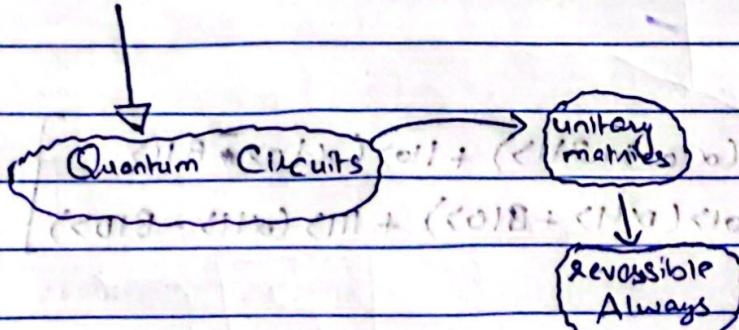
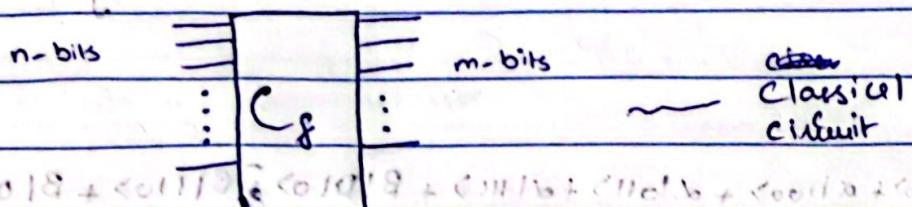


Circuit Classical \rightarrow Quantum Circuit

$$f: \{0,1\}^n \rightarrow \{0,1\}^m \text{ ~~~ classical function}$$



Universal gate set.

{And, NOT}

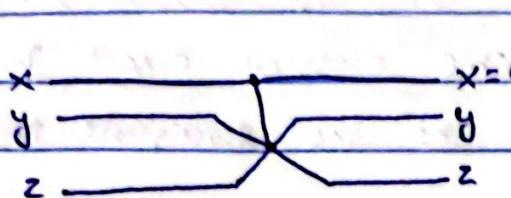
All classical circuit can be drawn only using And & NOT.



And gate is not reversible.

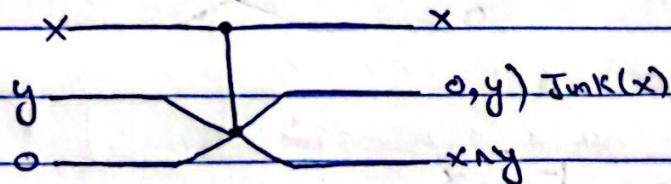
CHAP 6 - CNOT + CPHASE

Controlled swap gate (To create reversible And gate)



x	y	$x=0$	$x=0$	$x=1$	x
z	z	y	z	y	z

Now to create And gate using C-Swap gate, we set $z=0$, In this case our first output will be x , the second will be either 0 or y depending on the value of x .



We often refer to the second output as junk which is a function of x Junk(x)

Truth Table		xNy	x	y	
0	0	0	0	0	
0	1	0	0	1	
1	0	1	1	0	
1	1	1	1	1	

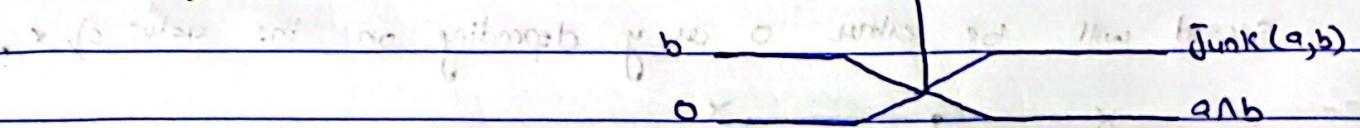
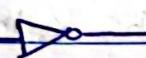
Notice that when x is 0, our output is 0 regardless of the value of y . In contrast when $x=1$ our output will be equal to y . Thus our output will be equal to 1 if and only if x and y are 1, which precisely

make the behaviour of AND gate.

now we have our universal set of classical reversible gates. We can create a reversible classical circuit for function f , denoted as R_f using the reversible NOT and Control swap gate

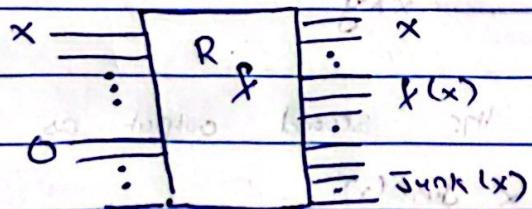
NOT

Control swap



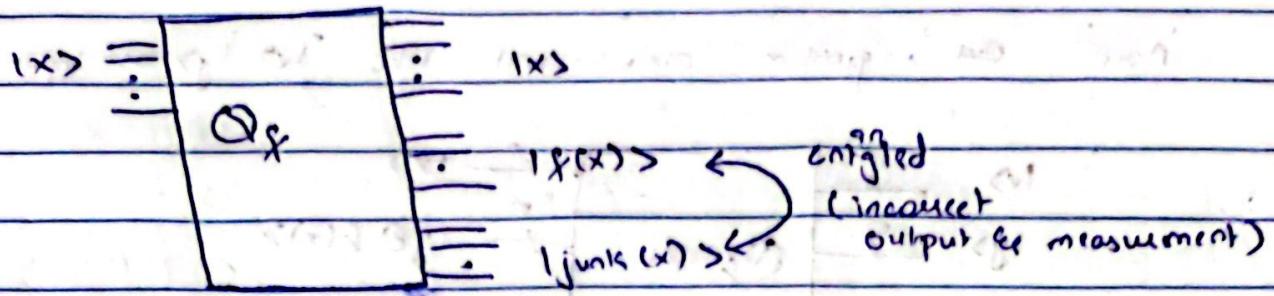
Junk(a,b)

a**b**



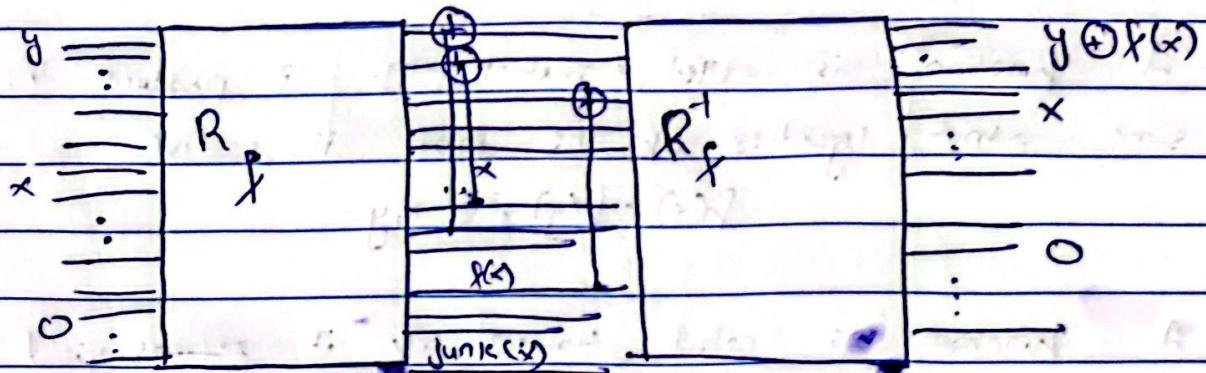
The input for circuit R_f will be x along with some 0s. The output will consist of x , $f(x)$ and what we were been referring to as junk. which is also a function of x .

The junk can be problematic if we convert a classical reversible circuit with this junk into quantum circuit. It's called it of the junk could become entangled with the output, causing either positive or negative interference.



Therefore we must ~~also~~ modify our classical available circuit to eliminate the junk to ensure the quantum circuit function correctly.

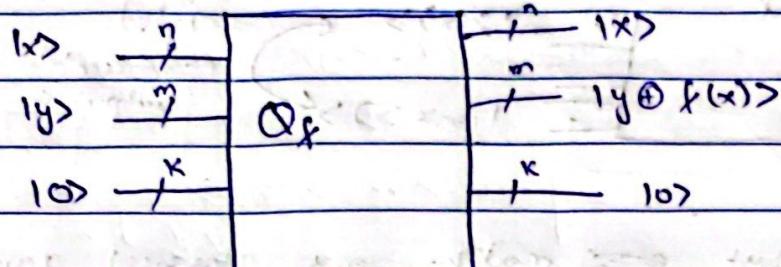
In our redesign process to eliminate junk we feed the output of R_f into its inverse R_f^{-1} . Thus converting the output back to original input, and getting rid of junk.



However this process has also eliminated $f(x)$ to prevent the value of $f(x)$ we introduce controlled NOT gate (\oplus)



Now our quantum circuit will take $|x\rangle |y\rangle$ as input



Deutsch Algorithm

Problem Definition:

Given a function $f : \{0,1\} \rightarrow \{0,1\}$ and yields a single bit output $f(0,1) \rightarrow \{0,1\}$
 our goal to determine whether the function f is constant or balanced.

A function is called constant if it produce the same output regardless of the input it receive

$$f(x) = f(y), \forall x, y$$

A function is called balance if it ~~yields~~ yield 0 for one input and 1 for other half

$f(x) = 0$ for half of input

$f(x) = 1$ for other

Constant

x	f(x)
0	0
1	0

Case 1

x	f(x)
0	1
1	1

Case 2

Balanced :-

x	f(x)
0	0
1	1

Case 3

x	f(x)
0	1
1	0

Case 4

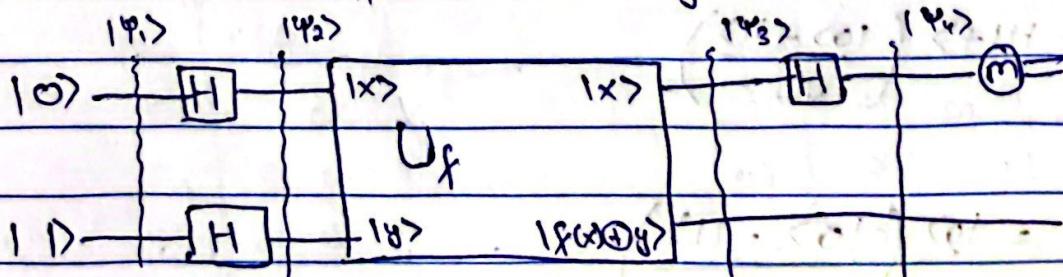
Quantum

- 1 cell to $\gamma(x)$
- Speed-up via superposition
- of all possible input.

Classical

- 2 cell to $\gamma(x) \rightarrow$ key
- If both are same then f is constant otherwise balanced.

Quantum Circuit for deutsch algorithm.



- Claim we measur. 0 on first qubit the Junction must
- be constant, other balanced

$$|14_1\rangle = |10\rangle |11\rangle = |101\rangle$$

$$|14_2\rangle = H|10\rangle \quad H|11\rangle$$

$$= \frac{|10\rangle + |11\rangle}{\sqrt{2}} \quad \frac{|10\rangle - |11\rangle}{\sqrt{2}}$$

$$= \frac{1}{2} [|100\rangle \downarrow \downarrow \downarrow \downarrow + |101\rangle \downarrow \downarrow \downarrow \downarrow - |110\rangle \downarrow \downarrow \downarrow \downarrow - |111\rangle \downarrow \downarrow \downarrow \downarrow]$$

$$|14_3\rangle = \frac{1}{2} \left[|10\rangle |f(0)\oplus 0\rangle - |10\rangle |f(0)\oplus 1\rangle \right]$$

$$+ |11\rangle |f(1)\oplus 0\rangle - |11\rangle |f(1)\oplus 1\rangle]$$

$$\therefore U_f |1x\rangle |1y\rangle = |1x\rangle |f(x)\oplus y\rangle$$

Case - 1

$$\text{Case - 1 } |14_3\rangle = \frac{1}{2} \left[|10\rangle |0\rangle - |10\rangle |1\rangle + |11\rangle |0\rangle - |11\rangle |1\rangle \right] \quad f(0) = f(1) = 0$$

$$= \frac{1}{2} \left[|10\rangle (|0\rangle - |1\rangle) + |11\rangle (|0\rangle - |1\rangle) \right]$$

$$|14_4\rangle = H \left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) + \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right)$$

$$= H |1+\rangle \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right)$$

$$= |10\rangle \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right)$$

(M) 1 bit = 100 probability.

$\text{H1 } 1+> = 10>$ so $\textcircled{M} 10>$

Constant = $14_3> = \pm 1+>1->$

$\text{17 } \text{Balance in } 14_3> = \pm 1->1->$

$\text{H1}-> = 11>$ so $\textcircled{M} 11>$

Deutsch - Jozsa Algorithm

Problem..

Key difference is:-

$f: \{0,1\}^n \rightarrow \{0,1\}$ take n bits input and produce single bit output.

Find

$\text{17 } \text{is Constant}$

$\text{17 } \text{is balanced}$

Example $n=2$:

x	$f(x)$
00	0 or 1
01	0 or 1
10	0 or 1
11	0 or 1

Constant

x	$f(x)$
00	0 or 1 or 1 or 0 or 0 or 0
01	0 or 1 or 1 or 0 or 0 or 0
10	1 or 0 or 0 or 1 or 0 or 0
11	1 or 0 or 0 or 0 or 1 or 0

Classical

- Must check half plus one inputs

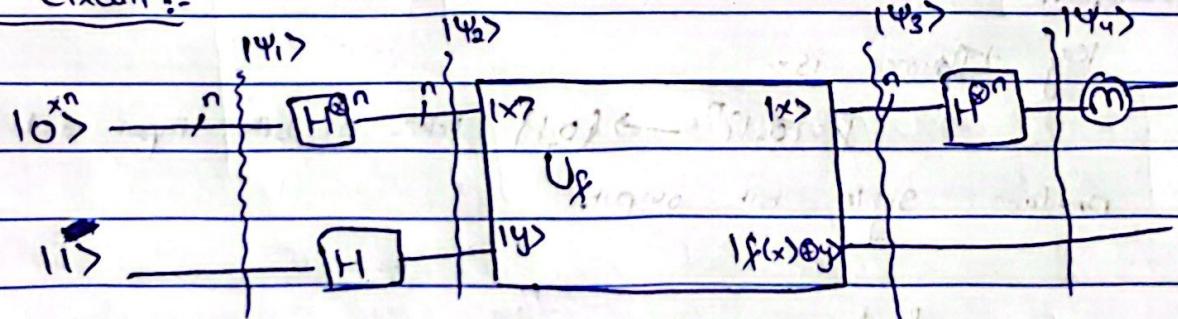
$$\hookrightarrow \frac{2^n}{2} + 1 = O(2^n)$$

Quantum

- Call oracle once via superposition
- Exponential speedup.

- If all are input produce some result is will be constant otherwise balanced.

Quantum Circuit:-



$$|\Psi_1\rangle = |10^n\rangle|11\rangle, |\Psi_2\rangle = H^{\otimes n}|10^n\rangle|1111\rangle$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |1\rangle$$

$$|\Psi_3\rangle = U_f |\Psi_2\rangle$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x) \oplus 0\rangle - |f(x) \oplus 1\rangle$$

$$\therefore f(x) \oplus 0 = f(x)$$

$$\therefore f(x) \oplus 1 = f(x)'$$



$$|\Psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle (|f(x)\rangle - |\bar{f}(x)\rangle)$$

when $f(x) = 0$

$\frac{1|0\rangle - 1|1\rangle}{\sqrt{2}}$

when $f(x) = 1$

$\frac{1|1\rangle - 1|0\rangle}{\sqrt{2}}$

So,

$$|\Psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle (-1)^{f(x)} |x\rangle$$

$$|\Psi_4\rangle = H^{\otimes n} \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} H^{\otimes n} |x\rangle$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2^n}} \times \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{f(x) + x \cdot y} |y\rangle$$

Case 1 f is constant : Count $|y\rangle - |0\rangle$ with 1 prob

Fact #1 $x \cdot y = 0$

Fact #2 $f(x) = 0 \Rightarrow (-1)^{f(x)} = 1$

Constant
Always
 $|1\rangle$

$n=2$

x	$f(x)$
00	1
01	0
10	0
11	-1

Balanced
 $m \neq 10^n$

$$|\Psi_1\rangle = |00\rangle|11\rangle, |\Psi_2\rangle = H^{\otimes 2}|00\rangle|111\rangle$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2^2}} [|00\rangle + |01\rangle + |10\rangle + |11\rangle] |1\rangle$$

$$|\Psi_3\rangle = U_f |\Psi_2\rangle$$

Since the second register has $|1\rangle$,
Uf gate will change the sign of
all the inputs which are
providing an output of 1, Hence,

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}} [-|00\rangle + |01\rangle + |10\rangle - |11\rangle]$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2}} [-|100\rangle + |101\rangle + |110\rangle - |111\rangle]$$

There will be 16 terms but we can calculate
them easily with using tabular method.

	γ_1, γ_2
$x_1 x_2$	$(-1)^{x_1 + x_2}$
- 100	*
101	*
110	*
- 111	*

$$= \frac{1}{2} \cdot \frac{1}{2} [-4|111\rangle] = -|111\rangle$$

(M) $|111\rangle$ with prob $(-1)^2 = 1$ as not (M) $|100\rangle$ since y is not balanced

Quantum Fourier Transform

Primitive root of unity

$z^n = 1$, z is complex number

n roots (Solutions)

Solutions are $\{w_n^0, w_n^1, w_n^2, \dots, w_n^{n-1}\}$

$$w_n^k = e^{2\pi i k/n}$$

1) w_n^k lie on unit circle.

$$\begin{aligned} |w_n^k| &= 1 = \sqrt{w_n^{k*} \cdot w_n^k} \\ &= \sqrt{e^{-2\pi i k/n} \cdot e^{2\pi i k/n}} \end{aligned}$$

$$= \sqrt{e^0} = 1$$

2) w_n^k is a periodic function

$$w_n^k = w_n^{k \bmod n}$$

Example :-

write all root

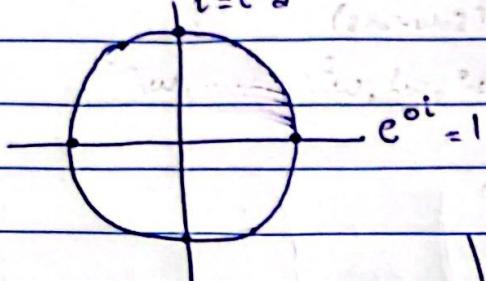
$$z^4 = 1$$

$$\{w_4^0, w_4^1, w_4^2, w_4^3\}$$

$$\{e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}\}$$

$$= \{e^{0i}, e^{\frac{\pi i}{2}}, e^{\pi i}, e^{\frac{3\pi i}{2}}\}$$

$$\{1, i, -1, -i\}$$



$$\omega_4^7 = \omega_4^{7 \bmod 4} = \omega_4^3$$

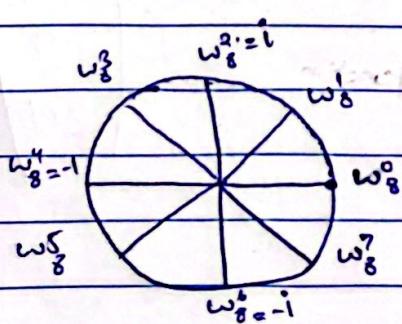
$$z^4 \rightarrow = 1 \quad \{1, -1, i, -i\}$$

$$\text{so for periodic } = \omega_4^4 = \omega_4^{4 \bmod 4} = \omega_4^0 \neq$$

Example:-

~~$$z^{28} \cdot z^8 = 1$$~~

previous 3 are calculate in previous question ~~see next~~



$$\frac{360^\circ}{8} = 45^\circ$$

$$\{1, w_8, i, iw_8, -1, -w_8, -i, -iw_8\}$$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} w_n^{0x0} & w_n^{0x1} & w_n^{0x2} & \cdots & w_n^{0xn-1} \\ w_n^{Kx0} & w_n^{Kx1} & \cdots & \cdots & w_n^{K(n-1)} \\ \vdots & \vdots & \ddots & \ddots & w_n^{(n-1)(n-1)} \end{pmatrix}$$

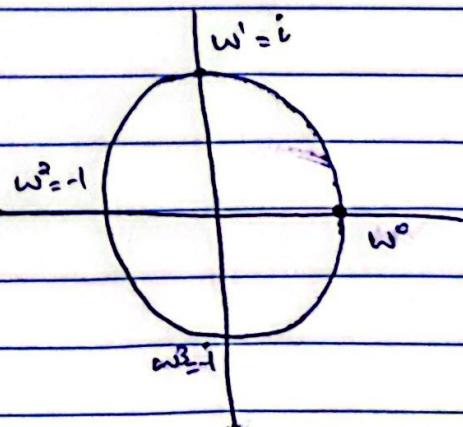
$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdots & 1 \\ 1 & w_n & w_n^2 & w_n^3 & \cdots & \cdots & \cdots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & w_n^{12} & \cdots & \cdots & \cdots & w_n^{2(n-1)} \\ 1 & w_n^3 & w_n^6 & w_n^9 & w_n^{18} & \cdots & \cdots & \cdots \\ & & & & & & & \end{pmatrix} \quad \begin{matrix} \text{Table of 1} \\ \text{Table of 2} \\ \vdots \\ \vdots \end{matrix}$$

Example

$$\text{Transform } |14\rangle = \frac{|10\rangle + |15\rangle}{\sqrt{2}} \quad \text{using } F_4$$

$$F_4 |14\rangle = |1\phi\rangle$$

$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{pmatrix}$$

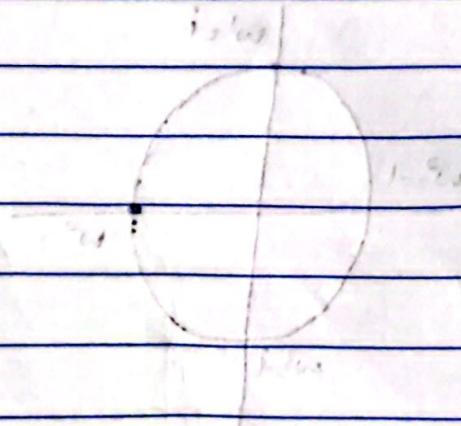


$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^0 & w^2 \\ 1 & w^3 & w^1 & w \end{pmatrix}$$

$$= \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & i & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 1-i \\ 0 \\ 1+i \end{pmatrix} = \phi$$

$$\langle \phi | = \langle \psi | \alpha$$



$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ w & w^2 & w^3 & w \\ w^2 & w^0 & w^1 & w^2 \\ w^3 & w^1 & w^2 & w \end{pmatrix}$$

Properties of DFT

① DFT is unitary:-

There are different ways to prove this transformation but the simple way is to show that the columns of DFT are orthogonal.

It implies the each column is normalised as its magnitude is equals to one and each column is orthogonal from all other columns.

So if we have two column of DFT say that column j and k and we take dot product of those column then our dot product must be either equals to zero or equal to 1.

$$\langle \tilde{c}_j | \tilde{c}_k \rangle = \begin{cases} 0 & ; j \neq k (\text{orthogonal}) \\ 1 & ; j = k (\text{normalised}) \end{cases}$$

Our result will be equal to 0 when j is not equal to k and it shows column are orthogonal.

Our result will be equal to 1 when j is equal to k and it represent our column are normalised.

$$\vec{c}_j = \frac{1}{\sqrt{n}} \begin{pmatrix} w^{0xj} \\ w^{1xj} \\ w^{2xj} \\ \vdots \\ w^{(n-1)xj} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} w_j \\ w^{0j} \\ w^{1j} \\ w^{2j} \\ \vdots \\ w^{(n-1)j} \end{pmatrix}$$

$$\vec{c}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} w^k \\ w^{2k} \\ \vdots \\ w^{(n-1)k} \end{pmatrix}$$

now dot product of column j with k , that we have to take conjugate of the first column and multiple it with the corresponding entry of second column.

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \overline{w^{mxj}} w^{mk}$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} w^{m(k-j)}$$

Case 1: $j = k$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} w^{m(0-0)}$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} w^0$$

$$= \frac{1}{n} [1+1+\dots+1]$$

$$= \frac{n}{n} = 1 \text{ (each column is normalized or each column is a unit vector)}$$

Case 2 :- $j \neq k$

Geometric series :-

$$a + \gamma a + \gamma^2 a + \dots + \gamma^{n-1} a$$

$$S = a(\gamma^n - 1) \quad \therefore a = 1$$

$$\gamma - 1 \quad \therefore \gamma = \omega^{k-j}$$

Case :- 2 $j \neq k$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(k-j)} = 0$$

$$\langle c_j | c_k \rangle = \omega^{(k-j)n} = 1$$

Now whenever we have ω_n raised to power any multiple say

$\alpha n + kt$ say:-

$$\omega_n^{\alpha n} = 1 \quad \omega_n^{\alpha n} = \omega_n^{\alpha n \bmod n} = \omega_n^0$$

$$S_0 = \frac{1 - 1}{w^{k-j} - 1} = 0$$

[Explanation :-]

② Convolution \leftrightarrow multiplication :-

So the basic idea is we linearly shift my input vector and then apply QFT on that input vector and my output vector has phase shift in it.

Before linear shift

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & w^3 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & w^6 & \dots & w^{2(n-1)} \\ 1 & w^3 & w^6 & w^9 & \dots & w^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & w^{3(n-1)} & \dots & w^{(k-1)(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_{n-1} \end{pmatrix}$$

measure our output
(result is an index of the vector)

$$|B_k|^2$$

Let say it is linearly shift by 1

After linear shift

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-2} \end{pmatrix} = \begin{pmatrix} B_0 \\ wB_1 \\ w^2B_2 \\ \vdots \\ w^{n-1}B_{n-1} \end{pmatrix}$$

$|B_k|^2$

$|w|^2 = |w^3| = 1$

so magnitude of all powers of w is 1.
so we don't have to write w^n .

to write $|w^n B_k|^2$.

we can just ignore it. And write

$|B_k|^2$



Before linear shift

$$B_K = \alpha_0 + w^K \alpha_1 + w^{2K} \alpha_2 + \dots + w^{(n-1)K} \alpha_{n-1}$$

After linear shift by 1

$$\alpha_m = \alpha_{m+1}$$

K^{th} output after linear shift

$$= \alpha_{n-1} + w^K \alpha_0 + w^{2K} \alpha_1 + w^{3K} \alpha_2 + \dots + w^{(n-1)K} \alpha_{(n-2)}$$

Now we have to show is that output before linear shift but only has the phase attach to it.

$$w^K B_K = w^K \alpha_0 + w^{2K} \alpha_1 + \dots + w^{(n-1)K} \alpha_{(n-1)}$$

$$\therefore \frac{w^{nK \bmod n}}{w^{(n-1)K}} = 0$$

$$w^{nK \bmod n} = 0$$

$$= w^K \alpha_0 + w^{2K} \alpha_1 + \dots + w^{(n-1)K} \alpha_{n-1}$$

$$\therefore w^0 = 1$$

③ Period / wave length Relationship:

a function f take n bits as input and it produces n bits output.

$$f: \{0,1\}^n \rightarrow \{0,1\}^n$$

and we apply QFT on our function f

$$\text{QFT } f(x) = \hat{f}(x)$$

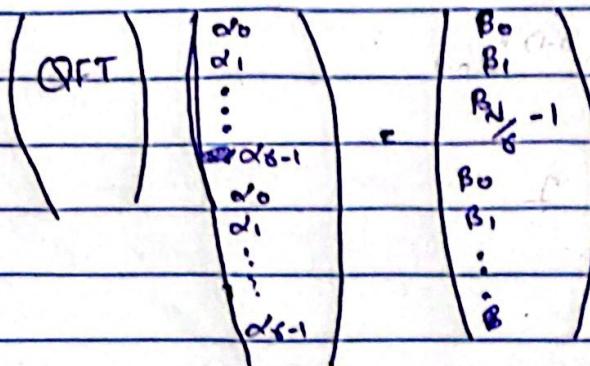
if our function f

is periodic and

has a per

$$\text{period} = \delta$$

$$\frac{n}{\delta} = 2^{\eta}$$



1st field next to a_0

2nd field next to a_1

3rd field next to a_2

4th field next to a_3

5th field next to a_4

6th field next to a_5

7th field next to a_6

O = obtain

O = obtain $a_0, a_1, \dots, a_{N_g-1}$

O = obtain a_g

1st field

2nd field $a_0, a_1, \dots, a_{N_g-1}$

3rd field

4th field

5th field

6th field

7th field

8th field

9th field

10th field

11th field

12th field

13th field

14th field

15th field

16th field

17th field

18th field

19th field

20th field

21st field

22nd field

23rd field

24th field

25th field

26th field

27th field

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103rd field

104th field

105th field

106th field

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256th field

Period Finding Algorithm

Problem Definition :-

Given a periodic function $f: \{0,1\}^n \rightarrow \{0,1\}^n$

Find a period σ such that $f(x) = f(x + k\sigma) \forall x \in \{0,1\}^n$

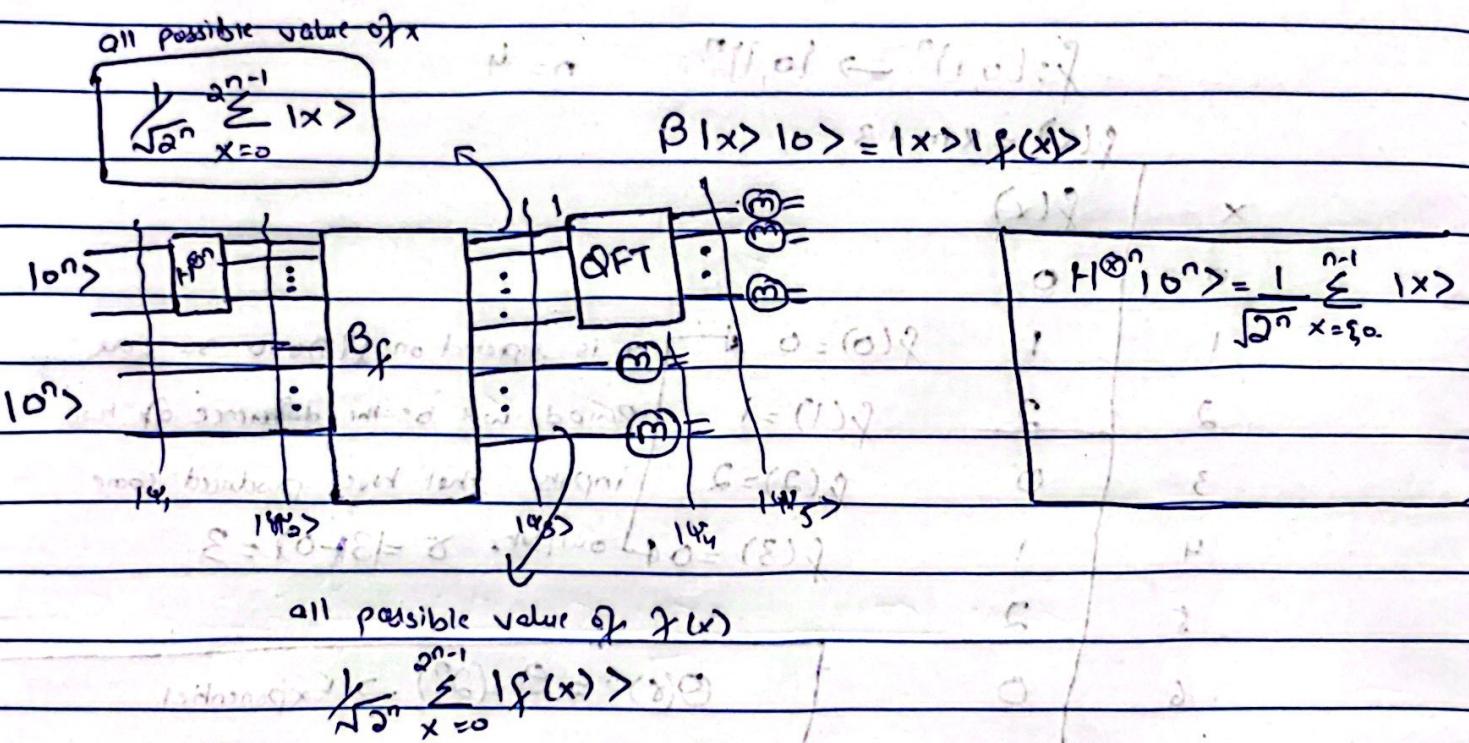
- Similar to Simon's Algo.

$$f: \{0,1\}^n \rightarrow \{0,1\}^n \quad n=4$$

$$f(x) = x \bmod 3$$

x	f(x)	
0	0	
1	1	$f(0) = 0 \leftarrow 0$ is repeated on $f(3) = 0$ so one
2	2	$f(1) = 1$ period will be the difference of two
3	0	$f(2) = 2$ inputs that here produced same
4	1	$f(3) = 0 \leftarrow$ output. $\sigma = 3 - 0 = 3$
5	2	
6	0	$\Theta(\sigma) = \Theta(2^n) \rightarrow$ Exponential
7	1	
8	2	
9	0	if no does passed 2 steps will reflect
10	1	then after taking long time
11	2	other steps will return same
12	0	
13	1	
14	2	
15	0	

The function f whose period we want to find might not be unitary. Thus we might not be able to use function f directly in the quantum circuit. instead we have to develop a quantum wrapper over the function f i call the quantum implementation of function f .



So if we measure the second register then it will transform both the register because each of these two register are no physical operated with each other their state are tensor product with each other.

So combine we can write

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x> |f(x)>$$

So if we measure the second register according to the rule of partial measurement content of the first register will also change.

let say if my measurement of the first register is $|f(x_i)\rangle$ where x_i is some specific value then the first register will have only those basis that can produce this specific output of $|f(x_i)\rangle$

The input that can produce $|f(x_i)\rangle$

$$|x_1\rangle + |x_1+s\rangle + |x_1+2s\rangle + \dots + |x_1 + \left(\frac{2^n}{s} - 1\right)s\rangle$$

$$\sqrt{\frac{2^n}{s}}$$

	2nd Register measurement	First Register superposition
$ f(0)\rangle$	$ 0\rangle$	$ 0\rangle + 0+s\rangle + 0+2s\rangle + \dots$
$ f(1)\rangle$		$ 1\rangle + 1+s\rangle + 1+2s\rangle + \dots$
$ f(2)\rangle$		$ 2\rangle + 2+s\rangle + 2+2s\rangle + \dots$
\vdots		\vdots
$ f(k)\rangle$		$ k\rangle + k+s\rangle + k+2s\rangle + \dots$
\vdots		\vdots



Working logic of Period finding Algorithm circuit on previous page

Example:-

Find period(s) given $f: \{0,1\}^3 \rightarrow \{0,1\}^3$

{ For use in Bf, $f(x) = x \bmod 2$ meant "odd or even". }

$$14,7 = 10^3 \cdot 10^3 \cdot 1000 = 1000 \cdot 1000$$

$$1427 = K \left(\sum_{x=10}^7 (1 - 1/x) \cdot 1000 \right) + (C_{\text{GP},x} + R_{\text{GP},x}) + K \cdot x$$

$$1427 = \frac{1}{1000} (107 + 117 + 127 + \dots + 177) \quad 1000 >$$

$$14_3 > \underline{\underline{B_f}} 14_2 > + < + > \quad \text{and} \quad B_f 1 \times 10^3 = 1 \times 1.86 >$$

$$\frac{2^3 - 1}{2 - 1} = \sum_{x=0}^{2^3-1} (1 \times x)$$

1940-1941

$$|\Psi_3\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|0\rangle + |3\rangle|1\rangle + |4\rangle|0\rangle + |5\rangle|1\rangle + |6\rangle|0\rangle + |7\rangle|1\rangle$$

$\sqrt{8}$

Assume that we have to measure $|1\rangle$

$$|\Psi_4\rangle = (|1\rangle + |3\rangle + |5\rangle + |7\rangle) |1\rangle$$

$\sqrt{4}$

Assume that we measure $|0\rangle$

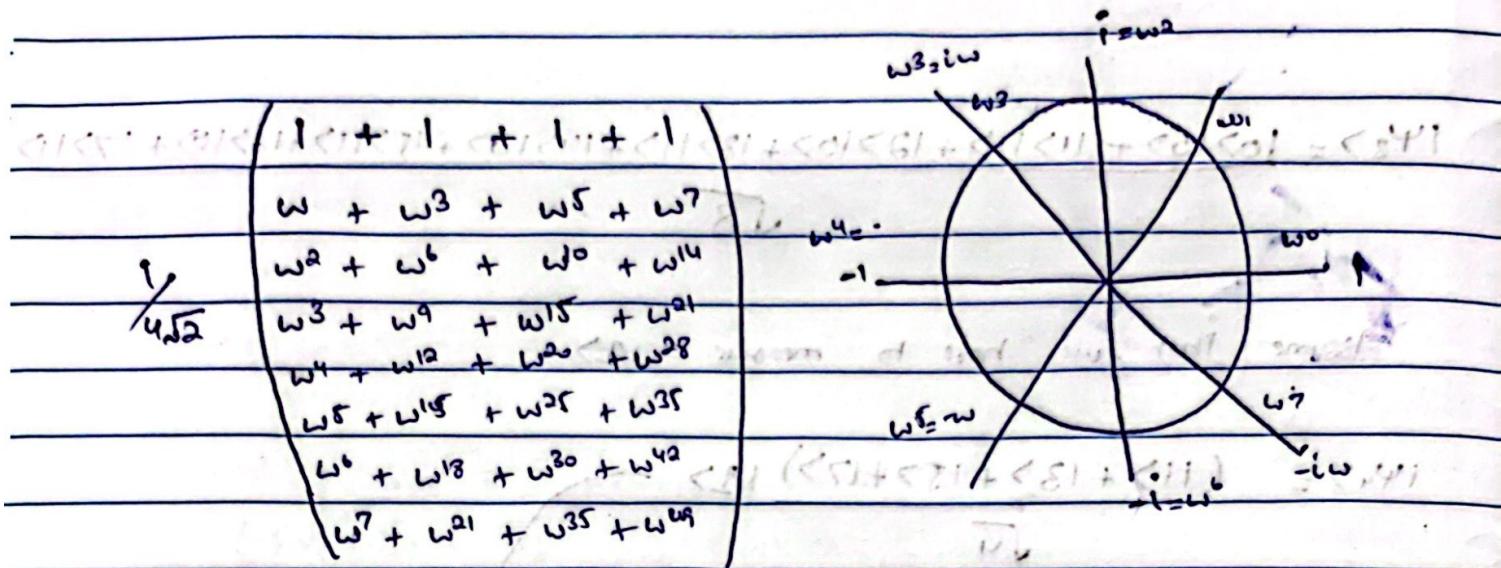
$$|\Psi_5\rangle = (|0\rangle + |2\rangle + |4\rangle + |6\rangle) |0\rangle$$

$\sqrt{4}$
nam

$$|\Psi_5\rangle = QFT_B |\Psi_4\rangle$$

$$|\Psi_5\rangle = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} \\ 1 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} \\ 1 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\sqrt{2}/\sqrt{3}$



$$|4\rangle = \frac{1}{4\sqrt{2}} \begin{pmatrix} 4 \\ 0 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{10\langle -14 \rangle_{21}}{\sqrt{2}} = \underline{(m)}$$

$$\langle \psi_1 | \psi_7 \rangle = \delta_{17}$$

$$\langle \psi_1 | \psi_7 \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

$$|14_4\rangle = (|10\rangle + |12\rangle + |14\rangle + |16\rangle) / \sqrt{4}$$

✓4

$$145 = \frac{107 + 147}{\sqrt{2}}$$

When we have 2 result with different phase then their measure remains the same.

Run the circuit

$$\underline{O(\log N)} \\ = O(\log^3 d)$$

$$\text{GCD of our mod} = \frac{N}{6} = 4 \text{ and we will start with } 24$$

$$= \frac{8}{8} = 4$$

$$\gamma = 2 \quad \left\{ \frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^3}, 1 \right\}$$

A-level Complex Number

$$\text{COL} (\text{COMPLEX NUMBER}) = \text{CMI}$$

Root of Unity :-

$$z^n = 1$$

$$\{w_0^n, w_1^n, w_2^n, \dots, w_{n-1}^n\}$$

$$\text{CMI} + \text{COL} = \text{CMI}$$

$$w_n^k = e^{\frac{2\pi i k}{n}}$$

and all these numbers are there to root are needed

$$z^3 = 1$$

mod and argument

$$\{w_0^3, w_1^3, w_2^3\}$$

$e^{\frac{2\pi i k}{3}} = 1$

$e^{\frac{2\pi i}{3}}$

$e^{\frac{4\pi i}{3}}$

$$(1, 0)$$

$$(\frac{1}{2}, \frac{\sqrt{3}}{2})$$

We have three different form of Complex number

Exponential Form

$$\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$$

Now we want to express them in modular argument form

Modular Argument - Form

In this we express complex number as

$$z = r \cos \theta + r i \sin \theta$$

$$e^{\frac{2\pi i}{3}} = e^{i\theta}$$

$$\left\{ 1, \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right\}$$

lastly we need to write it in Cartesian form.

~~Step - Method~~, coming to [Euler's Formulas]

$$\left\{ 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right\}$$

$$\begin{aligned} z_1 &= 1 \\ z_2 &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ z_3 &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

Example:-

$$(2\cos\theta, 2\sin\theta)$$

$$z^3 = 8$$

Find root

$$z = \sqrt[3]{8}$$

$$z = 2$$

exponential form :-

$$\{2, 2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}}\}$$

Cartesian form :-

$$\{1, -1+i\sqrt{3}, -1-i\sqrt{3}\}$$

Example :-

$$(z-5)^3 = 8$$

Find root

noticing the basic required steps

noticing noticing the fact, polaric

noticing noticing the fact, noticin

noticing to solve sum of four

noticing solving lots of 2D problems which are needed in solving

Let $(\sqrt[3]{8}) \cos i + (\sqrt[3]{8}) \sin i = (\sqrt[3]{8}) \cos i + (\sqrt[3]{8}) \sin i$

$$z - 5 = y$$

$y^3 = 8$ roots are $\omega_1, \omega_2, \omega_3$ of form $\omega_1 = 2$

$\{2, -1+i\sqrt{3}, -1-i\sqrt{3}\}$ as previous

$$y = z - 5$$

$$z = y + 5$$

$\{7, 4+i\sqrt{3}, 4-i\sqrt{3}\}$

$\rightarrow \text{Imaginary}$

$8\omega^2$

Modulus Argument Form

from brief

Cartesian form

Modulus from

$8\omega^2$

$6\omega^2$

$$z = x + iy \longrightarrow z = r \cos \theta + i r \sin \theta$$

imaginary axis (z)

Modulus : $|z| = \sqrt{x^2 + y^2}$

(magnitude)

2nd quad

1st quad

imagine axis (z)

3rd

4th

real axis (x)

Argument (Principle argument)
 $-\pi < \theta \leq \pi$

Angle between x -axis and position

vector, but my position vector

can be in any quadrant. we

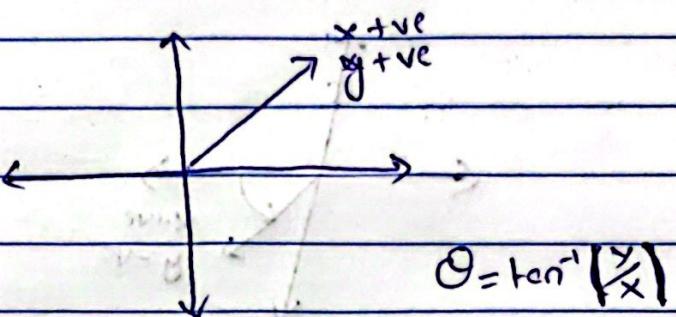
want to have value of argument

which is between -180° and 180° and is called principle argument

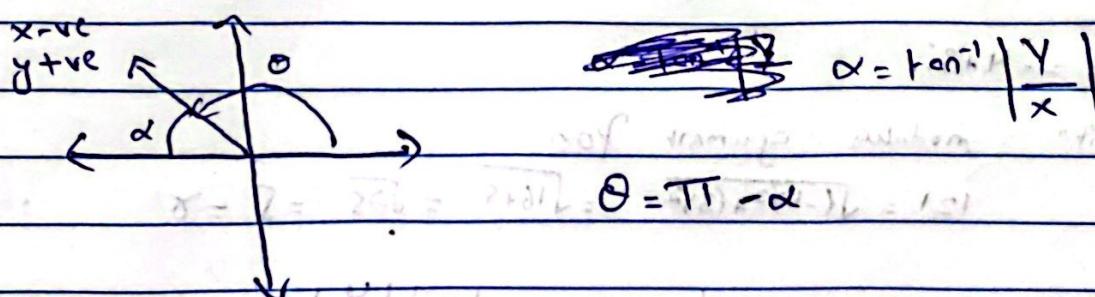
$\delta = \theta (2-s)$

from brief

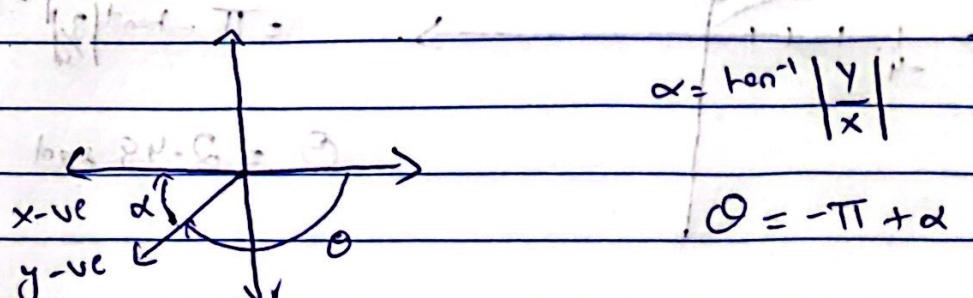
case 1 :- we are in first quadrant.



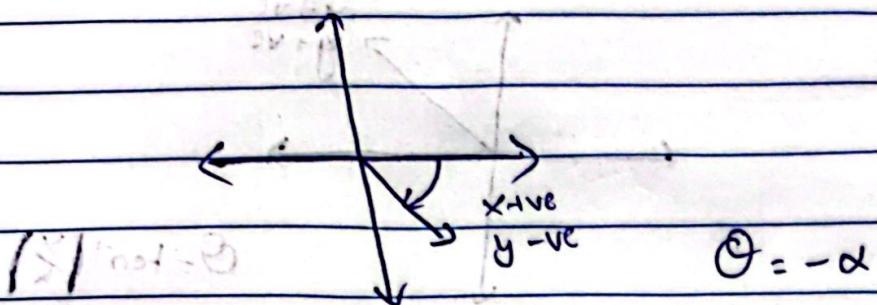
Case 2 :- we are in second quadrant:-



Case 3 :- we are in third quadrant



Case 4: We are fourth quadrant

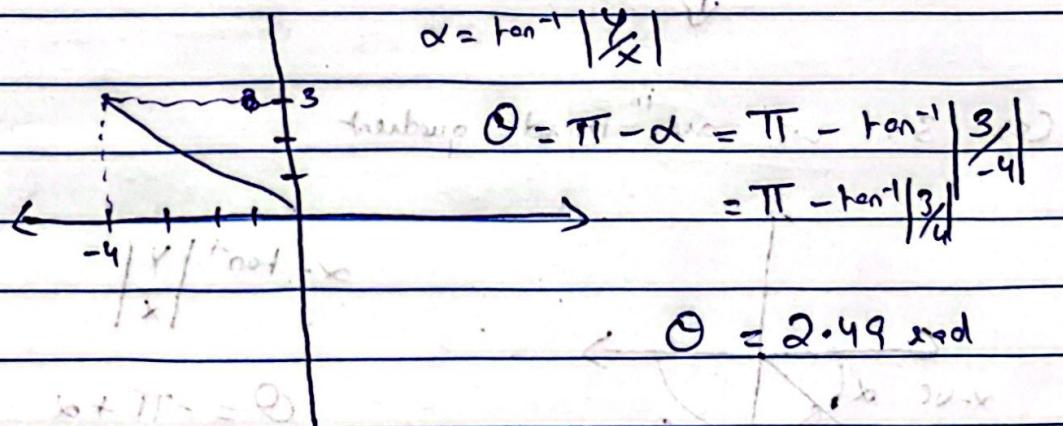


Example:-

$$z = -4 + 3i$$

write modulus argument form :

$$|z| = \sqrt{(-4)^2 + (3)^2} = \theta = \sqrt{16+9} = \sqrt{25} = 5 = r$$



Exponential Form

Cartesian Form

$$\underline{z = x + iy}$$

Modulus - Argument Form

$$z = r \cos \theta + i r \sin \theta$$

Exponential Form

$$z = r e^{i\theta}$$

Example

$$z = -4 + 23i$$

$$= |z| = 5$$

$$\theta = 2.49$$

$$z = 5e^{i2.49}$$

Solving Equation

$$ax^2 + bx + c = 0 \quad \text{and} \quad a \neq 0 \quad \text{int. no. } 0 \leq ax^2 + bx + c$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\sqrt{b^2 - 4ac}$ tells us if my solution will be real or it will have complex root or my solution will be distinct or not

Case 1 :- $b^2 - 4ac > 0$ we will has (real solution)

$$\alpha, \beta \in \mathbb{R}$$

$$\alpha \neq \beta$$

Case 2 :- $b^2 - 4ac = 0$

$$\alpha, \beta \in \mathbb{R}$$

$$\alpha = \beta$$

Case 3 :- $b^2 - 4ac < 0$

$$\alpha, \beta \in \mathbb{C} \text{ complex number}$$

$$\beta = \alpha^*$$

Example:- using this fact solve of the given quadratic equations in \mathbb{R}

$$x^2 + 2x + 5 = 0$$

Find roots

$$b^2 - 4ac = 4 - 4(1)(2) = -4$$

$b^2 - 4ac < 0$, so the root will be complex $x + id$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

rearranged over $\frac{2a}{2a}$

so now calculate for no. root

$$= -2 \pm \sqrt{-4} = -2 \pm 2i \quad -1 \pm i$$

(positive $2i$) and $11w \geq 0 \Rightarrow P_0 - M_0 C > 0$

For poly degree 3

$$ax^3 + bx^2 + cx + d = 0$$

~~a~~ $a \neq 0$

① 3 real root

$\beta \neq \alpha$

② 1 real, 2 complex root

$\alpha \beta - \gamma^2 = 0$

Poly deg 4

minimum signs $\beta \beta \alpha \alpha$

① 4 real root

$\alpha = \beta$

② 2 real, 2 complex root (where these complex root will be conjugate to each other)

③ 4 complex (where these root will be in pairs and each pair will be conjugate of other complex root. $\alpha, \alpha^*, \beta, \beta^*$)

Stock book

Example

$$z^4 + 18z^2 + 36 = 0$$

Given $z = 2i$ is a root Find remaining 3 roots.

Sol:-

$$z = 2i, z = (-2i)i + (2i) = -2i + 2i$$

$$\text{Factor } (z-2i)(z+2i)$$

$$= z^2 - (2i)^2 = z^2 + 4$$

$$z^2 + 4 = \sqrt{z^2 + 18z^2 + 36} \Rightarrow z^2 + 4 = (z^2 + 9)(z^2 + 4)$$

$$-z^4 + 4z^2$$

$$9z^2 + 36 = 0 \quad (z^2 + 4)(z^2 + 9) = 0$$

$$9z^2 + 36 = 0$$

X

The equation has two factors which are $z^2 + 4$ and $z^2 + 9$

$$\text{root of } (z^2 + 4)(z^2 + 9) = 0$$

$$z^2 + 9 = 0$$

$$z^2 = 9$$

$$z^2 = \pm \sqrt{9} = \pm 3$$

$$2i - 2i + 3i - 3i$$



Arithmetic Operation

$$z_1 = a+ib, z_2 = c+id$$

Addition / Subtraction

$$z_1 + z_2 = (a+c) + i(b+d)$$

$$z_1 - z_2 = (a-c) + i(b-d)$$

• Multiplication:-

$$z_1 \cdot z_2 = (ac - bd) + i(ad + bc)$$

$$\frac{z_1}{z_2} = \frac{(ac+bd) + i(ad+bc)}{c^2 + d^2}$$

In exponential form

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$$

Addition / Division

$$z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$$

$$z_1 - z_2 = r_1 e^{i\theta_1} + -r_2 e^{i\theta_2}$$

Multiply :-

$$z_1 \cdot z_2 = r_1 \cdot r_2 e^{i(\theta_1 + \theta_2)}$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$


Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Result :- In exponential form

$$\textcircled{1} \quad \arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

$$\textcircled{2} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

$$\textcircled{3} \quad \arg\left|\frac{z_1}{z_2}\right| = \frac{\theta_1 - \theta_2}{2} = \frac{|z_1|}{|z_2|}$$

$$\textcircled{4} \quad |z_1 \cdot z_2| = r_1 \cdot r_2 = |z_1| \cdot |z_2|$$

Square Root

Qno Find square root of $7 + i6\sqrt{2}$

Sol :-

$$x + iy = \sqrt{7 + i6\sqrt{2}}$$

$$(x+iy)^2 = 7 + i6\sqrt{2}$$

$$x^2 + 2ixy - y^2 = 7 + i6\sqrt{2}$$

$$(x^2 - y^2) + i(2xy) = 7 + i6\sqrt{2}$$

$$(x^2 - y^2) = 7 \quad \textcircled{1}$$

~~$$2xy = 6\sqrt{2}$$~~

$$\boxed{xy = 3\sqrt{2}} \quad \textcircled{2}$$

$$y = \frac{3\sqrt{2}}{x} \quad \text{put in } \textcircled{1}$$

$$\text{put in } \textcircled{1} \quad (x^2 - \frac{9}{x^2}) + i(\frac{18}{x}) = 7 + i6\sqrt{2} \quad \textcircled{3}$$

$$x^2 - \frac{(3\sqrt{2})^2}{x^2} = 7$$

$$\frac{x^4 - 18}{x^2} = 7 \quad \text{or} \quad x^4 - 18 = 7x^2$$

$$x^4 - 18 = 7x^2 \quad \text{or} \quad x^4 - 7x^2 - 18 = 0 \quad \textcircled{4}$$

$$\text{Let } u = x^2$$

$$u^2 - 7u - 18 = 0$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{7 \pm 11}{2}$$

$$u = 9, u = -2$$



$$x^2 = 9$$

$$x = \pm 3$$

$$(x^2 = -2)$$

not use because it give imaginary value

$$\sqrt{-2} = \pm i\sqrt{2}$$

$$y = \frac{3\sqrt{2}}{x}$$

$$= 3\sqrt{2} = -\sqrt{2}$$

both signs are not feasible because solution is unit

$$= \frac{3\sqrt{2}}{3} = \sqrt{2} \text{ in right side of brief or we}$$

$$\left(\frac{1688\pi}{6}\right) \cos \pi i + \left(\frac{1688\pi}{6}\right) 200 \pi i = \text{?}$$

so Find :-

$$x+iy = 3+i\sqrt{2}$$

$$(x+iy)^2 = -3+i\sqrt{2}$$

x

$$(3\sqrt{2}-8 = \text{?})$$

Square Root \rightarrow De-moivre

Q no Find Square root of

$$7+i6\sqrt{2}$$

$$|z| = r = \sqrt{7^2 + (6\sqrt{2})^2}$$

$$= \sqrt{49+72}$$

$$= \sqrt{121} = 11$$

$$\theta = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{6\sqrt{2}}{7} \right| = 0.88 \text{ rad}$$

$$z = r \cos \theta + i r \sin \theta$$

$$z^n = r^n \cos n\theta + i r^n \sin(n\theta)$$

$$z^{1/2} = \sqrt{r} \cos \frac{\theta}{2} + i \sqrt{r} \sin \frac{\theta}{2}$$

$$z^{\frac{1}{2}} = \sqrt{8} \cos \frac{\theta}{2} + i\sqrt{8} \sin \frac{\theta}{2}$$

$$= \sqrt{11} \cos \frac{0.88}{2} + i\sqrt{11} \sin \frac{0.88}{2}$$

$$\boxed{z^{\frac{1}{2}} = 3 + i\sqrt{2}}$$

there is another square root of this complex number that we can find by adding 2π in my θ .

$$z^{\frac{1}{2}} = \sqrt{11} \cos \left(\frac{0.88 + 2\pi}{2} \right) + i\sqrt{11} \sin \left(\frac{0.88 + 2\pi}{2} \right)$$

$$\boxed{z^{\frac{1}{2}} = -3 - i\sqrt{2}}$$

→ *Two roots*

$$\sin 2\theta + \cos 2\theta = 5$$

$$(0.88)^2 + (0.6)^2 = 1.44$$

$$\sin 2\theta + \cos 2\theta = 4.5$$

$$\tan 2\theta = \text{not defined}$$

$$2\theta = 90^\circ$$

$$\sqrt{(5^2 + 4.5^2)} = 8.51$$

$$R^2 = P^2 + Q^2$$

$$R = \sqrt{P^2 + Q^2}$$

$$\tan 88^\circ = \frac{5}{4.5} \text{ not } = \frac{1.11}{1.11} = 1$$