Exam Rev Retry

December 18, 2022 1:26 PM

1 Short Questions

1. What is the minimal sampling frequency rounded to an integer that is needed such that we can sample and reconstruct the signal

$$x(t) = \cos(50\pi t) + \cos(100\pi t)$$

$(f_s)_{min} > 2f_0$

$f_0 = \frac{\omega_0}{2\pi} = \frac{100\pi}{2\pi} = 50$

$(f_s)_{min} > 2(50)$

$(f_s)_{min} > 100$

2 Difference Equations to State Space Equation

Given a system by its frequency response

$$H(\omega) = \frac{1}{1 - .5e^{-i\omega}}$$

Determine the $[A, B, C, D]$ representation of the system.

$H - \frac{1}{2}He^{-i\omega} = 1$

$y(n) - \frac{1}{2}y(n-1) = x(n)$

$y(n) = x(n) + \frac{1}{2}\underbrace{y(n-1)}_{S_1}$

$S_1 = y(n-1) \rightarrow S_1(n+1) = y(n) = x(n) - \frac{1}{2}S_1$

$S(n+1) = \left(\frac{1}{2}\right)S(n) + (1)\,X(n)$

$y(n) = \left(\frac{1}{2}\right)S(n) + (1)\,x(n)$

3 Z and Laplace Transform

1. Determine if the system

$$\ddot{y} + 4\dot{y} - 2y = x$$

is stable. (Compute the Laplace transform, compute the poles, and decide if stable).

2. The Z-transform is given by

$$\hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Compute the Z-transform of the signal $x(n) = \delta(n) + \delta(n-1)$.

1) $y'' + 4y' - 2y = x$

$\sqrt{3} \hat{=} 1.72$

$s^2Y + 4Ys' - 2 = Xs$

$= \dfrac{s}{s^2+4s-2}$

$s_{1,2} = \dfrac{-4 \pm \sqrt{16+8}}{2} = \dfrac{-4 \pm \sqrt{24}}{2} = -1 \pm \sqrt{3}$

$s_1 = 0.72 \qquad s_2 = -2.72$

unstable b/c $s_1 > 0$

2. The Z-transform is given by

$$\hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Compute the Z-transform of the signal $x(n) = \delta(n) + \delta(n-1)$.

$x(n) = \delta(n) + \delta(n-1)$

$\hat{X}(z) = \delta(0)z^{-0} + \delta(1-1)z^{-1}$

$= 1 + z^{-1}$

4 Filter Design

A digital system is running at the sampling frequency $f_s = 2'000Hz$. Construct a minimal order FIR filter (give the difference equation) that filters (removes) sinodials with the frequency $f = 500Hz$ and filters any constant signal. The filter should be normalized to have a gain of 1 at $f = 1'000Hz$.

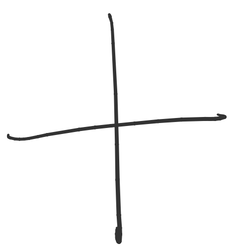
$f_s = 2kHz \rightarrow f_0 = \dfrac{500}{} = \dfrac{1}{}$ $\qquad \omega_0 = \frac{1}{4}2\pi = \pi/2$

$f_s = 2kHz \rightarrow f_0 = \frac{500}{2000} = \frac{1}{4}$ $\quad \omega_0 = \frac{1}{4}2\pi = \pi/2$

$H(\pi/2) = H(-\pi/2) = \emptyset$ $\quad f_0 = \frac{1000}{2000} = \frac{1}{2}2\pi = \pi$

$H(0) = \emptyset$

$H(\pi) = 1$



$$\tilde{H}(\omega) = (e^{-i\omega} - e^{i\pi/2})(e^{-i\omega} - e^{-i\pi/2})(e^{-i\omega} - 1)$$

$$= (e^{-2i\omega} + 1)(e^{-i\omega} - 1)$$

$$= e^{-3i\omega} - e^{-2i\omega} + e^{-i\omega} - 1$$

$$H(\pi) = 1 = e^{-3i\pi} - e^{-2i\pi} + e^{-i\pi} - 1$$

$$\tilde{H}(\pi) = -1 - 1 - 1 - 1$$

$$= -4$$

$$H(\omega) = -\frac{1}{4}(e^{-3i\pi} - e^{-2i\pi} + e^{-i\pi} - 1)$$

5 Frequency response to Differential equation - Impulse response

After designing a filter we obtained the frequency response

$$H(\omega) = \frac{1}{1 + \frac{1}{2}e^{-i\omega}}$$

What is the corresponding difference equation and impulse response ?

$$y(n) = x(n) - \frac{1}{2}y(n-1)$$

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq \emptyset \\ \text{else} & \end{cases}$$

6 Impulse Response

Given the impulse response

$$h(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 3 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine the difference equation of the system, then use convolution to compute the output to the input

$$x(n) = \delta(n) + \delta(n-1) + \delta(n-3)$$

$h(n) = x(n) + 2x(n-1) + 3x(n-2)$

n	0	1	2	3	4	5	6	7
$\delta(n)$	1	0	...					
$h(n)$	1	2	3					
$\delta(n-1)$	0	1						
$h(n)$	0	1	2	3				
$\delta(n-3)$	0	0	0	1				
$h(n)$				1	2	3		

$y(n) = \{1, 3, 5, 4, 2, 3\}$

7 Fourier Series

The Fourier Series of a continuous p-periodic signal is given my

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{i\omega_0 kt}$$

where

$$X_k = \frac{1}{p}\int_0^p x(t)e^{-i\omega_0 kt}dt$$

Compute X_0, X_1 for the signal

$$x(t) = \begin{cases} 1 & 0 \leq t < \pi \\ -1 & \pi \leq t < 2\pi \end{cases}$$

Make sure you have the right ω_0!

$\omega_0 = \frac{2\pi}{2\pi} = 1$

$p \to 2\pi$

$p \to 2\pi$

$$X_0 = \frac{1}{2\pi}\left[\int_0^{\pi} e^{-i0t}\,dt + \int_{\pi}^{2\pi} -e^{-i0t}\,dt\right] = \frac{1}{2\pi}\left[(\pi-0)-(\pi-2\pi)\right] = 0$$

$$X_1 = \frac{1}{2\pi}\left[\int_0^{\pi} e^{-it} - \int_{\pi}^{2\pi} e^{-it}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{-i}\left[e^{-i\pi}-1\right] - \frac{1}{-i}\left[e^{-i2\pi}-e^{-i\pi}\right]\right]$$

$$= \frac{-1}{2\pi i}\left[(-1-1)-(1+1)\right]$$

$$= \frac{4}{2\pi i} = \frac{2}{\pi i}$$

8 Brick-wall Filter

The CTFT if given by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

Given a continuous system with the frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Filter the signal $x(t) = \delta(t-2)$. (You first have to compute $X(\omega)$ then filter and then transform back!)

$$X(\omega) = \int \delta(t-2)e^{-i\omega t}\,dt$$

$$= e^{-2i\omega}$$

$$-2 \le \omega \le 2$$

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2} e^{i\omega t}e^{-2i\omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{2} e^{i\omega(t-2)}\,d\omega$$

$$= \frac{1}{2\pi(t-2)i}\left[e^{i2(t-2)} - e^{-i2(t-2)}\right]$$

$$= \sin(2(t-2))/\pi(t-2)$$

$$= \sin(2(t-2))/\pi(t-2)$$

9 Impulse Response to Frequency Response

(10 points) Use the DTFT or the CTFT to determine the frequency response that corresponds to the following impulse responses:

$$h(t) = \delta(t-1) + 2\delta(t-2)$$

$$h(n) = \begin{cases} \frac{1}{3} & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

1) $h(t) = \delta(t-1) + 2\delta(t-2)$

$$H(\omega) = \int \delta(t-1)e^{-i\omega t} + 2\delta(t-2)e^{-i\omega t}\,dt$$

$$= e^{-i\omega} + 2e^{-2i\omega}$$

2) $h(n) = \begin{cases} \frac{1}{3} & x = 0, 1, 2) \\ 0 & \text{else} \end{cases}$

$$= \frac{1}{3}\left(\delta(n) + \delta(n-1) + \delta(n-2)\right)$$

10) Find DFT & DTFT of $x(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$ $(N=4)$

DTFT

$$X(\omega) = \sum \delta(n)e^{-i\omega n}$$

$$= e^{-i\omega(0)} = 1$$

DFT pick N

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

DFT pick N

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$X_0 = \sum_{n=0}^{3} \delta(n) e^{-i\omega_0 n k}$$

$$X_0 = \delta(0) e^{-i\frac{\pi}{2}(0)(0)} = 1$$

$$X_1 = \delta(0) e^{-i\frac{\pi}{2}(0)(1)} = 1$$

$$X_{N-1} = 1$$