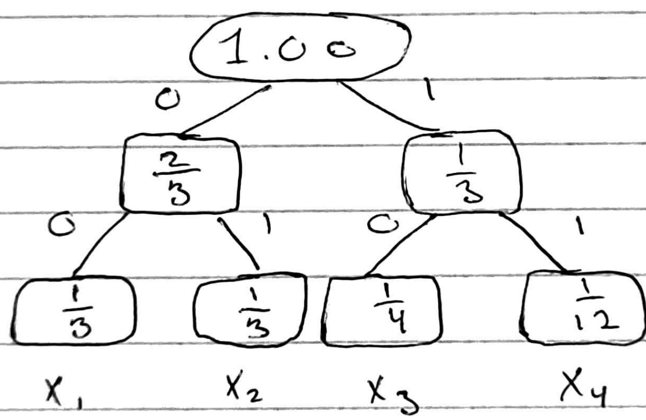
1) $X \sim \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right) = \vec{p}_1$

Codeword	X	Probabilities			
00	x_1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1.00
01	x_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
10	x_3	$\frac{1}{4}$	$\frac{1}{3}$		
11	x_4	$\frac{1}{12}$			



2) Consider the encoding

Codeword	X	Probability
1	x_1	$\frac{1}{3}$
01	x_2	$\frac{1}{3}$
001	x_3	$\frac{1}{4}$
000	x_4	$\frac{1}{12}$

It is prefix free. We wish to show it's optimal. Let L_1 be the expected code length for the Huffman encoding we found/constructed.

$$L_1 = 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{12} = 2$$

It has been shown that this encoding is optimal. Thus, it suffices to show that L_2, where L_2 is the expected code length for this encoding above, is equal to L_1.

Consider,

$$L_2 = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} + 3 \cdot \frac{1}{12} = 2$$

Therefore,

$$L_1 = L_2$$

which shows that L_2 is optimal length and this encoding is optimal. These are clearly two distinct sets of encoding. Therefore, we have shown that there exists two different sets of optimal lengths for the codewords of X. $\square$

3.) As we can see from the encoding with lengths $(1, 2, 3, 3)$, the symbol x_3 has a corresponding codeword length of 3, but $\lceil \log_2(3) \rceil = 2$. But it's an optimal encoding. Therefore, we've shown that there are optimal codes w/ codeword lengths for some symbols that exceed the Shannon code length $\lceil \log \frac{1}{p(x)} \rceil$. $\square$