Problem 5

Lagrange Basis

$(x_0, f(x_0)) = (1, 3)$
$(x_1, f(x_1)) = (3/2, 13/4)$
$(x_2, f(x_2)) = (0, 3)$
$(x_3, f(x_3)) = (2, 5/3)$

$$\boxed{l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}}$$

$l_0(x)$: denom $= (1 - 3/2)(1-0)(1-2) = 0.5 = 1/2$
num $= (x - 3/2)(x)(x-2) = x^3 - \frac{7}{2}x^2 + 3x$

$l_1(x)$: denom $= (\frac{3}{2} - 1)(\frac{3}{2})(-\frac{1}{2}) = -0.375 = -3/8$
num $= (x-1)(x)(x-2) = x^3 - 3x^2 + 2x$

$l_2(x)$: denom $= (-1)(-3/2)(-2) = -3$
num $= (x-1)(x - 3/2)(x-2) = x^3 - \frac{9}{2}x^2 + \frac{13}{2}x - 3$

$l_3(x)$: denom $= (2-1)(2 - 3/2)(2) = 1$
num $= (x-1)(x-3/2)(x) = x^3 - \frac{5}{2}x^2 + \frac{3}{2}x$

$$l_0(x) = \frac{x^3 - \frac{7}{2}x^2 + 3x}{\frac{1}{2}} = 2x^3 - 7x^2 + 6x$$

$$l_1(x) = \frac{x^3 - 3x^2 + 2x}{-\frac{3}{8}} = -\frac{8}{3}(x^3 - 3x^2 + 2x) = -\frac{8}{3}x^3 + 8x^2 - \frac{16}{3}x$$

$$l_2(x) = \frac{x^3 - \frac{9}{2}x^2 + \frac{13}{2}x - 3}{-3} = -\frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{13}{6}x + 1$$

$$l_3(x) = \frac{x^3 - \frac{5}{2}x^2 + \frac{3}{2}x}{1} = x^3 - 5/2\,x^2 + 3/2\,x$$

Lagrange Formula

$$p_3(x) = \sum_{k=0}^{3} f(x_k)\, l_k(x) \quad \text{, where}$$

$f(x_0)l_0(x) = 6x^3 - 21x^2 + 18x$, $f(x_0) = 3$

$f(x_1)l_1(x) = -\frac{26}{3}x^3 + 26x^2 - \frac{52}{3}x$, $f(x_0) = 13/4$

$f(x_2)l_2(x) = -x^3 + \frac{9}{2}x^2 - \frac{13}{2}x + 3$, $f(x_1) = 3$

$f(x_3)l_3(x) = \frac{5}{3}x^3 - \frac{25}{6}x^2 + \frac{5}{2}x$, $f(x_2) = 5/3$

Simplifying,

$$6x^3 - 21x^2 + 18x - \frac{26}{3}x^3 + 26x^2 - \frac{52}{3}x$$

$$-x^3 + \frac{9}{2}x^2 - \frac{13}{2}x + 3 + \frac{5}{3}x^3 - \frac{25}{6}x^2 + \frac{5}{2}x$$

$$= \left(6 - 1 - \frac{26}{3} + \frac{5}{3}\right)x^3 + \left(-21 + 26 + \frac{9}{2} - \frac{25}{6}\right)x^2$$

$$+ \left(18 - \frac{52}{3} - \frac{13}{2} + \frac{5}{2}\right)x + 3$$

$$= -2x^3 + \frac{16}{3}x^2 - \frac{10}{3}x + 3.$$

$$p_3(x) = -2x^3 + \frac{16}{3}x^2 - \frac{10}{3}x + 3$$

Newton's Divided Difference

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{\frac{13}{4} - 3}{\frac{3}{2} - 1} = \frac{1}{2}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{3 - 13/4}{-3/2} = \frac{-1/4}{-3/2} = \frac{1}{6}$$

So, $f[x_0, x_1, x_2] = \dfrac{\frac{1}{6} - \frac{1}{2}}{-1} = 1/3$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{\frac{5}{3} - 3}{2} = -\frac{2}{3}$$

So, $f[x_1, x_2, x_3] = \dfrac{-\frac{2}{3} - \frac{1}{6}}{2 - \frac{3}{2}} = -\dfrac{5}{3}$

Thus,

$$f[x_0, x_1, x_2, x_3] = \frac{-\frac{5}{3} - \frac{1}{3}}{2 - 1} = -2$$

Divided Difference Summary

$$f[x_0,x_1] = \frac{13/4 - 3}{1/2} = 1/2$$

$$f[x_1,x_2] = \frac{3 - 13/4}{-3/2} = 1/6$$

$$f[x_2,x_3] = \frac{5/3 - 3}{2} = -2/3$$

$$f[x_0,x_1,x_2] = \frac{\frac{1}{6} - \frac{1}{2}}{-1} = 1/3$$

$$f[x_1,x_2,x_3] = \frac{-\frac{2}{3} - \frac{1}{6}}{2 - 3/2} = -5/3$$

$$f[x_0,x_1,x_2,x_3] = \frac{-\frac{5}{3} - \frac{1}{3}}{2-1} = -2.$$

Newton's Formula

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

- $(x-x_0) = (x-1)$
- $(x-x_0)(x-x_1) = (x-1)(x-\frac{3}{2}) = x^2 - \frac{5}{2}x + \frac{3}{2}$
- $(x-x_0)(x-x_1)(x-x_2) = (x^2 - \frac{5}{2}x + \frac{3}{2})(x) = x^3 - \frac{5}{2}x^2 + \frac{3}{2}x$

$$p_3(x) = 3 + \left(\frac{1}{2}\right)(x-1) + \left(\frac{1}{3}\right)\left(x^2 - \frac{5}{2}x + \frac{3}{2}\right)$$
$$+ (-2)\left(x^3 - \frac{5}{2}x^2 + \frac{3}{2}x\right)$$

$$= 3 + \frac{1}{2}x - \frac{1}{2} + \frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{2} - 2x^3 + 5x^2 - 3x$$
$$= -2x^3 + 16/3\,x^2 - 10/3\,x + 3$$

Lagrange Polynomial and Newton are equivalent