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Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

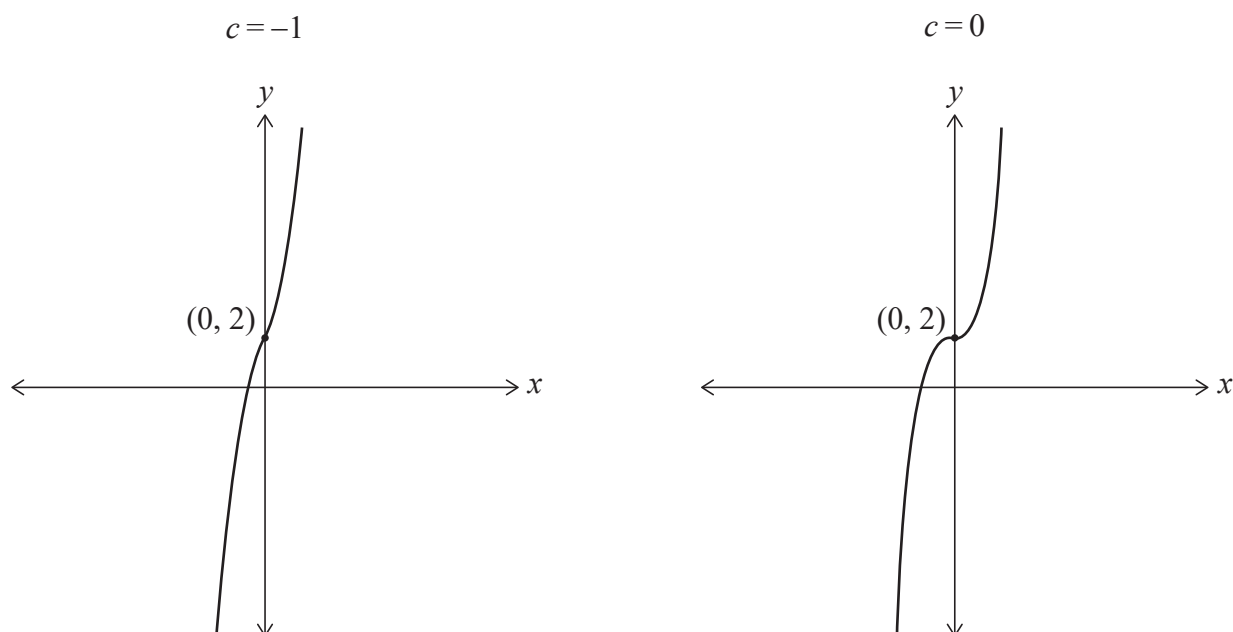
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y -intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has
- (i) a point of inflexion with zero gradient; [1]
 - (ii) one local maximum point and one local minimum point; [2]
 - (iii) no points where the gradient is equal to zero. [1]
- (d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that
- (i) the y -coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]
 - (ii) the y -coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]
- (e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has
- (i) exactly one x -axis intercept; [2]
 - (ii) exactly two x -axis intercepts; [2]
 - (iii) exactly three x -axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c, d \in \mathbb{R}$.

- (f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x -axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

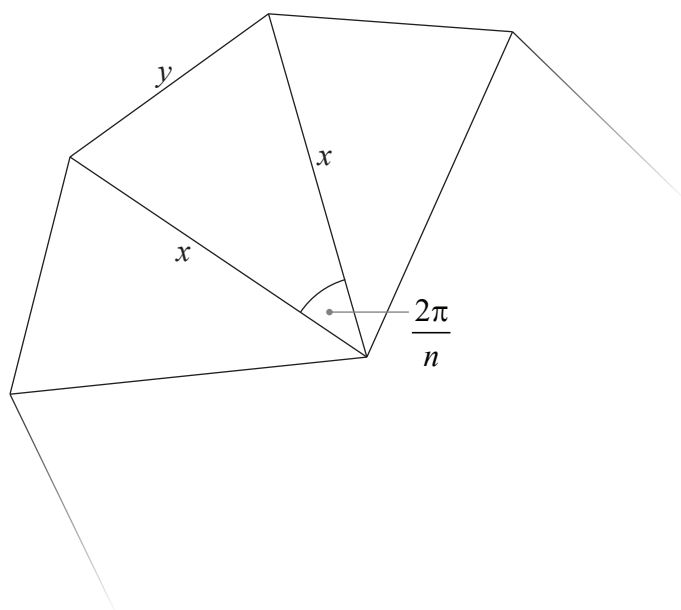
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P .

- (a) Find the side length, s , where $s > 0$, of a square such that $A = P$. [3]

An n -sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n -sided regular polygon is shown in the following diagram.



- (b) Write down, in terms of x and n , an expression for the area, A_T , of one of these isosceles triangles. [1]

- (c) Show that $y = 2x \sin \frac{\pi}{n}$. [2]

Consider a n -sided regular polygon such that $A = P$.

- (d) Use the results from parts (b) and (c) to show that $A = P = 4n \tan \frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a , b and $\sqrt{a^2 + b^2}$, where $a \geq b$, such that $A = P$.

(f) Show that $a = \frac{8}{b-4} + 4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a, b, A, P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References: