Question 7

- $f(a + h/2) = f(a) + f'(a)(a + h/2 - a)$

$$+ \frac{f^{(2)}(a)}{2}(a + h/2 - a)^2 + \dots$$

$$= f(a) + f'(a)\frac{h}{2} + \frac{f^{(2)}(a)}{2!}\frac{h^2}{2^2} + \frac{f^{(3)}(a)}{3!}\frac{h^3}{2^3} + \dots$$

$$= f(a) + \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + \dots$$

- $f(a - h/2)$

$$= f(a) + f'(a)(a - \tfrac{h}{2} - a) + \frac{f^{(2)}(a)}{2!}(a - \tfrac{h}{2} - a)^2 + \dots$$

$$= f(a) - \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + \dots$$

So,

$$f(a + h/2) = f(a) + \tfrac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + O(h^4)$$

$$f(a - h/2) = f(a) - \tfrac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + O(h^4)$$

$$f(a + h/2) - f(a - h/2) = hf'(a) + \frac{h^3}{4}\frac{f^{(3)}(a)}{3!} + O(h^5)$$

$$= hf'(a) + \frac{h^3}{24}f^{(3)}(a) + O(h^5)$$

Thus, $$\frac{f(a + h/2) - f(a - h/2)}{2h} = \frac{1}{2}f'(a) + \frac{h^2}{48}f^{(3)}(a) + O(h^4)$$

That is,

$$\frac{f(a+h/2) - f(a-h/2)}{2h} = \frac{1}{2} f'(a) + \frac{h^2}{48} f^{(2)}(a) + O(h^4)$$

Hence, the error between $\frac{1}{2} f'(a)$ and the approximation $\frac{1}{2h}\left(f(a+h/2) - f(a-h/2)\right)$ is

$$\left| \frac{1}{2} f'(a) - \frac{f(a+h/2) - f(a-h/2)}{2h} \right|$$

$$= \left| \frac{h^2}{48} f^{(2)}(a) + O(h^4) \right| = \frac{h^2}{48} \left| f^{(2)}(a) \right| + O(h^4)$$

The dominant term is h^2, thus, the order of the error is

$$O(h^2).$$