HW 3

Question 1 Let $\{P_i(x)\}_{i=0}^{\infty}$ be a sequence of orthogonal polynomials and let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $P_{k+1}(x)$.

Prove that the Lagrange polynomials $l_i(x) = \prod_{j \neq i} (x - x_j)/(x_i - x_j)$, $i = 0, \ldots, k$ for these points are orthogonal to each other.

Hint: Show that for $i \neq j$, $l_i(x) l_j(x) = P_{k+1}(x) g(x)$, where $g(x)$ is some polynomial of degree $\leq k$.

Solution

$$l_i(x) = \prod_{\substack{t=0 \\ t \neq i}}^{k} \frac{x - x_t}{x_i - x_t} \quad \text{and} \quad l_j(x) = \prod_{\substack{s=0 \\ s \neq j}}^{k} \frac{x - x_s}{x_j - x_s}$$

$$l_i(x_t) = \begin{cases} 1 & t = i \\ 0 & t \neq i \end{cases} \qquad l_j(x_s) = \begin{cases} 1 & s = j \\ 0 & s \neq j \end{cases}$$

$\deg(l_i(x)) = \deg(l_j(x)) = k$. $\deg(l_i(x) l_j(x)) \leq 2k$

$$l_i(x) l_j(x) = \prod_{\substack{t=0 \\ t \neq i}}^{k} \frac{x - x_t}{x_i - x_t} \prod_{\substack{s=0 \\ s \neq j}}^{k} \frac{x - x_s}{x_j - x_s}$$

$$l_i(x_t) l_j(x_t) = \begin{cases} (1)(0) & t = i \\ (0)(1) & t = j \\ (0)(0) & t \neq i \text{ and } t \neq j \end{cases}$$

Thus, $l_i(x_t) l_j(x_t) = 0$ $\quad x_0, \ldots, x_k$

So $l_i(x) l_j(x)$ is a degree $\leq 2k$ polynomial with $k+1$ distinct roots at $x_0, \ldots, x_k$.

This means, for some $k-1$ degree or less polynomial $g(x)$

$l_i(x) l_j(x) = P_{k+1}(x) g(x)$

But $P_{k+1}(x)$ is orthogonal $k+1$ degree, so $\langle P_{k+1}, g \rangle = 0$

Hence $\langle l_i, l_j \rangle = 0$ ∎

Question 2

Approximation by Orthogonal Projection

The best polynomial $p^*(x)$ is given by projecting onto the basis of:

- Legendre polynomials for $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$
- Chebyshev polynomials for $\langle f,g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{(1-x^2)^{1/2}}\,dx$
- Hermite polynomials for $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$

For up to degree 2 with our objective being to minimize

$$\int_{-1}^{1}(\cos(\pi x) - p(x))^2\,dx$$

we note this corresponds to $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ inner product.

$$p_2^*(x) = \sum_{i=0}^{2} \frac{\langle \cos(\pi x), L_i\rangle}{\langle L_i, L_i\rangle} L_i(x)$$

$$L_0 = 1,\quad L_1(x) = x,\quad L_2(x) = \frac{3xL_1(x) - L_0(x)}{2} = \frac{3}{2}x^2 - \frac{1}{2}$$

As $L_{n+1}(x) = \frac{(2n+1)xL_n(x) - nL_{n-1}(x)}{n+1}$ is Legendre polynomial general form.

Further, $\langle L_i, L_i\rangle = \frac{2}{2i+1}$. Thus,

$$p_2^*(x) = \frac{\langle \cos(\pi x), 1\rangle}{2} + \frac{\langle \cos(\pi x), x\rangle}{\frac{2}{3}} x$$

$$+ \frac{\langle \cos(\pi x), \frac{3}{2}x^2 + \frac{1}{2}\rangle}{\frac{2}{5}}\left(\frac{3}{2}x^2 + \frac{1}{2}\right)$$

- $\langle \cos(\pi x), 1 \rangle = \int_{-1}^{1} \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)\Big|_{-1}^{1}$

$= \frac{1}{\pi}\left(\sin(\pi) - \sin(-\pi)\right) = \frac{1}{\pi}\left(\sin(\pi) + \sin(\pi)\right) = 0$

- $\langle \cos(\pi x), x \rangle = \int_{-1}^{1} \cos(\pi x)\,x\,dx = 0$ Steps below

$\int u\,dv = uv - \int v\,du$ $\quad u = x,\ du = dx \quad v = \frac{1}{\pi}\sin(\pi x),\ dv = \cos(\pi x)\,dx$

$\int x\cos(\pi x)\,dx = \frac{x}{\pi}\sin(\pi x) - \frac{1}{\pi}\int \sin(\pi x)\,dx$

$\int_{-1}^{1} x\cos(\pi x)\,dx = \frac{x}{\pi}\sin(\pi x)\Big|_{-1}^{1} + \frac{1}{\pi^2}\left(\cos(\pi x)\Big|_{-1}^{1}\right)$

$= \frac{1}{\pi^2}\left(\cos(\pi) - \cos(-\pi)\right) = \frac{1}{\pi^2}(0) = 0$

- $\langle \cos(\pi x), \frac{3}{2}x^2 - \frac{1}{2} \rangle = \frac{3}{2}\int_{-1}^{1} x^2\cos(\pi x)\,dx - \frac{1}{2}\int_{-1}^{1}\cos(\pi x)\,dx$

- $-\frac{1}{2}\int_{-1}^{1}\cos(\pi x)\,dx = 0$
- $\frac{3}{2}\int_{-1}^{1} x^2\cos(\pi x)\,dx = \frac{3}{2}\left(-\frac{4}{\pi^2}\right) = -\frac{12}{2\pi^2} = \underline{\underline{-6\pi^{-2}}}$

$u = x^2 \quad v = \frac{1}{\pi}\sin(\pi x)$
$du = 2x\,dx \quad dv = \cos(\pi x)\,dx$

$\int_{-1}^{1} x^2\cos(\pi x)\,dx = \frac{x^2}{\pi}\sin(\pi x)\Big|_{-1}^{1} - \frac{1}{\pi}\int_{-1}^{1}\sin(\pi x)\,2x\,dx = -\frac{1}{\pi}\int_{-1}^{1}\sin(\pi x)\,2x\,dx$

$-\frac{1}{\pi}\int_{-1}^{1}\sin(\pi x)\,2x\,dx = -\frac{2}{\pi}\left(\frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2}\right)\Big|_{-1}^{1} = -\frac{2}{\pi}\left(\frac{2}{\pi}\right)$

$= -4/\pi^2$

- $\langle \cos(\pi x), 1 \rangle = 0$
- $\langle \cos(\pi x), x \rangle = 0$
- $\langle \cos(\pi x), \frac{3}{2}x^2 - \frac{1}{2} \rangle = -6\pi^{-2}$

$$p_2^*(x) = \frac{-6\pi^{-2}}{\frac{2}{5}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{-30}{2\pi^2}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$= -15\pi^{-2}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{-45\pi^{-2}}{2}x^2 + \frac{15\pi^{-2}}{2}$$

$$p_2^*(x) = -\frac{45}{2\pi^2}x^2 + \frac{15}{2\pi^2}. \quad \blacksquare$$

Question 3 If $f(x)$ is a 2π periodic function, prove that $g_\alpha(x) = f(x+\alpha)$ is also 2π-periodic.

Proof.
We want to show $g_\alpha(x+2\pi) = g_\alpha(x) \quad \forall x$.

$$g_\alpha(x+2\pi) = f((x+2\pi)+\alpha) = f(x+\alpha+2\pi)$$

Since f is 2π periodic, $f((x+\alpha)+2\pi) = f(x+\alpha)$

Hence, $g_\alpha(x+2\pi) = g_\alpha(x)$, as $f(x+\alpha) = g_\alpha(x)$.

Therefore, $g_\alpha(x)$ is also 2π-periodic $\blacksquare$

$$\hat{f}(j) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ijx}\,dx \qquad \hat{g}_\alpha(j) = \frac{1}{2\pi}\int_0^{2\pi} g_\alpha(x)\, e^{-ijx}\,dx$$

$$\hat{g}_\alpha(j) = \frac{1}{2\pi}\int_0^{2\pi} f(x+\alpha)\, e^{-ijx}\,dx$$

$u = x+\alpha \qquad du = dx \qquad$ so $\quad 0 \mapsto \alpha, \quad 2\pi \mapsto 2\pi+\alpha$

$x = u - \alpha$

$$\hat{g}_\alpha(j) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} f(u)\, e^{-ij(u-\alpha)}\,du$$

$f(u)$ is 2π-periodic, so $\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} f(u) e^{-ij(u-\alpha)}\,du = \frac{1}{2\pi}\int_0^{2\pi} f(u) e^{-ij(u-\alpha)}\,du$

And thus, $\hat{g}_\alpha(j) = \frac{1}{2\pi}\int_0^{2\pi} f(u)\, e^{-iju}\, e^{ij\alpha}\,du$

which is

$$\hat{g}_\alpha(j) = \frac{e^{ij\alpha}}{2\pi}\int_0^{2\pi} f(u)\, e^{-iju}\,du$$

Thus, $\hat{g}_\alpha(j) = e^{ij\alpha}\hat{f}(j)$. $\blacksquare$

Question 4 Verify that the 2π-periodic function $f(x)$ whose values on $[0, 2\pi)$ are given by

$$f(x) = \begin{cases} (x/\pi)^2 - x/\pi & 0 \leq x < \pi \\ (x-\pi)/\pi - \left(\frac{x-\pi}{\pi}\right)^2 & \pi \leq x < 2\pi \end{cases}$$

is continuous and has a continuous first derivative (as a 2π-periodic) but has jumps in the second derivative. Then construct the spectrum of $f(x)$ and show that it decays like j^{-3}, $j \to \infty$.

Solution

$$f'(x) = \begin{cases} \frac{2x}{\pi^2} - \frac{1}{\pi}, & x \in [0, \pi) \\ \frac{1}{\pi} - \frac{2(x-\pi)}{\pi^2}, & x \in [\pi, 2\pi) \end{cases}$$

$$f''(x) = \begin{cases} \frac{2}{\pi^2}, & x \in [0, \pi) \\ -\frac{2}{\pi^2}, & x \in [\pi, 2\pi) \end{cases}$$

- $\lim_{x \to \pi^-} f(x) = \left(\frac{\pi}{\pi}\right)^2 - \frac{\pi}{\pi} = 1 - 1 = 0$
- $\lim_{x \to \pi^+} f(x) = \frac{\pi - \pi}{\pi} - \left(\frac{\pi - \pi}{\pi}\right)^2 = 0$

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$, f is continuous over $[0, 2\pi)$

- $\lim_{x \to \pi^-} f'(x) = \frac{2\pi}{\pi^2} - \frac{1}{\pi} = \frac{1}{\pi}$
- $\lim_{x \to \pi^+} f'(x) = \frac{1}{\pi} - 0 = \frac{1}{\pi}$

$\lim_{x \to \pi^-} f'(x) = \lim_{x \to \pi^+} f'(x)$, f' is continuous over $[0, 2\pi)$

- $\lim_{x \to \pi^-} f''(x) = \frac{2}{\pi^2} \neq \lim_{x \to \pi^+} f''(x) = -\frac{2}{\pi^2}$ f'' is not contin. over $[0, 2\pi)$.

$$\hat{f}(j) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ijx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \left(\frac{x}{\pi}\right)^2 e^{-ijx} - \frac{x}{\pi} e^{-ijx}\,dx + \int_0^{\pi} \frac{x-\pi}{\pi} e^{-ijx} - \frac{(x-\pi)^2}{\pi^2} e^{-ijx}\,dx\right]$$

Or if using $p^*(x) = \sum_{j=1}^{n} a_j \sin jx + \sum_{j=0}^{n} b_j \cos jx$ form instead, where:

$a_j = \langle f, \sin jx \rangle$

$b_j = \langle f, \cos jx \rangle$

$$a_j = \frac{1}{2\pi}\int_0^{2\pi} f(x) \sin jx\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} \left(\frac{x^2}{\pi^2} - \frac{x}{\pi}\right) \sin jx\,dx + \int_{\pi}^{2\pi} \left(\frac{x-\pi}{\pi} - \frac{(x-\pi)^2}{\pi^2}\right) \sin jx\,dx\right]$$

$$b_j = \frac{1}{2\pi}\int_0^{2\pi} f(x) \cos jx\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} \left(\frac{x^2}{\pi^2} - \frac{x}{\pi}\right) \cos jx\,dx + \int_{\pi}^{2\pi} \left(\frac{x-\pi}{\pi} - \frac{(x-\pi)^2}{\pi^2}\right) \cos jx\,dx\right]$$ 3

$f \in C^1[0, 2\pi]$ since $f^{(2)}(x)$ is not continuous over $[0, 2\pi]$ but $f, f^{(1)}$ are. -1

Thus, $|\hat{f}(j)| = O(|j|^{-2})$ ∎

Question 6

First need to determine the orthogonal polynomials associated with $\omega(x) = -x$ over $[-1, 0]$.

Satisfy: $\int_{-1}^{0} P_m(x) P_n(x) \omega(x)\,dx = 0$ for $m \neq n$.

These are the Legendre polynomials shifted by $x \mapsto -2x - 1$

Let $\tilde{P}_i(x)$ be non-shifted Legendre poly. $[-1, 0] \longrightarrow [-1, 1]$

So, $P_0(x) = 1$, $P_1(x) = \tilde{P}_1(2x+1) = 2x + 1$

$$P_2(x) = \tilde{P}_2(2x+1) = \tfrac{1}{2}(3(2x+1)^2 - 1) = \tfrac{3}{2}(4x^2 + 4x + 1) - \tfrac{1}{2}$$
$$= 6x^2 + 6x + 1$$

(n=1) One-Point Quadrature

$P_1(x) = 2x + 1$. The quadrature node is the root of $P_1(x)$, which is $x_0 = -1/2$

The weight is $A_0 = \int_{-1}^{0} (-x) l_0(x)\,dx = \int_{-1}^{0} -x\,dx = -\frac{x^2}{2}\Big|_{-1}^{0} = 1/2$

The one-point Gaussian quadrature rule is

$$I \approx \tfrac{1}{2} f(x_0) = \tfrac{1}{2} f(-1/2)$$

(n=2) Two-Point Quadrature

$P_2(x) = 6x^2 + 6x + 1$. Solving $6x^2 + 6x + 1 = 0$ we find,

$$x = \frac{-6 \pm \sqrt{36 - 24}}{12} = \frac{-6 \pm \sqrt{12}}{12} = \frac{-6 \pm 2\sqrt{3}}{12} = -\frac{1}{2} \pm \frac{1}{6}\sqrt{3}$$

$$x_0 = -\frac{1}{2} - \frac{1}{6}\sqrt{3}, \quad x_1 = -\frac{1}{2} + \frac{1}{6}\sqrt{3}$$

$$A_0 = \int_{-1}^{0} -x\, l_0(x)\,dx = \int_{-1}^{0} -x \frac{x - x_1}{x_0 - x_1}\,dx = \frac{1}{x_0 - x_1}\int_{-1}^{0} -x^2 + x x_1\,dx = \frac{1}{x_0 - x_1}\left[\frac{-x^3}{3} + \frac{x^2}{2}x_1\right]_{-1}^{0}$$

$$A_0 = \frac{\frac{1}{3} + \frac{1}{2}x_1}{x_0 - x_1}$$

$$A_1 = \int_{-1}^{0} -x\, l_1(x)\,dx = \int_{-1}^{0} -x \frac{x - x_0}{x_1 - x_0}\,dx = \frac{1}{x_1 - x_0}\left[\frac{-x^3}{3} + \frac{x^2}{2}x_0\right]_{-1}^{0} = \frac{\frac{1}{3} + \frac{1}{2}x_0}{x_1 - x_0}$$

The Two-Point Quadrature Rule is thus,

$$A_0 f(x_0) + A_1 f(x_1) = \frac{\frac{1}{3} + \frac{1}{2}x_1}{x_0 - x_1} f(x_0) + \frac{\frac{1}{3} + \frac{1}{2}x_0}{x_1 - x_0} f(x_1)$$

- $\frac{1}{3} + \frac{1}{2}x_1 = \frac{1}{3} - \frac{1}{4} + \frac{\sqrt{3}}{12} = \frac{1+\sqrt{3}}{12}$
- $\frac{1}{3} + \frac{1}{2}x_0 = \frac{1}{3} - \frac{1}{4} - \frac{\sqrt{3}}{12} = \frac{1-\sqrt{3}}{12}$
- $x_0 - x_1 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{6}\right) - \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}\right) = -\sqrt{3}/3$
- $x_1 - x_0 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}\right) - \left(-\frac{1}{2} - \frac{\sqrt{3}}{6}\right) = \sqrt{3}/3$

Final Answer:

$$\frac{1 + \frac{\sqrt{3}}{12}}{-\frac{\sqrt{3}}{3}} f\left(-\frac{1}{2} - \frac{\sqrt{3}}{6}\right) + \frac{1 - \frac{\sqrt{3}}{12}}{\frac{\sqrt{3}}{3}} f\left(-\frac{1}{2} + \frac{\sqrt{3}}{6}\right)$$

Question 7

The corrected trapezoidal rule is

$$I(f) = \int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\big(f(a)+f(b)\big) + \frac{(b-a)^2}{12}\big(f'(a)-f'(b)\big)$$

For evenly spaced points we let $x_i = x_0 + ih$ for $x_0, \dots, x_N$ and $h = \frac{b-a}{N}$.

So $x_{i+1} - x_i = h$, $\forall i = 0, \dots, N-1$. Now, we want to compute

$$I(f) = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

But for the corrected trapezoidal rule,

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}(x_{i+1}-x_i)\big(f(x_i)+f(x_{i+1})\big) + \frac{(x_{i+1}-x_i)^2}{12}\big(f'(x_i)-f'(x_{i+1})\big)$$

$$= \frac{1}{2}h\big(f(x_i)+f(x_{i+1})\big) + \frac{h^2}{12}\big(f'(x_i)-f'(x_{i+1})\big)$$

So,

$$I(f) \approx \sum_{i=0}^{N-1}\left[\frac{h}{2}\big(f(x_i)+f(x_{i+1})\big) + \frac{h^2}{12}\big(f'(x_i)-f'(x_{i+1})\big)\right]$$

$$= \frac{h}{2}\sum_{i=0}^{N-1}\big[f(x_i)+f(x_{i+1})\big] + \frac{h^2}{12}\sum_{i=0}^{N-1}\big[f'(x_i)-f'(x_{i+1})\big]$$

Answer $\Rightarrow$ $$= \frac{h}{2}\big(f(a)+f(b)\big) + h\sum_{i=1}^{N-1} f(x_i) + \frac{h^2}{12}\big(f'(a)-f'(b)\big)$$

Since $\sum_{i=0}^{N-1}\big[f'(x_i)-f'(x_{i+1})\big]$

$$= f'(x_0) - f'(x_1) + f'(x_1) - f'(x_2) + \dots - f'(x_{N-1}) + f'(x_{N-1}) - f'(x_N)$$
$$= f'(x_0) - f'(x_N) = f'(a) - f'(b)$$