Diploma Programme
Programme du diplôme
Programa del Diploma

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Friday 7 May 2021 (morning)

2 hours

Candidate session number

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221–7107

14 pages

16EP01

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

   The following table shows the data collected from an experiment.

x	3.3	6.9	11.9	13.4	17.8	19.6	21.8	25.3
y	6.3	8.1	8.4	11.6	10.3	12.9	13.1	17.3

   The data is also represented on the following scatter diagram.
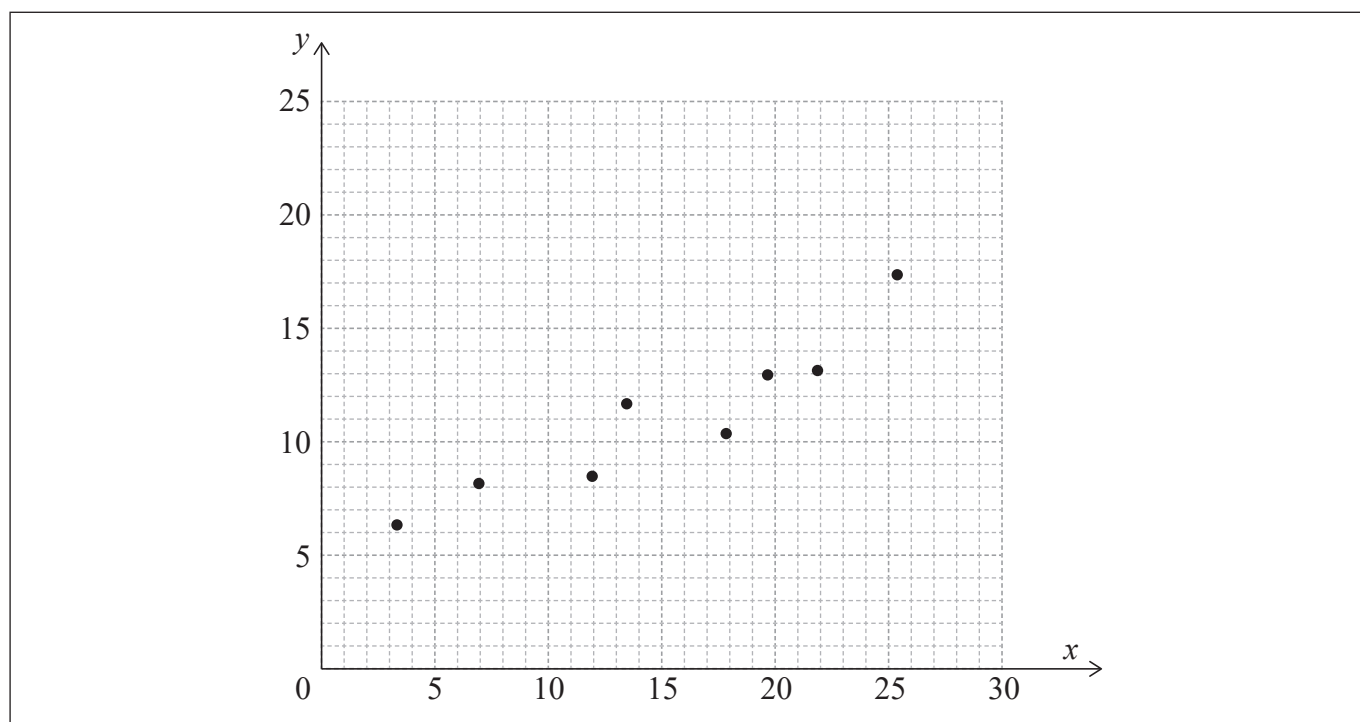
(This question continues on the following page)

(Question 1 continued)

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$, where a, $b \in \mathbb{R}$.

(a)   Write down the value of a and the value of b.   [2]

(b)   Use this model to predict the value of y when $x = 18$.   [2]

(c)   Write down the value of $\overline{x}$ and the value of $\overline{y}$.   [1]

(d)   Draw the line of best fit on the scatter diagram.   [2]

2. [Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of 100 bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. [3]

3. [Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale
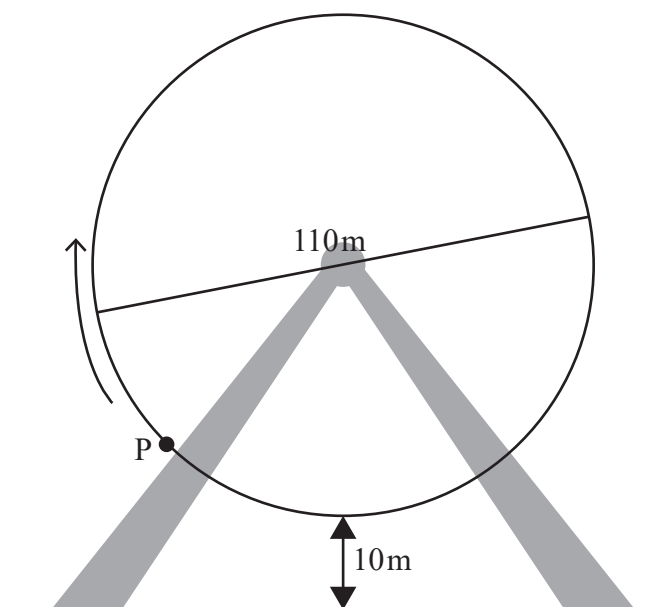
The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where a, b, $c \in \mathbb{R}$.

Find the values of a, b and c.

4. [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\,\mathrm{ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.
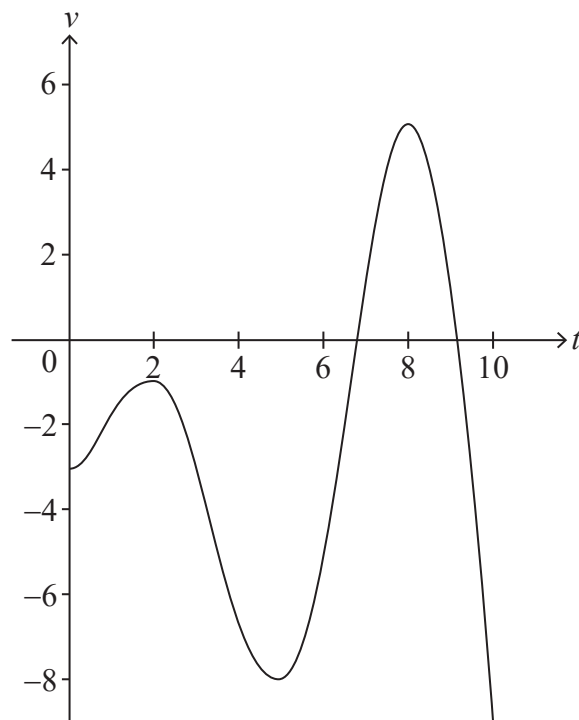
(a) Find the smallest value of t for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

5. [Maximum mark: 5]

Consider the expansion of $(3 + x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is $20\,412$, find the value of n.

6. [Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1: 3x + 2y + z = 6$

$\Pi_2: x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{(x^2+k)^3}} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}^+$.

(a) Show that $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of k. [2]

8. [Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\dfrac{\pi}{5} + \mathrm{i}\sin\dfrac{\pi}{5}\right)$ and $w = 8\left(\cos\dfrac{2k\pi}{5} - \mathrm{i}\sin\dfrac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of zw. [1]

(b) Find the argument of zw in terms of k. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of k.

(ii) For the value of k found in part (i), find the value of zw. [4]

9. [Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.

diagram not to scale
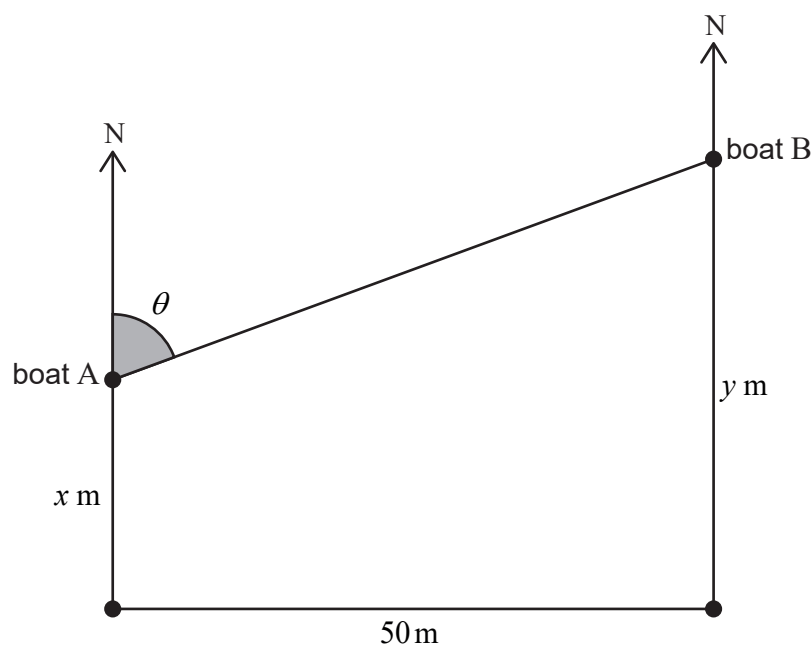
(a)   Show that $y = x + 50 \cot \theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b)   Find the speed of boat A at time T. [6]

....

....

....

....

....

....

Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.
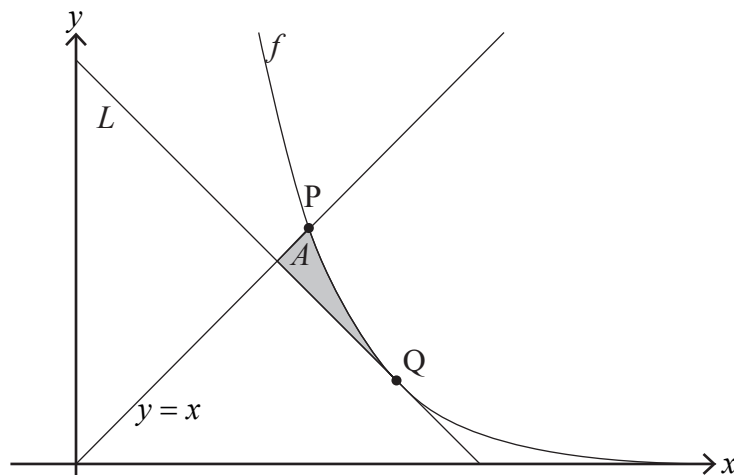
(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.

(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.
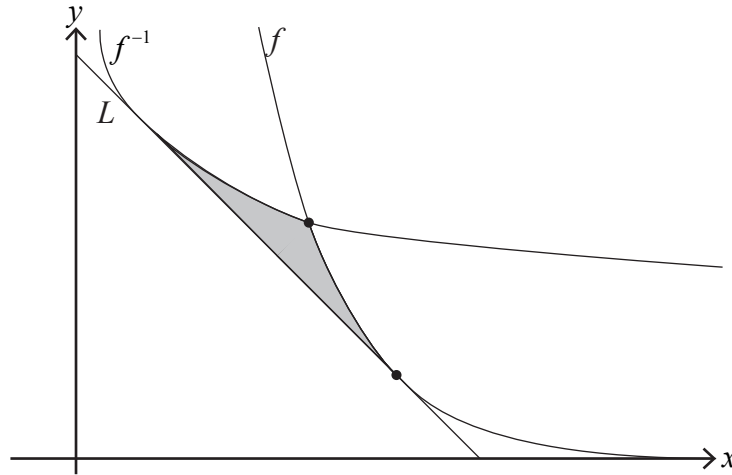
(ii) Hence, find the area of A. [5]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1}.

(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

11. [Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 20]

The function f has a derivative given by $f'(x) = \dfrac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\dfrac{a}{x} + \dfrac{b}{k-x}$, where a, $b \in \mathbb{R}$. Find a and b in terms of k. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P = \dfrac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k, giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

Please **do not** write on this page.

Answers written on this page
will not be marked.

Please **do not** write on this page.

Answers written on this page
will not be marked.