(b) For $\mathbf{B} = \hat{\mathbf{z}}10\cos x \cos 10^3 t$ (T),

$$V_{emf} = -100\frac{d}{dt}\left(10\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x\,dx\,dy\right) = 62.3\sin 10^3 t \quad \text{(kV)}.$$

(c) For $\mathbf{B} = \hat{\mathbf{z}}10\cos x \sin 2y \cos 10^3 t$ (T),

$$V_{emf} = -100\frac{d}{dt}\left(10\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x \sin 2y\,dx\,dy\right) = 0.$$

Problem 6.4 A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 2.5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2-Ω resistor is connected across its open ends?

Solution: V_{emf} is independent of the resistance which is in the loop. Therefore, when the loop is intact and the internal resistance is only 0.5 Ω,

$$V_{emf} = 2.5 \text{ A} \times 0.5\ \Omega = 1.25 \text{ V}.$$

When the small gap is created, the total resistance in the loop is infinite and the current flow is zero. With a 2-Ω resistor in the gap,

$$I = V_{emf}/(2\ \Omega + 0.5\ \Omega) = 1.25 \text{ V}/2.5\ \Omega = 0.5 \quad \text{(A)}.$$

Problem 6.5 A circular-loop TV antenna with 0.01 m^2 area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 20 (mV). What is the peak magnitude of $\mathbf{B}$ of the incident wave?

Solution: TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to $\Phi = \int \mathbf{B} \cdot d\mathbf{s} = \pm BA$ for a loop of area A and a uniform magnetic field with magnitude $B = |\mathbf{B}|$. Since we know the frequency of the field is $f = 300$ MHz, we can express B as $B = B_0 \cos(\omega t + \alpha_0)$ with $\omega = 2\pi \times 300 \times 10^6$ rad/s and α_0 an arbitrary reference phase. From Eq. (6.6),

$$V_{emf} = -N\frac{d\Phi}{dt} = -A\frac{d}{dt}[B_0\cos(\omega t + \alpha_0)] = AB_0\omega\sin(\omega t + \alpha_0).$$

V_{emf} is maximum when $\sin(\omega t + \alpha_0) = 1$. Hence,

$$20 \times 10^{-3} = AB_0\omega = 10^{-2} \times B_0 \times 6\pi \times 10^8,$$

which yields $B_0 =$ 1.06 (nA/m)

is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{5\text{ cm}}^{15\text{ cm}} \left(-\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y}\right) \cdot [-\hat{\mathbf{x}}\, 10\text{ (cm)}]\, dy$$
$$= \frac{\mu_0 I \times 10^{-1}}{2\pi} \ln \frac{15}{5}$$
$$= \frac{4\pi \times 10^{-7} \times 2.5\cos(2\pi \times 10^4 t) \times 10^{-1}}{2\pi} \times 1.1$$
$$= 0.55 \times 10^{-7} \cos(2\pi \times 10^4 t) \quad \text{(Wb)}.$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 0.55 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7}$$
$$= 3.45 \times 10^{-3} \sin(2\pi \times 10^4 t) \quad \text{(V)}.$$

(b)

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{4+1} = \frac{3.45 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 0.69 \sin(2\pi \times 10^4 t) \quad \text{(mA)}.$$

At $t = 0$, **B** is a maximum, it points in $-\hat{\mathbf{x}}$-direction, and since it varies as $\cos(2\pi \times 10^4 t)$, it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.

Problem 6.7 The rectangular conducting loop shown in Fig. 6-20 (P6.7) rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}\, 50 \quad \text{(mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

Solution:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \hat{\mathbf{y}}\, 50 \times 10^{-3} \cdot \hat{\mathbf{y}}(2 \times 3 \times 10^{-4}) \cos\phi(t) = 3 \times 10^{-5} \cos\phi(t),$$
$$\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad \text{(rad/s)},$$
$$\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad \text{(Wb)},$$
$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad \text{(V)},$$
$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad \text{(mA)}.$$

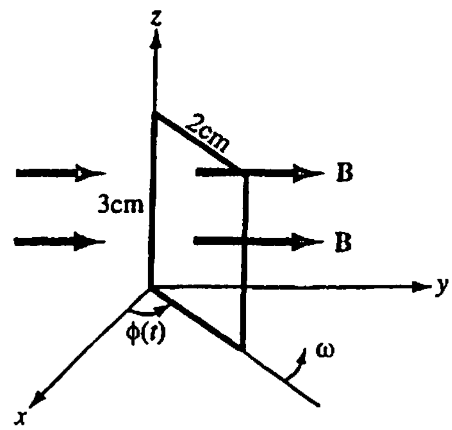

Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \le \phi \le \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.

Problem 6.8 A rectangular conducting loop 5 cm ×10 cm with a small air gap in one of its sides is spinning at 7200 revolutions per minute. If the field **B** is normal to the loop axis and its magnitude is 5×10^{-6} T, what is the peak voltage induced across the air gap?

Solution:

$$\omega = \frac{2\pi \text{ rad/cycle} \times 7200 \text{ cycles/min}}{60 \text{ s/min}} = 240\pi \text{ rad/s},$$

$$A = 5 \text{ cm} \times 10 \text{ cm}/(100 \text{ cm/m})^2 = 5.0 \times 10^{-3} \text{ m}^2.$$

From Eqs. (6.36) or (6.38), $V_{\text{emf}} = A\omega B_0 \sin\omega t$; it can be seen that the peak voltage is

$$V_{\text{emf}}^{\text{peak}} = A\omega B_0 = 5.0 \times 10^{-3} \times 240\pi \times 5 \times 10^{-6} = 18.85 \quad (\mu\text{V}).$$

Problem 6.9 A 50-cm-long metal rod rotates about the z-axis at 180 revolutions per minute, with end 1 fixed at the origin as shown in Fig. 6-21 (P6.9). Determine the induced emf V_{12} if $\mathbf{B} = \hat{\mathbf{z}}3 \times 10^{-4}$ T.

Solution: Since **B** is constant, $V_{\text{emf}} = V_{\text{emf}}^{\text{m}}$. The velocity **u** for any point on the bar is given by $\mathbf{u} = \hat{\boldsymbol{\phi}} r\omega$, where

$$\omega = \frac{2\pi \text{ rad/cycle} \times (180 \text{ cycles/min})}{(60 \text{ s/min})} = 6\pi \text{ rad/s}.$$

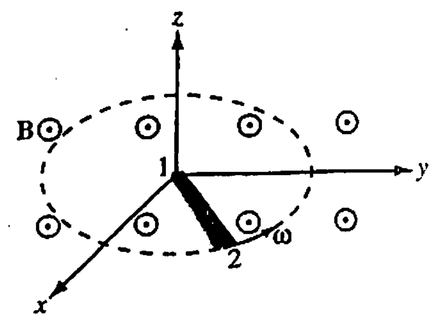

Figure P6.9: Rotating rod of Problem 6.9.

From Eq. (6.24),

$$V_{12} = V_{\text{emf}}^{\text{m}} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{r=0.5}^{0} (\hat{\boldsymbol{\phi}} 6\pi r \times \hat{\mathbf{z}} 3 \times 10^{-4}) \cdot \hat{\mathbf{r}}\, dr$$
$$= 18\pi \times 10^{-4} \int_{r=0.5}^{0} r\, dr$$
$$= 9\pi \times 10^{-4} r^2 \Big|_{0.5}^{0}$$
$$= -9\pi \times 10^{-4} \times 0.25 = -707 \quad (\mu\text{V}).$$

Problem 6.10 The loop shown in Fig. 6-22 (P6.10) moves away from a wire carrying a current $I_1 = 10$ (A) at a constant velocity $\mathbf{u} = \hat{\mathbf{y}}5$ (m/s). If $R = 10\ \Omega$ and the direction of I_2 is as defined in the figure, find I_2 as a function of y_0, the distance between the wire and the loop. Ignore the internal resistance of the loop.

Solution: Assume that the wire carrying current I_1 is in the same plane as the loop. The two identical resistors are in series, so $I_2 = V_{\text{emf}}/2R$, where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

$$V_{\text{emf}} = V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

The magnetic field $\mathbf{B}$ is created by the wire carrying I_1. Choosing $\hat{\mathbf{z}}$ to coincide with the direction of I_1, Eq. (5.30) gives the external magnetic field of a long wire to be

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r}.$$

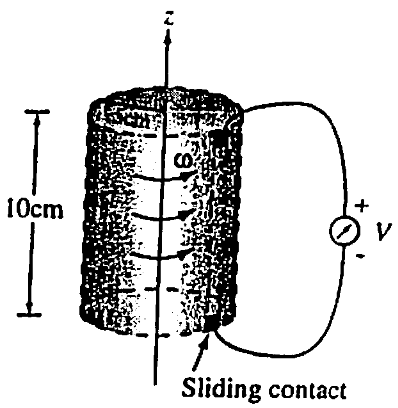

Figure P6.11: Rotating cylinder in a magnetic field (Problem 6.11).

The cylinder, whose radius is 5 cm and height 10 cm, has sliding contacts at its top and bottom connected to a voltmeter. Determine the induced voltage.

Solution: The surface of the cylinder has velocity **u** given by

$$\mathbf{u} = \hat{\boldsymbol{\phi}}\,\omega r = \hat{\boldsymbol{\phi}}\,2\pi \times \frac{1,200}{60} \times 5 \times 10^{-2} = \hat{\boldsymbol{\phi}}\,2\pi \quad \text{(m/s)},$$

$$V_{12} = \int_0^L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^{0.1} (\hat{\boldsymbol{\phi}}\,2\pi \times \hat{\mathbf{r}}\,6) \cdot \hat{\mathbf{z}}\,dz = -3.77 \quad \text{(V)}.$$

Problem 6.12 The electromagnetic generator shown in Fig. 6-12 is connected to an electric bulb with a resistance of 100 Ω. If the loop area is 0.1 m^2 and it rotates at 3,600 revolutions per minute in a uniform magnetic flux density $B_0 = 0.2$ T, determine the amplitude of the current generated in the light bulb.

Solution: From Eq. (6.38), the sinusoidal voltage generated by the a-c generator is $V_{\text{emf}} = A\omega B_0 \sin(\omega t + C_0) = V_0 \sin(\omega t + C_0)$. Hence,

$$V_0 = A\omega B_0 = 0.1 \times \frac{2\pi \times 3,600}{60} \times 0.2 = 7.54 \quad \text{(V)},$$

$$I = \frac{V_0}{R} = \frac{7.54}{100} = 75.4 \quad \text{(mA)}.$$

Problem 6.13 The circular disk shown in Fig. 6-24 (P6.13) lies in the x–y plane and rotates with uniform angular velocity ω about the z-axis. The disk is of radius a and is present in a uniform magnetic flux density $\mathbf{B} = \hat{\mathbf{z}}B_0$. Obtain an expression for the emf induced at the rim relative to the center of the disk.

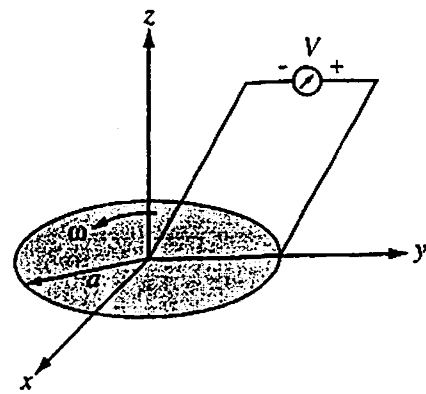

Figure P6.13: Rotating circular disk in a magnetic field (Problem 6.13).

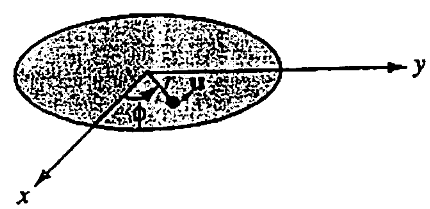

Figure P6.13: (a) Velocity vector **u**.

Solution: At a radial distance r, the velocity is

$$\mathbf{u} = \hat{\boldsymbol{\phi}}\,\omega r$$

where ϕ is the angle in the x–y plane shown in the figure. The induced voltage is

$$V = \int_0^a (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} = \int_0^a [(\hat{\boldsymbol{\phi}}\,\omega r)\times\hat{\mathbf{z}}B_0]\cdot\hat{\mathbf{r}}\,dr.$$

$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}$ is along $\hat{\mathbf{r}}$. Hence,

$$V = \omega B_0 \int_0^a r\,dr = \frac{\omega B_0 a^2}{2}.$$

Section 6-7: Displacement Current

Problem 6.14 The plates of a parallel-plate capacitor have areas 10 cm^2 each and are separated by 1 cm. The capacitor is filled with a dielectric material with

$\varepsilon = 4\varepsilon_0$, and the voltage across it is given by $V(t) = 20\cos 2\pi \times 10^6 t$ (V). Find the displacement current.

Solution: Since the voltage is of the form given by Eq. (6.46) with $V_0 = 20$ V and $\omega = 2\pi \times 10^6$ rad/s, the displacement current is given by Eq. (6.49):

$$\begin{aligned} I_d &= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \\ &= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{1 \times 10^{-2}} \times 20 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t) \\ &= -445 \sin(2\pi \times 10^6 t) \quad (\mu\text{A}). \end{aligned}$$

Problem 6.15 A coaxial capacitor of length $l = 6$ cm uses an insulating dielectric material with $\varepsilon_r = 9$. The radii of the cylindrical conductors are 0.5 cm and 1 cm. If the voltage applied across the capacitor is

$$V(t) = 100 \sin(120\pi t) \quad \text{(V)},$$

what is the displacement current?

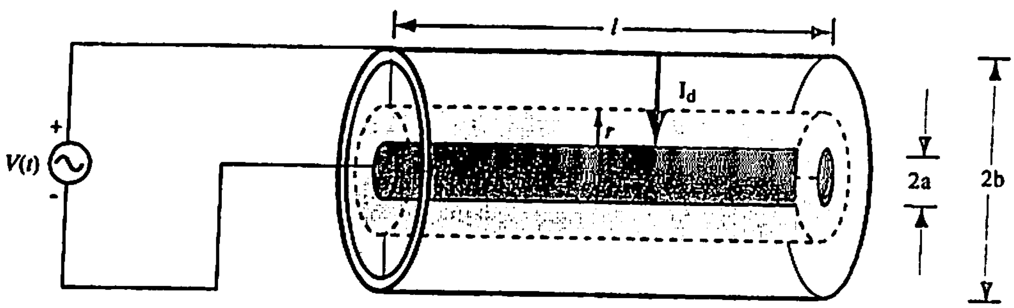

Figure P6.15:

Solution: To find the displacement current, we need to know **E** in the dielectric space between the cylindrical conductors. From Eqs. (4.114) and (4.115),

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi\varepsilon r l},$$

$$V = \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right).$$

Hence,

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{V}{r \ln\left(\frac{b}{a}\right)} = -\hat{\mathbf{r}} \frac{100 \sin(120\pi t)}{r \ln 2} = -\hat{\mathbf{r}} \frac{144.3}{r} \sin(120\pi t) \quad \text{(V/m)},$$

$$
\begin{aligned}
\mathbf{D} &= \varepsilon\mathbf{E} \\
&= \varepsilon_r\varepsilon_0\mathbf{E} \\
&= -\hat{\mathbf{r}}9 \times 8.85 \times 10^{-12} \times \frac{144.3}{r}\sin(120\pi t) \\
&= -\hat{\mathbf{r}}\frac{1.15 \times 10^{-8}}{r}\sin(120\pi t) \quad (\text{C/m}^2).
\end{aligned}
$$

The displacement current flows between the conductors through an imaginary cylindrical surface of length l and radius r. The current flowing from the outer conductor to the inner conductor along $-\hat{\mathbf{r}}$ crosses surface $\mathbf{S}$ where

$$\mathbf{S} = -\hat{\mathbf{r}}2\pi rl.$$

Hence,

$$
\begin{aligned}
I_d = \frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{S} &= -\hat{\mathbf{r}}\frac{\partial}{\partial t}\left(\frac{1.15 \times 10^{-8}}{r}\sin(120\pi t)\right)\cdot(-\hat{\mathbf{r}}2\pi rl) \\
&= 1.15 \times 10^{-8} \times 120\pi \times 2\pi l\cos(120\pi t) \\
&= 1.63\cos(120\pi t) \quad (\mu\text{A}).
\end{aligned}
$$

Alternatively, since the coaxial capacitor is lossless, its displacement current has to be equal to the conduction current flowing through the wires connected to the voltage sources. The capacitance of a coaxial capacitor is given by (4.116) as

$$C = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}.$$

The current is

$$I = C\frac{dV}{dt} = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}[120\pi \times 100\cos(120\pi t)] = 1.63\cos(120\pi t) \quad (\mu\text{A}),$$

which is the same answer we obtained before.

Problem 6.16 The parallel-plate capacitor shown in Fig. 6-25 (P6.16) is filled with a lossy dielectric material of relative permittivity ε_r and conductivity σ. The separation between the plates is d and each plate is of area A. The capacitor is connected to a time-varying voltage source $V(t)$.

(a) Obtain an expression for I_c, the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.

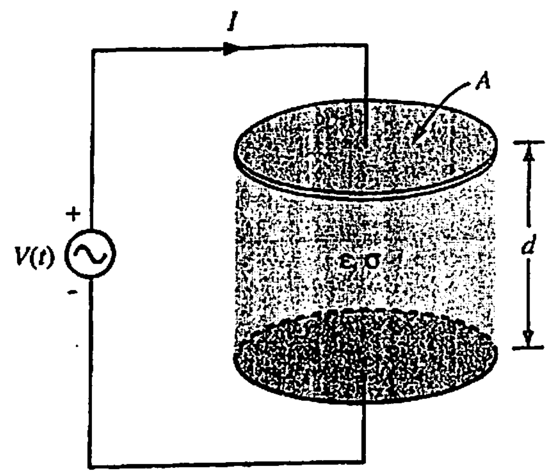

Figure P6.16: Parallel-plate capacitor containing a lossy dielectric material (Problem 6.16).

(b) Obtain an expression for I_d, the displacement current flowing inside the capacitor.

(c) Based on your expression for parts (a) and (b), give an equivalent-circuit representation for the capacitor.

(d) Evaluate the values of the circuit elements for $A = 2$ cm^2, $d = 0.5$ cm, $\varepsilon_r = 4$, $\sigma = 2.5$ (S/m), and $V(t) = 10\cos(3\pi \times 10^3 t)$ (V).

Solution:

(a)

$$R = \frac{d}{\sigma A}, \qquad I_c = \frac{V}{R} = \frac{V\sigma A}{d}.$$

(b)

$$E = \frac{V}{d}, \qquad I_d = \frac{\partial D}{\partial t} \cdot A = \varepsilon A \frac{\partial E}{\partial t} = \frac{\varepsilon A}{d} \frac{\partial V}{\partial t}.$$

(c) The conduction current is directly proportional to V, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence,

$$R = \frac{d}{\sigma A} \qquad \text{and} \qquad C = \frac{\varepsilon A}{d}.$$

(d)

$$R = \frac{0.5 \times 10^{-2}}{2.5 \times 2 \times 10^{-4}} = 10\ \Omega,$$

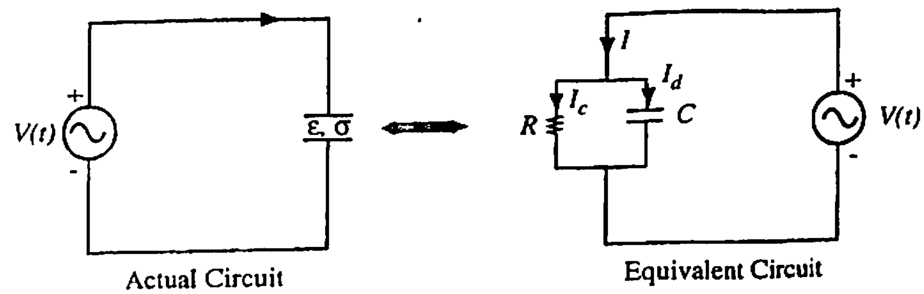

Figure P6.16: (a) Equivalent circuit.

$$C = \frac{4 \times 8.85 \times 10^{-12} \times 2 \times 10^{-4}}{0.5 \times 10^{-2}} = 1.42 \times 10^{-12} \text{ F}.$$

Problem 6.17 An electromagnetic wave propagating in seawater has an electric field with a time variation given by $\mathbf{E} = \hat{\mathbf{z}}E_0 \cos \omega t$. If the permittivity of water is $81\varepsilon_0$ and its conductivity is 4 (S/m), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies: (a) 1 kHz, (b) 1 MHz, (c) 1 GHz, (d) 100 GHz.

Solution: From Eq. (6.44), the displacement current density is given by

$$\mathbf{J}_d = \frac{\partial}{\partial t}\mathbf{D} = \varepsilon \frac{\partial}{\partial t}\mathbf{E}$$

and, from Eq. (4.67), the conduction current is $\mathbf{J} = \sigma \mathbf{E}$. Converting to phasors and taking the ratio of the magnitudes,

$$\left|\frac{\tilde{\mathbf{J}}}{\tilde{\mathbf{J}}_d}\right| = \left|\frac{\sigma \tilde{\mathbf{E}}}{j\omega\varepsilon_r\varepsilon_0 \tilde{\mathbf{E}}}\right| = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0}.$$

(a) At $f = 1$ kHz, $\omega = 2\pi \times 10^3$ rad/s, and

$$\left|\frac{\tilde{\mathbf{J}}}{\tilde{\mathbf{J}}_d}\right| = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.$$

The displacement current is negligible.

(b) At $f = 1$ MHz, $\omega = 2\pi \times 10^6$ rad/s, and

$$\left|\frac{\tilde{\mathbf{J}}}{\tilde{\mathbf{J}}_d}\right| = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.$$

The displacement current is practically negligible.

(c) At $f = 1$ GHz, $\omega = 2\pi \times 10^9$ rad/s, and

$$\left|\frac{\tilde{\mathbf{J}}}{\tilde{\mathbf{J}}_d}\right| = \frac{4}{2\pi \times 10^9 \times 81 \times 8.854 \times 10^{-12}} = 0.888.$$

Neither the displacement current nor the conduction current are negligible.

(d) At $f = 100$ GHz, $\omega = 2\pi \times 10^{11}$ rad/s, and

$$\left|\frac{\tilde{\mathbf{J}}}{\tilde{\mathbf{J}}_d}\right| = \frac{4}{2\pi \times 10^{11} \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^{-3}.$$

The conduction current is practically negligible.

Sections 6-9 and 6-10: Continuity Equation and Charge Dissipation

Problem 6.18 At $t = 0$, charge density ρ_{v0} was introduced into the interior of a material with a relative permittivity $\varepsilon_r = 4\varepsilon_0$. If at $t = 1$ μs the charge density has dissipated down to $10^{-3}\rho_{v0}$, what is the conductivity of the material?

Solution: We start by using Eq. (6.61) to find τ_r:

$$\rho_v(t) = \rho_{v0}e^{-t/\tau_r},$$

or

$$10^{-3}\rho_{v0} = \rho_{v0}e^{-10^{-6}/\tau_r},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_r},$$

or

$$\tau_r = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \quad \text{(s)}.$$

But $\tau_r = \varepsilon/\sigma = 4\varepsilon_0/\sigma$. Hence

$$\sigma = \frac{4\varepsilon_0}{\tau_r} = \frac{4 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 2.44 \times 10^{-4} \quad \text{(S/m)}.$$