

1) Ket-Notation:- $|1\rangle$

Define as a two dimensional vector who 0th location has one in it.

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\langle 001, 0011, 0101, 0001 \rangle + 6000$$

comprised of base binary vector functionality and so on

Similarly:-

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example:- $|101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

as it is 5 in decimal
so it has 1 unit
 $\hookrightarrow 2^3 = 8$ Dimensional

5th location will have 1 as 101 in decimal is 5

Example:- $|111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\hookrightarrow 2^2 = 4$ Dimensional

3rd location will have 1 as 11 in decimal is 3

2 Dimension:- $\{|0\rangle, |1\rangle\}$ Standard Basis

$$\langle 111 | \langle 001 | (1+8) = \langle 111 |$$

$$113 + 103(12-8) + 1241 = 143$$

$$\begin{pmatrix} 1 \\ 3+5i \end{pmatrix} = 7|0\rangle + (3+5i)|1\rangle$$

$$\langle 111 | \langle 001 | (12-8) = 141 - (12) = 141$$

$$\underline{40} \quad \begin{pmatrix} 7 \\ 0 \\ i+3 \\ 0 \end{pmatrix} = 7|00\rangle + i|31\rangle$$

Basis: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

2 Bra-Notation :-

is a two dimensional vector whose zero location has one in it.

$$\langle 01 | [1, 0] , \langle 11 | = [0, 1]$$

$$|\Psi\rangle^+ = \langle \Psi | , \langle \Psi |^+ = |\Psi\rangle$$

Conjugate Transpose (+) :-

Comprise of two operation the

$+$ = Conjugate & Transpose
(*)

Example:-

$$|\Psi\rangle = (3+5i)|0\rangle + 7|1\rangle$$

$$\langle \Psi | = |\Psi\rangle^+ = (3-5i)\langle 01 | + 7\langle 11 |$$

Change the sign of iota term

$$|\Psi\rangle = \begin{pmatrix} 3+5i \\ 7 \end{pmatrix} , \langle \Psi | = [3-5i \ 7]$$

4D $\{3 \ 0 \ i \ 7\}$

$$= 3|001\rangle + i|101\rangle + 7|111\rangle$$

3 Ket-Ket Notation :- $(\cancel{\downarrow} \cancel{\uparrow} + 1 - \rightarrow 1 - \rightarrow) : = f \mid = \langle 001 |$

gives us the tensor product of two column vector
creating a column vector of high dimension.

Tensor Product

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 \cdot B & 0 \cdot B \\ 0 \cdot B & 2 \cdot B \end{pmatrix} = \begin{pmatrix} 123 & 000 \\ 047 & 000 \\ 000 & 246 \\ 000 & 0814 \end{pmatrix}$$

$$|\Psi\rangle \otimes |\Phi\rangle = |\Psi\rangle |\Phi\rangle = |\Psi\Phi\rangle \quad \text{[} \rightarrow \text{ simple way to write it]}$$

Example:-

$$|\Psi\rangle = i|10\rangle + 7|11\rangle$$

$$|\Phi\rangle = |00\rangle + 3|10\rangle + 7|11\rangle$$

$$\begin{aligned} |\Psi\Phi\rangle &= i|100\rangle + 3i|101\rangle + 7i|1011\rangle \\ &\quad + 7|1100\rangle + 21|1110\rangle + 49|1111\rangle \end{aligned}$$

in term of vector:-

$$|\psi\rangle = \begin{bmatrix} i \\ 7 \end{bmatrix}, |\phi\rangle = \begin{bmatrix} 1 \\ 0 \\ \frac{3}{7} \end{bmatrix}$$

$$|\psi\phi\rangle = \begin{bmatrix} i & 7 & 0 \\ 0 & 3i & 7i \\ 7 & 0 & 21 \\ 0 & 0 & 49+7 \end{bmatrix}$$

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4 Box-Box Notation ($\langle -1 | -1 | -1 \rangle = 8$, $|0 1\rangle = 4$) gives us the tensor product of two row vector of higher dimension.

$$\text{Ex } |\psi\rangle = 3|001\rangle + 7|111\rangle$$

$$|\phi\rangle = |01\rangle + |11\rangle$$

$$|\psi\phi\rangle = 3|001\rangle + 3i|011\rangle + 7|101\rangle + 7i|111\rangle$$

in terms of vector:-

$$|\psi\rangle = [3 \ 7]$$

$$|\phi\rangle = [0, i]/8 + |001\rangle = |\phi\rangle$$

$$|\psi_01\rangle = \begin{bmatrix} 3 \\ 3i \\ 7 \\ 7i \end{bmatrix}$$

- 0m
- 4m

$$|001\rangle = e^{i\phi_1}|0\rangle + e^{i\phi_2}|1\rangle$$

$$|011\rangle = e^{i\phi_1}|0\rangle + e^{i\phi_2}|1\rangle$$

$$|101\rangle = e^{i\phi_1}|0\rangle + e^{i\phi_2}|1\rangle$$

5// Ket/Bra Notation (1×1) :-

$| \alpha \rangle \langle \beta | \rightarrow$ used to write it this way

So we write is as $\langle \beta | \alpha \rangle$

" my result will always be a matrix and no matter what if this operation will always be valid."

(many rows but one column)

(many column but one row)
↑

as we know $| \alpha \rangle$ represent column vector and $\langle \beta |$ represent row vector. $m \times 1$

$$(|1\rangle\langle 1| + |0\rangle\langle 0| + |2\rangle\langle 2|) (|1\rangle\langle 1| + |0\rangle\langle 0|) = 1 \times 1$$

~~zero~~ 1×1

$$\begin{matrix} \downarrow \\ m \times 1 \end{matrix} \quad \begin{matrix} \downarrow \\ n \times 1 \end{matrix}$$

$$|1\rangle\langle 1| + |0\rangle\langle 0| + |2\rangle\langle 2|$$

$$= m \times n$$

Example:-

$$100 \times 101$$

commutes

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [0010]$$

$$\begin{matrix} \downarrow \\ 100 \end{matrix} \quad \begin{matrix} \downarrow \\ 101 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex

$\therefore (I - X - I)$ relates to $A^2 + A + I$.

$$| \alpha \rangle = 3|0\rangle + i|1\rangle \text{ (given in basis) } |A\rangle = |00\rangle$$

$$\langle \beta | = |00\rangle + |10\rangle \quad | \beta \rangle = |00\rangle + 2|10\rangle + 7|11\rangle \text{ (given in basis)}$$

To find relation on basis vector is to make the three given

$$|\alpha \times \beta|$$

"basis and operate the matrices and"

(and find matrix form)

(matrix and basis form)

$$|\alpha \times \beta|^t = |B| = |001 + 2|101 + 7|111|_{\text{row}} \quad (x_1 \text{ row} \text{ row} \text{ row})$$

$m \times 1$

$1 \times m$ (row row row)

$$|\alpha \times \beta| = (|0\rangle + i|1\rangle)(|00\rangle + 2|10\rangle + 7|11\rangle)$$

$1 \times 1 \times 1$

$$= 3|0 \times 00| + 6|0 \times 10| + 2|10 \times 11| \\ + i|1 \times 00| + 2i|1 \times 10| + 7i|1 \times 11|$$

$n \times m =$

Matrix:-

= 2 rows

= 2 columns

= 4 columns

$|001001|$

$$\begin{bmatrix} 3 & 0 & 6 & 2i \\ i & 0 & 2i & 7i \end{bmatrix} \quad \begin{matrix} 0 \text{ m row} \\ 1 \text{ st row} \end{matrix}$$

1st column 2nd column 3rd column

$$[0|100] \begin{bmatrix} | \\ 0 \\ 0 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} | & 0 & 0 & 0 \\ 0 & | & 0 & 0 \\ 0 & 0 & | & 0 \\ 0 & 0 & 0 & | \end{bmatrix} =$$

Example (Ket-Bra)

$$A = \begin{vmatrix} 0 & 1 \\ 3 & i \\ 7 & 0 \\ 0 & 13 \end{vmatrix} \quad \begin{matrix} 0 \text{ row} \\ 3 \text{ row} \end{matrix}$$

$$= 100 \times 1 + 3101 \times 0 + i101 \times 1 + 7120 \times 0 + 1311 \times 1$$

6/ Bra-Ket Notation:-

$$\cancel{\langle \alpha | \beta \rangle}$$

number.
 $\alpha = 1 \times 0 \quad \beta = [0]$
 $\underbrace{1 \times n}_{n \times m} \quad \underbrace{m \times 1}_{1 \times 1}$ (inner product of two vectors)

condition $n = m$ needs to fulfill otherwise we can't multiply.

Example:-

$$\langle 0 | 0 \rangle = [1 \ 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 1 \times 1 + 0 \times 1 = 1$$

\therefore if the inner product of a vector ~~is equal~~ with itself is equals to one then it implies that its magnitude will also be equal to 1. because magnitude or Norm of any vector say Ket is equal to square root of inner product of Ket Alpha with itself



norm

$$|\alpha\rangle$$

is

$$\sqrt{\langle \alpha | \alpha \rangle}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = A$$

(28-73) sign $\times 3$

$$|\alpha\rangle = \sqrt{\langle \alpha | \alpha \rangle} = 1 \text{ (then it will be called unit vector.)}$$

$$|1x01\rangle |01\rangle + |0x01\rangle |1\rangle + |1x10\rangle |1\rangle + |0x10\rangle |0\rangle + |1x00\rangle |0\rangle =$$

Orthogonal vector:-

$$\text{When } \langle \alpha | \beta \rangle = 0$$

$$\langle 01 | \alpha \rangle = 0$$

$$\text{and } \langle 01 | \beta \rangle = [1, 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \times 0 + 0 \times 1 = 0$$

several types of basis can be defined

Example:-

$$|\alpha\rangle = |10\rangle + |11\rangle$$

$$|\beta\rangle = |10\rangle + |11\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \langle 0 | 0 \rangle = 1$$

$$\langle \alpha | \beta \rangle = ?$$

$$= 1 \times 0 + 1 \times 1 = 1$$

if that is true then $|\alpha\rangle$ is orthogonal to $|\beta\rangle$ because $|\alpha\rangle$ and $|\beta\rangle$ are linearly independent.

1st check the dimension, as in this, the dimensions are same

for any reason we can't say that $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal.

so if $|\alpha\rangle$ and $|\beta\rangle$ are linearly independent then $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal.

Fest 11 May

Sol:

$$\langle \alpha | = |\alpha \rangle^* = -i\langle 01| + 7\langle 11|$$

$$\langle \alpha | B \rangle = (-i\langle 01| + 7\langle 11|) \cdot (3|0\rangle + 1|1\rangle)$$

$$= -3i\langle 01|0\rangle - i\langle 01|1\rangle + 21\langle 11|0\rangle + 7\langle 11|1\rangle$$

$\begin{matrix} = 1 \\ \text{unit vector} \end{matrix}$

$\begin{matrix} = 0 \\ \text{orthogonal vector} \end{matrix}$

$\begin{matrix} = 0 \\ \text{unit vector} \end{matrix}$

$\begin{matrix} = 0 \\ \text{unit vector} \end{matrix}$

$$= \boxed{-3i + 7} \rightarrow \text{inner product}$$

Real number: $\langle \alpha | B \rangle = \langle B | \alpha \rangle$

Complex number: $\langle \alpha | B \rangle = \langle B | \alpha \rangle^*$

$A \text{ has } 0$

$\begin{matrix} \text{middle} \\ \times 9+1 \end{matrix}$

$\begin{matrix} \text{middle} \\ \times 9 \end{matrix}$

$$\langle 110 \rangle + \langle 010 \rangle = \langle 101 \rangle$$

group of structures no 8,0 x (7)

group of structures no 8,0 x (7)

Lec #2 Qubits & Measurements

Classical Computer:-

$$(1|S\rangle + 1|S\rangle) \otimes (1|S\rangle + 1|S\rangle) = 1|S\rangle$$

- Basic unit of data on classical computer is called bit.

$$(1|0\rangle + 1|1\rangle) \otimes (1|0\rangle + 1|1\rangle) = 1|0\rangle$$

- A bit can store either a value of 0 or a value of 1.

- In quantum computer we have quantum bits.

- In quantum computer, we have qubit.

Qubit states:-

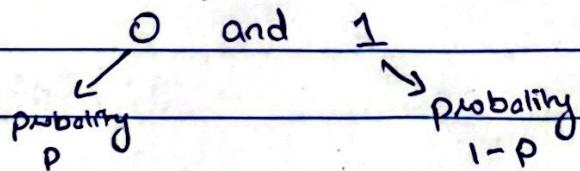
$$1|0\rangle + 1|1\rangle$$

① Pure state:-

In pure state qubit behave exactly like classical bit so it can have the value of 0 or 1.

0 or 1

② Super Position

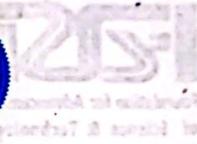


Mathematical Representation

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

α, β are amplitude of qubits.

$(\alpha, \beta \in \mathbb{C})$ $\Rightarrow \alpha, \beta$ are complex number that can also take negative values



$$\text{prob of } 0 = |\alpha|^2$$

$$\text{prob of } 1 = |\beta|^2$$

$$|\alpha|^2 = \alpha^* \alpha$$

$$|\beta|^2 = \beta^* \beta$$

Normalization constraint :-

- We know sum of probability is always equal to 1 that is why

$|\alpha|^2 + |\beta|^2 = 1$ this is called normalization constraint

$$|\Psi\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\Psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

← unit vector.

$$1 = \frac{|\alpha|^2}{|\beta|^2} + \frac{|\beta|^2}{|\alpha|^2}$$

It implies the magnitude or norm of this vector is equal to 1

$$||\Psi\rangle|| = 1$$

or

$$\langle \Psi | \Psi \rangle = 1$$

$$\left(\frac{i}{\sqrt{2}} - x \frac{i}{\sqrt{2}} \right) =$$

$$(i^2 + x^2) = 1$$

$$1 = P + \bar{P} =$$

Example (Single Qubit)

$$|1\rangle = -\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$$

State of B and

State of A and

$$\alpha^* \beta = \beta^* \alpha$$

(a) Is $|1\rangle$ a valid qubit?

$$\alpha^* \beta = \beta^* \alpha$$

(b) What is the probability of $|1\rangle$ equal to zero, and minimum

prob of 1 being equal to 1 if we want 0.5.

Part a) Normalization of basis of basis $|0\rangle = \frac{1}{5}|0\rangle + \frac{3}{5}|1\rangle$

- ✓ $\cdot |\alpha|^2 + |\beta|^2 = 1$ } method.
- $|1| |1\rangle |1| = 1$ } we can
- $\langle 1 | 1\rangle = 1$ } use

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\left| -\frac{4}{5} \right|^2 + \left| \frac{3}{5} \right|^2 = 1$$

$$\left| \frac{3}{5} \right|^2 = \left| \frac{4}{5} \right|^2$$

$$\alpha \downarrow \quad \beta \downarrow$$

$$|\alpha|^2 = \alpha^* \alpha$$

$$= \left(\frac{4}{5} i \times -\frac{4}{5} i \right)$$

$$1 = \langle 1 | 1 \rangle = 1$$

$$|\beta|^2 = \left(\frac{3}{5} i \times \frac{3}{5} i \right)$$

$$= \frac{16}{25} + \frac{9}{25} = 1$$

Part (b) :-

$$\text{Prob of } 0 = \left| -\frac{4}{5} i \right|^2$$

$$= \frac{16}{25} = 0.64$$

$$\text{Prob of } 1 = 1 - 0.64 = 0.36$$

$$= \left| \frac{3}{5} \right|^2 = 0.36$$

(Multiple Qubit):

• Tensor Product

$$|\Psi\rangle = -\frac{4}{5}|10\rangle + \frac{3}{5}|11\rangle$$

$$|\Phi\rangle = \frac{|10\rangle}{\sqrt{2}} + \frac{|11\rangle}{\sqrt{2}}$$

$$|\Psi \otimes \Phi\rangle = \left(-\frac{4}{5}i|10\rangle + \frac{3}{5}|11\rangle \right) \left(\frac{|10\rangle}{\sqrt{2}} + \frac{|11\rangle}{\sqrt{2}} \right)$$

$$= \frac{(A)}{\sqrt{2}}|100\rangle + \frac{(B)}{\sqrt{2}}|101\rangle + \frac{(C)}{\sqrt{2}}|110\rangle + \frac{(D)}{\sqrt{2}}|111\rangle$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

Prob 00

$$|A|^2 = \left| \frac{-4}{5\sqrt{2}} i \right|^2$$

$$\left| \frac{i}{\sqrt{2}} \right|^2 = 0.50 \text{ daB}$$

$$= \left(\frac{4i}{5\sqrt{2}} \times \frac{-4i}{5\sqrt{2}} \right) = \frac{16}{50} = 0.32$$

Prob 01

$$|B|^2 = \left| \frac{-4}{5\sqrt{2}} i \right|^2$$

$$0.50 = \left| \frac{i}{\sqrt{2}} \right|^2$$

$$= \frac{16}{50} = 0.32$$

Prob 10

$$|C|^2 = \left| \frac{3}{5\sqrt{2}} \right|^2$$

$$CII + COI - CBI$$

$$= \left(\frac{9}{50} = 0.180 \right) \left(CII + COI - CBI \right) = 0.180$$

Prob 11

$$|D|^2 = \left| \frac{3}{5\sqrt{2}} \right|^2 = \frac{9}{50} = 0.18$$

$$0.32 + 0.32 + 0.18 + 0.18 = 1$$



Operation on Qubit:-

- Measure qubit.
 - Transform using quantum gates.

Now on Washington 10/1/14 2011 + 2014 = 2015 10/1/14 10/1/14

Full - Partial

Full measurement:-

It implies that given n -qubits register you measure all the n -qubits.

Partial measurement

It implies that given n -qubit register you measure a subset of these n -qubits.

Ruler

i) Superposition

m → pure state

(2) Normalization constraint must always uphold.

Ex 1)

$$|14\rangle = \frac{1}{\sqrt{2}} |100\rangle - \frac{i}{\sqrt{2}} |110\rangle + \frac{1}{\sqrt{2}} |111\rangle \quad \text{Cost: } (817 + 741) = 1558$$

Full measurement in

Measurement	Probability	Resultant State
00	$ \psi_2 ^2 = \gamma_4 = 0.25$	$ \Psi\rangle = 00\rangle$
10	$ \psi_3 ^2 = \gamma_4 = 0.25$	$ \Psi\rangle = 10\rangle$
11	$ \psi_5 ^2 = \frac{1}{2} = 0.5$	$ \Psi\rangle = 11\rangle$

$$\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\left(-\frac{i}{2} + \frac{1}{\sqrt{2}} \right) \times \frac{\sqrt{4}}{\sqrt{3}}$$

$$\frac{2}{\sqrt{2} \times \sqrt{3}}$$

Partial measurement:-

Measurement	Probability	Resultant State
Isqubit = 1	$\left -\frac{i}{2} \right ^2 + \left \frac{1}{\sqrt{2}} \right ^2$ $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$	$ 4\rangle = \frac{-i}{2} 10\rangle + \frac{1}{\sqrt{2}} 11\rangle$ (not normalized so we divide it with the norm) $\frac{1}{\sqrt{3}}(-\frac{i}{2} 10\rangle + \frac{1}{\sqrt{2}} 11\rangle) \rightarrow$ it will have the same of probability $= \frac{-i}{2} 10\rangle + \frac{1}{\sqrt{3}} 11\rangle$
Isqubit = 0	$\left \frac{1}{2} \right ^2 + \left -\frac{i}{2} \right ^2$ $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$	$ 4\rangle = \frac{1}{2} 100\rangle - \frac{i}{2} 110\rangle$ $= \frac{1}{\sqrt{2}} 100\rangle - \frac{i}{\sqrt{2}} 110\rangle$

Example:-

$$\frac{1}{\sqrt{5}}|10000\rangle - \sqrt{\frac{2}{5}}|10100\rangle + \sqrt{\frac{1}{5}}|1111\rangle + \sqrt{\frac{1}{5}}|10110\rangle$$

What is the prob and resultant state if 1st and 4th qubit are 0.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$\text{Prob} = \left| \frac{1}{\sqrt{5}} \right|^2 + \left| -\sqrt{\frac{2}{5}} \right|^2 + \left| \sqrt{\frac{1}{5}} \right|^2 = \frac{4}{5}$$

$$\text{Resultant state} = |4\rangle + \frac{1}{\sqrt{5}}|10000\rangle - \sqrt{\frac{2}{5}}|10100\rangle + \frac{1}{\sqrt{5}}|10110\rangle$$

$$\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}$$

$$= \frac{1}{2} |10000\rangle - \frac{1}{\sqrt{2}} |0100\rangle + \frac{1}{2} |0110\rangle$$

Measuring from basis vector (0110) & (0100)

Measuring in Orthonormal Basis

1) Standard Basis

is a set of linearly independent vectors where each vector has exactly one 1 in $\langle i|i \rangle$ where i is the index of the elements of that vector are all zeros.

$$\text{2D: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\left(\frac{\langle 1|1 \rangle + \langle 0|0 \rangle}{\sqrt{2}}, \frac{\langle 1|0 \rangle + \langle 0|1 \rangle}{\sqrt{2}} \right)$$

$$= \left[|10\rangle, |11\rangle \right] \quad \left[\langle 11|11 + \langle 01|01 + \langle 11|01 + \langle 01|01 \right] =$$

$$3D: \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$I = 6 \times 1 =$$

$$4D: \left\{ |100\rangle, |101\rangle, |110\rangle, |111\rangle \right\}$$

$$I = 4 \times 3 = 12 = 11 \times 11$$

2) Orthonormal Basis

Composed of two different words orthogonal and normalized. That bases must have vectors which are orthogonal and as well normalized.

$$|111\rangle = \frac{1}{\sqrt{3}} (|111\rangle + |011\rangle - |110\rangle - |010\rangle)$$

1) Normalized "unit vector"

$$\langle 001 \rangle + \langle 010 \rangle = \sqrt{1+1} = \sqrt{2}$$

2) Orthonormal " \perp "

Example:-

$$\langle 1+>, 1-> \quad 1+> = \frac{\langle 0> + \langle 1>}{\sqrt{2}}, \quad 1-> = \frac{\langle 0> - \langle 1>}{\sqrt{2}}$$

unit vector $\langle 1+>$ $\langle 1->$

$$\|\langle 1+>\| = \sqrt{\langle 1+> \cdot \langle 1+>} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1}$$

$$= \sqrt{\left(\frac{\langle 01 \rangle + \langle 11 \rangle}{\sqrt{2}} \right) \left(\frac{\langle 10 \rangle + \langle 11 \rangle}{\sqrt{2}} \right)}$$

$$= \frac{1}{2} \left[\langle 010 \rangle + \langle 011 \rangle + \langle 110 \rangle + \langle 111 \rangle \right]$$

$$= \frac{1}{2} \times 2 = 1$$

unit vector

$$\|\langle 1->\| = \sqrt{\langle 1-> \cdot \langle 1->}$$

$$= \sqrt{\left(\frac{\langle 01 \rangle - \langle 11 \rangle}{\sqrt{2}} \right) \left(\frac{\langle 10 \rangle - \langle 11 \rangle}{\sqrt{2}} \right)}$$

$$= \frac{1}{2} \left[\langle 010 \rangle - \langle 011 \rangle - \langle 110 \rangle + \langle 111 \rangle \right] = \sqrt{\frac{1}{2} \times 2} = \sqrt{1} = 1$$

orthogonal

If they are orthogonal their inner product will be zero

$$\langle +1- \rangle = \left(\frac{\langle 01 + 11 \rangle}{\sqrt{2}} \right) \left(\frac{\langle 10 - 11 \rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2} \left[\frac{\langle 010 \rangle}{1} - \frac{\langle 011 \rangle}{0} + \frac{\langle 110 \rangle}{0} - \frac{\langle 111 \rangle}{1} \right]$$

$$= \frac{1}{2} \times 0 = 0$$

3/ Formula :-

To write any vector in the orthogonal basis of your liking.

$$\{ | \alpha \rangle, | \beta \rangle \}$$

$$| \Psi \rangle = \langle \Psi | \alpha \rangle | \alpha \rangle + \langle \Psi | \beta \rangle | \beta \rangle$$

amplitude
number

amplitude
number

Probability of measuring

$$| \alpha \rangle = 1 \langle \Psi | \alpha \rangle | \alpha \rangle$$

if $|\Psi\rangle = |\Psi\rangle_1 + |\Psi\rangle_2$ then $| \alpha \rangle = (|\alpha\rangle_1 + |\alpha\rangle_2) / \sqrt{2}$

Probability of measuring

$$| \beta \rangle = 1 \langle \Psi | \beta \rangle | \beta \rangle$$

$$(|\beta\rangle_1 - |\beta\rangle_2) / \sqrt{2}$$

Example

14) $|10\rangle$ is incident upon with superposition in part of

Standard basis

$$\text{probability } m_0 = |11|^2 = \left| \frac{1+1}{\sqrt{2}} \right|^2 = \left(\frac{1+1}{\sqrt{2}} \right)^2 = \frac{1+1}{2}$$

$$\text{probability } m_1 = 0$$

Hadamard Basis:-

$$|1+\rangle, |1-\rangle$$

$$\begin{pmatrix} |11\rangle - |00\rangle \\ |10\rangle + |01\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |11\rangle - |00\rangle \\ |10\rangle + |01\rangle \end{pmatrix}$$

$$\text{prob of } m_{1+} = |\langle 1+\rangle|^2$$

$$\langle 1+\rangle = \left(\frac{|01\rangle}{\sqrt{2}} \right) \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}} \right)$$

$$= \frac{\langle 010\rangle + \langle 011\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}}$$

$$\langle 01| \langle 01| + \langle 01| \langle 01| = \langle 01|$$

$$|\langle 1+\rangle|^2 = \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right) = \frac{1}{2} = 0.5$$

$$\text{prob of } m_{1-} = |\langle 1-\rangle|^2$$

$$= \left(\frac{\langle 01\rangle}{\sqrt{2}} \right) \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left(\frac{\langle 010\rangle - \langle 011\rangle}{\sqrt{2}} \right) = \frac{1}{2} \cancel{\frac{1}{\sqrt{2}}}$$

$$1 < \psi_1 - \psi_2 = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{6}}$$

Exemple

$$|1\rangle = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) |+\rangle + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \right) |-\rangle$$

Solv Hadamard Basis

$$\text{Prob of } 1+>= \left| \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right)^2 \right| = \frac{1}{6} + \frac{1}{3} - 2 \cdot \frac{1}{\sqrt{6}\sqrt{3}} = \frac{1}{6} + \frac{1}{3} - \frac{\sqrt{2}}{3}$$

$$= 1+2-2\sqrt{2} = 3-2\sqrt{2}$$

$$6<11>\left[\frac{6}{11} \left(\frac{11}{6} - \frac{11}{11} \right) + 11 \cdot \frac{1}{11} \left(\frac{11}{11} - \frac{11}{11} \right) \right] = 6<11>$$

$$\text{prob of } 1 \rightarrow = 1 - \frac{3-2\sqrt{2}}{4} = \frac{3+2\sqrt{2}}{4}$$

$$|+10\rangle = \frac{(|01\rangle + |11\rangle)}{\sqrt{2}} |10\rangle$$

$$= \overline{\langle 0|0\rangle} + \overline{\langle 1|1\rangle}$$

Standard Basis $\{10>, 11>\}$

$$P_{\text{prob}} \propto |\langle \psi | \hat{M} | 10 \rangle|^2$$

$$\langle -10 \rangle \left(\frac{\langle 01 \rangle - \langle 11 \rangle}{\sqrt{2}} \right) \langle 10 \rangle$$

$$\langle \Psi | \alpha \rangle = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \alpha L + 1 + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \right) \alpha L - 1 \cdot [10\rangle = \frac{1}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \langle +10 \rangle + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \right) \langle -10 \rangle$$

$$= \left(\frac{y_1}{\sqrt{6}} - \frac{y_2}{\sqrt{2}} \right) \left(\frac{y_1}{\sqrt{2}} \right) + \left(\frac{y_1}{\sqrt{6}} + \frac{y_2}{\sqrt{2}} \right) \left(\frac{y_2}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \right] = \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2}}{\sqrt{3}}$$

$$|\psi_{10}\rangle|^2 = \frac{1}{3} \quad \langle -1 \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) + \langle +1 \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \right) = |\psi_1\rangle$$

now $\langle +1 \rangle = \langle 01 \rangle + \langle 10 \rangle = \frac{1}{2} (\langle 01 \rangle + \langle 10 \rangle) / = \langle +1 \rangle$ for doing

$$\text{Prob } (m=0) \langle 11 \rangle = |\psi_{11}\rangle|^2$$

$$\langle \psi_{11} \rangle = \left[\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \langle +1 \rangle + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \langle -1 \rangle \right] \langle 11 \rangle$$

$$\langle 011 \rangle = \left[\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \underline{\underline{\langle +11 \rangle}} + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \right) \underline{\underline{\langle -11 \rangle}} \right]$$

$$\langle +11 \rangle = \langle 01 \rangle + \langle 10 \rangle \langle 11 \rangle$$

$$\therefore \langle 011 \rangle + \langle 111 \rangle = \frac{1}{\sqrt{2}}$$

$$\langle -11 \rangle = \frac{\langle 01 \rangle - \langle 10 \rangle}{\sqrt{2}} \langle 11 \rangle$$

$$= -\frac{1}{\sqrt{2}}$$

~~$$|\psi_{11}\rangle^2 = \frac{1}{2} \langle 011 \rangle + \langle 111 \rangle$$~~

Quantum Gate

$U|\alpha\rangle = |\beta\rangle$ (no logic or trap signs) \rightarrow U is unitary matrix
 qubits \rightarrow transform qubits

The operation U must be special which preserve the sum of probability that means this operation U must preserve the norm of a vector and that why we know that there only 1. Single type of matrix that preserve norm of a vector is called unitary matrix.

- $U^+U = UU^+ = I$ (identity)
- $U^+ = U^{-1}$
- $H^+ = H_0/2 + (H \otimes I)$ (Hermitian matrix)

Example - $(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}) \cdot (\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}) = (\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix})$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$Y^+Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -ixi & 0 \\ 0 & -ixi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Gate :-

Pauli-x: \boxed{x} Pauli-x (single qubit as input and single qubit as output)
 like all quantum gates it is reversible and it is inverse of itself. It is because this gate is both (unitary and Hermitian). All the other gates like U & H are inverse of themselves.

Matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example:

$$x|0\rangle = |1\rangle \quad I = U^\dagger U$$

$$x|1\rangle = |0\rangle \quad U^\dagger = U$$

$$x(\alpha|0\rangle + \beta|1\rangle) = \alpha x|0\rangle + \beta x|1\rangle$$

$$(x|0\rangle = \alpha|1\rangle + \beta|0\rangle) \quad U^\dagger = U$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0\alpha + 1\beta \\ 1\alpha + 0\beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \beta|0\rangle + \alpha|1\rangle$$

Pauli-z: \boxed{z} is also a single qubit gate so it

takes a single qubit as input and produces a single qubit output, and it is also U & H gate that means

it is inverse of itself. So if we apply two - pauli-z gate

in sequence they will cancel each other effect and my input will become equal to my - output.

Matrix:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example:

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

Superposition:-

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

$$\begin{pmatrix} 3|0\rangle & 3|0\rangle \\ 3|0\rangle & 3|0\rangle \end{pmatrix} = \alpha|0\rangle - \beta|1\rangle$$

Hadamard Gate:

\boxed{H} it is a single qubit gate which is also $U \in H$.

Matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Example:

$$H|0\rangle = |0\rangle + \frac{|1\rangle}{\sqrt{2}} = |+\rangle$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$H|1\rangle = |0\rangle - \frac{|1\rangle}{\sqrt{2}} = |- \rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H|+\rangle = |0\rangle$$

$$H|-\rangle = |1\rangle$$



Rotation Gate:

R_θ — this is unitary gate but not Hermitian
 U but not H .

Matrix:

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$\cos \theta - \cos \theta = (\cos \theta + \cos \theta) \cdot S$$

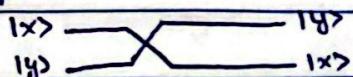
Example:

$$R_\theta + R_\theta^*$$

$$R_\theta^* \cdot R_\theta = I$$

Two Qubit gates:

Swap gate



$$U \equiv H \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sqrt{2} = H$$

Matrix:

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$CH = CII + SII = SII \cdot H$$

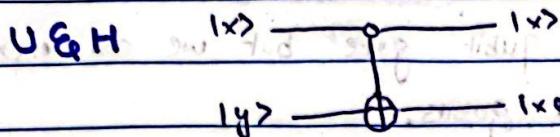
$$CII = SII \cdot CII = SII \cdot H$$

Example:

$$\text{Swap } |01\rangle = |10\rangle \rightarrow \text{test}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

CNOT..



Matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example:

$$|CNOT|110> = 111>$$

$$|CNOT|101> = 101>$$

(control bit is 1 if it is 0 then the other will be flip)

$$|CNOT|011> = 011>$$

$$|CNOT|001> = 011>$$

normalizing factor

$$\sqrt{2} |CNOT|001> = \sqrt{2}|011>$$

$\frac{1}{\sqrt{2}}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} |011>$$

$$= \frac{1}{\sqrt{2}} |011>$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} |011> = \frac{1}{\sqrt{2}} |011>$$

Hadamard (Single Qubit)

Hadamard gate is a single qubit gate but we can expand it and apply it on multiple qubits.

$$\text{matrix of } H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

because it is a two by two matrix so we can apply this matrix on single qubit.

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \text{complex number}$$

$$H|\psi\rangle = |\psi'\rangle$$

$$|\psi'\rangle = \langle 01|T_{00}\rangle$$

Example:

$$H|0\rangle = |0\rangle + |1\rangle = \frac{1}{\sqrt{2}} \quad \text{Equal Superposition}$$

$$H|1\rangle = |0\rangle - |1\rangle = \frac{1}{\sqrt{2}}|-\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1+0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} |+\rangle$$

(initial) state (final) state

$$= \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle] - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

\therefore Hadamard gate is both unitary and Hermitian.

unitary gate :-

$$HH^\dagger = \text{Identity}$$

Hermitian :-

$$[|0\rangle\langle 0|]^\dagger H = [|0\rangle\langle 0|]H$$

$$H = H^\dagger |0\rangle\langle 0|$$

$$\langle 0|^\dagger \langle 0| = \langle 0| \langle 0|$$

Combining both we can write:-

$$\langle 0|^\dagger \langle 0| + \langle 1|^\dagger \langle 1| + H^\dagger H = H$$

Example:-

$$H \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{H|0\rangle}{\sqrt{2}} - \frac{H|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

2

$$= \cancel{\frac{|1\rangle}{\sqrt{2}}} - \cancel{\frac{|1\rangle}{\sqrt{2}}}$$

$$H^\dagger H = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} I_4$$

$$\begin{array}{c} \text{Root} \\ \hline \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \end{array} \quad \begin{array}{c} \text{Root} \\ \hline \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \end{array} =$$

$$\begin{array}{c} \text{Root} \\ \hline \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \end{array}$$

Hadamard (Multiple Qubit)

example:- $\xrightarrow{\text{Tensil product}}$

$$H^{\otimes 2} |100\rangle$$

$$\langle 111+001 | \quad [111+001] \quad 64$$

1 method (matrix)

$$H^{\otimes 2} |100\rangle$$

$$H^{*2} = H \otimes H$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

often used method:-

~~$H^{\otimes 2}$~~ $\hat{H}^{\otimes 2} = +I + H$

$H^{\otimes 2} |100\rangle = H^{\otimes 2} [10|10\rangle]$

$= H|10\rangle H|10\rangle^H = H$

$= |10\rangle + |01\rangle \quad |10\rangle + |11\rangle$

$\rightarrow \frac{\sqrt{2}}{2} |100\rangle + \frac{\sqrt{2}}{2} |110\rangle$

$H = |100\rangle + |101\rangle + |110\rangle + |111\rangle$

$(\hat{H}^{\otimes 2} |100\rangle)$

$\langle 111 | \quad \langle 001 | =$

$\langle 111-001 | \quad \langle 101+011 | =$

$\cancel{\langle 111-001 |} \quad \cancel{\langle 101+011 |} =$

$\langle 111+001 | \quad \langle 111+001 | =$

6

$\cancel{\langle 111-001 |} \quad \cancel{\langle 111+001 |} =$

Hadamard (For n qubit)

$$H|0\rangle = |0\rangle + |1\rangle, H|1\rangle = |0\rangle - |1\rangle$$

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \rightarrow$$

$$\text{Dwider} \rightarrow \frac{\sqrt{2}}{\sqrt{2}} |0\rangle + \frac{\sqrt{2}}{\sqrt{2}} |1\rangle$$

$$H|1\rangle = |0\rangle + (-1)^x |1\rangle$$

$$x \in \{0,1\} \rightarrow \sqrt{2}$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle - \textcircled{1}$$

$$H^{\otimes n} |0\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=1}^{2^n} |y\rangle$$

↓

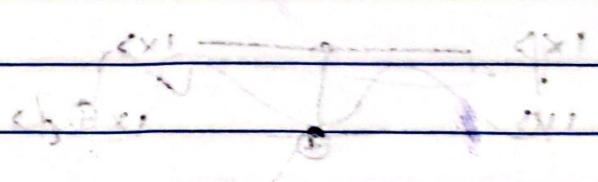
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = I_{4 \times 4}$$

Dwider

$H^{\otimes 2}$

and $\sqrt{2}$ times did taught in combination of it. Now for next few things will explain what is meant by $|0\rangle$ and $|1\rangle$ and how they are related to each other.

> Local & Local ports
$ 000\rangle$ & $ 001\rangle$ ports
$ 010\rangle$ & $ 011\rangle$ ports
$ 100\rangle$ & $ 101\rangle$ ports



$|00\rangle$ & $|01\rangle$ ports

$|10\rangle$ & $|11\rangle$ ports

$|000\rangle$ & $|001\rangle$ ports

$|010\rangle$ & $|011\rangle$ ports

Controlled NOT

Unitary

$$CNOT \times CNOT^* = I \quad \therefore CNOT^* = CNOT^{-1} = CNOT^T$$

Hamilton

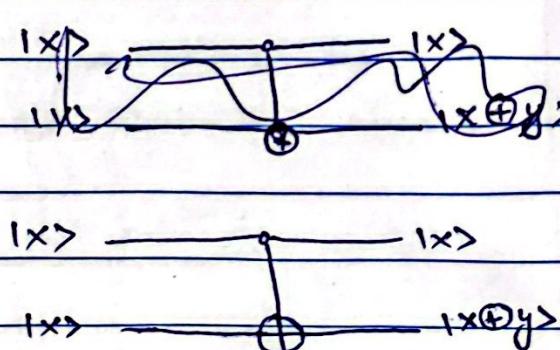
$$CNOT = CNOT^*$$

$$CNOT^* = CNOT$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4x4

First qubit is represented as a control bit and if the first qubit is 1 then it flips the second qubit. If the first qubit is 0 then the second qubit remain the same.



$$CNOT|00\rangle = |00\rangle$$

$$CNOT|10\rangle = |11\rangle$$

$$CNOT|01\rangle = |01\rangle$$

$$CNOT|11\rangle = |10\rangle$$

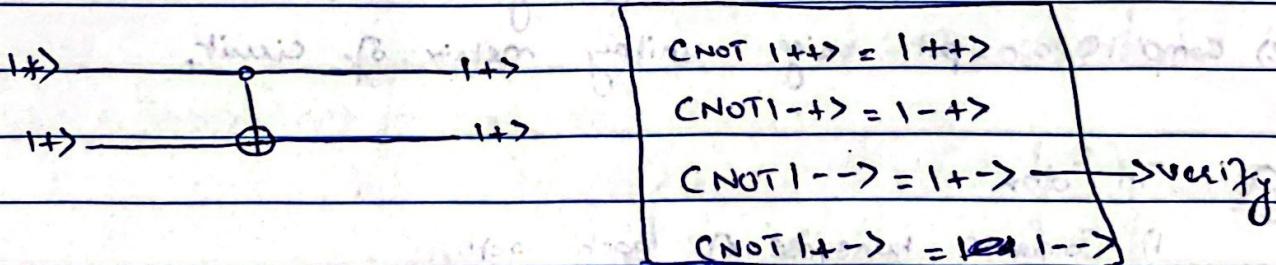
example:-

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}|111\rangle$$

example

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |- \rangle$$

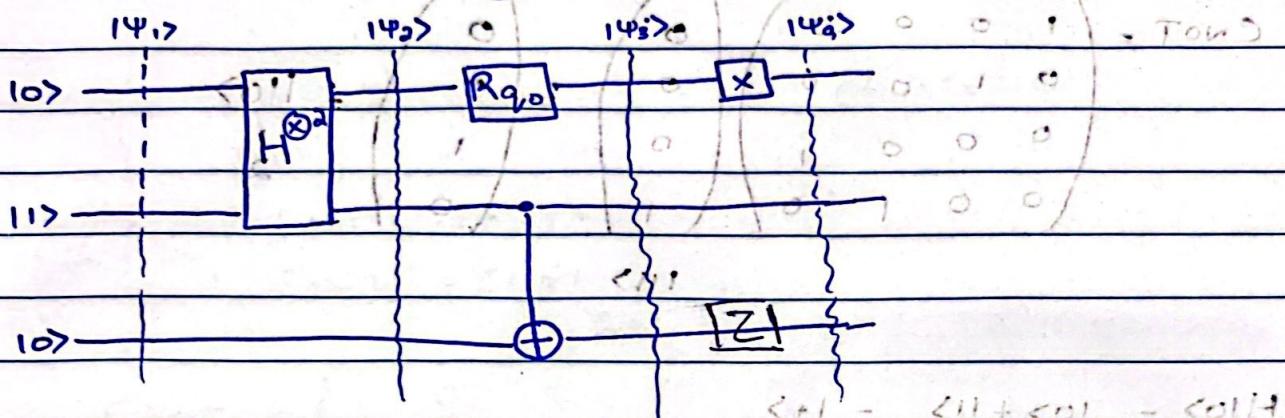


$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{\sqrt{0} - \sqrt{1}}{\sqrt{2}}$$

$$= |00\rangle - |01\rangle - |10\rangle + |11\rangle$$

$$= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

Quantum Circuit



$$\langle \psi | = \langle 11 | \psi | = \langle 01 |$$

a) make reverse circuit

b) write output of the circuit.

$$\langle \psi | = \langle 11 | \psi | = \langle 111 |$$

c) write complete circuit as unitary matrix.

$$\langle \psi | = \langle 11 | \psi | = \langle 111 |$$

d) write reverse circuit as unitary matrix.

$$\langle \psi | = \langle 11 | \psi | = \langle 111 |$$

e) complete output using unitary matrix of circuit.

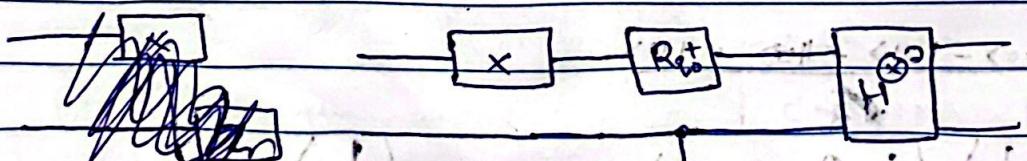
$$\langle \psi | = \langle 11 | \psi | = \langle 111 |$$

a)

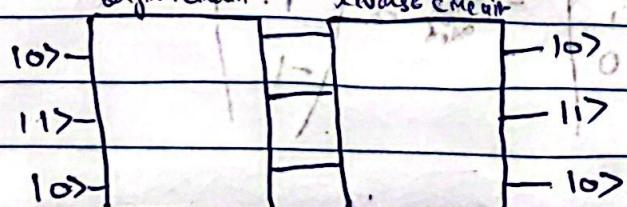
i) Conjugate transpose of each gate

$$U^{-1} = U^+$$

$$H^+ = H$$



original circuit. | reverse circuit



b)

$$|1111\rangle = |0011\rangle - |1101\rangle + |0001\rangle - |1110\rangle$$

$$|\Psi_1\rangle = |0\rangle |1\rangle |0\rangle$$

$$|\Psi_2\rangle = H^{\otimes 3}(|0\rangle |1\rangle) |0\rangle$$

$$= \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \quad |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \quad |0\rangle$$

$$= \frac{1}{2} (|00\rangle - |01\rangle + |10\rangle - |11\rangle) |0\rangle$$

$$= \frac{1}{2} (|00\rangle - |01\rangle + |10\rangle - |11\rangle) |0\rangle$$

$$|\Psi_3\rangle = R_{90^\circ} |\Psi_2\rangle$$

$$\times \text{ rotation } \theta \text{ around } z \text{-axis} : R_z \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} (|100\rangle - |110\rangle - |000\rangle + |010\rangle)$$

$$\times \text{ rotation } \theta \text{ around } z \text{-axis} : R_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle$$

= CNOT gate on second bit

$$|\Psi_3\rangle = \frac{1}{2} (|100\rangle - |111\rangle - |100\rangle + |011\rangle) + |000\rangle + |011\rangle - |100\rangle$$

$$= \frac{1}{2} (|111\rangle - |111\rangle)$$

$$|\Psi_4\rangle = \text{Pauli-X} \therefore \text{flip 1 to 0 and 0 to 1}$$

$$\text{Pauli-Z} \therefore \text{change sign of qubit 1 if qubit is 11}$$

$$|100\rangle - |000\rangle + |011\rangle - |100\rangle \mp |111\rangle$$

2

$$|010\rangle |011\rangle |010\rangle - |111\rangle$$

$$|010\rangle |011\rangle |010\rangle \otimes H = |011\rangle$$

(c)

$$\text{Stage}_1 = H \otimes H \otimes I_{8 \times 8}$$

$$|001\rangle |011\rangle |011\rangle - |001\rangle$$

$$|111\rangle |111\rangle |111\rangle$$

$$\text{Stage}_2 = R_y \otimes CNOT$$

6

$$|00111\rangle + |00011\rangle + |00101\rangle - |00111\rangle$$

$$\text{Stage}_3 = X \otimes I \otimes Z_{8 \times 8}$$

6

~~OP = 0~~

$$(S_1 \times S_2 \times S_3) \otimes S_1 \times S_2 \times S_3 \times |11111\rangle = |11111\rangle$$

$$(S_3 \times S_2 \times S_1)$$

$$S_3 \times S_2 \times S_1 \checkmark$$

$$|01011\rangle + |00011\rangle - |00111\rangle - |00111\rangle$$

$$C1. (i) = (S_3 (S_2 (S_1 1010)))$$

6

(d)

Following answer step 7003 =

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~Step 7003~~

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6

$$x - 11001 = 00111$$

$$|11111\rangle + |11001\rangle - |10111\rangle - |10001\rangle : x - 11001 = 00111$$

$$|11111\rangle + |11001\rangle - |10111\rangle - |10001\rangle : x - 11001 = 00111$$

$$S_2 = R_{q_0} \otimes CNOT$$

(mid) (incorrect)

$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \{0000\} & : -1000: \\ \{0000\} & : 0100: \\ \{0000\} & : 0001: \\ \{0000\} & : 0010: \end{array} \right)$$

$$\left(\begin{array}{c|c} \{1000\} & : 0000: \\ \{0100\} & : 0000: \\ \{0001\} & : 0000: \\ \{0010\} & : 0000: \end{array} \right)$$

~~$S_3 = \text{X} \otimes \text{X} \otimes \text{I} \otimes \text{Z}$~~

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \{00\} & \{10\} \\ \{00\} & \{01\} \end{array} \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{c|c} \{10\} & \{00\} \\ \{01\} & \{00\} \end{array} \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_3 \times S_2 \times S_1$$

e) $(S_3 \times S_2 \times S_1) (D_{10})$

$$(S_3 \times S_2 \times S_1)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_3 \otimes I \otimes S_1 \times \text{Rotations}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} (0, 1) & (0, 0) \\ (0, 0) & (0, 1) \end{pmatrix}$$

Entanglement

entanglement can be describe using two different equivalent definitions.

① Given a state of multiple entangled qubits one cannot express individual qubits separately.

Example:-

$$\frac{100}{\sqrt{2}} + \frac{111}{\sqrt{2}} = (\alpha|10\rangle + \beta|11\rangle) \otimes (|\psi\rangle + |\phi\rangle)$$

so they are entangled.

Separable

$$\frac{101}{\sqrt{2}} + \frac{100}{\sqrt{2}} = |10\rangle \otimes |10\rangle + |11\rangle$$

② Given a state of multiple entangled qubits, measuring any qubit individually reveals all other qubits.

Example:- (Entanglement)

$$\frac{100}{\sqrt{2}} + \frac{111}{\sqrt{2}} \quad \begin{array}{l} \text{1st qubit } m = 1/0 \\ \text{2nd qubit will also be 1} \end{array}$$

$$2nd \text{ qubit } m = 1/0$$

Separable qubit

$$|101\rangle + |100\rangle \quad \text{1st qubit } |10\rangle$$

forms $\sqrt{2}$

$$\text{2nd qubit } : |10\rangle - |11\rangle$$

$$|11\rangle \text{ prob } \frac{1}{2}$$

Bell State (EPR States)

We have 4 entangled qubit to describe this bell state.

$$\textcircled{1} \quad |\Psi^+\rangle = |B_{00}\rangle = |100\rangle + |111\rangle$$

$\sqrt{2}$

$$\textcircled{2} \quad |\Psi^-\rangle = |B_{10}\rangle = |100\rangle - |111\rangle$$

$\sqrt{2}$

$$\textcircled{3} \quad |\Psi^+\rangle = |B_{01}\rangle = |101\rangle + |110\rangle$$

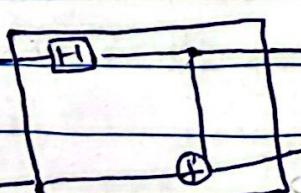
$\sqrt{2}$

$$\textcircled{4} \quad |\Psi^-\rangle = |B_{11}\rangle = |101\rangle - |110\rangle$$

$\sqrt{2}$

quantum circuit to create this bell state.

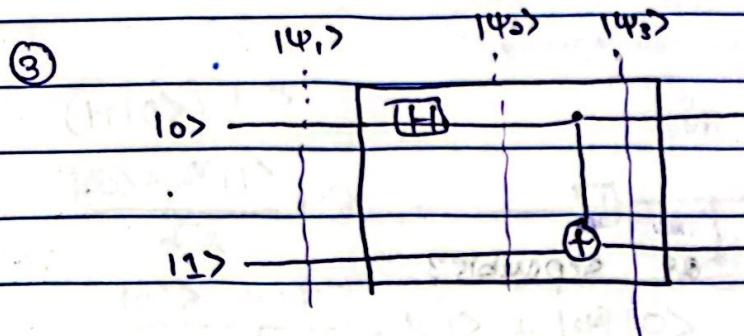
|i>



$|B_{ij}\rangle$

|j>

$|11\rangle - |00\rangle$



$$|101\rangle = |10\rangle \otimes |11\rangle$$

$$\text{CNOT}_{Bij} \quad B_i$$

$$|101\rangle = |10\rangle H |11\rangle = |101\rangle$$

$$(CNOT + CNOT) \otimes (CNOT + CNOT) = |101\rangle - |101\rangle$$

$$|101\rangle = (|10\rangle H) |11\rangle$$

$$|101\rangle = \frac{(|10\rangle + |11\rangle) - (|11\rangle + |10\rangle)}{\sqrt{2}} = \frac{|11\rangle - |10\rangle}{\sqrt{2}}$$

$$|101\rangle = |101\rangle + |111\rangle$$

$$|101\rangle = \frac{|101\rangle + |111\rangle}{\sqrt{2}}$$

$$|101\rangle = |101\rangle$$

$$|101\rangle$$

Given :-

$$|\psi\rangle = |\psi_1\rangle - |\psi_2\rangle$$

$\sqrt{2}$

$\langle \psi | \psi \rangle$

$\langle \psi_1 | \psi_1 \rangle$

$\langle \psi_1 |$

$\langle \psi_1 |$

$\langle \psi_1 |$

(1)

Q:- Find $|\psi\rangle$ is entangled or separable?

Proof :- Proof by contradiction.

$$\langle \psi_1 | \psi_1 \rangle = \langle \psi_1 | \psi_1 \rangle = \langle \psi_1 | \psi_1 \rangle$$

$$|\psi\rangle = (\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \otimes (\delta|\psi_1\rangle + \gamma|\psi_2\rangle)$$

$\sqrt{2}$

$$(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \otimes (\delta|\psi_1\rangle + \gamma|\psi_2\rangle)$$

$$|\psi\rangle = (\alpha\delta|\psi_1\rangle + \alpha\gamma|\psi_2\rangle + \beta\delta|\psi_1\rangle + \beta\gamma|\psi_2\rangle)$$

$\sqrt{2}$

~~So~~ B and γ cannot be equal to $\alpha\delta + \beta\gamma$

$$\alpha\delta = 0$$

But $B \neq 0$ $\gamma \neq 0$ so this equation cannot exist this is ~~contradiction~~.

$$\alpha\delta = \frac{1}{\sqrt{2}}$$

Contradiction L.H.S \neq R.H.S

$$\text{right hand side} = \frac{\beta\gamma}{\sqrt{2}}$$

Thus $|\psi\rangle$ is entangled.

$$\alpha\delta = 0$$

either $B = 0$ or $\gamma = 0$

not present

at right hand side

β cannot be equal

to 0

otherwise (3) will be zero

$\beta \neq 0$

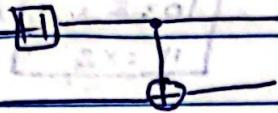
$\gamma \neq 0$

How to create Bell state
(Einstein-Podolsky-Rosen pair)

$$|H10\rangle |11\rangle$$

\downarrow
EPR

$$\frac{|H\rangle + \sqrt{2}|1\rangle}{\sqrt{2}} |11\rangle$$

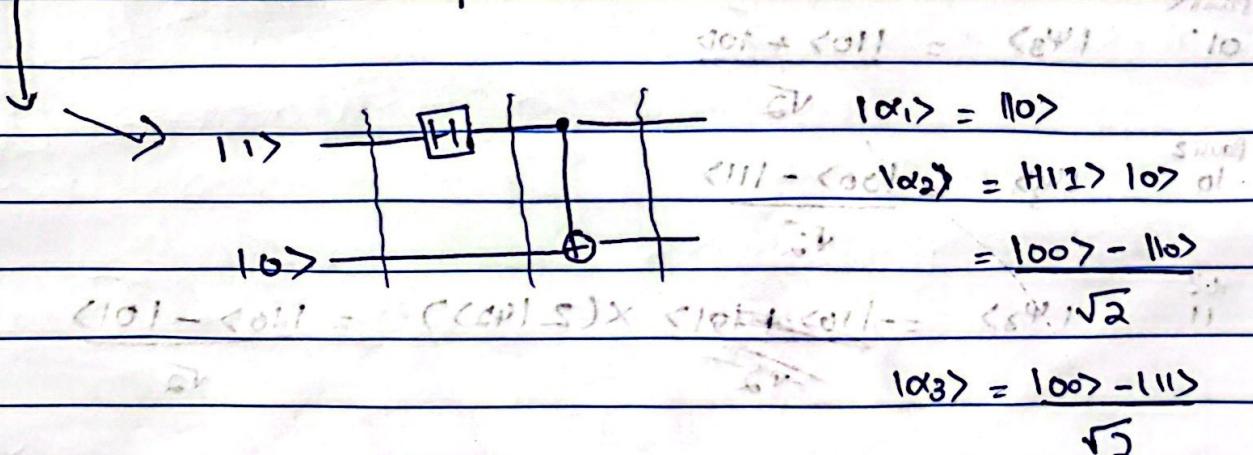


$$\frac{|10\rangle + |11\rangle}{\sqrt{2}} = \frac{|10\rangle + |11\rangle}{\sqrt{2}}$$

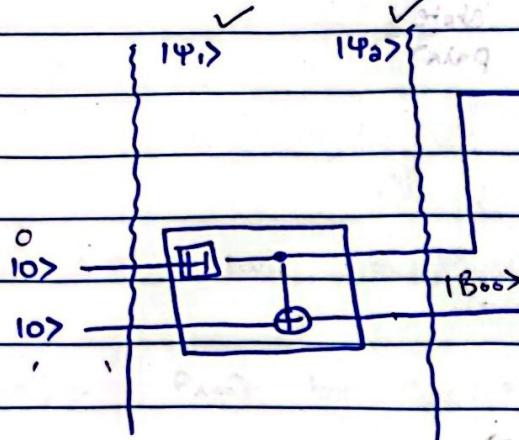
General formula:

$$|xy\rangle \rightarrow \frac{|xy\rangle + (-1)^x |y\rangle}{\sqrt{2}}$$

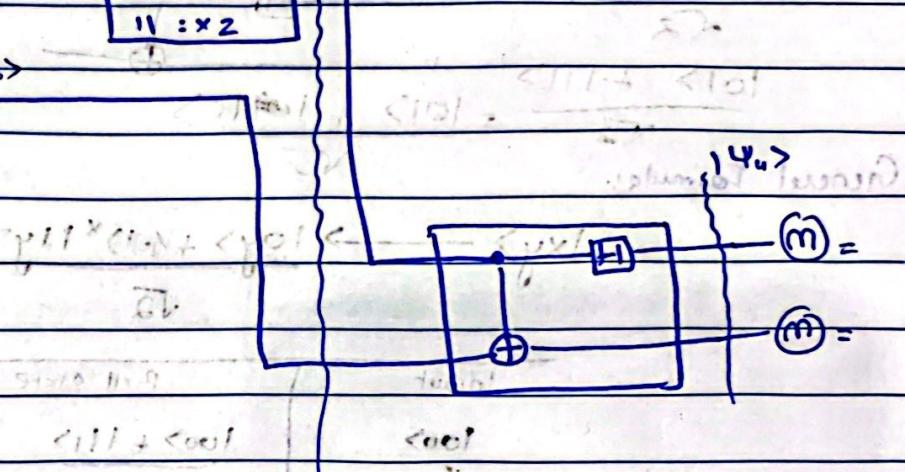
Input	Bell State
$ 00\rangle$	$\frac{ 00\rangle + 11\rangle}{\sqrt{2}} = \Psi^+\rangle$
$ 01\rangle$	$\frac{ 01\rangle + 10\rangle}{\sqrt{2}} = \Psi^+\rangle$
$ 10\rangle$	$\frac{ 00\rangle + (-1)^0 11\rangle}{\sqrt{2}} = \frac{ 00\rangle - 11\rangle}{\sqrt{2}} = \Psi^-\rangle$
$ 11\rangle$	$\frac{ 01\rangle - 10\rangle}{\sqrt{2}} = \frac{ 01\rangle - 10\rangle}{\sqrt{2}} = \Psi^-\rangle$



Superdense Coding



00 : T	$ 1\Psi_3\rangle$
01 : X	
10 : Z	
11 : $X \otimes Z$	



$$|1\Psi_1\rangle = |00\rangle$$

$$\langle 1\Psi_1 |$$

$$\langle 1\Psi_3 |$$

$$|1\Psi_2\rangle = |100\rangle + |110\rangle = |Q\rangle$$

$$\langle 1\Psi_1 | - \langle 1\Psi_2 | = \sqrt{2} |111\rangle$$

$$\langle 1\Psi_1 |$$

$$\langle 1\Psi_3 |$$

$$|00\rangle \quad |1\Psi_3\rangle = \frac{|100\rangle + |111\rangle}{\sqrt{2}} \quad \langle 1\Psi_1 | - \langle 1\Psi_2 | \quad \langle 1\Psi_3 |$$

Pauli_x

$$|01\rangle \quad |1\Psi_3\rangle = |110\rangle + |101\rangle$$

$$\langle 1\Psi_1 | - \langle 1\Psi_2 | = \sqrt{2}$$

$$|10\rangle \quad |1\Psi_3\rangle = \frac{|100\rangle - |111\rangle}{\sqrt{2}}$$

$$\langle 1\Psi_1 | - \langle 1\Psi_2 | = \sqrt{2}$$

$$|11\rangle \quad |1\Psi_3\rangle = -|110\rangle + |101\rangle \times (2|1\Psi_2\rangle) = |110\rangle - |101\rangle$$

$$\langle 1\Psi_1 | - \langle 1\Psi_2 | = \frac{\sqrt{2}}{2}$$

$$\sqrt{2}$$

$$00 \quad |4_4\rangle = |00\rangle + |11\rangle \xrightarrow{\text{CNOT}} H(|0\rangle + |1\rangle)$$

$$= \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

$$= \frac{|00\rangle + |10\rangle}{\sqrt{2}} = |0\rangle + |1\rangle \quad |0\rangle + |1\rangle - |2\rangle \quad |0\rangle - |2\rangle$$

$$= |00\rangle + |10\rangle + |00\rangle - |10\rangle$$

12

$$= |00\rangle = |00\rangle$$

$$01 \quad |4_4\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$= \frac{|11\rangle + |01\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} |1\rangle + \frac{|0\rangle + |1\rangle}{\sqrt{2}} |1\rangle$$

~~$|00\rangle + |10\rangle - |00\rangle - |10\rangle$~~

$$\approx |01\rangle - |11\rangle + |01\rangle + |11\rangle$$

2

~~$|01\rangle - |11\rangle + |01\rangle + |11\rangle$~~

~~So it's (011) & (101)~~

So on for 10 and 11

similarly for 100 and 111

~~111 + 100 =~~

~~|1011\rangle + |011\rangle + |110\rangle + |001\rangle = |101\rangle~~

~~111~~

~~111 + 100 = |001\rangle~~

Quantum Teleportation

NY (New York)

$|B_{AB}\rangle$

Alice

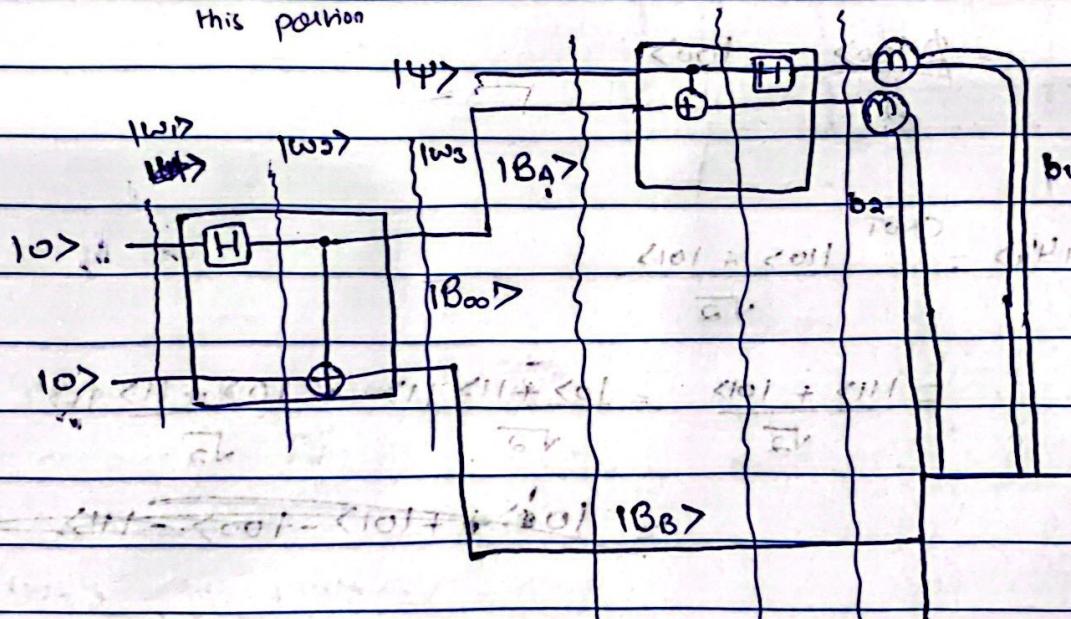
$|B_{00}\rangle$

Bob

Bob

$|B_B\rangle$

to move
moved and take
this position



$$|w_1\rangle = |00\rangle$$

$$|w_2\rangle = (H|0\rangle)|0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$|w_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) |B_{00}\rangle$$

$$|w_4\rangle = \alpha|10\rangle + \beta|11\rangle$$

$$|w_4\rangle = |1\rangle |B_{00}\rangle = (\alpha|10\rangle + \beta|11\rangle) \left(\frac{|100\rangle + |111\rangle}{\sqrt{2}} \right)$$

$$= \alpha|100\rangle + \alpha|101\rangle + \beta|110\rangle + \beta|111\rangle$$

$$|w_5\rangle = \alpha|100\rangle + \alpha|101\rangle + \beta|110\rangle + \beta|111\rangle$$

$$|w_5\rangle = \alpha|1000\rangle + \alpha|1011\rangle + \beta|1110\rangle + \beta|1101\rangle$$

$$= \frac{1}{\sqrt{2}}(|1000\rangle + |1011\rangle + |1110\rangle + |1101\rangle)$$



$$|\psi_2|^2$$

$$|\psi_6\rangle = \left[\frac{\alpha(10\rangle + 11\rangle)}{\sqrt{2}} |00\rangle + \frac{\alpha(10\rangle + 11\rangle)}{\sqrt{2}} |11\rangle + \frac{\beta(10\rangle - 11\rangle)}{\sqrt{2}} |10\rangle \right]$$

$$\frac{1}{\sqrt{2}}$$

$$+ \frac{\beta(10\rangle - 10\rangle)}{\sqrt{2}} |01\rangle$$

$$= \frac{1}{2}$$

$$\alpha|1000\rangle + \alpha|1100\rangle + \alpha|1011\rangle + \alpha|1111\rangle + \beta|1010\rangle + \beta|1110\rangle + \beta|0010\rangle - \beta|0110\rangle$$

$$|\psi_6\rangle = \frac{1}{2} \left[|00\rangle (\alpha|10\rangle + \beta|11\rangle) + |10\rangle (\alpha|10\rangle + \beta|11\rangle) + |01\rangle (\alpha|11\rangle + \beta|10\rangle) + |11\rangle (\alpha|11\rangle - \beta|10\rangle) \right]$$

Measure qubit

m	Prob	Resultant state	Operation
00	$\frac{1}{4}$	$\alpha 10\rangle + \beta 11\rangle$	no operation
10	$\frac{1}{4}$	$\alpha 10\rangle - \beta 11\rangle$	Pauli-Z
01	$\frac{1}{4}$	$\alpha 11\rangle + \beta 10\rangle$	Pauli-X
11	$\frac{1}{4}$	$\alpha 11\rangle - \beta 10\rangle$	Pauli-X (Z($ 11\rangle$))

$$|\psi\rangle = \alpha|10\rangle + \beta|11\rangle$$