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**Mathematics: analysis and approaches**  
**Higher level**  
**Paper 3**

Tuesday 11 May 2021 (morning)

1 hour

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

**This question asks you to explore the behaviour and some key features of the function  $f_n(x) = x^n(a - x)^n$ , where  $a \in \mathbb{R}^+$  and  $n \in \mathbb{Z}^+$ .**

In parts (a) and (b), **only** consider the case where  $a = 2$ .

Consider  $f_1(x) = x(2 - x)$ .

- (a) Sketch the graph of  $y = f_1(x)$ , stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider  $f_n(x) = x^n(2 - x)^n$ , where  $n \in \mathbb{Z}^+$ ,  $n > 1$ .

- (b) Use your graphic display calculator to explore the graph of  $y = f_n(x)$  for
- the odd values  $n = 3$  and  $n = 5$ ;
  - the even values  $n = 2$  and  $n = 4$ .

Hence, copy and complete the following table. [6]

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider  $f_n(x) = x^n(a - x)^n$  where  $a \in \mathbb{R}^+$  and  $n \in \mathbb{Z}^+$ ,  $n > 1$ .

- (c) Show that  $f'_n(x) = nx^{n-1}(a - 2x)(a - x)^{n-1}$ . [5]
- (d) State the three solutions to the equation  $f'_n(x) = 0$ . [2]
- (e) Show that the point  $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$  on the graph of  $y = f_n(x)$  is always above the horizontal axis. [3]

(This question continues on the following page)

**(Question 1 continued)**

(f) Hence, or otherwise, show that  $f'_n\left(\frac{a}{4}\right) > 0$ , for  $n \in \mathbb{Z}^+$ . [2]

(g) By using the result from part (f) and considering the sign of  $f'_n(-1)$ , show that the point  $(0, 0)$  on the graph of  $y = f_n(x)$  is

(i) a local minimum point for even values of  $n$ , where  $n > 1$  and  $a \in \mathbb{R}^+$ ; [3]

(ii) a point of inflexion with zero gradient for odd values of  $n$ , where  $n > 1$  and  $a \in \mathbb{R}^+$ . [2]

Consider the graph of  $y = x^n(a - x)^n - k$ , where  $n \in \mathbb{Z}^+$ ,  $a \in \mathbb{R}^+$  and  $k \in \mathbb{R}$ .

(h) State the conditions on  $n$  and  $k$  such that the equation  $x^n(a - x)^n = k$  has four solutions for  $x$ . [5]

## 2. [Maximum mark: 24]

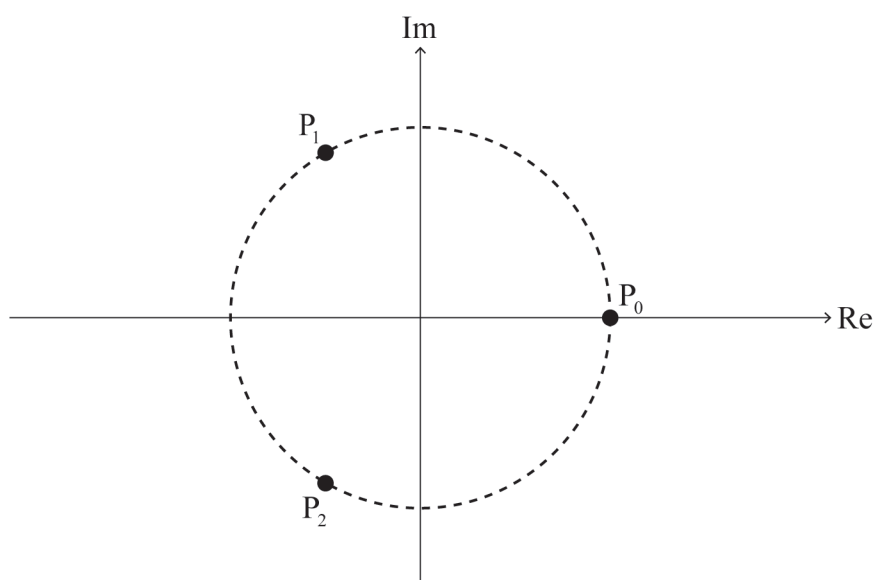
**This question asks you to investigate and prove a geometric property involving the roots of the equation  $z^n = 1$  where  $z \in \mathbb{C}$  for integers  $n$ , where  $n \geq 2$ .**

The roots of the equation  $z^n = 1$  where  $z \in \mathbb{C}$  are  $1, \omega, \omega^2, \dots, \omega^{n-1}$ , where  $\omega = e^{\frac{2\pi i}{n}}$ . Each root can be represented by a point  $P_0, P_1, P_2, \dots, P_{n-1}$ , respectively, on an Argand diagram.

For example, the roots of the equation  $z^2 = 1$  where  $z \in \mathbb{C}$  are  $1$  and  $\omega$ . On an Argand diagram, the root  $1$  can be represented by a point  $P_0$  and the root  $\omega$  can be represented by a point  $P_1$ .

Consider the case where  $n = 3$ .

The roots of the equation  $z^3 = 1$  where  $z \in \mathbb{C}$  are  $1, \omega$  and  $\omega^2$ . On the following Argand diagram, the points  $P_0, P_1$  and  $P_2$  lie on a circle of radius 1 unit with centre  $O(0, 0)$ .



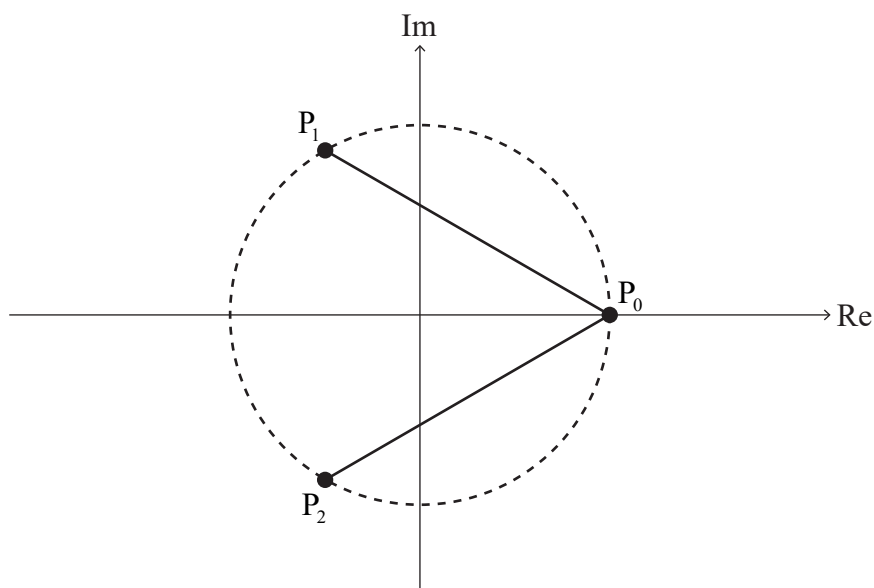
(a) (i) Show that  $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$ . [2]

(ii) Hence, deduce that  $\omega^2 + \omega + 1 = 0$ . [2]

**(This question continues on the following page)**

**(Question 2 continued)**

Line segments  $[P_0P_1]$  and  $[P_0P_2]$  are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



$P_0P_1$  is the length of  $[P_0P_1]$  and  $P_0P_2$  is the length of  $[P_0P_2]$ .

(b) Show that  $P_0P_1 \times P_0P_2 = 3$ . [3]

Consider the case where  $n = 4$ .

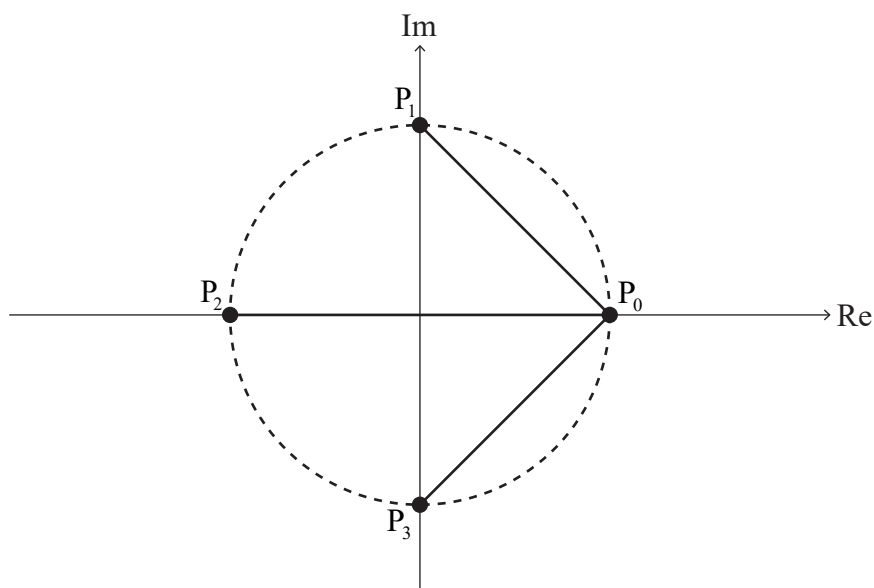
The roots of the equation  $z^4 = 1$  where  $z \in \mathbb{C}$  are  $1, \omega, \omega^2$  and  $\omega^3$ .

(c) By factorizing  $z^4 - 1$ , or otherwise, deduce that  $\omega^3 + \omega^2 + \omega + 1 = 0$ . [2]

**(This question continues on the following page)**

**(Question 2 continued)**

On the following Argand diagram, the points  $P_0, P_1, P_2$  and  $P_3$  lie on a circle of radius 1 unit with centre  $O(0, 0)$ .  $[P_0P_1]$ ,  $[P_0P_2]$  and  $[P_0P_3]$  are line segments.



- (d) Show that  $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ . [4]

For the case where  $n = 5$ , the equation  $z^5 = 1$  where  $z \in \mathbb{C}$  has roots  $1, \omega, \omega^2, \omega^3$  and  $\omega^4$ .

It can be shown that  $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$ .

Now consider the general case for integer values of  $n$ , where  $n \geq 2$ .

The roots of the equation  $z^n = 1$  where  $z \in \mathbb{C}$  are  $1, \omega, \omega^2, \dots, \omega^{n-1}$ . On an Argand diagram, these roots can be represented by the points  $P_0, P_1, P_2, \dots, P_{n-1}$  respectively where  $[P_0P_1], [P_0P_2], \dots, [P_0P_{n-1}]$  are line segments. The roots lie on a circle of radius 1 unit with centre  $O(0, 0)$ .

- (e) Suggest a value for  $P_0P_1 \times P_0P_2 \times \dots \times P_0P_{n-1}$ . [1]

$P_0P_1$  can be expressed as  $|1 - \omega|$ .

- (f) (i) Write down expressions for  $P_0P_2$  and  $P_0P_3$  in terms of  $\omega$ . [2]

- (ii) Hence, write down an expression for  $P_0P_{n-1}$  in terms of  $n$  and  $\omega$ . [1]

Consider  $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$  where  $z \in \mathbb{C}$ .

- (g) (i) Express  $z^{n-1} + z^{n-2} + \dots + z + 1$  as a product of linear factors over the set  $\mathbb{C}$ . [3]

- (ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

**References:**