Q1 $\quad p(x) = x^3 - 2x - 3$

$p(0) = -3$

$p(1) = -4$

$p(2) = 1$

There must be a root in $[0,2]$ since $p(0)p(2) < 0$.

p is continuous on $[0,2]$.

$a = 0, \quad b = 2$

$a_0 = a, \quad b_0 = b$

$$m_0 = \frac{a_0 + b_0}{2} = \frac{0+2}{2} = 1$$

$p(m_0) = -4 \qquad p(a_0)p(m_0) > 0, \quad p(m_0)p(b_0) < 0$

$a_1 = m_0, \quad b_1 = b_0$

$$m_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = \frac{3}{2}$$

$$p(m_1) = \left(\frac{3}{2}\right)^3 - 2\left(\frac{3}{2}\right) - 3 = \frac{27}{8} - 6 = \frac{27-48}{8} = -\frac{21}{8}$$

$p(a_1)p(m_1) > 0, \quad p(m_1)p(b_1) < 0$

$$a_2 = m_1, \quad b_2 = b_1, \quad m_2 = \frac{\frac{3}{2}+2}{2} = \frac{7}{4}$$

$$p(m_2) = \left(\frac{7}{4}\right)^3 - 2\,{}^{7}\!/_{4} - 3 = \frac{343}{64} - \frac{26}{4} = -{}^{73}\!/_{64}$$

$p(a_2)\,p(m_2) > 0 \quad , \quad p(m_2)\,p(b_2) < 0$

$a_3 = m_2 \,,\; b_3 = b_2 \qquad m_3 = \dfrac{\frac{7}{4} + 2}{2} = \dfrac{15}{8}$

$p(m_3) = \left(\dfrac{15}{8}\right)^3 - 2\left(\dfrac{15}{8}\right) - 3 = \dfrac{3375}{512} - \dfrac{30}{8} - 3$

$= \dfrac{3375}{512} - \dfrac{54}{8} = \dfrac{3375 - 3456}{512} = -\dfrac{81}{512}$

$p(a_3)\,p(m_3) > 0$ so,

$a_4 = m_3 \,,\; b_4 = b_3 \qquad m_4 = \dfrac{\frac{15}{8} + 2}{2} = \dfrac{31}{16}$

$p(m_4) = \left(\dfrac{31}{16}\right)^3 - 2\left(\dfrac{31}{16}\right) - 3 = \dfrac{29791}{4096} - \dfrac{62}{16} - \dfrac{48}{16}$

$= \dfrac{29791}{4096} - \dfrac{110}{16} = \dfrac{29791 - 28160}{4096} = \dfrac{1631}{4096}$

$p(a_4)\,p(m_4) < 0$, so

$a_5 = a_4 \,,\; b_5 = m_4 \qquad m_5 = \dfrac{\frac{15}{8} + \frac{31}{16}}{2} = \dfrac{\frac{61}{16}}{2}$

$m_5 = \dfrac{61}{32}$

$p(m_5) = \left(\dfrac{61}{32}\right)^3 - 2\left(\dfrac{61}{32}\right) - 3 = \dfrac{226981}{32768} - \dfrac{122}{32} - \dfrac{96}{32}$

$= \dfrac{226981}{32768} - \dfrac{218}{32} = \dfrac{226981 - 223232}{32768}$

$= \dfrac{3749}{32768}$

$p(a_5)p(m_5) < 0$

$a_6 = a_5, \quad b_6 = m_5$

$$m_6 = \frac{\frac{15}{8} + \frac{61}{32}}{2} = \frac{121}{64}$$

Exact $x^* = 1.8932891963045\,0$

m_6 is within 3 significant figures of x^*.

Answer $\hat{x}^* = \frac{121}{64} \approx x^*$ (w/ 3 sig figs)

$\left[\frac{121}{64} = 1.890625 \right]$

Analysis

A tolerance less or equal to $1/16$ would result in us being at least within 3 significant digits.

Q2

(a) $f(x) = e^x - 3x^2$ $\qquad$ $\frac{d^2}{dx^2} f(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(e^x - 6x) = e^x - 6$

So, $f \in C^2$.

Define: $\quad e_n = x_n - x^*$

We know that since $f \in C^2$, for some neighborhood of x^*, if we start Newton's x_0 in that neighborhood, the convergence is quadratic.

(b) $f(x) = x - 1 - 0.2\sin(x)$
$f'(x) = 1 - 0.2\cos(x)$
$f''(x) = 0.2\sin(x)$
So, $f \in C^2$.

Thus, Newton's method has quadratic convergence if started w/in some neighborhood of x^*.

(c) $G(x) = x^4 + 2x + 1$
Define $\quad f(x) = \nabla G(x) = 4x^3 + 2x$
To minimize G, we find roots of f using Newton's Method
$f'(x) = 12x^2 + 2$
$f''(x) = 24x$

So $f \in C^2$. Thus, Newton's method has quadratic convergence if started w/in some neighborhood of x^*.

Q3 $x_{n+1} = 2 - (1+c)x_n + cx_n^3$, $\alpha = 1$ is a fixed point.
$F(\alpha) = \alpha$

$x_{n+1} = F(x_n)$
F is differentiable and need $|F'(\alpha)| < 1$ to ensure local convergence.

Now, $F(x) = 2 - (1+c)x + cx^3$

$F'(x) = -1 - c + 3cx^2$
Take the Taylor expansion of F about α:
$x_{n+1} = F(x_n) = \alpha + F'(\alpha)e_n + \frac{1}{2}F''(\alpha)e_n^2 + \ldots$
meaning that...
$e_{n+1} = x_{n+1} - \alpha = F'(\alpha)e_n + \frac{1}{2}F''(\alpha)e_n^2 + \ldots$
Now, consider:
$F'(\alpha) = F'(1) = -1 - c + 3c = -1 + 2c$

We need $|F'(\alpha)| < 1 \Rightarrow -1 < F'(\alpha) < 1$

$\Leftrightarrow -1 < -1 + 2c < 1 \Rightarrow 0 < 2c < 2$
$\Rightarrow 0 < c < 1$

$\boxed{\text{So } c \in (0,1)}$

But as we know this is for linear convergence.
To know what c grants quadratic convergence, we need the value of c such that $F'(\alpha) = 0$

That is, $F'(1) = -1 + 2c = 0 \Rightarrow c = 1/2$

Answer: $c = 1/2$ will grant quadratic convergence. Why?
$e_{n+1} = \frac{1}{2}F''(\alpha)e_n^2 + \ldots$ in this case, since $F'(\alpha) = 0$.

Q4

(a) $x_{n+1} = -16 + 6x_n + 12/x_n \quad , \quad \alpha = 2$

$x_{n+1} = F(x_n)$, $F(2) = -16 + 12 + 6 = 2$

$F'(x) = 6 - 12/x^2$ so $F \in C'$ near α.

$F'(\alpha) = 6 - 12/4 = 6 - 3 = 3$

$|F'(\alpha)| > 1 \Rightarrow$ does not converge.

(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2} \quad , \quad \alpha = 3^{1/3}$

$x_{n+1} = F(x_n)$, $F(3^{1/3}) = \frac{2}{3} 3^{1/3} + \frac{1}{3^{2/3}} =$

$F(3^{1/3})\, 3^{2/3} = \frac{2}{3}(3) + \frac{3^{2/3}}{3^{2/3}} = 2 + 1 = 3$

$F(3^{1/3})\, 3^{2/3} = 3 \Rightarrow F(3^{1/3}) = 3^{1/3}$

$F'(x) = \frac{2}{3} - \frac{2}{x^3}$ so $F \in C'$ near α.

$F'(\alpha) = \frac{2}{3} - \frac{2}{(3^{1/3})^3} = 0$

$|F'(\alpha)| < 1$, does converge.

$F''(x) = 6/x^4 \quad , \quad F''(\alpha) = \frac{6}{3^{4/3}} > 0$

Since $F'(\alpha) = 0$, $F''(\alpha) \neq 0$ this iteration has quadratic convergence.

(c) $x_{n+1} = \frac{12}{1+x_n}$, $\alpha = 3$

$x_{n+1} = F(x_n)$ $\qquad F(3) = \frac{12}{1+3} = \frac{12}{4} = 3$

$F'(x) = \frac{d}{dx}(12)(1+x)^{-1} = (-12)(1+x)^{-2} = -\frac{12}{(1+x)^2}$

$F \in C^1$ near 3.

$F'(\alpha) = -\frac{12}{(1+3)^2} = -\frac{12}{16} = -\frac{3}{4}$

$|F'(\alpha)| < 1$, does converge.

Since $|F'(\alpha)| > 0$, the linear term in the Taylor expansion

$$e_{n+1} = x_{n+1} + \alpha = F'(\alpha) e_n + \frac{1}{2} F''(\alpha) e_n^2 + \dots$$

dominates. So $|e_{n+1}| \leq \delta |e_n|$ $\qquad \delta \in (0,1)$

Therefore, this iteration converges linearly.

$\boxed{\mathcal{Z}}$

Q5 $(n=3)$

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 2$

$u_{12} = -1$

$u_{13} = 0$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$2l_{21} = -1 \Rightarrow l_{21} = -1/2$

$2l_{31} = 0 \Rightarrow l_{31} = 0$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$\frac{1}{2} + u_{22} = 2$

$u_{22} = 3/2$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\frac{3}{2} l_{32} = -1$

$l_{32} = -\frac{2}{3}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$u_{23} = -1$

$\frac{2}{3} + u_{33} = 2$

$u_{33} = 4/3$

For $n=3$ $\quad A_3 = LU$,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

Notes from $n=3$ LU factorization

$u_{11} = A_{11} = 2$, $u_{12} = A_{12} = -1$, $u_{13} = A_{13} = 0$

$l_{21} = -1/2 = -1/u_{11}$

$l_{31} = 0$ $\qquad l_{32} = -2/3 = -1/u_{22}$

$u_{22} = \frac{3}{2} = 2 + l_{21}$ $\qquad l_{i\,i-1} = -1/u_{i-1\,i-1}$

$u_{23} = -1$

$u_{33} = 4/3 = 2 + l_{32}$ $\qquad u_{ii} = 2 + l_{i\,i-1}$

$u_{12} = u_{23} = -1$ $\qquad u_{i\,i+1} = -1$

Rule / Formula

$u_{ii} = 2 + l_{i\,i-1}$

$u_{i\,i+1} = -1$

$l_{i\,i-1} = -1/u_{i-1\,i-1}$

Checking for $n=4$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Then our formula says

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

The pattern follows:

$$\begin{cases} l_{i,i-1} = -\frac{i-1}{i}, \quad l_{ii} = 1, \quad \text{otherwise } 0 \\ u_{i,i+1} = -1, \quad u_{ii} = \frac{i+1}{i}, \quad \text{otherwise } 0 \end{cases}$$

Final answer for formula for L, U where $A_n = LU$

Q7

$$\begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}$$

Let $v^{(0)} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

$k=1$: $w = Av^{(0)} = \begin{bmatrix} 3+2+2+1/2 \\ 2+3+1/2+2 \\ 2+1/2+3+2 \\ 1/2+2+2+3 \end{bmatrix} = \begin{bmatrix} 15/2 \\ 15/2 \\ 15/2 \\ 15/2 \end{bmatrix}$

$$v^{(1)} = w/\|w\| = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

It just so happens that $v^{(0)}$ is eigenvector.

The associated eigenvalue is 15, since

$$\lambda^{(1)} = (v^{(1)})^T A v^{(1)} = 15, \text{ and}$$

we see that $v^{(0)} = v^{(1)}$, so $\lambda^{(1)}$ is the dominant eigenvalue corresponding to eigenvector $v^{(1)}$.

$\boxed{4}$

Q8 $\begin{pmatrix} -2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$

I'm going to use Gershgorin Circle Theorem which states:

Every eigenvalue of a square matrix lies within at least one Gershgorin disk defined from its rows.

Each disk is centered at a_{ii}, with radius equal to the $\sum_{\substack{j=1 \\ j\neq i}} |a_{ij}|$

That is,

$$D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \}$$

- $a_{11} = -2$ and $\sum_{\substack{j=1 \\ j\neq 1}}^{3} |a_{1j}| = |1| + |1| = 2$

This disk is $D_1 = [-4, 0]$.

- $a_{22} = 3$ and $\sum_{\substack{j=1 \\ j\neq 2}}^{3} |a_{2j}| = |1| + |1| = 2$

This disk is $D_2 = [1, 5]$

- $a_{33} = 3$ and $\sum_{\substack{j=1 \\ j\neq 3}}^{3} |a_{3j}| = |1| + |-1| = 2$

This disk is $D_3 = [1, 5]$

Eigenvalues lie within $[-4, 0] \cup [1, 5]$

That is, $-4 \leq \lambda_i \leq 0$ or $1 \leq \lambda_i \leq 5$ for each λ_i $\boxed{\Xi}$