Math 4640, Numerical Analysis I
Homework 3

(1) Let $p_0(x), p_1(x), p_2(x), \cdots,$ be a sequence of orthogonal polynomial and let $x_0, ..., x_k$ be the $k+1$ distinct zeros of $p_{k+1}(x)$. Prove that the Lagrange polynomials $l_i(x) = \prod_{j\neq i}(x-x_j)/(x_i-x_j), i = 0, \cdots, k$, for these points are orthogonal to each other. (Hint: show that for $i \neq j$, $l_i(x)l_j(x) = p_{k+1}(x)g(x)$, where $g(x)$ is some polynomial of degree $\leq k$).

(2) Calculate the polynomial $p(x)$ of degree ≤ 2 which minimizes

$$\int_{-1}^{1} (\cos(\pi x) - p(x))^2 dx$$

over all polynomials of degree ≤ 2. Carry out the calculations to five decimal places.

(3) If $f(x)$ is a 2π-periodic function, prove that $g_\alpha(x) = f(x+\alpha)$ is also a 2π-periodic function for any number α. What is the relationship between the $\hat{f}(j)$ and the $\hat{g}_\alpha(j)$.

(4) Verify that the 2π-periodic function $f(x)$ whose values on $[0, 2\pi)$ are given by

$$f(x) = \begin{cases} (x/\pi)^2 - x/\pi & 0 \leq x < \pi \\ (x-\pi)/\pi - ((x-\pi)/\pi)^2 & \pi \leq x < 2\pi \end{cases}$$

is continuous and has a continuous first derivative (as a 2π-periodic function), but has jumps in the second derivative. Then construct the spectrum of $f(x)$ and show that it decays like j^{-3} as $j \to \infty$.

(5) (a) Let $f(x) = \sin^2(5x)$, $0 \leq x \leq 2\pi$. Compute its Fourier coefficients ($N = 200$)

$$\hat{f}_N(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(x_k)e^{-ix_kj}, \quad \text{with} \quad x_k = \frac{2\pi k}{N}, k = 0, \cdots N-1$$

by FFT (You may use any existing FFT package, such as the ones in Matlab or Mathematica. You need to be aware that they may have an equivalent, but different definition for Fourier transform. In that case, you need to find the answers corresponding to our definitions in class). Reconstruct an approximation of $f(x)$ by

$$g_5(x) = \sum_{j=-5}^{5} \hat{f}_N(j)e^{ijx}.$$

Plot $f(x)$ and $g_5(x)$ in the same picture and compare them.

(b) Repeat (a) for another function $f(x) = \cos^3(3x) + \cos(\sqrt{2}x)$. What conclusion can you draw by comparing the results in (a) and (b)?

(6) Derive the one- and two-point Gaussian quadrature formulas for

$$I = \int_{-1}^{0} -xf(x)dx \approx \sum_{j=1}^{n} A_j f(x_j)$$

with weight function $w(x) = -x$.

(7) Derive the composite corrected trapezoidal rule and verify that the interior derivatives $f'(x_i)$, $i = 1, \cdots, N-1$, cancel out in the sum. (You must show your steps of derivation).

(8) Write your code to evaluate $I = \int_a^b f(x)dx$ using the composite trapezoidal rule with n subdivisions. Use the program to calculate the following integals with $n = 128, 256, 512$ respectively.

(a) $\int_0^1 e^{-\frac{x^2}{2}} dx$,

(b) $\int_0^{2\pi} \frac{1}{2+\sin x} dx$.