Question 2

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + \dots + f[x_0,x_1,x_2,x_3](x-x_0) \cdot (x-x_1)(x-x_2)$$

$$f(x) = p_3(x) + f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$f(x) - p_3(x) = f[x_0,x_1,x_2,x_3,x]\,\psi_3(x)$$

$$f[x_0,x_1,x_2,x_3,x] = \frac{f^{(4)}(\xi)}{4!} \quad \text{for some } \xi \in [x_0,x_3].$$

Thus,

$$\left(f(x) - p_3(x)\right)' = f[x_0,x_1,x_2,x_3,x]'\,\psi_3(x) + f[x_0,x_1,x_2,x_3,x]\,\psi_3'(x)$$

by Chain rule. And further,

$$\frac{d}{dx} f[x_0,x_1,x_2,x_3,x] = f[x_0,x_1,x_2,x_3,x,x]$$

and $$f[x_0,x_1,x_2,x_3,x] = \frac{f^{(4)}(\xi)}{4!}$$

$$f[x_0,x_1,x_2,x_3,x,x] = \frac{f^{(5)}(\eta)}{5!}$$

for some $\xi, \eta \in [x_0, x_3]$.

$$\psi_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$\psi_3'(x) = (x-x_1)(x-x_2)(x-x_3) + (x-x_0)(x-x_2)(x-x_3) + (x-x_0)(x-x_1)(x-x_3) + (x-x_0)(x-x_1)(x-x_2)$$

So, $$\left|f'(x) - p_3'(x)\right| = \left|\frac{f^{(5)}(\eta)}{5!}\psi_3(x) + \frac{f^{(4)}(\xi)}{4!}\psi_3'(x)\right|$$

$$\left| \frac{f^{(5)}(\eta)}{5!} \ell_3(x) + \frac{f^{(4)}(\xi)}{4!} \ell_3'(x) \right|$$

$$= \frac{1}{120} \left| f^{(5)}(\eta) \ell_3(x) \right| + \frac{1}{24} \left| f^{(4)}(\xi) \ell_3'(x) \right|$$

Now, the nodes are evenly spaced, so

$$\ell_3(x) = (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$$

Then, $\max\limits_{x \in [x_0, x_3]} |\ell_3(x)| = (\frac{3}{2}h)(\frac{1}{2}h)(\frac{1}{2}h)(\frac{3}{2}h)$

$$= \frac{9}{16} h^4$$

And $\ell_3'(x) = (x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$

$+ (x - x_0)(x - (x_0 + 2h))(x - (x_0 + 3h))$

$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 3h))$

$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))$

$\max\limits_{x \in [x_0, x_3]} |\ell_3'(x)| \le (h)(2h)(3h) + (h)(h)(2h) + (2h)(h)(h)$
$+ (3h)(2h)(h) = 13h^3$

Where we took the $\max\limits_{x \in [x_0, x_3]}$ over each term.

$$|f'(x) - p_3'(x)| \le \frac{9}{16 \cdot 5!} h^4 \max_{\eta \in [x_0, x_3]} |f^{(5)}(\eta)| + \frac{13}{4!} h^3 \max_{\xi \in [x_0, x_3]} |f^{(4)}(\xi)|$$

$$= \frac{9}{1920} h^4 \max_{\eta \in [x_0, x_3]} |f^{(5)}(\eta)| + \frac{13}{24} h^3 \max_{\xi \in [x_0, x_3]} |f^{(4)}(\xi)|$$