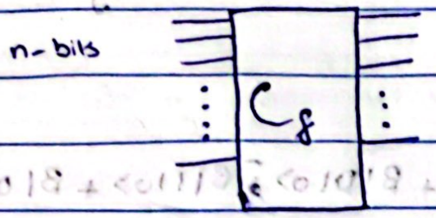
Classical Circuit to Quantum Circuit

$f : \{0,1\}^n \longrightarrow \{0,1\}^m$ —— classical function



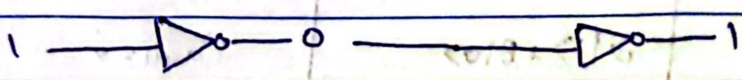
~ Classical circuit

Quantum Circuits → unitary matrices → reversible Always

Universal gate set.

{And, NOT}

All classical circuit can be drawn only using And & NOT.



And gate is not reversible.

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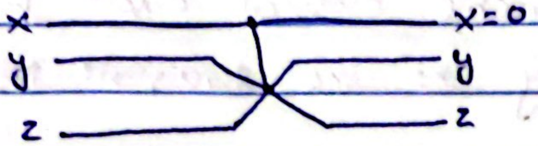
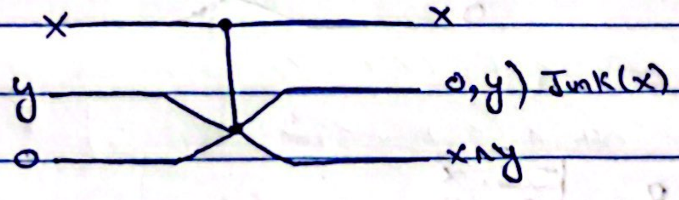
Controlled swap gate (To create reversible And gate)



if	x=0	order x=0 y z	if x=1	x z y

Now to create And gate using C-Swap gate, we set z = 0, In this case our first output will be x, the second will be either 0 or y depending on the value of x.



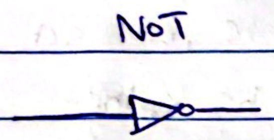
We often refer to the second output as junk which is a function of x Junk(x)

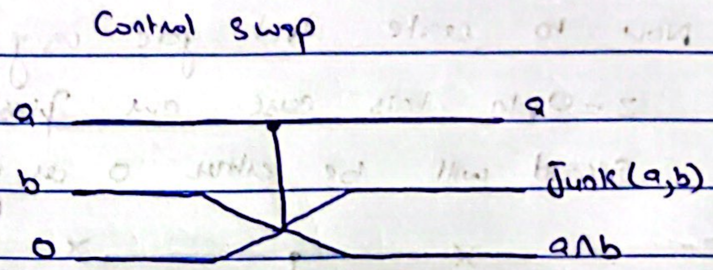
Truth Table $x \wedge y$

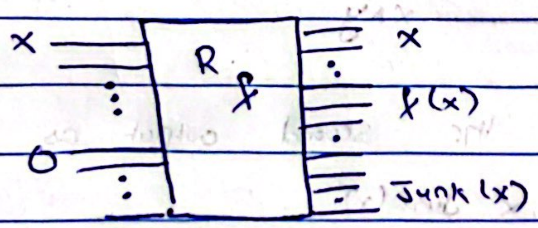
x	y	
0	0	0
0	1	0
1	0	y
1	1	y

Notice that when x is 0, our output is 0 regardless of the value of y. In contrast when x=1 our output will be equal to y. Thus our output will be equal to 1 if and only if x and y are 1, which precisely

make the behaviour of and gate.
now we have our univarsle set of classical reversible gates. we can create a reversible classical circuit for function f, denoted as R_f using the the revuisible NOT and Control swape gate

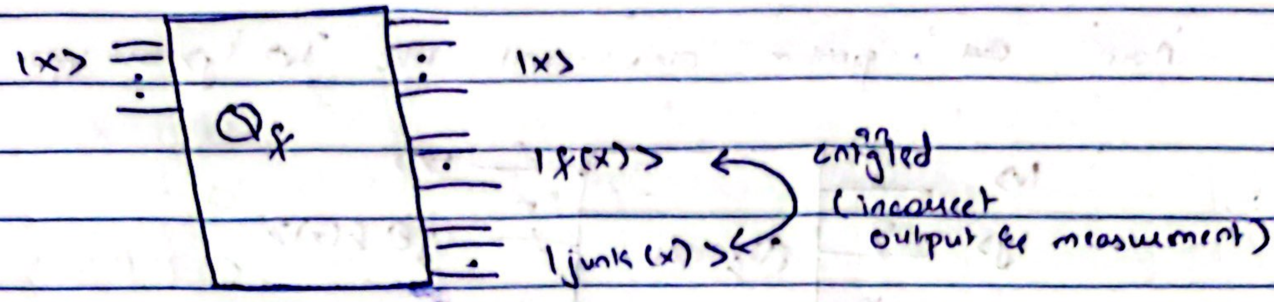






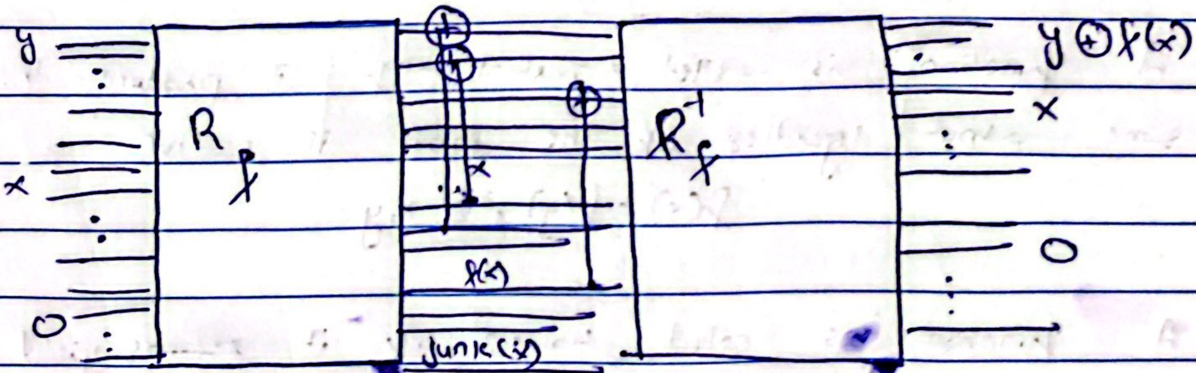
The input for circuit R_f will be x along with some 0s. The output will consist of x, f(x) and what we were been referring to as junk. which is also a function of x.

The junk can be problematic if we convert a classical reversible circuit with this junk into quantum circuit lets called it Q_f the junk could become entengled with the output, causing either positive or negative interference



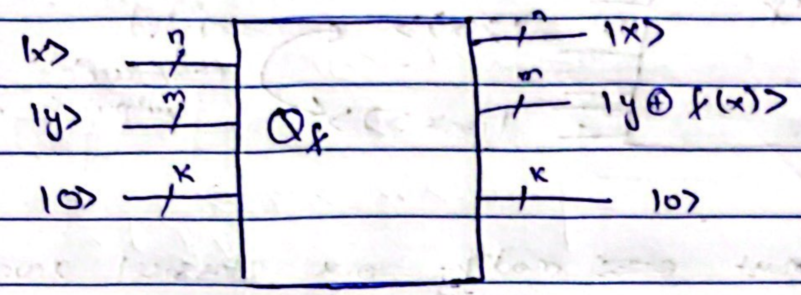
Therefore we must ~~classi~~ modify our classical reversible circuit to eliminate the junk to ensure the quantum circuit function correctly.

In our redesign process to eliminate junk we feed the output of R_f into its inverse R^{-1}_f. Thus converting the output back to original input and getting rid of junk.



However this process has also eliminated $f(x)$ to preserve the value of $f(x)$ we introduce Controlled NOT gate $\oplus$

now our quantum circuit will take $|x\rangle\, |y\rangle$ as input



Deutsch Algorithm

Problem Definition:

Given a function $f: \{0,1\}$ and yields a single bit output $f: \{0,1\} \rightarrow \{0,1\}$
our goal to ~~do~~ determine whether the function f is constant or balanced.

A function is called constant if it produce the same output regardless of the input it receive

$$f(x) = f(y), \ \forall\, x, y$$

A function is called balance if it ~~receives~~ yield 0 for one half and 1 for other half

$$f(x) = 0 \text{ for half of input}$$
$$f(x) = 1 \text{ for other}$$

Constant

x	f(x)
0	0
1	0

Case 1

x	f(x)
0	1
1	1

Case 2

Balanced :-

x	f(x)
0	0
1	1

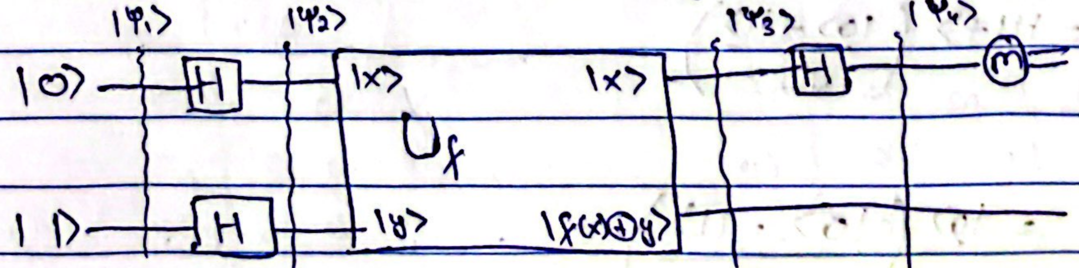
Case 3

x	f(x)
0	1
1	0

Case - 4

Quantum	Classical
- 1 call to f(x)	- 2 call to f(x) → key
- Speed-up via superposition of all possible input.	- If both are same then f is constant otherwise balance.

Quantum Circuit for deutsch algorithm:-



Claim we mesure, 0 on first qubit the function must
• ~~bet~~ be constent, other balanced

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$|\psi_1\rangle = |0\rangle\,|1\rangle = |01\rangle$

$|\psi_2\rangle = H|0\rangle \quad H|1\rangle$

$= \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

$= \frac{1}{2}\left[\,|00\rangle - |01\rangle + |10\rangle - |11\rangle\,\right]$

x y x y x y x y

$|\psi_3\rangle = \frac{1}{2}\left\{|0\rangle|f(0)\oplus 0\rangle - |0\rangle|f(0)\oplus 1\rangle\right.$ $\quad \therefore U_f\,|x\rangle|y\rangle = |x\rangle\,|f(x)\oplus y\rangle$

$\left. + |1\rangle|f(1)\oplus 0\rangle - |1\rangle|f(1)\oplus 1\rangle\,\right]$

Case -1

Case-1 $\quad |\psi_3\rangle = \frac{1}{2}\left[|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right]$ $\quad f(0) = f(1) = 0$

$= \frac{1}{2}\left[\,|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)\,\right]$

$|\psi_4 \quad = H\left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \left(\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$= H|+\rangle \left(\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$= |0\rangle \left(\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

(M) 1 bit = 100 probaliy.

$H|+\rangle = |0\rangle$ so (m) $|0\rangle$

Constant $= |\psi_3\rangle = \pm |+\rangle|-\rangle$

If Balance in $|\psi_3\rangle = \pm|-\rangle|-\rangle$

$H|-\rangle = |1\rangle$ so (m) $|1\rangle$

Deutsch - Jozsa Algorithm

Problem:

Key difference is:-

$f:\{0,1\}^n \longrightarrow \{0,1\}$ take n bits input and produce single bit output.

find

If is Constant

or

If is balanced

Example $n=2$:-

x	$f(x)$		all 1's
00	0	or	1
01	0	or	1
10	0	or	1
11	0	or	1

$\underbrace{\qquad\qquad}_{\text{Constant}}$

x	$f(x)$										
00	0	or	1	or	1	or	0	or	0	or	0
01	0		1		0		1		1		
10	1		0		0		1		0		
11	1		0		01		0		1		0

Classical

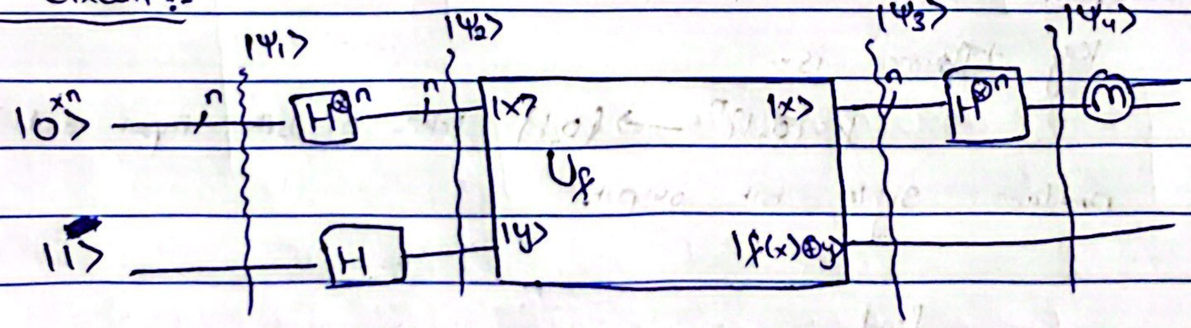
- Must check half plus one inputs
  $\hookrightarrow \frac{2^n}{2} + 1 = O(2^n)$

Quantum :

- Call oracle once via superposition $O(1)$.
- Exponential speedup.

- If all are input produce same result is will be constant otherwise balanced.

Quantum Circuit :-



$$|\psi_1\rangle = |0^n\rangle|1\rangle, \quad |\psi_2\rangle = H^{\otimes n}|0^n\rangle \otimes H|1\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |-\rangle$$

$$|\psi_3\rangle = U_f |\psi_2\rangle$$

$\therefore f(x) \oplus 0 = f(x)$

$\therefore f(x) \oplus 1 = f(x)'$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}} |x\rangle \frac{|f(x) \oplus 0\rangle - |f(x) \oplus 1\rangle}{\sqrt{2}}$$

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$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \frac{|f(x)\rangle - |\overline{f(x)}\rangle}{\sqrt{2}}$$

when $f(x)=0$: $\underbrace{\frac{|0\rangle - |1\rangle}{\sqrt{2}}}_{|-\rangle}$

when $f(x)=1$: $\underbrace{\frac{|1\rangle - |0\rangle}{\sqrt{2}}}_{-|-\rangle}$

So,

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle (-1)^{f(x)} |-\rangle$$

$$|\psi_4\rangle = H^{\otimes n}\left(\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} \cancel{|-\rangle} (-1)^{f(x)} |x\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} H^{\otimes n} |x\rangle$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \overbrace{\sum_{y\in\{0,1\}} (-1)^{x\cdot y} |y\rangle}^{1/\sqrt{2^n}}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2^n}} \times \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} \sum_{y\in\{0,1\}^n} (-1)^{f(x)+x\cdot y} |y\rangle$$

Case 1 f is constant: Claim $|y\rangle - |0^n\rangle$ with 1 prob

Fact #1 $x\cdot y = 0$

Fact #2 $f(x) = 0 \Rightarrow (-1)^{f(x)} = 1$

$\boxed{\text{Constant Always (m) } |0^n\rangle}$

$n=2$

x	$f(x)$
00	1
01	0
10	0
11	1

Balanced

$m \neq |0^n\rangle$

$|\psi_1\rangle = |00\rangle|1\rangle, \; |\psi_2\rangle = H^{\otimes 2}|00\rangle \, H|1\rangle$

$|\psi_2 = \frac{1}{\sqrt{2^2}} \left[|00\rangle + |01\rangle + |10\rangle + |11\rangle \right] |-\rangle$

$|\psi_3\rangle = U_f \, |\psi_2\rangle$

Since the second register has $|-\rangle$ U_f gate will change the sign of all the inputs which are providing an output of 1, Hence

$|\psi_3\rangle = \frac{1}{2} \left[-|00\rangle + |01\rangle + |10\rangle - |11\rangle \right]$

$|\psi_4\rangle = \frac{1}{2} \left[-H|00\rangle + H|01\rangle + H|10\rangle - H|11\rangle \right]$

There will be 16 terms but we can calculate them easily with using tabular method.

	$y_1 y_2$
$x_1 x_2$	$(-1)^{x y_1 + x}$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$-\|00\rangle$	+	+	+	−
$\|01\rangle$	+	+	+	−
$\|10\rangle$	+	+	+	−
$-\|11\rangle$	+	+	+	−

$= \frac{1}{2} \cdot \frac{1}{2} \left[-4|11\rangle \right] = -|11\rangle$

m $|11\rangle$ with prob $(-1)^2 = 1$ as not m $|00\rangle$ since y is not

Quantum Fourier Transform

Primitive root of unity

$z^n = 1$, z is Comple number

n roots (Solutions)

Solution are $\{ w_n^0, w_n^1, w_n^2, \ldots, w_n^{n-1}$

$$w_n^k = e^{2\pi i k/n}$$

1) w_n^k lie on unit circle.

$$|w_n^k| = 1 = \sqrt{w_n^{k*} \cdot w_n^k}$$

$$= \sqrt{e^{-2\pi i k/n} \cdot e^{2\pi i k/n}}$$

$$= \sqrt{e^0} = 1$$

② w_n^k is a periodic function

$$w_n^k = w_n^{k \bmod n}$$

Example :-

write all root

$z^4 = 1$

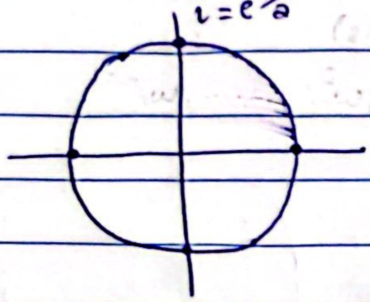
$\{ w_4^0, w_4^1, w_4^2, w_4^3 \}$

$\{e^{2\pi i 0/4}, e^{2\pi i 1/4}, e^{2\pi i 2/4}, e^{2\pi i 3/4}\}$

$= \{e^{0i}, e^{\frac{\pi}{2}i}, e^{\pi i}, e^{\frac{3}{2}\pi i}\}$

$e^{i\theta}$

$\{1, i, -1, -i\}$



$w_4^7 = w_4^{7 \bmod 4} = w_4^3$

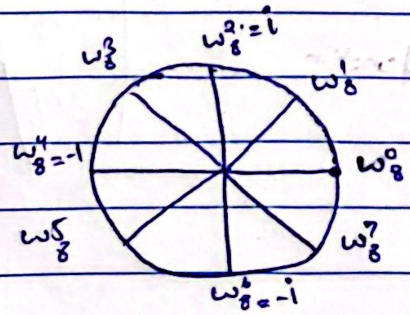
$z^4 \longrightarrow = 1 \quad \{1, -1, i, -i\}$

So for periodic $= w_4^4 = w_4^{4 \bmod 4} = w_4^0$

Example:-

$~~z8~~ \cdot z^8 = 1$

previous 3 are calculate in previous question ~~so that~~



$360/8 = 45$

$\{1, w_8, i, iw_8, -1, -w_8, -i, -iw_8\}$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega_n^{0\times 0} & \omega_n^{0\times 1} & \omega_n^{0\times 2} & \cdots & \omega_n^{0\times n-1} \\ \omega_n^{k\times 0} & \omega_n^{k\times 1} & \cdots & \cdots & \omega^{k(n-1)} \\ \vdots & & & & \omega_n^{(n-1)(n-1)} \end{pmatrix}$$

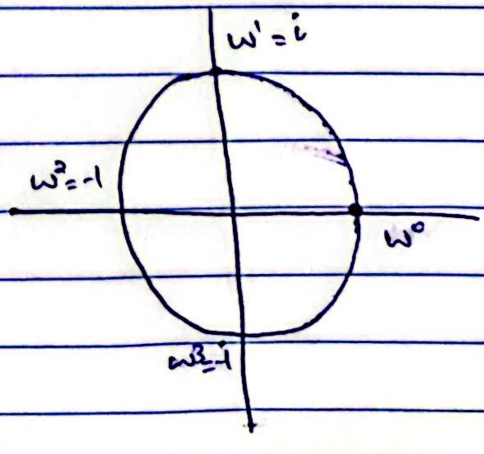
$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 \cdots & \omega^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 \quad \omega_n^{12} & \\ \vdots & & & & \\ 1 & \cdots & \cdots & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{array}{l} \\ \rightarrow \text{Table of 1} \\ \rightarrow \text{Table of 2} \\ \\ \\ \text{row } n \end{array}$$

Example

Transform $|\psi\rangle = \frac{|0\rangle + |3\rangle}{\sqrt{2}}$ using F_4

$F_4 |\psi\rangle = |\phi\rangle$

$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}$$



$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^0 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix}$$

$$= \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 1-i \\ 0 \\ 1+i \end{pmatrix} = \phi$$

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Properties of DFT

① DFT is unitary:-

There are different ways to prove this transformation but the simple way is to show that the columns of dFT are orthonormal.

It implies the each column is normalised or it magnitude is equals to one and each column is orthogonal from all other columns

So if we have two column of DFT say that column j and K and we take dot product of those column then ~~or~~ our of dot product must be either equals to ~~ze~~ 0 or equal to 1.

$$\langle \bar{c}_j | \bar{c}_k \rangle = \begin{cases} 0 & ; j \neq k \text{ (orthogonal)} \\ 1 & ; j = k \text{ (normalised)} \end{cases}$$

- our result will be equal to 0 when j is not equal to K and it show column are orthogonal.

Our result will be equal to 1 when j is equal to K and it represent our column are normalised

$$\vec{c}_j = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega^{0\times j} \\ \omega^{1\times j} \\ \omega^{2\times j} \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{j} \\ \omega^{2j} \\ \omega^{2j} \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix}$$

$$\vec{c}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix}$$

now dot product of column j with k. that we have to take conjugate of the first colum and multiple it with the corresponding entry of second colum.

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \overline{\omega^{m\times j}}\, \omega^{m\times k}$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(k-j)}$$

Case 1: $j=k$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(0-0)}$$

$$= \frac{1}{n} \sum_{m=0}^{n-1} \omega^{0}$$

$= \frac{1}{n} [1+1+1 \ldots\ldots +1]$

$= \frac{n}{n} = 1$ (each column is normalized or each colum is a unit vector)

~~Case 2 : $j \neq k$~~

Geometric series:-

$a + ra + r^2 a + \ldots + r^{n-1} a$

$S = \frac{a(r^n - 1)}{r - 1}$ $\therefore a = 1$

$\therefore r = \omega^{k-j}$

Case :-2 $j \neq k$

$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(k-j)} = 0$

$\langle c_j | c_k \rangle = \frac{\omega^{(k-j)n} - 1}{\omega^{k-j} - 1}$

Now whenever we have ω_n raised to power any multiple ~~any~~ of n let say:-

$\omega_n^{ln} = 1$ $\quad \omega_n^{ln} = \omega_n^{ln \bmod n}$ $= \omega_n^0$

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$$So = \frac{1 - 1}{\omega^{k-j} - 1} = 0$$

(2) Convolution ⟷ multiplication :-

So the basic idea is we linearly shift my input vector and then apply QFT on that input vector and my output vector has phase shift in it.

Before linear shift

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 \ \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 \ \omega^6 & \cdots & \omega^{2(n-1)} \\ \vdots & \omega^3 & \omega^6 \ \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & & \cdots & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha^{n-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$$

measure my output (result is an index of the vector)

$|k\rangle \longrightarrow |\beta_k|^2$

let say it is linearly shift by 1

After linear shift

$$\frac{1}{\sqrt{n}} \begin{pmatrix} & '' & \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-2} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \omega\beta_1 \\ \omega^2\beta_2 \\ \vdots \\ \omega^{n-1}\beta_{n-1} \end{pmatrix}$$

$|k\rangle \longrightarrow |\beta_k|^2$

$|\omega^2| = |\omega^3| = 1$

∴ magnitude of all power of ω is 1. So we donot have ~~to~~ to write $|\omega^k \beta_k|^2$. we can just ignore it. and write $|\beta_k|^2$.

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Before linear shift

$B_K = \alpha_0 + \omega^K \alpha_1 + \omega^{2K} \alpha_2 + \dots \omega^{(n-1)K} \alpha_{n-1}$

After linear shift by 1

$\alpha m = \alpha m+1$

K^{th} output after linear shift

$= \alpha_{n-1} + \omega^K \alpha_0 + \omega^{2K} \alpha_1 + \omega^{3K} \alpha_2 + \dots \omega^{(n-1)K} \alpha_{(n-2)}$

• Now we have to show is that output before linear shift but only has the phase attach to it.

$\omega^K B_K = \omega^K \alpha_0 + \omega^{2K} \alpha_1 \dots \omega^{nK} \alpha_{(n-1)}$ $\quad\quad \therefore \omega^{nK \bmod n}_{\alpha(n-1)} = 0$

$nK \bmod n = 0$

$= \omega^K \alpha_0 + \omega^{2K} \alpha_1 \dots \dots \alpha_{n-1}$ $\quad\quad \therefore \omega^0 = 1$

(3) Period / wave length Relationship:

a function f take n bits as input and it produces n bits output.

$$f: \{0,1\}^n \to \{0,1\}^n$$

and we apply QFT on our function f

$$QFT\, f(x) = \hat{f}(x)$$

If our function f is periodic and has a ~~per~~ → period $= r$ → $\frac{N}{r} = \frac{2^n}{r}$

$$\begin{pmatrix} QFT \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_{N/r - 1} \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta \end{pmatrix}$$

Period Finding Algorithm

Problem Definition :-

Given a periodic function $f: \{0,1\}^n \longrightarrow \{0,1\}^n$
find period r such that $f(x) = f(x + kr)$ $\forall$ $\in Z - \{0\}$

- Similar to Simon's Algo.

$f: \{0,1\}^n \rightarrow \{0,1\}^n$ $n = 4$

$f(x) = x \bmod 3$

x	$f(x)$
0	0
1	1
2	2
3	0
4	1
5	2
6	0
7	1
8	2
9	0
10	1
11	2
12	0
13	1
14	2
15	0

$f(0) = 0$
$f(1) = 1$
$f(2) = 2$
$f(3) = 0$

0 is repeated on $f(3) = 0$ so our period will be the difference of two inputs that have produced same output. $r = (3 - 0) = 3$

$\Theta(r) = \Theta(2^n) \rightarrow$ Exponential

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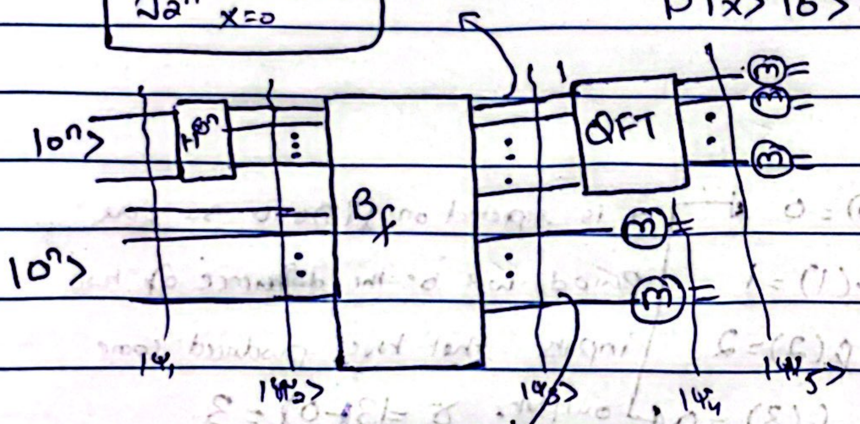
The function f whose period we want to find might not be unitary. Thus we might not be able to use function f directly in the quantum circuit. instead we have to develop a quantum wrapper over the function f i call the quantum implementation of function f B_f.

all possible value of x

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

$$B|x\rangle|0\rangle = |x\rangle|f(x)\rangle$$



$$H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{n-1} |x\rangle$$

all possible value of $f(x)$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |f(x)\rangle$$

So if we measure the second register then it will transform both the register because each of these two register are no physical seperated with each other their state are tensor product with each other

So combine we can write

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle|f(x)\rangle$$

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So we if we measure the second register according to the rule of partial measurement content of the first register will also change.

let say ~~&~~ my measurement of the first register is $|f(x_1)\rangle$ where x_1 is some specific value then the first register will have only those axes that can produce this specific output of $|f(x_1)\rangle$

The input that can produce $|f(x_1)\rangle$

$$\frac{|x_1\rangle + |x_1+r\rangle + |x_1+2r\rangle + \dots + |x_1 + (\frac{2^n}{r} - 1)r\rangle}{\sqrt{\frac{2^n}{r}}}$$

∴ So if we use QFT on First register will remove the linear shift, we will has phase shift but phase shift donot effect the result of measurement.

2nd Register measure	First Register superposition
$\|f(0)\rangle$	$\|0\rangle + \|0+r\rangle + \|0+2r\rangle \dots$
	L.shift of 1
$\|f(1)\rangle$	$\|1\rangle + \|1+r\rangle + \|1+2r\rangle + \dots$
	Linear shift of 2
$\|f(2)\rangle$	$\|2\rangle + \|2+r\rangle + \|2+2r\rangle + \dots$
⋮	
$\|f(k)\rangle$	$\|k\rangle + \|k+r\rangle + \|k+2r\rangle + \dots$
⋮	

Working logic of Period Finding Algorithm circuit on privous page:-

Example:-

Find period of given $f:\{0,1\}^3 \rightarrow \{0,1\}^3$
[For use in B_f $\quad f(x) = x \bmod 2$

$$|\psi_1\rangle = |0^3\rangle |0^3\rangle = |000\rangle |000\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^3}} \sum_{x=0}^{7} |x\rangle |000\rangle$$

$$|\psi_2\rangle = \frac{|0\rangle + |1\rangle + |2\rangle + \ldots + |7\rangle}{\sqrt{8}} |000\rangle$$

$$|\psi_3\rangle = B_f |\psi_2\rangle$$

$$B_f |x\rangle |0\rangle = |x\rangle |f(x)\rangle$$

$$= \frac{1}{\sqrt{8}} \sum_{x=0}^{2^3-1} |x\rangle |f(x)\rangle$$

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$$|\psi_3\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|0\rangle + |3\rangle|1\rangle + |4\rangle|0\rangle + |5\rangle|1\rangle + |6\rangle|0\rangle + |7\rangle|1\rangle}{\sqrt{8}}$$

Assume that we have to measure $|1\rangle$

$$|\psi_4\rangle = \frac{(|1\rangle + |3\rangle + |5\rangle + |7\rangle)}{\sqrt{4}}|1\rangle$$

Assume the we measure $|0\rangle$

$$|\psi_4\rangle = \frac{(|0\rangle + |2\rangle + |4\rangle + |6\rangle)}{\sqrt{4}}|0\rangle$$

norm

$$|\psi_5\rangle = QFT_8\,|\psi_4\rangle$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2^3}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{pmatrix} \frac{1}{\sqrt{4}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

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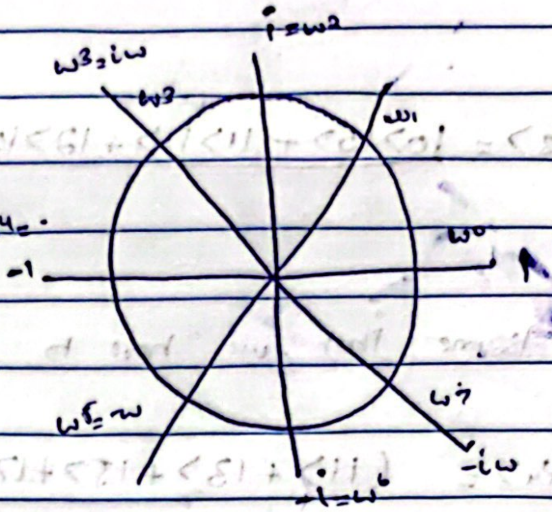
$$\frac{1}{4\sqrt{2}}\begin{pmatrix} 1 + 1 + 1 + 1 \\ w + w^3 + w^5 + w^7 \\ w^2 + w^6 + w^{10} + w^{14} \\ w^3 + w^9 + w^{15} + w^{21} \\ w^4 + w^{12} + w^{20} + w^{28} \\ w^5 + w^{15} + w^{25} + w^{35} \\ w^6 + w^{18} + w^{30} + w^{42} \\ w^7 + w^{21} + w^{35} + w^{49} \end{pmatrix}$$



$$|\psi_5\rangle = \frac{1}{4\sqrt{2}}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{|0\rangle - |4\rangle}{\sqrt{2}} = m$$

$|\psi_5\rangle$. $|0\rangle$ $|4\rangle$

$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ $\left|\frac{1}{\sqrt{2}}\right|^2$

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$$|\psi_4\rangle = \frac{(|0\rangle + |2\rangle + |4\rangle + |6\rangle)}{\sqrt{4}} |0\rangle$$

$$|\psi_5 = \frac{|0\rangle + |4\rangle}{\sqrt{2}}$$

When we have a result with different phase then there measure remains the same.

Return the circuit

$$O(\log N)$$
$$= O(\log 2^3)$$

GCD of our measure $= \frac{N}{r} = 4$

$$= \frac{8}{r} = 4$$

$$r = 2$$

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A-level Complex Number

Root of Unity:-

$$z^n = 1$$

$$\{w_n^0, w_n^1, w_n^2, \dots, w_3^{n-1}\}$$

$$w_n^k = e^{\frac{2\pi i k}{2n}}$$

$$z^3 = 1$$

$$\{w_3^0, w_3^1, w_3^2\}$$

$$e^{\frac{2\pi i(0)}{3}} = 1 \qquad e^{\frac{2\pi i}{3}} \qquad e^{\frac{4\pi i}{3}}$$

we have three different form of Complex number

Exponential form

$$\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$$

now we want to express them in modular argument form

Modular argument - form

In this we express complex number as

$$z = r\cos\theta + r i \sin\theta$$

$$e^{\frac{2\pi i}{3}} = e^{i\theta}$$

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$$\left\{1, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right\}$$

lastly we need to write it in Cartesian form.

$$\left\{1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}\right\}$$

Example:-

$z^3 = 8$

Find root

$z = \sqrt[3]{8}$

$z = 2$

exponential form :-

$$\left\{2, 2e^{\frac{2\pi i}{3}}, 2e^{\frac{4\pi i}{3}}\right\}$$

cartisian form :-

$$\{1, -1+i\sqrt{3}, -1-i\sqrt{3}$$

Example :-

$(z-5)^3 = 8$

Find root

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Let

$z-5=y$

$y^3=8$

$\{2, -1+i\sqrt{3}, -1-i\sqrt{3}\}$ as previous

$y=z-5$

$z=y+5$

$\{7, 4+i\sqrt{3}, 4-i\sqrt{3}\}$

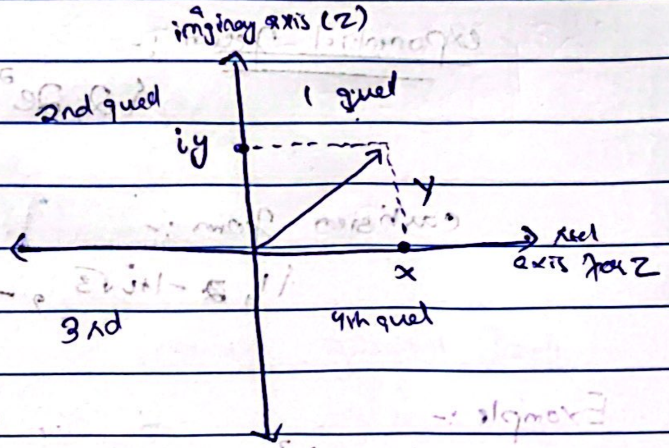
Modulus Argument Form

Cartision form $\qquad$ Modulus form

$z=x+iy \longrightarrow z=r\cos\theta+ir\sin\theta$

Modulus : $|z|=\sqrt{x^2+y^2}$

(magnitude)

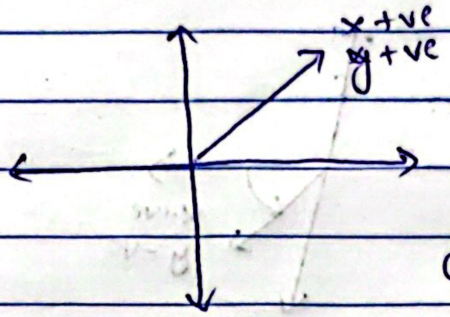


Argument (Principle argument)

$-\pi < \theta \leq \pi$

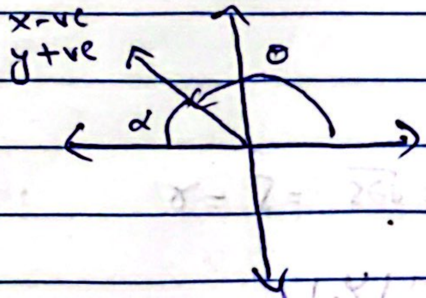
Angle between x-axis and position vector, but my position vector can be in any quadrant, we want to have value of argument which is between -180 and 180 and is called principle argument

Case 1 :- we are in first quadrant.



$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

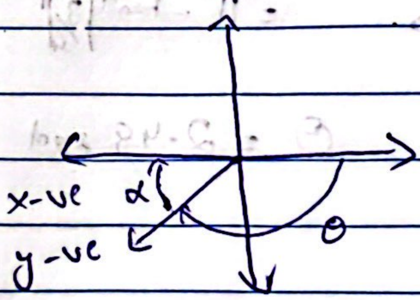
Case 2 :- we are in second quadrant:-



$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\theta = \pi - \alpha$

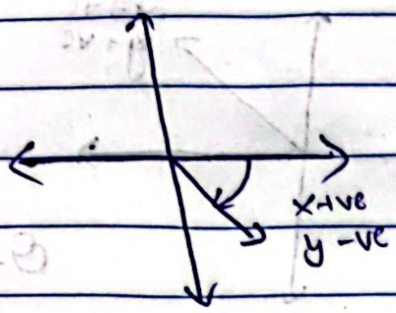
Case 3 :- we are in third quadrant



$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\theta = -\pi + \alpha$

Case 4:- we are fourth quad



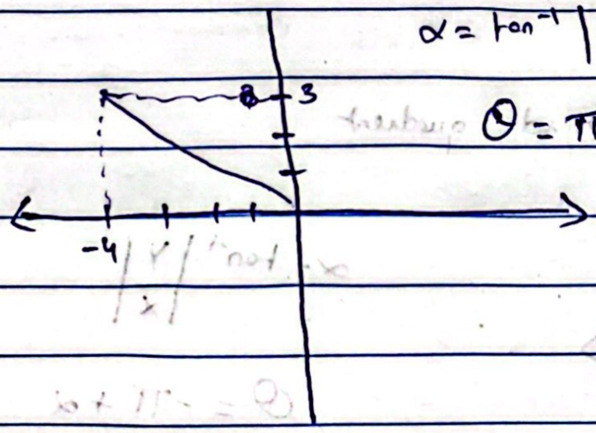
$\theta = -\alpha$

Example:-

$z = -4+3i$

write modulus argument form

$|z| = \sqrt{(-4)^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5 = r$



$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\theta = \pi - \alpha = \pi - \tan^{-1}\left|\frac{3}{-4}\right|$

$= \pi - \tan^{-1}\left|\frac{3}{4}\right|$

$\theta = 2.49 \text{ rad}$

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Exponentia form

Cartision form

$z = x + iy$

Modulus – Argument form

$z = r\cos\theta + ir\sin\theta$

Exponential form

$$z = re^{i\theta}$$

Example $z = -4 + 3i$

$= |z| = 5$

$\theta = 2.49$

$= 5e^{i2.49}$

Solving Equation

$ax^2+bx+c=0 \qquad a \neq 0$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\sqrt{b^2-4ac}$ tells us if my solution will be real or it will have complex root or my solution will be distinct or not

Case 1:- $b^2-4ac > 0$ we will has (real solution)

$\alpha, \beta \in R$

$\alpha \neq \beta$

Case 2:- $b^2-4ac = 0$

$\alpha, \beta \in R$

$\alpha = \beta$

Case 3:- $b^2-4ac < 0$

$\alpha \beta \in \mathbb{C}$ complex number

$\beta = \alpha^*$

Example:-

$x^2+2x+2=0$

Find roots

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$b^2-4ac = 4-4(1)(2) = -4$

$b^2-4ac < 0$, so the root will be complex

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} \quad -1 \pm i$$

For poly degree 3

$ax^3 + bx^2 + cx + d = 0$

$a \neq 0$

① 3 real root

② 1 real, 2 complex root

Poly deg 4

① 4 real root

② 2 real, 2 complex root (where these complex root will be conjugate to each other)

③ 4 complex (where these root will be in pairs and each pair will be conjugate of other complex root. $\alpha, \alpha^*, \beta, \beta^*$

Example

$$z^4 + 13z^2 + 36 = 0$$

Given $z = 2i$ is a root find remaining 3 roots.

Sol:-

$$z = 2i, \quad z = -2i$$

Factor $(z-2i)(z+2i)$

$$= z^2 - (2i)^2 = z^2 + 4$$

$$z^2 + 4 = \sqrt{z^2 + 13z^2 + 36} \; z^2 + 9$$

$$-z^4 \quad \pm 4z^2$$

$$9z^2 + 36$$

$$9z^2 + 36$$

$$\times$$

The equation has two factor which are z^2+4 and z^2+9

root of $(z^2+4)(z^2+9) = 0$

$z^2 + 9 = 0$ $\quad z^2 = 9$

$$z^2 = \pm\sqrt{-9} = \pm i3$$

- $2i - 2i + 3i - 3i$

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Arthimetic Operation

$z_1 = a+ib$, $z_2 = c+id$

~~Add~~ • **Addition / Subtraction**

$$z_1 + z_2 = (a+c) + i(b+d)$$

$$z_1 - z_2 = (a-c) + i(b-d)$$

• Multiplication:-

$$z_1 \cdot z_2 = (ac - bd) i (ad + bc)$$

• Dividsion:- $$\frac{z_1}{z_2} = \frac{(ac+bd) + i(ad+bc)}{c^2+d^2}$$

In exponential form

$$z_1 = r_1 e^{i\theta_1} , z_2 = r e^{i\theta_1}$$

Addition / Division

$$z_1 + z_2 = r_1 e^{i\theta} + r_2 e^{i\theta_2}$$

$$z - z_2 = r_1 e^{i\theta} + - r_2 e^{i\theta_2}$$

Multiply :-

$$z_1 \cdot z_2 = r_1 \cdot r_2 e^{i\theta + i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

4 Result :- In exponential form

① $\text{argument}(z_1 \cdot z_2) = \theta_1 + \theta_2 = \text{argument}(z_1) + \text{argument}(z_2)$

② $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

③ $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$

④ $|z_1 z_2| = r_1 \cdot r_2 = |z_1| \cdot |z_2|$

Square Root

Qno Find square root of $7 + i6\sqrt{2}$

Sol :-

$$x + iy = \sqrt{7 + i6\sqrt{2}}$$

$$(x+iy)^2 = 7 + i6\sqrt{2}$$

$$x^2 + 2ixy - y^2 = 7 + i6\sqrt{2}$$

$(x^2-y^2)+i(2xy) = 7+i6\sqrt{2}$

$\boxed{(x^2-y^2)=7}$ — ①

$2xy = 6\sqrt{2}$

$\boxed{xy = 3\sqrt{2}}$ — ②

$y = \frac{3\sqrt{2}}{x}$

put in ①

$x^2 - \left(\frac{3\sqrt{2}}{x}\right)^2 = 7$

$x^2 - \frac{18}{x^2} = 7$

$x^4 - 18 = 7x^2$

$x^4 - 7x^2 - 18 = 0$

Let $u = x^2$

$u^2 - 7u - 18$

$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{7 \pm 11}{2}$ in $+ = 9$

in $- = -2$

$u = 9,\ u = -2$

$x^2 = 9$ $\quad$ $x^2 = -2$ → not use because it give imginary value

$x = \pm 3$

$y = \frac{3\sqrt{2}}{x}$

$= \frac{3\sqrt{2}}{-3} = -\sqrt{2}$

$= \frac{3\sqrt{2}}{3} = \sqrt{2}$

so Find :-

$x + iy \quad 3 + i\sqrt{2}$

$(x+iy)^2 = -3 - i\sqrt{2}$

Square Root → De-moivre

Q no Find square root of

$7 + i6\sqrt{2}$

$|z| = r = \sqrt{7^2 + (6\sqrt{2})^2}$

$= \sqrt{49 + 72}$

$= \sqrt{121} = 11$

$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{6\sqrt{2}}{7}\right| = 0.88$ rad

$z = r\cos\theta + ir\sin\theta$

$z^n = r^n\cos n\theta + ir^n\sin(n\theta)$

$z^{1/2} = \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\sin\frac{\theta}{2}$

$z^{1/2} = \sqrt{r} \cos \theta/2 + i\sqrt{r} \sin \theta/2$

$= \sqrt{11} \cos 0.88/2 + i\sqrt{11} \sin 0.88/2$

$\boxed{z^{1/2} = 3 + i\sqrt{2}}$

there is another square root of this complex number that we can find by adding 2pi in my θ

$z^{1/2} = \sqrt{11} \cos\left(\frac{0.88+2\pi}{2}\right) + i\sqrt{11} \sin\left(\frac{0.88\,2\pi}{2}\right)$

$\boxed{z^{1/2} = -3 - i\sqrt{2}}$