Question 8

Recall: $\frac{d}{dx} f[x_0, \ldots, x_k, x] = f[x_0, \ldots, x_k, x, x]$

$$f(x) = p_k(x) + f[x_0, x_1, \ldots, x_k, x]\, \psi_k(x)$$

$$\left[f[x_0, \ldots, x_k, x]\, \psi_k(x)\right]' = f[x_0, \ldots, x_k, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x]\, \psi_k'(x)$$

which we've shown before.

$$\left[f[x_0, \ldots, x_k, x]\, \psi_k(x)\right]^{(2)}$$

$$= \left[f[x_0, \ldots, x_k, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x]\, \psi_k'(x)\right]'$$

$$= f[x_0, \ldots, x_k, x, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x, x]\, \psi_k'(x)$$
$$+ f[x_0, \ldots, x_k, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x]\, \psi_k''(x)$$

$$\left[f[x_0, \ldots, x_k, x]\, \psi_k(x)\right]^{(3)}$$

$$= \Big[f[x_0, \ldots, x_k, x, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x, x]\, \psi_k'(x)$$
$$+ f[x_0, \ldots, x_k, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x]\, \psi_k''(x) \Big]'$$

$$= f[x_0, \ldots, x_k, x, x, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x, x, x]\, \psi_k'(x)$$
$$+ f[x_0, \ldots, x_k, x, x, x]\, \psi_k'(x) + f[x_0, \ldots, x_k, x, x]\, \psi_k''(x)$$
$$+ f[x_0, \ldots, x_k, x, x, x]\, \psi_k(x) + f[x_0, \ldots, x_k, x, x]\, \psi_k'(x)$$
$$+ f[x_0, \ldots, x_k, x, x]\, \psi_k''(x) + f[x_0, \ldots, x_k, x]\, \psi_k'''(x)$$

$$= f[x_0, \dots, x_k, x, x, x, x]\, \ell_k(x)$$

$$+ f[x_0, \dots, x_k, x, x, x]\left(\ell_k(x) + 2\ell_k'(x)\right)$$

$$+ f[x_0, \dots, x_k, x, x]\left(\ell_k'(x) + 2\ell_k''(x)\right)$$

$$+ f[x_0, \dots, x_k, x]\, \ell_k^{(3)}(x)$$

$$= \frac{f^{(k+4)}(\xi)}{(k+4)!}\ell_k(x) + \frac{f^{(k+3)}(\eta)}{(k+3)!}\left(\ell_k(x) + 2\ell_k'(x)\right)$$

$$+ \frac{f^{(k+2)}(\gamma)}{(k+2)!}\left(\ell_k'(x) + 2\ell_k''(x)\right)$$

$$+ \frac{f^{(k+1)}(\rho)}{(k+1)!}\ell^{(3)}(x) = \psi(x)$$

for some, $\xi, \eta, \gamma, \rho \in [\min\{x_0, \dots, x_k, x\}, \max\{x_0, \dots, x_k, x\}]$

So,

$$f^{(3)}(x) = p_k^{(3)}(x) + \psi(x),$$ where $\psi(x)$ is

defined as above. $\square$