

Mechatronics 3DX4

Slides 8: Root Locus Techniques

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Material based on *Control Systems Engineering* by N. Nise.

Introduction

- ▶ The **root locus** technique shows graphically how the closed-loop poles change as a system parameter is varied.
- ▶ Used to analyze and design systems for stability and transient response.
- ▶ Shows graphically the effect of varying the gain on things like percent overshoot, and settling time.
- ▶ Also shows graphically how stable a system is; shows ranges of stability, instability, and when system will start oscillating.

The Control System Problem

- ▶ The poles of the open-loop transfer function are typically easy to find and do not depend on the gain, K .
- ▶ It is thus easy to determine stability and transient response for an open-loop system.
- ▶ Let $G(s) = \frac{N_G(s)}{D_G(s)}$ and $H(s) = \frac{N_H(s)}{D_H(s)}$.

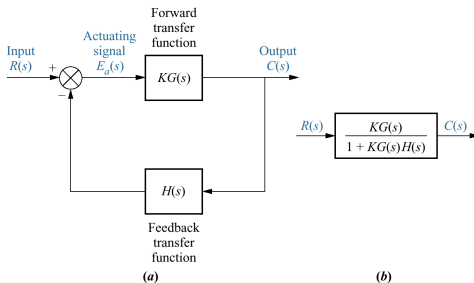


Figure 8.1.

The Control System Problem - II

- Our closed transfer function is thus

$$T(s) = \frac{K \frac{N_G(s)}{D_G(s)}}{1 + K \frac{N_G(s)}{D_G(s)} \frac{N_H(s)}{D_H(s)}} \quad (1)$$

$$= \frac{K N_G(s) D_H(s)}{D_G(s) D_H(s) + K N_G(s) N_H(s)} \quad (2)$$

- We thus see that we have to factor the denominator of $T(s)$ to find the closed-loop poles, and they will be a function of K .

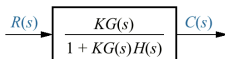


Figure 8.1(b).

The Control System Problem - III

- ▶ For example, if $G(s) = \frac{s+1}{s(s+2)}$ and $H(s) = \frac{s+3}{s+4}$, our closed-loop transfer function is:

$$T(s) = \frac{K(s+1)(s+4)}{s(s+2)(s+4) + K(s+1)(s+3)} \quad (3)$$

$$= \frac{K(s+1)(s+4)}{s^3 + (6+K)s^2 + (8+4K)s + 3K} \quad (4)$$

- ▶ To find the poles, we would have to factor the polynomial for a specific value of K .
- ▶ The root-locus will give us a picture of how the poles will vary with K .

Vector Representation of Complex Numbers

- ▶ Any complex number, $\sigma + j\omega$, can be represented as a vector.
- ▶ It can be represented in polar form with magnitude M , and an angle θ , as $M\angle\theta$.
- ▶ If $F(s)$ is a complex function, setting $s = \sigma + j\omega$ produces a complex number. For $F(s) = (s + a)$, we would get $(\sigma + a) + j\omega$.

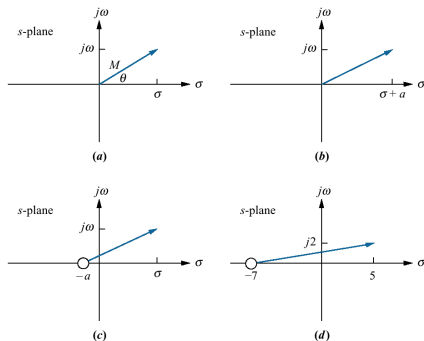


Figure 8.2.

Vector Representation of Complex Numbers - II

- ▶ If we note that function $F(s) = (s + a)$ has a zero at $s = -a$, we can alternately represent $F(\sigma + j\omega)$ as originating at $s = -a$, and terminating at $\sigma + j\omega$.
- ▶ To multiply and divide the polar form complex numbers, $z_1 = M_1\angle\theta_1$ and $z_2 = M_2\angle\theta_2$, we get

$$z_1 z_2 = M_1 M_2 \angle(\theta_1 + \theta_2) \qquad \frac{z_1}{z_2} = \frac{M_1}{M_2} \angle(\theta_1 - \theta_2) \qquad (5)$$

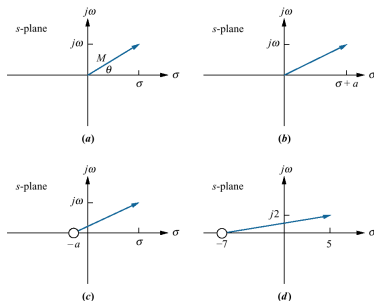


Figure 8.2.

Polar Form and Transfer Functions

- For a transfer function, we have:

$$G(s) = \frac{(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)} = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{i=1}^n (s + p_i)} = M_G \angle \theta_G \quad (6)$$

where

$$M_G = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{i=1}^n |(s + p_i)|} = \frac{\prod_{i=1}^m M_{z_i}}{\prod_{i=1}^n M_{p_i}} \quad (7)$$

and

$$\theta_G = \Sigma \text{zero angles} - \Sigma \text{pole angles} \quad (8)$$

$$= \Sigma_{i=1}^m \angle(s + z_i) - \Sigma_{j=1}^n \angle(s + p_j) \quad (9)$$

Polar Form and Transfer Functions eg.

- Use Equation 6 to evaluate $F(s) = \frac{(s+1)}{s(s+2)}$ at $s = -3 + j4$.

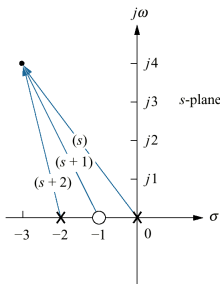


Figure 8.3.

Polar Form and Transfer Functions eg.

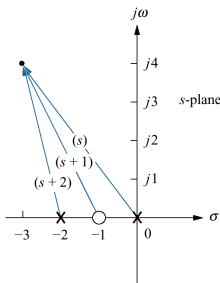


Figure 8.3.

Check answer using matlab:

```
1 s=tf('s')
2 F=(s+1)/(s*(s+2))
3 s1=evalfr(F,-3+j*4)
4 M=abs(s1)
5 theta=angle(s1)
6 theta*180/pi
```

Root Locus Introduction

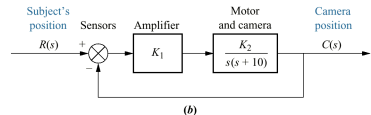
- ▶ System below can automatically track subject wearing infrared sensors.
- ▶ Solving for the poles using the quadratic equation, we can create the table below for different values of K .

Table 8.1.

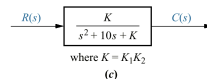
K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



(a)



(b)



(c)

Figure: 8.4

Root Locus Introduction - II

- We can plot the poles from Table 8.1. labelled by their corresponding gain.

Table 8.1.

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

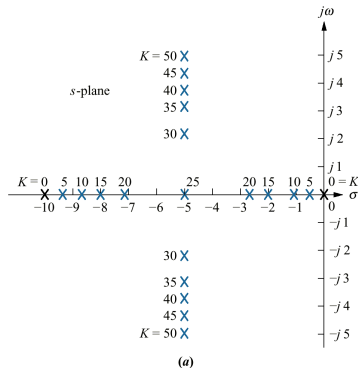


Figure: 8.5

Root Locus Introduction - III

- ▶ We can go a step further, and replace the individual poles with their paths.
- ▶ We refer to this graphical representation of the path of the poles as we vary the gain, as the **root locus**.
- ▶ We will focus our discussion on $K \geq 0$.
- ▶ For pole $\sigma_D + j\omega_D$, $T_s = \frac{4}{\sigma_D}$, $T_p = \frac{\pi}{\omega_D}$, and $\zeta = \frac{|\sigma_D|}{\omega_n}$.

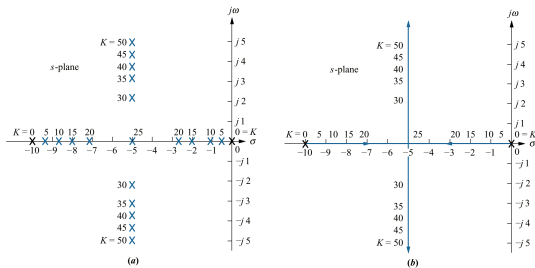


Figure 8.5.

Root Locus Properties

- ▶ For second-order systems, we can easily factor a system and draw the root locus.
- ▶ We do not want to have to factor for higher-order systems (5th, 10th etc.) for multiple values of K !
- ▶ We will develop properties of the root locus that will allow us to rapidly **sketch** the root locus of higher-order systems.
- ▶ Consider the closed-loop transfer function below:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- ▶ A pole of $T(s)$ exists when

$$KG(s)H(s) = -1 = 1\angle(2k + 1)180^\circ \quad k = 0, \pm 1, \pm 2, \dots \quad (10)$$

Root Locus Properties - II

- ▶ Equation 10 is equivalent to

$$|KG(s)H(s)| = 1 \quad (11)$$

and

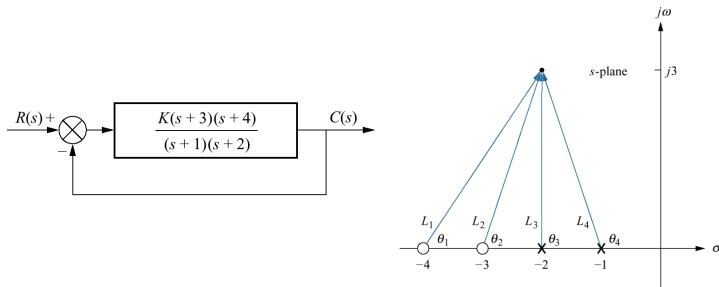
$$\angle KG(s)H(s) = (2k + 1)180^\circ \quad (12)$$

- ▶ Equation 12 says that any s' that makes the angle of $KG(s)H(s)$ be an odd multiple of 180° is a pole for some value of K .
- ▶ Given s' above, the value of K that s' is a pole of $T(s)$ for is found from Equation 11 as follows:

$$K = \frac{1}{|G(s)||H(s)|} \quad (13)$$

Root Locus Properties eg.

- For system below, consider $s = -2 + j3$ and $s = -2 + j(\sqrt{2}/2)$.



Figures 8.6 and 8.7.

Sketching Root Locus

- Now give a set of rules so that we can quickly sketch a root locus, and then we can calculate exactly just those points of particular interest.
 - 1. Number of branches:** a **branch** is the path a single pole traverses. *The number of branches thus equals the number of poles.*
 - 2. Symmetry:** As complex poles occur in conjugate pairs, *a root locus must be symmetric about the real axis.*

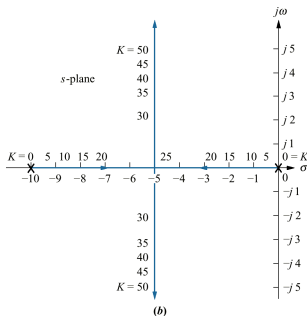


Figure 8.5.

Sketching Root Locus - II

- 3. Real-axis segments:** *For $K > 0$, the root locus only exists on the real axis to the left of an odd number of finite open-loop poles and/or zeros, that are also on the real axis.*

Why? By Equation 12, the angles must add up to an odd multiple of 180.

- ▶ A complex conjugate pair of open-loop zeros or poles will contribute zero to this angle.
- ▶ An open-loop pole or zero on the real axis, but to the left of the respective point, contributes zero to the angle.
- ▶ The number must be odd, so they add to an odd multiple of 180, not an even one.

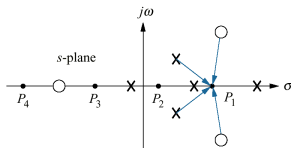


Figure 8.8.

Sketching Root Locus - III

4. **Starting and ending points:** *The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.*

Why? Consider the transfer function below

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

- ▶ The root locus begins at zero gain, thus for small K , our denominator is

$$D_G(s)D_H(s) + \epsilon \tag{14}$$

- ▶ The root locus ends as K approaches infinity, thus our denominator becomes

$$\epsilon + KN_G(s)N_H(s)$$

Infinite Poles and Zeros

- ▶ Consider the open-loop transfer function below

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \quad (15)$$

- ▶ From *point 4*, we would expect our three poles to terminate at three zeros, but there are no finite zeros.
- ▶ A function can have an **infinite zero** if the function approaches zero as s approaches infinity. ie. $G(s) = \frac{1}{s}$.
- ▶ A function can have an **infinite pole** if the function approaches infinity as s approaches infinity. ie. $G(s) = s$.
- ▶ When we include infinite poles and zeros, every function has an equal number of poles and zeros

$$\lim_{s \rightarrow \infty} KG(s)H(s) = \lim_{s \rightarrow \infty} \frac{K}{s(s+1)(s+2)} \approx \frac{K}{s \cdot s \cdot s} \quad (16)$$

How do we locate where these zeros at infinity are so we can terminate our root locus?

Sketching Root Locus - IV

- 5. Behavior at Infinity:** *As the locus approaches infinity, it approaches straight lines as asymptotes.*

The asymptotes intersect the real-axis at σ_a , and depart at angles θ_a , as follows:

$$\sigma_a = \frac{\Sigma \text{finite poles} - \Sigma \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (17)$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (18)$$

where $k = 0, \pm 1, \pm 2, \pm 3$, and the angle is in radians relative to the positive real axis.

Sketching Root Locus eg. 1

- Sketch the root locus for system below.

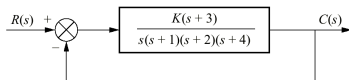


Figure 8.11.

Real-axis Breakaway and Break-in Points

- ▶ Consider root locus below.
- ▶ We want to be able to calculate at what points on the real axis does the locus leave the real-axis (**breakaway point**), and at what point we return to the real-axis (**break-in point**).
- ▶ At breakaway/break-in points, the branches form an angle of $180^\circ/n$ with the real axis where n is number of poles converging on the point.

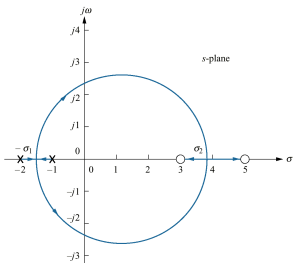


Figure 8.13.

Real-axis Breakaway and Break-in Points - II

- ▶ Breakaway points occur at maximums in the gain for that part of the real-axis.
- ▶ Break-in points occur at minimums in the gain for that part of the real-axis.
- ▶ We can thus determine the breakaway and break-in points by setting $s = \sigma$, and setting the derivative of equation below equal to zero:

$$K = \frac{-1}{G(\sigma)H(\sigma)} \quad (19)$$

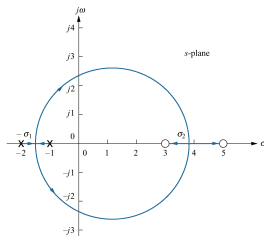


Figure 8.13.

Real-axis Breakaway and Break-in Points - III

An alternative method for computing the real-axis breakaway and break-in points without differentiation is to solve the equation:

$$\sum_{i=1}^m \frac{1}{\sigma + z_i} = \sum_{i=1}^n \frac{1}{\sigma + p_i} \quad (8.37)$$

where z_i and p_i are the negative of the zeros and poles, respectively, of $G(s)H(s)$.

i.e.

$$G(s)H(s) = \frac{K_{GH}(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)}$$

The $j\omega$ -Axis Crossings

- ▶ For systems like the one below, finding the $j\omega$ -axis crossing is important as it is the value of the gain where the system goes from stable to unstable.
- ▶ Can use the Routh-Hurwitz criteria to find crossing:
 1. Force a row of zeros to get gain
 2. Determine polynomial for row above to get ω , the frequency of oscillation.

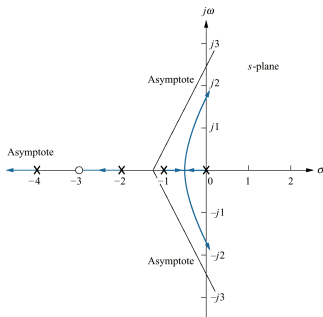
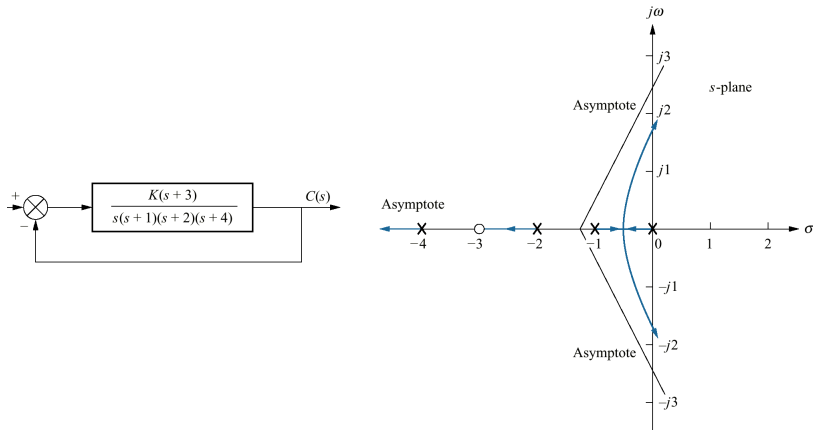


Figure 8.12.

The $j\omega$ -Axis Crossing eg.

- For system below, find the frequency and gain for which the system crosses the $j\omega$ -axis.



Figures 8.11 and 8.12.

Angles of Departure and Arrival

- ▶ We can refine our sketch by determining at which angles we depart from complex poles, and arrive at complex zeros.
- ▶ Net angle from all open-loop poles and zeros to a point on root access must satisfy:

$$\Sigma \text{zero angles} - \Sigma \text{pole angles} = (2k + 1)180^\circ \quad (20)$$

- ▶ To find angle θ_1 , we choose a point ϵ on root locus near complex pole, and assume all angles except θ_1 are to the complex pole instead of ϵ . Can then use Equation 20 to solve for θ_1 .

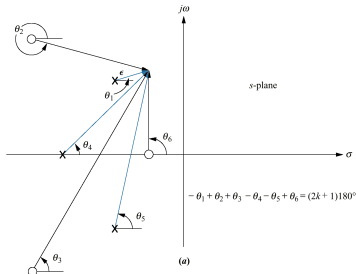


Figure 8.15.

Angles of Departure and Arrival - II

- For example in Figure 8.15a, we can solve for θ_1 in equation below:

$$\theta_2 + \theta_3 + \theta_6 - (\theta_1 + \theta_4 + \theta_5) = (2k + 1)180^\circ \quad (21)$$

- Similar approach can be used to find angle of arrival of complex zero in figure below.
- Simply solve for θ_2 in Equation 21.

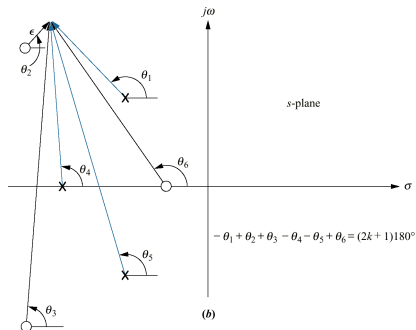
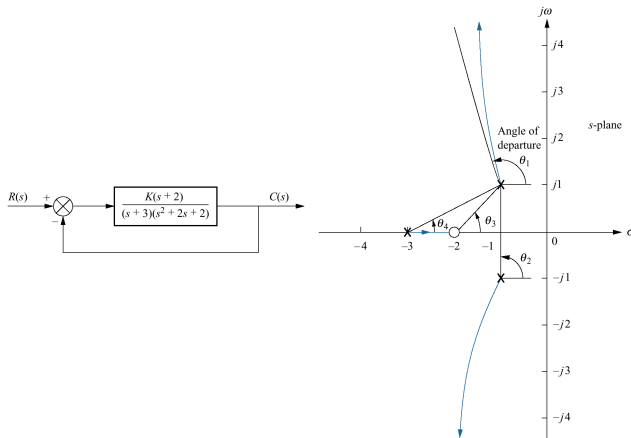


Figure 8.15.

Angles of Departure and Arrival eg.

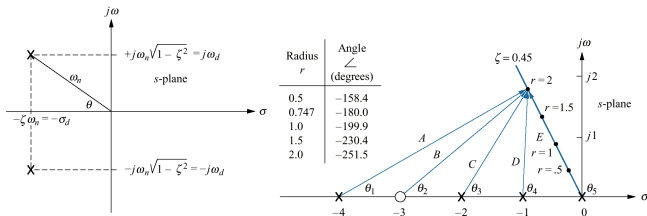
- Find angle of departure for complex poles, and sketch root locus.



Figures 8.16 and 8.17.

Plotting and Calibrating Root Locus

- ▶ Once sketched, we may wish to accurately locate certain points and their associated gain.
- ▶ For example, we may wish to determine the exact point the locus crosses the 0.45 damping ratio line in figure below.
- ▶ From Figure 4.17, we see that $\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\zeta\omega_n}{\omega_n} = \zeta$.
- ▶ We then use computer program to try sample radiuses, calculate the value of s at that point, and then test if point satisfies angle requirement.



Figures 4.17 and 8.18.

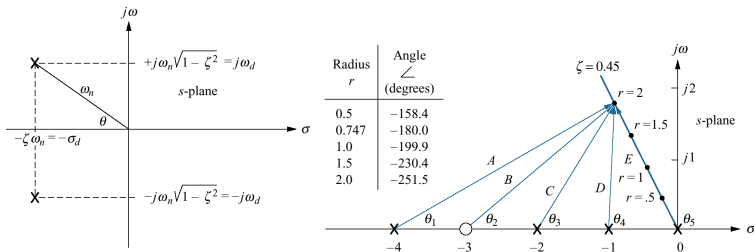
Plotting and Calibrating Root Locus - II

- Once we have found our point we can use the equation below to solve for the required gain, K .

$$K = \frac{1}{|G(s)||H(s)|} = \frac{\prod_{i=1}^m M_{p_i}}{\prod_{i=1}^n M_{z_i}} \quad (22)$$

- Uses labels in Figure 8.18, we would have for our example:

$$K = \frac{ACDE}{B} \quad (23)$$



Figures 4.17 and 8.18.

Transient Response Design via Gain Adjustment

- ▶ We want to be able to apply our transient response parameters and equations for second-order underdamped systems to our root locuses.
- ▶ These are only accurate for second-order systems with no finite zeros, or systems that can be approximated by them.
- ▶ What are the conditions that must be true for a 2nd order approximation to be “close” to the higher order system?

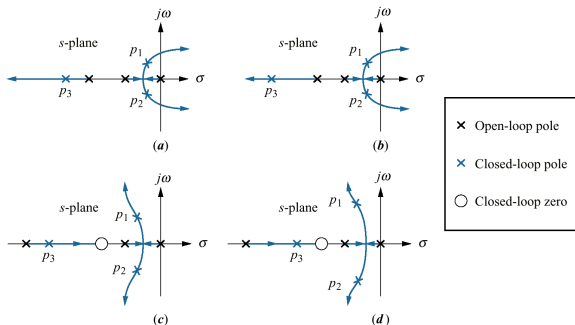
Recall if $G(s) = \frac{N_G(s)}{D_G(s)}$ and $H(s) = \frac{N_H(s)}{D_H(s)}$, closed loop TF is:

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)} \quad (24)$$

Transient Response Design via Gain Adjustment II

Conditions for justifying 2nd order approx of a higher order systems

1. Higher order poles are much farther left (e.g. $> 5\times$) of the s -plane dominant closed loop poles. (Holds for (b), not (a))
2. The closed-loop zeros near the two dominant closed-loop poles must be nearly canceled by higher-order poles near them. (Holds for (d), not (c))
3. Closed-loop zeros not cancelled, must be far away from the two dominant closed-loop poles.



Defining Parameters on Root Locus

- ▶ We have already seen that as $\zeta = \cos \theta$, vectors from the origin are lines of constant damping ratio.
- ▶ As percent overshoot is solely a function of ζ , these lines are also lines of constant %OS.
- ▶ From diagram we can see that the real part of a pole is $\sigma_d = \zeta \omega_n$, and the imaginary part is $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.
- ▶ As $T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d}$, vertical lines have constant values of T_s .

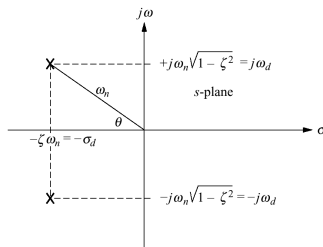


Figure 4.17.

Defining Parameters on Root Locus - II

- ▶ As $T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d}$, horizontal lines thus have constant peak time.
- ▶ We thus choose a line with the desired property, and test to find where it intersects our root locus.

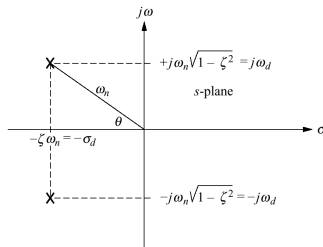


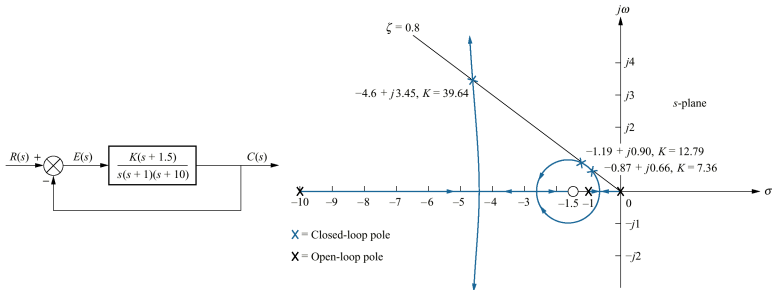
Figure 4.17.

Design Procedure For Higher-order Systems

1. Sketch root locus for system.
2. Assume system has no zeros and is second-order. Find gain that gives desired transient response.
3. Check that systems satisfies criteria to justify our approximation.
4. Simulate system to make sure transient response is acceptable.

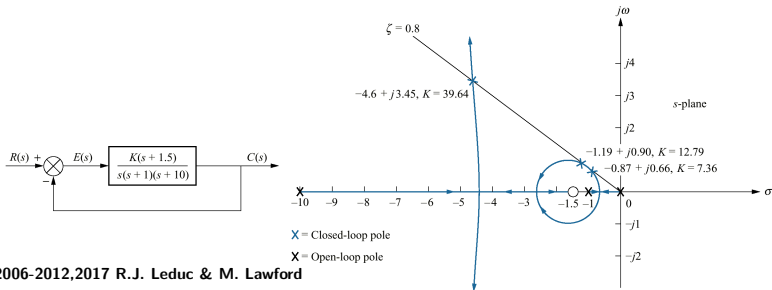
Third-order System Gain Design eg.

- ▶ For system below, design the value of gain, K , that will give 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.
- ▶ First step is to sketch the root locus below.
- ▶ We next assume system can be approximated by second-order system, and solve for ζ using $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$.



Third-order System Gain Design eg. - II

- ▶ This gives $\zeta = 0.8$. Our angle is thus $\theta = \cos^{-1}(0.8) = 36.87^\circ$.
- ▶ We then use root locus to search values along this line to see if they satisfy the angle requirement.
- ▶ The program finds three conjugate pairs on the locus and our $\zeta = 0.8$ line. They are $-0.87 \pm j0.66$, $-1.19 \pm j0.90$, $-4.6 \pm j3.45$ with respective gains of $K = 7.36$, 12.79 , and 39.64 .
- ▶ We will use $T_p = \frac{\pi}{\omega_d}$, and $T_s = \frac{4}{\sigma_d}$.



Third-order System Gain Design eg. - III

- For steady-state error, we have:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{K(s + 1.5)}{s(s + 1)(s + 10)} = \frac{K(1.5)}{(1)(10)} \quad (25)$$

- To test to see if our approximation of a second-order system is valid, we calculate the location of the third pole for each value of K we found.
- The table below shows the results of our calculations.

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9

Table 8.4.

Third-order System Gain Design eg. - IV

- We now simulate to see how good our result is:

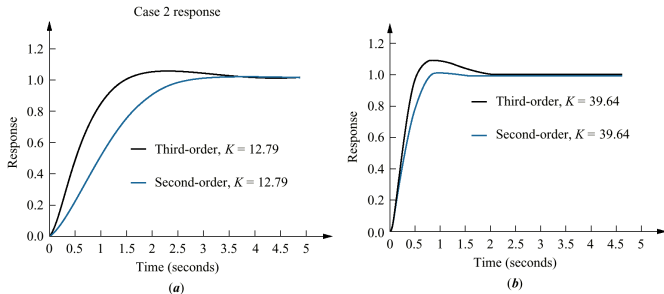


Figure 8.23.