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# Mathematics: analysis and approaches

## Higher level

### Paper 3

31 October 2023

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 24]

**This question asks you to explore some properties of the family of curves**

$y = x^3 + ax^2 + b$  **where**  $x \in \mathbb{R}$  **and**  $a, b$  **are real parameters.**

Consider the family of curves  $y = x^3 + ax^2 + b$  for  $x \in \mathbb{R}$ , where  $a \in \mathbb{R}$ ,  $a \neq 0$  and  $b \in \mathbb{R}$ .

First consider the case where  $a = 3$  and  $b \in \mathbb{R}$ .

- (a) By systematically varying the value of  $b$ , or otherwise, find the two values of  $b$  such that the curve  $y = x^3 + 3x^2 + b$  has exactly two  $x$ -axis intercepts. [2]
- (b) Write down the set of values of  $b$  such that the curve  $y = x^3 + 3x^2 + b$  has exactly
- (i) one  $x$ -axis intercept; [1]
- (ii) three  $x$ -axis intercepts. [1]

Now consider the case where  $a = -3$  and  $b \in \mathbb{R}$ .

- (c) Write down the set of values of  $b$  such that the curve  $y = x^3 - 3x^2 + b$  has exactly
- (i) two  $x$ -axis intercepts; [1]
- (ii) one  $x$ -axis intercept; [1]
- (iii) three  $x$ -axis intercepts. [1]

**(This question continues on the following page)**

**(Question 1 continued)**

For the following parts of this question, consider the curve  $y = x^3 + ax^2 + b$  for  $a \in \mathbb{R}$ ,  $a \neq 0$  and  $b \in \mathbb{R}$ .

- (d) Consider the case where the curve has exactly three  $x$ -axis intercepts. State whether each point of zero gradient is located above or below the  $x$ -axis. [1]
- (e) Show that the curve has a point of zero gradient at  $P(0, b)$  and a point of zero gradient at  $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ . [5]
- (f) Consider the points  $P$  and  $Q$  for  $a > 0$  and  $b > 0$ .
- (i) Find an expression for  $\frac{d^2y}{dx^2}$  and hence determine whether each point is a local maximum or a local minimum. [3]
- (ii) Determine whether each point is located above or below the  $x$ -axis. [1]
- (g) Consider the points  $P$  and  $Q$  for  $a < 0$  and  $b > 0$ .
- (i) State whether  $P$  is a local maximum or a local minimum and whether it is above or below the  $x$ -axis. [1]
- (ii) State the conditions on  $a$  and  $b$  that determine when  $Q$  is below the  $x$ -axis. [1]
- (h) Prove that if  $4a^3b + 27b^2 < 0$  then the curve,  $y = x^3 + ax^2 + b$ , has exactly three  $x$ -axis intercepts. [5]

## 2. [Maximum mark: 31]

**This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle,  $\alpha$ .**

- (a) Consider a family of straight lines,  $L$ , with equation  $y = mx$ , where  $m$  is a parameter. Each member of  $L$  intersects every member of a family of curves,  $C$ , at right-angles.

**Note:** In parts (i), (ii) and (iii), you are not required to consider the case where  $x = 0$ .

- (i) Write down an expression for the gradient of  $L$  in terms of  $x$  and  $y$ . [1]

- (ii) Hence show that the gradient of  $C$  is given by  $\frac{dy}{dx} = -\frac{x}{y}$ . [1]

- (iii) By solving the differential equation  $\frac{dy}{dx} = -\frac{x}{y}$ , show that the family of curves,  $C$ , has equation  $x^2 + y^2 = k$  where  $k$  is a parameter. [2]

A family of curves has equation  $y^2 = 4a^2 - 4ax$  where  $a$  is a positive real parameter.

A second family of curves has equation  $y^2 = 4b^2 + 4bx$  where  $b$  is a positive real parameter.

- (b) Consider the case where  $a = 2$  and  $b = 1$ . On the same set of axes, sketch the curves  $y^2 = 16 - 8x$  and  $y^2 = 4 + 4x$ . On your sketch, clearly label each curve and any  $x$ -intercepts.

**Note:** You are not required to find the coordinates of any points of intersection of the two curves. [3]

- (c) By solving  $y^2 = 4a^2 - 4ax$  and  $y^2 = 4b^2 + 4bx$  simultaneously, show that these curves intersect at the points  $M(a-b, 2\sqrt{ab})$  and  $N(a-b, -2\sqrt{ab})$ . [6]

- (d) At point  $M$ , show that the curves  $y^2 = 4a^2 - 4ax$  and  $y^2 = 4b^2 + 4bx$  intersect at right-angles. [5]

**(This question continues on the following page)**

**(Question 2 continued)**

Consider two families of curves,  $F$  and  $G$ .

The gradient of  $F$  is denoted by  $f(x, y)$ .

The gradient of  $G$  is denoted by  $g(x, y)$ .

Each member of  $F$  intersects every member of  $G$  at an acute angle,  $\alpha$ .

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha}.$$

In part (e), consider the specific case where  $f(x, y) = -\frac{x}{y}$ , for  $x \neq 0$ ,  $y \neq 0$  and  $\alpha = \frac{\pi}{4}$ .

(e) (i) Show that  $g(x, y) = \frac{y-x}{y+x}$ . [2]

(ii) Hence, by solving the homogeneous differential equation  $\frac{dy}{dx} = \frac{y-x}{y+x}$ , find a general equation that represents this family of curves,  $G$ . Give your answer in the form  $h(x, y) = d$  where  $d$  is a parameter. [9]

(f) By considering  $\lim_{\alpha \rightarrow \frac{\pi}{2}} \tan \alpha$ , show that, for all finite  $f(x, y)$ ,

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}. \quad [2]$$


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