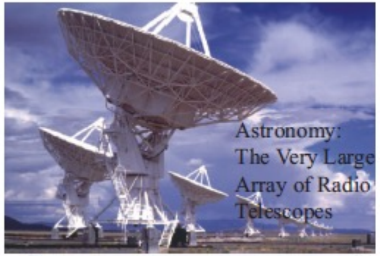
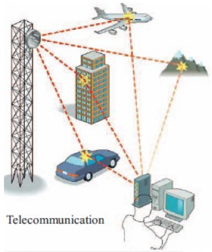
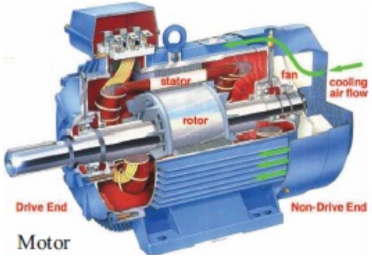


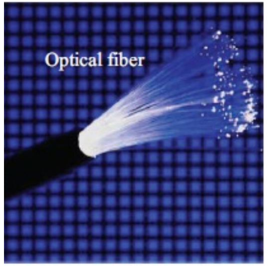


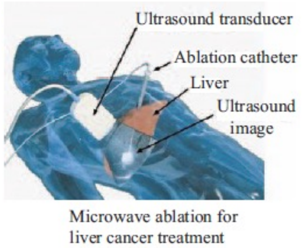


Electricity and Magnetism

Engineering Physics 2A04

Summary





Electricity & Magnetism

The fundamental problem the theory of Electricity and Magnetism hopes to solve is:

I hold up a bunch of electric charges *here* (and maybe shake them around); what happens to some other charges over *here*?

- Griffiths

The Three Branches of Electricity & Magnetism

Branch	Condition	Field Quantities [Units]
Electrostatics	Stationary charges $\frac{\partial q}{\partial t} = 0$	Electric field intensity $\vec{\boldsymbol{E}}$ [V/m] Electric flux density $\vec{\boldsymbol{D}}$ [C/m^2]
Magnetostatics	Steady currents $\frac{\partial I}{\partial t} = 0$	Magnetic field intensity $\vec{\boldsymbol{H}}$ [A/m] Magnetic flux density $\vec{\boldsymbol{B}}$ [T]
Dynamics (Time-varying fields)	Time-varying currents $\frac{\partial I}{\partial t} \neq 0$	$\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{D}}$, $\vec{\boldsymbol{H}}$, and $\vec{\boldsymbol{B}}$ $(\vec{\boldsymbol{E}}, \vec{\boldsymbol{D}})$ coupled to $(\vec{\boldsymbol{H}}, \vec{\boldsymbol{B}})$

Maxwell's Equations

Differential Form	Integral Form	
1) $\nabla \cdot \vec{\boldsymbol{D}} = \rho_v(x,y,z)$	$\oint_S \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}}' = Q$	Gauss's law
2) $\nabla \times \vec{\boldsymbol{E}} = -\dfrac{\partial \vec{\boldsymbol{B}}}{\partial t}$	$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}' = -\int_S \dfrac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot d\vec{\boldsymbol{s}}'$	Faraday's law (stationary surface *S*)
3) $\nabla \cdot \vec{\boldsymbol{B}} = 0$	$\oint_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}}' = 0$	Gauss's law, magnetism (no magnetic charges)
4) $\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \dfrac{\partial \vec{\boldsymbol{D}}}{\partial t}$	$\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}}' = \int_S \left(\vec{\boldsymbol{J}} + \dfrac{\partial \vec{\boldsymbol{D}}}{\partial t}\right) \cdot d\vec{\boldsymbol{s}}'$	Ampere's law

- An electric field can be produced either by charges or changing magnetic fields.
- A magnetic field can be produced either by currents or changing electric fields.

EM Toolbox

Electrostatics

Used	Eqn
Coulomb's law	$\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{\boldsymbol{R}}')(\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}')}{4\pi\varepsilon\|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\|^3} d\mathcal{V}'$
Material properties	$\vec{\boldsymbol{D}} = \varepsilon \vec{\boldsymbol{E}}$
Gauss's law	$\nabla \cdot \vec{\boldsymbol{D}} = \rho_V(x,y,z)$, $\int_S \varepsilon \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \rho_V(x,y,z)$
Scalar potential function	$\vec{\boldsymbol{E}} = -\nabla V$, $V(\vec{\boldsymbol{R}}) = \int_{\mathcal{V}'} \frac{\rho_v(\vec{\boldsymbol{R}}')}{4\pi\varepsilon\|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\|} d\mathcal{V}'$
Energy density	$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 d\mathcal{V}$

Magnetostatics

Used	Eqn
Biot-Savart law	$\vec{\boldsymbol{B}}(\vec{\boldsymbol{R}}) = \int_{\mathcal{V}'} \frac{\mu\vec{\boldsymbol{J}}(\vec{\boldsymbol{R}}')\times(\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}')}{4\pi\|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\|^3} d\mathcal{V}'$
Material properties	$\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}}$
Ampere's law	$\nabla\times\vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}$, $\int_C \vec{\boldsymbol{H}} \cdot d\boldsymbol{l}' = I$
Vector potential function	$\vec{\boldsymbol{B}} = \nabla\times\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{A}}(\vec{\boldsymbol{R}}) = \int_{\mathcal{V}'} \frac{\mu\vec{\boldsymbol{J}}(\vec{\boldsymbol{R}}')}{4\pi\|\vec{\boldsymbol{R}} - \vec{\boldsymbol{R}}'\|} d\mathcal{V}'$
Energy density	$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 d\mathcal{V}$

EM Toolbox

Electrostatics

Used	Eqn
Electric current	$I\ [A]$
Current density	$\vec{J}\left[\frac{A}{m^2}\right]$
Electric field strength	$\vec{\boldsymbol{E}}\left[\frac{V}{m}\right]$
Electrical conductivity and permittivity	$\sigma\left[\frac{m}{W}\right],\ \varepsilon\left[\frac{F}{m}\right]$
Capacitance	$C\ [F]$
Electric scalar potential	$V\ [V]$

Magnetostatics

Used	Eqn
Magnetic flux	$\Phi_m\ [Wb]$
Magnetic flux density	$\vec{\boldsymbol{B}}\left[T\ or\ \frac{Wb}{m^2}\right]$
Magnetic field strength	$\vec{\boldsymbol{H}}\left[\frac{A}{m}\right]$
Magnetic permeability	$\mu\left[\frac{H}{m}\right]$
Inductance	$L\ [H]$
Magnetic vector potential	$A\left[\frac{Wb}{m}\right]$

Constitutive Parameters of Materials

Parameter	Units	Free-Space Value
Electrical permittivity, ε	F/m	$\varepsilon_0 = 8.854\times10^{-12}$ F/m $\simeq 1/36\pi\times10^{-9}$ F/m
Magnetic permeability, μ	H/m	$\mu_0 = 4\pi\times10^{-7}$ H/m
Conductivity, σ	S/m	0

Electric and magnetic fields are connected through the speed of light:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times10^{8}\ \text{m/s}$$

Polarization, $\vec{\boldsymbol{P}}$ and Magnetization, $\vec{\boldsymbol{M}}$,

Electric flux

In free space: $\vec{\boldsymbol{D}} = \varepsilon_{\boldsymbol{o}}\vec{\boldsymbol{E}}$

In a dielectric material: $\vec{\boldsymbol{D}} = \varepsilon_{\boldsymbol{o}}\vec{\boldsymbol{E}} + \vec{\boldsymbol{P}}$

$\vec{\boldsymbol{P}}$ = the electric flux density induced by the applied field $\vec{\boldsymbol{E}}$

$$\vec{\boldsymbol{P}} = \varepsilon_o \chi_e \vec{\boldsymbol{E}}$$

Where χ_e is the electric susceptibility

$$\vec{\boldsymbol{D}} = \varepsilon_{\boldsymbol{o}}\vec{\boldsymbol{E}} + \varepsilon_o \chi_e \vec{\boldsymbol{E}}$$

$$= \varepsilon_o \underbrace{(1 + \chi_e)}_{\varepsilon_r} \vec{\boldsymbol{E}}$$

relative permittivity

$$\vec{\boldsymbol{D}} = \varepsilon \vec{\boldsymbol{E}}$$

($\chi_{e,m}$ is unitless)

Magnetic flux

In free space: $\vec{\boldsymbol{B}} = \mu_{\boldsymbol{o}}\vec{\boldsymbol{H}}$

In a magnetic material: $\vec{\boldsymbol{B}} = \mu_{\boldsymbol{o}}\vec{\boldsymbol{H}} + \mu_{\boldsymbol{o}}\vec{\boldsymbol{M}}$

$\vec{\boldsymbol{M}}$ = vector sum of magnetic dipole moments in medium

$$\vec{\boldsymbol{M}} = \chi_m \vec{\boldsymbol{H}}$$

Where χ_m is the magnetic susceptibility

$$\vec{\boldsymbol{B}} = \mu_o \vec{\boldsymbol{H}} + \mu_o \chi_m \vec{\boldsymbol{H}}$$

$$= \mu_o \underbrace{(1 + \chi_m)}_{\mu_r} \vec{\boldsymbol{H}}$$

relative permeability

$$\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}}$$

General Boundary Conditions

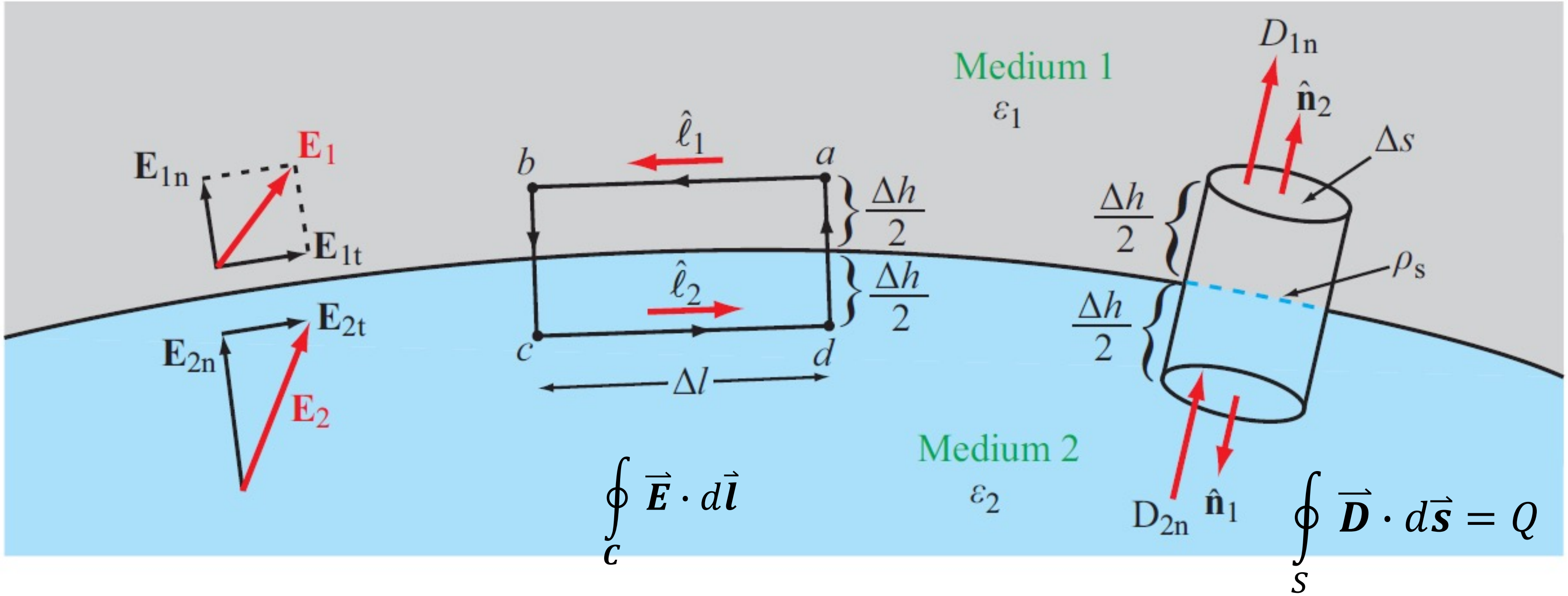


Figure 4-18: Interface between two dielectric media.

Electrostatic boundary conditions

From our general boundary conditions for electrostatic case:

Conservative property of $\vec{\boldsymbol{E}}$ leads to continuous tangential component across a boundary

$$\nabla\times\vec{\boldsymbol{E}} = 0 \qquad \oint_C \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{l}} = 0 \longrightarrow \boxed{\vec{\mathbf{E}}_{1t} = \vec{\mathbf{E}}_{2t}}$$

Divergent property of $\vec{\boldsymbol{D}}$ leads to discontinuous normal component across a boundary

$$\nabla\cdot\vec{\boldsymbol{D}} = \rho_v \qquad \oint_S \vec{\boldsymbol{D}}\cdot d\vec{\boldsymbol{s}} = Q \longrightarrow \boxed{\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s}$$

If $\rho_s = 0, \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

Summary of Boundary Conditions

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Remember $\vec{\mathbf{E}} = 0$ in a good conductor

Electric and Magnetic Forces

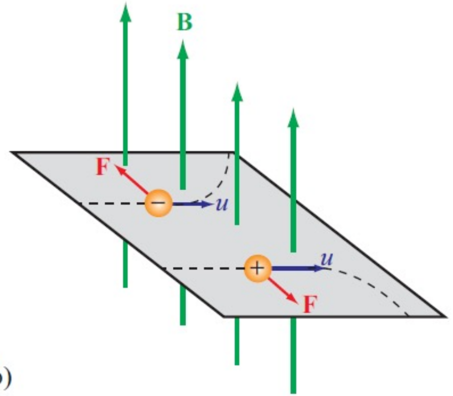
Electromagnetic (Lorentz) force

$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B}$$

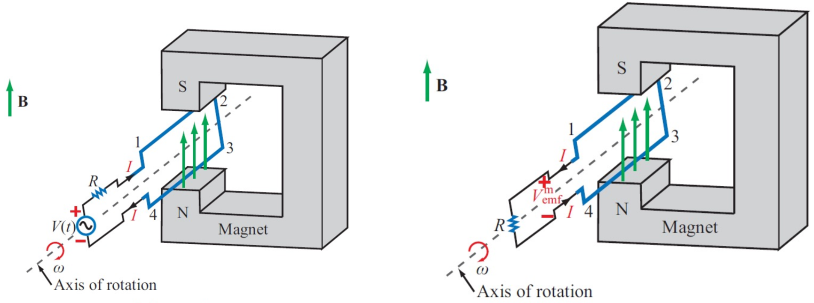
Torque

$$\vec{T} = \vec{m} \times \vec{B}$$

Magnetic moment, m $\quad \vec{m} = \hat{n}NIA$



(b)



(a) ac motor (b) ac generator

Generator: Mechanical to electrical energy conversion

Motor: Electrical to mechanical energy conversion

Ohm's law in multiple forms

- Phasor form

Resistor: $\tilde{\mathbf{V}}_R = R\tilde{\mathbf{I}}_R$ $\qquad \dfrac{\tilde{\mathbf{V}}_R}{\tilde{\mathbf{I}}_R} = R$

Capacitor: $\tilde{\mathbf{V}}_C = \dfrac{1}{j\omega C}\tilde{\mathbf{I}}_C$ $\qquad \dfrac{\tilde{\mathbf{V}}_C}{\tilde{\mathbf{I}}_C} = \dfrac{1}{j\omega C}$

Inductor: $\tilde{\mathbf{V}}_L = j\omega L\tilde{\mathbf{I}}_L$ $\qquad \dfrac{\tilde{\mathbf{V}}_L}{\tilde{\mathbf{I}}_L} = j\omega L$

- Conductivity form

$$\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$$

- Fields form

$$R = \frac{V}{I} = \frac{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}{\int_S \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{s}}} = \frac{-\int_{l'} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}{\int_S \sigma\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}}$$

EM Toolbox (math and simplifications)

E&M vectors related concepts

- Mathematical tools are needed to manipulate vector quantities in different coordinate systems (Cartesian, cylindrical and spherical)
- **Vector Algebra**: addition, subtraction and multiplication(dot and cross) of vectors.
- **Vector Calculus:**
  - gradients: vector pointing in the direction a scalar field is most rapidly increasing with the scalar component showing rate of change. The gradient of a scalar field ∇f gives a vector
  - divergence: calculate the flux per unit volume assuming an infinitesimally small point, The divergence of a vector field $\nabla \cdot \vec{\boldsymbol{v}}$ gives a scalar
  - curl: measure that quantifies the circulation of the field, The curl of a vector field $\nabla \times \vec{\boldsymbol{v}}$ gives a vector

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta\cos\phi + \hat{\mathbf{y}} \sin\theta\sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta\cos\phi + \hat{\mathbf{y}} \cos\theta\sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

[Play around!]()
Interactive Module 3.1

Vector calculus: differential variables

Differential length vector

$$d\vec{\boldsymbol{l}} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$$

Differential area vectors

$$ds_x = \hat{\boldsymbol{x}}dxdy$$

Differential volume vectors

$$dV = dxdydz$$

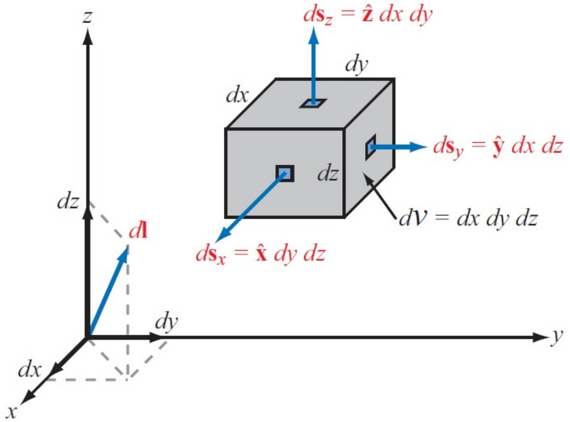


Table 3-1: Summary of vector relations.

Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{R}}\, dR + \hat{\boldsymbol{\theta}} R\, d\theta + \hat{\boldsymbol{\phi}} R \sin\theta\, d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\, dy\, dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy$	$d\mathbf{s}_r = \hat{\mathbf{r}} r\, d\phi\, dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz$ $d\mathbf{s}_z = \hat{\mathbf{z}} r\, dr\, d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}} R \sin\theta\, dR\, d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}} R\, dR\, d\theta$
Differential volume $dV =$	$dx\, dy\, dz$	$r\, dr\, d\phi\, dz$	$R^2 \sin\theta\, dR\, d\theta\, d\phi$

Summary

[Gradient, divergence, curl -- Math @ Libretexts]

Cartesian

(x, y, z): Scalar function F; Vector field $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

- gradient : $\nabla F = \dfrac{\partial F}{\partial x}\mathbf{i} + \dfrac{\partial F}{\partial y}\mathbf{j} + \dfrac{\partial F}{\partial z}\mathbf{k}$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right)\mathbf{k}$
- Laplacian : $\Delta F = \dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} + \dfrac{\partial^2 F}{\partial z^2}$

Cylindrical

(r, θ, z): Scalar function F; Vector field $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$

- gradient : $\nabla F = \dfrac{\partial F}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{\partial F}{\partial z}\mathbf{e}_z$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r f_r) + \dfrac{1}{r}\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{\partial f_z}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{1}{r}\dfrac{\partial f_z}{\partial \theta} - \dfrac{\partial f_\theta}{\partial z}\right)\mathbf{e}_r + \left(\dfrac{\partial f_r}{\partial z} - \dfrac{\partial f_z}{\partial r}\right)\mathbf{e}_\theta + \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(r f_\theta) - \dfrac{\partial f_r}{\partial \theta}\right)\mathbf{e}_z$
- Laplacian : $\Delta F = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{\partial^2 F}{\partial z^2}$

Spherical

(ρ, θ, φ): Scalar function F; Vector field $\mathbf{f} = f_\rho\mathbf{e}_\rho + f_\theta\mathbf{e}_\theta + f_\varphi\mathbf{e}_\varphi$

- gradient : $\nabla F = \dfrac{\partial F}{\partial \rho}\mathbf{e}_\rho + \dfrac{1}{\rho\sin\varphi}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{1}{\rho}\dfrac{\partial F}{\partial \varphi}\mathbf{e}_\varphi$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}(\rho^2 f_\rho) + \dfrac{1}{\rho}\sin\varphi\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{1}{\rho\sin\varphi}\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta)$
- curl : $\nabla \times \mathbf{f} = \dfrac{1}{\rho\sin\varphi}\left(\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta) - \dfrac{\partial f_\varphi}{\partial \theta}\right)\mathbf{e}_\rho + \dfrac{1}{\rho}\left(\dfrac{\partial}{\partial \rho}(\rho f_\varphi) - \dfrac{\partial f_\rho}{\partial \varphi}\right)\mathbf{e}_\theta + \left(\dfrac{1}{\rho\sin\varphi}\dfrac{\partial f_\rho}{\partial \theta} - \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho f_\theta)\right)\mathbf{e}_\varphi$
- Laplacian : $\Delta F = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}\left(\rho^2\dfrac{\partial F}{\partial \rho}\right) + \dfrac{1}{\rho^2\sin^2\varphi}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{1}{\rho^2\sin\varphi}\dfrac{\partial}{\partial \varphi}\left(\sin\varphi\dfrac{\partial F}{\partial \varphi}\right)$

E&M vector transformation related concepts

- Divergence Theorem

$$\int_V \nabla \cdot \vec{\boldsymbol{E}} dV = \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

- Stoke's Theorem

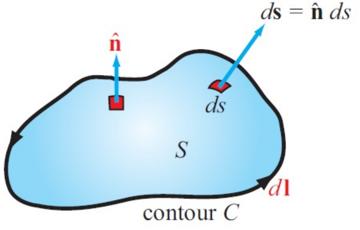


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

$$\int_S \nabla \times \vec{\boldsymbol{B}} \cdot ds = \oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}}$$

Relations for Complex Numbers

Euler's identity:

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Rectangular and polar form:

$$\mathbf{z} = x + jy = |\mathbf{z}|e^{j\theta} \qquad \mathbf{z}^* = x - jy = |\mathbf{z}|e^{-j\theta}$$

$$x = \mathfrak{Re}(\mathbf{z}) = |\mathbf{z}|\cos\theta \qquad |\mathbf{z}| = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$$

$$y = \mathfrak{Im}(\mathbf{z}) = |\mathbf{z}|\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

$$\mathbf{z}^n = |\mathbf{z}|^n e^{jn\theta} \qquad \mathbf{z}^{1/2} = \pm|\mathbf{z}|^{1/2}e^{j\theta/2}$$

Complex algebra:

$$\mathbf{z}_1 = x_1 + jy_1 \qquad \mathbf{z}_2 = x_2 + jy_2$$

$$\mathbf{z}_1 = \mathbf{z}_2 \text{ iff } x_1 = x_2 \text{ and } y_1 = y_2 \qquad \mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$\mathbf{z}_1\mathbf{z}_2 = |\mathbf{z}_1||\mathbf{z}_2|e^{j(\theta_1+\theta_2)} \qquad \frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{|\mathbf{z}_1|}{|\mathbf{z}_2|}e^{j(\theta_1-\theta_2)}$$

Useful relations:

$$-1 = e^{j\pi} = e^{-j\pi} = 1\angle{\pm 180^\circ}$$

$$j = e^{j\pi/2} = 1\angle{90^\circ} \qquad -j = e^{-j\pi/2} = 1\angle{-90^\circ}$$

$$\sqrt{j} = \pm e^{j\pi/4} = \pm\frac{(1+j)}{\sqrt{2}} \qquad \sqrt{-j} = \pm e^{-j\pi/4} = \pm\frac{(1-j)}{\sqrt{2}}$$

Learn how to perform these with your *McMaster Casio fx-991MS* calculator

Time and Phasor Domain

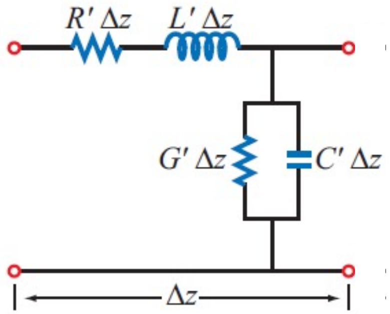
$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}(x(t))$	$\leftrightarrow$	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\leftrightarrow$	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t+\phi)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

Ulaby Table 1-5

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain

We just need to track magnitude & phase, knowing that everything is at frequency ω

Transmission Line Model



- R': The combined ***resistance*** of both conductors per unit length, in Ω/m,
- L': The combined ***inductance*** of both conductors per unit length, in H/m,
- C': The ***capacitance*** of the two conductors per unit length, in F/m.
- G': The ***conductance*** of the insulation medium between the two conductors per unit length, in S/m, and

characteristic impedance, Z_0

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \frac{\sqrt{(R' + j\omega L')}}{\sqrt{(G' + j\omega C')}}$$

Complex propagation constant

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

Transmission Line Parameters

Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

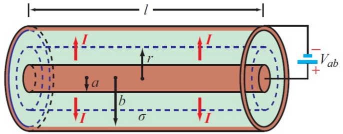
Expressions will be derived in later chapters

Critical to keep in mind: R_s is the surface resistance of the conductors

[Play around!](#)

Interactive Module 2.1-2.2

$$R_S = \sqrt{\pi f \, \mu_C / \sigma_C}$$



Telegraphers and Maxwell's Equations

- Maxwell's equations in free space form a set of coupled, first order, partial differential equations

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\nabla \times \vec{\boldsymbol{B}} = \varepsilon_0 \mu_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

- Similar to the telegrapher's equations that we saw for transmission lines (L9-L12)

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z)$$

$$-\frac{dI(z)}{dz} = (G' + j\omega C')\tilde{V}(z)$$

Wave Equations: separation of variables

Derive the *wave equations* by separating variables, using the second derivative

For the telegrapher's equations, one dimensional (L9)

$$\frac{d^2\tilde{V}}{dz^2} - \gamma^2\tilde{V}(z) = 0$$

$$\frac{d^2\tilde{I}}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

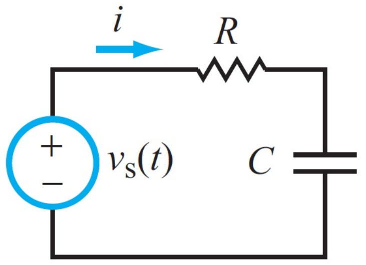
For the Maxwell's equations, use the curl of a curl and the Laplacian

$$\nabla^2\vec{\boldsymbol{E}} = \varepsilon_0\mu_0\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2}$$

$$\nabla^2\vec{\boldsymbol{B}} = \varepsilon_0\mu_0\frac{\partial^2\vec{\boldsymbol{B}}}{\partial t^2}$$

ac Phasor Analysis: General Procedu

Step 1
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12\sin(\omega t - 45°)$ (V)

Step 2
Transfer to Phasor Domain

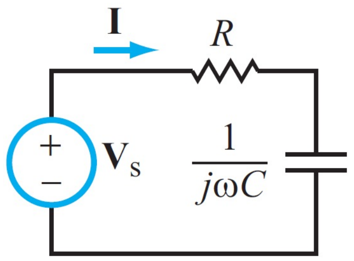
$i \longrightarrow \mathbf{I}$

$v \longrightarrow \mathbf{V}$

$R \longrightarrow \mathbf{Z}_R = R$

$L \longrightarrow \mathbf{Z}_L = j\omega L$

$C \longrightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135°}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

(apply Ohm's and Kirchoff's laws)

Step 4
Solve for Unknown Variable
(Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

Step 5
Transform Solution
Back to Time Domain

$$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}]$$

$$= I_0\cos(\omega t - \phi_i) \text{ (A)}$$