Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

Specimen paper

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Let $A$ and $B$ be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

**2.** [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

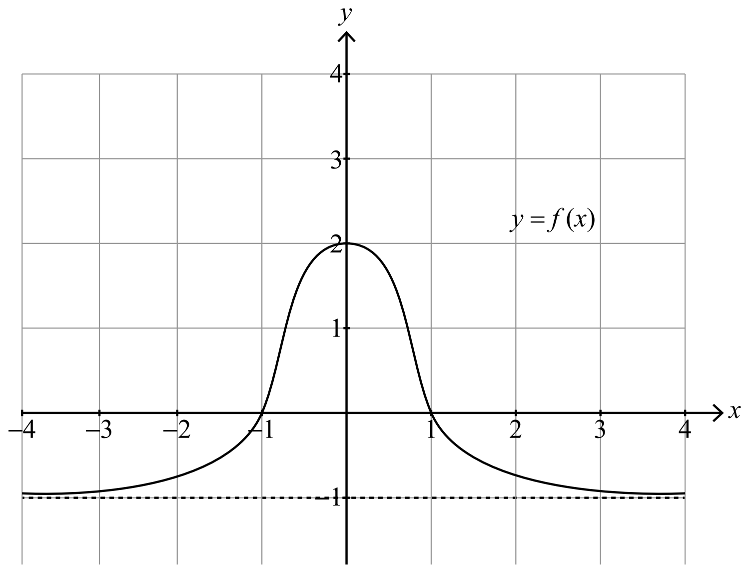
(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

**3.** [Maximum mark: 5]

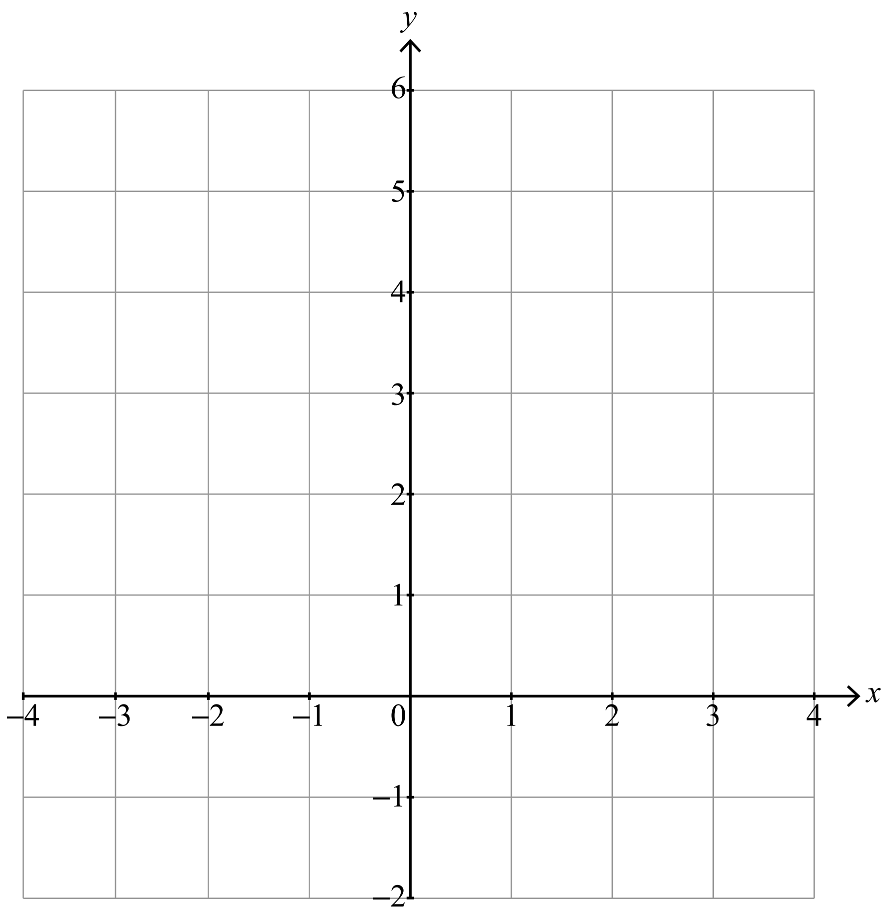
Let $f'(x) = \dfrac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

**4.** [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the $x$-axis at $x = -1$ and $x = 1$, and the $y$-axis at $y = 2$.




On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



**5.** [Maximum mark: 5]

The functions $f$ and $g$ are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of $a$. [3]

**6.** [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3 \sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

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**7.** [Maximum mark: 7]

A continuous random variable $X$ has the probability density function $f$ given by

$$f(x) = \begin{cases} \dfrac{\pi x}{36}\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

**8.** [Maximum mark: 7]

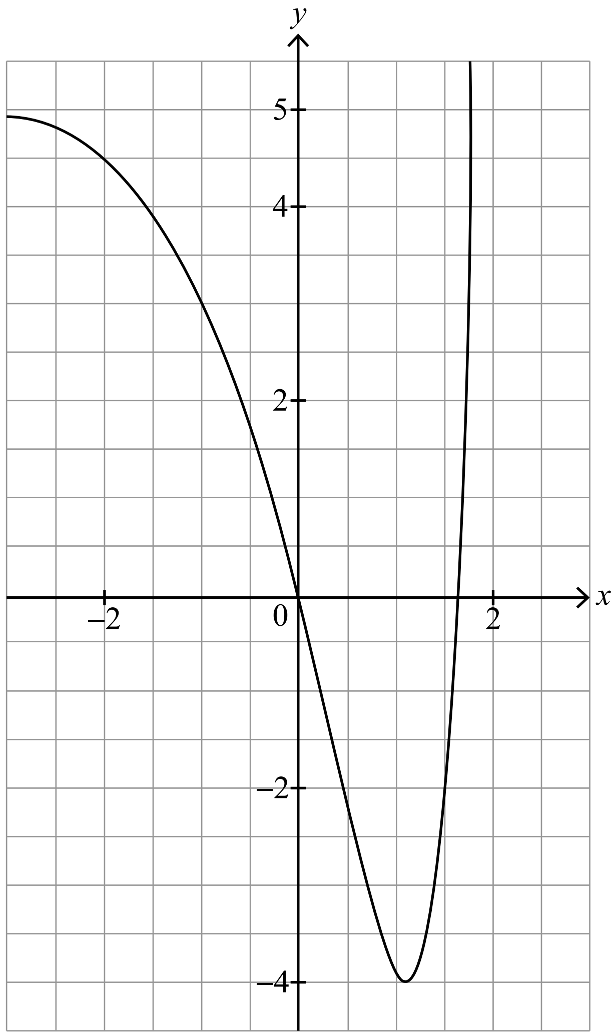
The plane $\Pi$ has the Cartesian equation $2x + y + 2z = 3$.

The line $L$ has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line $L$ and the plane $\Pi$ is $30^\circ$.

Find the possible values of $p$.

**9.** [Maximum mark: 8]

The function $f$ is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.




(a) Find the largest value of $a$ such that $f$ has an inverse function. [3]

(b) For this value of $a$, find an expression for $f^{-1}(x)$, stating its domain. [5]

**(This question continues on the following page)**

**(Question 9 continued)**

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

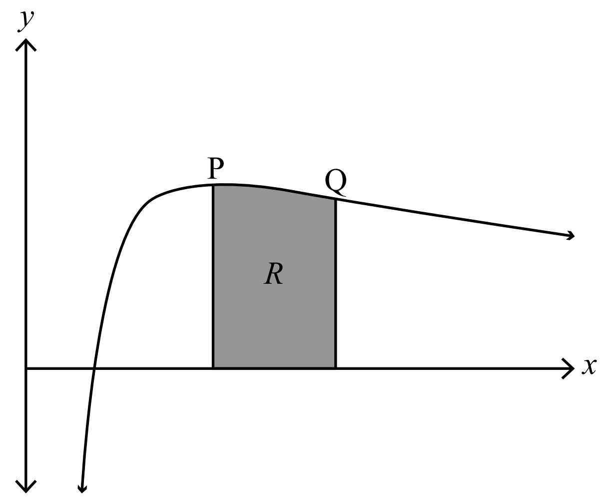
The graph of $f$ has exactly one maximum point $\mathrm{P}$.

(b) Find the $x$-coordinate of $\mathrm{P}$. [3]

The second derivative of $f$ is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of $f$ has exactly one point of inflexion $\mathrm{Q}$.

(c) Show that the $x$-coordinate of $\mathrm{Q}$ is $\dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region $R$ is enclosed by the graph of $f$, the $x$-axis, and the vertical lines through the maximum point $\mathrm{P}$ and the point of inflexion $\mathrm{Q}$.




(d) Given that the area of $R$ is $3$, find the value of $k$. [7]

Do **not** write solutions on this page.

**11.** [Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be $u$, $v$ and $w$.

(b) Find $u$, $v$ and $w$ expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

On an Argand diagram, $u$, $v$ and $w$ are represented by the points $\mathrm{U}$, $\mathrm{V}$ and $\mathrm{W}$ respectively.

(c) Find the area of triangle $\mathrm{UVW}$. [4]

(d) By considering the sum of the roots $u$, $v$ and $w$, show that
$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$ [4]

**12.** [Maximum mark: 21]

The function $f$ is defined by $f(x)=\mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the $x^2$ term. [8]

(b) Show that the coefficient of $x^3$ in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x}-1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x}-1\right)$ up to and including the $x^3$ term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x\to 0}\frac{f(x)-1}{\arctan\left(\mathrm{e}^{3x}-1\right)}$. [3]