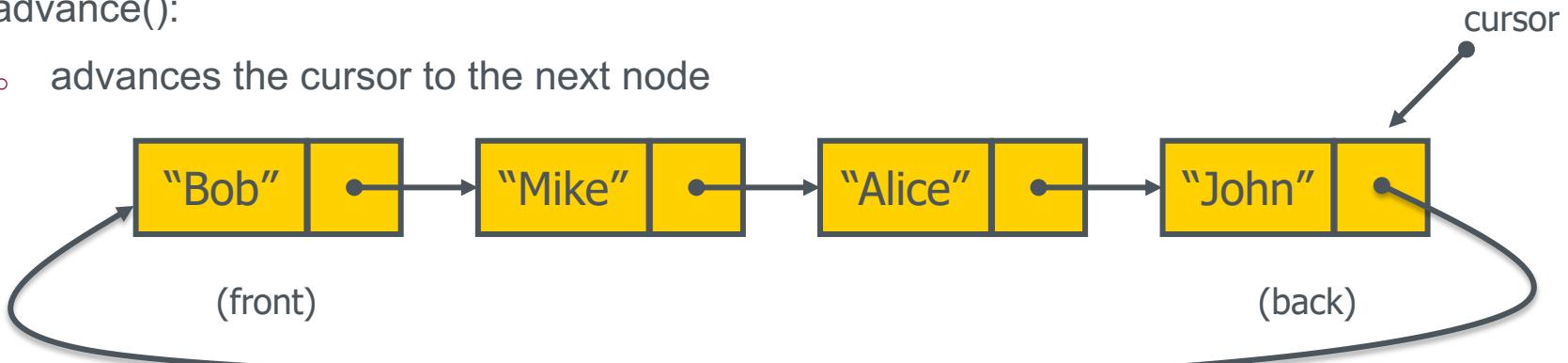
CircleList::~CircleList()
{ while (!empty()) remove(); }

bool CircleList::empty() const
{ return cursor == NULL; }

const Elem& CircleList::back() const
{ return cursor->elem; }

const Elem& CircleList::front() const
{ return cursor->next->elem; }

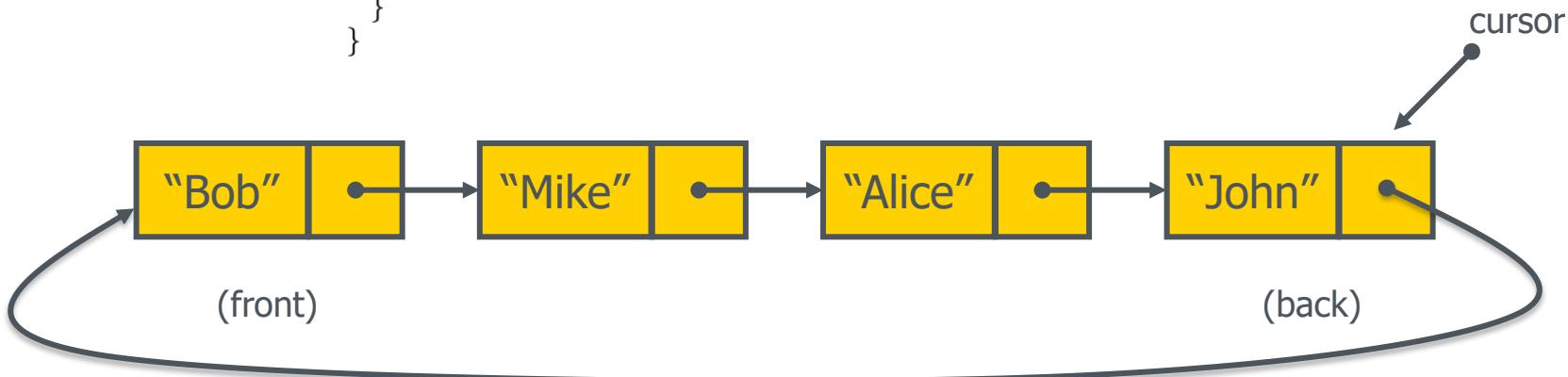
void CircleList::advance()
{ cursor = cursor->next; }
```



Circular Linked List - add

- add an element
 - Insert a new node with element **e** immediately after the cursor; if the list is empty, then this node becomes the cursor and its next pointer points to itself.

```
void CircleList::add(const Elem& e) {           // add after cursor
    CNode* v = new CNode;                         // create a new node
    v->elem = e;
    if (cursor == NULL) {                          // list is empty?
        v->next = v;                            // v points to itself
        cursor = v;                            // cursor points to v
    }
    else {                                         // list is nonempty?
        v->next = cursor->next;                  // link in v after cursor
        cursor->next = v;
    }
}
```



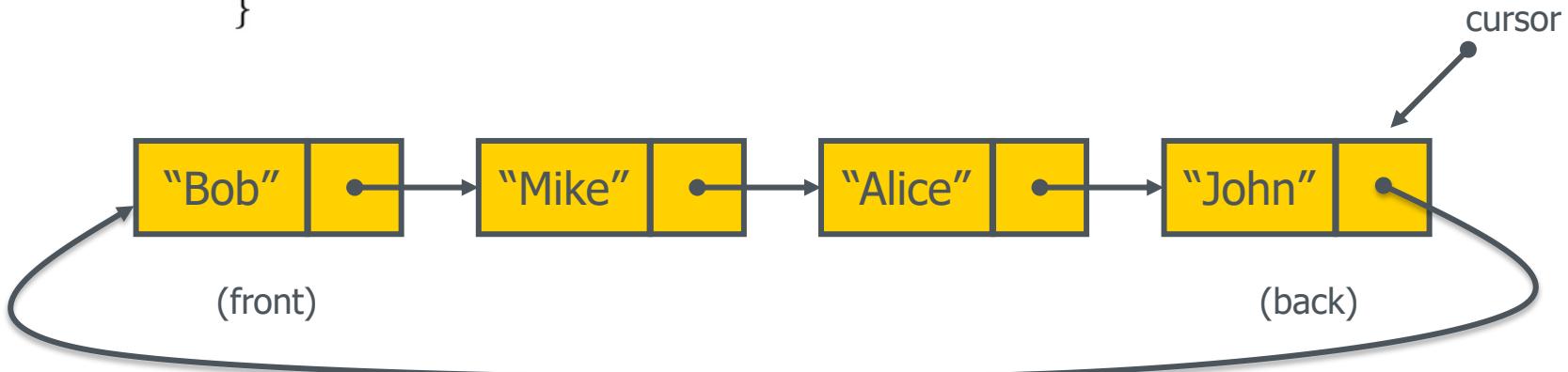
Circular Linked List - remove

- remove the element after cursor
 - Remove the node immediately after the cursor (not the cursor itself, unless it is the only node); if the list becomes empty, the cursor is set to null.

```
void CircleList::remove() {  
    CNode* old = cursor->next;  
    if (old == cursor)  
        cursor = NULL;  
    else  
        cursor->next = old->next;  
    delete old;  
}
```

// remove node after cursor
// the node being removed
// removing the only node?
// list is now empty

// link out the old node
// delete the old node



Circularly Linked List - An Example

- A music playlist!

```
int main() {
    CircleList playList;           // []
    playList.add("Stayin Alive");   // [Stayin Alive*]
    playList.add("Le Freak");      // [Le Freak, Stayin Alive*]
    playList.add("Jive Talkin");    // [Jive Talkin, Le Freak, Stayin Alive*]

    playList.advance();           // [Le Freak, Stayin Alive, Jive Talkin*]
    playList.advance();           // [Stayin Alive, Jive Talkin, Le Freak*]
    playList.remove();            // [Jive Talkin, Le Freak*]
    playList.add("Disco Inferno"); // [Disco Inferno, Jive Talkin, Le Freak*]
    return EXIT_SUCCESS;
}
```

Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

13 Recursion

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

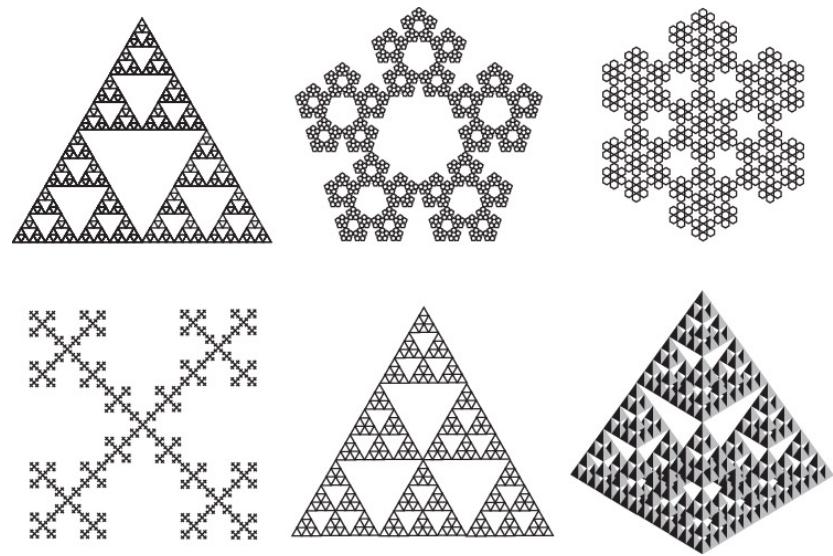
February 10, 2022

Administration

- Today's lecture is shorter
 - I will stay until 2:30 to answer questions of students who can not attend my office hour.

What is Recursion

- **Recursion** is the concept of defining a function that makes a call to itself.
- We call a function **Recursive** if it calls itself.
- When a function calls itself, we refer to this as a **Recursive Call**.
 - if function M calls another function that ultimately leads to a call back to M, this is also a recursive call and function M is recursive.
- A lot of use-cases in real-life:
 - Nature : fractals
 - Mathematics: recursive functions



Recursive Algorithms

- The idea is to avoid loops!
- A recursive implementation can be significantly simpler and easier to understand than an iterative implementation.
- Types of Recursion:
 - Linear Recursion
 - Binary Recursion
 - Tail Recursion
 - Multiple Recursion

Recursive Algorithms - Example

- Factorial of a whole number n is defined as:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 & \text{if } n \geq 1. \end{cases}$$

- \circ $\text{factorial}(4) = 4 * (3 * 2 * 1) = 4 * \text{factorial}(3)$
- \circ $\text{factorial}(4)$ can be defined in terms of $\text{factorial}(3)$
- Recursive definition of the factorial function is:

$$\text{factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{factorial}(n-1) & \text{if } n \geq 1. \end{cases}$$

- \circ base case
 - \bullet are non-recursive
- \circ recursive case
 - \bullet no circularity (function terminates)

Recursive Algorithms - Example

- Iterative Factorial

```
int factorial_iter(int n){  
    int factorial = 1;  
    for (int i = 2; i <= n; i++){  
        factorial = factorial * i;  
    }  
    return factorial;  
}
```

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 & \text{if } n \geq 1. \end{cases}$$

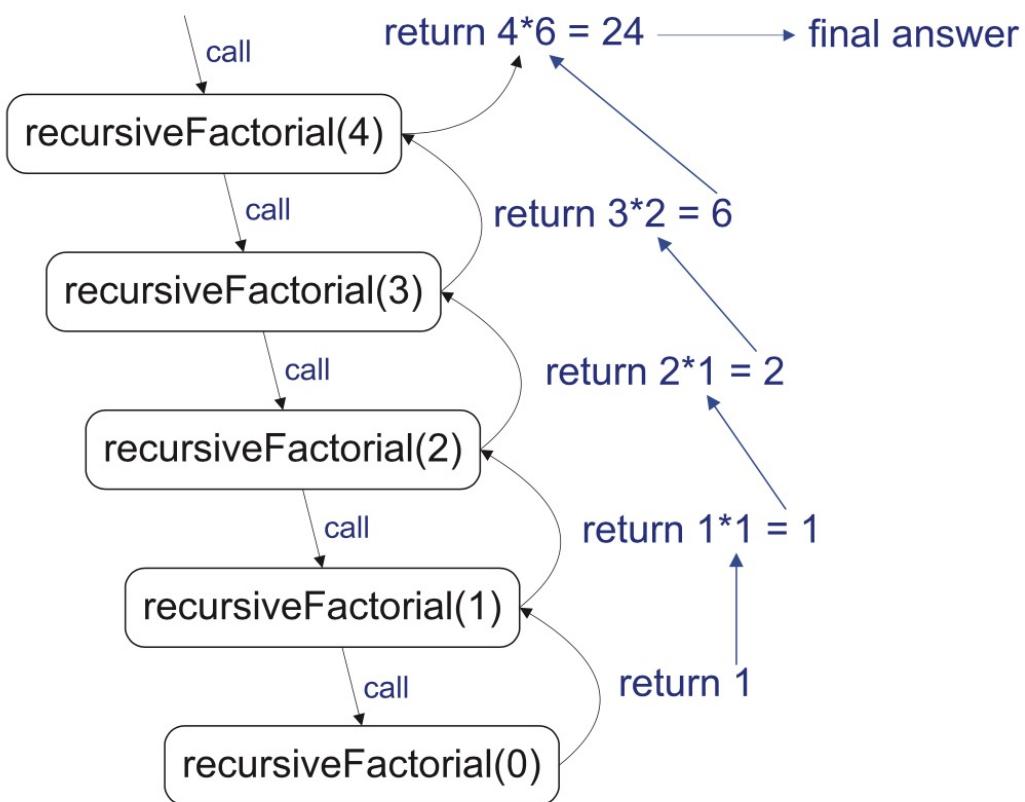
- Recursive Factorial:

```
int factorial_rec(int n){  
    if (n==0){  
        return 1;  
    }  
    else{  
        return n * factorial_rec(n - 1);  
    }  
}
```

$$\text{factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{factorial}(n-1) & \text{if } n \geq 1. \end{cases}$$

Recursive Algorithms - Example

- Recursive Factorial
- **Recursion Trace:**



```
int factorial_rec(int n){  
    if (n==0){  
        return 1;  
    }  
    else{  
        return n * factorial_rec(n - 1);  
    }  
}
```

$$\text{factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{factorial}(n - 1) & \text{if } n \geq 1. \end{cases}$$

Linear Recursion

- The simplest form of recursion
- function makes at most one recursive call each time it is invoked
- When we have a sequence, we can view some problems in terms of:
 - a first or last element
 - a remaining sequence that has the same structure as the original sequence

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```
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    return LinearSum( $A, n - 1$ ) +  $A[n - 1]$ 
```

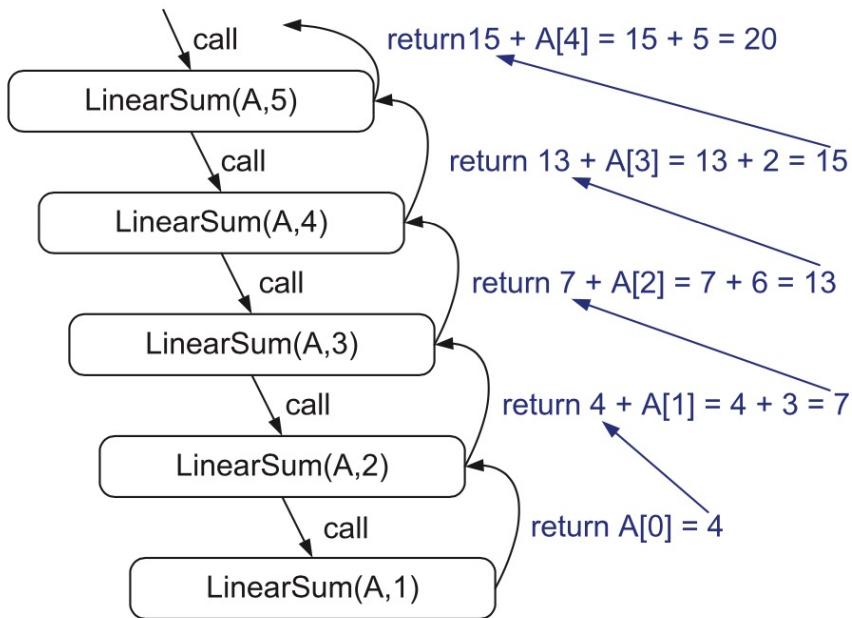
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```



```
int linearSum_iter(int data[], int n){  
    int sum = 0;  
    for (int i = 0; i < n; i++){  
        sum += data[i];  
    }  
    return sum;  
}
```

```
int linearSum(int data[], int n){  
    if (n <= 0)  
        return 0;  
    return (linearSum(data, n - 1) + data[n - 1]);  
}
```

Recursion in Computer Science

- Recursion is heavily used in Functional Programming
 - In Pure Functional Languages it is the only way to iterate!
 - Like Haskell
 - Non-pure programming languages allow iteration
 - Like Scala
- Recursion has overheads in Imperative Programming Languages
 - Like in C++ and Java
- Pure Functional Language use tail recursion conversions to reduce significant impact on memory consumption of heavy recursive calls
- It has many applications in Big Data Tools
 - Laziness!

Questions?

Those who can not attend the office hours can stay and ask
questions for mid-term

If you don't have questions, feel free to leave ☺

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

14 Recursion - continued

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

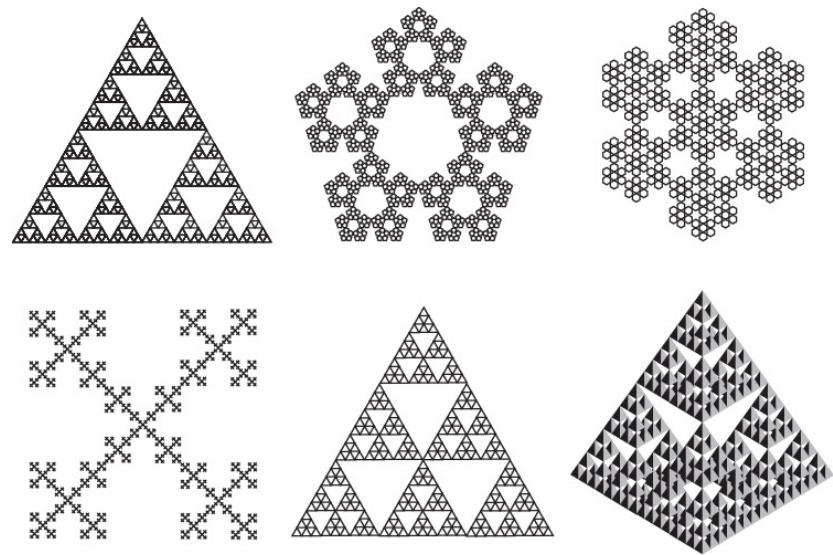
February 16, 2022

Administration

- I will return assignments today! finally!
 - Sorry about the delay
 - TAs did great, I delayed
- My Office Hour:
 - Today at 15:00 in **ITB-159** in-person (or virtually using teams as usual)
- We review the recursion quickly

What is Recursion

- **Recursion** is the concept of defining a function that makes a call to itself.
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    }  
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}
```

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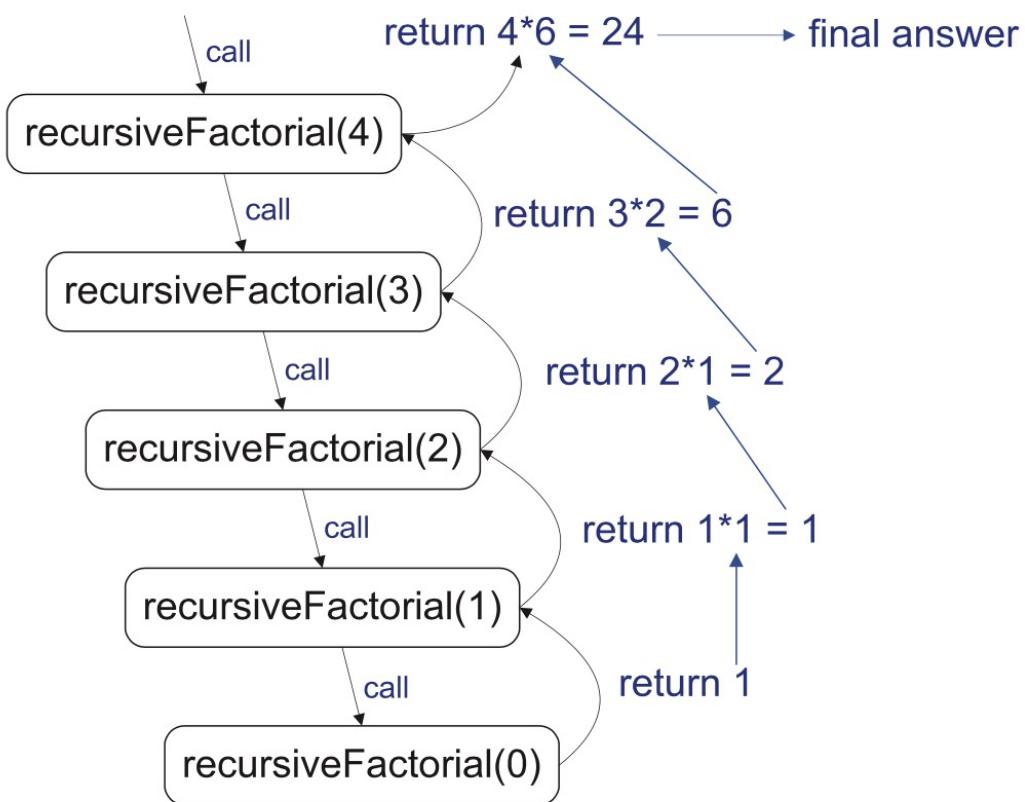
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```
int factorial_rec(int n){  
    if (n==0){  
        return 1;  
    }  
    else{  
        return n * factorial_rec(n - 1);  
    }  
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```

$$\text{factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{factorial}(n-1) & \text{if } n \geq 1. \end{cases}$$

Recursive Algorithms - Example

- Recursive Factorial
- **Recursion Trace:**



```
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    if (n==0){  
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    }  
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Algorithm LinearSum(A, n):

Input: A integer array A and an integer $n \geq 1$, such that A has at least n elements

Output: The sum of the first n integers in A

```
if  $n = 1$  then
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else
    return LinearSum( $A, n - 1$ ) +  $A[n - 1]$ 
```

Linear Recursion

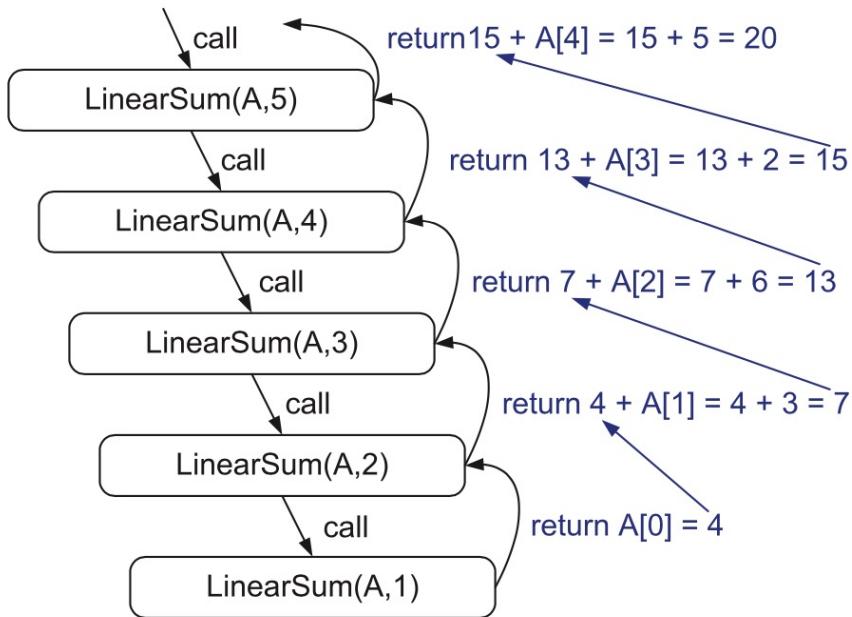
$$A = \{4, 3, 6, 2, 5\}$$

Algorithm LinearSum(A, n):

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```
int linearSum_iter(int data[], int n){  
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 - Laziness!

Linear Recursion - Reversing an Array

- Reversing the n elements of an array
- Reversal: by swapping the first and last elements and then recursively reversing the remaining elements in the array.
 - Note how subproblems are defined! (initial call ReverseArray(A, 0, n-1)
 - other than array, we have two other index inputs
- Base cases?

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

 Swap $A[i]$ and $A[j]$

 ReverseArray($A, i + 1, j - 1$)

return

$$A = \{4,3,6,2,5\}$$

ReverseArray($A, 0, 4$)

$$A = \{5,3,6,2,4\}$$

call

ReverseArray($A, 1, 3$)

$$A = \{5,2,6,3,4\}$$

call

ReverseArray($A, 2, 2$)

$$A = \{5,2,6,3,4\}$$

Linear Recursion - Reversing an Array

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 - if $i = j$ or $i > j$
 - in either of cases algorithm terminates

Algorithm `ReverseArray(A, i, j)`:

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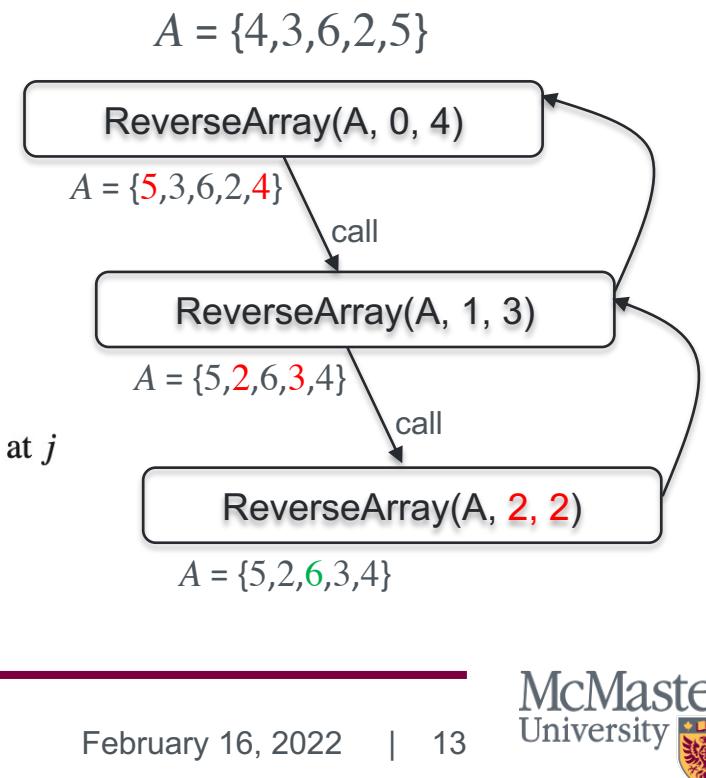
Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

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- Does algorithm terminate?

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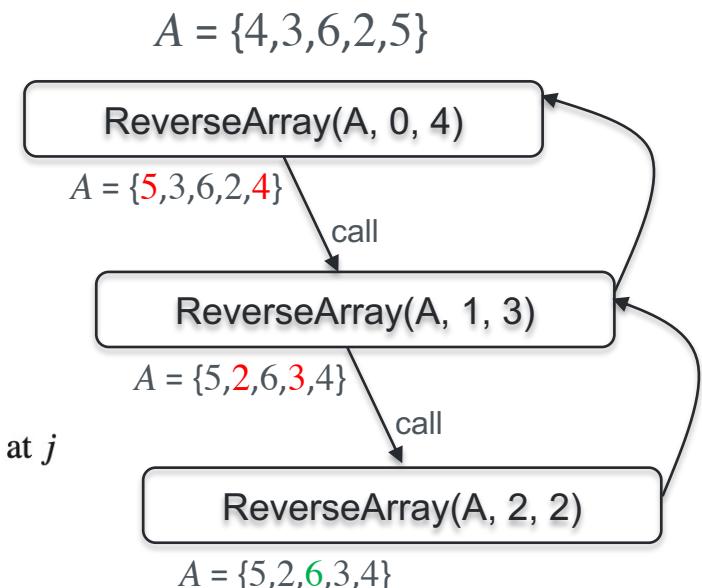
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if $i < j$ **then**

 Swap $A[i]$ and $A[j]$

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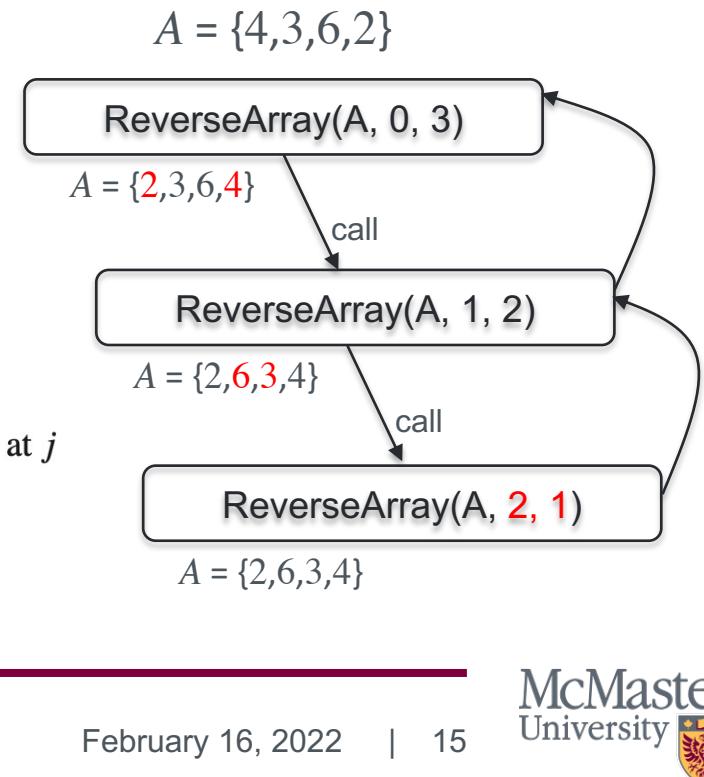
Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

 Swap $A[i]$ and $A[j]$

`ReverseArray(A, i + 1, j - 1)`

return



Tail Recursion

- Recursive calls are costly
 - Stack memory keeps track of the state of each active recursive call
- We can convert some recursive algorithms into non-recursive version
 - Suitable for imperative languages like C++
- There are two conditions:
 - The recursion is Linear
 - The recursive call is the last thing that function does

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

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if $i < j$ **then**

 Swap $A[i]$ and $A[j]$

 ReverseArray($A, i + 1, j - 1$)

Recursive call is the last thing!

return

Tail Recursion

- Recursive calls are costly
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- We can convert some recursive algorithms into non-recursive version
 - Suitable for imperative languages like C++
- There are two conditions:
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Algorithm `LinearSum(A, n):`

Input: A integer array A and an integer $n \geq 1$, such that A has at least n elements

Output: The sum of the first n integers in A

```
if  $n = 1$  then  
    return  $A[0]$   
else
```

```
    return LinearSum(A, n - 1) + A[n - 1]
```

Recursive call is NOT the last thing!

Tail Recursion

- There are two conditions:
 - The recursion is Linear
 - The recursive call is the last thing that function does
 - **Iterate** through the recursive calls rather than calling them explicitly
 - Linear Recursive version:
 - Tail recursion conversion:
 - (iterative)
- Algorithm** ReverseArray(A, i, j):
Input: An array A and nonnegative integer indices i and j
Output: The reversal of the elements in A starting at index i and ending at j
- ```
if $i < j$ then
 Swap $A[i]$ and $A[j]$
 ReverseArray($A, i + 1, j - 1$)
return
```
- Algorithm** IterativeReverseArray( $A, i, j$ ):  
*Input:* An array  $A$  and nonnegative integer indices  $i$  and  $j$   
*Output:* The reversal of the elements in  $A$  starting at index  $i$  and ending at  $j$
- ```
while  $i < j$  do
    Swap  $A[i]$  and  $A[j]$ 
     $i \leftarrow i + 1$ 
     $j \leftarrow j - 1$ 
return
```

Binary Recursion

- When an algorithm makes **two** recursive calls
 - Useful to solve two similar halves of a problem
- Again! summing the **n** elements of an integer array A:
 - recursively
 - summing first half
 - summing second half
 - adding the two

Algorithm BinarySum(A, i, n):

Input: An array A and integers i and n

Output: The sum of the n integers in A starting at index i

if $n = 1$ **then**

return $A[i]$

return $\text{BinarySum}(A, i, \lceil n/2 \rceil) + \text{BinarySum}(A, i + \lceil n/2 \rceil, \lfloor n/2 \rfloor)$

Binary Recursion

- Again! summing the n elements of an integer array A :

- recursively
 - summing first half
 - summing second half
 - adding the two

- Analysis of the algorithm:

- We assume n is a power of 2
- $\text{BinarySum}(A, 0, n)$
 - $\text{BinarySum}(A, 0, 8)$
- n is halved at each recursive call

Algorithm $\text{BinarySum}(A, i, n)$:

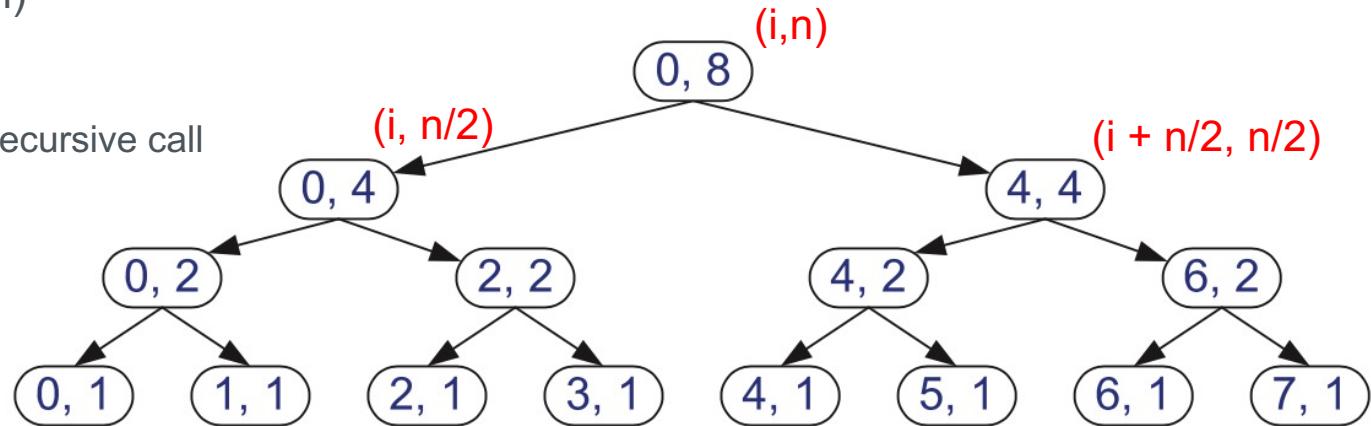
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Binary Recursion

- Analysis of the algorithm:
 - We assume n is a power of 2
 - $\text{BinarySum}(A, 0, n)$
 - $\text{BinarySum}(A, 0, 8)$
 - n is halved at each recursive call
- The depth of the recursion, that is, the maximum number of function instances that are active at the same time, is $1 + \log_2 n$

Algorithm $\text{BinarySum}(A, i, n)$:

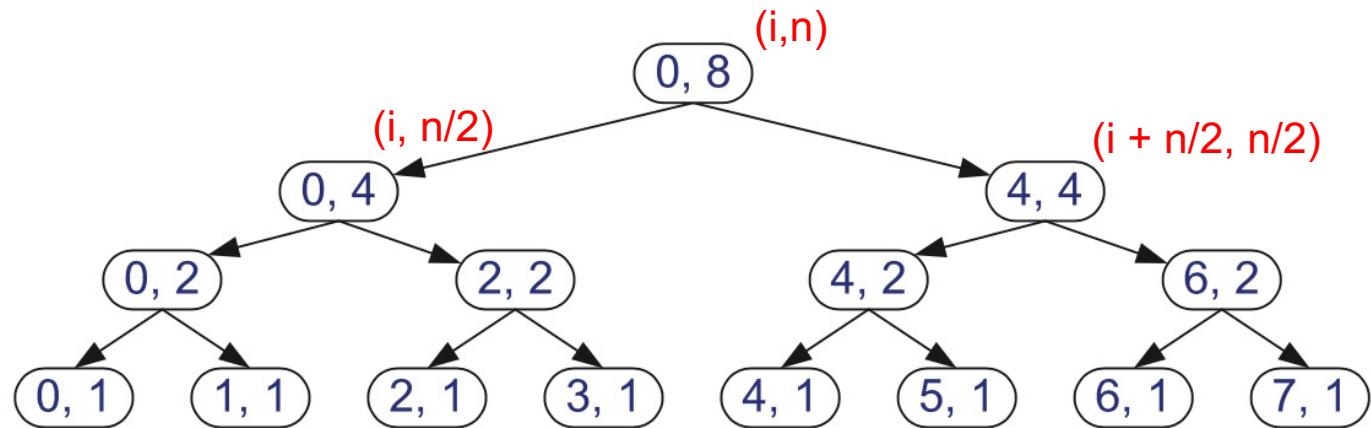
Input: An array A and integers i and n

Output: The sum of the n integers in A starting at index i

if $n = 1$ **then**

return $A[i]$

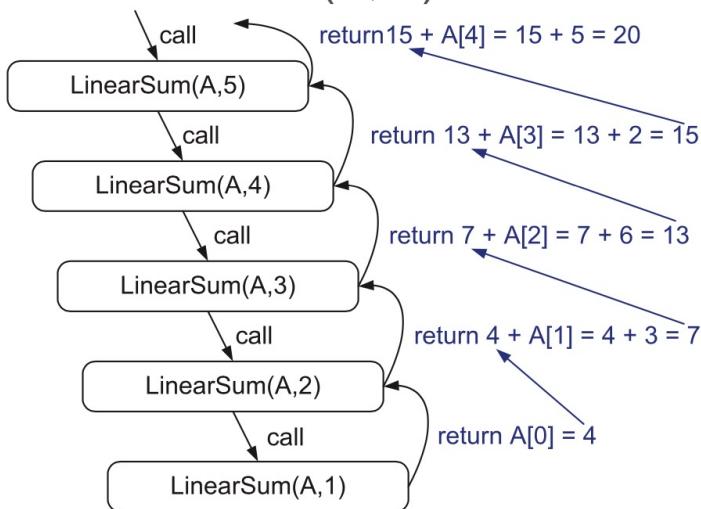
return $\text{BinarySum}(A, i, \lceil n/2 \rceil) + \text{BinarySum}(A, i + \lceil n/2 \rceil, \lfloor n/2 \rfloor)$



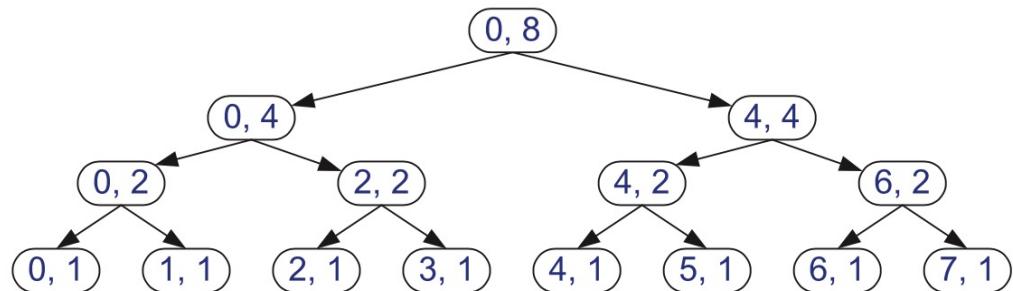
Binary Recursion

- Analysis of the algorithm:
 - We assume n is a power of 2
- **Memory consumption** (Space complexity)
 - The depth of the recursion, that is, the maximum number of function instances that are active at the same time, is $1 + \log_2 n$
 - Remember that this depth was n for **LinearSum**

LinearSum(A, 5)

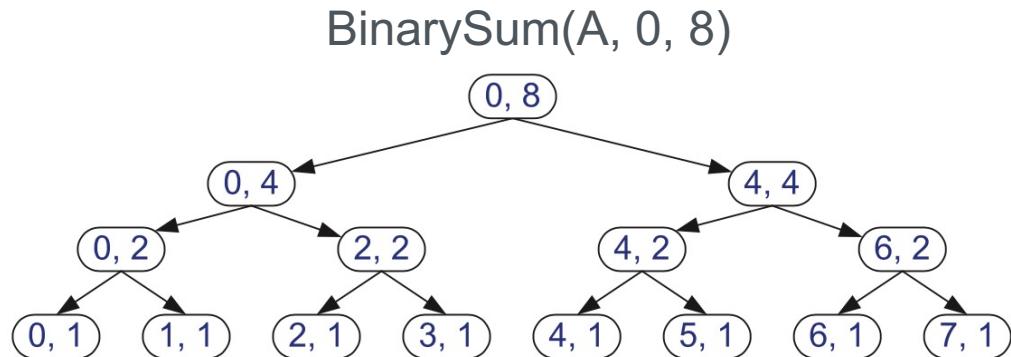
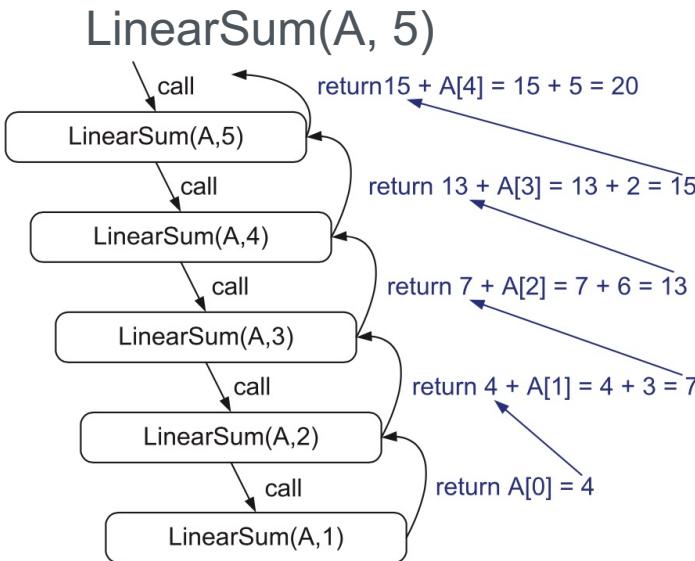


BinarySum(A, 0, 8)



Binary Recursion

- Analysis of the algorithm:
 - We assume n is a power of 2
- **Runtime** (Time complexity)
 - There are $2n-1$ boxes (calls) in **BinarySum**
 - There are n boxes (calls) in **LinearSum**
- Assume each calls is visited in a constant time



Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

15 Algorithms Analysis

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

February 17, 2022

Binary Recursion (from last lecture)

- Again! summing the n elements of an integer array A :

- recursively
 - summing first half
 - summing second half
 - adding the two

- Analysis of the algorithm:

- We assume n is a power of 2
- $\text{BinarySum}(A, 0, n)$
 - $\text{BinarySum}(A, 0, 8)$
- n is halved at each recursive call

Algorithm $\text{BinarySum}(A, i, n)$:

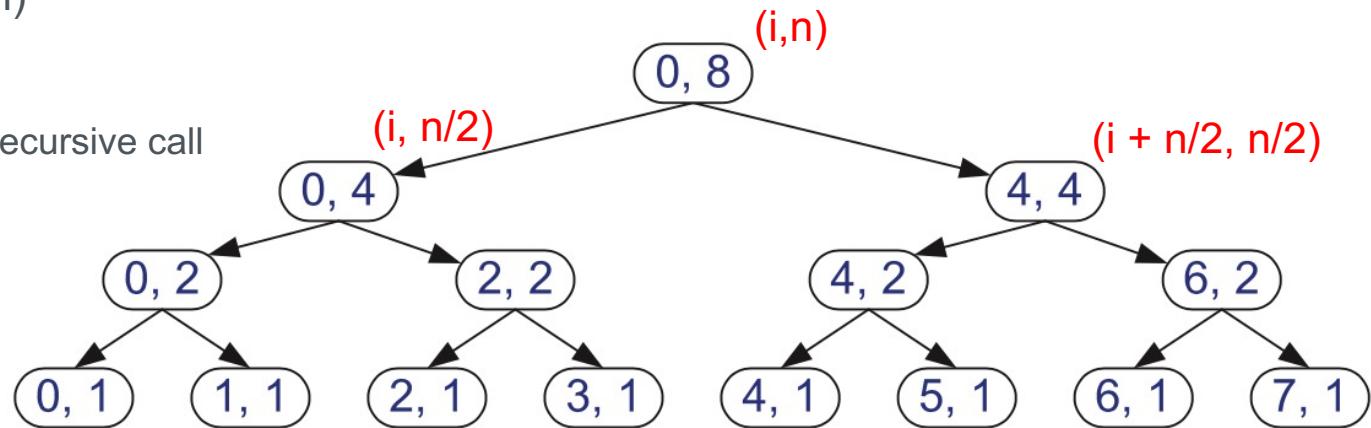
Input: An array A and integers i and n

Output: The sum of the n integers in A starting at index i

if $n = 1$ **then**

return $A[i]$

return $\text{BinarySum}(A, i, \lceil n/2 \rceil) + \text{BinarySum}(A, i + \lceil n/2 \rceil, \lfloor n/2 \rfloor)$



Binary Recursion (from last lecture)

- Analysis of the algorithm:
 - We assume n is a power of 2
 - $\text{BinarySum}(A, 0, n)$
 - $\text{BinarySum}(A, 0, 8)$
 - n is halved at each recursive call
- The depth of the recursion, that is, the maximum number of function instances that are active at the same time, is $1 + \log_2 n$

Algorithm $\text{BinarySum}(A, i, n)$:

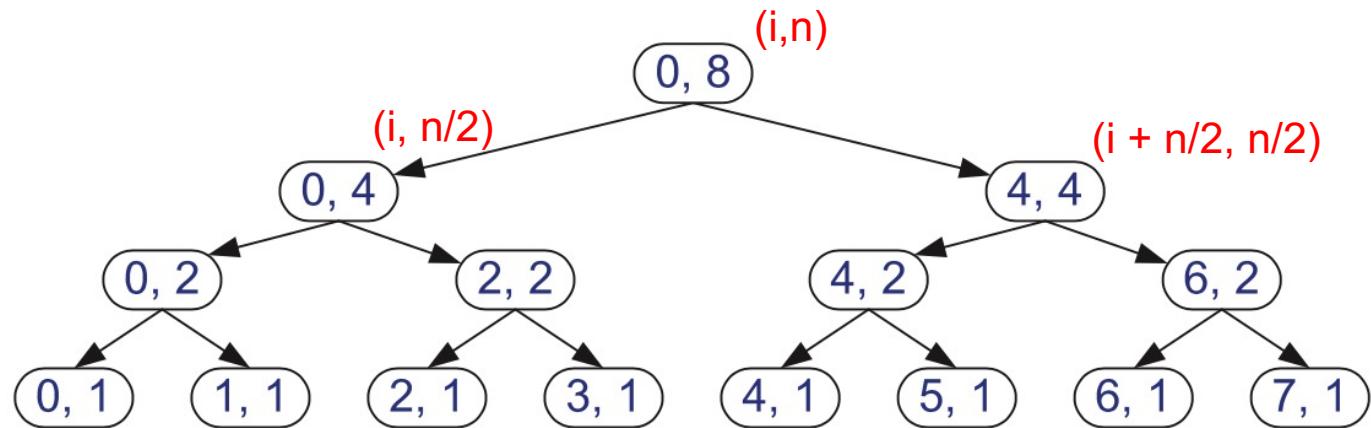
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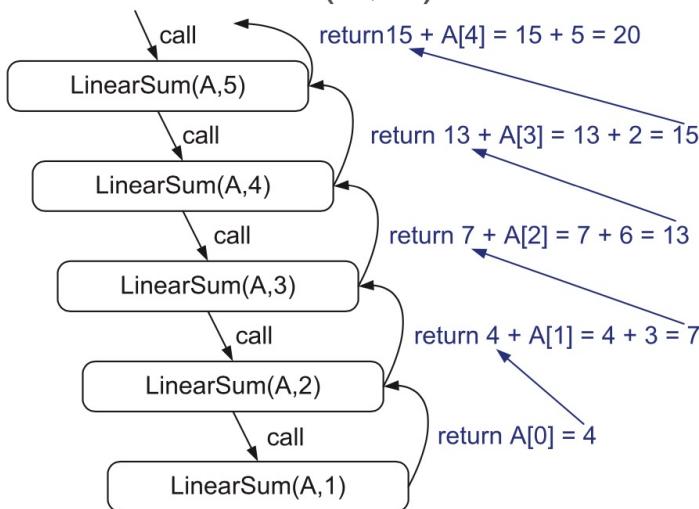
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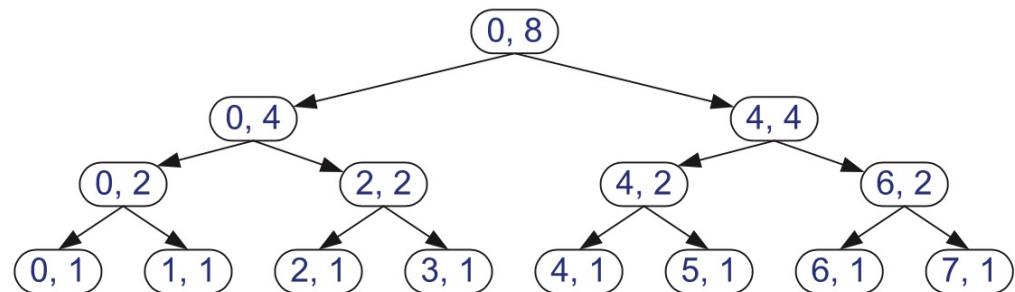
Binary Recursion (from last lecture)

- Analysis of the algorithm:
 - We assume n is a power of 2
- **Memory consumption** (Space complexity)
 - The depth of the recursion, that is, the maximum number of function instances that are active at the same time, is $1 + \log_2 n$
 - Remember that this depth was n for **LinearSum**

LinearSum(A, 5)

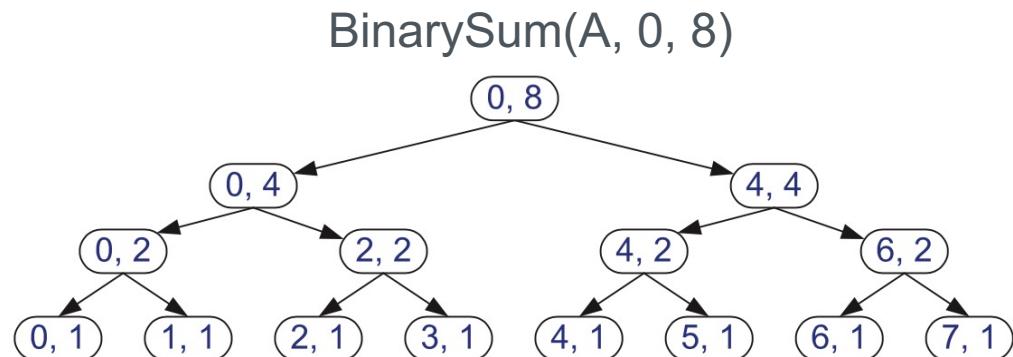
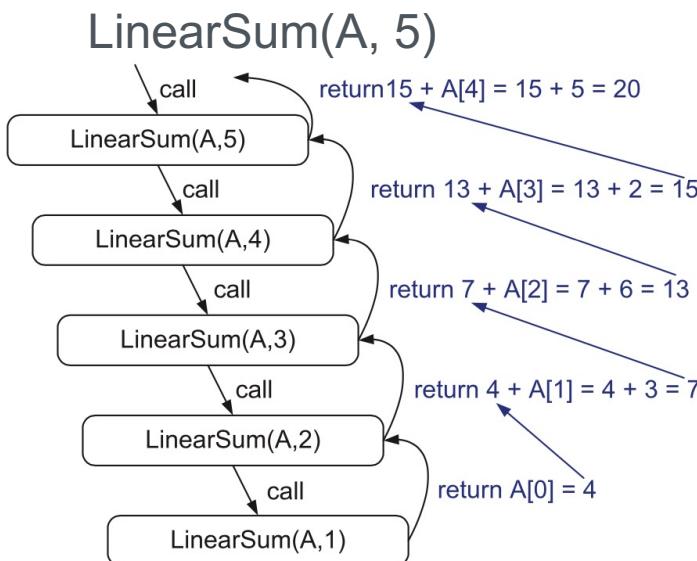


BinarySum(A, 0, 8)



Binary Recursion (from last lecture)

- Analysis of the algorithm:
 - We assume n is a power of 2
- **Runtime** (Time complexity)
 - There are $2n-1$ boxes (calls) in **BinarySum**
 - There are n boxes (calls) in **LinearSum**
- Assume each calls is visited in a constant time

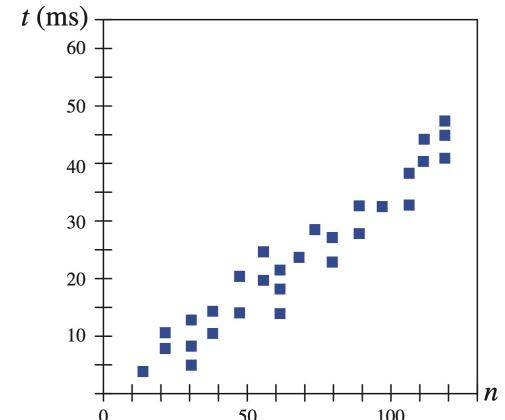


Comparing Algorithms

- Algorithm: An algorithm is a step-by-step procedure for solving a problem in a finite amount of time.
- There are different solutions for a problem
- We need to identify which solution is **better** than the other.
- Better in the sense of:
 - Running time
 - Memory space required
 - structure of programs (readability, simplicity, design)

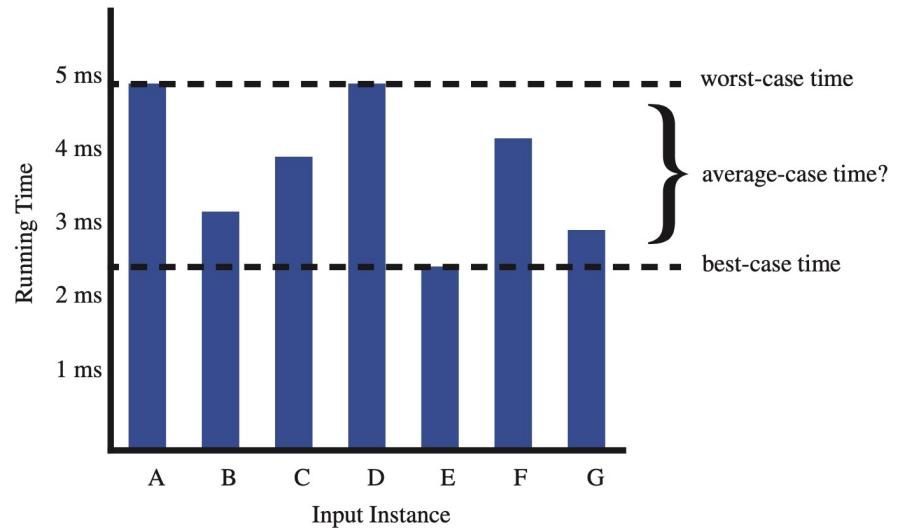
Running Time

- Most algorithms transform input objects into output objects.
 - We need to think about input
 - Insertion sort: array of size n
 - recursive factorial: n
- The running time of an algorithm typically grows with the input size.
 - best cases
 - average cases
 - worst cases
- We focus on the worst-case running time.
 - Easier to analyze
 - Crucial to applications such as games, embedded systems
- Average-case running time is often difficult to determine.
 - Why?



Average vs. Worst Case

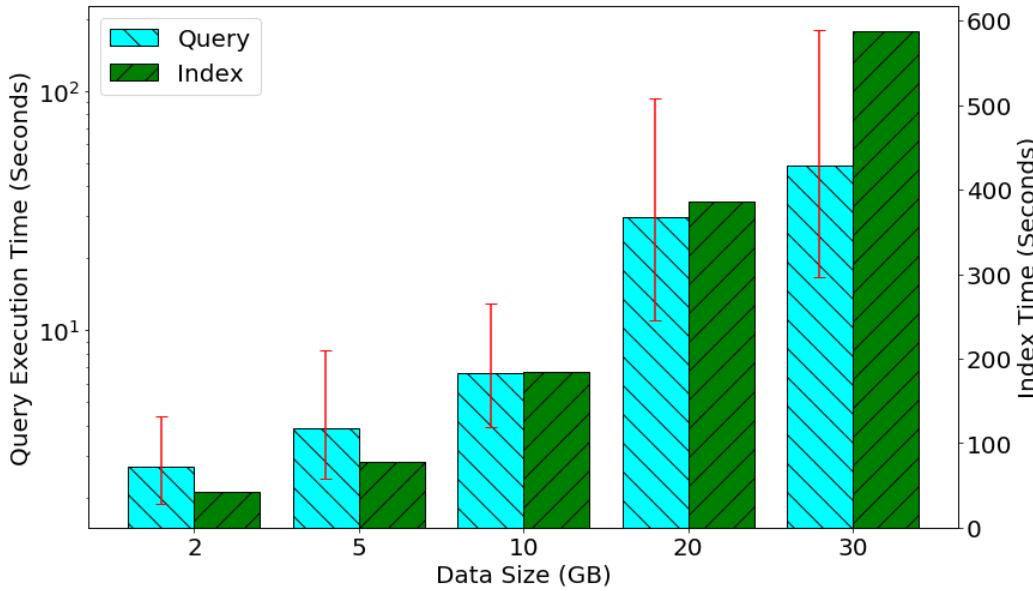
- The average case running time is harder to analyze because you need to know the probability distribution of the input.



- knowing the worst-case time is important in many applications
 - real-time systems
 - nuclear reactor
 - air traffic control

Experimental Approach

- Implement your algorithm
- Run the program with inputs of varying size and composition
- Use a wall clock to get an accurate measure of the actual running time
- Plot the results



Experimental Approach - Limitations

- It is necessary to implement your algorithm, which may be difficult and often time-consuming
 - Sometimes you just ideate!
- Results may not be indicative of the running time on other inputs not included in the experiment.
 - Your input may not be inclusive enough of all possible inputs
- Competing algorithms!
 - In order to compare your algorithm with a competing algorithm, the same hardware and software environments must be used
 - Your competitor may have ran on a high-end machine to which you don't have access
 - You need to implement it!
 - Or buy the same machine

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
 - The algorithm receives a worth of data
 - Performs a few operations on the data
- Characterizes running time as a function of the input
 - Size of input: array of length n
 - input: n for factorial
- Considers all possible inputs
- Allows us to evaluate the **relative efficiency** of any two algorithms **independent of** the hardware/software environment
- For each algorithm, we will end up with a function $f(n)$ that characterizes the running time of the algorithm as a function of the input size n

Primitive Operations

- Primitive operation corresponds to a low-level instruction with an execution time that is constant
 - Basic computations performed by an algorithm
 - Identifiable in pseudocode
 - Largely independent of the programming language
 - Exact definition is not important
- We define a set of primitive operations such as the following:
 - Assigning a value to a variable
 - Calling a function
 - Performing an arithmetic operation
 - Comparing two numbers
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Algorithm `arrayMax(A, n)`:

Input: An array A storing $n \geq 1$ integers.

Output: The maximum element in A .

$currMax \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n - 1$ **do**

if $currMax < A[i]$ **then**
 $currMax \leftarrow A[i]$

return $currMax$

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1. accessing $A[0]$ (indexing in array)
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The for loop repeats **n** times, why?
Each time it has **2** operations, why?

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The for loop will repeat n times, why?

We account for the last increment to n in which the for loop identifies it should exit before entering next iteration

for example: $n = 4$

for i from 1 to 3 → in the last iteration i becomes 4 and will be compared to 3

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currMax ← A[0]           <----- 2 operations
for i ← 1 to n – 1 do    <----- 2n operations
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        currMax ← A[i]      <-- 2(n-1) operations
    i++                      <-- 2(n-1) operations
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The body of for loop will repeat **n-1** times

The increment of **i** is performed at the end of each iteration

Primitive Operations

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        i++ currMax ← A[i] <-- 2(n-1) operations
        <-- 2(n-1) operations
return currMax <-- 1 operation
```

We will have a total of $8n - 2$ operations

n is the input size!

Estimating Runtime

- Algorithm **arrayMax** executes $8n - 2$ primitive operations in total

- Suppose:

- a = Time taken by the fastest primitive operation
 - b = Time taken by the slowest primitive operation

- The running time of **arrayMax** is bounded by two linear functions

- $a(8n - 2) \leq T(n) \leq b(8n - 2)$

- Changing the hardware / software environment

- Affects $T(n)$ by a constant factor, but does not alter the growth rate of $T(n)$

- The linear growth rate of the running time $T(n)$ is an **intrinsic property** of algorithm **arrayMax**

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Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

16 Algorithms Analysis (cont.)

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

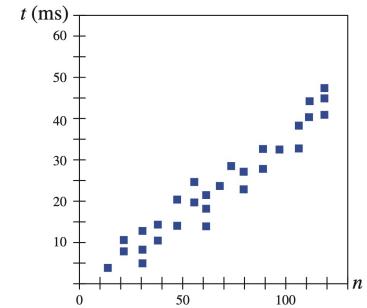
February 28, 2022

Administration

- Please pay special attention to this week's tutorial
- Start the assignment as early as possible. I will not be able to answer questions during coming weekend!
 - If you have questions, ask early.

Review

- Runtime Complexity Analysis
 - Experimental approach
 - Implement, run with varying input sizes, and measure run-times



- Theoretical Approach
 - Allows us to evaluate the **relative efficiency** of any two algorithms **independent of** the hardware/software environment
 - For each algorithm, we will end up with a function $f(n)$ that characterizes the running time of the algorithm as a function of the input size n .
 - We started by looking at primitive operations to get an idea of what operations an algorithm perform on input

Primitive Operations

- We define a set of primitive operations such as the following:
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The for loop repeats **n** times, why?
Each time it has **2** operations, why?

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for example: $n = 4$

for i from 1 to 3 → in the last iteration i becomes 4 and will be compared to 3

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        currMax ←  $A[i]$   <- 2( $n-1$ ) operations
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The body of for loop will repeat **$n-1$** times

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        <-- 2(n-1) operations
return currMax <-- 1 operation
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We will have a total of $8n - 3$ operations

n is the input size!

Estimating Runtime

- Algorithm **arrayMax** executes $8n - 3$ primitive operations in total
- Suppose:
 - a = Time taken by the fastest primitive operation
 - b = Time taken by the slowest primitive operation
- The running time of **arrayMax** is bounded by two linear functions
 - $a(8n - 3) \leq T(n) \leq b(8n - 3)$
- Changing the hardware / software environment
 - Affects $T(n)$ by a constant factor, but does not alter the growth rate of $T(n)$
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```

Asymptotic Notation

- Big-picture approach
 - In algorithm analysis, we focus on the growth rate of the running time as a function of the input size n .
 - It is often enough just to know that the running time of an algorithm such as arrayMax, grows proportionally to n , with its true running time being n times a **constant factor** that depends on the specific computer. (was $8n - 3$)
 - We characterize the running times of algorithms by using functions that map the size of the input, n , to values that correspond to the main factor that determines the growth rate in terms of n .

n	$\log n$	n	$n \log n$	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	1.84×10^{19}
128	7	128	896	16,384	2,097,152	3.40×10^{38}
256	8	256	2,048	65,536	16,777,216	1.15×10^{77}
512	9	512	4,608	262,144	134,217,728	1.34×10^{154}

Why Growth Rate Matters

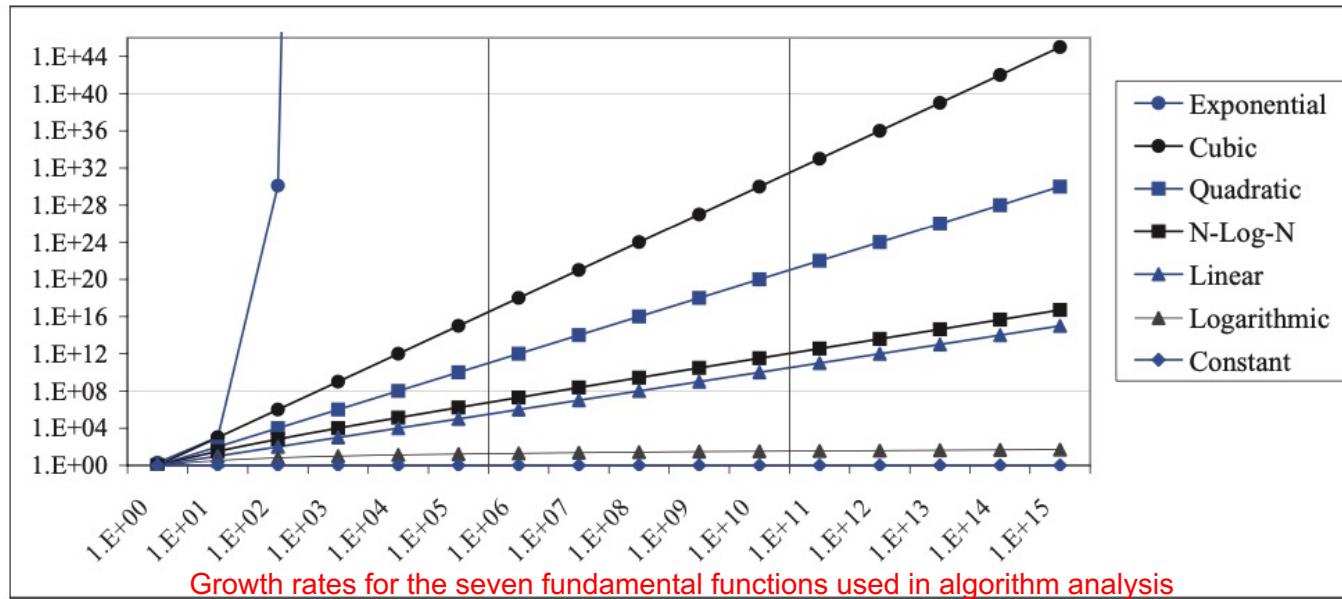
- **Classes of functions**

<i>constant</i>	<i>logarithm</i>	<i>linear</i>	<i>n-log-n</i>	<i>quadratic</i>	<i>cubic</i>	<i>exponential</i>
1	$\log n$	n	$n \log n$	n^2	n^3	a^n

- Ideally, we would like:

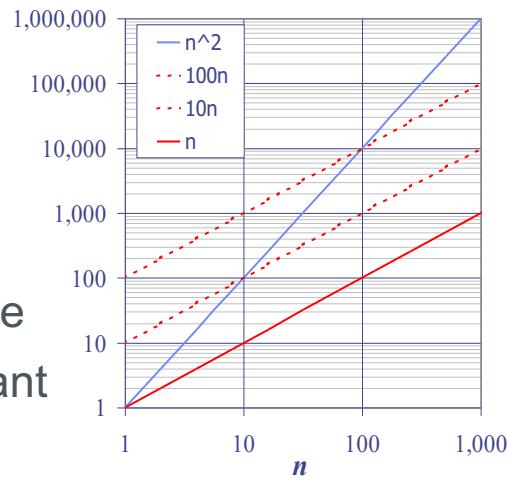
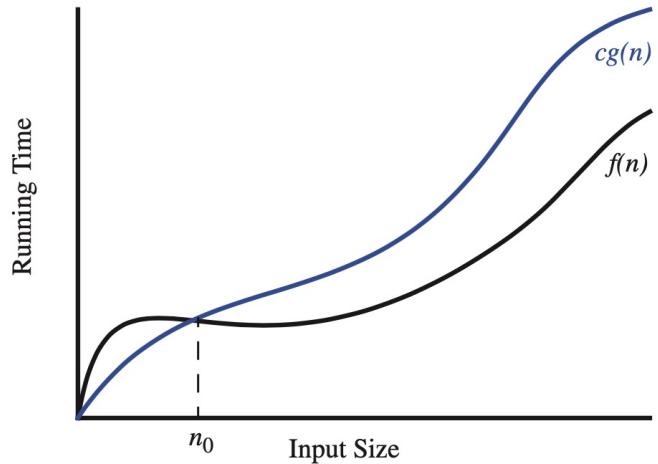
- data structure operations to run in times proportional to the **constant** or **logarithm** function.
- algorithms to run in linear or $n\text{-log-}n$ time.
- Algorithms with quadratic or cubic running times are less practical, but algorithms with exponential running times are infeasible for all but the small-sized inputs.

log-log chart



Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that: $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
 - $2n + 10 \leq cn$
 - $(c - 2)n \geq 10$
 - $n \geq 10/(c - 2)$
 - Pick $c = 3$ and $n_0 = 10$
- Example: ArrayMax: $8n - 3$ is $O(n)$
 - $8n - 3 \leq cn$
 - Pick $c = 8$, and $n_0 = 1$
- Example: n^2 is not $O(n)$
 - $n^2 \leq cn$
 - $n \leq c$
 - The above inequality cannot be satisfied since c must be a constant



Asymptotic Analysis of Algorithms

- Now we can write the following mathematically precise statement on the running time of algorithm `arrayMax` for **any** computer:
 - The Algorithm `arrayMax`, for computing the maximum element in an array of n integers, runs in $O(n)$ time.
 - proof: The number of primitive operations executed by algorithm `arrayMax` in each iteration is a constant. Hence, since each primitive operation runs in constant time, we can say that the running time of algorithm `arrayMax` on an input of size n is **at most a constant times n** , that is, we may conclude that the **running time of algorithm `arrayMax` is $O(n)$** .
- The asymptotic analysis
 - identify the running time in Big-Oh notation
 - We find the worst-case number of primitive operations executed as a function of the input size
 - We express this function with Big-Oh notation

Algorithm `arrayMax(A, n):`

Input: An array A storing $n \geq 1$ integers.

Output: The maximum element in A .

```
currMax ←  $A[0]$ 
for  $i \leftarrow 1$  to  $n - 1$  do
    if  $currMax < A[i]$  then
        currMax ←  $A[i]$ 
return currMax
```

Big-Oh Notation Rules

- If $f(n)$ is a polynomial of degree d , then $f(n)$ is $O(n^d)$:
 - Drop lower-order terms
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 - $3n^3 + 20n^2 + 5$ is $O(n^3)$
 - need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$
this is true for $c = 4$ and $n_0 = 21$
 - Use the **smallest possible class** of functions
 - Say “ $2n$ is $O(n)$ ” instead of “ $2n$ is $O(n^2)$ ”
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The big-Oh notation gives an upper bound
on the growth rate of a function

Asymptotic Analysis - Example

- Prefix Averages
- The i -th prefix average of an array X is average of the first $(i + 1)$ elements of X :
$$A[i] = (X[0] + X[1] + \dots + X[i])/(i+1)$$

$$A[i] = \frac{\sum_{j=0}^i X[j]}{i+1}.$$

Algorithm prefixAverages1(X):

Input: An n -element array X of numbers.

Output: An n -element array A of numbers such that $A[i]$ is
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Let A be an array of n numbers.

for $i \leftarrow 0$ **to** $n - 1$ **do**

$a \leftarrow 0$

for $j \leftarrow 0$ **to** i **do**

$a \leftarrow a + X[j]$

$A[i] \leftarrow a/(i+1)$

return array A

Algorithm prefixAverages2(X):

Input: An n -element array X of numbers.

Output: An n -element array A of numbers such that $A[i]$ is
the average of elements $X[0], \dots, X[i]$.

Let A be an array of n numbers.

$s \leftarrow 0$

for $i \leftarrow 0$ **to** $n - 1$ **do**

$s \leftarrow s + X[i]$

$A[i] \leftarrow s/(i+1)$

return array A

Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

17 Algorithms Analysis (cont.2)

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

March 2, 2022

Administration

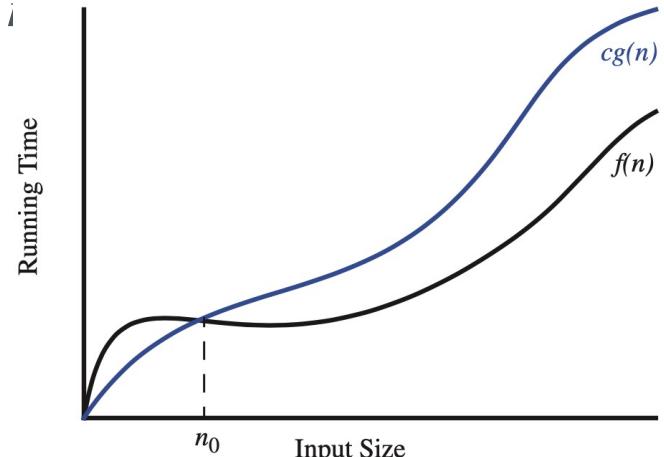
- My Office Hour:
 - Today at 15:00 in **ITB-159** in-person (or virtually using teams)
- Please **read the questions carefully** and watch the video if needed for the assignment 2

Review

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that: $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
- Example: ArrayMax: $8n - 3$ is $O(n)$
- Example: n^2 is not $O(n)$
- Known class of functions:

constant	logarithm	linear	$n\text{-log-}n$	quadratic	cubic	exponential
1	$\log n$	n	$n \log n$	n^2	n^3	a^n

- Desired complexities:
 - Linear or $n\text{-log-}n$ for algorithms
 - sort, search
 - Constant or Logarithm operations for DS
 - add, remove, indexing



Asymptotic Analysis of Algorithms

- Now we can write the following mathematically precise statement on the running time of algorithm `arrayMax` for **any** computer:
 - The Algorithm `arrayMax`, for computing the maximum element in an array of n integers, runs in $O(n)$ time.
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Algorithm `arrayMax(A, n):`

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return currMax
```

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this is true for $c = 4$ and $n_0 = 21$

```
for (i = 0; i < n; i++) {  
    sequence of statements  
}
```

is $O(n)$

Big-Oh Notation Rules

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this is true for $c = 4$ and $n_0 = 21$

```
for (i = 0; i < n; i++) {  
    for (j = 0; j < n; j++) {  
        sequence of statements  
    }  
}
```

is $O(n^2)$

Big-Oh Notation Rules

- If $f(n)$ is a polynomial of degree d , then $f(n)$ is $O(n^d)$:
 - Drop lower-order terms
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this is true for $c = 4$ and $n_0 = 21$

```
for (i = 0; i < n; i++) {  
    for (j = 0; j < i; j++) {  
        sequence of statements  
    }  
}
```

- The outer loops is $O(n)$
 - The statements in the inner loop executed $i+1$ times: i.e.: $1 + 2 + 3 + \dots + n$ times which is $n(n+1)/2$ which is $O(n^2)$
- So, this is $O(n^2)$

Big-Oh Notation Rules

- If $f(n)$ is a polynomial of degree d , then $f(n)$ is $O(n^d)$:
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 - $3n^3 + 20n^2 + 5$ is $O(n^3)$
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this is true for $c = 4$ and $n_0 = 21$

```
for (i = 0; i < n; i++) {  
    for (j = 0; j < n; j++) {  
        sequence of statements  
    }  
}  
for (k = 0; k < n; k++) {  
    sequence of statements  
}
```

is $O(n^2+n)$ which is $O(n^2)$
we select the main component that affects the growth

Big-Oh Notation Rules

- If $f(n)$ is a polynomial of degree d , then $f(n)$ is $O(n^d)$:
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this is true for $c = 4$ and $n_0 = 21$

```
for (j = 0; j < n; j++)  
    f(j);
```

is $O(n)$

- function f takes a constant time

```
for (j = 0; j < n; j++)  
    g(j);
```

is $O(n^2)$

- function g takes a linear time proportional to its parameter

```
for (j = 0; j < n; j++)  
    g(k);
```

is $O(k \cdot n)$

or $O(n)$, if k is not very large or has a relative size to n

Big-Oh Notation Rules

- Use the **smallest possible class** of functions
 - Say “ $2n$ is $O(n)$ ” instead of “ $2n$ is $O(n^2)$ ”
- Use the **simplest expression of the class**
 - Say “ $3n + 5$ is $O(n)$ ” instead of “ $3n + 5$ is $O(3n)$ ”
- Think about hidden constant factors!

The big-Oh notation gives an upper bound
on the growth rate of a function

Complex Examples on Growth Rates

- You need some math to reason about some functions
- math you need to review
 - properties of logarithms:
 - $\log_b(x \cdot y) = \log_b x + \log_b y$
 - $\log_b(x / y) = \log_b x - \log_b y$
 - $\log_b x^a = a \log_b x$
 - $\log_b a = \log_x a / \log_x b$
 - properties of exponentials:
 - $a^{(b+c)} = a^b \cdot a^c$
 - $a^{bc} = (a^b)^c$
 - $a^b / a^c = a^{(b-c)}$
 - $b = a^{\log_a b}$
 - $b^c = a^{c \log_a b}$

for example:

$$(\sqrt{2})^{\log n} = (2^{0.5})^{\log n} = (2^{\log n})^{0.5} = \sqrt{n}$$

$$2^{100\log n} = 2^{\log n^{100}} = n^{100}$$

$$\begin{aligned}\log(n!) &= \log(n(n-1)(n-2)\dots) = \\ &= \log(n) + \log(n-1) + \dots = \\ &\text{is } O(\log(n))\end{aligned}$$

Asymptotic Analysis - Example

- Prefix Averages
- The i -th prefix average of an array X is average of the first $(i + 1)$ elements of X :
$$A[i] = (X[0] + X[1] + \dots + X[i]) / (i+1)$$

$$A[i] = \frac{\sum_{j=0}^i X[j]}{i+1}.$$

Input Array X :

0	1	2	3	4
2	8	5	13	7

Prefix Average Array A :

0	1	2	3	4
2	5	5	7	7

Asymptotic Analysis - Example

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$$A[i] = (X[0] + X[1] + \dots + X[i]) / (i+1)$$

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Input: An n -element array X of numbers.

Output: An n -element array A of numbers such that $A[i]$ is
the average of elements $X[0], \dots, X[i]$.

Let A be an array of n numbers.

for $i \leftarrow 0$ **to** $n - 1$ **do**

$a \leftarrow 0$

for $j \leftarrow 0$ **to** i **do**

$a \leftarrow a + X[j]$

$A[i] \leftarrow a / (i + 1)$

return array A

$i = 0$

$a = 0$

after inner for loop ($i + 1$ times)

$a = 2$

$A[0] = 2 / 1 = 2$

X	0	1	2	3	4
	2	8	5	13	7

A	0	1	2	3	4
	2				

Asymptotic Analysis - Example

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for $j \leftarrow 0$ **to** i **do**

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$A[i] \leftarrow a / (i + 1)$

return array A

$i = 1$

$a = 0$

after inner for loop ($i + 1$ times)

$a = 10$

$A[0] = 10 / 2 = 5$

X	0	1	2	3	4
	2	8	5	13	7

A	0	1	2	3	4
	2	5			

Asymptotic Analysis - Example

- Prefix Averages
- The i -th prefix average of an array X is average of the first $(i + 1)$ elements of X :
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for $j \leftarrow 0$ **to** i **do**

$a \leftarrow a + X[j]$

$A[i] \leftarrow a / (i + 1)$

return array A

$i = 2$

$a = 0$

after inner for loop ($i + 1$ times)

$a = 15$

$A[0] = 15 / 3 = 5$

X	0	1	2	3	4
	2	8	5	13	7

A	0	1	2	3	4
	2	5	5		

Asymptotic Analysis - Example

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$$A[i] = (X[0] + X[1] + \dots + X[i]) / (i+1)$$

$$A[i] = \frac{\sum_{j=0}^i X[j]}{i+1}.$$

Algorithm `prefixAverages1(X)`:

Input: An n -element array X of numbers.

Output: An n -element array A of numbers such that $A[i]$ is
the average of elements $X[0], \dots, X[i]$.

```
Let  $A$  be an array of  $n$  numbers. <----- O(n)
for  $i \leftarrow 0$  to  $n - 1$  do <----- O(n) operations
     $a \leftarrow 0$  <----- O(n) operations
    for  $j \leftarrow 0$  to  $i$  do <-- O( $n^2$ ) operations
         $a \leftarrow a + X[j]$  <-- O( $n^2$ ) operations
     $A[i] \leftarrow a/(i + 1)$  <----- O(n) operations
return array  $A$  <-- O(1) operation (in the textbook it is O(n) considering return by value)
```

so `prefixAverages1` is $O(n^2)$

Asymptotic Analysis - Example

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- The i -th prefix average of an array X is average of the first $(i + 1)$ elements of X :
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for $j \leftarrow 0$ **to** i **do**

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return array A

$$A[i-1] = (X[0]+X[1]+\cdots+X[i-1]) / i$$
$$A[i] = (X[0]+X[1]+\cdots+X[i-1]+X[i]) / (i+1)$$

Asymptotic Analysis - Example

- Prefix Averages
- The i -th prefix average of an array X is average of the first $(i + 1)$ elements of X :
$$A[i] = (X[0] + X[1] + \dots + X[i]) / (i+1)$$

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$$A[i] = (X[0]+X[1]+\dots+X[i-1]+X[i]) / (i+1)$$

$$A[i] = \frac{\sum_{j=0}^i X[j]}{i+1}.$$

Algorithm `prefixAverages2(X)`:

Input: An n -element array X of numbers.

Output: An n -element array A of numbers such that $A[i]$ is
the average of elements $X[0], \dots, X[i]$.

Let A be an array of n numbers. <----- $O(n)$

$s \leftarrow 0$ <----- $O(1)$ operations

for $i \leftarrow 0$ **to** $n - 1$ **do** <-- $O(n)$ operations

$s \leftarrow s + X[i]$ <----- $O(n)$ operations

$A[i] \leftarrow s/(i+1)$ <----- $O(n)$ operations

return array A <-- $O(1)$ operation (in the textbook it is $O(n)$ considering return by value)

so `prefixAverages2` is $O(n)$

Using Sorting as a Problem-Solving Tool

- Sometimes we can use sorting as a tool to ease our problem-solving and even make the algorithm faster
- Element uniqueness problem:
 - Given an array, identify if all elements are unique (there are no duplicates)
 - Naïve algorithm:
 - compare each pair of elements in the array
 - $O(n^2)$
 - Better algorithm
 - sort the array first, then traverse the array once, look for duplicates among consecutive elements.
 - Best sorting algorithm runs in $O(n \log n)$
 - The algorithm will be $O(n \log n + n)$ which is $(n \log n)$

Final Remarks

- Given an algorithm
 - What is the order of this algorithm? (runtime complexity)
- Try to reduce the running time as much as possible
 - Sometimes based on some observations that we already had we can easily reduce the complexity (like previous slide)
- This analysis technique is not the whole story!
 - There are situations where a worst-case $O(n^2)$ is faster than $O(n \log n)$
 - If you want, you can study more.
- There are some problems for which there is **NO** polynomial-time algorithm found (up until now)
 - We say that they “NP-hard” or “NP-complete”
 - If someone can find a polynomial-time solution one of them, that solution may work for many others as well.

Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

18 Stacks

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

March 3, 2022

Stack

- A stack is a container of objects that are inserted and removed according to the Last-In First-Out (LIFO) principle.

“push” adds a new item on top of the stack.

push (A)

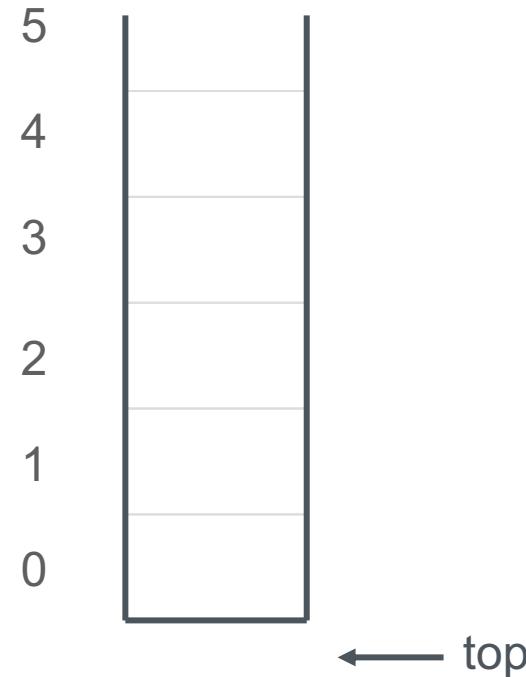
push (B)

push (C)

push (D)

push (E)

pop ()

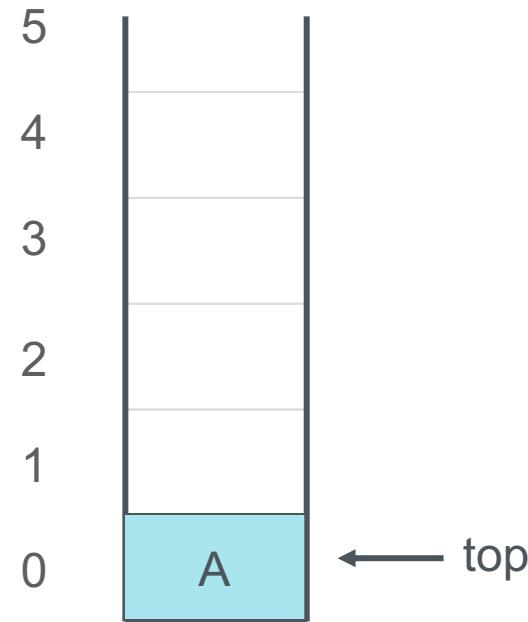


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- ▶ push (A)
- push (B)
- push (C)
- push (D)
- push (E)
- pop ()

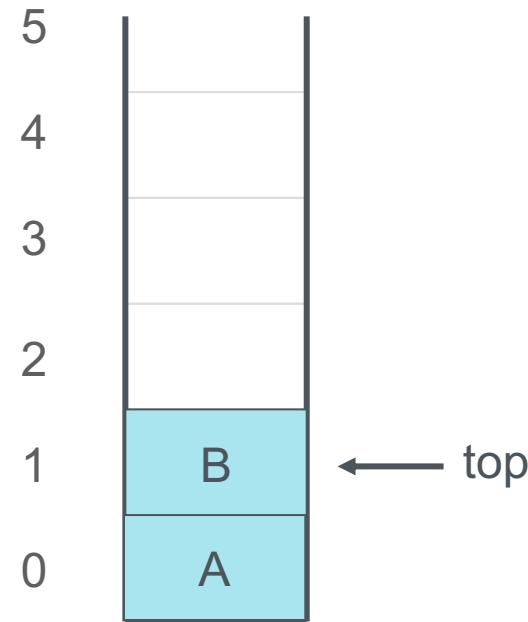


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- push (D)
- push (E)
- pop ()

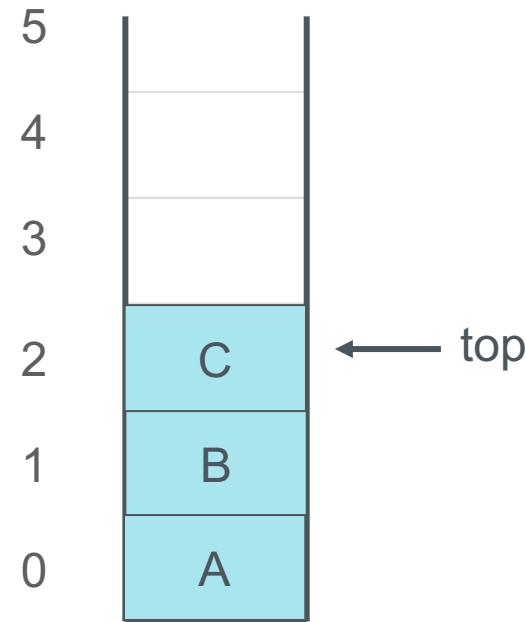


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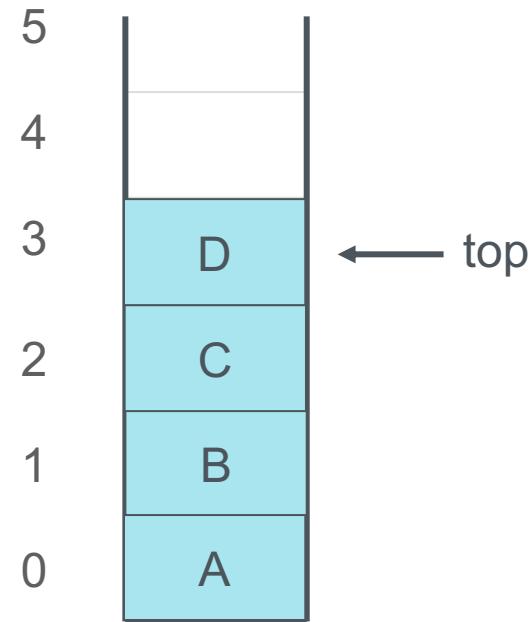


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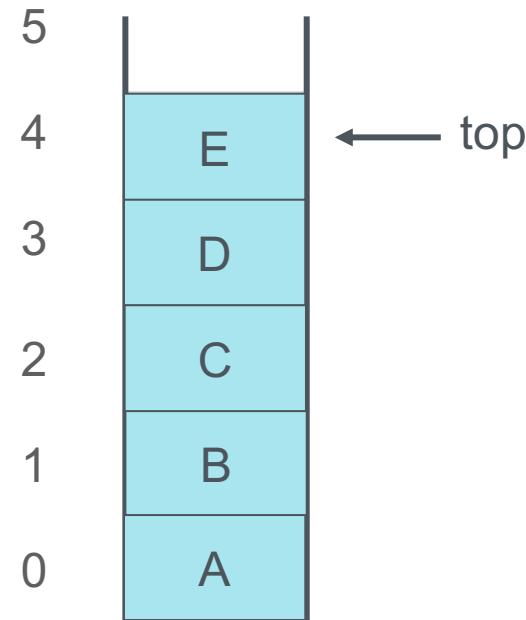


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“push” adds a new item on top of the stack.

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- push (C)
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- pop ()



Stack

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“push” adds a new item on top of the stack.

push (A)

push (B)

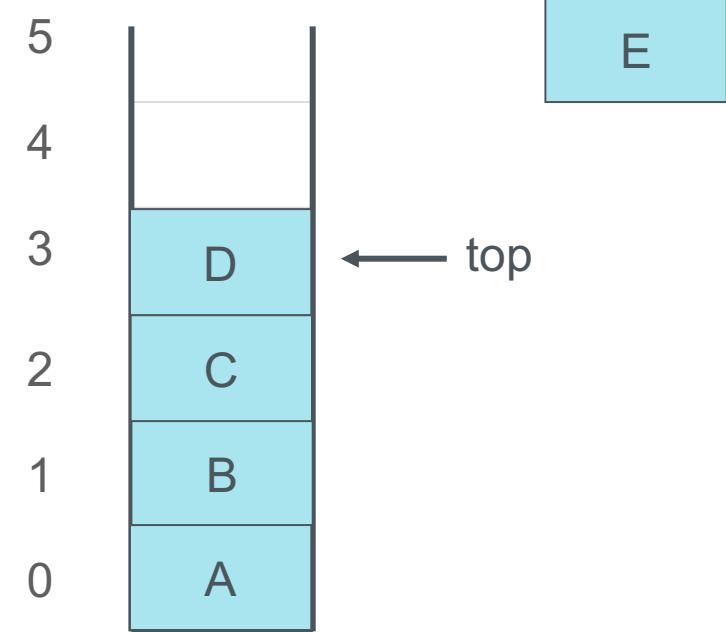
push (C)

push (D)

push (E)

▶ pop ()

Remove an item from the top of stack.



Stack and its Applications

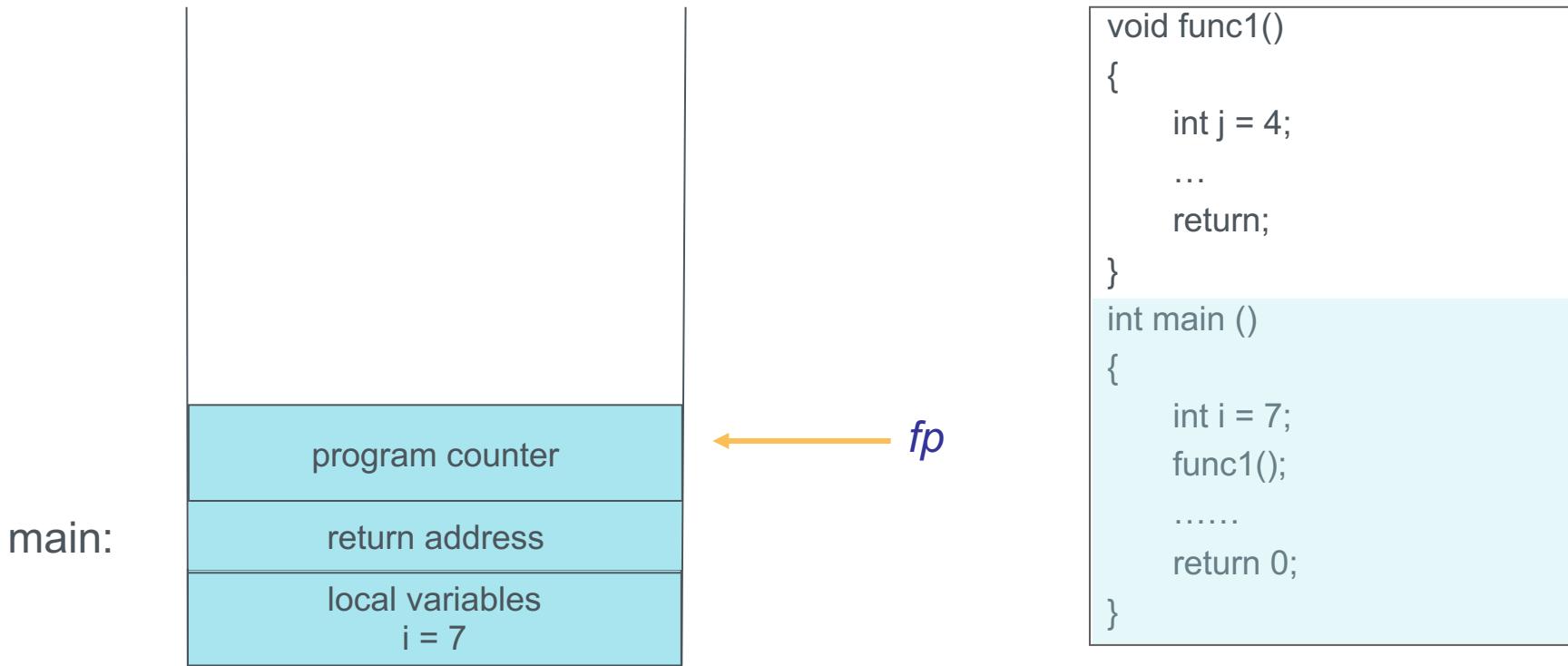
- Applications
 - History of visited pages in a web browser
 - Sequence of Undo operations in a text editor
 - Keep track of function calls in the C++ run-time system (System Stack)
 - As a data structure for algorithms to solve problems
 - Example: Reversing a list
 - Component of other data structures

Stack and its Applications

- System Stack
 - The C++ run-time system keeps track of the chain of active function calls with a stack.
- When a function is called, the system pushes a “**stack frame**” onto the stack which contains:
 - Local variables and return value
 - Program counter, keeping track of the statement being executed
- When the function ends, its frame is popped from the stack and control is passed to the function on top of the stack
- Allows for **recursion**

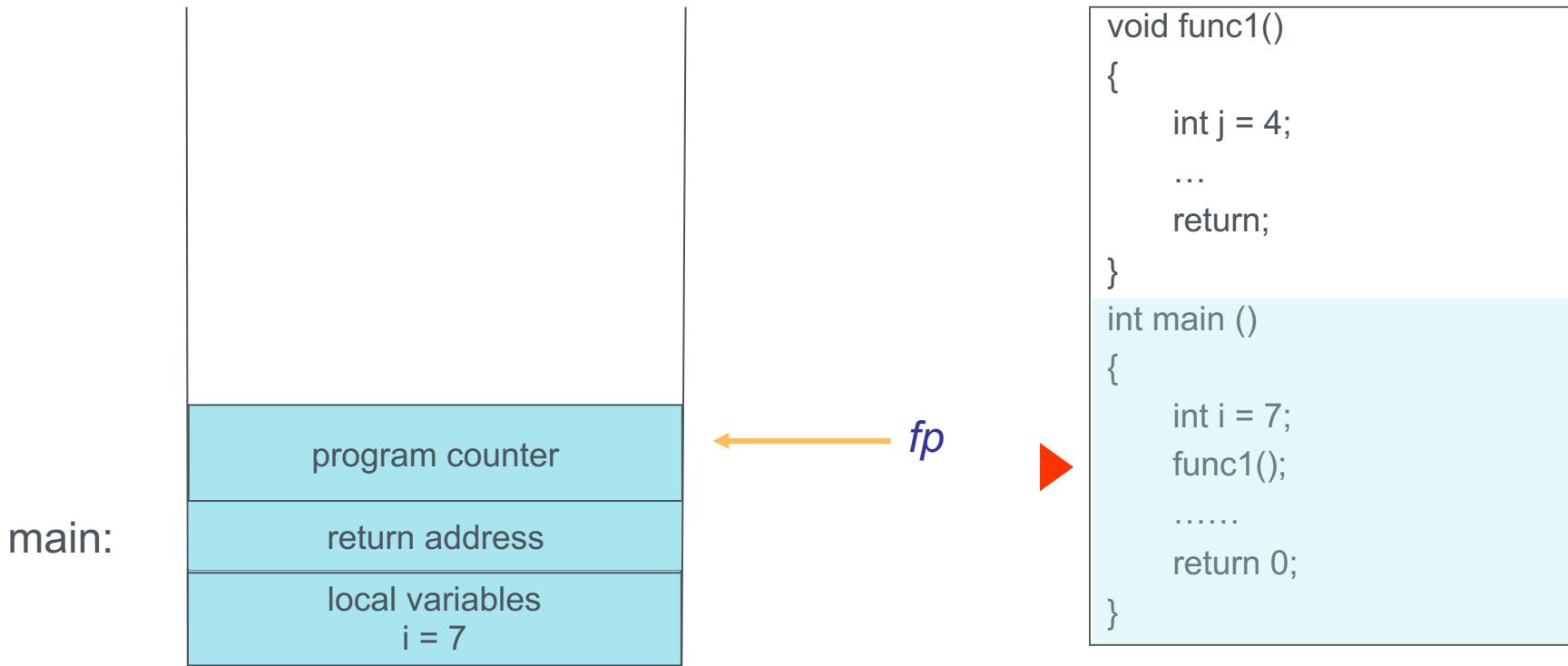
Stack and its Applications

- System Stack
 - “**stack frame**” contains:
 - Local variables and return value
 - Program counter, keeping track of the statement being executed



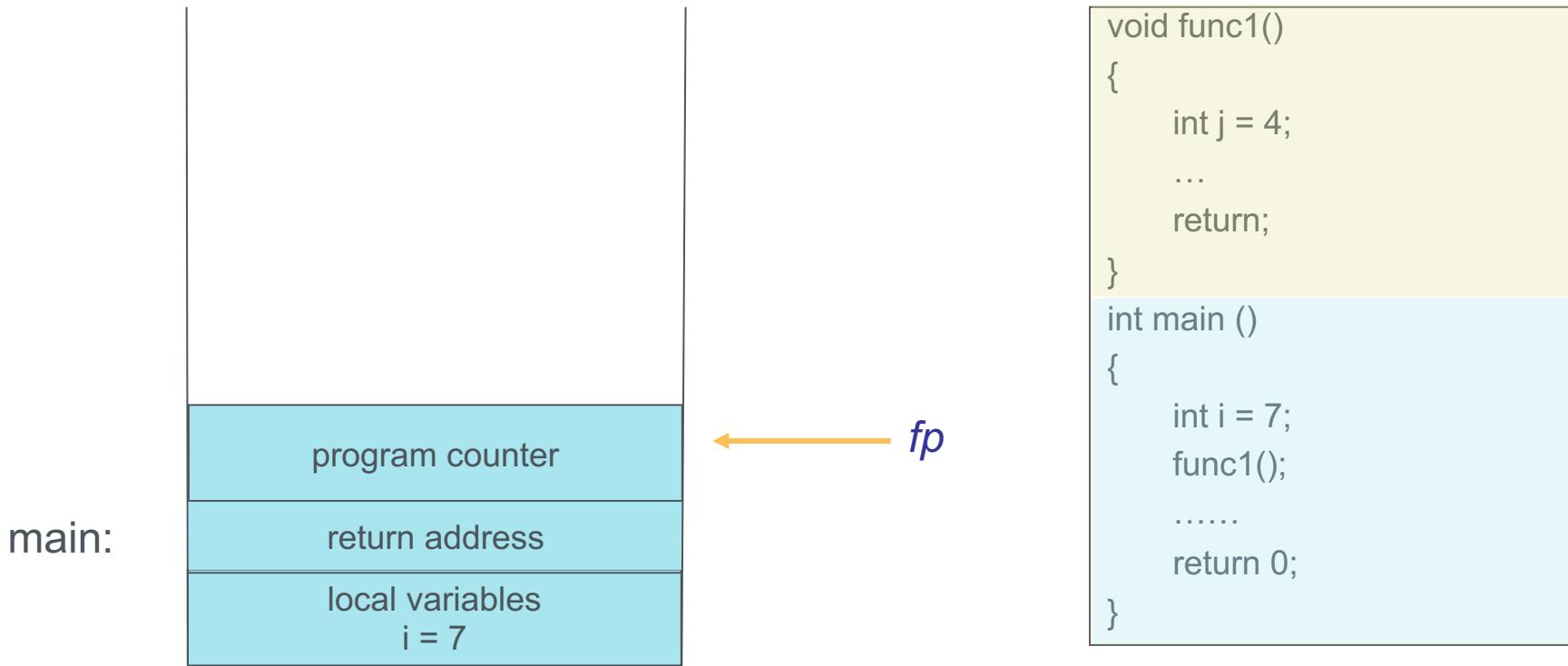
Stack and its Applications

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 - “**stack frame**” contains:
 - Local variables and return value
 - Program counter, keeping track of the statement being executed



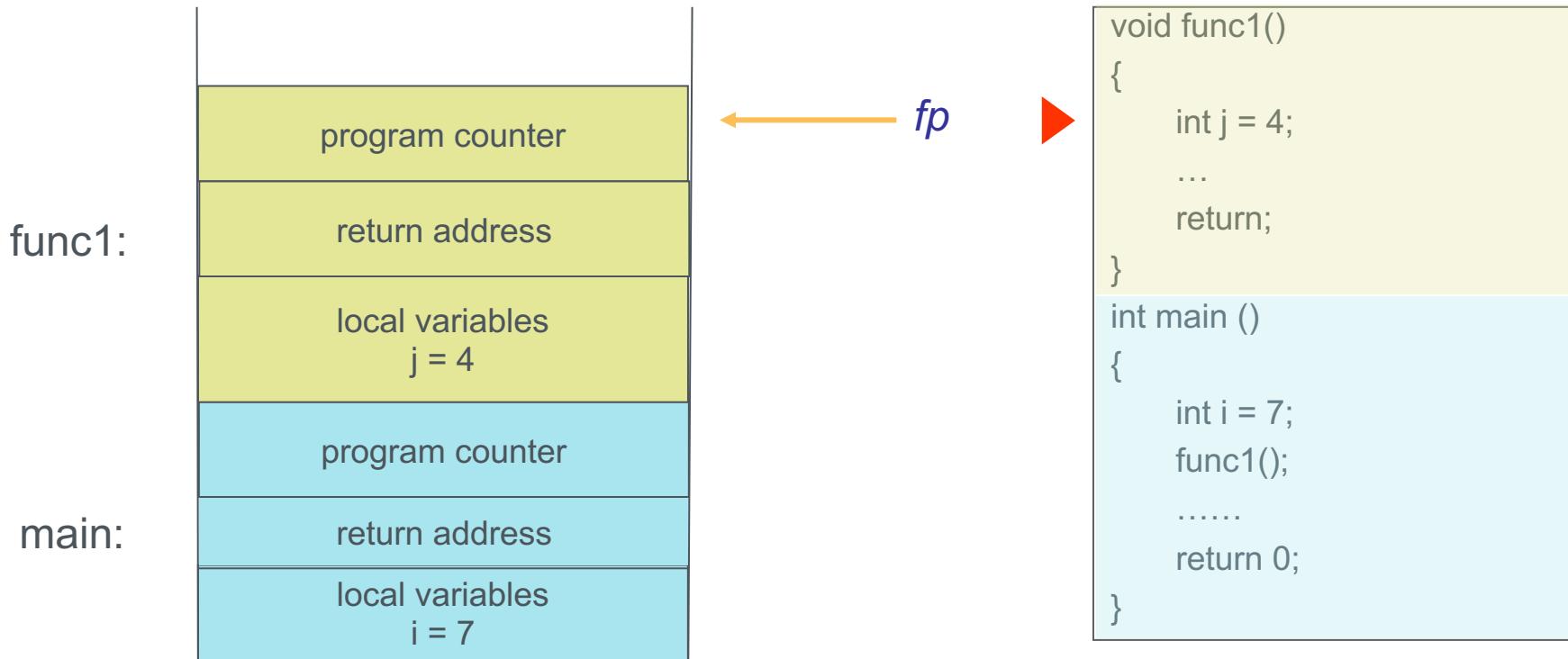
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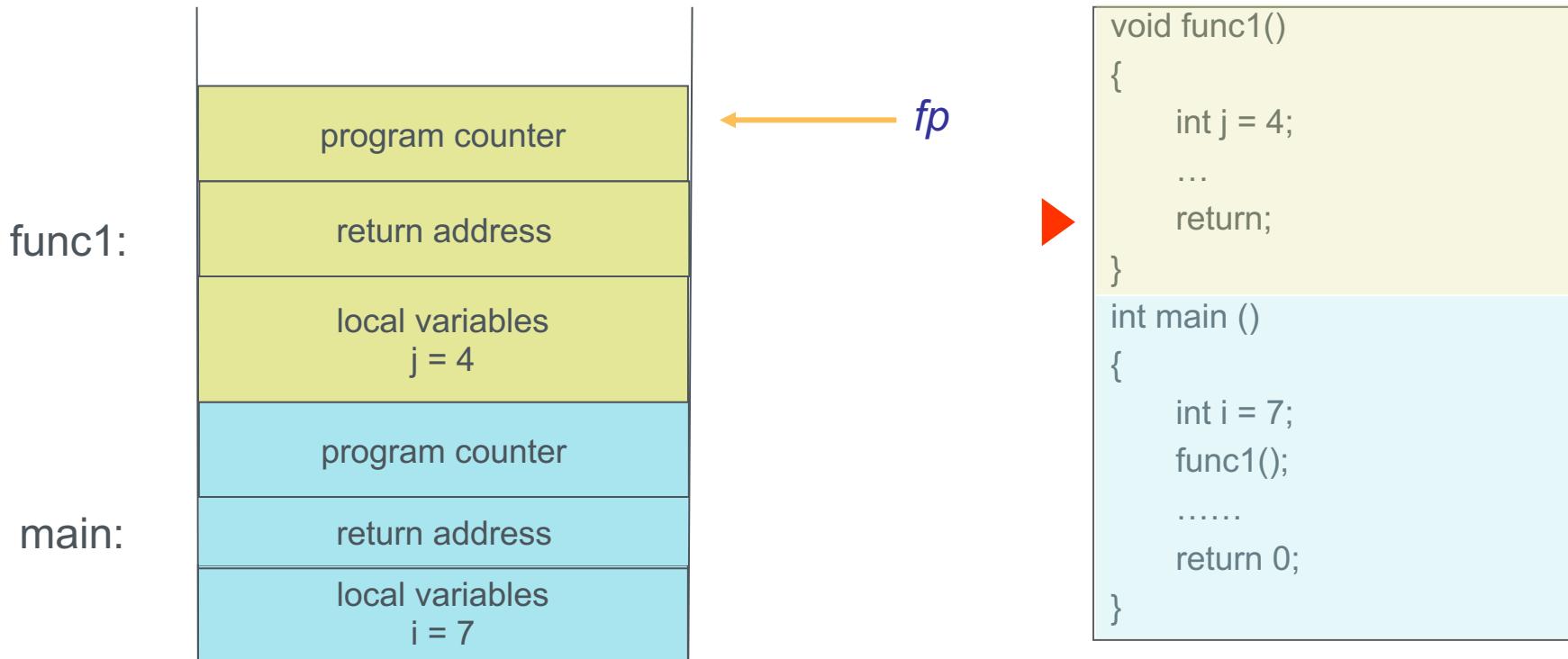
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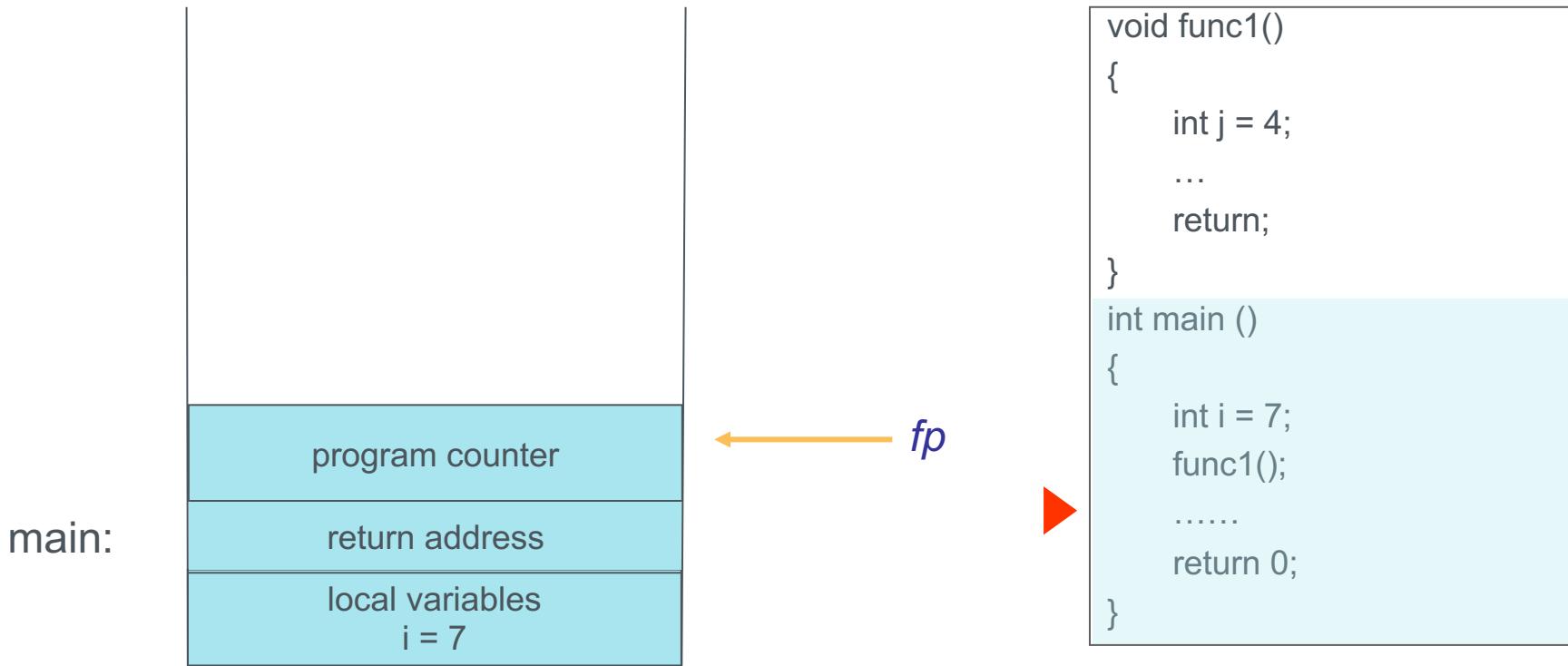
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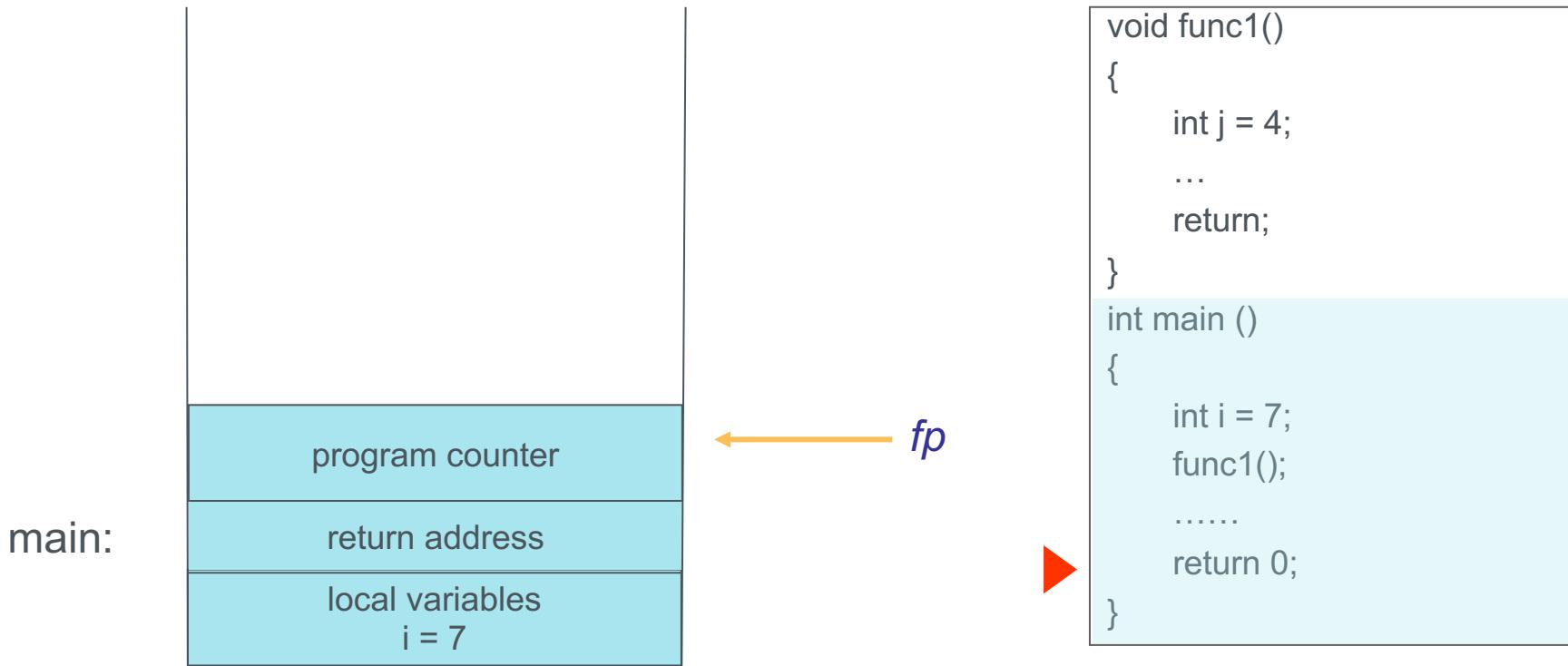
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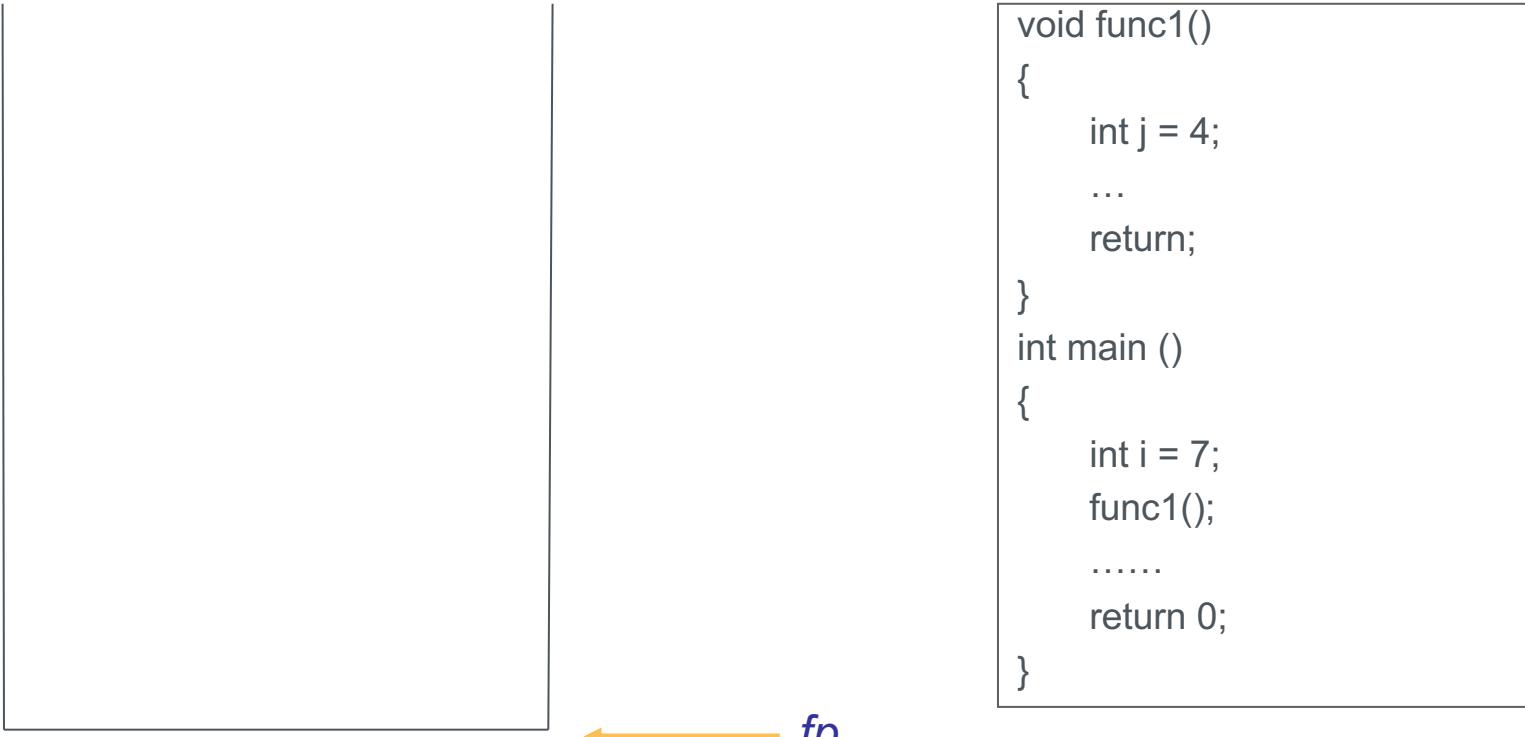
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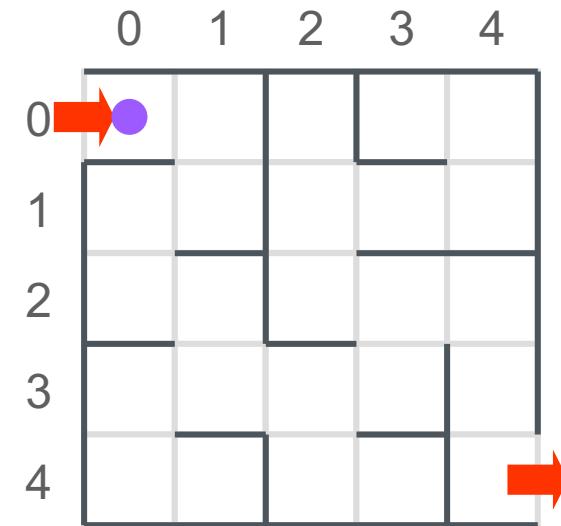


Stack and its Applications

- Path Planning for a Robot

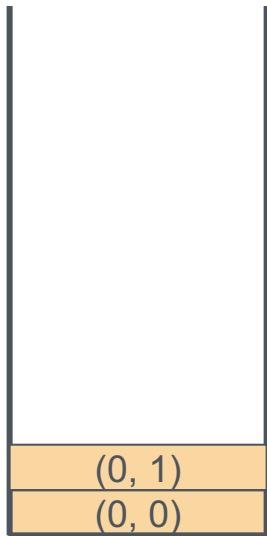


Stack

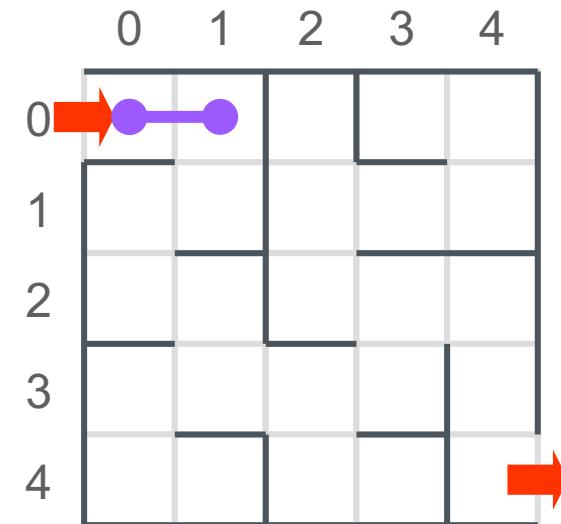


Stack and its Applications

- Path Planning for a Robot



Stack

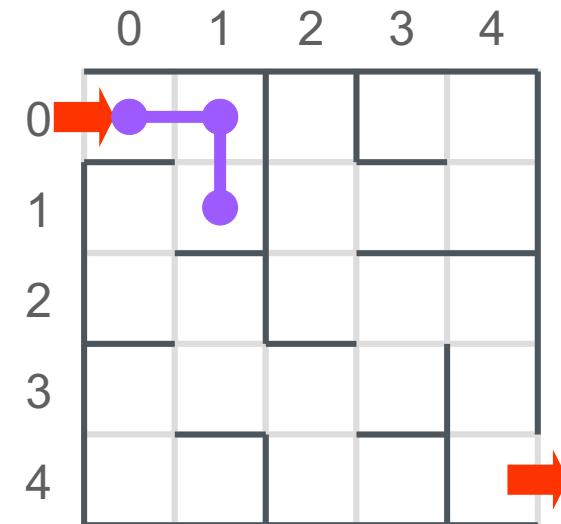


Stack and its Applications

- Path Planning for a Robot

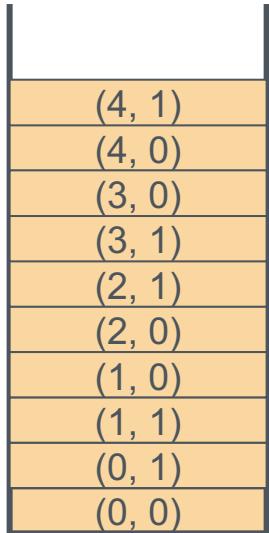


Stack

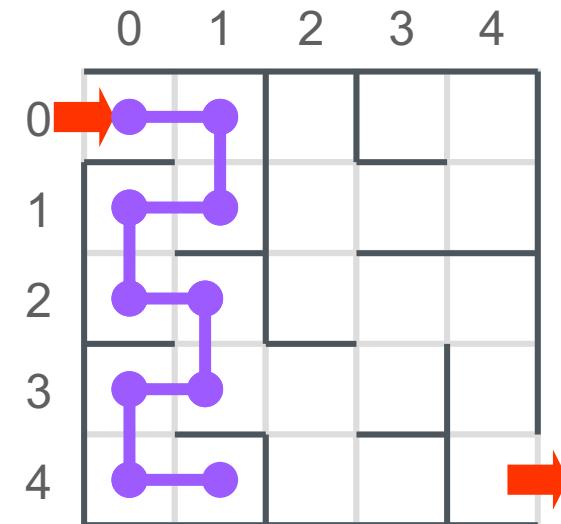


Stack and its Applications

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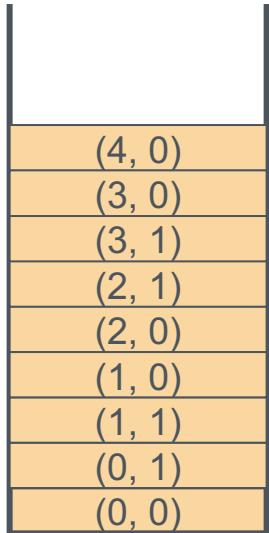


Stack

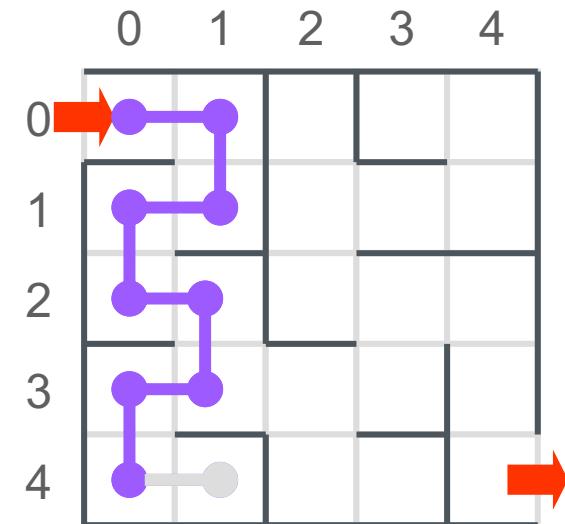


Stack and its Applications

- Path Planning for a Robot

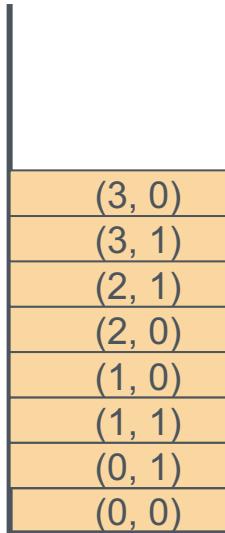


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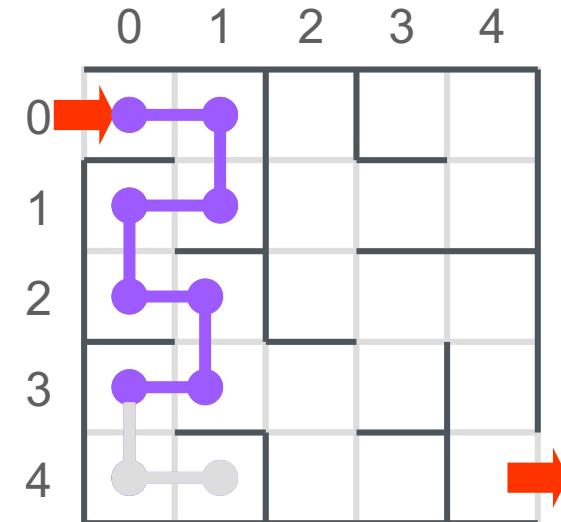


Stack and its Applications

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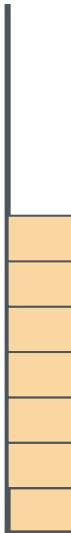


Stack

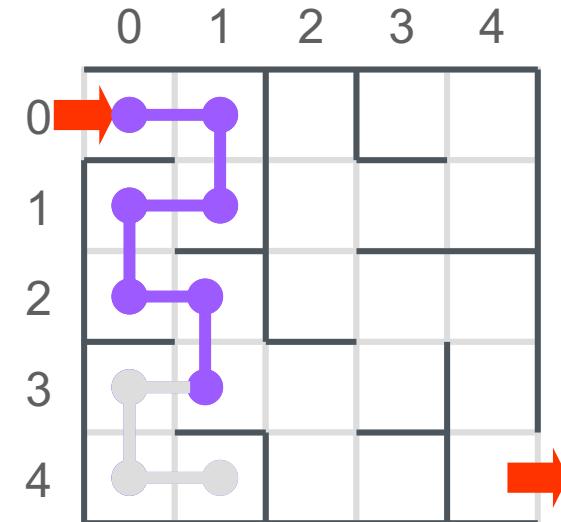


Stack and its Applications

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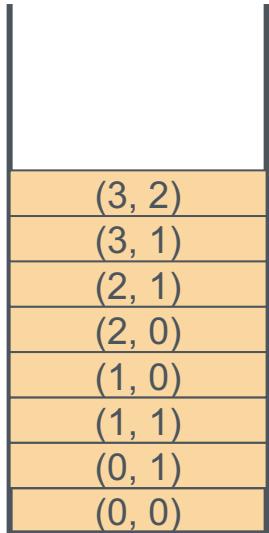


Stack

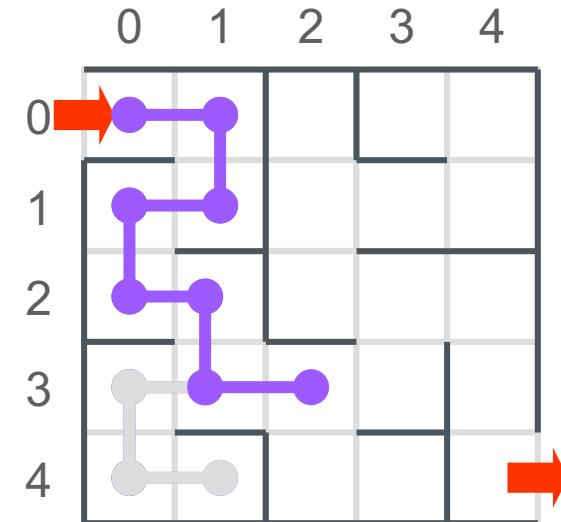


Stack and its Applications

- Path Planning for a Robot



Stack



ADT

- An ADT is a mathematical model (abstraction) of a data structure that specifies
 - the type of the data stored
 - the operations supported on them
 - the types of the parameters of the operations.
 - Error conditions associated with operations
- An **ADT specifies what each operation does, but not how it does it.**

The Stack ADT

- The Stack ADT stores arbitrary objects
- Insertions and deletions follow the last-in first-out scheme
- Main stack operations:
 - `push(object)`: inserts an element object
 - `pop()`: removes the top element
 - `object top()`: returns the last inserted element without removing it
 - `integer size()`: returns the number of elements stored in stack
 - Boolean `empty()`: indicates whether no elements are stored

```
template <typename E>
class Stack {                                // an interface for a stack
public:
    int size() const;                         // number of items in stack
    bool empty() const;                        // is the stack empty?
    const E& top() const throw(StackEmpty);   // the top element
    void push(const E& e);                   // push x onto the stack
    void pop() throw(StackEmpty);             // remove the top element
};
```

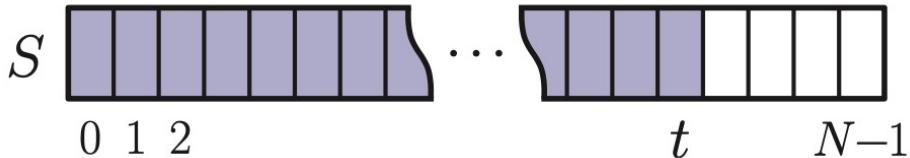
Exceptions for an ADT

- Attempting the execution of an operation of ADT may sometimes cause an error condition, called an exception
- Exceptions are said to be “**thrown**” by an operation that cannot be executed
- In the Stack ADT, operations **pop** and **top** cannot be performed if the stack is empty
- Attempting **pop** or **top** on an empty stack throws a **StackEmpty** exception

```
// Exception thrown on performing top or pop of an empty stack.  
class StackEmpty : public RuntimeException {  
public:  
    StackEmpty(const string& err) : RuntimeException(err) {}  
};
```

Array-Based Stack

- A simple way of implementing the Stack ADT uses an array
- We add elements from left to right
- A variable keeps track of the index of the top element



- The array storing the stack elements may become **full**
- A push operation will then throw a **StackFull** exception:
 - Limitation of the array-based implementation
 - Not intrinsic to the Stack ADT

```
Algorithm size():
    return  $t + 1$ 
Algorithm empty():
    return ( $t < 0$ )
Algorithm top():
    if empty() then
        throw StackEmpty exception
    return  $S[t]$ 
Algorithm push( $e$ ):
    if size() =  $N$  then
        throw StackFull exception
     $t \leftarrow t + 1$ 
     $S[t] \leftarrow e$ 
Algorithm pop():
    if empty() then
        throw StackEmpty exception
     $t \leftarrow t - 1$ 
```

Array-Based Stack - C++ Implementation

```
template <typename E>
class ArrayStack {
    enum { DEF_CAPACITY = 100 };           // default stack capacity
public:
    ArrayStack(int cap = DEF_CAPACITY);    // constructor from capacity
    int size() const;                      // number of items in the stack
    bool empty() const;                    // is the stack empty?
    const E& top() const throw(StackEmpty); // get the top element
    void push(const E& e) throw(StackFull); // push element onto stack
    void pop() throw(StackEmpty);          // pop the stack
    // ...housekeeping functions omitted
private:
    E* S;                                // member data
    int capacity;                         // array of stack elements
    int t;                                // stack capacity
                                         // index of the top of the stack
};
```

Array-Based Stack - C++ Implementation

```
template <typename E> ArrayStack<E>::ArrayStack(int cap)
: S(new E[cap]), capacity(cap), t(-1) { } // constructor from capacity

template <typename E> int ArrayStack<E>::size() const
{ return (t + 1); } // number of items in the stack

template <typename E> bool ArrayStack<E>::empty() const
{ return (t < 0); } // is the stack empty?

template <typename E> // return top of stack
const E& ArrayStack<E>::top() const throw(StackEmpty) {
    if (empty()) throw StackEmpty("Top of empty stack");
    return S[t];
}

template <typename E> // push element onto the stack
void ArrayStack<E>::push(const E& e) throw(StackFull) {
    if (size() == capacity) throw StackFull("Push to full stack");
    S[++t] = e;
}

template <typename E> // pop the stack
void ArrayStack<E>::pop() throw(StackEmpty) {
    if (empty()) throw StackEmpty("Pop from empty stack");
    --t;
}
```

Array-Based Stack - C++ Implementation

- Example use:

```
ArrayStack<int> A;
A.push(7);
A.push(13);
cout << A.top() << endl; A.pop();
A.push(9);
cout << A.top() << endl;
cout << A.top() << endl; A.pop();
ArrayStack<string> B(10);
B.push("Bob");
B.push("Alice");
cout << B.top() << endl; B.pop();
B.push("Eve");
```

// A = [], size = 0
// A = [7*], size = 1
// A = [7, 13*], size = 2
// A = [7*], outputs: 13
// A = [7, 9*], size = 2
// A = [7, 9*], outputs: 9
// A = [7*], outputs: 9
// B = [], size = 0
// B = [Bob*], size = 1
// B = [Bob, Alice*], size = 2
// B = [Bob*], outputs: Alice
// B = [Bob, Eve*], size = 2

Array-Based Stack - Performance

- Performance:
 - Let n be the number of elements in the stack
 - The space used is $O(n)$
 - Each operation runs in time $O(1)$
- Limitations:
 - The maximum size of the stack must be defined a priori and cannot be changed
 - Trying to push a new element into a full stack causes an implementation-specific exception

<i>Operation</i>	<i>Time</i>
size	$O(1)$
empty	$O(1)$
top	$O(1)$
push	$O(1)$
pop	$O(1)$

Generic Linked List-Based Implementation

- We have already developed:
 - Generic Singly Linked List (Lecture of Feb 7)
- We use that SLinkedList class
 - **S** stores the stack values
 - **n** stores the number of elements on stack

```
typedef string Elem;
class LinkedStack {
public:
    LinkedStack();
    int size() const;
    bool empty() const;
    const Elem& top() const throw(StackEmpty); // the top element
    void push(const Elem& e);
    void pop() throw(StackEmpty);
private:
    SLinkedList<Elem> S;
    int n;
};
```

// stack element type
// stack as a linked list

// constructor
// number of items in the stack
// is the stack empty?
// the top element
// push element onto stack
// pop the stack
// member data
// linked list of elements
// number of elements

Generic Singly Linked List

- We assumed elements were strings, now we want arbitrary element types
- It is easy using C++'s **Template** mechanism

```
class StringNode {
private:
    string elem; // element value
    StringNode* next; // next item in the list
    friend class StringLinkedList; // provide StringLinkedList access
};
```

Before

```
template <typename E>
class SNode { // singly linked list node
private:
    E elem; // linked list element value
    SNode<E>* next; // next item in the list
    friend class SLinkedList<E>; // provide SLinkedList access
};
```

After

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Generic Linked List-Based Implementation

```
LinkedStack::LinkedStack()
    : S(), n(0) { }                                // constructor

int LinkedStack::size() const
{ return n; }                                     // number of items in the stack

bool LinkedStack::empty() const
{ return n == 0; }                                // is the stack empty?

                                                // get the top element
const Elem& LinkedStack::top() const throw(StackEmpty) {
    if (empty()) throw StackEmpty("Top of empty stack");
    return S.front();
}

void LinkedStack::push(const Elem& e) {           // push element onto stack
    ++n;
    S.addFront(e);
}
                                                // pop the stack
void LinkedStack::pop() throw(StackEmpty) {
    if (empty()) throw StackEmpty("Pop from empty stack");
    --n;
    S.removeFront();
}
```

Parentheses Matching Problem

- Each "(", "{", or "[" must be paired with a matching ")" , "}" , or "[
 - correct: ()(()){([()])}
 - correct: ((()(()){([()])})
 - incorrect:)(()){([()])}
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Algorithm ParenMatch(X, n):

Input: An array X of n tokens, each of which is either a grouping symbol, a variable, an arithmetic operator, or a number

Output: true if and only if all the grouping symbols in X match

Let S be an empty stack

for $i \leftarrow 0$ to $n - 1$ **do**

if $X[i]$ is an opening grouping symbol **then**

$S.push(X[i])$

else if $X[i]$ is a closing grouping symbol **then**

if $S.empty()$ **then**

return false {nothing to match with}

if $S.top()$ does not match the type of $X[i]$ **then**

return false {wrong type}

$S.pop()$

if $S.empty()$ **then**

return true {every symbol matched}

else

return false {some symbols were never matched}

Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

19 Queues

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

March 7, 2022

Stack Example - Parentheses Matching Problem

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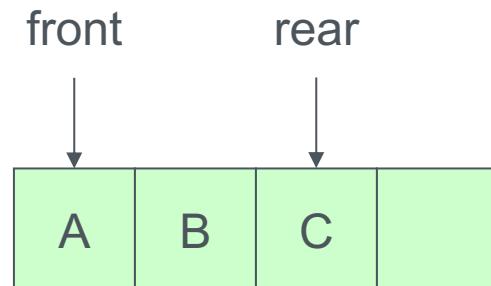
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Queue

- A queue is a container of elements that are inserted and removed according to the First-In First-Out (FIFO) principle.
- Elements enter queue at the **rear** and *are removed from the front*
“push” adds a new item on top of the stack.



Queue and its Applications

- Applications
 - Waiting lists, bureaucracy
 - Access to shared resources (e.g., printer)
 - Multiprogramming
 - As a data structure for algorithms to solve problems
 - BFS(Breadth First Search) for graphs
 - Component of other data structures

The Queue ADT

- The Queue ADT stores arbitrary objects
- Main stack operations:
 - enqueue(**e**): Insert element **e** at the rear of the queue
 - dequeue(): Remove element at the front of the queue; an error occurs if the queue is empty.
 - front(): Return, **but do not remove**, a reference to the front element in the queue; an error occurs if the queue is empty.
 - size(): Return the number of elements in the queue
 - empty(): Return **true** if the queue is empty and **false** otherwise.

```
template <typename E>
class Queue {                                     // an interface for a queue
public:
    int size() const;                           // number of items in queue
    bool empty() const;                         // is the queue empty?
    const E& front() const throw(QueueEmpty);   // the front element
    void enqueue (const E& e);                  // enqueue element at rear
    void dequeue() throw(QueueEmpty);           // dequeue element at front
};
```

Exceptions for Queue ADT

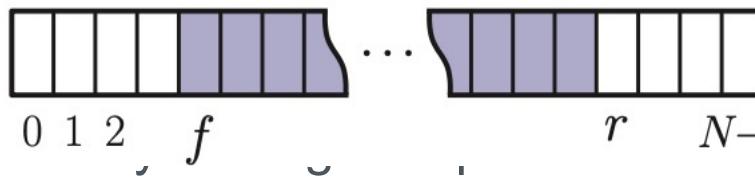
- In the Queue ADT, operations **dequeue** and **front** cannot be performed if the Queue is empty
- Attempting **dequeue** or **front** on an empty queue throws a **QueueEmpty** exception

```
class QueueEmpty : public RuntimeException {  
public:  
    QueueEmpty(const string& err) : RuntimeException(err) { }  
};
```

```
class RuntimeException {           // generic run-time exception  
private:  
    string errorMsg;  
public:  
    RuntimeException(const string& err) { errorMsg = err; }  
    string getMessage() const { return errorMsg; }  
};
```

Array-Based Queue

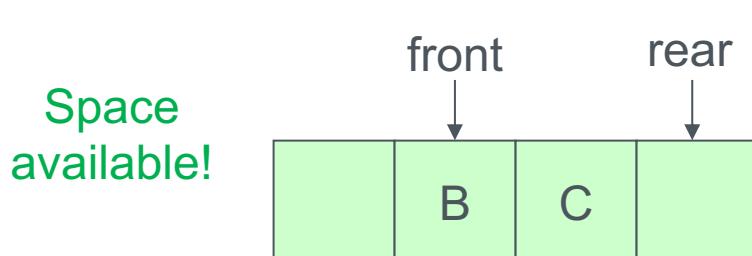
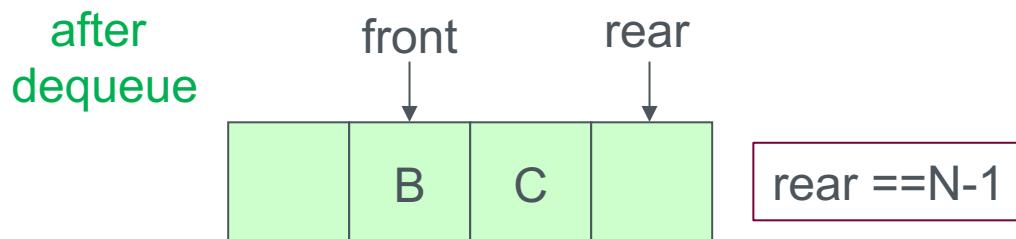
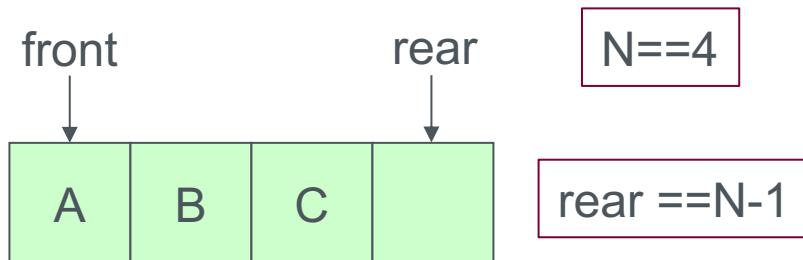
- An array with capacity N stores elements
- Two variables keeps track of the index of the front and rear element;
- f : index of front element
- r : index of element following rear element



- The indices $0, 1, 2, \dots, f, \dots, r, \dots, N-1$ may become **full**
- A **enqueue** operation will then throw a **QueueFull** exception

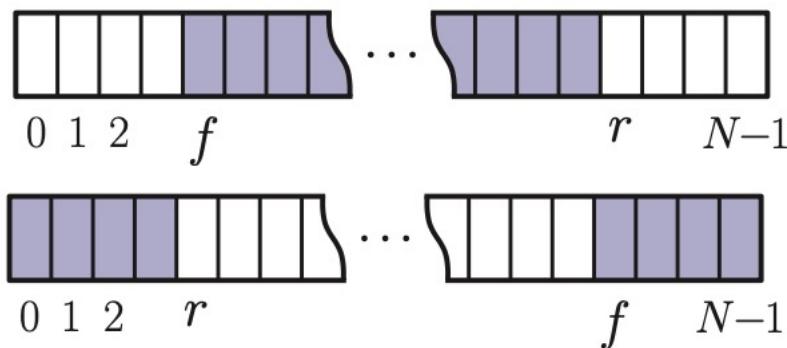
Array-Based Queue - An Issue

- f : index of front element
- r : index of element following rear element
- We can run into problem



Array-Based Queue - Circular Array

- An array with capacity N stores elements
- f : index of front element
- r : index of element following rear element
- Each time we increment f or r , we simply need to compute this increment as:
 - $(f + 1) \bmod N$
 - $(r + 1) \bmod N$
- modulo operator in C++ is `%`



Algorithm size():

return n

Algorithm empty():

return ($n = 0$)

Algorithm front():

if empty() **then**
 throw QueueEmpty exception
 return $Q[f]$

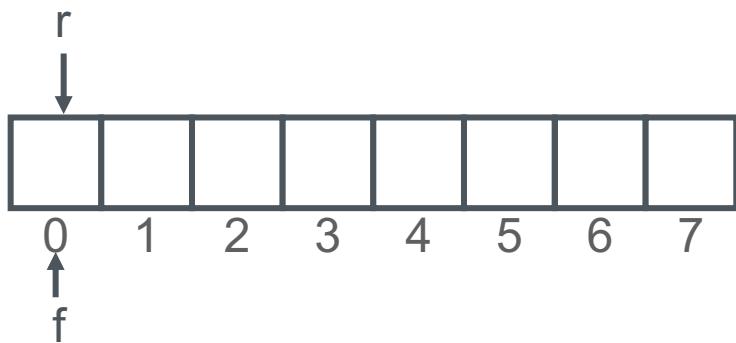
Algorithm dequeue():

if empty() **then**
 throw QueueEmpty exception
 $f \leftarrow (f + 1) \bmod N$
 $n = n - 1$

Algorithm enqueue(e):

if size() = N **then**
 throw QueueFull exception
 $Q[r] \leftarrow e$
 $r \leftarrow (r + 1) \bmod N$
 $n = n + 1$

Array-Based Queue - Circular Array



Enq (A)

Enq (B)

Enq (C)

Enq (D)

Deq ()

Deq ()

Deq ()

Enq (E)

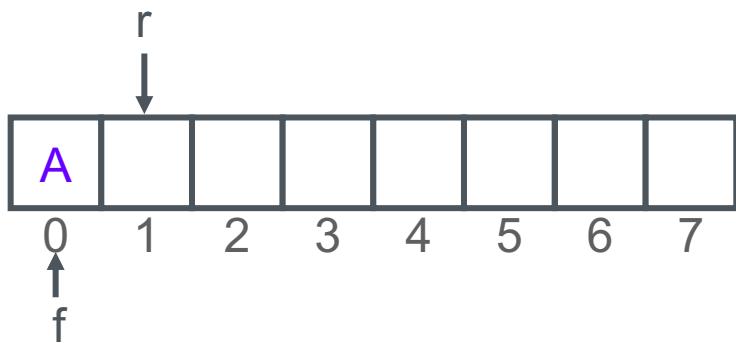
Enq (F)

Enq (G)

Enq (H)

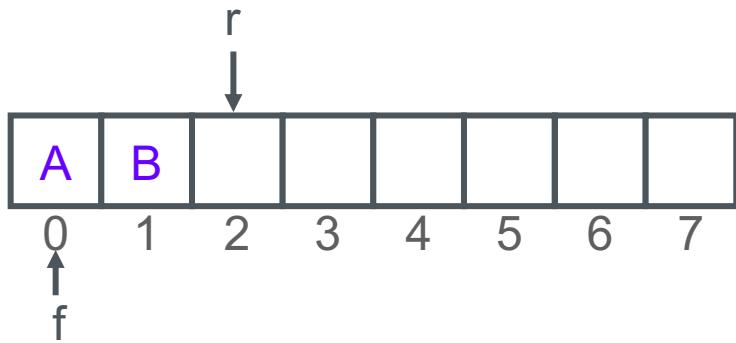
Enq (I)

Array-Based Queue - Circular Array



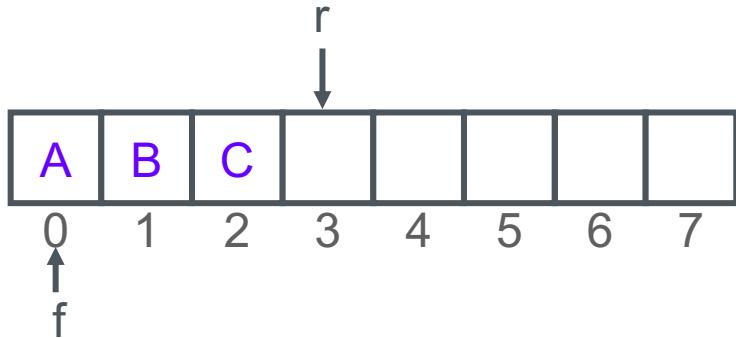
- ▶ Enq (A)
- Enq (B)
- Enq (C)
- Enq (D)
- Deq ()
- Deq ()
- Deq ()
- Enq (E)
- Enq (F)
- Enq (G)
- Enq (H)
- Enq (I)

Array-Based Queue - Circular Array



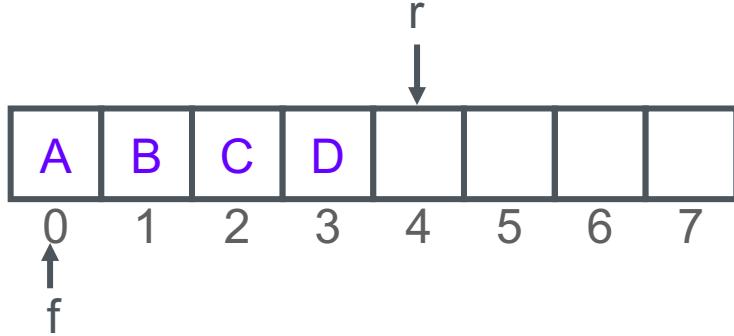
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Array-Based Queue - Circular Array



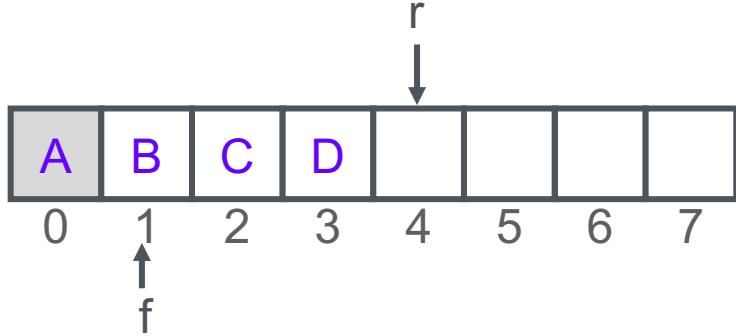
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Array-Based Queue - Circular Array



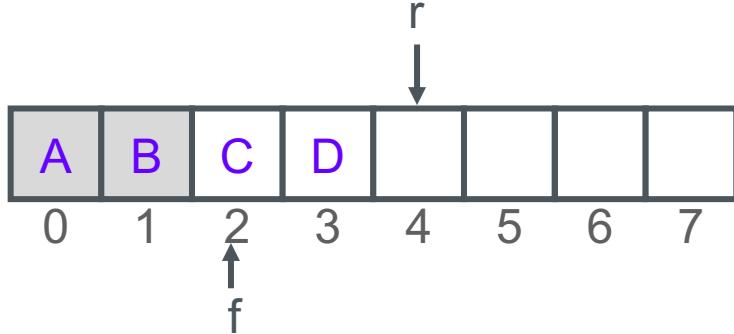
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Array-Based Queue - Circular Array



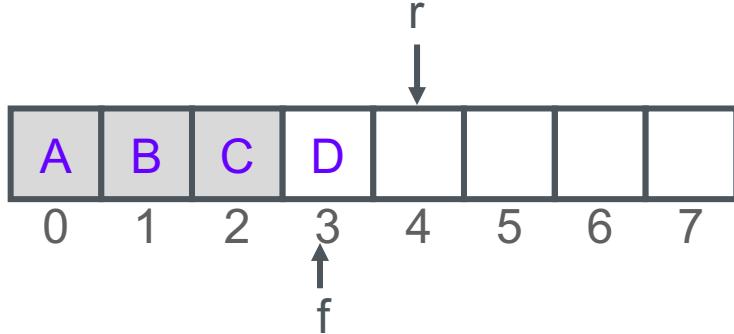
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- Enq (E)
- Enq (F)
- Enq (G)
- Enq (H)
- Enq (I)

Array-Based Queue - Circular Array



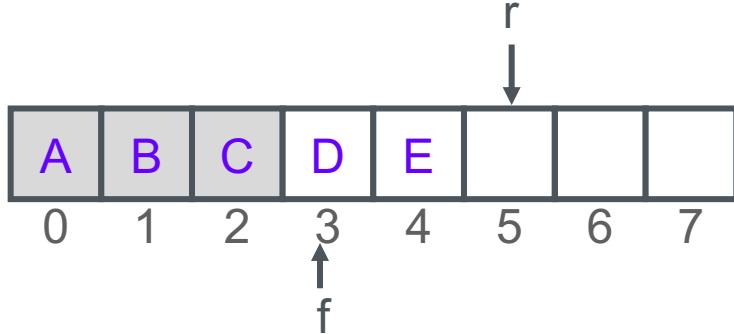
- Enq (A)
- Enq (B)
- Enq (C)
- Enq (D)
- Deq ()
- Deq ()
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Array-Based Queue - Circular Array



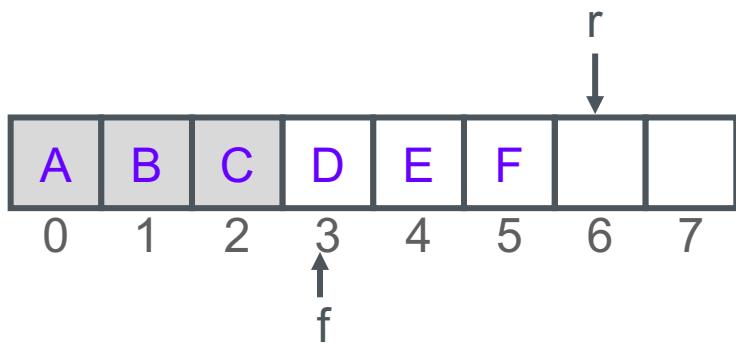
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Array-Based Queue - Circular Array



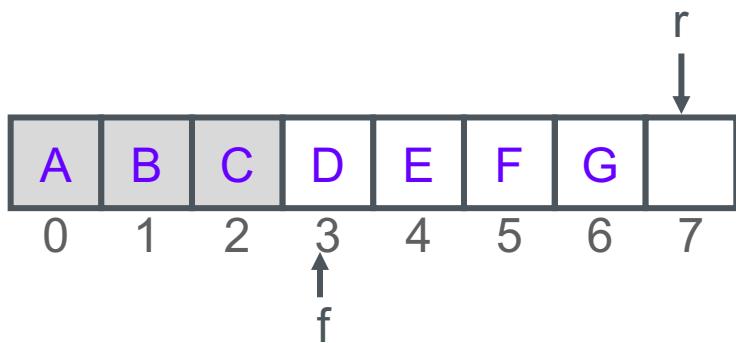
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Array-Based Queue - Circular Array



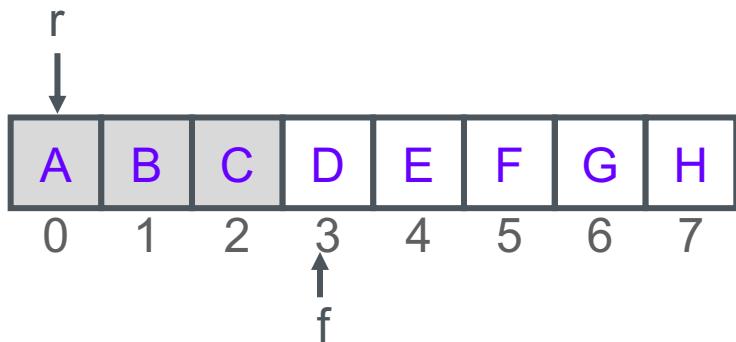
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Array-Based Queue - Circular Array



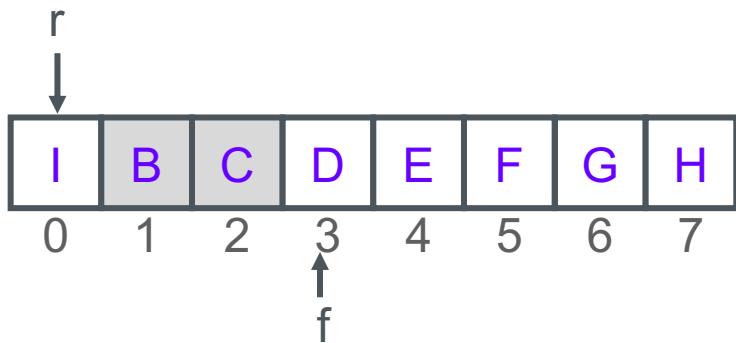
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Array-Based Queue - Circular Array



- Enq (A)
- Enq (B)
- Enq (C)
- Enq (D)
- Deq ()
- Deq ()
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- Enq (E)
- Enq (F)
- Enq (G)
- ▶ Enq (H)
- Enq (I)

Array-Based Queue - Circular Array



Enq (A)

Enq (B)

Enq (C)

Enq (D)

Deq ()

Deq ()

Deq ()

Enq (E)

Enq (F)

Enq (G)

Enq (H)

► Enq (I)

Array-Based Queue - Performance

- Performance:
 - Let n be the number of elements in the Queue
 - The space used is $O(N)$
 - this is independent of the number of elements n
 - Each operation runs in time $O(1)$
- Limitations:
 - The maximum size of the Queue must be defined beforehand and cannot be changed
 - Trying to enqueue a new element into a full queue causes an implementation-specific exception (QueueFull Exception)

Circularly Linked List-Based Implementation

- We use that CircleList class
 - C stores the Queue elements
 - n stores the number of elements on Queue

```
typedef string Elem;                                // queue element type
class LinkedQueue {                                 // queue as doubly linked list
public:
    LinkedQueue();                                  // constructor
    int size() const;                             // number of items in the queue
    bool empty() const;                           // is the queue empty?
    const Elem& front() const throw(QueueEmpty); // the front element
    void enqueue(const Elem& e);                // enqueue element at rear
    void dequeue() throw(QueueEmpty);             // dequeue element at front
private:
    CircleList C;                                // member data
    int n;                                       // circular list of elements
};                                              // number of elements
```

Circularly Linked List-Based Implementation

```
LinkedQueue::LinkedQueue()           // constructor
    : C(), n(0) { }

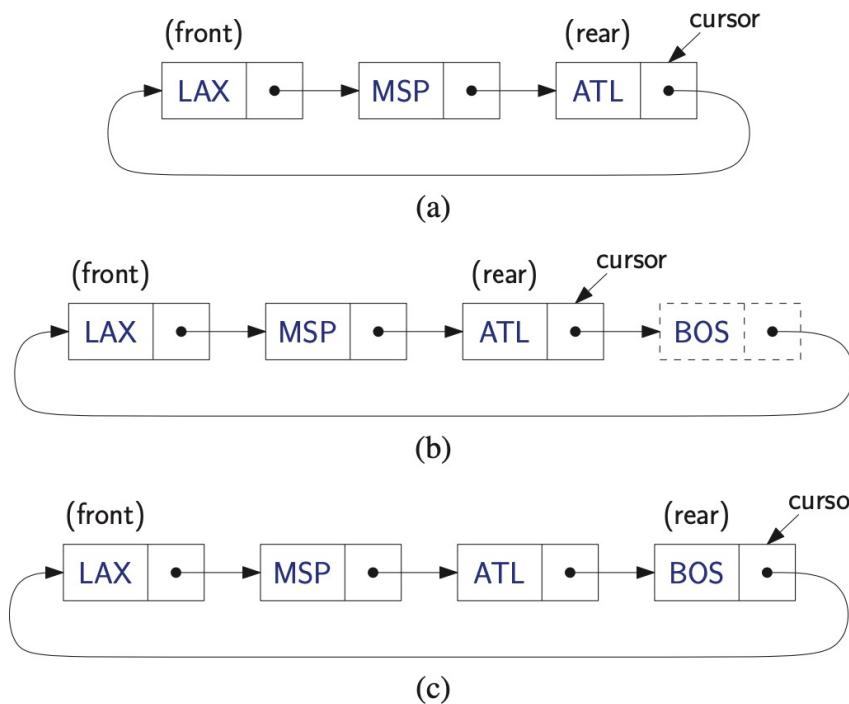
int LinkedQueue::size() const        // number of items in the queue
{ return n; }

bool LinkedQueue::empty() const      // is the queue empty?
{ return n == 0; }

const Elem& LinkedQueue::front() const throw(QueueEmpty) {
    if (empty())
        throw QueueEmpty("front of empty queue");
    return C.front();                  // list front is queue front
}
```

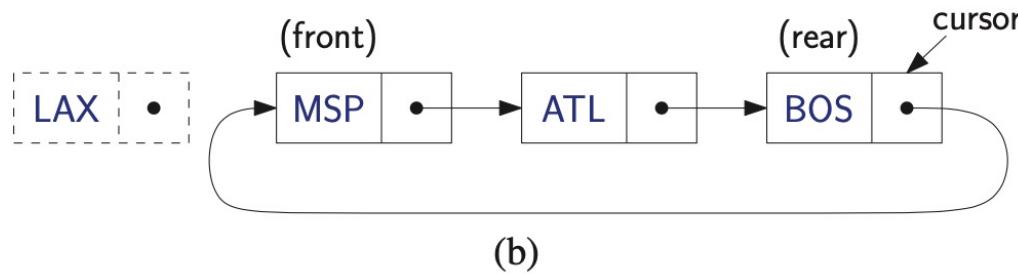
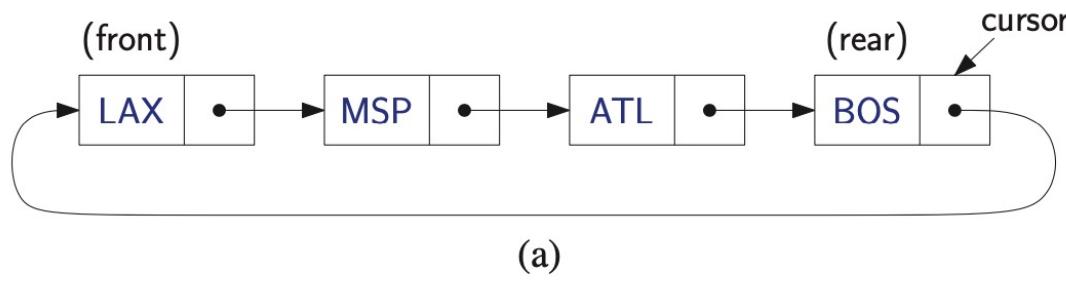
Circularly Linked List-Based Implementation

```
// enqueue element at rear  
void LinkedQueue::enqueue(const Elem& e) {  
    C.add(e);  
    C.advance();  
    n++;  
}
```



Circularly Linked List-Based Implementation

```
// dequeue element at front  
void LinkedQueue::dequeue() throw(QueueEmpty) {  
    if (empty())  
        throw QueueEmpty("dequeue of empty queue");  
    C.remove(); // remove from list front  
    n--;  
}
```



Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

19 Queues

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

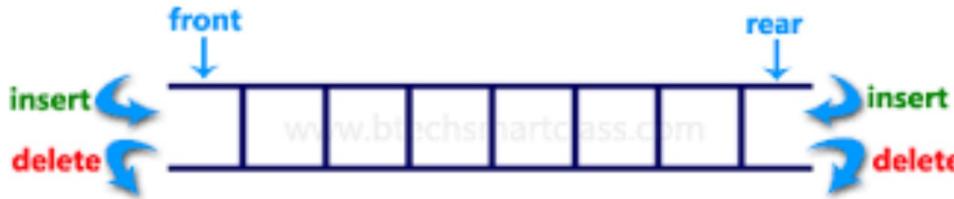
March 7, 2022

Admin.

- One more day for the assignment 2!

Double-Ended Queues

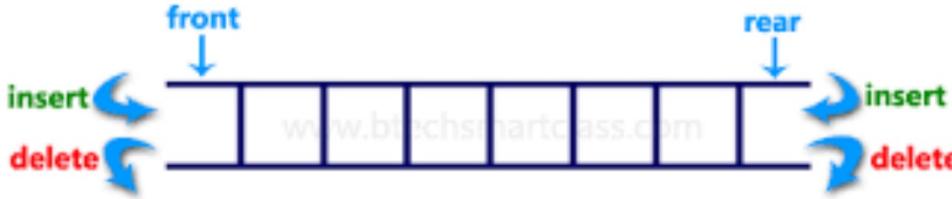
- Double-Ended Queues (sometimes pronounced like "deck")
 - supports insertion and deletion at both the front and the rear of the queue



- **insertFront(e)**: Insert a new element e at the beginning of the deque.
- **insertBack(e)**: Insert a new element e at the end of the deque.
- **eraseFront()**: Remove the first element of the deque; an error occurs if the deque is empty.
- **eraseBack()**: Remove the last element of the deque; an error occurs if the deque is empty.
- **front()**: Return the first element of the deque; an error occurs if the deque is empty.
- **back()**: Return the last element of the deque; an error occurs if the deque is empty.
- **size()**: Return the number of elements of the deque.
- **empty()**: Return true if the deque is empty and false otherwise.

Double-Ended Queues

- Double-Ended Queues (sometimes pronounced like "deck")
 - supports insertion and deletion at both the front and the rear of the queue



- A running example:

<i>Operation</i>	<i>Output</i>	<i>D</i>
insertFront(3)	–	(3)
insertFront(5)	–	(5, 3)
front()	5	(5, 3)
eraseFront()	–	(3)
insertBack(7)	–	(3, 7)
back()	7	(3, 7)
eraseFront()	–	(7)
eraseBack()	–	()

Implementation with Doubly Linked List



- We will use the functionalities provided by the DLL to implement `LinkedDeque`'s functions. We have seen this pattern a few times before.

```
typedef string Elem;                                // deque element type
class LinkedDeque {                                 // deque as doubly linked list
public:
    LinkedDeque();                                  // constructor
    int size() const;                             // number of items in the deque
    bool empty() const;                           // is the deque empty?
    const Elem& front() const throw(DequeEmpty); // the first element
    const Elem& back() const throw(DequeEmpty); // the last element
    void insertFront(const Elem& e);           // insert new first element
    void insertBack(const Elem& e);            // insert new last element
    void removeFront() throw(DequeEmpty);        // remove first element
    void removeBack() throw(DequeEmpty);         // remove last element
private:
    DLinkedList D;                               // member data
    int n;                                     // linked list of elements
};                                              // number of elements
```

Implementation with Doubly Linked List



- We will use the functionalities provided by the DLL to implement LinkedDeque's functions. We have seen this pattern a few times before.
- Performance of a deque realized by a doubly linked list.

<i>Operation</i>	<i>Time</i>
size	$O(1)$
empty	$O(1)$
front, back	$O(1)$
insertFront, insertBack	$O(1)$
eraseFront, eraseBack	$O(1)$

- The space used usage is $O(n)$

```
// insert new first element
void LinkedDeque::insertFront(const Elem& e) {
    D.addFront(e); ←
    n++;
}

// insert new last element
void LinkedDeque::insertBack(const Elem& e) {
    D.addBack(e); ←
    n++;
}

// remove first element
void LinkedDeque::removeFront() throw(DequeEmpty) {
    if (empty())
        throw DequeEmpty("removeFront of empty deque");
    D.removeFront(); ←
    n--;
}

// remove last element
void LinkedDeque::removeBack() throw(DequeEmpty) {
    if (empty())
        throw DequeEmpty("removeBack of empty deque");
    D.removeBack(); ←
    n--;
}
```

Adapter Design Pattern

- Design pattern: which describes a solution to a “typical” software design problem.
 - provides a general template for a solution that can be applied in many different situations.
 - describes the main elements of a solution in an abstract way that can be specialized for a specific problem at hand.
 - In Algorithms:
 - Recursion
 - Using Stack to solve problems
 - In Software Engineering
 - Adapter pattern
 - Iterator pattern
- You remember from previous LinkedDeque, and also Circular Linked List-based implementation of Queue

Adapter Design Pattern

- You remember from previous LinkedDeque, and also Circular Linked List-based implementation of Queue that we took an existing data structure and **adapted** it
 - E.g. we added size **n**
 - We added operations that are meaningful for the new data structure
- For the operations, we have simply **mapped** each deque operation to the corresponding operation of DLinkedList.
- An adapter (also called a wrapper) is a data structure that translates one interface to another.
 - e.g.: In the LinkedDeque implementation:
 - deque operation **insertFront** is mapped to the corresponding operation of DLinkedList **addFront**

Adapter Design Pattern

- Implementing a Stack using Deque:

```
typedef string Elem;                                // element type
class DequeStack {                                     // stack as a deque
public:
    DequeStack();                                     // constructor
    int size() const;                                 // number of elements
    bool empty() const;                             // is the stack empty?
    const Elem& top() const throw(StackEmpty); // the top element
    void push(const Elem& e);                      // push element onto stack
    void pop() throw(StackEmpty);                   // pop the stack
private:
    LinkedDeque D;                                  // deque of elements
};
```

Stack Method	Deque Implementation
size()	size()
empty()	empty()
top()	front()
push(<i>o</i>)	insertFront(<i>o</i>)
pop()	eraseFront()

```
DequeStack::DequeStack()                                // constructor
    : D() { }

int DequeStack::size() const                         // number of elements
{ return D.size(); }

bool DequeStack::empty() const                       // is the stack empty?
{ return D.empty(); }

const Elem& DequeStack::top() const throw(StackEmpty) // the top element
{
    if (empty())
        throw StackEmpty("top of empty stack");
    return D.front();
}

void DequeStack::push(const Elem& e)                // push element onto stack
{ D.insertFront(e); }

void DequeStack::pop() throw(StackEmpty)             // pop the stack
{
    if (empty())
        throw StackEmpty("pop of empty stack");
    D.removeFront();
}
```

Adapter Design Pattern

- Implementing a Queue using Deque:

<i>Queue Method</i>	<i>Deque Implementation</i>
<code>size()</code>	<code>size()</code>
<code>empty()</code>	<code>empty()</code>
<code>front()</code>	<code>front()</code>
<code>enqueue(<i>e</i>)</code>	<code>insertBack(<i>e</i>)</code>
<code>dequeue()</code>	<code>eraseFront()</code>

- The operations are equally efficient.
- We have used and will use this design pattern many times.

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.
 - string (String class with all operations)
 - stack (Container with last-in, first-out access)
 - queue (Container with first-in, first-out access)
 - deque (Double-ended queue)
 - vector (Resizable array)
 - list (Doubly linked list)
 - priority queue (Queue ordered by value)
 - set (Set)
 - map Associative array (dictionary)

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.

- o string:

```
#include <string>
using std::string;
// ...
string s = "to be";
string t = "not " + s;           // t = "not to be"
string u = s + " or " + t;       // u = "to be or not to be"
if (s > t)                      // true: "to be" > "not to be"
    cout << u;                  // outputs "to be or not to be"

string s = "John";
int i = s.size();                // s = "John"
char c = s[3];                  // i = 4
c = 'n';                        // c = 'n'
s += " Smith";                 // now s = "John Smith"
```

s.find(p)	Return the index of first occurrence of string <i>p</i> in <i>s</i>
s.find(p, i)	Return the index of first occurrence of string <i>p</i> in <i>s</i> on or after position <i>i</i>
s.substr(i, m)	Return the substring starting at position <i>i</i> of <i>s</i> and consisting of <i>m</i> characters
s.insert(i, p)	Insert string <i>p</i> just prior to index <i>i</i> in <i>s</i>
s.erase(i, m)	Remove the substring of length <i>m</i> starting at index <i>i</i>
s.replace(i, m, p)	Replace the substring of length <i>m</i> starting at index <i>i</i> with <i>p</i>
getline(is, s)	Read a single line from the input stream <i>is</i> and store the result in <i>s</i>

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.
 - vector:

```
#include <vector>
using namespace std;                                // make std accessible

vector<int> scores(100);                          // 100 integer scores
vector<char> buffer(500);                         // buffer of 500 characters
vector<Passenger> passenList(20);                 // list of 20 Passengers
```

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.
 - stack:

```
#include <stack>
using std::stack;                                // make stack accessible
stack<int> myStack;                             // a stack of integers
```

`size()`: Return the number of elements in the stack.

`empty()`: Return true if the stack is empty and false otherwise.

`push(e)`: Push *e* onto the top of the stack.

`pop()`: Pop the element at the top of the stack.

`top()`: Return a reference to the element at the top of the stack.

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.
 - queue:

```
#include <queue>
using std::queue;
queue<float> myQueue;           // make queue accessible
                                // a queue of floats
```

`size()`: Return the number of elements in the queue.

`empty()`: Return true if the queue is empty and false otherwise.

`push(e)`: Enqueue *e* at the rear of the queue.

`pop()`: Dequeue the element at the front of the queue.

`front()`: Return a reference to the element at the queue's front.

`back()`: Return a reference to the element at the queue's rear.

Standard Template Library (STL)

- The Standard Template Library (STL) is a collection of classes for common data structures. In addition to the string class, which we have seen many times, it also provides data structures for the following standard containers.
 - deque:

```
#include <deque>
using std::deque;                                // make deque accessible
deque<string> myDeque;                          // a deque of strings
```

- `size()`: Return the number of elements in the deque.
- `empty()`: Return true if the deque is empty and false otherwise.
- `push_front(e)`: Insert *e* at the beginning the deque.
- `push_back(e)`: Insert *e* at the end of the deque.
- `pop_front()`: Remove the first element of the deque.
- `pop_back()`: Remove the last element of the deque.
- `front()`: Return a reference to the deque's first element.
- `back()`: Return a reference to the deque's last element.

List and Sequence Containers

- Vector (also called Array List)
 - Access each element using a notion of index in $[0, n-1]$
 - Index of element e: the number of elements that are before e
 - Typically we use the “index” (e.g., `[]`)
 - A more general ADT than “array”
- List
 - Not using an index to access, but use a node to access
 - Insert a new element e before some “position” p
 - A more general ADT than “linked list”
- Sequence
 - Can access an element as vector and list (using both index and position)

The Vector ADT

- The Vector or Array List ADT extends the notion of array by storing a sequence of objects
- An element can be accessed, inserted or removed by specifying its index (number of elements preceding it)
- An exception is thrown if an incorrect index is given (e.g., a negative index)
- Main methods:
 - $\text{at}(i)$: Return the element of V with index i ; an error condition occurs if i is out of range.
 - $\text{set}(i, e)$: Replace the element at index i with e ; an error condition occurs if i is out of range.
 - $\text{insert}(i, e)$: Insert a new element e into V to have index i ; an error condition occurs if i is out of range.
 - $\text{erase}(i)$: Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector

- Use an array A of size N
- A variable n keeps track of the size of the array list (number of elements stored)
- Operation **at(i)** is implemented in O(1) time by returning $A[i]$
- Operation **set(i,o)** is implemented in O(1) time by performing $A[i] = o$



at(i): Return the element of V with index i ; an error condition occurs if i is out of range.

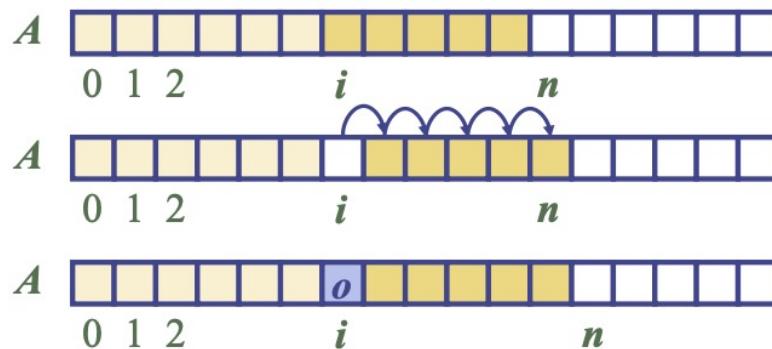
set(i, e): Replace the element at index i with e ; an error condition occurs if i is out of range.

insert(i, e): Insert a new element e into V to have index i ; an error condition occurs if i is out of range.

erase(i): Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector - Insertion

- In operation $\text{insert}(i, o)$, we need to make room for the new element by shifting forward the $n - i$ elements $A[i], \dots, A[n - 1]$
 - In the worst case ($i = 0$), this takes $O(n)$ time



$\text{at}(i)$: Return the element of V with index i ; an error condition occurs if i is out of range.

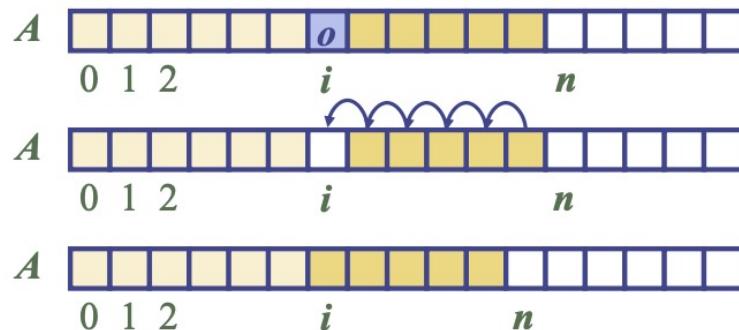
$\text{set}(i, e)$: Replace the element at index i with e ; an error condition occurs if i is out of range.

$\text{insert}(i, e)$: Insert a new element e into V to have index i ; an error condition occurs if i is out of range.

$\text{erase}(i)$: Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector - Removal

- In operation `erase(i)`, we need to fill the hole left by the removed element by shifting backward the $n - i - 1$ elements $A[i + 1], \dots, A[n - 1]$
 - In the worst case ($i = 0$), this takes $O(n)$ time



`at(i)`: Return the element of V with index i ; an error condition occurs if i is out of range.

`set(i, e)`: Replace the element at index i with e ; an error condition occurs if i is out of range.

`insert(i, e)`: Insert a new element e into V to have index i ; an error condition occurs if i is out of range.

`erase(i)`: Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector - Performance

- In the array-based implementation of an array list:
 - The space used by the data structure is $O(n)$
 - `size`, `empty`, `at` and `set` run in $O(1)$ time
 - `insert` and `erase` run in $O(n)$ time in worst case
- If we use the array in a circular fashion, operations `insert(0, x)` and `erase(0, x)` run in $O(1)$ time
- In an insert operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one

<i>Operation</i>	<i>Time</i>
<code>size()</code>	$O(1)$
<code>empty()</code>	$O(1)$
<code>at(i)</code>	$O(1)$
<code>set(i, e)</code>	$O(1)$
<code>insert(i, e)</code>	$O(n)$
<code>erase(i)</code>	$O(n)$

Comparison of the Strategies

- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n insert(o) operations
- We assume that we start with an empty stack represented by an array of size 1
- We call amortized time of an insert operation the average time taken by an insert over the series of operations, i.e., $T(n)/n$

Incremental Strategy Analysis

- We replace the array $k = n/c$ times
- The total time $T(n)$ of a series of n insert operations is proportional to

$$n + c + 2c + 3c + 4c + \dots + kc =$$

$$n + c(1 + 2 + 3 + \dots + k) =$$

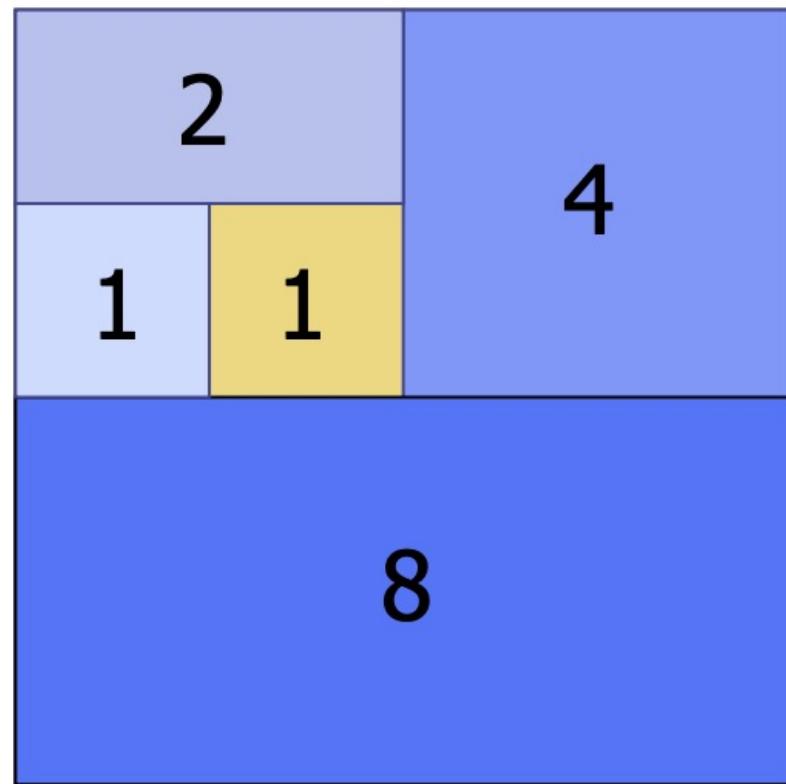
$$n + ck(k + 1)/2$$

- Since c is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$
- The amortized time of an insert operation is $O(n)$

Doubling Strategy Analysis

- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n insert operations is proportional to
- $$n + 1 + 2 + 4 + 8 + \dots + 2^k = \\ n + 2^{k+1} - 1 =$$
- $3n - 1$
- $T(n)$ is $O(n)$
- The amortized time of an insert operation is $O(1)$

geometric series



Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

21 Containers

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

March 10, 2022

List and Sequence Containers

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- Operation **set(i,o)** is implemented in O(1) time by performing $A[i] = o$



at(i): Return the element of V with index i ; an error condition occurs if i is out of range.

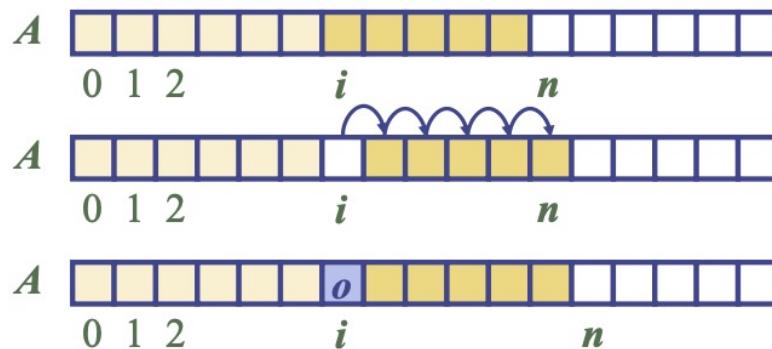
set(i, e): Replace the element at index i with e ; an error condition occurs if i is out of range.

insert(i, e): Insert a new element e into V to have index i ; an error condition occurs if i is out of range.

erase(i): Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector - Insertion

- In operation $\text{insert}(i, o)$, we need to make room for the new element by shifting forward the $n - i$ elements $A[i], \dots, A[n - 1]$
 - In the worst case ($i = 0$), this takes $O(n)$ time



$\text{at}(i)$: Return the element of V with index i ; an error condition occurs if i is out of range.

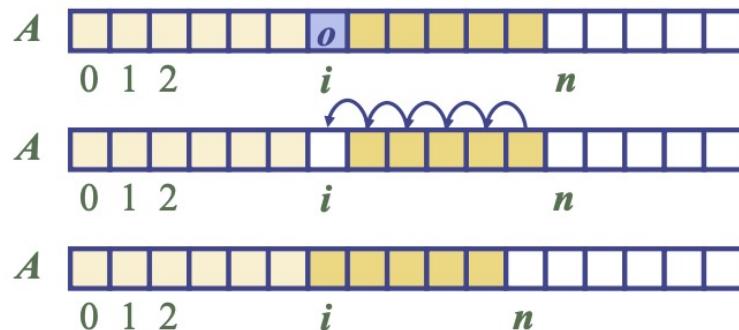
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$\text{erase}(i)$: Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector - Removal

- In operation `erase(i)`, we need to fill the hole left by the removed element by shifting backward the $n - i - 1$ elements $A[i + 1], \dots, A[n - 1]$
 - In the worst case ($i = 0$), this takes $O(n)$ time



`at(i)`: Return the element of V with index i ; an error condition occurs if i is out of range.

`set(i, e)`: Replace the element at index i with e ; an error condition occurs if i is out of range.

`insert(i, e)`: Insert a new element e into V to have index i ; an error condition occurs if i is out of range.

`erase(i)`: Remove from V the element at index i ; an error condition occurs if i is out of range.

Array-based Implementation of Vector in C++

```
typedef int Elem;                                // base element type
class ArrayVector {
public:
    ArrayVector();                                // constructor
    int size() const;                            // number of elements
    bool empty() const;                          // is vector empty?
    Elem& operator[](int i);                    // element at index
    Elem& at(int i) throw(IndexOutOfBoundsException); // element at index
    void erase(int i);                           // remove element at index
    void insert(int i, const Elem& e);        // insert element at index
    void reserve(int N);                        // reserve at least N spots
    // ... (housekeeping functions omitted)
private:
    int capacity;                             // current array size
    int n;                                    // number of elements in vector
    Elem* A;                                  // array storing the elements
};
```

Array-based Implementation of Vector in C++

```
ArrayVector::ArrayVector()           // constructor
: capacity(0), n(0), A(NULL) { }

int ArrayVector::size() const      // number of elements
{ return n; }

bool ArrayVector::empty() const    // is vector empty?
{ return size() == 0; }

Elem& ArrayVector::operator[](int i)      // element at index
{ return A[i]; }

Elem& ArrayVector::at(int i) throw(IndexOutOfBoundsException) {
    if (i < 0 || i >= n)
        throw IndexOutOfBoundsException("illegal index in function at()");
    return A[i];
}

void ArrayVector::erase(int i) {          // remove element at index
    for (int j = i+1; j < n; j++)
        A[j - 1] = A[j];
    n--;
}
```

Array-based Implementation of Vector in C++

- The reserve function first checks whether the capacity already exceeds n , in which case nothing needs to be done.
- The insert function first checks whether there is sufficient capacity for one more element. If not, it sets the capacity to the maximum of 1 and twice the current capacity.

```
void ArrayVector::reserve(int N) {  
    if (capacity >= N) return;  
    Elem* B = new Elem[N];  
    for (int j = 0; j < n; j++)  
        B[j] = A[j];  
    if (A != NULL) delete [] A;  
    A = B;  
    capacity = N;  
}  
  
void ArrayVector::insert(int i, const Elem& e) {  
    if (n >= capacity) // overflow?  
        reserve(max(1, 2 * capacity));  
    for (int j = n - 1; j >= i; j--) // double array size  
        A[j+1] = A[j];  
    A[i] = e; // shift elements up  
    n++; // put in empty slot  
    // one more element  
}
```

Array-based Implementation of Vector - Performance

- In the array-based implementation of an array list:
 - The space used by the data structure is $O(n)$
 - `size`, `empty`, `at` and `set` run in $O(1)$ time
 - `insert` and `erase` run in $O(n)$ time in worst case
- If we use the array in a circular fashion, operations `insert(0, x)` and `erase(0, x)` run in $O(1)$ time
- In an insert operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one

<i>Operation</i>	<i>Time</i>
<code>size()</code>	$O(1)$
<code>empty()</code>	$O(1)$
<code>at(i)</code>	$O(1)$
<code>set(i, e)</code>	$O(1)$
<code>insert(i, e)</code>	$O(n)$
<code>erase(i)</code>	$O(n)$

Comparison of the Strategies to Resize Array

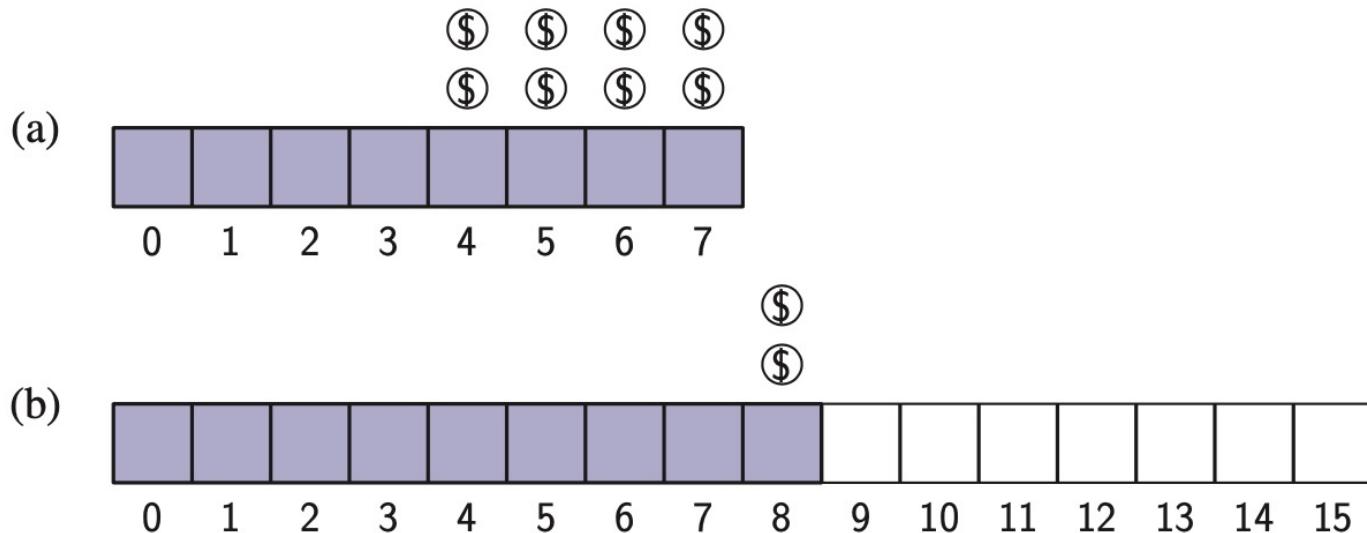
- Suppose we are doing only push operations
- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n insert(o) operations
- How large should the new array be?
 - Incremental strategy: increase the size by a constant c
 - Doubling strategy: double the size

```
Algorithm insert( $o$ )
  if  $t = S.length - 1$  then
     $A \leftarrow$  new array of
      size ...
    for  $i \leftarrow 0$  to  $n-1$  do
       $A[i] \leftarrow S[i]$ 
     $S \leftarrow A$ 
     $n \leftarrow n + 1$ 
     $S[n-1] \leftarrow o$ 
```

- We assume that we start with an empty array of size 2
- We call **amortized time** of an insert operation the average time taken by an insert over the series of operations, i.e., $T(n)/n$

Comparison of the Strategies to Resize Array

- Amortizations:
 - Certain operations may be extremely costly
 - But they cannot occur frequently enough to slow down the entire program
 - The less costly operations far outnumber the costly one
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Comparison of the Strategies to Resize Array

- Suppose we are doing only push operations
- total time $T(n)$: to perform a series of n insert(o) operations
- Remember that every push (storing an element) takes 1 unit of time.
After all we will have n pushes.
- Each array resize needs a time proportional to the size of the old array
- What is the time for n push operation
- For simplicity, we start with an array of capacity 2 and size zero and grow it dynamically
- We have to identify how many times the array resizes
- We call amortized time of an insert operation the average time taken by an insert over the series of operations, i.e., $T(n)/n$

Incremental Strategy Analysis

- We replace the array $k = n/c$ times
 - Suppose if you want to increment its size by 2, adding two more spaces.
- We suppose $c = 2$. There will be $k=n/2$ array resizes.
- For the incremental approach, each time we go past our capacity ($k=(n/c)=(n/2)$) times, we will increase capacity by **$c=2$** and we will have to copy the stuff already in the array into the new array.
- Assuming each item we copy requires 1 time unit.
- For 2 items: 2 units of time
For 4 items: 4 units of time
For 6 items: 6 units of time
- We the have need $2 + 4 + 6 + 8 + \dots + 2^*k$ units of time
- Total time = $n + 2 + 4 + 6 + 8 + \dots + 2^*k = n + c(1+2+\dots+k)$.
- Since c is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$
- The amortized time of an insert operation is $O(n^2) / n$ which is $O(n)$

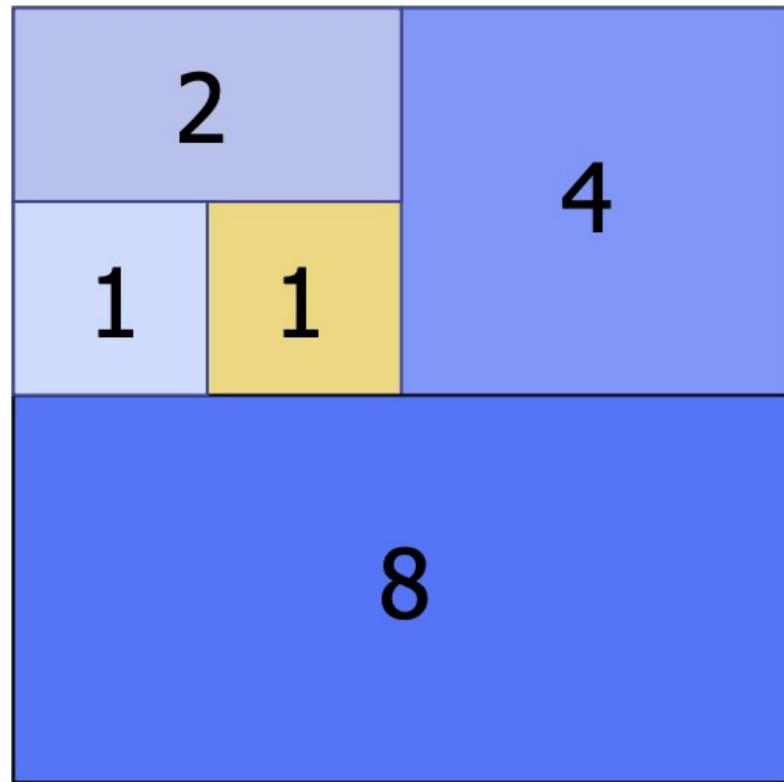
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$$n + c + 2c + 3c + 4c + \dots + kc =$$
$$n + c(1 + 2 + 3 + \dots + k) =$$
$$n + ck(k + 1)/2$$
- Since c is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$
- The amortized time of an insert operation is $O(n)$

Doubling Strategy Analysis

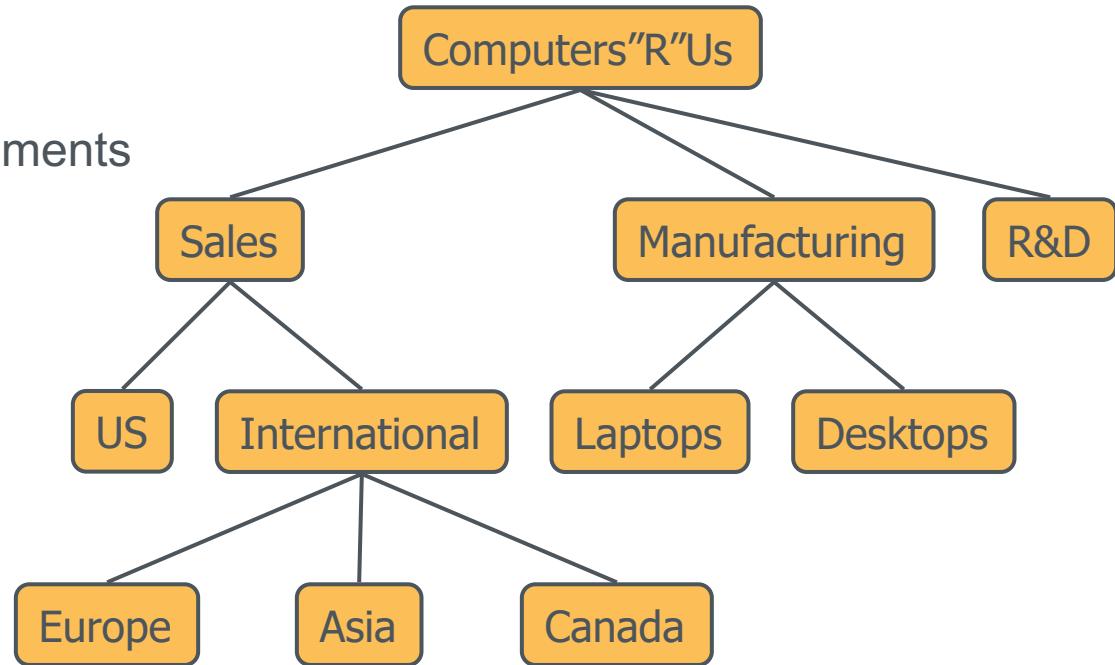
- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n insert operations is proportional to
- $$n + 2 + 4 + 8 + \dots + 2^k =$$
$$n + 2^{k+1} - 1 =$$
- $3n - 1$
- $T(n)$ is $O(n)$
- The amortized time of an insert operation is $O(1)$

geometric series



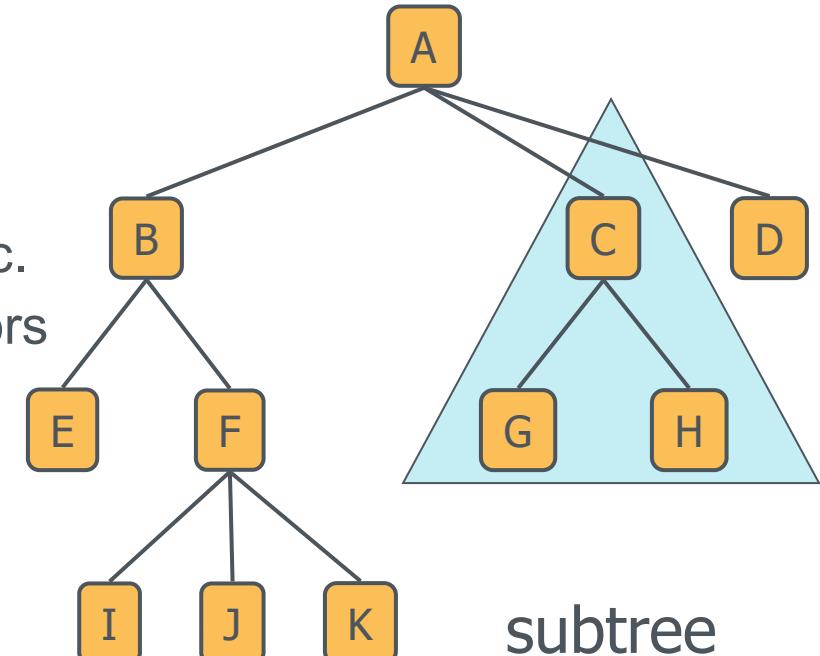
Trees

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
 - Organization charts
 - File systems
 - Programming environments



Trees

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics
Winter 2022

22 Lists, Positions, and Trees

Department of Computing and Software

Instructor:

Omid Isfahanialamdar

March 14, 2022

Admin

- Mid-Term 2:
 - Wednesday 23 March 2022
 - Duration: **1 hour**
 - Location: TBA
 - Covers: Topics from “Doubly Linked Lists” until the lecture of Wednesday 16 March 2022 (inclusive)

Array-based Implementation of Vector in C++ - Review

- The reserve function first checks whether the capacity already exceeds n , in which case nothing needs to be done.
- The insert function first checks whether there is sufficient capacity for one more element. If not, it sets the capacity to the maximum of 1 and twice the current capacity.

```
void ArrayVector::reserve(int N) {  
    if (capacity >= N) return;  
    Elem* B = new Elem[N];  
    for (int j = 0; j < n; j++)  
        B[j] = A[j];  
    if (A != NULL) delete [] A;  
    A = B;  
    capacity = N;  
}  
  
void ArrayVector::insert(int i, const Elem& e) {  
    if (n >= capacity)  
        reserve(max(1, 2 * capacity));  
    for (int j = n - 1; j >= i; j--)  
        A[j+1] = A[j];  
    A[i] = e;  
    n++;  
}
```

// reserve at least N spots
// already big enough
// allocate bigger array
// copy contents to new array
// discard old array
// make B the new array
// set new capacity
// overflow?
// double array size
// shift elements up
// put in empty slot
// one more element

Comparison of the Strategies to Resize Array

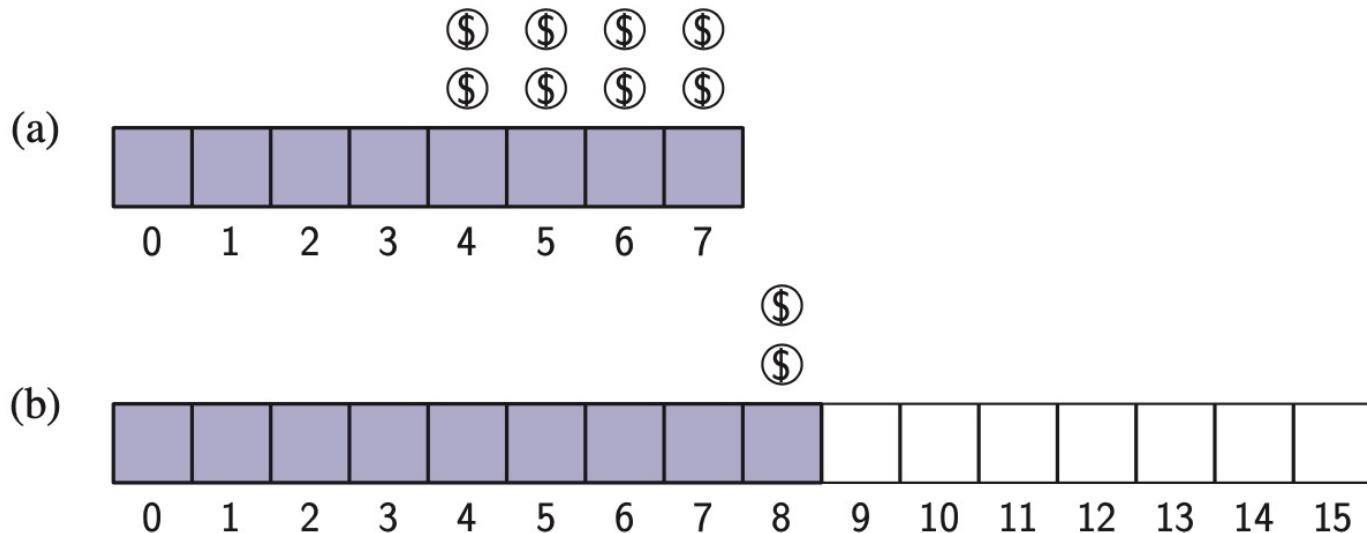
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     $n \leftarrow n + 1$ 
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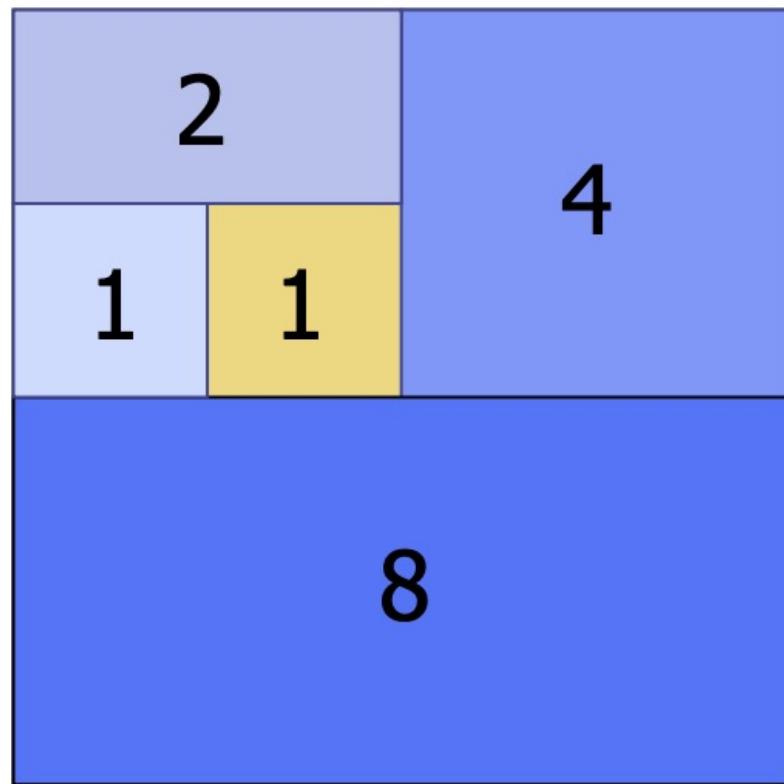
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Doubling Strategy Analysis

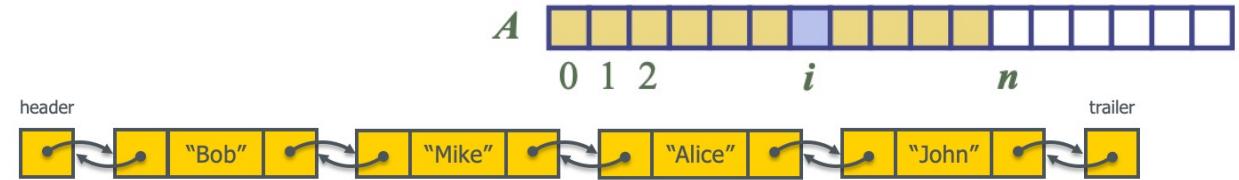
- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n insert operations is proportional to
- $$n + 2 + 4 + 8 + \dots + 2^k =$$
$$n + 2^{k+1} - 1 =$$
- $3n - 1$
- $T(n)$ is $O(n)$
- The amortized time of an insert operation is $O(1)$

geometric series



Linear Data Structures and Positions

- We have seen many data structures
 - Vector
 - Linked-Lists
- Position: Where a data element is stored
- The **Position ADT** models the notion of place within a data structure where a single object is stored
 - abstracts the notion of the relative position or place of an element within a list
- It gives a unified view of diverse ways of storing data, such as:
 - a cell of an array
 - a node of a linked list
- Just one method:
 - *object p.element()*: returns the element at position
 - In C++ it is convenient to implement this as **p*



(Node) List ADT

- The **Node List ADT** models
 - a sequence of positions storing objects
 - It establishes a **before/after** relation between positions
 - Generic methods:
 - `size()`, `empty()`
 - Iterators:
 - `begin()`, `end()`
 - Update methods:
 - `insertFront(e)`, `insertBack(e)`, `removeFront()`, `removeBack()`
 - Iterator-based update:
 - `insert(p, e)`, `remove(p)`

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 - `insertFront(e)`, `insertBack(e)`, `removeFront()`, `removeBack()`
 - Iterator-based update:
 - `insert(p, e)`, `remove(p)`
 - **No method for accessing a specific node?**

(Node) List ADT

- This implementation is basically a Doubly Linked-List
- **n:**
 - The number of data elements
- **Iterator** is an implementation of the Position ADT
 - We use iterator instead of a pointer to a specific node in this implementation

```
typedef int Elem; // list base element type
class NodeList { // node-based list
private:
    // insert Node declaration here...
public:
    // insert Iterator declaration here...
public:
    NodeList(); // default constructor
    int size() const; // list size
    bool empty() const; // is the list empty?
    Iterator begin() const; // beginning position
    Iterator end() const; // (just beyond) last position
    void insertFront(const Elem& e); // insert at front
    void insertBack(const Elem& e); // insert at rear
    void insert(const Iterator& p, const Elem& e); // insert e before p
    void eraseFront(); // remove first
    void eraseBack(); // remove last
    void erase(const Iterator& p); // remove p
    // housekeeping functions omitted...
private:
    int n; // data members
    Node* header; // number of items
    Node* trailer; // head-of-list sentinel
}; // tail-of-list sentinel
```

Containers and Iterators

- An **iterator** abstracts the process of scanning through a collection of elements
- A **container** is an abstract data structure that supports element access through iterators
 - Examples include Stack, Queue, Vector, List
 - **begin()**: returns an iterator to the first element
 - **end()**: return an iterator to an imaginary position just after the last element
- An iterator behaves like a pointer to an element
 - ***p**: returns the element referenced by this iterator
 - **++p**: advances to the next element
 - Iterator extends the concept of **position** by adding a traversal capability

Iterating through a Container

- Let **C** be a **container** and **p** be an **iterator** for **C**

```
for (p = C.begin(); p != C.end(); ++p)
    loop_body
```
- Example: (with an STL vector)
 - Notice how for loop is implemented
 - Notice how we access the element of iterator

Output:

Iterator is on data 1
Iterator is on data 2
Iterator is on data 3
Iterator is on data 4
Iterator is on data 5
The sum is :15

```
#include <iostream>
#include <vector>
using namespace std;

int main(){
    vector<int> v1;
    for (int i = 1; i <= 5; i++)
        v1.push_back(i);

    typedef vector<int>::iterator Iterator;
    int sum = 0;
    for (Iterator p = v1.begin(); p != v1.end(); ++p){
        cout << "Iterator is on data " << *p << endl;
        sum += *p;
    }
    cout << "The sum is :" << sum << endl;
}
```

Implementing Iterators

- Array-based
 - **Array A of the n elements**
 - **index i** that keeps track of the cursor
 - **begin() = 0**
 - **end() = n** (index following the last element)
- Linked list-based
 - For example: A doubly linked-list **L** storing the elements, with sentinels for header and trailer
 - **pointer to node** containing the current element
 - **begin() = front node** (immediately after the header)
 - **end() = trailer node** (immediately after last node)

Implementation of Iterator for (Node) List

- This is an implementation of the iterator for our NodeList (aka DLL)

```
class Iterator {                                     // an iterator for the list
public:
    Elem& operator*();                         // reference to the element
    bool operator==(const Iterator& p) const; // compare positions
    bool operator!=(const Iterator& p) const;
    Iterator& operator++();                   // move to next position
    Iterator& operator--();                   // move to previous position
    friend class NodeList;                    // give NodeList access
private:
    Node* v;                                  // pointer to the node
    Iterator(Node* u);                       // create from node
};
```

```
NodeList::Iterator::Iterator(Node* u)           // constructor from Node*
{ v = u; }
```

```
Elem& NodeList::Iterator::operator*()          // reference to the element
{ return v->elem; }
// compare positions
bool NodeList::Iterator::operator==(const Iterator& p) const
{ return v == p.v; }
```

```
bool NodeList::Iterator::operator!=(const Iterator& p) const
{ return v != p.v; }
// move to next position
NodeList::Iterator& NodeList::Iterator::operator++()
{ v = v->next; return *this; }
// move to previous position
NodeList::Iterator& NodeList::Iterator::operator--()
{ v = v->prev; return *this; }
```

```
struct Node {
    Elem elem;
    Node* prev;
    Node* next;
};
```

(Node) List ADT (duplicate slide)

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    void eraseFront(); // remove first
    void eraseBack(); // remove last
    void erase(const Iterator& p); // remove p
    // housekeeping functions omitted...
private:
    int n; // data members
    Node* header; // number of items
    Node* trailer; // head-of-list sentinel
}; // tail-of-list sentinel
```

STL Iterators in C++

- Each STL container type **C** supports iterators:
 - **C::iterator** – read/write iterator type
 - **C::const_iterator** – read-only iterator type
 - **C.begin()**, **C.end()** – return start/end iterators
- This iterator-based operators and methods:
 - ***p**: access current element
 - **++p, --p**: advance to next/previous element
 - **C.assign(p, q)**: replace C with contents referenced by the iterator range [p, q) (from p up to, but not including, q)
 - **insert(p, e)**: insert e prior to position p
 - **erase(p)**: remove element at position p
 - **erase(p, q)**: remove elements in the iterator range [p, q)

STL List in C++

```
#include <list>
using std::list;                                // make list accessible
list<float> myList;                            // an empty list of floats
```

list(*n*): Construct a list with *n* elements; if no argument list is given, an empty list is created.

size(): Return the number of elements in *L*.

empty(): Return true if *L* is empty and false otherwise.

front(): Return a reference to the first element of *L*.

back(): Return a reference to the last element of *L*.

push_front(*e*): Insert a copy of *e* at the beginning of *L*.

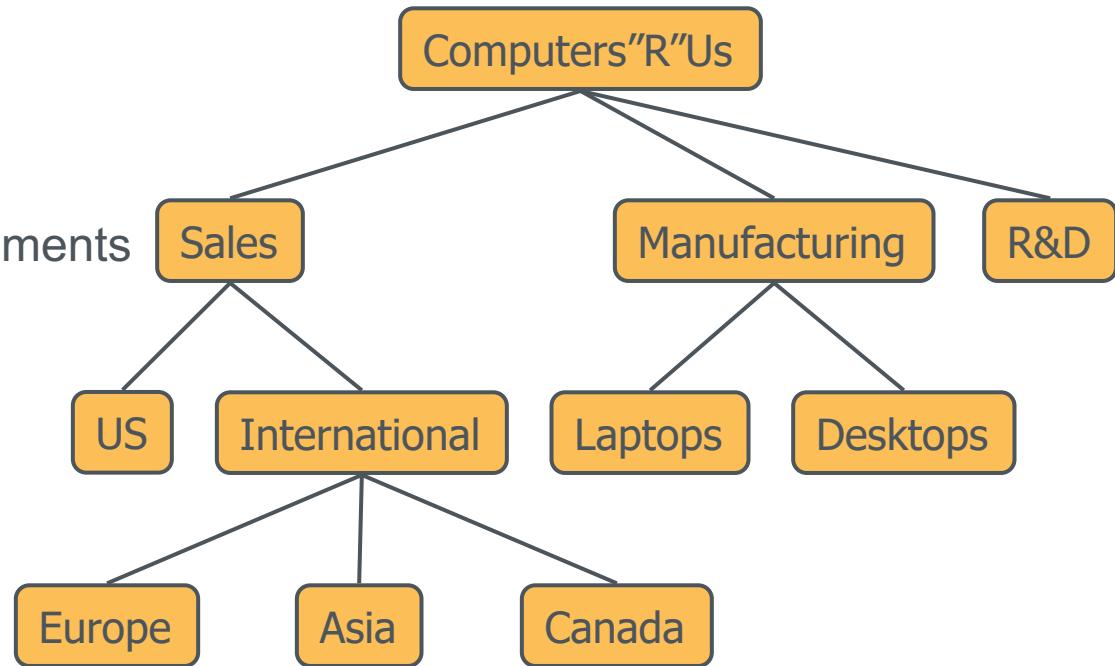
push_back(*e*): Insert a copy of *e* at the end of *L*.

pop_front(): Remove the fist element of *L*.

pop_back(): Remove the last element of *L*.

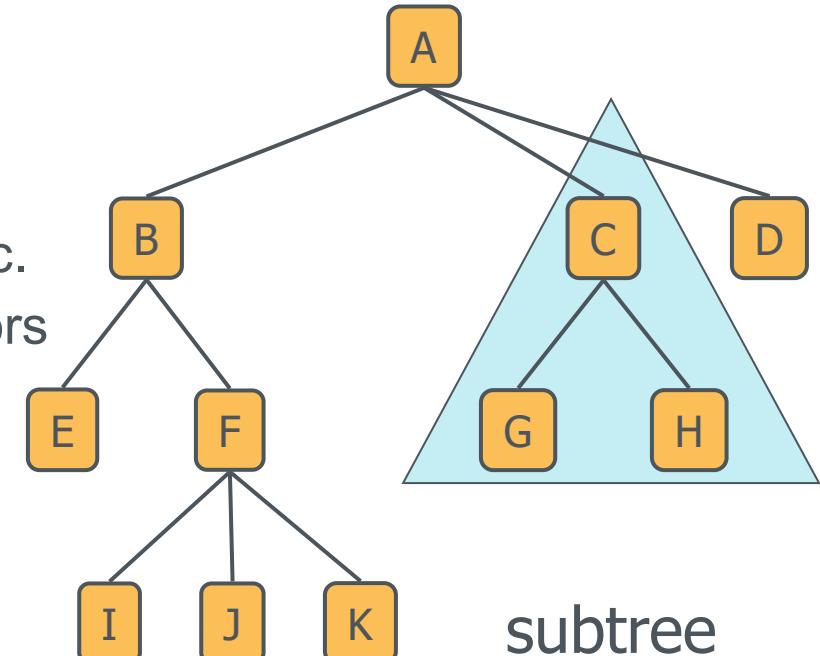
Trees

- In computer science, a tree is an abstract model of a hierarchical structure
- The relation between elements is non-linear
- A tree consists of nodes with a parent-child relation
- Applications:
 - Organization charts
 - File systems
 - Programming environments



Trees

- Root: node without parent (A)
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Questions?

MECHTRON 2MD3

Data Structures and Algorithms for Mechatronics

Winter 2022

23 Trees

Department of Computing and Software

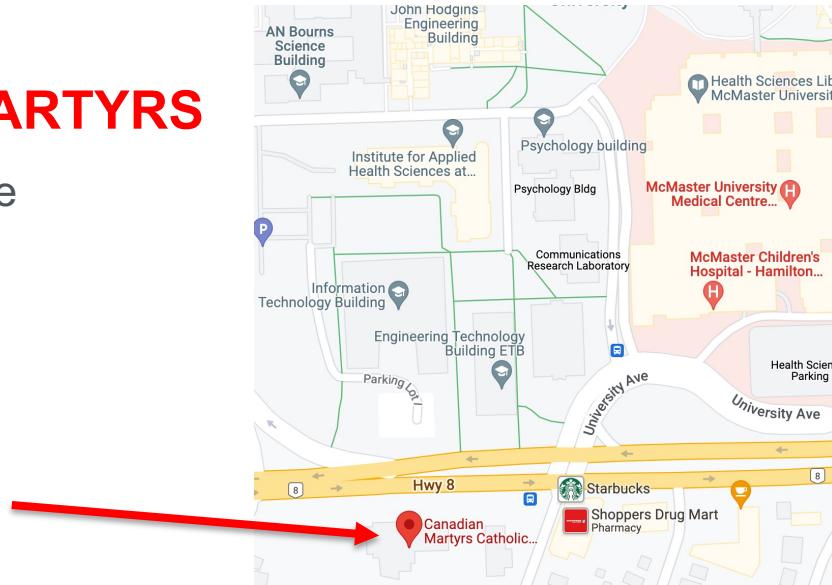
Instructor:

Omid Isfahanialamdar

March 16, 2022

Admin

- Mid-Term 2:
 - Wednesday 23 March 2022
 - Duration: **1 hour**
 - Location: **MCMST CDN_MARTYRS**
 - Seems to be here, I am not sure



- Covers: Topics from “Doubly Linked Lists” until the lecture of Wednesday 16 March 2022 (inclusive)

Iterating through a Container (Review)

- Let **C** be a **container** and **p** be an **iterator** for **C**

```
for (p = C.begin(); p != C.end(); ++p)
    loop_body
```
- Example: (with an STL vector)
 - Notice how for loop is implemented
 - Notice how we access the element of iterator

Output:

Iterator is on data 1
Iterator is on data 2
Iterator is on data 3
Iterator is on data 4
Iterator is on data 5
The sum is :15

```
#include <iostream>
#include <vector>
using namespace std;

int main(){
    vector<int> v1;
    for (int i = 1; i <= 5; i++)
        v1.push_back(i);

    typedef vector<int>::iterator Iterator;
    int sum = 0;
    for (Iterator p = v1.begin(); p != v1.end(); ++p){
        cout << "Iterator is on data " << *p << endl;
        sum += *p;
    }
    cout << "The sum is :" << sum << endl;
}
```

Implementing Iterators (remained from pre. lecture)

- Array-based
 - **Array A of the n elements**
 - **index i** that keeps track of the cursor
 - **begin() = 0**
 - **end() = n** (index following the last element)
- Linked list-based
 - For example: A doubly linked-list **L** storing the elements, with sentinels for header and trailer
 - **pointer to node** containing the current element
 - **begin() = front node** (immediately after the header)
 - **end() = trailer node** (immediately after last node)

Implementation of Iterator for (Node) List

- This is an implementation of the iterator for our NodeList (aka DLL)

```
class Iterator {                                     // an iterator for the list
public:
    Elem& operator*();                         // reference to the element
    bool operator==(const Iterator& p) const; // compare positions
    bool operator!=(const Iterator& p) const;
    Iterator& operator++();                   // move to next position
    Iterator& operator--();                   // move to previous position
    friend class NodeList;                    // give NodeList access
private:
    Node* v;                                  // pointer to the node
    Iterator(Node* u);                       // create from node
};
```

```
NodeList::Iterator::Iterator(Node* u)           // constructor from Node*
{ v = u; }
```

```
Elem& NodeList::Iterator::operator*()          // reference to the element
{ return v->elem; }
// compare positions
bool NodeList::Iterator::operator==(const Iterator& p) const
{ return v == p.v; }
```

```
bool NodeList::Iterator::operator!=(const Iterator& p) const
{ return v != p.v; }
// move to next position
NodeList::Iterator& NodeList::Iterator::operator++()
{ v = v->next; return *this; }
// move to previous position
NodeList::Iterator& NodeList::Iterator::operator--()
{ v = v->prev; return *this; }
```

```
struct Node {
    Elem elem;
    Node* prev;
    Node* next;
};
```

(Node) List ADT (duplicate slide)

- This implementation is basically a Doubly Linked-List
- **n:**
 - The number of data elements
- **Iterator** is an implementation of the Position ADT
 - We use iterator instead of a pointer to a specific node in this implementation

```
typedef int Elem; // list base element type
class NodeList { // node-based list
private:
    // insert Node declaration here...
public:
    // insert Iterator declaration here...
public:
    NodeList(); // default constructor
    int size() const; // list size
    bool empty() const; // is the list empty?
    Iterator begin() const; // beginning position
    Iterator end() const; // (just beyond) last position
    void insertFront(const Elem& e); // insert at front
    void insertBack(const Elem& e); // insert at rear
    void insert(const Iterator& p, const Elem& e); // insert e before p
    void eraseFront(); // remove first
    void eraseBack(); // remove last
    void erase(const Iterator& p); // remove p
    // housekeeping functions omitted...
private:
    int n; // data members
    Node* header; // number of items
    Node* trailer; // head-of-list sentinel
}; // tail-of-list sentinel
```

STL Iterators in C++

- Each STL container type **C** supports iterators:
 - **C::iterator** – read/write iterator type
 - **C::const_iterator** – read-only iterator type
 - **C.begin()**, **C.end()** – return start/end iterators
- This iterator-based operators and methods:
 - ***p**: access current element
 - **++p, --p**: advance to next/previous element
 - **C.assign(p, q)**: replace C with contents referenced by the iterator range [p, q) (from p up to, but not including, q)
 - **insert(p, e)**: insert e prior to position p
 - **erase(p)**: remove element at position p
 - **erase(p, q)**: remove elements in the iterator range [p, q)

STL List in C++

```
#include <list>
using std::list;                                // make list accessible
list<float> myList;                            // an empty list of floats
```

`list(n)`: Construct a list with *n* elements; if no argument list is given, an empty list is created.

`size()`: Return the number of elements in *L*.

`empty()`: Return true if *L* is empty and false otherwise.

`front()`: Return a reference to the first element of *L*.

`back()`: Return a reference to the last element of *L*.

`push_front(e)`: Insert a copy of *e* at the beginning of *L*.

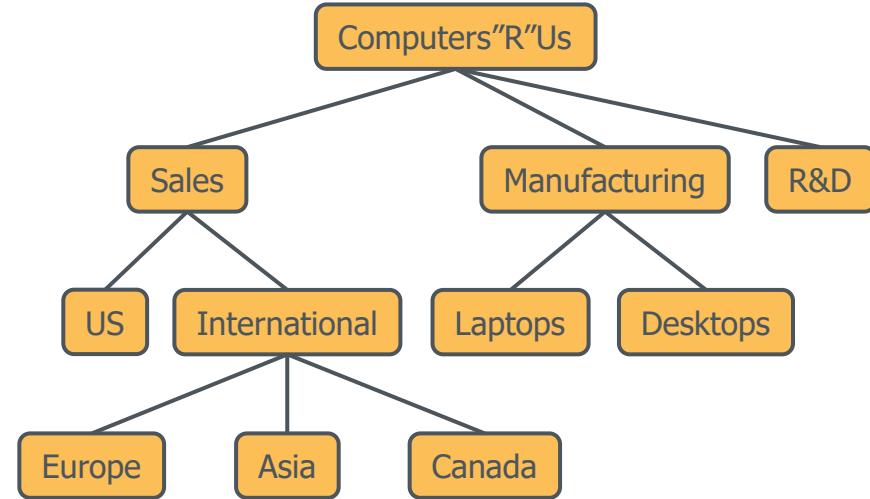
`push_back(e)`: Insert a copy of *e* at the end of *L*.

`pop_front()`: Remove the first element of *L*.

`pop_back()`: Remove the last element of *L*.

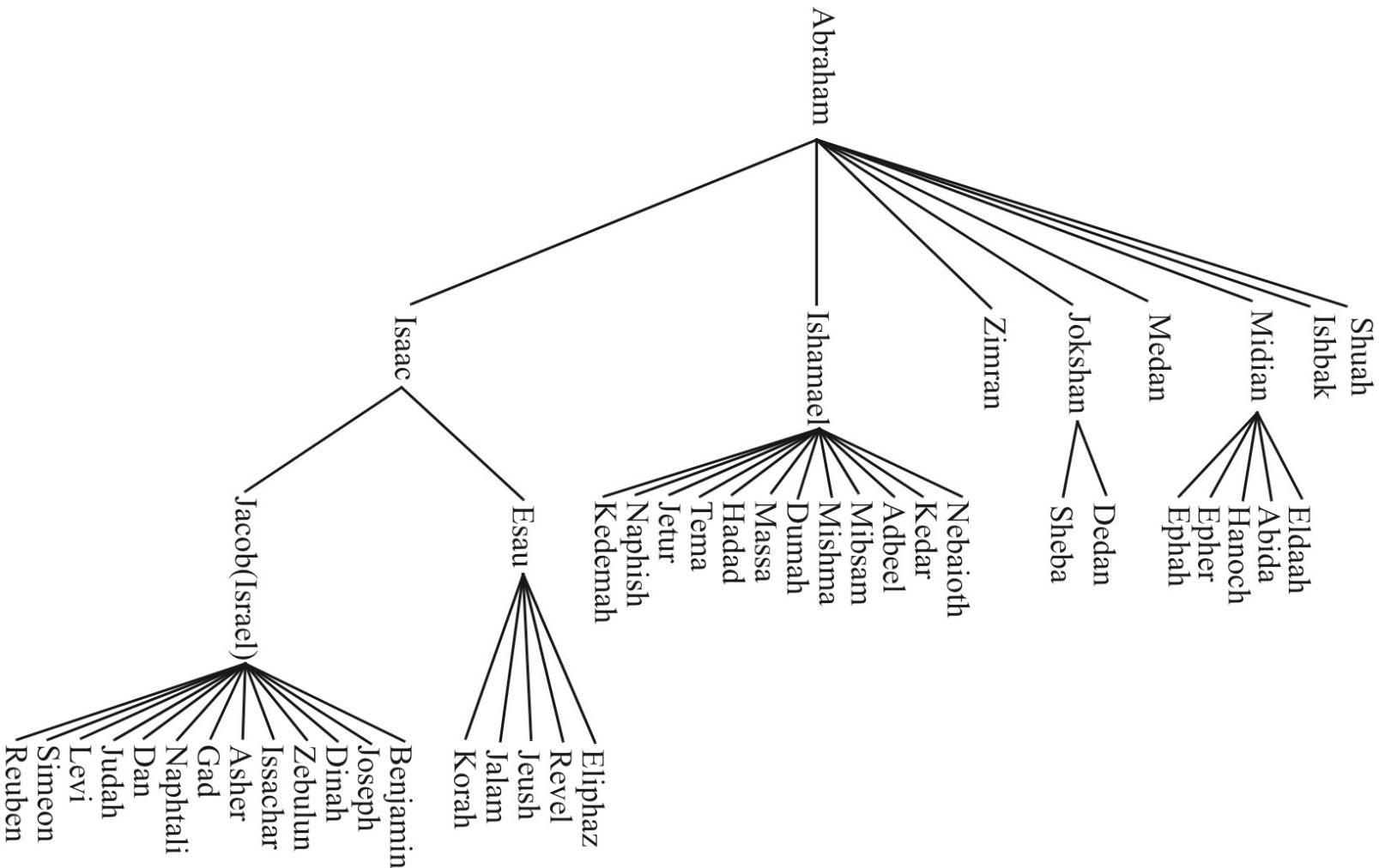
Trees

- So far you are familiar with:
 - Storing elements in a linear fashion
 - Containers and iterators
- In computer science, a tree is an abstract model of a hierarchical structure
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- Applications:
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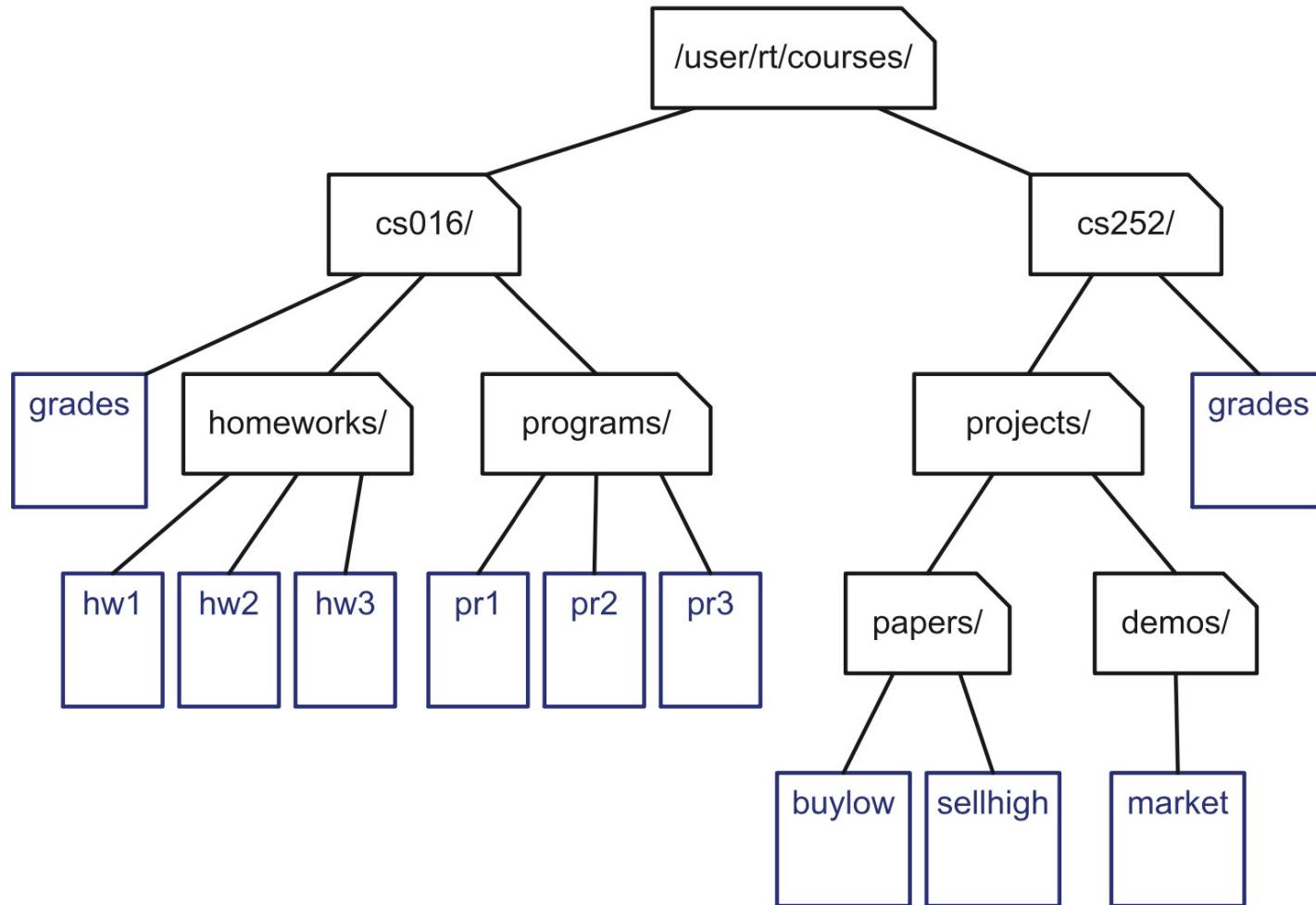
Tree Examples

- Family Tree



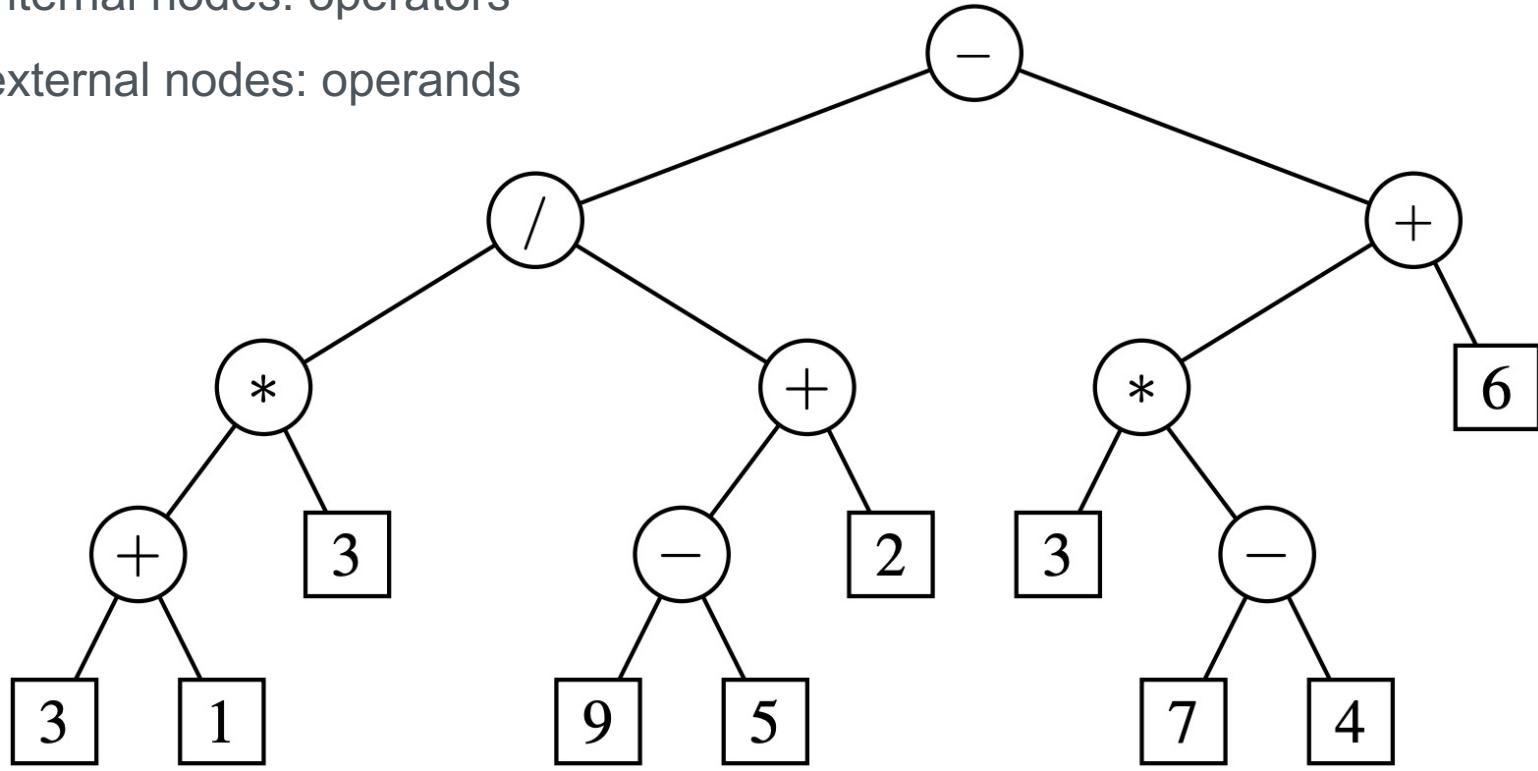
Tree Examples

- File System Tree



Tree Examples

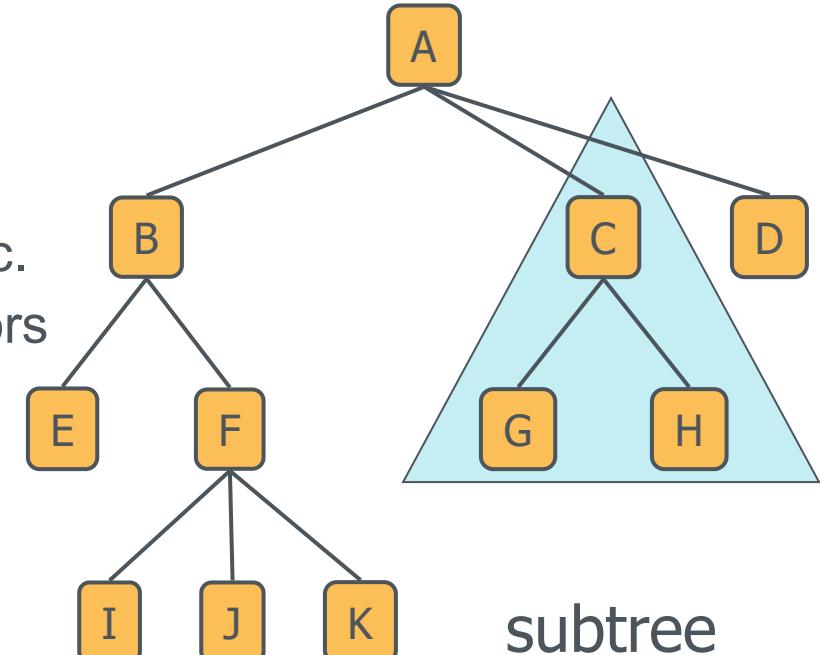
- Expression Tree
 - Binary tree associated with an arithmetic expression
 - internal nodes: operators
 - external nodes: operands



Tree represents: $\left(\left(\left(3 + 1 \right) * 3 \right) / \left(\left(9 - 5 \right) + 2 \right) \right) - \left(\left(3 * \left(7 - 4 \right) \right) + 6 \right)$

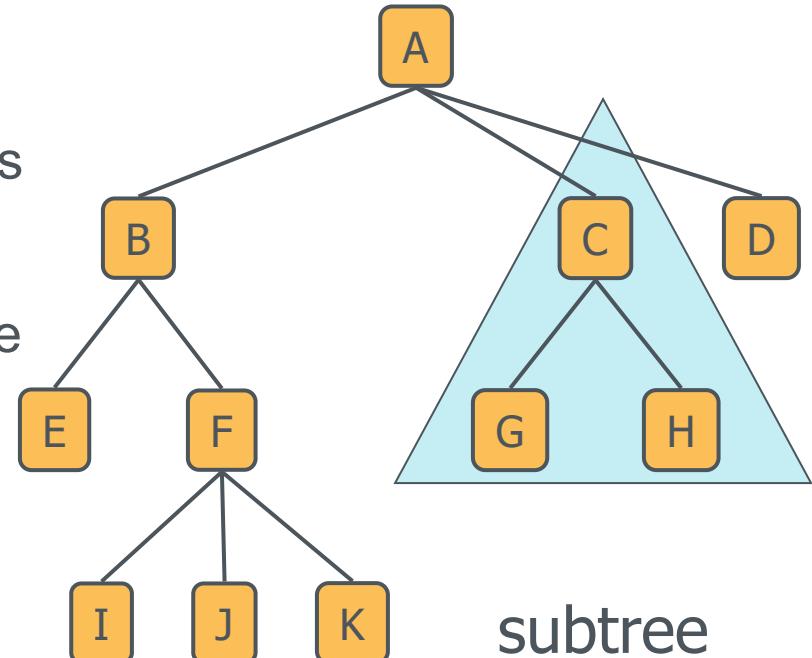
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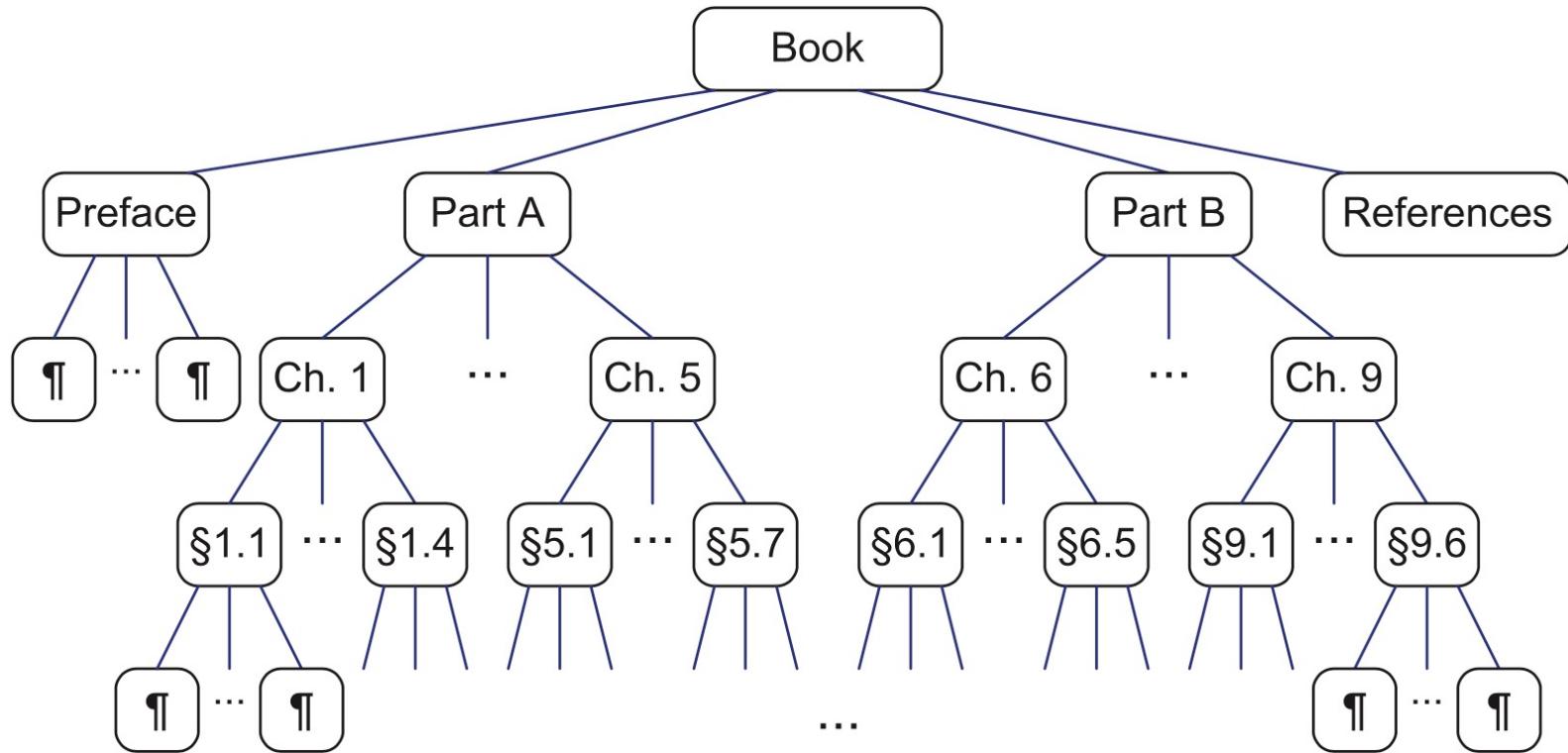
Formal Definition of Tree

- we define tree T to be a **set of nodes** storing elements in a **parent-child** relationship with the following properties:
 - If T is *nonempty*, it has a special node, called the **root** of T , that has no parent.
 - Each node v of T different from the root has a unique **parent** node w ; every node with parent w is a **child** of w .
- A tree can be empty!
- So, with the above definition we can define it recursively:
 - a tree T is either empty or consists of a node r , called the root of T , and a (possibly empty) set of trees whose roots are the children of r



Ordered Tree Examples

- Ordered Tree: A tree is ordered if there is a linear ordering defined for the children of each node
 - linear order relationship existing between siblings
- Book Organization Tree



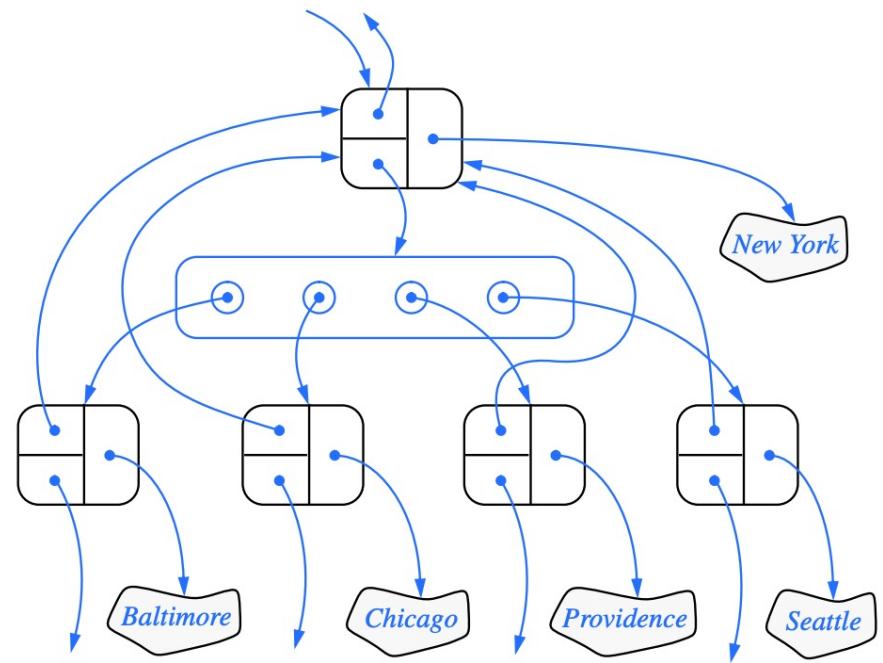
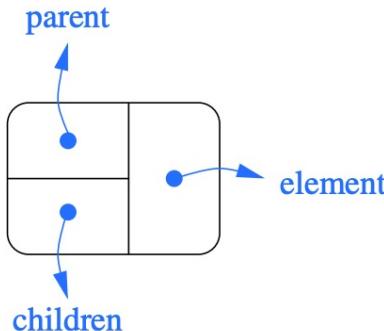
Tree ADT

- We use **positions** to abstract nodes
- Generic methods:
 - integer **size()**
 - boolean **empty()**
- Accessor methods:
 - position **root()**
 - list<position> **positions()**
- Position-based methods:
 - position **p.parent()**
 - list<position> **p.children()**
- Query methods:
 - boolean **p.isRoot()**
 - boolean **p.isExternal()**
- Additional update methods may be defined by data structures implementing the Tree ADT

Tree ADT

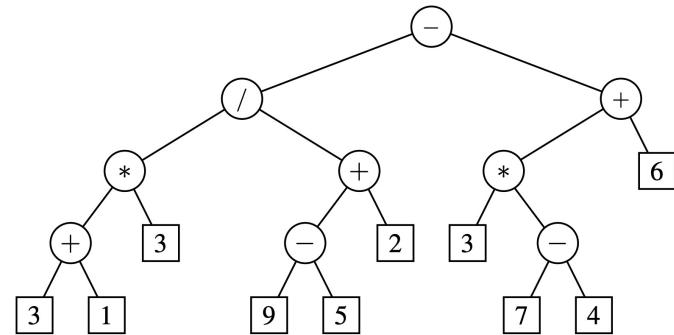
- General Tree Implementation

```
template <typename E> // base element type
class Position<E> { // a node position
public:
    E& operator*(); // get element
    Position parent() const; // get parent
    PositionList children() const; // get node's children
    bool isRoot() const; // root node?
    bool isExternal() const; // external node?
};
```



Tree Traversals

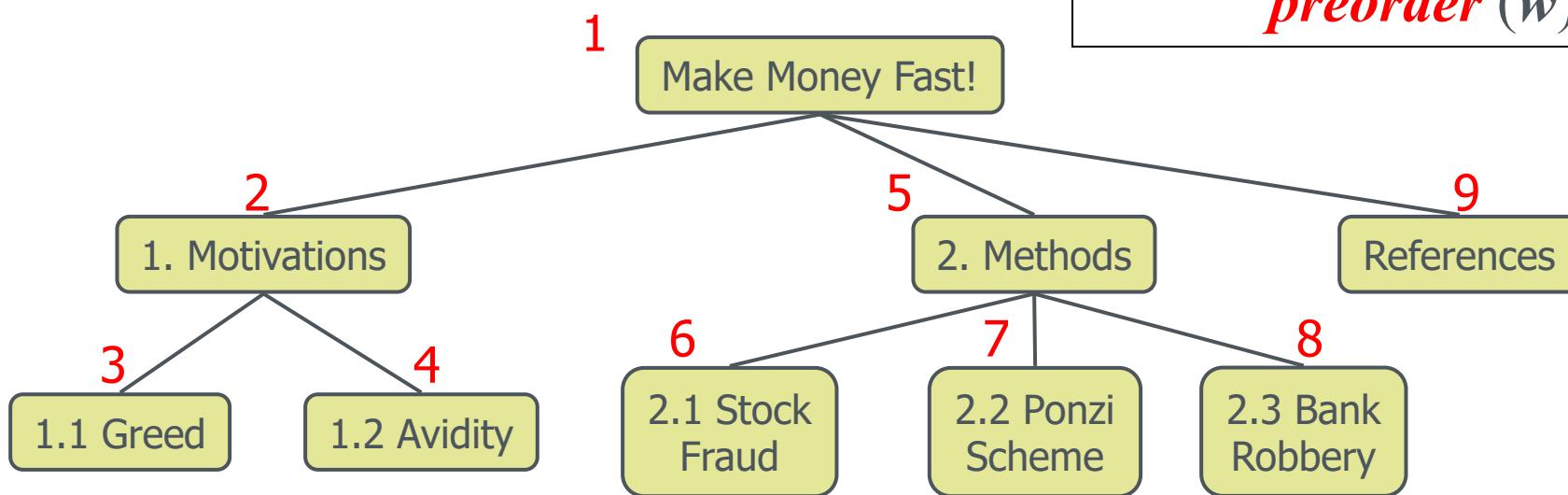
- We were able to easily traverse a linear data structure
 - The traversal of a tree is not trivial
- A **traversal** of a tree T is a systematic way of accessing or visiting all the nodes of T .
 - Preorder traversal
 - Inorder traversal
 - Postorder traversal
 - Breadth-first traversal



Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

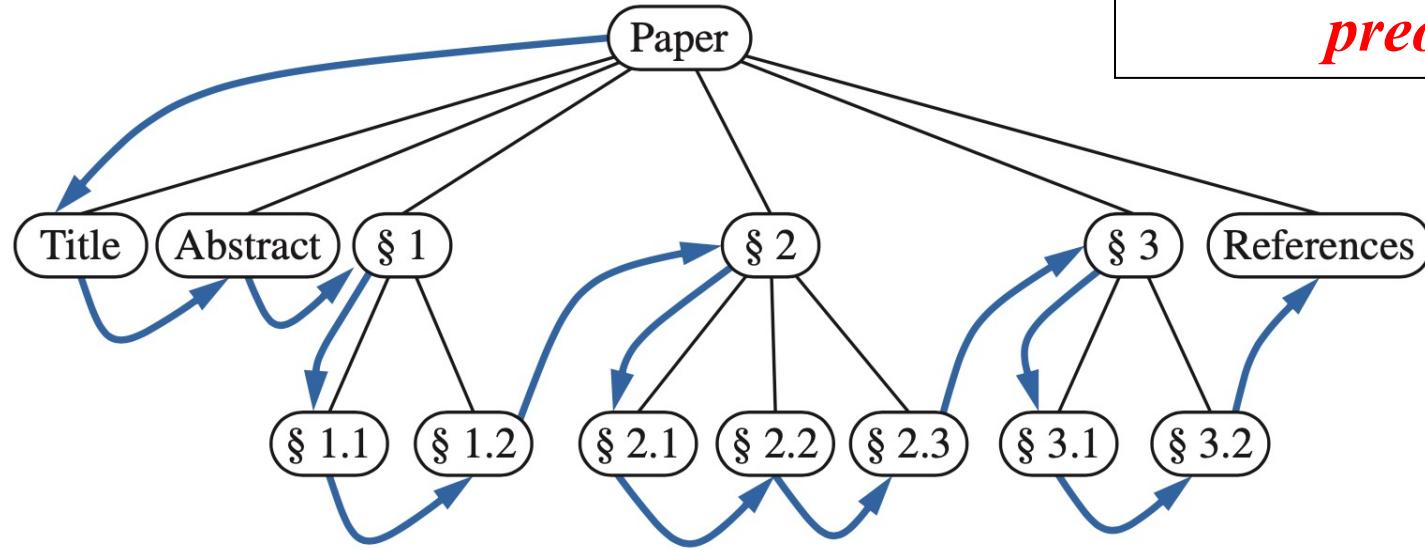
```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preorder (w)
```



Preorder Traversal

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Preorder Traversal

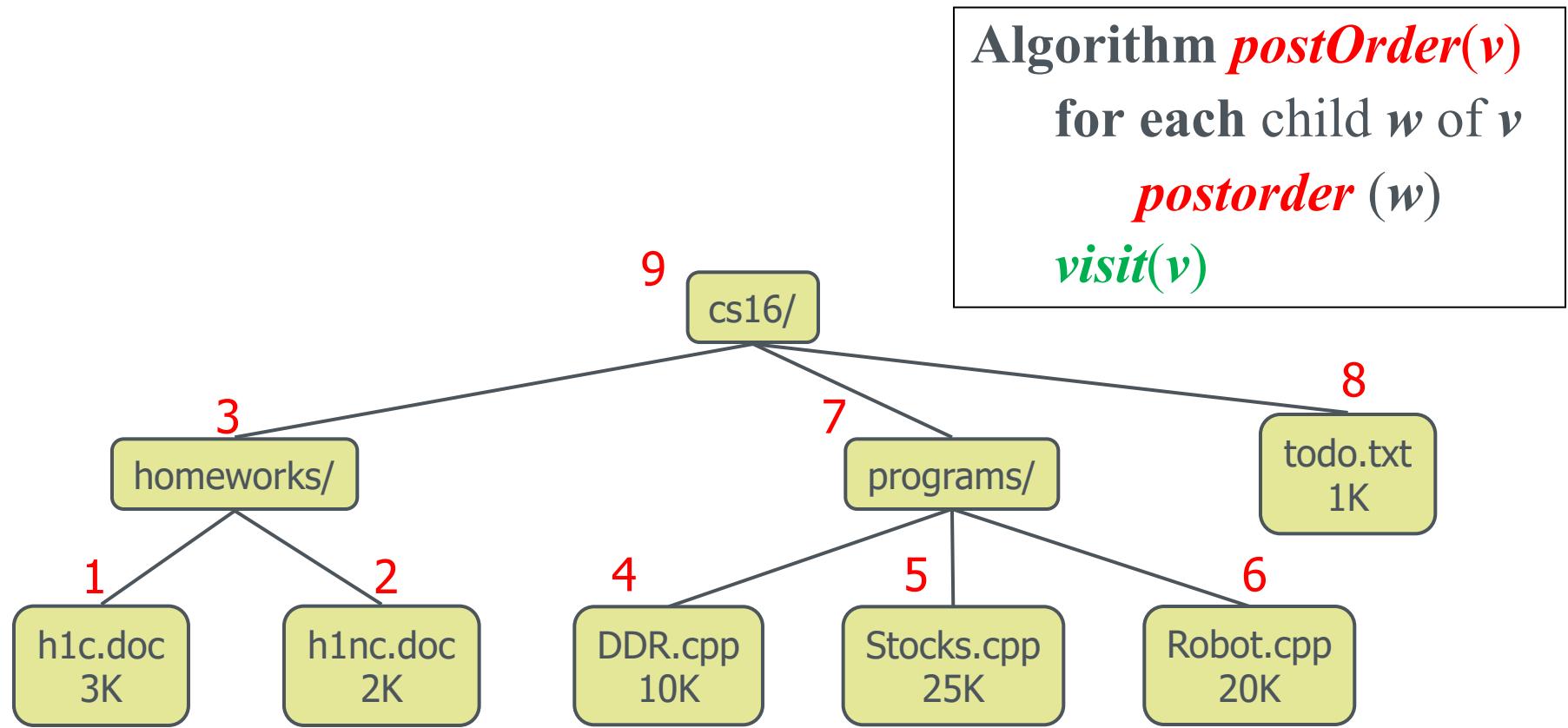
- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preorder (w)
```

```
void preorderPrint(const Tree& T, const Position& p) {
    cout << *p;                                // print element
    PositionList ch = p.children();             // list of children
    for (Iterator q = ch.begin(); q != ch.end(); ++q) {
        cout << " ";
        preorderPrint(T, *q);
    }
}
```

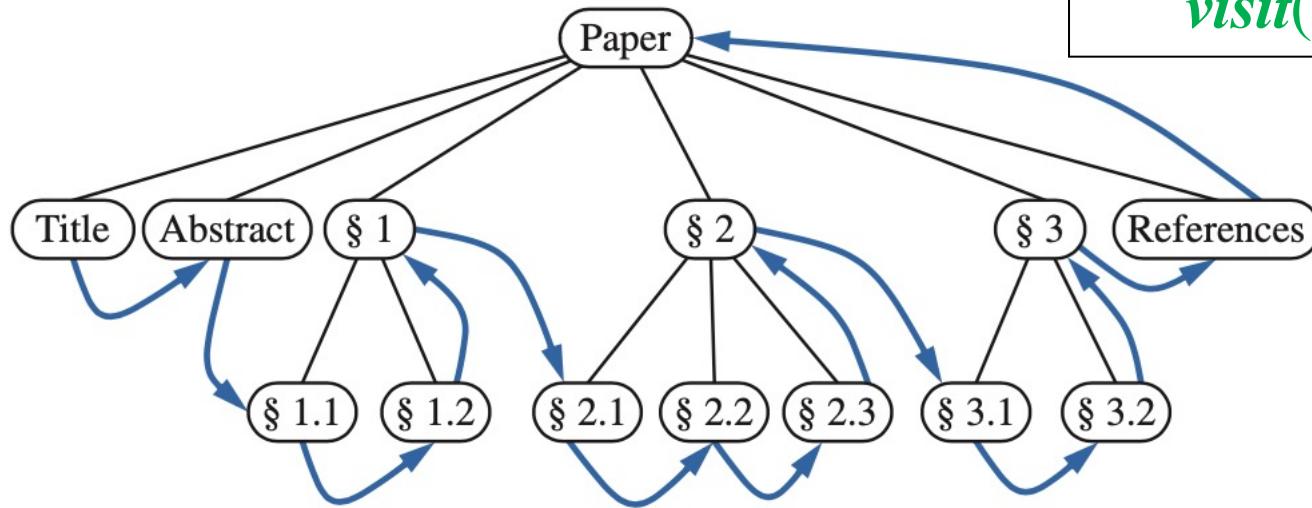
Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories



Algorithm *postOrder*(v)
for each child w of v
postorder (w)
visit(v)

Postorder Traversal

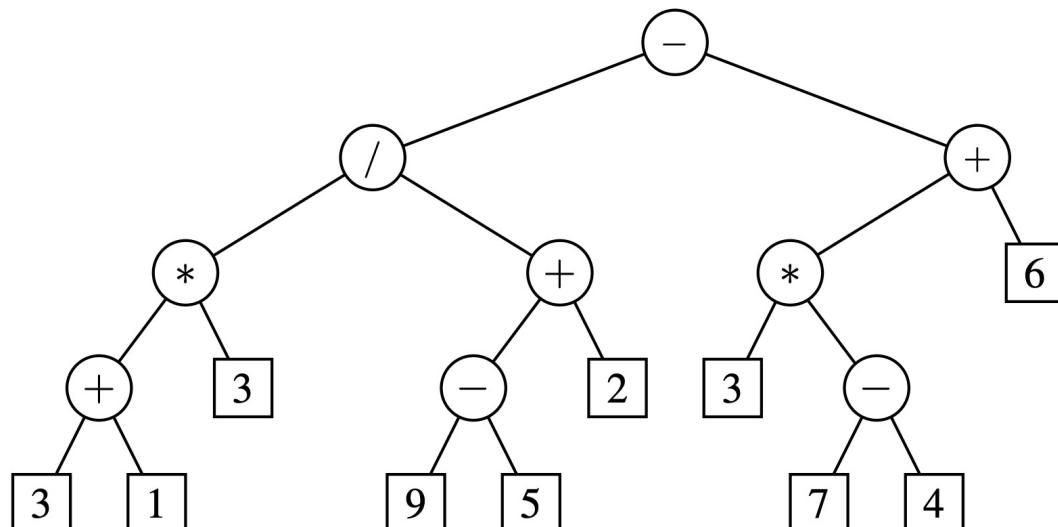
- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm *postOrder(v)*
for each child w of v
postorder (w)
visit(v)

```
void postorderPrint(const Tree& T, const Position& p) {  
    PositionList ch = p.children();           // list of children  
    for (Iterator q = ch.begin(); q != ch.end(); ++q) {  
        postorderPrint(T, *q);  
        cout << " ";  
    }  
    cout << *p;                            // print element  
}
```

Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories



$$(((3 + 1) * 3) / ((9 - 5) + 2)) - ((3 * (7 - 4)) + 6))$$

Postorder?

Algorithm *postOrder(v)*
for each child *w* of *v*
postorder (*w*)
visit(v)

Questions?