

MATH 4640 Numerical Analysis - HW 2 Solutions

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February 27, 2025

Problem 1:

$$y_0 = -1 \quad y_4 = 1$$

$$\Delta x_i = 1 \quad \forall i \in \{0, 1, 2, 3, 4\}$$

$$\Delta x_i + \Delta x_{i+1} = 2 \quad \forall i \in \{0, 1, 2, 3\}$$

Note: This is important to understand the simplifications to the general form below

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$M = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}$$

$$D = \begin{bmatrix} f_1 - f_0 - y_0 \\ f_2 - f_1 - (f_1 - f_0) \\ f_3 - f_2 - (f_2 - f_1) \\ f_4 - f_3 - (f_3 - f_2) \\ y_4 - (f_4 - f_3) \end{bmatrix} = \begin{bmatrix} -0.3 + 1 \\ -0.3 + 0.3 \\ -0.2 + 0.3 \\ 0.2 + 0.2 \\ 1 - 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0 \\ 0.1 \\ 0.4 \\ 0.8 \end{bmatrix}$$

We want to find M s.t.

$AM = D$, below is it expressed as a linear system of equations

$$\textcircled{1} \quad \frac{1}{3}M_0 + \frac{1}{6}M_1 = 0.7$$

$$\textcircled{2} \quad \frac{1}{6}M_0 + \frac{2}{3}M_1 + \frac{1}{6}M_2 = 0$$

$$\textcircled{3} \quad \frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = 0.1$$

$$\textcircled{4} \quad \frac{1}{6}M_2 + \frac{2}{3}M_3 + \frac{1}{6}M_4 = 0.4$$

$$\textcircled{5} \quad \frac{1}{6}M_3 + \frac{1}{3}M_4 = 0.8$$

$$AM = \begin{bmatrix} \frac{2M_0 + M_1}{6} \\ \frac{4M_1 + M_0 + M_2}{6} \\ \frac{4M_2 + M_1 + M_3}{6} \\ \frac{4M_3 + M_2 + M_4}{6} \\ \frac{2M_4 + M_3}{6} \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0 \\ 0.1 \\ 0.4 \\ 0.8 \end{bmatrix}$$

$$\frac{2M_4 + M_3}{6} = 0.8 \Rightarrow 2M_4 + M_3 = 4.8$$

$$\Rightarrow M_4 = 2.4 - \frac{1}{2}M_3$$

$$\frac{4M_3 + M_2 + M_4}{6} = 0.4 \Rightarrow 4M_3 + M_2 + M_4 = 2.4$$

$$\Rightarrow M_2 = 2.4 - M_4 - 4M_3 = 2.4 - (2.4 - \frac{1}{2}M_3) - 4M_3 \\ = \frac{1}{2}M_3 - 4M_3 = -\frac{7}{2}M_3 \\ M_2 = -\frac{7}{2}M_3$$

$$\frac{4M_2 + M_1 + M_3}{6} = 0.1 \Rightarrow 4M_2 + M_1 + M_3 = 0.6$$

$$\Rightarrow M_1 = 0.6 - 4M_2 - M_3 = 0.6 - 4(-\frac{7}{2}M_3) - M_3$$

$$= 0.6 + 14M_3 - M_3 = 0.6 + 13M_3 \\ M_1 = 0.6 + 13M_3$$

$$\frac{4M_1 + M_0 + M_2}{6} = 0 \Rightarrow M_0 = -4M_1 - M_2$$

$$= -4(0.6 + 13M_3) - (-\frac{7}{2}M_3) = -2.4 - 52M_3 + \frac{7}{2}M_3$$

$$M_0 = -2.4 - 52M_3 + \frac{7}{2}M_3 = -2.4 - \frac{97}{2}M_3$$

$$\frac{2M_0 + M_1}{6} = 0.7 = \frac{2(-2.4 - \frac{97}{2}M_3) + (0.6 + 13M_3)}{6}$$

$$= \frac{-4.8 - 97M_3 + 0.6 + 13M_3}{6}$$

$$\Rightarrow 4.2 = -4.2 - 84M_3 \Rightarrow 8.4 = -84M_3$$

$$\Rightarrow -0.1 = M_3$$

$M_3 = -0.1$, thus,

$$M_0 = -2.4 - 52(-0.1) + \frac{7}{2}(-0.1) = 2.45$$

$$M_1 = 0.6 + 13(-0.1) = -0.7$$

$$M_2 = -\frac{7}{2}(-0.1) = 0.35$$

$$M_4 = 2.4 - \frac{1}{2}(-0.1) = 2.45$$

$$M = \begin{bmatrix} 2.45 \\ -0.7 \\ 0.35 \\ -0.1 \\ 2.45 \end{bmatrix}$$

$$P_0(x) = \frac{(1-x)^3 M_0 + x^3 M_1}{6} + (1-x)(1.1) + x0.8$$

$$- \frac{1}{6} ((1-x)M_0 + xM_1), \quad x \in [0, 1]$$

$$P_1(x) = \frac{(2-x)^3 M_1 + (x-1)^3 M_2}{6} + (2-x)0.8 + (x-1)0.5$$

$$- \frac{1}{6} ((2-x)M_1 + (x-1)M_2), \quad x \in [1, 2]$$

$$P_2(x) = \frac{(3-x)^3 M_2 + (x-2)^3 M_3}{6} + (3-x)0.5 + (x-2)0.3$$

$$- \frac{1}{6} ((3-x)M_2 + (x-2)M_3), \quad x \in [2, 3]$$

$$P_3(x) = \frac{(4-x)^3 M_3 + (x-3)^3 M_4}{6} + (4-x)0.3 + (x-3)0.5$$

$$- \frac{1}{6} ((4-x)M_3 + (x-3)M_4), \quad x \in [3, 4]$$

$$\text{where } M_0 = 2.45$$

$$M_1 = -0.7$$

$$M_2 = 0.35$$

$$M_3 = -0.1$$

$$M_4 = 2.45$$

Problem 2:

Question 2

$$p_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, x_2, x_3](x - x_0) \\ \cdot (x - x_1)(x - x_2)$$

$$f(x) = p_3(x) + f[x_0, x_1, x_2, x_3, x](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$f(x) - p_3(x) = f[x_0, x_1, x_2, x_3, x] \epsilon_3(x)$$

$$f[x_0, x_1, x_2, x_3, x] = \frac{f^{(4)}(\xi)}{4!} \quad \text{for some } \xi \in [x_0, x_3].$$

Thus,

$$(f(x) - p_3(x))' = f[x_0, x_1, x_2, x_3, x]' \epsilon_3(x) + f[x_0, x_1, x_2, x_3, x] \epsilon_3'(x)$$

by Chain rule. And further,

$$\frac{d}{dx} f[x_0, x_1, x_2, x_3, x] = f[x_0, x_1, x_2, x_3, x, x]$$

$$\text{and } f[x_0, x_1, x_2, x_3, x] = \frac{f^{(4)}(\xi)}{4!}$$

$$f[x_0, x_1, x_2, x_3, x, x] = \frac{f^{(5)}(\eta)}{5!}$$

for some $\xi, \eta \in [x_0, x_3]$.

$$\epsilon_3(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$\epsilon_3'(x) = (x - x_1)(x - x_2)(x - x_3) + (x - x_0)(x - x_2)(x - x_3)$$

$$+ (x - x_0)(x - x_1)(x - x_3) + (x - x_0)(x - x_1)(x - x_2)$$

$$\text{So, } |f(x) - p_3(x)| = \left| \frac{f^{(5)}(\eta)}{5!} \epsilon_3(x) + \frac{f^{(4)}(\xi)}{4!} \epsilon_3'(x) \right|$$

$$\left| \frac{f^{(5)}(\eta)}{5!} \varphi_3(x) + \frac{f^{(4)}(\xi)}{4!} \varphi_3'(x) \right| \\ = \frac{1}{120} \left| f^{(5)}(\eta) \varphi_3(x) \right| + \frac{1}{24} \left| f^{(4)}(\xi) \varphi_3'(x) \right|$$

Now, the nodes are evenly spaced, so

$$\varphi_3(x) = (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$$

$$\text{Then, } \max_{x \in [x_0, x_3]} |\varphi_3(x)| = \left(\frac{3}{2}h\right)\left(\frac{1}{2}h\right)\left(\frac{1}{2}h\right)\left(\frac{3}{2}h\right) \\ = \frac{9}{16}h^4$$

$$\text{And } \varphi_3'(x) = (x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$$

$$+ (x - x_0)(x - (x_0 + 2h))(x - (x_0 + 3h))$$

$$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 3h))$$

$$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))$$

$$\max_{x \in [x_0, x_3]} |\varphi_3'(x)| \leq (h)(2h)(3h) + (h)(h)(2h) + (2h)(h)(h) \\ + (3h)(2h)(h) = 13h^3$$

where we took the $\max_{x \in [x_0, x_3]}$ over each term.

$$|F'(x) - p_3'(x)| \leq \frac{9}{16 \cdot 5!} h^4 \max_{\eta \in [x_0, x_3]} |f^{(5)}(\eta)| + \frac{13}{4!} h^3 \max_{\xi \in [x_0, x_3]} |f^{(4)}(\xi)|$$

$$= \frac{9}{1920} h^4 \max_{\eta \in [x_0, x_3]} |F^{(5)}(\eta)| + \frac{13}{24} h^3 \max_{\xi \in [x_0, x_3]} |F^{(4)}(\xi)|$$

Problem 3:

$$\text{Question 3} \quad p(x) = Ax^2 + Bx + C$$

Solving for A, B, C :

$$p_1(x_0) = Ax_0^2 + Bx_0 + C = a$$

$$p_2(x_1) = Ax_1^2 + Bx_1 + C = b$$

$$p_2'(x_1) = 2Ax_1 + B = c$$

$$\bullet \quad A = \frac{c - B}{2x_1}$$

$$\bullet \quad \frac{c - B}{2x_1} x_1^2 + Bx_1 + C = \frac{cx_1}{2} + \frac{Bx_1}{2} + C = b$$

$$\Rightarrow C = b - \frac{cx_1}{2} - \frac{Bx_1}{2}$$

$$\bullet \quad Ax_0^2 + Bx_0 + C = Ax_0^2 + Bx_0 + b - \frac{cx_1}{2} - \frac{Bx_1}{2} = a$$

$$= \frac{c - B}{2x_1} x_0^2 + Bx_0 + b - \frac{cx_1}{2} - \frac{Bx_1}{2} = a$$

$$\Rightarrow B(x_0 - x_1/2) + \frac{c}{2x_1} x_0^2 - \frac{B}{2x_1} x_0^2 + b - \frac{cx_1}{2}$$

$$= B(x_0 - x_1/2 - x_0^2/2x_1) + \frac{c}{2x_1} x_0^2 - \frac{cx_1}{2} + b$$

$$\Rightarrow B = \frac{a - \frac{c}{2x_1} x_0^2 - \frac{cx_1}{2} - b}{x_0 - x_1/2 - x_0^2/2x_1}$$

$$= \frac{2(a - b) - c(\frac{x_0^2}{x_1} - x_1)}{2x_0 - x_1 - x_0^2/x_1}$$

$$A = \frac{c - \left(\frac{2(a - b) - c(\frac{x_0^2}{x_1} - x_1)}{2x_0 - x_1 - x_0^2/x_1} \right)}{2x_1}$$

$$C = b - \frac{cx_1}{2} - \frac{\left(\frac{2(a - b) - c(\frac{x_0^2}{x_1} - x_1)}{2x_0 - x_1 - x_0^2/x_1} \right)}{2}$$

Problem 4:

Figures 1-3 show the algorithms of DivDif and Interpolate implemented in Rust, using these algorithms on the data from the problem, and the output of the code when executed. The coefficients obtained from the DivDif algorithm are: $f[x_0] = 7, f[x_0, x_1] = -5, f[x_0, x_1, x_2] = 2, f[x_0, x_1, x_2, x_3] = -2/3, f[x_0, x_1, x_2, x_3, x_4] = 0.29167, f[x_0, x_1, x_2, x_3, x_4, x_5] = -0.125$.

In the output is a test of the Interpolate algorithm on values within the interval of $[-2, 3]$, including interpolated points and not interpolated points, and some points outside of the interval.

```

1 // Newton's Divided Difference Algorithm
2 pub fn ndd(x: &[f64], y: &[f64]) -> Vec<f64> {
3     println!("Beginning Newton's Divided Difference algorithm...");
4     let mut d = vec![0.0; n];
5     assert_eq!(n, y.len(), "x and y must have the same length.");
6
7     // Div Diff table
8     let mut d = vec![vec![0.0; n]; n];
9
10    // First column is y
11    for i in 0..n {
12        d[i][0] = y[i];
13    }
14
15    println!("Calculating divided difference table...");
16    for j in 1..n {
17        for i in 0..(n - j) {
18            let numerator: f64 = d[i + 1][i] - d[i][i - 1];
19            let denominator: f64 = x[i + j] - x[i];
20            d[i][j] = numerator / denominator;
21            println!("({},{}, ..., {}) = ({}) / ({}) = {}", i, j, x[i], x[i + 1], ..., x[i + j - 1], numerator, denominator);
22        }
23    }
24    println!("Success! Divided difference table:");
25    for row in 0..n {
26        println!("{}: ", d[row]);
27    }
28
29    d[0].clone()
30}
31
32 // Interpolation given NDD coefficients
33 pub fn interpolate_poly(x: &[f64], t: f64, d: &[f64], n: usize) -> f64 {
34     // println!("Beginning interpolation of polynomial at point {}:", t);
35     let mut p = d[n - 1];
36     for i in 0..(n - 1).rev() {
37         p = d[i] + (t - x[i]) * p;
38     }
39     //println!("Interpolation succeeded.");
40
41     p
42 }
43
44 #[cfg(test)]
45 Normal | rust [0+0] -> /Documents/Academics/Math/Numerical/numerical_analysis_math4640/solutions/rust/hw2/src/math/interpolation.rs
46 numerical@nvim: 1:nvim-

```

Figure 1: Screenshot of code for the Newton's Divided Difference (NDD/DivDif) algorithm (upper) and the Interpolate using the NDD coefficients (lower).

```

2 use crate::math::approx::round_coefficients;
3 use crate::math::interpolation::interpolate_polynomial, ndd;
4
5 #[derive(Debug)]
6 struct DataPoints {
7     x: Vec<f64>,
8     y: Vec<f64>,
9 }
10
11 impl DataPoints {
12     fn new(x: Vec<f64>, y: Vec<f64>) -> Result<Self, &'static str> {
13         if !Self::is_valid(&x, &y) {
14             return Err("x and y must have the same length");
15         }
16         Ok(DataPoints { x, y })
17     }
18
19     fn is_valid(x: &Vec<f64>, y: &Vec<f64>) -> bool {
20         x.len() == y.len()
21     }
22
23     fn solve_pddata(d: &DataPoints) {
24         let d = ndd(d.x, &d.y);
25         let d_r = round_coefficients(&d, 1e-3);
26
27         let n = d.x.len();
28         println!("Newton's Divided Difference Coefficients: {:?}", d_r);
29
30         let t: Vec<f64> = vec![3., 2., -1., 0., 0.5, 1., 1.5, 2., 3., 4.];
31         println!("Interpolating values: {:?}", t);
32         for &t in t.values() {
33             let f_t = interpolate_polynomial(&d.x, &d.r, &t, &n);
34             println!("p({}) = {:?}", t, f_t);
35         }
36     }
37
38     pub fn main() {
39         let x: Vec<f64> = vec![0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5];
40         let y: Vec<f64> = vec![1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0, 3.25, 3.5];
41
42         match DataPoints::new(x, y) {
43             Ok(data) => solve_pddata(data),
44             Err(e) => println!("Error creating DataPoints: {}", e),
45         }
46     }
47 }

```

Figure 2: Screenshot of code using the DivDif and Interpolate algorithms. See Figure 1 for implementation of such algorithms.

Running solution to question 4...
Beginning Newton's Divided Difference algorithm...
Calculating divided difference table...
 $f(x_0, \dots, x_5) = (x - 7) / (-1 - 1) = -1$
 $f(x_2, \dots, x_3) = (x - 1) / (1 - 0) = -1$
 $f(x_3, \dots, x_4) = (2 - 0) / (2 - 1) = 2$
 $f(x_4, \dots, x_5) = (2 - 2) / (3 - 2) = 0$
 $f(x_0, \dots, x_2) = (x - 2) / (0 - 2) = 2$
 $f(x_1, \dots, x_3) = (-1 - 1) / (1 - 1) = 0$
 $f(x_2, \dots, x_4) = (2 - 1) / (2 - 0) = 1.5$
 $f(x_3, \dots, x_5) = (0 - 2) / (1 - 2) = -1$
 $f(x_0, \dots, x_3) = (0 - 2) / (1 - 2) = -0.6666666666666666$
 $f(x_1, \dots, x_4) = (1.5 - 0) / (2 - 1) = 0.5$
 $f(x_2, \dots, x_5) = (-1 - 1.5) / (3 - 0) = -0.8333333333333334$
 $f(x_0, \dots, x_4) = (-0.6666666666666666 - 0.5) / (2 - 2) = -0.2916666666666663$. Can you find the polynomial interpolation?

(5) Find the polynomial interpolation forms respectively. Verify that they forms.

Success! Divided difference table:

[7.0, -5.0, 2.0, -0.6666666666666666, -0.125]
[2.0, -0.6, 0.5, 0.3333333333333337, 0.0]
[0.0, -1.0, 0.0, 0.3333333333333334, 0.6, 0.0]
[2.0, -1.0, 0.0, 0.0, 0.0, 0.0]
[2.0, 0.0, 0.0, 0.0, 0.0, 0.0]

Newton's Divided Difference Coefficients: [7.0, -5.0, 2.0, -0.66667, 0.29167, -0.125].

Interpolating values: [-3.0, -2.0, -1.0, 0.0, 0.5, 1.0, 1.5, 2.0, 3.0, 4.0]:

(7) Assume that $f(x) \in C^k(a, b)$, and $x_0 < \xi < x_1$.
 $f[x_0, \dots, x_k] = \frac{f(x_k) - f(x_0)}{k!}$.

where $\xi \in (a, b)$.

(8) Find a piecewise linear interpolation

Math 4640, Numerical Analysis I
Homework 2

(1) Given $f'(0) = -1$ and $f'(4) = 4$, find the cubic spline interpolation for the data given in the following table.

x	0	1	2	3	4
$f(x)$	1.4	0.8	0.5	0.3	0.2

(2) Let $p_3(x)$ denote the cubic polynomial interpolation $f(x)$ at the evenly spaced points $x_j = x_0 + j\Delta x$, $j = 0, 1, 2, 3$. Assuming $f(x)$ is sufficiently differentiable, bound the error in using $p_3(x)$ as an approximation to $f(x)$, $x_0 \leq x \leq x_3$.

(3) Find a polynomial $p(x)$ of degree ≤ 2 that satisfies

$$p(x_0) = a, \quad p'(x_0) = b, \quad p''(x_0) = c,$$
 where a, b, c are given constants and x_0, x_1 are two different points.

(4) Write your own code to perform the algorithms *Divid* and *Interp* given in the class. Test your code on the following data. Give a polynomial of degree 5 that interpolates the given data.

x	-2	-1	0	1	2	3
$p(x)$	w	y	z	1	0	2

where w, y and z are the last three digits of your student ID number.

(5) For $f(x) = 2/(1+x^2)$, $-5 \leq x \leq 5$, use your code *Divid* and *Interp* to produce a 10-degree polynomial interpolation $p_{10}(x)$ using $n=11$ evenly spaced nodes on $[-5, 5]$. Evaluate $p_{10}(x)$ at a large number of points (you decide how large), and graph the results and the error on $[-5, 5]$. Repeat the problem for $p_4(x)$.

(6) Find values of (a, b, c, d) that make the following function a cubic spline?

$$p(x) = \begin{cases} x^3 + 1 & x \in [-1, 0] \\ ax + bx^2 + cx^3 + dx^4 & x \in [0, 1] \end{cases}$$

(7) Use Taylor expansion to find the error term for central difference scheme

$$\frac{1}{2}f''(a) \approx \frac{f(a+h/2) - f(a-h/2)}{2h}$$

(8) Derive a formula for $f''(a)$ by differentiating $f(x) = p_3(x) + f[x_0, \dots, x_3] \phi_4(x)$ three times. Choose $k = 3$ and set $x_0 = a, x_1 = a - h, x_2 = a + h, x_3 = a + 2h$. Also derive the error term for this formula.

"numerical-nvnum-1-tmx.m"

"cabin" 30:44 27-Feb-2023

Figure 3: Screenshot of output from code ran in Figure 2. You can run this code for yourself to verify. Documentation in README file with instructions if needed.

Problem 5:

Solution shared below in Figures 3, 4, 5, 6. Figure 5 corresponds to the plot for degree 2 and Figure 6 corresponds to plot for degree 4. The algorithms used are shown above in Figure 1 implemented in Rust.

```

Thursday February 27, 09:06
fn evaluate_on_rand(
    a: f64,
    b: f64,
    n: i32,
    random_points: &Vec<f64>,
    p_random_points: &Vec<f64>,
) -> (Vec<f64>, Vec<f64>, Vec<f64>) {
    let mut p_random_points: Vec<f64> = Vec::new();
    let mut error: Vec<f64> = Vec::new();

    let mut rng = rand::rngs::ThreadRng::new();
    let random_points: Vec<f64> = (0..n).map(|_| rng.gen_range(-5.0..=5.0))
        .collect();

    let f_random_points: Vec<f64> = random_points.iter().map(|&r| function_f(r)).collect();
    for (i, &t) in random_points.iter().enumerate() {
        p_random_points.push(interpolate_polynomial(&x_k, &d, &t, &x_k.len()));
        error.push((p_random_points[i] - f_random_points[i]).abs());
    }
}

fn solve_q5(a: f64, b: f64, n: i32) {
    let step = (b - a) / (n + 2) as f64;
    println!("Polynomial will have degree of {}", n);
    let x_k: Vec<f64> = (1..n + 2).map(|i| a + (i as f64) * step).collect();
    println!("Evenly spaced points on [{}], [{}]: {:?}", a, b, x_k);

    let f_x_k: Vec<f64> = x_k.iter().map(|x| function_f(x)).collect();
    println!("Data Table: nx = {:?} nf(x) = {:?}", x_k, f_x_k);

    let d = nodd(&x_k, &f_x_k);
    println!("Newton Divided Difference coefficients for polynomial of degree {}: {:?}", n, d);
}

166 let (random_points, f_random_points, p_random_points, error): (
    Vec<f64>,
    Vec<f64>,
    Vec<f64>,
    Vec<f64>,
) = evaluate_on_rand(&x_k, &d, 2000);

match plot_interpolation_results(
    random_points,
    &f_random_points,
    &p_random_points,
    &error,
    &x_k,
    &f_x_k,
) {
    Ok(_) => println!("Plot saved as 'interpolation_plot_{}.png", n),
    Err(e) => println!("Failed to create plot: {}", e),
}
}

pub fn main() {
    solve_q5(5., 5., 2);
    solve_q5(-5., 5., 4);
}

```

Figure 4: Screenshot of code using DivDif and Interpolate algorithms. See Figure 1 for implementations of such algorithms.

```

Thursday February 27, 09:05
fn plot_interpolation_results(
    r: &[f64],
    f: &[f64],
    p_r: &[f64],
    error: &[f64],
    x_k: &[f64],
    f_x_k: &[f64],
    degree: i32,
) -> Result<(), Box> {
    let filename: String = format!("interpolation_plot_degree{}.png", degree);
    let root = BldMapBackend::new(&filename, (1200, 800)).into_drawing_area();
    root.fill(WHITE)?;

    let upper, lower = root.split_vertically(600);

    let y_max = f_r
        .iter()
        .chain(r.iter())
        .fold(f64::NEG_INFINITY, |a, &b| a.max(b));
    let y_min = f_r
        .iter()
        .chain(r.iter())
        .fold(f64::POSITIVE_INFINITY, |a, &b| a.min(b));

    let mut main_chart = ChartBuilder::on(&upper)
        .caption(
            "Function vs. Interpolated Polynomial",
            ("sans-serif", 30).into_font(),
        )
        .margin(5)
        .x_label_area_size(30)
        .y_label_area_size(30)
        .x_labels(&x_k)
        .y_labels(&f_x_k)
        .x_labels_formatter(&x_k | format!("{}"), <)
        .y_labels_formatter(&f_x_k | format!("{}"), >)
        .draw()?;
    41 main_chart
        .configure_mesh()
        .x_desc("x")
        .y_desc("y")
        .x_labels(11)
        .y_labels(21)
        .x_labels_formatter(&x_k | format!("{}"), <)
        .y_labels_formatter(&f_x_k | format!("{}"), >)
        .draw();
}

main_chart
    .draw_series(
        x_k.iter()
            .zip(r.iter())
            .map(|(x, y)| Circle::new((x, y), 2, RED.filled())),
        &f_x_k,
        &label("Interpolation points"),
        &legend((x, y) | Circle::new((x, y), 5, RED.filled())));
    11
    12 main_chart
        .draw_series(
            x_k.iter()
                .zip(r.iter())
                .map(|(x, y)| Circle::new((x, y), 2, BLUE.mix(0.3))),
            &f_x_k,
            &label("f(x)"),
            &legend((x, y) | Circle::new((x, y), 5, BLUE.filled())));
    13
    14 main_chart
        .draw_series(
            r.iter()
                .zip(p_r.iter())
                .map(|(x, y)| Circle::new((x, y), 2, GREEN.mix(0.3))),
            &f_x_k,
            &label("p(x) (interpolated polynomial)"),
            &legend((x, y) | Circle::new((x, y), 5, GREEN.filled())));
    15
    16 main_chart
        .configure_series_labels()
        .background_style(WHITE.mix(0.8))
        .border_style(BLACK)
        .position(SeriesLabelPosition::UpperRight)
        .draw();
}

let error_max: f64 = error.r.iter().fold(0.0, |a, &b| a.max(b));
let mut error_chart = ChartBuilder::on(&lower)
    .caption("Error", ("sans-serif", 30).into_font())
    .margin(5)
    .x_label_area_size(30)
    .y_label_area_size(30)
    .build_cartesian(20..=f64..5f64, 0f64..(error_max + error_max * 0.1))?;

error_chart
    .configure_mesh()
    .x_desc("x")
    .y_desc("Error")
    .draw();
}

error_chart
    .draw_series(
        r.iter()
            .zip(error.r.iter())
            .map(|(x, e)| Circle::new((x, e), 2, RED.mix(0.5).filled())),
        &f_x_k,
        &label("Absolute Error"),
        &legend((x, y) | Circle::new((x, y), 2, RED.mix(0.5).filled())));
    31
    32 error_chart
        .configure_series_labels()
        .background_style(WHITE.mix(0.8))
        .border_style(BLACK)
        .position(SeriesLabelPosition::LowerRight)
        .draw();
}

error_chart
    .draw_series(
        r.iter()
            .zip(error.r.iter())
            .map(|(x, e)| Circle::new((x, e), 2, GREEN.mix(0.5).filled())),
        &f_x_k,
        &label("Relative Error"),
        &legend((x, y) | Circle::new((x, y), 2, GREEN.mix(0.5).filled())));
    37
    38 Ok(())
}

```

Figure 5: Screenshot of code plotting the true function, the interpolated polynomial, the interpolated points, and the error between the interpolated polynomial and true function.

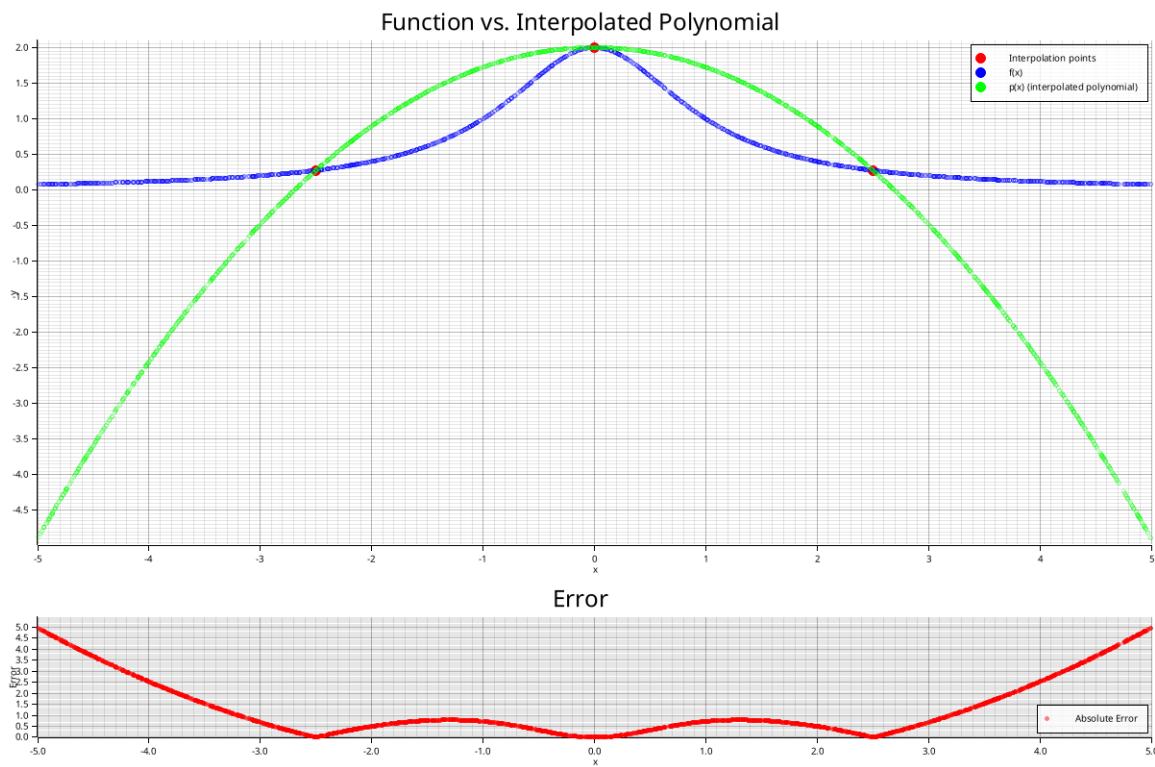


Figure 6: Plot for degree 2 polynomial and error.

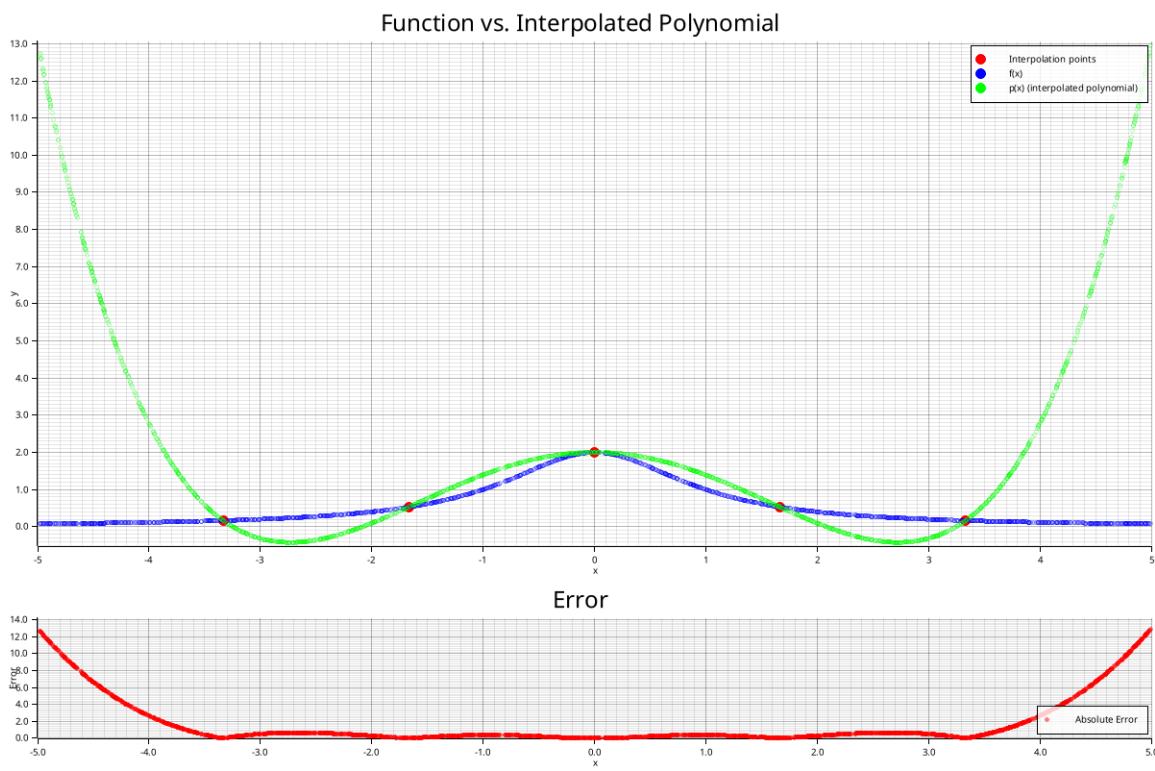


Figure 7: Plot for degree 4 polynomial and error.

Problem 6:

Question 6

For continuity, cubic splines require, in this case, that
 $p(0^+) = p(0^-)$ so

$$a = 1, \text{ since } p(x^-) = x^3 + 1$$

$$p(x^+) = a + bx + cx^2 + dx^3$$

$$\text{Now } p(x) = 1 + bx + cx^2 + dx^3, x \in [0, 1]$$

Additionally, the derivative must be continuous. Thus,

$$p'(0^+) = p'(0^-)$$

$$p'(x^-) = 3x^2$$

$$p'(x^+) = b + 2cx + 3dx^2$$

$$\text{Thus, } b = 0$$

$$\text{And further, } p''(0^+) = p''(0^-)$$

$$p''(x^-) = 6x$$

$$p''(x^+) = 2c + 6dx$$

$$\text{Thus, } c = 0.$$

$$\text{Hence } p(x) = 1 + dx^3 \quad x \in [0, 1].$$

Therefore, the values (a, b, c, d) are $(1, 0, 0, d)$
 where $d \in \mathbb{R}$.



Problem 7:

Question 7

$$\begin{aligned}
 f(a + \frac{h}{2}) &= f(a) + f'(a)(a + \frac{h}{2} - a) \\
 &\quad + \frac{f^{(2)}(a)}{2!} (a + \frac{h}{2} - a)^2 + \dots \\
 &= f(a) + f'(a) \frac{h}{2} + \frac{f^{(2)}(a)}{2!} \frac{h^2}{2^2} + \frac{f^{(3)}(a)}{3!} \frac{h^3}{2^3} + \dots \\
 &= f(a) + \frac{h}{2} f'(a) + \left(\frac{h}{2}\right)^2 \frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3 \frac{f^{(3)}(a)}{3!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 f(a - \frac{h}{2}) &= f(a) + f'(a)(a - \frac{h}{2} - a) + \frac{f^{(2)}(a)}{2!} (a - \frac{h}{2} - a)^2 + \dots \\
 &= f(a) - \frac{h}{2} f'(a) + \left(\frac{h}{2}\right)^2 \frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3 \frac{f^{(3)}(a)}{3!} + \dots
 \end{aligned}$$

So,

$$\begin{aligned}
 f(a + \frac{h}{2}) &= f(a) + \frac{h}{2} f'(a) + \left(\frac{h}{2}\right)^2 \frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3 \frac{f^{(3)}(a)}{3!} + O(h^4) \\
 f(a - \frac{h}{2}) &= f(a) - \frac{h}{2} f'(a) + \left(\frac{h}{2}\right)^2 \frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3 \frac{f^{(3)}(a)}{3!} + O(h^4) \\
 f(a + \frac{h}{2}) - f(a - \frac{h}{2}) &= h f'(a) + \frac{h^3}{3!} \frac{f^{(3)}(a)}{3!} + O(h^5) \\
 &= h f'(a) + \frac{h^3}{24} f^{(3)}(a) + O(h^5)
 \end{aligned}$$

$$\text{Thus, } \frac{f(a + \frac{h}{2}) - f(a - \frac{h}{2})}{2h} = \frac{1}{2} f'(a) + \frac{h^2}{48} f^{(3)}(a) + O(h^4)$$

That is,

$$\frac{f(a+h/2) - f(a-h/2)}{2h} = \frac{1}{2} f'(a) + \frac{h^2}{48} f''(a) + O(h^4)$$

Hence, the error between $\frac{1}{2} f'(a)$ and the approximation $\frac{1}{2h} (f(a+h/2) - f(a-h/2))$ is

$$\left| \frac{1}{2} f'(a) - \frac{f(a+h/2) - f(a-h/2)}{2h} \right|$$

$$= \left| \frac{h^2}{48} f''(a) + O(h^4) \right| = \frac{h^2}{48} |f''(a)| + O(h^4)$$

The dominant term is h^2 , thus, the order of the error is

$$O(h^2).$$

Problem 8:

Question 8

Recall $\frac{d}{dx} f[x_0, \dots, x_k, x] = f[x_0, \dots, x_k, x, x]$

$$f(x) = p_k(x) + f[x_0, x_1, \dots, x_k, x] \mathcal{C}_k(x)$$

$$[f[x_0, \dots, x_k, x] \mathcal{C}_k(x)]' = f[x_0, \dots, x_k, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x] \mathcal{C}'_k(x)$$

which we've shown before.

$$[f[x_0, \dots, x_k, x] \mathcal{C}_k(x)]^{(2)}$$

$$= [f[x_0, \dots, x_k, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x] \mathcal{C}'_k(x)]'$$

$$= f[x_0, \dots, x_k, x, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x, x] \mathcal{C}'_k(x)$$

$$+ f[x_0, \dots, x_k, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x] \mathcal{C}''_k(x)$$

$$[f[x_0, \dots, x_k, x] \mathcal{C}_k(x)]^{(3)}$$

$$= [f[x_0, \dots, x_k, x, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x, x] \mathcal{C}'_k(x)]'$$

$$+ f[x_0, \dots, x_k, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x] \mathcal{C}''_k(x)$$

$$= f[x_0, \dots, x_k, x, x, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x, x, x] \mathcal{C}'_k(x)$$

$$+ f[x_0, \dots, x_k, x, x, x] \mathcal{C}'_k(x) + f[x_0, \dots, x_k, x, x] \mathcal{C}''_k(x)$$

$$+ f[x_0, \dots, x_k, x, x, x] \mathcal{C}_k(x) + f[x_0, \dots, x_k, x, x] \mathcal{C}'_k(x)$$

$$+ f[x_0, \dots, x_k, x, x, x] \mathcal{C}''_k(x) + f[x_0, \dots, x_k, x] \mathcal{C}'''_k(x)$$

$$\begin{aligned}
&= f[x_0, \dots, x_k, x, x, x] \mathcal{C}_k(x) \\
&+ f[x_0, \dots, x_k, x, x, x] (\mathcal{C}_k(x) + 2\mathcal{C}'_k(x)) \\
&+ f[x_0, \dots, x_k, x, x] (\mathcal{C}'_k(x) + 2\mathcal{C}''_k(x)) \\
&+ f[x_0, \dots, x_k, x] \mathcal{C}_k^{(3)}(x) \\
&= \frac{f^{(k+4)}(\xi)}{(k+4)!} \mathcal{C}_k(x) + \frac{f^{(k+3)}(\eta)}{(k+3)!} (\mathcal{C}_k(x) + 2\mathcal{C}'_k(x)) \\
&+ \frac{f^{(k+2)}(\gamma)}{(k+2)!} (\mathcal{C}'_k(x) + 2\mathcal{C}''_k(x)) \\
&+ \frac{f^{(k+1)}(\rho)}{(k+1)!} \mathcal{C}^{(3)}(x) = \psi(x)
\end{aligned}$$

for some, $\xi, \eta, \gamma, \rho \in [\min\{x_0, \dots, x_k, x\}, \max\{x_0, \dots, x_k, x\}]$

So,

$f^{(3)}(x) = \beta_k^{(3)}(x) + \psi(x)$, where $\psi(x)$ is
defined as above. \square