LAST (family) NAME: ______________________

FIRST (given) NAME: ______________________

ID # : ___________________

MATHEMATICS 2ZZ3: FINAL EXAM SAMPLE B SOLUTIONS

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SOLUTIONS TO THE MUTIPLE-CHOICE PART OF THE EXAM

Part I: Multiple-choice questions: Enter your answers to Questions 1 to 16 on the scantron sheet provided, following the instructions given on page 2 and 3. You **do not** need to justify your answers.

1. *(4 pts.)* Consider the vector field

$$\mathbf{F} = (y^2 + \sin y)\,\mathbf{i} + (2xy + x\cos y)\,\mathbf{j} + \sin z\,\mathbf{k}.$$

Is $\mathbf{F}$ conservative? If so, then find a potential function ϕ for $\mathbf{F}$.

(A) not conservative

(B) $\phi(x,y,z) = 2x - x\sin y - \cos z$

→**(C)** $\phi(x,y,z) = xy^2 + x\sin y - \cos z$

(D) $\phi(x,y,z) = 2xy + x\sin y + \sin z$

(E) $\phi(x,y,z) = xy^2 - x\sin y + \cos z$

Solution. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2+\sin y) & (2xy + x\cos y) & \sin z \end{vmatrix} = \mathbf{0}$$

so $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla\phi$ for some function ϕ. We have $\frac{\partial\phi}{\partial x} = y^2 + \sin y$, so $\phi(x,y,z) = x\,y^2 + x\,\sin y + C(y,z)$. Thus $\frac{\partial\phi}{\partial y} = 2xy + x\cos y + \frac{\partial C}{\partial y} = 2xy + x\cos y$ and $\frac{\partial C}{\partial y} = 0$, so that $C(y,z) = C(z)$. Finally, $\frac{\partial\phi}{\partial z} = C'(z) = \sin z$ and $C(z) = -\cos z + C$. Therefore, $\phi(x,y,z) = x\,y^2 + x\,\sin y - \cos z + C$.

2. *(4 pts.)* Which of the following integrals gives the area of the part of the surface $z = x^2 + y + 1$ that lies over the triangle in the xy-plane with vertices $(0,0)$, $(1,1)$, and $(0,1)$?

(A) $\displaystyle\int_0^1\int_0^1 \sqrt{2x+2}\;dy\,dx$

(B) $\displaystyle\int_0^1\int_0^x \sqrt{4x^2+2}\;dy\,dx$

→**(C)** $\displaystyle\int_0^1\int_x^1 \sqrt{4x^2+2}\;dy\,dx$

(D) $\displaystyle\int_0^y\int_0^1 \sqrt{2x+2}\;dy\,dx$

(E) $\displaystyle\int_0^y\int_0^1 \sqrt{4x^2+2}\;dy\,dx$

Solution. We can parametrize S using $\mathbf{r}(x,y) = \langle x, y, x^2+y+1\rangle$, $(x,y) \in D$, where $D = \{(x,y), 0 \le x \le 1,\ x \le y \le 1\}$. We have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2\,x \\ 0 & 1 & 1 \end{vmatrix} = \langle -2\,x, -1, 1\rangle \quad \text{and} \quad \|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{4\,x^2+2}.$$

The area of S is thus computed as

$$\int_0^1\int_x^1 \sqrt{4\,x^2+2}\,dy\,dx.$$

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3. *(4 pts.)* Let C be the curve parametrized by

$$\mathbf{r}(t) = \left(te^{t(t-2)} + \frac{1}{2}t^2\cos\pi t\right)\,\mathbf{i} + \left(\ln(-t^2+2t+1) + t\sin\pi t\right)\,\mathbf{j}, \quad 0 \le t \le 2.$$

Let $\mathbf{G}(x,y) = \langle 2x+y, x\rangle$. Compute $\displaystyle\int_C \mathbf{G}\cdot d\mathbf{r}$. (Hint: is $\mathbf{G}$ conservative?)

(A) 0

(B) -8

(C) 8

(D) -16

$\rightarrow$**(E)** 16

Solution. We have $\mathbf{G}(x,y) = \nabla g(x,y)$, where $g(x,y) = x^2 + x\,y$ showing that $\mathbf{G}$ is conservative. Since $\mathbf{r}(0) = \langle 0,0\rangle$ and $\mathbf{r}(2) = \langle 4,0\rangle$,

$$\int_C \mathbf{G}\cdot d\mathbf{r} = g(4,0) - g(0,0) = 16.$$

4. *(4 pts.)* Rewrite the integral $\displaystyle\int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} F(x,y,z)\,dx\,dy\,dz$ in the order $dz\,dy\,dx$.

(A) $\displaystyle\int_{-2}^{2}\int_0^{x^2}\int_0^{2-x^2/2} F(x,y,z)\;dz\,dy\,dx$

(B) $\displaystyle\int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} F(x,y,z)\;dz\,dy\,dx$

(C) $\displaystyle\int_0^{2}\int_0^{x^2}\int_0^{2-y/2} F(x,y,z)\;dz\,dy\,dx$

(D) $\displaystyle\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{4-2z}\int_0^{2} F(x,y,z)\;dz\,dy\,dx$

$\rightarrow$**(E)** $\displaystyle\int_{-2}^{2}\int_{x^2}^{4}\int_0^{2-y/2} F(x,y,z)\;dz\,dy\,dx$

Solution. The region of integration is defined by the inequalities

$$0 \le z \le 2, \quad 0 \le y \le 4 - 2\,z, \quad -\sqrt{y} \le x \le \sqrt{y}$$

which can also be written as

$$-2 \le x \le 2, \quad x^2 \le y \le 4, \quad 0 \le z \le 2 - y/2.$$

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5. *(4 pts.)* Let

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$$

Find the Fourier series for the *even* 4-periodic extension of f.

(A) $1/2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \cos\left(\frac{n\pi}{2} x\right)$

(D) $1/2 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cos\left(\frac{n\pi}{2} x\right)$

→**(B)** $1/2 + \sum_{n=1}^{\infty} (-1)^{n} \frac{2}{(2n-1)\pi} \cos\left(\frac{(2n-1)\pi}{2} x\right)$

(E) $1/2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2} x\right)$

(C) $\sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} [(-1)^{n+1} + \cos(n\pi/2)] \cos\left(\frac{n\pi}{2} x\right)$

Solution. We have $L = 2$ and thus $f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos(n\pi x/2)$, where $a_0 = \frac{2}{2} \int_0^2 f(x)\,dx = \int_1^2 1\,dx = 1$, and

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx = \left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]_1^2 \\ &= -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n = 2k, \\ \frac{2}{(2k-1)\pi} (-1)^k, & \text{if } n = 2k-1. \end{cases} \end{aligned}$$

Thus, $f(x) \simeq \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{2}{(2k-1)\pi} \cos\left(\frac{(2k-1)\pi x}{2}\right)$

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6. *(4 pts.)* Compute the integral

$$I = \int_D \frac{1}{\sqrt{9 - x^2 - y^2}}\, dx\, dy,$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le -3\,x\}$ by passing to polar coordinates.

(A) $I = 3(\pi + 2)$

(B) $I = 3\pi$

→**(C)** $I = 3(\pi - 2)$

(D) $I = 2(\pi - 1)$

(E) $I = \pi$

Solution.
The region D can be expressed in polar coordinates as

$$D^* = \{(r, \theta), \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, 0 \le r \le -3\,\cos\theta\}.$$

Thus,

$$\begin{aligned} I &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-3\,\cos\theta} \frac{r}{\sqrt{9 - r^2}}\, dr\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[-(9 - r^2)^{1/2}\right]_{r=0}^{r=-3\,\cos\theta}\, d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3 - 3(1 - \cos^2\theta)^{1/2}\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3 - 3\,|\sin\theta|\, d\theta = 2\int_{\frac{\pi}{2}}^{\pi} 3 - 3\,\sin\theta\, d\theta \\ &= 6\,\left[\theta + \cos\theta\right]_{\frac{\pi}{2}}^{\pi} = 6\,(\frac{\pi}{2} - 1) = 3\,(\pi - 2). \end{aligned}$$

7. *(4 pts.)* Compute the curvature κ of the curve C parametrized by

$$\mathbf{r}(t) = \left\langle t, \frac{t^2}{2}, \frac{t^3}{3}\right\rangle, \quad -\infty < t < \infty,$$

at the point $(1, \frac{1}{2}, \frac{1}{3})$.

(A) $\kappa = \dfrac{\sqrt{3}}{2\sqrt{2}}$

→**(B)** $\kappa = \dfrac{\sqrt{2}}{3}$

(C) $\kappa = \dfrac{\sqrt{3}}{5\sqrt{5}}$

(D) $\kappa = \dfrac{4}{\sqrt{3}}$

(E) $\kappa = \dfrac{3}{\sqrt{5}}$

Solution. We have $\mathbf{r}'(t) = \langle 1, t, t^2\rangle$, $\mathbf{r}''(t) = \langle 0, 1, 2\,t\rangle$ and $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle t^2, -2\,t, 1\rangle$. Thus, when $t = 1$, $\|\mathbf{r}'(1)\| = \sqrt{3}$ and $\|\mathbf{r}'(t) \times \mathbf{r}''(1)\| = \sqrt{6}$ and $\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$

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8. *(4 pts.)* Let S be the part of the ellipsoid $x^2+y^2+\dfrac{z^2}{4}=9$ above the x,y plane oriented so that normal has a positive z component. Consider the vector field $\mathbf{F}(x,y,z)=-y^3\,\mathbf{i}+x^3\,\mathbf{j}+z^3\,\mathbf{k}$. Use Stokes' Theorem to compute the flux I of the vector field $\nabla\times\mathbf{F}$ through S.

(A) $I=2\,\pi$

(B) $I=\dfrac{3\,\pi}{4}$

(C) $I=\dfrac{316\,\pi}{7}$

(D) $I=\dfrac{284\,\pi}{3}$

$\rightarrow$ **(E)** $I=\dfrac{243\,\pi}{2}$

Solution. The boundary of the surface S is the circle $x^2+y^2=9$ on the x,y plane which must be oriented counterclockwise to match the orientation of the surface. We can parametrize C by $\mathbf{r}(t)=\langle 3\,\cos t, 3\,\sin t, 0\rangle$ for $0\le t\le 2\,\pi$. We have $\mathbf{r}'(t)=\big\langle -3\,\sin t, 3\,\cos t, 0\big\rangle$ and, using Stokes' theorem,

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\big\langle -3^3\sin^3 t, 3^3\,\cos^3 t, 0\big\rangle\cdot\big\langle -3\,\sin t, 3\,\cos t, 0\big\rangle\,dt$$
$$=3^4\int_0^{2\pi}\cos^4 t+\sin^4 t\,dt=\frac{3^4}{2^2}\int_0^{2\pi}(1+\cos(2t))^2+(1-\cos(2t))^2\,dt$$
$$=\frac{3^4}{2}\int_0^{2\pi}1+\cos^2(2t)\,dt=\frac{3^4}{4}\int_0^{2\pi}3+\cos(4t)\,dt=\frac{3^4}{4}\left[3\,t+\frac{\sin(4t)}{4}\right]_0^{2\pi}=\frac{243\,\pi}{2}$$

9. *(4 pts.)* The volume of the solid region bounded below by the part of the cone of $4\,z^2=x^2+y^2$ above the x,y plane and above by the sphere $x^2+y^2+z^2=5$ can be computed, after passing to cylindrical coordinates, by the integral

$$\int_0^{2\pi}\int_0^a\int_{\frac{r}{2}}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta$$

where a is the number

(A) $a=1$

$\rightarrow$ **(B)** $a=2$

(C) $a=3$

(D) $a=4$

(E) $a=5$

Solution. The solid region is described by the inequalities $\frac{\sqrt{x^2+y^2}}{2}\le z\le\sqrt{5-x^2-y^2}$ or, after passing to polar coordinates by $\frac{r}{2}\le z\le\sqrt{5-r^2}$. Since $\frac{r}{2}=\sqrt{5-r^2}$ when $r=2$, we must have $0\le r\le 2$.

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10. *(4 pts.)* Compute the value of the integral

$$I = \iiint_V \sqrt{x^2 + y^2 + z^2}\, dV,$$

where V is the solid region between the sphere $x^2 + y^2 + z^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$.

(A) $I = 13\,\pi$

(B) $I = 5\,\pi$

(C) $I = \frac{7\pi}{3}$

(D) $I = 12\,\pi$

$\rightarrow$ **(E)** $I = 65\,\pi$

Solution.

$$I = \int_0^{2\pi} \int_0^{\pi} \int_2^{3} \rho\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_2^{3} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\,\pi \left[-\cos\phi\right]_0^{\pi} \left[\frac{\rho^4}{4}\right]_2^3 = 2\,\pi\,(2)\left(\frac{65}{4}\right) = 65\,\pi.$$

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Part II: Provide all details and fully justify your answer in order to receive credit.

11. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the equation

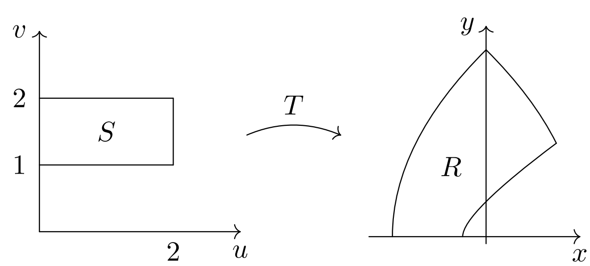
$$T(u,v) = \big(x(u,v), y(u,v)\big) = (u^2 - v^2, 2uv).$$

(a) *(3 pts.)* Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of T.

Solution. We have

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2\,u & -2\,v \\ 2\,v & 2\,u \end{vmatrix} \\ &= 4\,(u^2 + v^2).\end{aligned}$$

(b) *(3 pts.)* The image of the rectangle $S = [0,2] \times [1,2]$ under the transformation T is the region R bounded by the x-axis and the curves $y = 4\sqrt{4+x}$, $y = 2\sqrt{1+x}$, and $y = 4\sqrt{4-x}$.



Find the area of R by using the change of variables formula for double integrals.

Solution. We have

$$\begin{aligned} A = \text{area}(D) &= \iint_R 1\,dx\,dy \\ &= \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^2 \int_1^2 4\,(u^2+v^2)\,dv\,du \\ &= 4\int_0^2 \left[u^2\,v + \frac{v^3}{3}\right]_{v=1}^{v=2} du = 4\int_0^2 u^2 + \frac{7}{3}\,du = 4\left[\frac{u^3}{3} + \frac{7\,u}{3}\right]_0^2 \\ &= \frac{88}{3}. \end{aligned}$$

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(c) *(6 pts.)* Compute the centroid $(\bar{x}, \bar{y})$ of the region R using the change of variables formula. (Note that the centroid is the center of mass when the density function is $\rho(x, y) = 1$.)

Solution. We have

$$\begin{aligned}
M_y &= \iint_R x \, dx \, dy \\
&= \iint_S (u^2 - v^2) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv = \int_0^2 \int_1^2 4\,(u^4 - v^4) \, dv \, du \\
&= 4 \int_0^2 \left[u^4 \, v - \frac{v^5}{5} \right]_{v=1}^{v=2} du = 4 \int_0^2 u^4 - \frac{31}{5} \, du = 4 \left[\frac{u^5}{5} - \frac{31\,u}{5} \right]_0^2 \\
&= -24
\end{aligned}$$

and

$$\begin{aligned}
M_x &= \iint_R y \, dx \, dy \\
&= \iint_S (2\,u\,v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv = \int_0^2 \int_1^2 8\,(u^3\, v + u\, v^3) \, dv \, du \\
&= 8 \int_0^2 \left[u^3 \, \frac{v^2}{2} + u \, \frac{v^4}{4} \right]_{v=1}^{v=2} du = 8 \int_0^2 \frac{3}{2} \, u^3 + \frac{15}{4} \, u \, du = 8 \left[\frac{3}{8} \, u^4 + \frac{15}{8} \, u^2 \right]_0^2 \\
&= (48 + 60) = 108.
\end{aligned}$$

The centroid is thus

$$\begin{aligned}
(\bar{x}, \bar{y}) &= \left(\frac{M_y}{A}, \frac{M_x}{A} \right) \\
&= \frac{3}{88} \, (-24, 108) = 3 \left(-\frac{3}{11}, \frac{27}{22} \right) = \left(-\frac{9}{11}, \frac{81}{22} \right)
\end{aligned}$$

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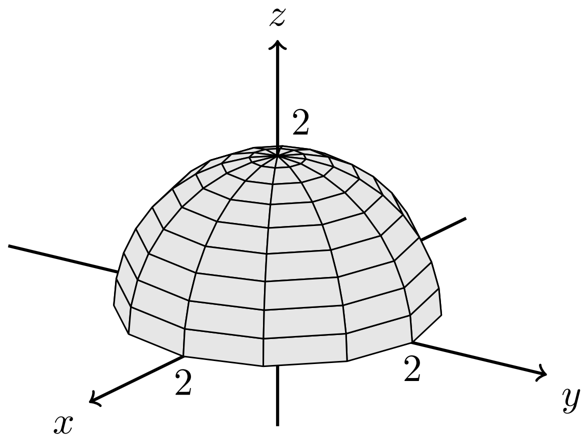
12. Let V be the solid region inside the sphere of radius 2 centered at the origin and above the x, y plane and let S be the boundary of V, oriented using the outward pointing normal. Note that the surface S can be decomposed into 2 surfaces: S_1 the hemisphere above the x, y plane and S_2 the part of the x, y plane below the hemisphere.



Consider the vector field $\mathbf{F}(x, y, z) = -y\ \mathbf{i} + x\ \mathbf{j} + z^3\ \mathbf{k}$.

(a) *(7 pts.)* Use the definition of surface integrals to compute the flux of the vector-field $\mathbf{F}$ through S.

Solution. We parametrize S_1 using

$$\mathbf{r}(x, y) = \langle x,\ y,\ \sqrt{4 - x^2 - y^2}\rangle, \quad \text{for } x^2 + y^2 \leq 4.$$

Letting $f(x, y) = \sqrt{4 - x^2 - y^2}$, we have

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x,\ -f_y,\ 1\rangle = \langle \frac{x}{\sqrt{4 - x^2 - y^2}},\ \frac{y}{\sqrt{4 - x^2 - y^2}},\ 1\rangle$$

This yields the correct orientation on S_1 since the normal has a positive z component.

We have

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_D \langle -y,\ x, (4 - x^2 + y^2)^{3/2}\rangle \cdot \left\langle \frac{x}{\sqrt{4 - x^2 + y^2}}, \frac{y}{\sqrt{4 - x^2 - y^2}}, 1 \right\rangle dA \\ &= \iint_D (4 - x^2 + y^2)^{3/2}\, dA \\ &= \int_0^2 \int_0^{2\pi} r\,(4 - r^2)^{3/2}\, d\theta\, dr = 2\,\pi \left[-\frac{1}{5}\,(4 - r^2)^{5/2} \right]_0^2 = \frac{64\pi}{5}.\end{aligned}$$

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We parametrize S_2 using
$$\mathbf{r}(x,y) = \langle x, y, 0\rangle, \quad (x,y) \in D,$$
where $D = \{(x,y), x^2+y^2 \leq 4\}$. We have
$$\mathbf{r}_x = \langle 1, 0, 0\rangle,$$
$$\mathbf{r}_y = 0, 1, 0\rangle,$$
and
$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1\rangle$$
This orientation is opposite to the one given by the outward pointing normal (and we need to multiply that vector by -1 we compute the surface integral). We have
$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_D \langle -y, x, 0\rangle \cdot \langle 0, 0, -1\rangle \; dx\, dy = 0$$
and
$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS = \frac{64\pi}{5} + 0 = \boxed{\frac{64\pi}{5}}.$$

(b) *(5 pts.)* Calculate the flux of the vector-field $\mathbf{F}$ through S using the divergence theorem by expressing it as a triple integral in spherical coordinates.

Solution. We have
$$\nabla \cdot F = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}\left(z^3\right) = 3\,z^2.$$
The solid region V can be expressed in spherical coordinates as
$$V^* = \{(\rho,\theta,\phi\},\ 0 \leq \rho \leq 2,\ 0 \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq 2\pi\}.$$
Using the divergence theorem,
$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= \iiint_T \nabla\cdot\mathbf{F}\, dV = \iiint_T 3\,z^2\, dV \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 3\,\rho^2\,\cos^2(\phi)\,\rho^2\,\sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= 6\,\pi \int_0^{\frac{\pi}{2}} \cos^2(\phi)\,\sin(\phi)\, d\phi \int_0^2 \rho^4\, d\rho = 6\,\pi\left[-\frac{\cos^3(\phi)}{3}\right]_0^{\frac{\pi}{2}}\left[\frac{\rho^5}{5}\right]_0^2 \\
&= 6\,\pi\left(\frac{1}{3}\right)\left(\frac{32}{5}\right) = \boxed{\frac{64\pi}{5}}.
\end{aligned}$$

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Some formulas you may use:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}, \qquad a_N = \kappa v^2 = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\operatorname{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\,\mathbf{b}$$

$$\frac{d}{dt}\left[u(t)\,\mathbf{r}(t)\right] = u(t)\,\mathbf{r}'(t) + u'(t)\,\mathbf{r}(t),$$

$$\frac{d}{dt}\left[\mathbf{r}_1\cdot\mathbf{r}_2\right] = \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) + \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t), \quad \frac{d}{dt}\left[\mathbf{r}_1\times\mathbf{r}_2\right] = \mathbf{r}_1'(t)\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\mathbf{r}_2'(t),$$

$$\frac{d}{dt}(\cos t) = -\sin t, \qquad \frac{d}{dt}(\sin t) = \cos t.$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t), \qquad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t).$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta), \quad 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad \cosh t = \frac{e^t + e^{-t}}{2} \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\rho = \sqrt{x^2+y^2+z^2}, \qquad x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi$$

$$r = \sqrt{x^2+y^2} = \rho\sin\phi$$

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_T \nabla\cdot\mathbf{F}\,dV$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \;\; 0<x<L.$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \;\; 0<x<L.$$

*** **THE END** ***