MATH 4640 Numerical Analysis - HW 2 Solutions

Austin Barton

February 27, 2025

Problem 1:

$y_0 = -1 \quad y_4 = 1$

$\Delta x_i = 1 \quad \forall i \in \{0,1,2,3,4\}$

$\Delta x_i + \Delta x_{i+1} = 2 \quad \forall i \in \{0,1,2,3\}$

Note: This is important to understand the simplifications to the general form below

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad M = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}$$

$$0 = \begin{bmatrix} f_1 - f_0 - y_0 \\ f_2 - f_1 - (f_1 - f_0) \\ f_3 - f_2 - (f_2 - f_1) \\ f_4 - f_3 - (f_3 - f_2) \\ y_4 - (f_4 - f_3) \end{bmatrix} = \begin{bmatrix} -0.3 + 1 \\ -0.3 + 0.3 \\ -0.2 + 0.3 \\ 0.2 + 0.2 \\ 1 - 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0 \\ 0.1 \\ 0.4 \\ 0.8 \end{bmatrix}$$

We want to find M s.t.

$AM = 0$, below is it expressed as a linear system of equations

① $\frac{1}{3}M_0 + \frac{1}{6}M_1 = 0.7$

② $\frac{1}{6}M_0 + \frac{2}{3}M_1 + \frac{1}{6}M_2 = 0$

③ $\frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = 0.1$

④ $\frac{1}{6}M_2 + \frac{2}{3}M_3 + \frac{1}{6}M_4 = 0.4$

⑤ $\frac{1}{6}M_3 + \frac{1}{3}M_4 = 0.8$

$$AM = \begin{bmatrix} \frac{2M_0 + M_1}{6} \\ \frac{4M_1 + M_0 + M_2}{6} \\ \frac{4M_2 + M_1 + M_3}{6} \\ \frac{4M_3 + M_2 + M_4}{6} \\ \frac{2M_4 + M_3}{6} \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0 \\ 0.1 \\ 0.4 \\ 0.8 \end{bmatrix}$$

- $\frac{2M_4 + M_3}{6} = 0.8 \Rightarrow 2M_4 + M_3 = 4.8$

$\Rightarrow M_4 = 2.4 - \frac{1}{2} M_3$

- $\frac{4M_3 + M_2 + M_4}{6} = 0.4 \Rightarrow 4M_3 + M_2 + M_4 = 2.4$

$\Rightarrow M_2 = 2.4 - M_4 - 4M_3 = 2.4 - (2.4 - \frac{1}{2}M_3) - 4M_3$

$= \frac{1}{2}M_3 - 4M_3 = -\frac{7}{2}M_3$

$M_2 = -7/2\, M_3$

- $\frac{4M_2 + M_1 + M_3}{6} = 0.1 \Rightarrow 4M_2 + M_1 + M_3 = 0.6$

$\Rightarrow M_1 = 0.6 - 4M_2 - M_3 = 0.6 - 4(-\frac{7}{2}M_3) - M_3$

$= 0.6 + 14M_3 - M_3 = 0.6 + 13M_3$

$M_1 = 0.6 + 13M_3$

- $\frac{4M_1 + M_0 + M_2}{6} = 0 \Rightarrow M_0 = -4M_1 - M_2$

$= -4(0.6 + 13M_3) - (-7/2\, M_3) = -2.4 - 52M_3 + \frac{7}{2}M_3$

$$M_0 = -2.4 - 52M_3 + \frac{7}{2}M_3 = -2.4 - \frac{97}{2}M_3$$

$$\frac{2M_0 + M_1}{6} = 0.7 = \frac{2\left(-2.4 - \frac{97}{2}M_3\right) + (0.6 + 13M_3)}{6}$$

$$= \frac{-4.8 - 97M_3 + 0.6 + 13M_3}{6}$$

$$\Rightarrow 4.2 = -4.2 - 84M_3 \Rightarrow 8.4 = -84M_3$$

$$\Rightarrow -0.1 = M_3$$

$M_3 = -0.1$, thus,

$$M_0 = -2.4 - 52(-0.1) + \frac{7}{2}(-0.1) = 2.45$$

$$M_1 = 0.6 + 13(-0.1) = -0.7$$

$$M_2 = -7/2(-0.1) = 0.35$$

$$M_4 = 2.4 - \frac{1}{2}(-0.1) = 2.45$$

$$M = \begin{bmatrix} 2.45 \\ -0.7 \\ 0.35 \\ -0.1 \\ 2.45 \end{bmatrix}$$

$$p_0(x) = \frac{(1-x)^3 M_0 + x^3 M_1}{6} + (1-x)(1.1) + x(0.8)$$

$$- \frac{1}{6}\left((1-x)M_0 + x M_1\right), \quad x \in [0, 1]$$

$$p_1(x) = \frac{(2-x)^3 M_1 + (x-1)^3 M_2}{6} + (2-x)0.8 + (x-1)(0.5)$$

$$- \frac{1}{6}\left((2-x)M_1 + (x-1)M_2\right), \quad x \in [1, 2]$$

$$p_2(x) = \frac{(3-x)^3 M_2 + (x-2)^3 M_3}{6} + (3-x)0.5 + (x-2)0.3$$

$$- \frac{1}{6}\left((3-x)M_2 + (x-2)M_3\right), \quad x \in [2, 3]$$

$$p_3(x) = \frac{(4-x)^3 M_3 + (x-3)^3 M_4}{6} + (4-x)0.3 + (x-3)0.5$$

$$- \frac{1}{6}\left((4-x)M_3 + (x-3)M_4\right), \quad x \in [3, 4]$$

where $M_0 = 2.45$
$M_1 = -0.7$
$M_2 = 0.35$
$M_3 = -0.1$
$M_4 = 2.45$

Problem 2:

Question 2

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + \dots + f[x_0,x_1,x_2,x_3](x-x_0)\cdot(x-x_1)(x-x_2)$$

$$f(x) = p_3(x) + f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$f(x) - p_3(x) = f[x_0,x_1,x_2,x_3,x]\,\ell_3(x)$$

$$f[x_0,x_1,x_2,x_3,x] = \frac{f^{(4)}(\xi)}{4!} \quad \text{for some } \xi \in [x_0,x_3].$$

Thus,

$$\left(f(x) - p_3(x)\right)' = f[x_0,x_1,x_2,x_3,x]'\,\ell_3(x) + f[x_0,x_1,x_2,x_3,x]\,\ell_3'(x)$$

by Chain rule. And further,

$$\frac{d}{dx} f[x_0,x_1,x_2,x_3,x] = f[x_0,x_1,x_2,x_3,x,x]$$

and $$f[x_0,x_1,x_2,x_3,x] = \frac{f^{(4)}(\xi)}{4!}$$

$$f[x_0,x_1,x_2,x_3,x,x] = \frac{f^{(5)}(\eta)}{5!}$$

for some $\xi, \eta \in [x_0,x_3]$.

$$\ell_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$\ell_3'(x) = (x-x_1)(x-x_2)(x-x_3) + (x-x_0)(x-x_2)(x-x_3) + (x-x_0)(x-x_1)(x-x_3) + (x-x_0)(x-x_1)(x-x_2)$$

So, $$\left|f'(x) - p_3'(x)\right| = \left|\frac{f^{(5)}(\eta)}{5!}\ell_3(x) + \frac{f^{(4)}(\xi)}{4!}\ell_3'(x)\right|$$

$$\left| \frac{f^{(5)}(\eta)}{5!} \ell_3(x) + \frac{f^{(4)}(\xi)}{4!} \ell_3'(x) \right|$$

$$= \frac{1}{120} \left| f^{(5)}(\eta) \ell_3(x) \right| + \frac{1}{24} \left| f^{(4)}(\xi) \ell_3'(x) \right|$$

Now, the nodes are evenly spaced, so

$$\ell_3(x) = (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$$

Then, $\max_{x \in [x_0, x_3]} |\ell_3(x)| = \left(\frac{3}{2}h\right)\left(\frac{1}{2}h\right)\left(\frac{1}{2}h\right)\left(\frac{3}{2}h\right)$

$$= \frac{9}{16} h^4$$

And $\ell_3'(x) = (x - (x_0 + h))(x - (x_0 + 2h))(x - (x_0 + 3h))$

$$+ (x - x_0)(x - (x_0 + 2h))(x - (x_0 + 3h))$$

$$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 3h))$$

$$+ (x - x_0)(x - (x_0 + h))(x - (x_0 + 2h))$$

$$\max_{x \in [x_0, x_3]} |\ell_3'(x)| \leq (h)(2h)(3h) + (h)(h)(2h) + (2h)(h)(h) + (3h)(2h)(h) = 13h^3$$

Where we took the $\max_{x \in [x_0, x_3]}$ over each term.

$$|f'(x) - p_3'(x)| \leq \frac{9}{16 \cdot 5!} h^4 \max_{\eta \in [x_0, x_3]} |f^{(5)}(\eta)| + \frac{13}{4!} h^3 \max_{\xi \in [x_0, x_3]} |f^{(4)}(\xi)|$$

$$= \frac{9}{1920} h^4 \max_{\eta \in [x_0, x_3]} |f^{(5)}(\eta)| + \frac{13}{24} h^3 \max_{\xi \in [x_0, x_3]} |f^{(4)}(\xi)|$$

Problem 3:

Question 3 $p(x) = Ax^2 + Bx + C$

Solving for A, B, C:

$$p_2(x_0) = Ax_0^2 + Bx_0 + C = a$$

$$p_2(x_1) = Ax_1^2 + Bx_1 + C = b$$

$$p_2'(x_1) = 2Ax_1 + B = c$$

- $A = \frac{c - B}{2x_1}$

- $\frac{c-B}{2x_1} x_1^2 + Bx_1 + C = \frac{cx_1}{2} + \frac{Bx_1}{2} + C = b$

$$\Rightarrow C = b - \frac{cx_1}{2} - \frac{Bx_1}{2}$$

- $Ax_0^2 + Bx_0 + C = Ax_0^2 + Bx_0 + b - \frac{cx_1}{2} - \frac{Bx_1}{2} = a$

$$= \frac{c-B}{2x_1} x_0^2 + Bx_0 + b - \frac{cx_1}{2} - \frac{Bx_1}{2} = a$$

$$\Rightarrow B(x_0 - x_1/2) + \frac{c}{2x_1} x_0^2 - \frac{B}{2x_1} x_0^2 + b - \frac{cx_1}{2}$$

$$= B(x_0 - x_1/2 - x_0^2/2x_1) + \frac{c}{2x_1} x_0^2 - \frac{cx_1}{2} + b$$

$$\Rightarrow B = \frac{a - \frac{c}{2x_1} x_0^2 - \frac{cx_1}{2} - b}{x_0 - x_1/2 - x_0^2/2x_1}$$

$$= \frac{2(a-b) - c\left(\frac{x_0^2}{x_1} - x_1\right)}{2x_0 - x_1 - x_0^2/x_1}$$

$$A = \frac{c - \left(\frac{2(a-b) - c(x_0^2/x_1 - x_1)}{2x_0 - x_1 - x_0^2/x_1}\right)}{2x_1}$$

$$C = b - \frac{cx_1}{2} - \frac{\left(\frac{2(a-b) - c(x_0^2/x_1 - x_1)}{2x_0 - x_1 - x_0^2/x_1}\right)}{2}$$

Problem 4:

Figures 1-3 show the algorithms of DivDif and Interpolate implemented in Rust, using these algorithms on the data from the problem, and the output of the code when executed. The coefficients obtained from the DivDif algorithm are: $f[x_0] = 7, f[x_0, x_1] = -5, f[x_0, x_1, x_2] = 2, f[x_0, x_1, x_2, x_3] = -2/3, f[x_0, x_1, x_2, x_3, x_4] = 0.29167, f[x_0, x_1, x_2, x_3, x_4, x_5] = -0.125$.

In the output is a test of the Interpolate algorithm on values within the interval of $[-2, 3]$, including interpolated points and not interpolated points, and some points outside of the interval.

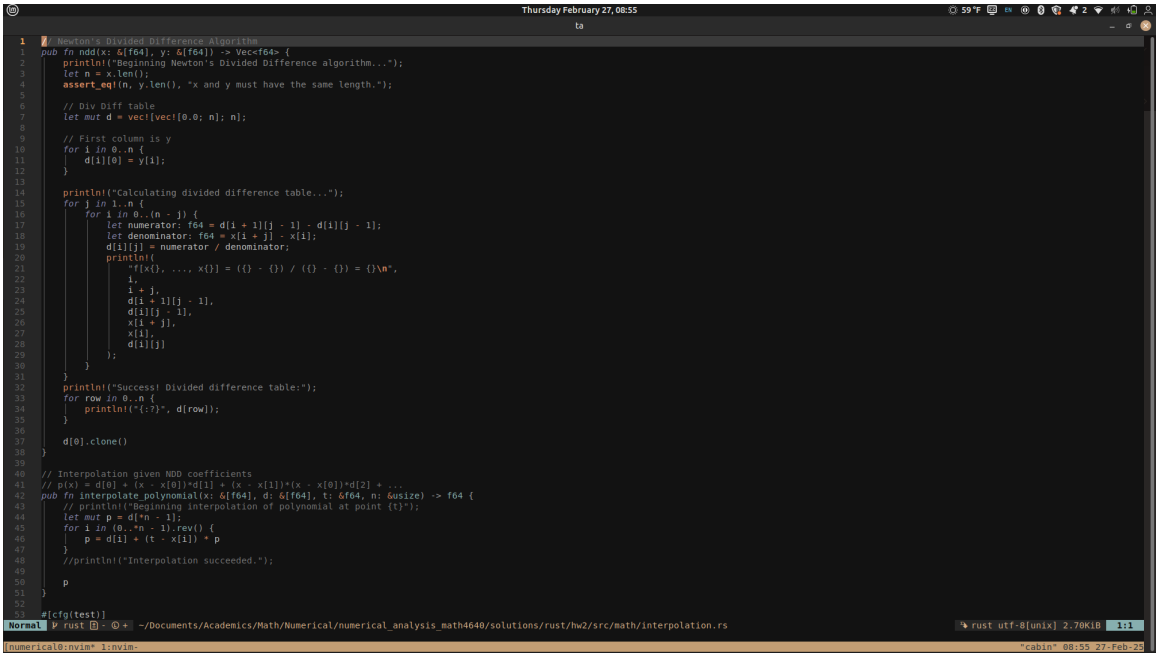

Figure 1: Screenshot of code for the Newton's Divided Difference (NDD/DivDif) algorithm (upper) and the Interpolate using the NDD coefficients (lower).

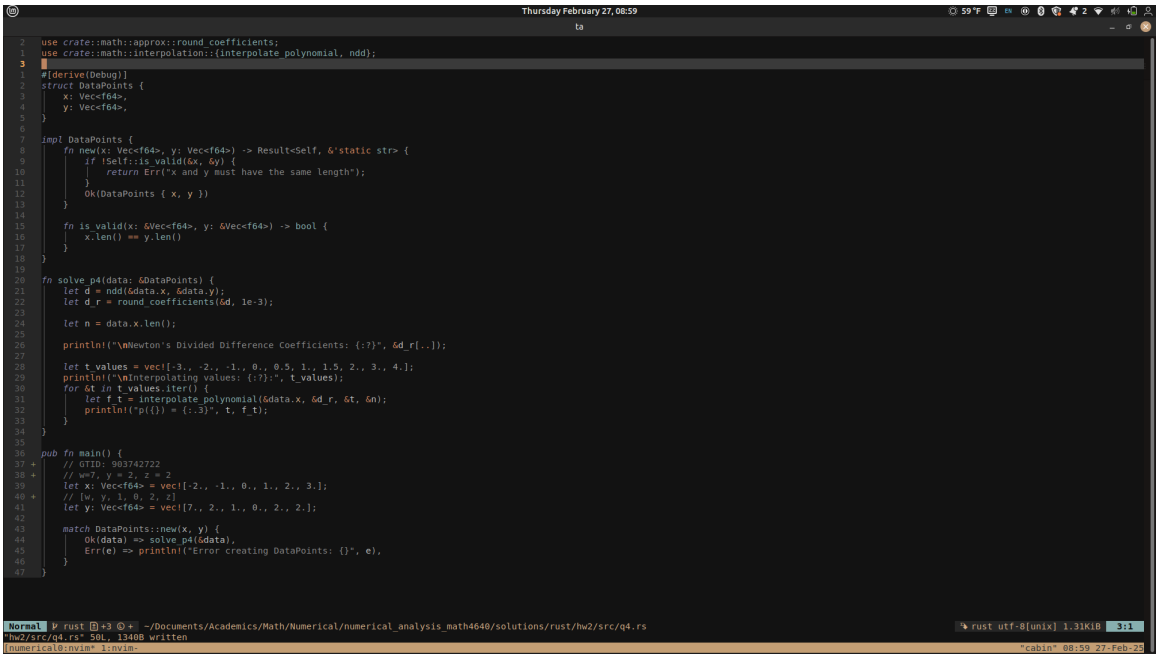

Figure 2: Screenshot of code using the DivDif and Interpolate algorithms. See Figure 1 for implementation of such algorithms.

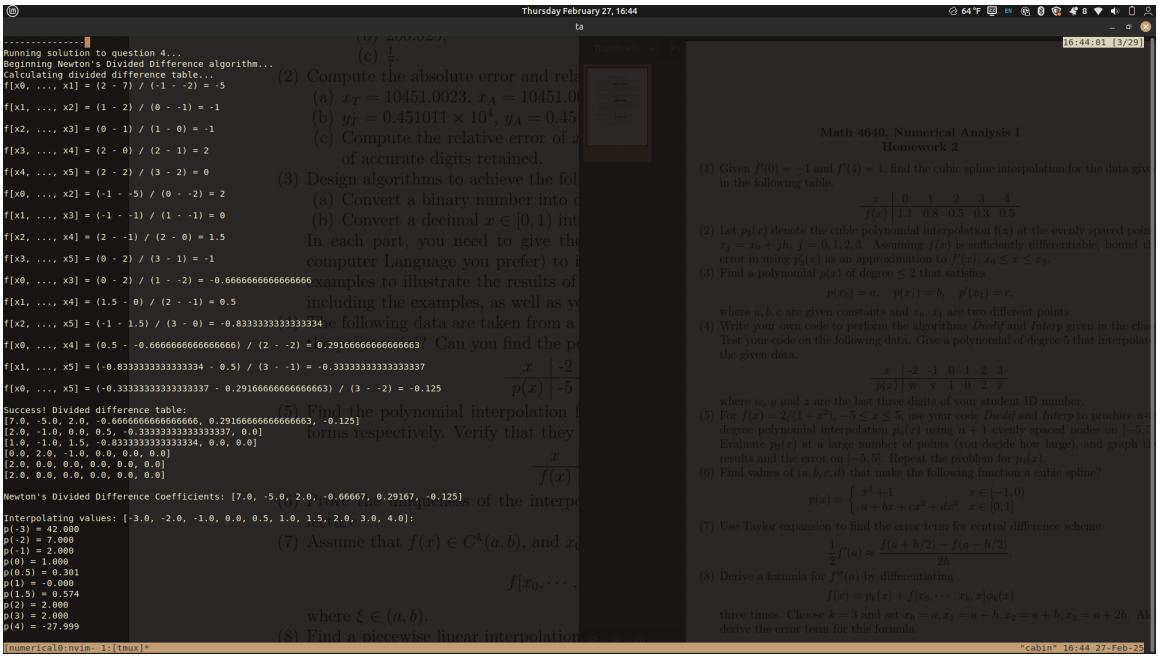

Figure 3: Screenshot of output from code ran in Figure 2. You can run this code for yourself to verify. Documentation in README file with instructions if needed.

Problem 5:

Solution shared below in Figures 3, 4, 5, 6. Figure 5 corresponds to the plot for degree 2 and Figure 6 corresponds to plot for degree 4. The algorithms used are shown above in Figure 1 implemented in Rust.

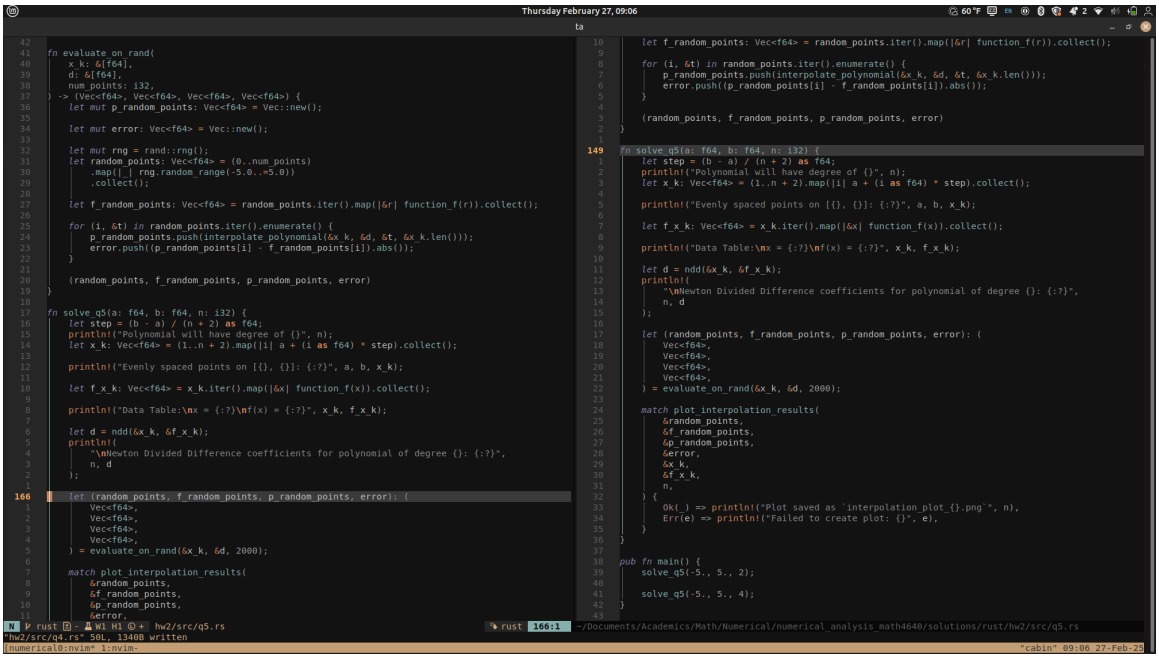

Figure 4: Screenshot of code using DivDif and Interpolate algorithms. See Figure 1 for implementations of such algorithms.

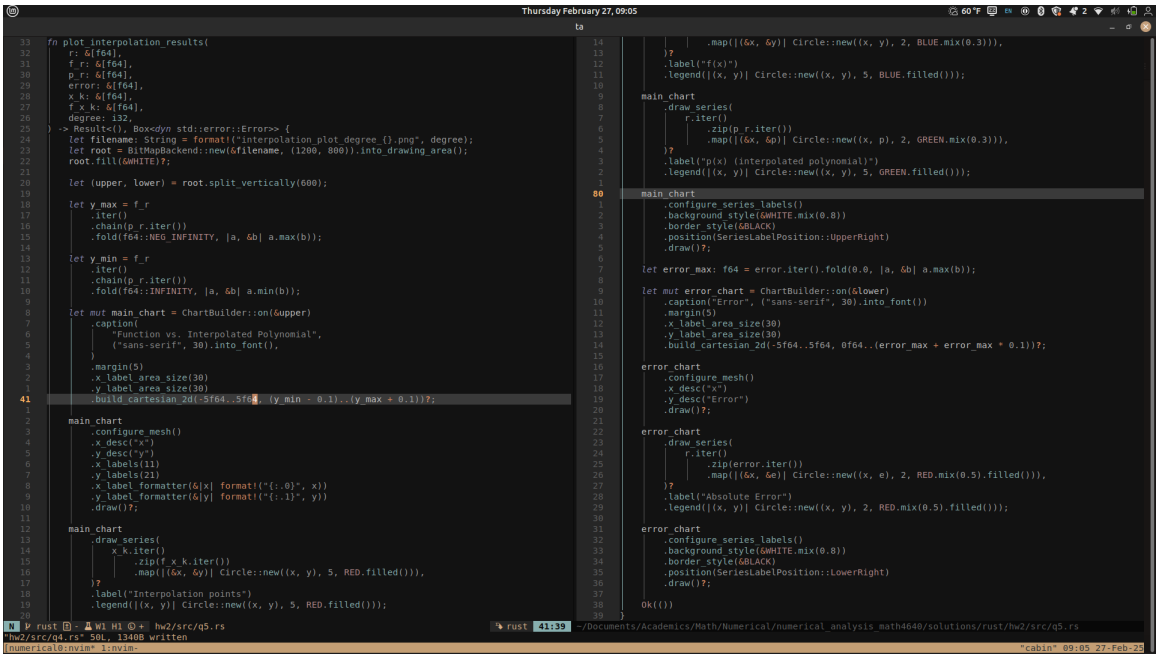

Figure 5: Screenshot of code plotting the true function, the interpolated polynomial, the interpolated points, and the error between the interpolated polynomial and true function.

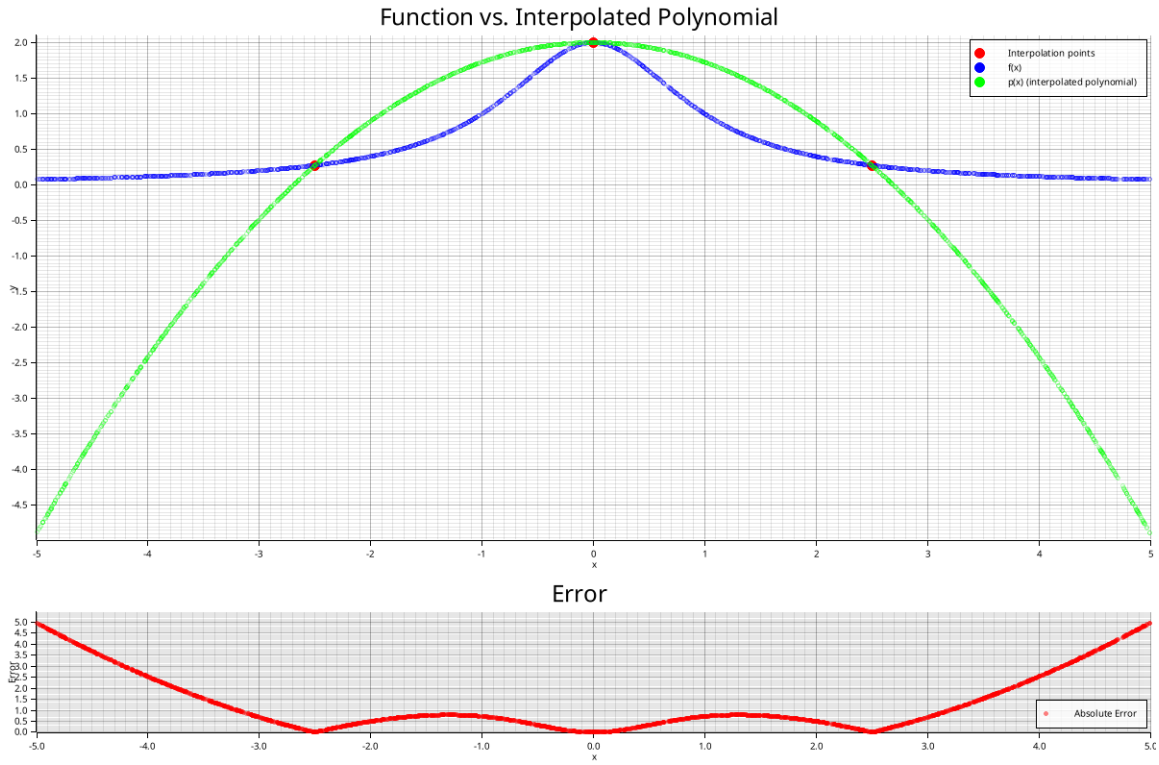

Figure 6: Plot for degree 2 polynomial and error.

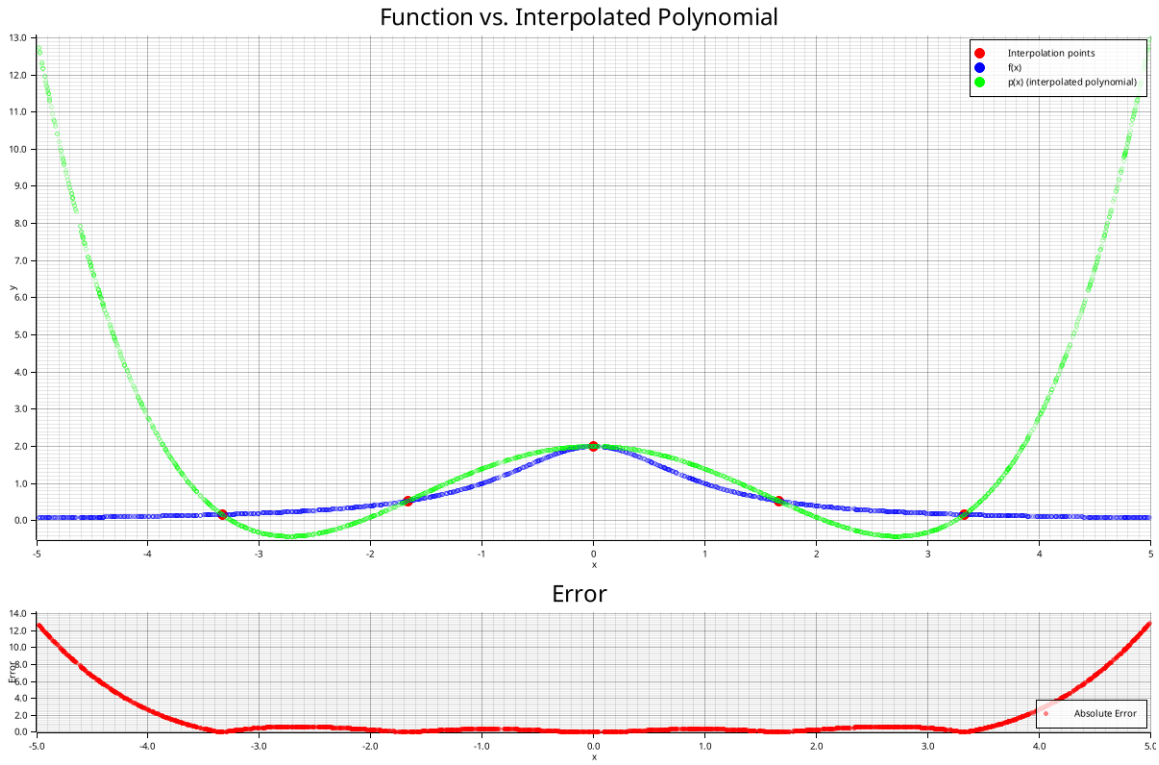

Figure 7: Plot for degree 4 polynomial and error.

Problem 6:

Question 6

For continuity, cubic splines require, in this case, that $p(0^+) = p(0^-)$ so

$a = 1$, since $p(x^-) = x^3 + 1$
$p(x^+) = a + bx + cx^2 + dx^3$

Now $p(x) = 1 + bx + cx^2 + dx^3$, $x \in [0,1]$

Additionally, the derivative must be continuous. Thus,

$p'(0^+) = p'(0^-)$
$p'(x^-) = 3x^2$
$p'(x^+) = b + 2cx + 3dx^2$

Thus, $b = 0$

And further, $p''(0^+) = p''(0^-)$

$p''(x^-) = 6x$
$p''(x^+) = 2c + 6dx$

Thus, $c = 0$.

Hence $p(x) = 1 + dx^3$ $x \in [0,1]$.

Therefore the values (a,b,c,d) are $(1,0,0,d)$ where $d \in \mathbb{R}$.

$\square$

Problem 7:

Question 7

$$f(a+h/2) = f(a) + f'(a)(a+h/2-a) + \frac{f^{(2)}(a)}{2}(a+h/2-a)^2 + \dots$$

$$= f(a) + f'(a)\frac{h}{2} + \frac{f^{(2)}(a)}{2!}\frac{h^2}{2^2} + \frac{f^{(3)}(a)}{3!}\frac{h^3}{2^3} + \dots$$

$$= f(a) + \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + \dots$$

$$f(a-h/2)$$

$$= f(a) + f'(a)\left(a-\frac{h}{2}-a\right) + \frac{f^{(2)}(a)}{2!}\left(a-\frac{h}{2}-a\right)^2 + \dots$$

$$= f(a) - \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + \dots$$

So,

$$f(a+h/2) = f(a) + \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} + \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + O(h^4)$$

$$f(a-h/2) = f(a) - \frac{h}{2}f'(a) + \left(\frac{h}{2}\right)^2\frac{f^{(2)}(a)}{2!} - \left(\frac{h}{2}\right)^3\frac{f^{(3)}(a)}{3!} + O(h^4)$$

$$f(a+h/2) - f(a-h/2) = hf'(a) + \frac{h^3}{4}\frac{f^{(3)}(a)}{3!} + O(h^5)$$

$$= hf'(a) + \frac{h^3}{24}f^{(3)}(a) + O(h^5)$$

Thus, $$\frac{f(a+h/2) - f(a-h/2)}{2h} = \frac{1}{2}f'(a) + \frac{h^2}{48}f^{(3)}(a) + O(h^4)$$

That is,

$$\frac{f(a+h/2) - f(a-h/2)}{2h} = \frac{1}{2}f'(a) + \frac{h^2}{48}f^{(2)}(a) + O(h^4)$$

Hence, the error between $\frac{1}{2}f'(a)$ and the approximation $\frac{1}{2h}\left(f(a+h/2) - f(a-h/2)\right)$ is

$$\left| \frac{1}{2}f'(a) - \frac{f(a+h/2) - f(a-h/2)}{2h} \right|$$

$$= \left| \frac{h^2}{48}f^{(2)}(a) + O(h^4) \right| = \frac{h^2}{48}\left|f^{(2)}(a)\right| + O(h^4)$$

The dominant term is h^2, thus, the order of the error is

$$O(h^2).$$

Problem 8:

Question 8

Recall $\frac{d}{dx} f[x_0, \dots, x_k, x] = f[x_0, \dots, x_k, x, x]$

$$f(x) = p_k(x) + f[x_0, x_1, \dots, x_k, x]\, e_k(x)$$

$$\left[f[x_0, \dots, x_k, x]\, e_k(x)\right]' = f[x_0, \dots, x_k, x, x]\, e_k(x) + f[x_0, \dots, x_k, x]\, e_k'(x)$$

which we've shown before.

$$\left[f[x_0, \dots, x_k, x]\, e_k(x)\right]^{(2)}$$

$$= \left[f[x_0, \dots, x_k, x, x]\, e_k(x) + f[x_0, \dots, x_k, x]\, e_k'(x)\right]'$$

$$= f[x_0, \dots, x_k, x, x, x]\, e_k(x) + f[x_0, \dots, x_k, x, x]\, e_k'(x)$$

$$+ f[x_0, \dots, x_k, x, x]\, e_k'(x) + f[x_0, \dots, x_k, x]\, e_k''(x)$$

$$\left[f[x_0, \dots, x_k, x]\, e_k(x)\right]^{(3)}$$

$$= \left[f[x_0, \dots, x_k, x, x, x]\, e_k(x) + f[x_0, \dots, x_k, x, x]\, e_k'(x) + f[x_0, \dots, x_k, x, x]\, e_k'(x) + f[x_0, \dots, x_k, x]\, e_k''(x)\right]'$$

$$= f[x_0, \dots, x_k, x, x, x, x]\, e_k(x) + f[x_0, \dots, x_k, x, x, x]\, e_k'(x)$$

$$+ f[x_0, \dots, x_k, x, x, x]\, e_k'(x) + f[x_0, \dots, x_k, x, x]\, e_k''(x)$$

$$+ f[x_0, \dots, x_k, x, x, x]\, e_k(x) + f[x_0, \dots, x_k, x, x]\, e_k'(x)$$

$$+ f[x_0, \dots, x_k, x, x]\, e_k''(x) + f[x_0, \dots, x_k, x]\, e_k'''(x)$$

$$= f[x_0, \ldots, x_k, x, x, x, x]\, e_k(x)$$

$$+ f[x_0, \ldots, x_k, x, x, x]\left(e_k(x) + 2e_k'(x)\right)$$

$$+ f[x_0, \ldots, x_k, x, x]\left(e_k'(x) + 2e_k''(x)\right)$$

$$+ f[x_0, \ldots, x_k, x]\, e_k^{(3)}(x)$$

$$= \frac{f^{(k+4)}(\xi)}{(k+4)!} e_k(x) + \frac{f^{(k+3)}(\eta)}{(k+3)!}\left(e_k(x) + 2e_k'(x)\right)$$

$$+ \frac{f^{(k+2)}(\gamma)}{(k+2)!}\left(e_k'(x) + 2e_k''(x)\right)$$

$$+ \frac{f^{(k+1)}(\rho)}{(k+1)!} e^{(3)}(x) = \psi(x)$$

for some $\xi, \eta, \gamma, \rho \in \left[\min\{x_0, \ldots, x_k, x\}, \max\{x_0, \ldots, x_k, x\}\right]$

So,

$f^{(3)}(x) = p_k^{(3)}(x) + \psi(x)$, where $\psi(x)$ is defined as above. ☐