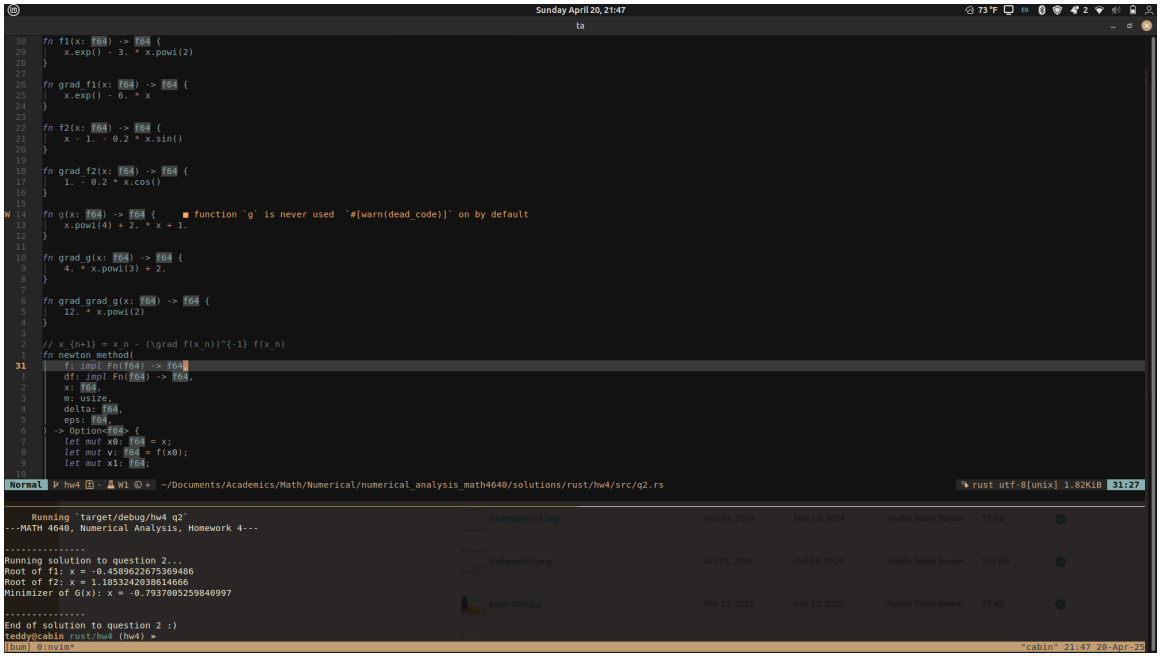
MATH 4640 Numerical Analysis - HW 3 Solutions

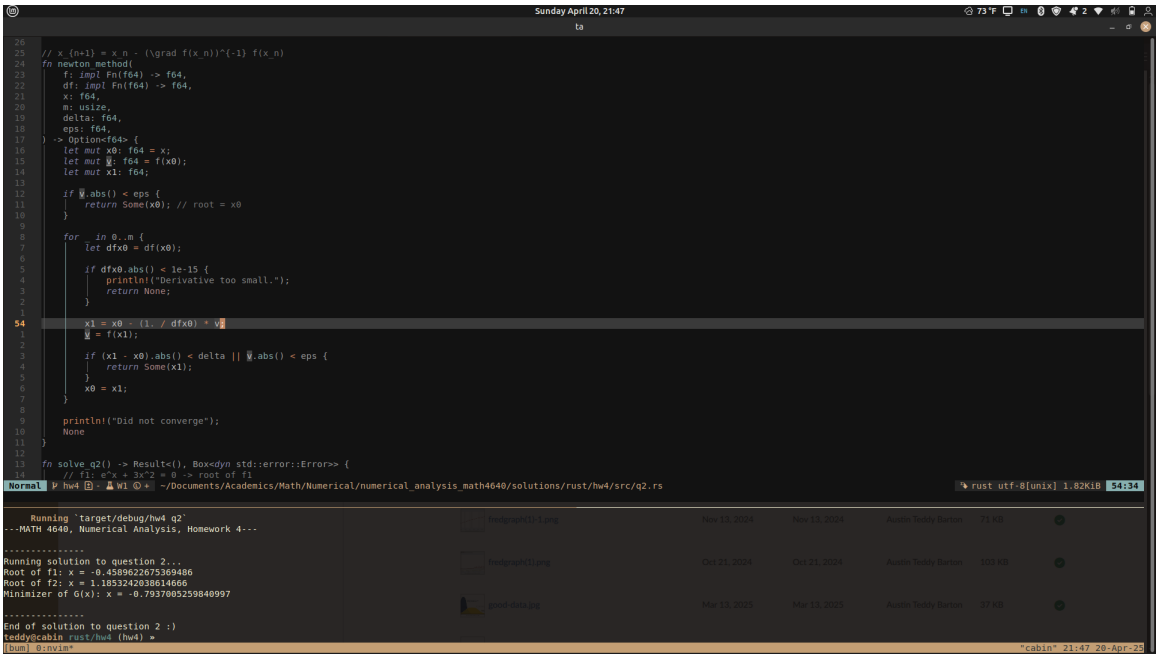
Austin Barton

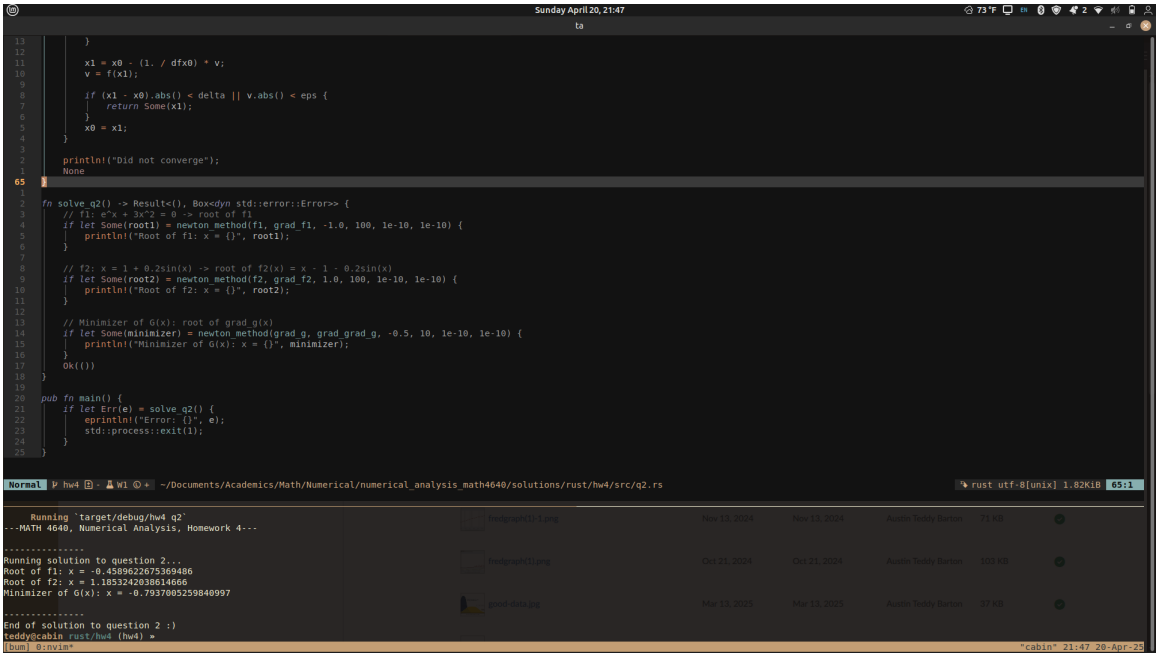
April 20, 2025

Problem 1: Please see attached PDF with all written solutions for questions 1-5 and 7-8.

Problem 2: This problem accompanies the attached PDF which has a dedicated section for question 2. With the code in the following screenshots are the outputs for the code.




Sunday April 20, 21:47
ta
// x_{n+1} = x_n - (\grad f(x_n))^{-1} f(x_n)
fn newton_method(
f: impl Fn(f64) -> f64,
df: impl Fn(f64) -> f64,
x: f64,
m: usize,
delta: f64,
eps: f64,
) -> Option<f64> {
let mut x0: f64 = x;
let mut v: f64 = f(x0);
let mut x1: f64;
if v.abs() < eps {
return Some(x0); // root = x0
}
for _ in 0..m {
let dfx0 = df(x0);
if dfx0.abs() < 1e-15 {
println!("Derivative too small.");
return None;
}
x1 = x0 - (1. / dfx0) * v;
v = f(x1);
if (x1 - x0).abs() < delta || v.abs() < eps {
return Some(x1);
}
x0 = x1;
}
println!("Did not converge");
None
}
fn solve_q2() -> Result<(), Box<dyn std::error::Error>> {
// f1: e^x + 3x^2 = 0 -> root of f1
Normal hw4 W1 ~/Documents/Academics/Math/Numerical/numerical_analysis_math4640/solutions/rust/hw4/src/q2.rs
rust utf-8[unix] 1.82KiB 54:34
Running `target/debug/hw4 q2`
---MATH 4640, Numerical Analysis, Homework 4---
Running solution to question 2...
Root of f1: x = -0.4589622675369486
Root of f2: x = 1.1853242038614666
Minimizer of G(x): x = -0.7937005259840997
End of solution to question 2 :)
teddy@cabin rust/hw4 (hw4) »
[bum] 0:nvim*
"cabin" 21:47 20-Apr-25


Sunday April 20, 21:47
ta
}
x1 = x0 - (1. / dfx0) * v;
v = f(x1);
if (x1 - x0).abs() < delta || v.abs() < eps {
return Some(x1);
}
x0 = x1;
}
println!("Did not converge");
None
}
fn solve_q2() -> Result<(), Box<dyn std::error::Error>> {
// f1: e^x + 3x^2 = 0 -> root of f1
if let Some(root1) = newton_method(f1, grad_f1, -1.0, 100, 1e-10, 1e-10) {
println!("Root of f1: x = {}", root1);
}
// f2: x = 1 + 0.2sin(x) -> root of f2(x) = x - 1 - 0.2sin(x)
if let Some(root2) = newton_method(f2, grad_f2, 1.0, 100, 1e-10, 1e-10) {
println!("Root of f2: x = {}", root2);
}
// Minimizer of G(x): root of grad_g(x)
if let Some(minimizer) = newton_method(grad_g, grad_grad_g, -0.5, 10, 1e-10, 1e-10) {
println!("Minimizer of G(x): x = {}", minimizer);
}
Ok(())
}
pub fn main() {
if let Err(e) = solve_q2() {
eprintln!("Error: {}", e);
std::process::exit(1);
}
}
Normal hw4 W1 ~/Documents/Academics/Math/Numerical/numerical_analysis_math4640/solutions/rust/hw4/src/q2.rs
rust utf-8[unix] 1.82KiB 65:1
Running `target/debug/hw4 q2`
---MATH 4640, Numerical Analysis, Homework 4---
Running solution to question 2...
Root of f1: x = -0.4589622675369486
Root of f2: x = 1.1853242038614666
Minimizer of G(x): x = -0.7937005259840997
End of solution to question 2 :)
teddy@cabin rust/hw4 (hw4) »
[bum] 0:nvim*
"cabin" 21:47 20-Apr-25

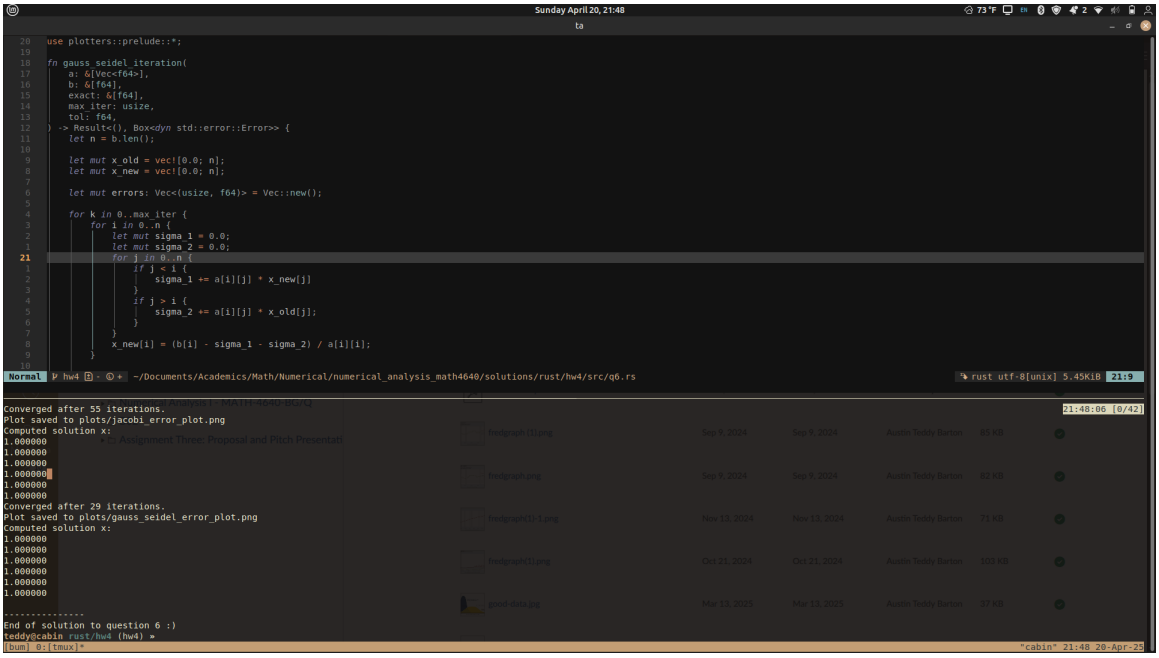
Problem 3: Please see attached PDF with all written solutions for questions 1-5 and 7-8.

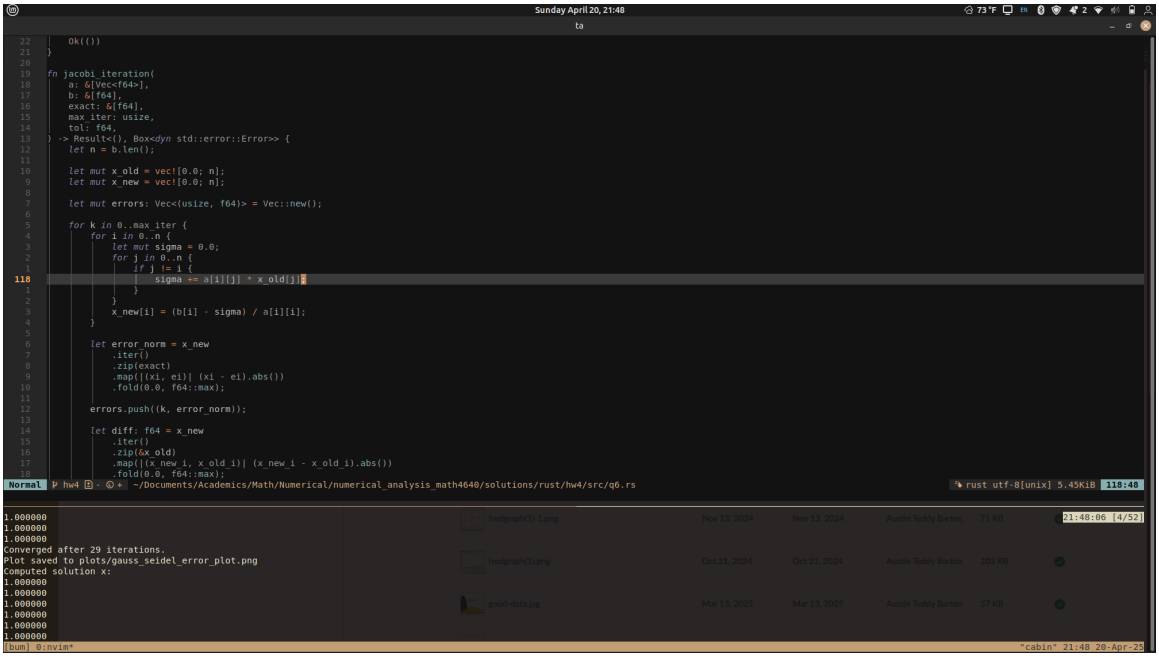
Problem 4: Please see attached PDF with all written solutions for questions 1-5 and 7-8.

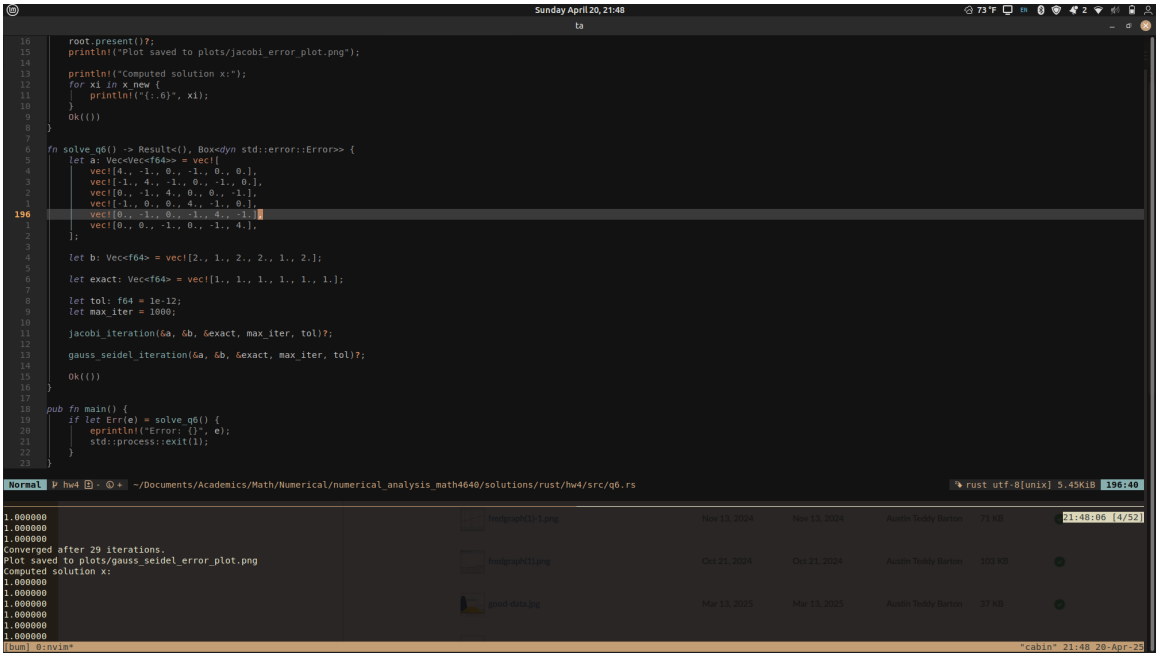
Problem 5: Please see attached PDF with all written solutions for questions 1-5 and 7-8.

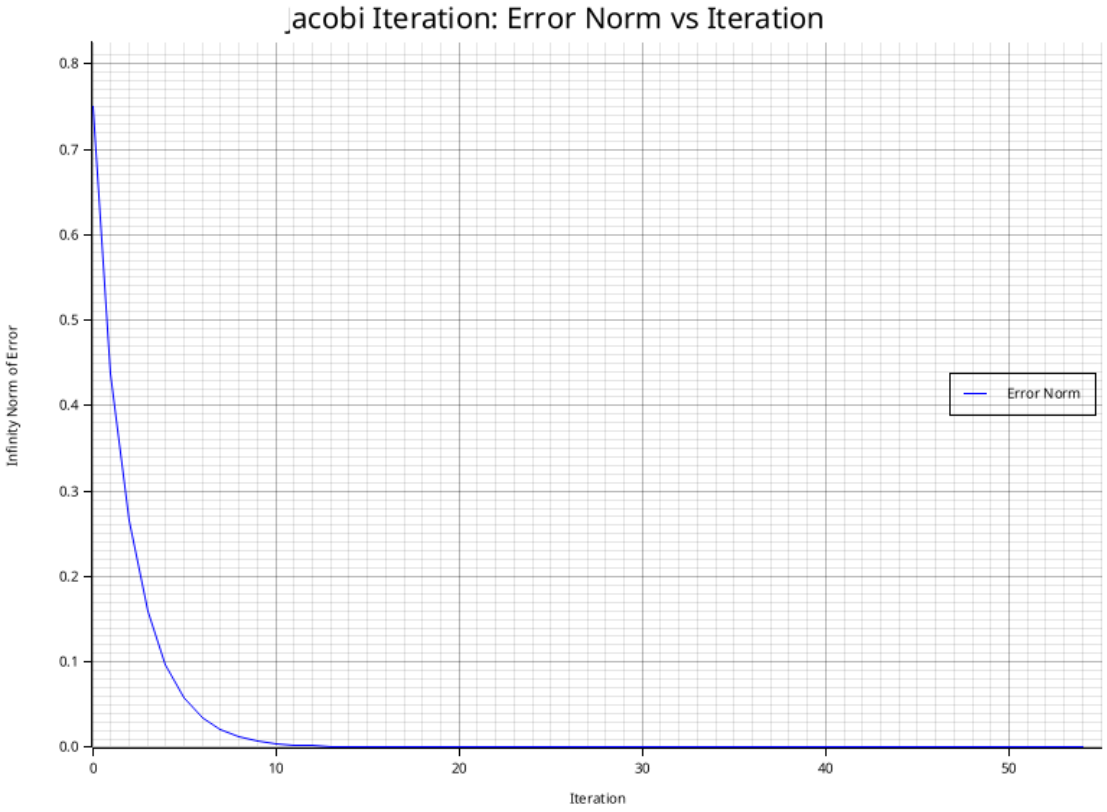
Problem 6:

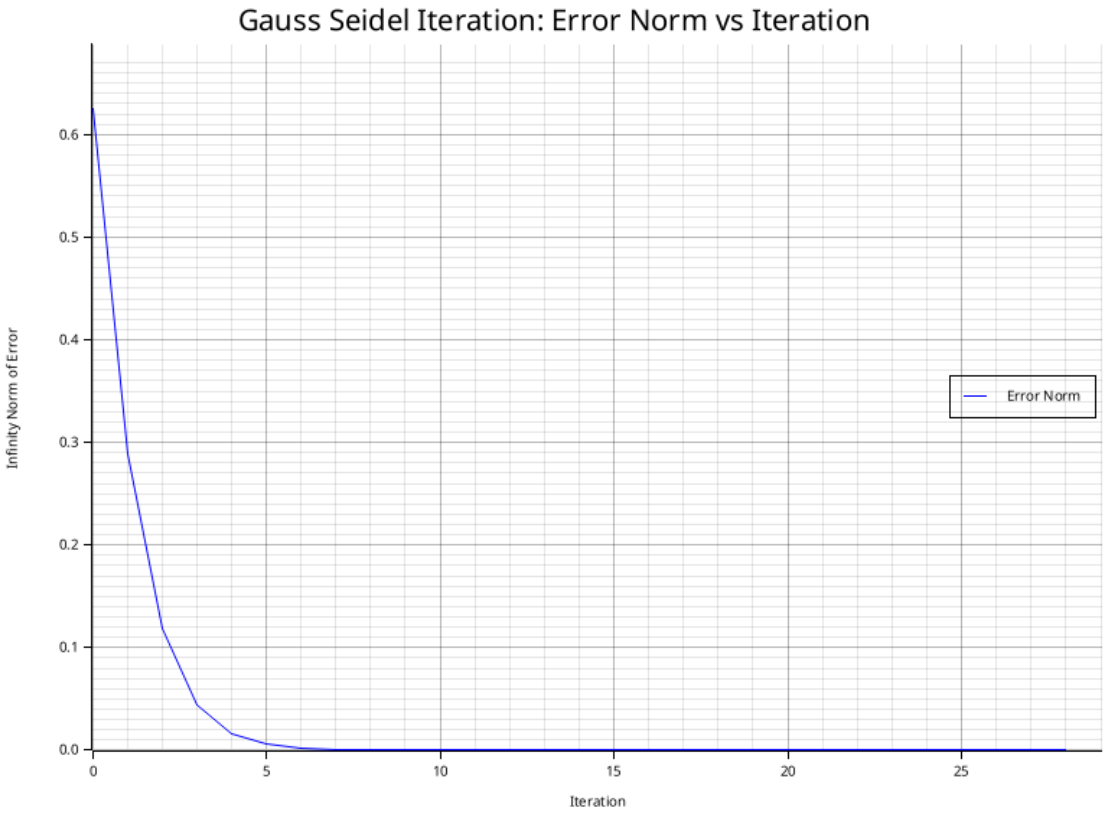
With the code in the following screenshots are the outputs for the code.






Sunday April 20, 21:48
73 °F
ta
root.present()?;
println!("Plot saved to plots/jacobi_error_plot.png");
println!("Computed solution x:");
for xi in x_new {
println!("{:.6}", xi);
}
Ok(())
}
fn solve_q6() -> Result<(), Box<dyn std::error::Error>> {
let a: Vec<Vec<f64>> = vec![
vec![4., -1., 0., -1., 0., 0.],
vec![-1., 4., -1., 0., -1., 0.],
vec![0., -1., 4., 0., 0., -1.],
vec![-1., 0., 0., 4., -1., 0.],
vec![0., -1., 0., -1., 4., -1.],
vec![0., 0., -1., 0., -1., 4.],
];
let b: Vec<f64> = vec![2., 1., 2., 2., 1., 2.];
let exact: Vec<f64> = vec![1., 1., 1., 1., 1., 1.];
let tol: f64 = 1e-12;
let max_iter = 1000;
jacobi_iteration(&a, &b, &exact, max_iter, tol)?;
gauss_seidel_iteration(&a, &b, &exact, max_iter, tol)?;
Ok(())
}
pub fn main() {
if let Err(e) = solve_q6() {
eprintln!("Error: {}", e);
std::process::exit(1);
}
}
Normal
hw4
~/Documents/Academics/Math/Numerical/numerical_analysis_math4640/solutions/rust/hw4/src/q6.rs
rust utf-8[unix] 5.45KiB
196:40
1.000000
1.000000
1.000000
Converged after 29 iterations.
Plot saved to plots/gauss_seidel_error_plot.png
Computed solution x:
1.000000
1.000000
1.000000
1.000000
1.000000
1.000000
[bum] 0:nvim*
"cabin" 21:48 20-Apr-25


Jacobi Iteration: Error Norm vs Iteration
Infinity Norm of Error
Iteration
Error Norm
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
10
20
30
40
50


Gauss Seidel Iteration: Error Norm vs Iteration
Infinity Norm of Error
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
5
10
15
20
25
Iteration
Error Norm

Problem 7: Please see attached PDF with all written solutions for questions 1-5 and 6-7.

Problem 8: Please see attached PDF with all written solutions for questions 1-5 and 7-8.

1 Written Solutions

Q1 $p(x) = x^3 - 2x - 3$

$p(0) = -3$

$p(1) = -4$

$p(2) = 1$

There must be a root in $[0,2]$ since $p(0)p(2) < 0$.

p is continuous on $[0,2]$.

$a = 0$, $b = 2$

$a_0 = a$, $b_0 = b$

$m_0 = \frac{a_0 + b_0}{2} = \frac{0+2}{2} = 1$

$p(m_0) = -4$ $\quad p(a_0)p(m_0) > 0$, $p(m_0)p(b_0) < 0$

$a_1 = m_0$, $b_1 = b_0$

$m_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = \frac{3}{2}$

$p(m_1) = \left(\frac{3}{2}\right)^3 - 2\left(\frac{3}{2}\right) - 3 = \frac{27}{8} - 6 = \frac{27-48}{8} = -\frac{21}{8}$

$p(a_1)p(m_1) > 0$, $p(m_1)p(b_1) < 0$

$a_2 = m_1$, $b_2 = b_1$, $m_2 = \frac{\frac{3}{2}+2}{2} = \frac{7}{4}$

$p(m_2) = \left(\frac{7}{4}\right)^3 - 2\,\frac{7}{4} - 3 = \frac{343}{64} - \frac{26}{4} = -\frac{73}{64}$

$p(a_2)\,p(m_2) > 0$, $p(m_2)\,p(b_2) < 0$

$a_3 = m_2$, $b_3 = b_2$ $\quad m_3 = \frac{\frac{7}{4}+2}{2} = \frac{15}{8}$

$p(m_3) = \left(\frac{15}{8}\right)^3 - 2\left(\frac{15}{8}\right) - 3 = \frac{3375}{512} - \frac{30}{8} - 3$

$= \frac{3375}{512} - \frac{54}{8} = \frac{3375 - 3456}{512} = -\frac{81}{512}$

$p(a_3)\,p(m_3) > 0$ so,

$a_4 = m_3$, $b_4 = b_3$ $\quad m_4 = \frac{\frac{15}{8}+2}{2} = \frac{31}{16}$

$p(m_4) = \left(\frac{31}{16}\right)^3 - 2\left(\frac{31}{16}\right) - 3 = \frac{29791}{4096} - \frac{62}{16} - \frac{48}{16}$

$= \frac{29791}{4096} - \frac{110}{16} = \frac{29791 - 28160}{4096} = \frac{1631}{4096}$

$p(a_4)\,p(m_4) < 0$, so

$a_5 = a_4$, $b_5 = m_4$ $\quad m_5 = \frac{\frac{15}{8}+\frac{31}{16}}{2} = \frac{\frac{61}{16}}{2}$

$m_5 = \frac{61}{32}$

$p(m_5) = \left(\frac{61}{32}\right)^3 - 2\left(\frac{61}{32}\right) - 3 = \frac{226981}{32768} - \frac{122}{32} - \frac{96}{32}$

$= \frac{226981}{32768} - \frac{218}{32} = \frac{226981 - 223232}{32768}$

$= \frac{3749}{32768}$

$p(a_5)\,p(m_5) < 0$

$a_6 = a_5, \quad b_6 = m_5$

$$m_6 = \frac{\frac{15}{8} + \frac{61}{32}}{2} = \frac{121}{64}$$

Exact $x^* = 1.893289196304\,50$

m_6 is within 3 significant figures of x^*.

Answer $\hat{x} = \frac{121}{64} \approx x^*$ (w/ 3 sig figs)

Analysis $\left[\frac{121}{64} = 1.890625\right]$

A tolerance less or equal to $1/16$ would result in us being at least within 3 significant digits.

Q2

(a) $f(x) = e^x - 3x^2$ $\quad \frac{d^2}{dx^2} f(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(e^x - 6x) = e^x - 6$

So $f \in C^2$.

Define: $e_n = x_n - x^*$

We know that since $f \in C^2$, for some neighborhood of x^*, if we start Newton's x_0 in that neighborhood, the convergence is quadratic.

(b) $f(x) = x - 1 - 0.2\sin(x)$
$f'(x) = 1 - 0.2\cos(x)$
$f''(x) = 0.2\sin(x)$
So, $f \in C^2$.

Thus, Newton's method has quadratic convergence if started w/in some neighborhood of x^*.

(c) $G(x) = x^4 + 2x + 1$
Define $f(x) = \nabla G(x) = 4x^3 + 2x$
To minimize G, we find roots of f using Newton's Method
$f'(x) = 12x^2 + 2$
$f''(x) = 24x$

So $f \in C^2$. Thus, Newton's method has quadratic convergence if started w/in some neighborhood of x^*.

Q3 $x_{n+1} = 2 - (1+c)x_n + cx_n^3$, $\alpha = 1$ is a fixed point.

$F(\alpha) = \alpha$

$x_{n+1} = F(x_n)$

F is differentiable and need $|F'(\alpha)| < 1$ to ensure local convergence.

Now, $F(x) = 2 - (1+c)x + cx^3$

$F'(x) = -1 - c + 3cx^2$

Take the Taylor expansion of F about α:

$x_{n+1} = F(x_n) = \alpha + F'(\alpha)e_n + \frac{1}{2}F''(\alpha)e_n^2 + \dots$

meaning that...

$e_{n+1} = x_{n+1} - \alpha = F'(\alpha)e_n + \frac{1}{2}F''(\alpha)e_n^2 + \dots$

Now, consider:

$F'(\alpha) = F'(1) = -1 - c + 3c = -1 + 2c$

We need $|F'(\alpha)| < 1 \Rightarrow -1 < F'(\alpha) < 1$

$\Leftrightarrow -1 < -1 + 2c < 1 \Rightarrow 0 < 2c < 2$

$\Rightarrow 0 < c < 1$

S. $c \in (0,1)$

But as we know this is for linear convergence.

To know what c grants quadratic convergence, we need the value of c such that $F'(\alpha) = 0$

That is, $F'(1) = -1 + 2c = 0 \Rightarrow c = 1/2$

Answer: $c = 1/2$ will grant quadratic convergence. Why?

$e_{n+1} = \frac{1}{2}F''(\alpha)e_n^2 + \dots$ in this case, since $F'(\alpha) = 0$.

Q4

(a) $x_{n+1} = -16 + 6x_n + {}^{12}/_{x_n}$, $\alpha = 2$

$x_{n+1} = F(x_n)$, $F(2) = -16 + 12 + 6 = 2$

$F'(x) = 6 + {}^{12}/_{x^2}$ so $F \in C^1$ near α.

$F'(\alpha) = 6 - {}^{12}/_{4} = 6 - 3 = 3$

$|F'(\alpha)| > 1 \Rightarrow$ does not converge.

(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $\alpha = 3^{1/3}$

$x_{n+1} = F(x_n)$, $F(3^{1/3}) = \frac{2}{3}3^{1/3} + \frac{1}{3^{2/3}} =$

$F(3^{1/3})\,3^{2/3} = \frac{2}{3}(3) + \frac{3^{2/3}}{3^{2/3}} = 2 + 1 = 3$

$F(3^{1/3})\,3^{2/3} = 3 \Rightarrow F(3^{1/3}) = 3^{1/3}$

$F'(x) = \frac{2}{3} - \frac{2}{x^3}$ so $F \in C^1$ near α.

$F'(\alpha) = \frac{2}{3} - \frac{2}{(3^{1/3})^3} = 0$

$|F'(\alpha)| < 1$, does converge.

$F''(x) = {}^{6}/_{x^4}$, $F''(\alpha) = \frac{6}{3^{4/3}} > 0$

Since $F'(\alpha) = 0$, $F''(\alpha) \neq 0$ this iteration has quadratic convergence.

(c) $x_{n+1} = \frac{12}{1+x_n}$, $\alpha = 3$

$x_{n+1} = F(x_n)$ $\quad F(3) = \frac{12}{1+3} = \frac{12}{4} = 3$

$F'(x) = \frac{d}{dx}(12)(1+x)^{-1} = (-12)(1+x)^{-2} = -\frac{12}{(1+x)^2}$

$F \in C^1$ near 3.

$F'(\alpha) = -\frac{12}{(1+3)^2} = -\frac{12}{16} = -\frac{3}{4}$

$|F'(\alpha)| < 1$, does converge.

Since $|F'(\alpha)| > 0$, the linear term in the Taylor expansion

$e_{n+1} = x_{n+1} + \alpha = F'(\alpha)e_n + \frac{1}{2}F''(\alpha)e_n^2 + \dots$

dominates. So $|e_{n+1}| \leq \delta |e_n| \quad \delta \in (0,1)$

Therefore, this iteration converges linearly.

$\square$

Q5 $(n=3)$

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 2$

$u_{12} = -1$

$u_{13} = 0$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$2l_{21} = -1 \Rightarrow l_{21} = -1/2$

$2l_{31} = 0 \Rightarrow l_{31} = 0$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$\frac{1}{2} + u_{22} = 2$

$u_{22} = 3/2$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\frac{3}{2} l_{32} = -1$

$l_{32} = -\frac{2}{3}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$u_{23} = -1$

$\frac{2}{3} + u_{33} = 2$

$u_{33} = 4/3$

For $n=3$ $\quad A_3 = LU$,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

Notes from $n=3$ LU factorization

$u_{11} = A_{11} = 2$, $u_{12} = A_{12} = -1$, $u_{13} = A_{13} = 0$

$l_{21} = -1/2 = -1/u_{11}$

$l_{31} = 0$ $\qquad l_{32} = -2/3 = -1/u_{22}$

$u_{22} = \frac{3}{2} = 2 + l_{21}$ $\qquad l_{i\,i-1} = -1/u_{i-1\,i-1}$

$u_{23} = -1$

$u_{33} = 4/3 = 2 + l_{32}$ $\qquad u_{ii} = 2 + l_{i\,i-1}$

$u_{12} = u_{23} = -1$ $\qquad u_{i\,i+1} = -1$

Rule / Formula

$u_{ii} = 2 + l_{i\,i-1}$

$u_{i\,i+1} = -1$

$l_{i\,i-1} = -1/u_{i-1\,i-1}$

Checking for $n=4$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Then our formula says

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

The pattern follows:

$$\begin{cases} l_{i,i-1} = -\frac{i-1}{i}, \; l_{ii} = 1, \text{ otherwise } 0 \\ u_{i,i+1} = -1, \; u_{ii} = \frac{i+1}{i}, \text{ otherwise } 0 \end{cases}$$

Final answer for formula for L, U where $A_n = LU$

Q7

$$\begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}$$

Let $v^{(0)} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

$k=1$: $w = Av^{(0)} = \begin{bmatrix} 3 + 2 + 2 + 1/2 \\ 2 + 3 + 1/2 + 2 \\ 2 + 1/2 + 3 + 2 \\ 1/2 + 2 + 2 + 3 \end{bmatrix} = \begin{bmatrix} 15/2 \\ 15/2 \\ 15/2 \\ 15/2 \end{bmatrix}$

$$v^{(1)} = w/\|w\| = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

It just so happens that $v^{(0)}$ is eigenvector.

The associated eigenvalue is 15, since

$$\lambda^{(1)} = (v^{(1)})^T A v^{(1)} = 15, \text{ and}$$

we see that $v^{(0)} = v^{(1)}$, so $\lambda^{(1)}$ is the dominant eigenvalue corresponding to eigenvector $v^{(1)}$.

G8 $\begin{pmatrix} -2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}$

I'm going to use Gershgorin Circle Theorem which states:

Every eigenvalue of a square matrix lies within at least one Gershgorin disk defined from its rows.

Each disk is centered at a_{ii}, with radius equal to the $\sum_{\substack{j=1 \\ j \neq i}}^{3} |a_{ij}|$

That is,

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{3} |a_{ij}| \right\}$$

- $a_{11} = -2$ and $\sum_{\substack{j=1 \\ j \neq 1}}^{3} |a_{1j}| = |1| + |1| = 2$

This disk is $D_1 = [-4, 0]$

- $a_{22} = 3$ and $\sum_{\substack{j=1 \\ j \neq 2}}^{3} |a_{2j}| = |1| + |1| = 2$

This disk is $D_2 = [1, 5]$

- $a_{33} = 3$ and $\sum_{\substack{j=1 \\ j \neq 3}}^{3} |a_{3j}| = |1| + |-1| = 2$

This disk is $D_3 = [1, 5]$

Eigenvalues lie within $[-4, 0] \cup [1, 5]$

That is, $-4 \leq \lambda_i \leq 0$ or $1 \leq \lambda_i \leq 5$ for each λ_i

☑