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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

30 October 2023

**Zone A** afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

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2 hours

## Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where $k$ is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of $k$. [3]

**2.** [Maximum mark: 4]

Events $A$ and $B$ are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cap B) = 0.6$.

(a) Find $\mathrm{P}(A \cup B)$. [2]

(b) Hence, or otherwise, find $\mathrm{P}(A' \cap B')$. [2]

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**3.** [Maximum mark: 7]

The sum of the first $n$ terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where $p$ and $q$ are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

(a) Find the value of $p$ and the value of $q$. [5]

(b) Find the value of $u_5$. [2]

**4.** [Maximum mark: 6]

In the following triangle $\mathrm{ABC}$, $\mathrm{AB} = \sqrt{6}$ cm, $\mathrm{AC} = 10\,\mathrm{cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.

**diagram not to scale**

Find the area of triangle $\mathrm{ABC}$.

**5.** [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + ... + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of $n$ and the value of $k$.

**6.** [Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by $17$ for all $n \in \mathbb{Z}^+$.

**7.** [Maximum mark: 5]

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation $z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p$, $q \in \mathbb{R}$.

Find the value of $p$ and the value of $q$.

**8.** [Maximum mark: 9]

(a) Find $\int x(\ln x)^2 \mathrm{d}x$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 \mathrm{d}x = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

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**9.** [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that $f$ is an even function. [2]

(b) Given that $\lim_{x \to 0} f(x) = 16$, find the value of $k$. [6]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

The functions $f$ and $g$ are defined by

$f(x) = \ln(2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of $d$. [9]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 10 continued)**

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.

The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$. [5]

Do **not** write solutions on this page.

**11.** [Maximum mark: 21]

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \le x \le \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in $x^4$.

(ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos 2x - 1}$, up to and including the term in $x^4$.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in $x^4$. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$. [3]

Do **not** write solutions on this page.

**12.** [Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where $a$ and $b$ are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 2

31 October 2023

**Zone A** afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

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2 hours

## Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

16 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

The following diagram shows a pyramid with vertex $\mathrm{V}$ and rectangular base $\mathrm{OABC}$.

Point $\mathrm{B}$ has coordinates $(6, 8, 0)$, point $\mathrm{C}$ has coordinates $(6, 0, 0)$ and point $\mathrm{V}$ has coordinates $(3, 4, 9)$.

**diagram not to scale**

(a) Find $\mathrm{BV}$. [2]

(b) Find the size of $\mathrm{B\hat{V}C}$. [4]

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**2.** [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of $f$ for $-4 \le x \le 3$. [3]

The function $g$ is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of $g$ is obtained from the graph of $f$ by a horizontal stretch with scale factor $k$, followed by a vertical translation of $c$ units.

Find the value of $k$ and the value of $c$. [2]

**(This question continues on the following page)**

**(Question 2 continued)**

**3.** [Maximum mark: 7]

A company is designing a new logo in the shape of a letter “C”.

The letter “C” is formed between two circles with centre $\mathrm{O}$.

The point $\mathrm{A}$ lies on the circumference of the inner circle with radius $r\,\mathrm{cm}$, where $r < 10$.

The point $\mathrm{B}$ lies on the circumference of the outer circle with radius $10\,\mathrm{cm}$.

The reflex angle $\mathrm{A\hat{O}B}$ is $5.2$ radians. The letter “C” is shown by the shaded area in the following diagram.

**diagram not to scale**

**(This question continues on the following page)**

**(Question 3 continued)**

(a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is $64\,\text{cm}^2$.

(b) (i) Find the value of $r$.

(ii) Find the perimeter of the "C". [5]

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**4.** [Maximum mark: 5]

A particle moves along a straight line. Its displacement, $s$ metres, from a fixed point $\mathrm{O}$ after time $t$ seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

The particle first comes to rest after $q$ seconds.

(a) Find the value of $q$. [2]

(b) Find the total distance that the particle travels in the first $q$ seconds. [3]

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**5.** [Maximum mark: 5]

The following table shows the probability distribution of a discrete random variable $X$, where $a, k \in \mathbb{R}^+$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $k$ | $k^2$ | $a$ | $k^3$ |

Given that $\mathrm{E}(X) = 2.3$, find the value of $a$.

**6.** [Maximum mark: 5]

The random variable $X$ is such that $X \sim B(25, p)$ and $\mathrm{Var}(X) = 5.75$.

(a) Find the possible values of $p$. [3]

The random variable $Y$ is such that $Y = 1 - 2X$.

(b) Find $\mathrm{Var}(Y)$. [2]

**7.** [Maximum mark: 6]

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

(a) In how many ways can the team members be placed if

(i) there are no restrictions;

(ii) the girls must be placed next to each other. [3]

(b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]

**8.** [Maximum mark: 9]

Three points are given by $\mathrm{A}(0, p, 2)$, $\mathrm{B}(1, 1, 1)$ and $\mathrm{C}(p, 0, 4)$, where $p$ is a positive constant.

(a) Show that $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

(b) Hence, find the smallest possible value of $\left|\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}\right|^2$. [3]

(c) Hence, find the smallest possible area of triangle $\mathrm{ABC}$. [2]

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**9.** [Maximum mark: 9]

Consider the differential equation $\frac{dy}{dx} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of $0.1$ to find an approximation for $y$ when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2e^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A farmer is growing a field of wheat plants. The height, $H\,\text{cm}$, of each plant can be modelled by a normal distribution with mean $\mu$ and standard deviation $\sigma$.

It is known that $\text{P}(H < 94.6) = 0.288$ and $\text{P}(H > 98.1) = 0.434$.

(a) Find the probability that the height of a randomly selected plant is between $94.6\,\text{cm}$ and $98.1\,\text{cm}$. [2]

(b) Find the value of $\mu$ and the value of $\sigma$. [5]

The farmer measures $100$ randomly selected plants. Any plant with a height greater than $98.1\,\text{cm}$ is considered ready to harvest. Heights of plants are independent of each other.

(c) (i) Find the probability that exactly $34$ plants are ready to harvest.

(ii) Given that fewer than $49$ plants are ready to harvest, find the probability that exactly $34$ plants are ready to harvest. [6]

In another field, the farmer is growing the same variety of wheat, but is using a different fertilizer. The heights of these plants, $F\,\text{cm}$, are normally distributed with mean $98.6$ and standard deviation $d$. The farmer finds the interquartile range to be $4.82\,\text{cm}$.

(d) Find the value of $d$. [3]

Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

Consider the function defined by $f(x) = \dfrac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of $f$. [1]

(b) Find the coordinates of the points where the graph of $f$ crosses the $x$-axis. [2]

The graph of $f$ also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of $a$ and the value of $b$. [4]

(d) Sketch the graph of $f$ for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of $f$. [4]

(f) Solve the inequality $f(x) > x$ . [4]

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

Line $L$ is given by the vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line $M$ is given by the vector equation $\boldsymbol{r}_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines $L$ and $M$ intersect at a point $\mathrm{A}$ and find the position vector of $\mathrm{A}$. [5]

(b) Verify that the lines $L$ and $M$ both lie in the plane $\Pi$ given by $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point $\mathrm{B}$ has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through $\mathrm{B}$ perpendicular to $\Pi$ intersects $\Pi$ at point $\mathrm{C}$.

(c) (i) Find the position vector of $\mathrm{C}$.

(ii) Hence, find $\left|\overrightarrow{\mathrm{BC}}\right|$. [7]

(d) Find the reflection of the point $\mathrm{B}$ in the plane $\Pi$. [3]

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

8 May 2023

**Zone A** afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

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| | | | | | | | | | |

2 hours

## Instructions to candidates

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- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

16 pages

2223 – 7106
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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 7]

The function $f$ is defined by $f(x)=\dfrac{7x+7}{2x-4}$ for $x \in \mathbb{R}$, $x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y=f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

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**2.** [Maximum mark: 6]

On a Monday at an amusement park, a sample of $40$ visitors was randomly selected as they were leaving the park. They were asked how many times that day they had been on a ride called *The Dragon*. This information is summarized in the following frequency table.

| **Number of times on *The Dragon*** | **Frequency** |
|---|---|
| 0 | 6 |
| 1 | 16 |
| 2 | 13 |
| 3 | 2 |
| 4 | 3 |

It can be assumed that this sample is representative of all visitors to the park for the following day.

(a) For the following day, Tuesday, estimate

(i) the probability that a randomly selected visitor will ride *The Dragon*;

(ii) the expected number of times a visitor will ride *The Dragon*. [4]

It is known that $1000$ visitors will attend the amusement park on Tuesday. *The Dragon* can carry a maximum of $10$ people each time it runs.

(b) Estimate the minimum number of times *The Dragon* must run to satisfy demand. [2]

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**3.** [Maximum mark: 6]

Solve $\cos 2x = \sin x$, where $-\pi \leq x \leq \pi$.

**4.** [Maximum mark: 6]

Find the range of possible values of $k$ such that $e^{2x} + \ln k = 3e^{x}$ has at least one real solution.

**5.** [Maximum mark: 6]

The function $f$ is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of $f$ for $0 \le x \le 4m$, where $x$ is in radians. There are $x$-intercepts at $x = 0$, $2m$ and $4m$.

(a) Find an expression for $m$ in terms of $q$. [2]

The function $g$ is defined by $g(x) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of $g$. [4]

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**6.** [Maximum mark: 5]

The side lengths, $x\,\text{cm}$, of an equilateral triangle are increasing at a rate of $4\,\text{cm}\,\text{s}^{-1}$.

Find the rate at which the area of the triangle, $A\,\text{cm}^2$, is increasing when the side lengths are $5\sqrt{3}$ cm.

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**7.** [Maximum mark: 6]

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$.

**8.** [Maximum mark: 7]

Part of the graph of a function, $f$, is shown in the following diagram. The graph of $y = f(x)$ has a $y$-intercept at $(0, 3)$, an $x$-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.

**(This question continues on the following page)**

**(Question 8 continued)**

Consider the function $g(x) = \left| f\left(|x|\right) \right|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]

(b) Find the possible values of $k$ such that $\left(g(x)\right)^2 = k$ has exactly two solutions. [3]

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**9.** [Maximum mark: 7]

The function $f$ is defined by $f(y) = \sqrt{r^2 - y^2}$ for $-r \le y \le r$.

The region enclosed by the graph of $x = f(y)$ and the $y$-axis is rotated by $360°$ about the $y$-axis to form a solid sphere. The sphere is drilled through along the $y$-axis, creating a cylindrical hole. The resulting spherical ring has height, $h$.

This information is shown in the following diagrams.

**diagram not to scale**

The spherical ring has a volume of $\pi$ cubic units. Find the value of $h$.

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first $n$ terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find $u_6$. [4]

(b) Find $u_1$. [2]

(c) Hence or otherwise, write an expression for $u_n$ in terms of $n$. [3]

Consider a geometric sequence, $v_n$, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, $r$. [3]

(e) Given that $v_{99} < 0$, find $v_5$. [2]

Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

**diagram not to scale**

A narrow passageway with width $\frac{3}{4}\,\mathrm{m}$ is perpendicular to a room of width $6\,\mathrm{m}$. There is a corner at point $\mathrm{C}$. Points $\mathrm{A}$ and $\mathrm{B}$ are variable points on the base of the walls such that $\mathrm{A}$, $\mathrm{C}$ and $\mathrm{B}$ lie on a straight line.

Let $L$ denote the length $\mathrm{AB}$ in metres.

Let $\alpha$ be the angle that $[\mathrm{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$. [2]

(b) (i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$.

(ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan 2$. [5]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 11 continued)**

(c) (i) Find $\frac{d^2L}{d\alpha^2}$.

(ii) When $\alpha = \arctan 2$, show that $\frac{d^2L}{d\alpha^2} = \frac{45}{4}\sqrt{5}$. [7]

(d) (i) Hence, justify that $L$ is a minimum when $\alpha = \arctan 2$.

(ii) Determine this minimum value of $L$. [3]

Two people need to carry a pole of length $11.25\,\text{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

Two lines, $L_1$ and $L_2$, intersect at point $\mathrm{P}$. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on $L_2$. This is shown in the following diagram.

**diagram not to scale**

The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of $L_1$ is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\overrightarrow{\mathrm{PA}} = \begin{pmatrix} 2t \\ 0 \\ 3+t \end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of $t$. [4]

(c) Hence or otherwise, find the shortest distance from $\mathrm{A}$ to $L_1$. [4]

A plane, $\Pi$, contains $L_1$ and $L_2$.

(d) Find a normal vector to $\Pi$. [2]

The base of a right cone lies in $\Pi$, centred at $\mathrm{A}$ such that $L_1$ is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

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Diploma Programme
Programme du diplôme
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# Mathematics: analysis and approaches
# Higher level
# Paper 1

8 May 2023

**Zone A** afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

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2 hours

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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
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15 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

The following diagram shows a circle with centre $\mathrm{O}$ and radius $4\,\mathrm{cm}$.

**diagram not to scale**

The points $\mathrm{P}$, $\mathrm{Q}$ and $\mathrm{R}$ lie on the circumference of the circle and $\mathrm{P\hat{O}R} = \theta$, where $\theta$ is measured in radians.

The length of arc $\mathrm{PQR}$ is $10\,\mathrm{cm}$.

(a) Find the perimeter of the shaded sector. [2]

(b) Find $\theta$. [2]

(c) Find the area of the shaded sector. [2]

**(This question continues on the following page)**

**(Question 1 continued)**

**2.** [Maximum mark: 5]

A function $f$ is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the $y$-axis;

(ii) the $x$-axis. [2]

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**(This question continues on the following page)**

**(Question 2 continued)**

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b). [1]

**3.** [Maximum mark: 5]

Events $A$ and $B$ are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A|B) = 0.25$ and $\mathrm{P}(A \cup B) = 0.55$.

Find $\mathrm{P}(B)$.

**4.** [Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2 + 2}$ for $x \geq 0$.

The shaded region $R$ is bounded by the curve, the $x$-axis and the line $x = c$.

The area of $R$ is $\ln 3$.

Find the value of $c$.

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**5.** [Maximum mark: 7]

The functions $f$ and $g$ are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b, \text{ where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3.$$

Find the two possible functions $f$ such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

**6.** [Maximum mark: 5]

A continuous random variable $X$ has probability density function $f$ defined by

$$f(x) = \begin{cases} \dfrac{1}{2a}, & a \le x \le 3a \\ 0, & \text{otherwise} \end{cases}$$

where $a$ is a positive real number.

(a) State $\mathrm{E}(X)$ in terms of $a$. [1]

(b) Use integration to find $\mathrm{Var}(X)$ in terms of $a$. [4]

**7.** [Maximum mark: 7]

Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

**8.** [Maximum mark: 7]

The functions $f$ and $g$ are defined by

$$f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \le x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point $\mathrm{P}$ whose $x$-coordinate is $k$, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at $\mathrm{P}$ and the tangent to the curve $y = g(x)$ at $\mathrm{P}$ intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a + \sqrt{b}}{c}$, where $a,\ c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

**9.** [Maximum mark: 9]

The following diagram shows parallelogram $\mathrm{OABC}$ with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OC}} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \neq 0$.

The angle between $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OC}}$ is $\theta$, where $0 < \theta < \pi$.

Point $\mathrm{M}$ is on $[\mathrm{AB}]$ such that $\overrightarrow{\mathrm{AM}} = k\,\overrightarrow{\mathrm{AB}}$, where $0 \leq k \leq 1$ and $\overrightarrow{\mathrm{OM}} \bullet \overrightarrow{\mathrm{MC}} = 0$.

(a) Express $\overrightarrow{\mathrm{OM}}$ and $\overrightarrow{\mathrm{MC}}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2\,(1-2k)\big(2\cos\theta - (1-2k)\big) = 0$. [3]

(c) Find the range of values for $\theta$ such that there are two possible positions for $\mathrm{M}$. [4]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius $3$.

A triangle, $\mathrm{PQR}$, is inscribed in the circle with its vertices at $\mathrm{P}(-3, 0)$, $\mathrm{Q}(x, y)$ and $\mathrm{R}(x, -y)$, where $\mathrm{Q}$ and $\mathrm{R}$ are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.

(a) For point $\mathrm{Q}$, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for $A$, the area of triangle $\mathrm{PQR}$, in terms of $x$. [3]

(c) Show that $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \dfrac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]

(d) Hence or otherwise, find the $y$-coordinate of $\mathrm{R}$ such that $A$ is a maximum. [6]

Do **not** write solutions on this page.

**11.** [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of $u$, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer $n$ such that $u^n$ is a real number.

(ii) Find the value of $u^n$ when $n$ takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that $u$ is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from $z$ to $w$, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing $z$ in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

Do **not** write solutions on this page.

**12.** [Maximum mark: 17]

(a) By using an appropriate substitution, show that $\int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos\sqrt{x}$ for $x \geq 0$.

The curve intersects the $x$-axis at $x_1$, $x_2$, $x_3$, $x_4$, ….

The $n$th $x$-intercept of the curve, $x_n$, is given by $x_n = \dfrac{(2n-1)^2\pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for $x_{n+1}$. [1]

The regions bounded by the curve and the $x$-axis are denoted by $R_1$, $R_2$, $R_3$, …, as shown on the above diagram.

(c) Calculate the area of region $R_n$.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the $x$-axis, $R_1$, $R_2$, $R_3$, …, form an arithmetic sequence. [3]

**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 2

9 May 2023

**Zone A** afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

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2 hours

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- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

2223–7107
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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

The cities Lucknow $(\mathrm{L})$, Jaipur $(\mathrm{J})$ and Delhi $(\mathrm{D})$ are represented in the following diagram. Lucknow lies $500\,\mathrm{km}$ directly east of Jaipur, and $\mathrm{JLD} = 25°$.

**diagram not to scale**

The bearing of $\mathrm{D}$ from $\mathrm{J}$ is $034°$.

(a) Find $\mathrm{J\hat{D}L}$. [2]

(b) Find the distance between Lucknow and Delhi. [3]

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**2.** [Maximum mark: 5]

The functions $f$ and $g$ are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \le x \le 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]

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**3.** [Maximum mark: 7]

The total number of children, $y$, visiting a park depends on the highest temperature, $T$, in degrees Celsius (°C). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \le T \le 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°\text{C}$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, $x$. The following table shows the data collected on five different days.

| **Total number of children** $(y)$ | 81 | 175 | 202 | 346 | 360 |
|---|---|---|---|---|---|
| **Ice creams sold** $(x)$ | 15 | 27 | 23 | 35 | 46 |

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are $y$ children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°\text{C}$. [2]

**4.** [Maximum mark: 5]

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, $D\,\text{mm}$, are normally distributed with mean $32$ and standard deviation $\sigma$. The interquartile range of the diameters is $0.28$.

Find the value of $\sigma$.

**5.** [Maximum mark: 7]

The coefficient of $x^6$ in the expansion of $(ax^3 + b)^8$ is $448$.

The coefficient of $x^6$ in the expansion of $(ax^3 + b)^{10}$ is $2880$.

Find the value of $a$ and the value of $b$, where $a, b > 0$.

**6.** [Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of $n$ that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to $z$ on the Argand diagram that results in $z^{10}$. [3]

**7.** [Maximum mark: 6]

A new test has been developed to identify whether a particular gene, $G$, is present in a population of parrots. The test returns a correct positive result $95\%$ of the time for parrots with the gene, and a false positive result $2\%$ of the time for parrots without the gene.

The proportion of the population with the gene is $p$.

(a) Complete the tree diagram below. [2]

(b) A random sample of the population was tested. It was found that $150$ tests returned a positive result. Out of the $150$ parrots with a positive test result, $18$ did not actually have the gene. Find an estimate for $p$. [4]

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**8.** [Maximum mark: 7]

The angle between a line and a plane is $\alpha$, where $\alpha \in \mathbb{R},\ 0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

Find the value of $\alpha$.

**9.** [Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

The gutter will be open at the top and is made by folding a piece of sheet metal $45\,\text{cm}$ wide and $600\,\text{cm}$ long.

**diagram not to scale**

The cross-section of the gutter is shaded in the following diagram.

**diagram not to scale**

The height of both vertical sides is $10\,\text{cm}$. The width of the gutter is $w\,\text{cm}$.

Arc $\text{AB}$ lies on the circumference of a circle with centre $\text{O}$ and radius $12\,\text{cm}$.

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 10 continued)**

Let $\mathrm{A\hat{O}B} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in $\mathrm{cm}^3$, of rainwater that enters the gutter can be modelled by a function $R(t)$, where $t$ is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000\,,\ t \geq 0\,.$$

During any $60$-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

A continuous random variable, $X$, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{6}{\pi\sqrt{16 - x^2}}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X < 0.5)$. [2]

A laboratory trial may require up to $2$ millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where $X$ is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X < 0.5$.

(c) Determine the least number of trials required to be $99\%$ sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider $n$ trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of $n$ such that the probability of three consecutive successful trials is more than $0.05$. [6]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of $0.03$, to find an approximate value for $y$ when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

(ii) Show that $\frac{d^2 y}{dx^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{d^3 y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for $y$.

(ii) Use the Maclaurin series to find an approximate value for $y$ when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2 - 1}{x^2 + 1} \equiv 1 - \frac{2}{x^2 + 1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of $y$ when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \le x < 1$, explain why the approximate value for $y$ obtained using Euler's method will always be less than the actual value for $y$. [2]

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**References:**

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 2

9 May 2023

**Zone A** afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

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2 hours

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15 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

A botanist is conducting an experiment which studies the growth of plants.

The heights of the plants are measured on seven different days.

The following table shows the number of days, $d$, that the experiment has been running and the average height, $h\,\text{cm}$, of the plants on each of those days.

| Number of days ($d$) | 2 | 5 | 13 | 24 | 33 | 37 | 42 |
|---|---|---|---|---|---|---|---|
| Average height ($h$) | 10 | 16 | 30 | 59 | 76 | 79 | 82 |

The value of Pearson's product-moment correlation coefficient, $r$, for this data is $0.991$, correct to three significant figures.

(a) The regression line of $h$ on $d$ for this data can be written in the form $h = ad + b$.

Find the value of $a$ and the value of $b$. [2]

(b) Use your regression line to estimate the average height of the plants when the experiment has been running for $20$ days. [2]

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**2.** [Maximum mark: 6]

The following diagram shows two buildings situated on level ground.

From point $\mathrm{P}$ on the ground directly between the two buildings, the angle of elevation to the top of each building is $\theta$.

**diagram not to scale**

The distance from point $\mathrm{P}$ to point $\mathrm{A}$ at the top of the taller building is $150$ metres.

The distance from point $\mathrm{P}$ to point $\mathrm{B}$ at the top of the shorter building is $90$ metres.

The distance between $\mathrm{A}$ and $\mathrm{B}$ is $154$ metres.

Find the height, $h$, of the taller building.

**3.** [Maximum mark: 8]

The weights, $W$ grams, of bags of rice packaged in a factory can be modelled by a normal distribution with mean $204$ grams and standard deviation $5$ grams.

(a) A bag of rice is selected at random.

Find the probability that it weighs more than $210$ grams. [2]

According to this model, $80\,\%$ of the bags of rice weigh between $w$ grams and $210$ grams.

(b) Find the probability that a randomly selected bag of rice weighs less than $w$ grams. [2]

(c) Find the value of $w$. [2]

(d) Ten bags of rice are selected at random.

Find the probability that exactly one of the bags weighs less than $w$ grams. [2]

**4.** [Maximum mark: 7]

The expansion of $(x+h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

Given that the coefficients, $a$, $b$ and $d$, are the first three terms of a geometric sequence, find the value of $h$.

**5.** [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time $t$ seconds is given by $v(t) = 4\mathrm{e}^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of $v$ is shown in the following diagram.

Let $t_1$ be the first time when the particle's **acceleration** is zero.

(a) Find the value of $t_1$. [2]

Let $t_2$ be the **second** time when the particle is instantaneously at rest.

(b) Find the value of $t_2$. [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

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**6.** [Maximum mark: 8]

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let $L$ be the line of intersection of $\Pi_1$ and $\Pi_2$.

(a) Verify that a vector equation of $L$ is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]

(b) Find the coordinates of the point $\mathrm{P}$ on $L$ that is nearest to the origin. [5]

**7.** [Maximum mark: 5]

A function $f$ is defined as $f(x) = \arctan(x-2)$, where $2 \le x \le 2+\sqrt{3}$.

The region bounded by the curve, the $y$-axis, the $x$-axis and the line $y = \frac{\pi}{3}$ is rotated $360^{\circ}$ about the $y$-axis to form a solid of revolution.

Find the volume of the solid.

**8.** [Maximum mark: 7]

A function $g$ is defined by $g(x) = \frac{2x-5}{x^2-3}$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$.

(a) Determine the range of $g$. [4]

A function $h$ is defined by $h(x) = g(|x|)\cos t$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$ and $t$ is a constant where $\frac{\pi}{2} < t \leq \pi$.

(b) Find the set of values of $x$ such that $h(x) \leq 0$. [3]

Please **do not** write on this page.

Answers written on this page
will not be marked.

**9.** [Maximum mark: 5]

Let $S$ be the set of $30$ positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from $S$ without replacement. He then adds them together and determines whether the sum is divisible by $3$.

Determine the total number of selections Raghu can make to obtain a sum that is divisible by $3$.

You may assume that order is not important, for example, $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{2, 1, 3\}, \{3, 1, 2\}, \{3, 2, 1\}$ are all considered to be the same selection. [5]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 13]

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

The height, $H$ metres, of the base of the weight above the ground can be modelled by the function $H(t) = a\cos(7.8t) + b$, for $a, b \in \mathbb{R}$ and $0 \le t \le 10$, where $t$ is the time in seconds after the weight is released.

(a) Find the period of the function. [2]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 10 continued)**

The weight is released when its base is at a minimum height of $1$ metre above the ground, and it reaches a maximum height of $1.8$ metres above the ground. The graph of $H$ is shown in the following diagram.

(b) Find the value of

(i) $a$;

(ii) $b$. [3]

(c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]

(d) Find the first time that the base of the weight reaches a height of $1.5$ metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

(e) Find the probability that the height of the base of the weight is greater than $1.5$ metres at the time the picture is taken. [4]

Do **not** write solutions on this page.

**11.** [Maximum mark: 20]

A game of chance involves drawing **two** balls at random out of a box without replacement. The box initially contains $r$ red balls and $y$ yellow balls.

Let $\mathrm{P}(YY)$ represent the probability of drawing two yellow balls from the box without replacement.

Consider a version of this game where it is known that $\mathrm{P}(YY) = \frac{1}{3}$.

(a) Show that $2y^2 - 2(r+1)y + r - r^2 = 0$. [4]

(b) By solving the equation in part (a), show that $y = \frac{(r+1) + \sqrt{3r^2+1}}{2}$. [4]

(c) Find two pairs of values for $r$ and $y$ that satisfy the condition $\mathrm{P}(YY) = \frac{1}{3}$. [4]

Now consider a similar game of chance that involves drawing **three** balls out of a box without replacement. The box initially contains $10$ red balls and $y$ yellow balls.

Let $\mathrm{P}(YYY)$ represent the probability of drawing three yellow balls from the box without replacement.

(d) Find an expression for $\mathrm{P}(YYY)$ in terms of $y$. [3]

A yellow ball is added so that the box now contains $10$ red balls and $(y+1)$ yellow balls. The probability of drawing three yellow balls from the box without replacement is now twice the probability expressed in part (d).

(e) Find the initial number of yellow balls in the box. [5]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0$, $y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length $0.1$ to find an approximate value of $y$ when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of $y$ when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \le x \le 1.1$, explain why the value of $y$ found in part (c) is greater than the approximate value of $y$ found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point $\mathrm{P}$.

(e) By sketching the graph of an appropriate derivative of $y$, determine the $x$-coordinate of $\mathrm{P}$. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0$, $y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point $\mathrm{P}$ lies on the straight line $y = mx$ where the exact value of $m$ is to be determined. [6]

**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

Monday 31 October 2022 (afternoon)

Candidate session number

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2 hours

## Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

8822 – 7101
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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

The function $g$ is defined by $g(x) = \mathrm{e}^{x^2+1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.

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**2.** [Maximum mark: 7]

Consider a circle with a diameter $AB$, where $A$ has coordinates $(1, 4, 0)$ and $B$ has coordinates $(-3, 2, -4)$.

(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex $V$ has coordinates $(-1, -1, 0)$.

**diagram not to scale**

(b) Find the exact volume of the cone. [3]

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**3.** [Maximum mark: 5]

Let $a$ be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2 - 1}{2}\right)^2 = \left(\frac{a^2 + 1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length $a$, $\left(\frac{a^2 - 1}{2}\right)$ and $\left(\frac{a^2 + 1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of $a$. [2]

**4.** [Maximum mark: 5]

The derivative of the function $f$ is given by $f'(x) = \frac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

**5.** [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, $\alpha$ and $\alpha^2$, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of $\alpha$. [4]

(b) Find the value of $p$. [3]

**6.** [Maximum mark: 6]

Events $A$ and $B$ are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events $A$ and $B$ are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

**7.** [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

**8.** [Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$ and $a, k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of $a$ for which the inverse function $f^{-1}$ exists. [2]

(b) Find the largest value of $a$ for which the inverse function $f^{-1}$ exists in the case where $k = \pi$. [1]

(c) Find the largest value of $a$ for which the inverse function $f^{-1}$ exists in the case where $\pi < k < 2\pi$. Give your answer in terms of $k$. [2]

**9.** [Maximum mark: 10]

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of $m$. [2]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 20]

The function $f$ is defined by $f(x) = \cos^2 x - 3\sin^2 x$, $0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

**11.** [Maximum mark: 16]

Consider a three-digit code $abc$, where each of $a$, $b$ and $c$ is assigned one of the values $1$, $2$, $3$, $4$ or $5$.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, $121$ or $444$);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of $a$, $b$ and $c$ is assigned one of the values $1$, $2$, $3$, $4$ or $5$. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for $b$ in terms of $a$.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

Let $z_n$ be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of $n$. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

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**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 2

Tuesday 1 November 2022 (morning)

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

The following table shows the Mathematics test scores $(x)$ and the Science test scores $(y)$ for a group of eight students.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Mathematics scores $(x)$ | 64 | 68 | 72 | 75 | 80 | 82 | 85 | 86 |
| Science scores $(y)$ | 67 | 72 | 77 | 76 | 84 | 83 | 89 | 91 |

The regression line of $y$ on $x$ for this data can be written in the form $y = ax + b$.

(a) Find the value of $a$ and the value of $b$. [2]

(b) Write down the value of the Pearson's product-moment correlation coefficient, $r$. [1]

(c) Use the equation of your regression line to predict the Science test score for a student who has a score of $78$ on the Mathematics test. Express your answer to the nearest integer. [2]

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**2.** [Maximum mark: 6]

The function $f$ is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 \le x \le 2$. The graph of $f$ is shown in the following diagram.

The graph of $f$ has a local maximum at point $\mathrm{A}$. The graph intersects the $x$-axis at the origin and at point $\mathrm{B}$.

(a) Find the coordinates of $\mathrm{A}$. [2]

(b) Find the $x$-coordinate of $\mathrm{B}$. [1]

(c) Find the total area enclosed by the graph of $f$, the $x$-axis and the line $x = 2$. [3]

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**3.** [Maximum mark: 5]

A geometric sequence has a first term of $50$ and a fourth term of $86.4$.

The sum of the first $n$ terms of the sequence is $S_n$.

Find the smallest value of $n$ such that $S_n > 33\,500$.

**4.** [Maximum mark: 7]

The population of a town $t$ years after 1 January 2014 can be modelled by the function

$$P(t) = 15\,000\mathrm{e}^{kt}, \text{ where } k < 0 \text{ and } t \geq 0.$$

It is known that between 1 January 2014 and 1 January 2022 the population decreased by $11\,\%$.

Use this model to estimate the population of this town on 1 January 2041.

**5.** [Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in $x^4$ is $\frac{8}{7}a^5$.

Find the value of $a$.

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**6.** [Maximum mark: 8]

The continuous random variable $X$ has a probability density function given by

$$f(x) = \begin{cases} axe^{x}, & 0 \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where $a, b \in \mathbb{R}^{+}$.

(a) Find an expression for $a$ in terms of $b$. [5]

(b) In the case where $a = b = 1$, find the median of $X$. [3]

**7.** [Maximum mark: 6]

Consider the vectors $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = \left(\cos\frac{1}{n}\right)\boldsymbol{i} + \left(\sin\frac{1}{n}\right)\boldsymbol{j}$, where $n \in \mathbb{Z}^+$.

Let $\theta$ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Find an expression for $\cos\theta$ in terms of $n$. [3]

(b) Find the exact value of the limit approached by $\theta$ as $n \to \infty$. [3]

**8.** [Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.

Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\text{cm}^3\text{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

The volume $V\,\text{cm}^3$ and the height $h\,\text{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\text{cm}^3$.

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**9.** [Maximum mark: 8]

Consider a quadrilateral $\mathrm{ABCD}$ such that $\mathrm{AB} = 2$, $\mathrm{BC} = 4$, $\mathrm{CD} = 6$ and $\mathrm{DA} = 8$, as shown in the following diagram. Let $\alpha = \mathrm{A\hat{B}C}$ and $\beta = \mathrm{A\hat{D}C}$.

**diagram not to scale**

(a) (i) Find $\mathrm{AC}$ in terms of $\alpha$ .

(ii) Find $\mathrm{AC}$ in terms of $\beta$ .

(iii) Hence or otherwise, find an expression for $\alpha$ in terms of $\beta$. [4]

(b) Find the maximum area of the quadrilateral $\mathrm{ABCD}$. [4]

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## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

The time worked, $T$, in hours per week by employees of a large company is normally distributed with a mean of $42$ and standard deviation $10.7$.

(a) Find the probability that an employee selected at random works more than $40$ hours per week. [2]

(b) A group of four employees is selected at random. Each employee is asked in turn whether they work more than $40$ hours per week. Find the probability that the fourth employee is the only one in the group who works more than $40$ hours per week. [3]

(c) A large group of employees work more than $40$ hours per week.

(i) An employee is selected at random from this large group.

Find the probability that this employee works less than $55$ hours per week.

(ii) Ten employees are selected at random from this large group.

Find the probability that exactly five of them work less than $55$ hours per week. [7]

It is known that $\mathrm{P}(a \le T \le b) = 0.904$ and that $\mathrm{P}(T > b) = 2\mathrm{P}(T < a)$, where $a$ and $b$ are numbers of hours worked per week. An employee who works fewer than $a$ hours per week is considered to be a part-time employee.

(d) Find the maximum time, in hours per week, that an employee can work and still be considered part-time. [4]

Do **not** write solutions on this page.

**11.** [Maximum mark: 15]

The function $f$ is defined by $f(x) = e^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the $x$-axis and the $y$-axis is rotated through $2\pi$ radians about the $x$-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function $g$ such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

**12.** [Maximum mark: 22]

Consider the points $\mathrm{A}(1, 2, 3)$, $\mathrm{B}(k, -2, 1)$ and $\mathrm{C}(5, 0, 2)$, where $k \in \mathbb{R}$.

(a) Write down $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$. [2]

(b) Given that the points $\mathrm{A}$, $\mathrm{B}$ and $\mathrm{C}$ lie on a straight line, show that $k = 9$. [1]

(c) For $k = 9$, let $L_1$ be the line passing through $\mathrm{A}$, $\mathrm{B}$ and $\mathrm{C}$.

(i) Find a vector equation of the line $L_1$.

(ii) Line $L_2$ has the equation $\frac{x-1}{2} = \frac{y}{3} = 1 - z$. Show that the lines $L_1$ and $L_2$ are skew. [10]

(d) For $k \neq 9$, let $\Pi$ be the plane containing $\mathrm{A}$, $\mathrm{B}$ and $\mathrm{C}$.

(i) Find the Cartesian equation of the plane $\Pi$.

(ii) Find the coordinates of the point on the plane $\Pi$ which is closest to the origin $(0, 0, 0)$. [9]

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
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- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Find the value of $\int_1^9 \left( \frac{3\sqrt{x} - 5}{\sqrt{x}} \right) \mathrm{d}x$ .

**2.** [Maximum mark: 7]

A survey at a swimming pool is given to one adult in each family. The age of the adult, $a$ years old, and of their eldest child, $c$ years old, are recorded.

The ages of the eldest child are summarized in the following box and whisker diagram.

**diagram not to scale**

(a) Find the largest value of $c$ that would not be considered an outlier. [3]

The regression line of $a$ on $c$ is $a = \frac{7}{4}c + 20$. The regression line of $c$ on $a$ is $c = \frac{1}{2}a - 9$.

(b) (i) One of the adults surveyed is 42 years old. Estimate the age of their eldest child.

(ii) Find the mean age of all the adults surveyed. [4]

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**3.** [Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

**4.** [Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)e^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5e^{k}x$.

Find the value of $k$.

**5.** [Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of $g$ is obtained by two transformations of the graph of $f$.

(a) Describe these two transformations. [2]

The $y$-intercept of the graph of $g$ is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of $r$. [5]

**6.** [Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of $n$ for which the expansion has a non-zero constant term.

**7.** [Maximum mark: 8]

The continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{4-3x^2}}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of $k$. [4]

(b) Find $\mathrm{E}(X)$ . [4]

**8.** [Maximum mark: 6]

Consider integers $a$ and $b$ such that $a^2 + b^2$ is exactly divisible by $4$. Prove by contradiction that $a$ and $b$ cannot both be odd.

**9.** [Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = (1 - b^2) - 2b\mathrm{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of $b$. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of $b$. [3]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 18]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm\frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of $x$. [6]

(b) Now consider the case where the series is arithmetic with common difference $d$.

(i) Show that $p = \frac{2}{3}$.

(ii) Write down $d$ in the form $k\ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first $n$ terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of $n$. [12]

**11.** [Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both $\Pi_1$ and $\Pi_2$.

(ii) Find a vector equation of $L$, the line of intersection of $\Pi_1$ and $\Pi_2$. [5]

(c) Find the distance between $L$ and $\Pi_3$. [6]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

The function $f$ is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the $x^3$ term. [4]

(b) Hence, find an approximate value for $\int_0^1 \mathrm{e}^{x^2} \sin(x^2)\,\mathrm{d}x$. [4]

The function $g$ is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2\big(g'(x) - g(x)\big)$.

(ii) Hence, deduce that $g^{(4)}(x) = 2\big(g'''(x) - g''(x)\big)$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the $x^4$ term. [5]

(e) Hence, or otherwise, determine the value of $\lim\limits_{x \to 0} \dfrac{\mathrm{e}^x \cos x - 1 - x}{x^3}$. [3]

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**References:**

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
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2 hours

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14 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

The $n^{\text{th}}$ term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, $u_1$. [1]

(b) Given that the $n^{\text{th}}$ term of this sequence is $-33$, find the value of $n$. [2]

(c) Find the common difference, $d$. [2]

**2.** [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, $n$ and $n+1$.

(a) Prove that the sum of these three integers is always divisible by $3$. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by $3$. [4]

**3.** [Maximum mark: 8]

A function $f$ is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

**(This question continues on the following page)**

**(Question 3 continued)**

**4.** [Maximum mark: 5]

Find the least positive value of $x$ for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

**5.** [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + ... + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of $x$. [5]

**6.** [Maximum mark: 8]

A function $f$ is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.

(a) Show that $f$ is an odd function. [2]

The range of $f$ is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of $a$ and the value of $b$. [6]

**7.** [Maximum mark: 6]

By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, \mathrm{d}x$ in terms of $n$, where $n$ is a non-zero real number.

**8.** [Maximum mark: 6]

A continuous random variable $X$ has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\[2ex] \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\\[2ex] 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.

Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of $X$ in terms of $a$, $b$ and $c$.

**9.** [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A biased four-sided die with faces labelled $1$, $2$, $3$ and $4$ is rolled and the result recorded. Let $X$ be the result obtained when the die is rolled. The probability distribution for $X$ is given in the following table where $p$ and $q$ are constants.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(X=x)$ | $p$ | 0.3 | $q$ | 0.1 |

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair.
Both yellow dice have faces labelled $1$, $2$, $3$ and $4$. Let $S$ represent the sum obtained by rolling the two yellow dice. The probability distribution for $S$ is shown below.

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\mathrm{P}(S=s)$ | $\frac{1}{16}$ | $\frac{2}{16}$ | $\frac{3}{16}$ | $\frac{4}{16}$ | $\frac{3}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ |

The first red die has faces labelled $1$, $2$, $2$ and $3$. The second red die has faces labelled $1$, $a$, $a$ and $b$, where $a<b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of $b$. [2]

(e) Find the value of $a$, providing evidence for your answer. [2]

Do **not** write solutions on this page.

**11.** [Maximum mark: 20]

A function $f$ is defined by $f(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function $g$ is defined by $g(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x > 3$.

(b) The inverse of $g$ is $g^{-1}$.

(i) Show that $g^{-1}(x)=1+\dfrac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of $g^{-1}$. [7]

A function $h$ is defined by $h(x)=\arctan\dfrac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a)=\dfrac{\pi}{4}$, find the value of $a$.

Give your answer in the form $p+\dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

In the following Argand diagram, the points $Z_1$, $O$ and $Z_2$ are the vertices of triangle $Z_1OZ_2$ described anticlockwise.

The point $Z_1$ represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point $Z_2$ represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles $\alpha$, $\theta$ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2^* = r_1 r_2 e^{i(\alpha - \theta)}$ where $z_2^*$ is the complex conjugate of $z_2$. [2]

(b) Given that $\mathrm{Re}(z_1 z_2^*) = 0$, show that $Z_1OZ_2$ is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where $Z_1OZ_2$ is an equilateral triangle.

(c) (i) Express $z_1$ in terms of $z_2$.

(ii) Hence show that $z_1^2 + z_2^2 = z_1 z_2$. [6]

Let $z_1$ and $z_2$ be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle $Z_1OZ_2$ can be formed from the point $O$ and the roots of this equation. [3]

**References:**

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 2

Monday 9 May 2022 (morning)

Candidate session number

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2 hours

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- Section A: answer all questions. Answers must be written within the answer boxes provided.
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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

The number of hours spent exercising each week by a group of students is shown in the following table.

| **Exercising time (in hours)** | **Number of students** |
|---|---|
| 2 | 5 |
| 3 | 1 |
| 4 | 4 |
| 5 | 3 |
| 6 | $x$ |

The median is $4.5$ hours.

(a) Find the value of $x$. [2]

(b) Find the standard deviation. [2]

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**2.** [Maximum mark: 6]

A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

**diagram not to scale**

The rectangle measures $5\,\text{cm}$ by $4\,\text{cm}$. The points $\text{A}$ and $\text{B}$ lie on a circle, with centre $\text{O}$ and radius $2\,\text{cm}$, such that $\text{A}\hat{\text{O}}\text{B} = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

**diagram not to scale**

(a) Find the area of one of the shaded segments in terms of $\theta$. [3]

(b) Given that the area of the logo is $13.4\,\text{cm}^2$, find the value of $\theta$. [3]

**(This question continues on the following page)**

**(Question 2 continued)**

**3.** [Maximum mark: 6]

A discrete random variable, $X$, has the following probability distribution:

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $0.41$ | $k - 0.28$ | $0.46$ | $0.29 - 2k^2$ |

(a) Show that $2k^2 - k + 0.12 = 0$. [1]

(b) Find the value of $k$, giving a reason for your answer. [3]

(c) Hence, find $\mathrm{E}(X)$. [2]

**4.** [Maximum mark: 7]

A particle moves along a straight line so that its velocity, $v\,\mathrm{m\,s^{-1}}$, after $t$ seconds is given by $v(t) = \mathrm{e}^{\sin t} + 4\sin t$ for $0 \le t \le 6$.

(a) Find the value of $t$ when the particle is at rest. [2]

(b) Find the acceleration of the particle when it changes direction. [3]

(c) Find the total distance travelled by the particle. [2]

**5.** [Maximum mark: 6]

Let $A$ and $B$ be two independent events such that $\mathrm{P}(A \cap B') = 0.16$ and $\mathrm{P}(A' \cap B) = 0.36$.

(a) Given that $\mathrm{P}(A \cap B) = x$, find the value of $x$. [4]

(b) Find $\mathrm{P}(A' \mid B')$. [2]

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**6.** [Maximum mark: 6]

Consider the function $f(x) = 2^x - \frac{1}{2^x}$, $x \in \mathbb{R}$.

(a) Show that $f$ is an odd function. [2]

The function $g$ is given by $g(x) = \frac{x-1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(b) Solve the inequality $f(x) \geq g(x)$. [4]

**7.** [Maximum mark: 9]

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\boldsymbol{b}| = 15$.

(a) Find the possible range of values for $|\boldsymbol{a} + \boldsymbol{b}|$. [2]

Consider the vector $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{p}$. [2]

Consider the vector $\boldsymbol{q}$ such that $\boldsymbol{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

(c) Find $\boldsymbol{q}$ such that $|\boldsymbol{q}| = |\boldsymbol{b}|$ and $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$. [5]

**8.** [Maximum mark: 4]

Consider the equation $kx^2 - (k+3)x + 2k + 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of $k$. [1]

(b) Hence or otherwise, determine the values of $k$ such that the equation has one positive and one negative real root. [3]

**9.** [Maximum mark: 7]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

(a) Find the number of ways these five people can be seated in this row. [3]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

(b) Find the number of ways these five people can now be seated in this row. [4]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 20]

Consider the function $f(x)=\sqrt{x^2-1}$, where $1 \le x \le 2$.

(a) Sketch the curve $y=f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of $f$ is given by $f^{-1}(x)=\sqrt{x^2+1}$.

(ii) State the domain and range of $f^{-1}$. [5]

The curve $y=f(x)$ is rotated $2\pi$ about the $y$-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\mathrm{m}^3$, of water in the container when it is filled to a height of $h$ metres is given by $V=\pi\left(\frac{1}{3}h^3+h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t=0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

Do **not** write solutions on this page.

**11.** [Maximum mark: 16]

A bakery makes two types of muffins: chocolate muffins and banana muffins.

The weights, $C$ grams, of the chocolate muffins are normally distributed with a mean of $62\,\text{g}$ and standard deviation of $2.9\,\text{g}$.

(a) Find the probability that a randomly selected chocolate muffin weighs less than $61\,\text{g}$. [2]

(b) In a random selection of 12 chocolate muffins, find the probability that exactly 5 weigh less than $61\,\text{g}$. [2]

The weights, $B$ grams, of the banana muffins are normally distributed with a mean of $68\,\text{g}$ and standard deviation of $3.4\,\text{g}$.

Each day $60\,\%$ of the muffins made are chocolate.

On a particular day, a muffin is randomly selected from all those made at the bakery.

(c) (i) Find the probability that the randomly selected muffin weighs less than $61\,\text{g}$.

(ii) Given that a randomly selected muffin weighs less than $61\,\text{g}$, find the probability that it is chocolate. [7]

The machine that makes the chocolate muffins is adjusted so that the mean weight of the chocolate muffins remains the same but their standard deviation changes to $\sigma\,\text{g}$. The machine that makes the banana muffins is not adjusted. The probability that the weight of a randomly selected muffin from these machines is less than $61\,\text{g}$ is now $0.157$.

(d) Find the value of $\sigma$. [5]

Do **not** write solutions on this page

**12.** [Maximum mark: 19]

Consider the differential equation $x^2 \frac{dy}{dx} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

(a) Use Euler's method, with a step length of $0.1$, to find an approximate value of $y$ when $x = 1.5$. [4]

(b) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = v^2 - v - 2$. [3]

(c) (i) By solving the differential equation, show that $y = \frac{8x + x^4}{4 - x^3}$.

(ii) Find the actual value of $y$ when $x = 1.5$.

(iii) Using the graph of $y = \frac{8x + x^4}{4 - x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of $y$ at $x = 1.5$. [12]

**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 2

Monday 9 May 2022 (morning)

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

The following diagram shows a circle with centre $\mathrm{O}$ and radius $5$ metres.

Points $\mathrm{A}$ and $\mathrm{B}$ lie on the circle and $\mathrm{A\hat{O}B} = 1.9$ radians.

**diagram not to scale**

(a) Find the length of the chord $[\mathrm{AB}]$. [3]

(b) Find the area of the shaded sector. [3]

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**2.** [Maximum mark: 5]

The derivative of a function $g$ is given by $g'(x) = 3x^2 + 5e^x$, where $x \in \mathbb{R}$. The graph of $g$ passes through the point $(0, 4)$. Find $g(x)$.

**3.** [Maximum mark: 6]

Events $A$ and $B$ are independent and $\mathrm{P}(A) = 3\mathrm{P}(B)$.

Given that $\mathrm{P}(A \cup B) = 0.68$, find $\mathrm{P}(B)$.

**4.** [Maximum mark: 6]

A random sample of nine adults were selected to see whether sleeping well affected their reaction times to a visual stimulus. Each adult's reaction time was measured twice.

The first measurement for reaction time was taken on a morning after the adult had slept well. The second measurement was taken on a morning after the same adult had not slept well.

The box and whisker diagrams for the reaction times, measured in seconds, are shown below.

Consider the box and whisker diagram representing the reaction times after sleeping well.

(a) State the median reaction time after sleeping well. [1]

(b) Verify that the measurement of $0.46$ seconds is not an outlier. [3]

(c) State why it appears that the mean reaction time is greater than the median reaction time. [1]

Now consider the two box and whisker diagrams.

(d) Comment on whether these box and whisker diagrams provide any evidence that might suggest that not sleeping well causes an increase in reaction time. [1]

**(This question continues on the following page)**

**(Question 4 continued)**

**5.** [Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time $t$ seconds is given by $v = \frac{\left(t^2+1\right)\cos t}{4}$, $0 \le t \le 3$.

(a) Determine when the particle changes its direction of motion. [2]

(b) Find the times when the particle's acceleration is $-1.9\,\mathrm{m\,s^{-2}}$. [3]

(c) Find the particle's acceleration when its speed is at its greatest. [2]

**6.** [Maximum mark: 5]

The following diagram shows the curve $\frac{x^2}{36}+\frac{(y-4)^2}{16}=1$, where $h \le y \le 4$.

**diagram not to scale**

The curve from point $\mathrm{Q}$ to point $\mathrm{B}$ is rotated $360°$ about the $y$-axis to form the interior surface of a bowl. The rectangle $\mathrm{OPQR}$, of height $h$ cm, is rotated $360°$ about the $y$-axis to form a solid base.

The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\,\mathrm{cm}^3$, determine the height of the base.

**7.** [Maximum mark: 8]

Consider $\lim_{x \to 0} \frac{\arctan(\cos x) - k}{x^2}$, where $k \in \mathbb{R}$.

(a) Show that a finite limit only exists for $k = \frac{\pi}{4}$. [2]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

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**8.** [Maximum mark: 7]

Rachel and Sophia are competing in a javelin-throwing competition.

The distances, $R$ metres, thrown by Rachel can be modelled by a normal distribution with mean $56.5$ and standard deviation $3$.

The distances, $S$ metres, thrown by Sophia can be modelled by a normal distribution with mean $57.5$ and standard deviation $1.8$.

In the first round of competition, each competitor must have five throws. To qualify for the next round of competition, a competitor must record at least one throw of $60$ metres or greater in the first round.

Find the probability that only one of Rachel or Sophia qualifies for the next round of competition.

**9.** [Maximum mark: 4]

Consider the set of six-digit positive integers that can be formed from the digits $0$, $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$ and $9$.

Find the total number of six-digit positive integers that can be formed such that

(a) the digits are distinct; [2]

(b) the digits are distinct and are in increasing order. [2]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, $A$ and $B$, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant $A$ was given fertilizer regularly, while Plant $B$ was not.

The scientist found that the height of Plant $A$, $h_A \text{cm}$, at time $t$ weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \le t \le 9$.

The scientist found that the height of Plant $B$, $h_B \text{cm}$, at time $t$ weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 \le t \le 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant $B$;

(ii) Plant $A$ correct to three significant figures. [3]

(b) Find the values of $t$ when $h_A(t) = h_B(t)$. [3]

(c) For $t > 6$, prove that Plant $A$ was always taller than Plant $B$. [3]

(d) For $0 \le t \le 9$, find the total amount of time when the rate of growth of Plant $B$ was greater than the rate of growth of Plant $A$. [6]

Do **not** write solutions on this page.

**11.** [Maximum mark: 20]

Two airplanes, $A$ and $B$, have position vectors with respect to an origin $\mathrm{O}$ given respectively by

$$\boldsymbol{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where $t$ represents the time in minutes and $0 \le t \le 2.5$.

Entries in each column vector give the displacement east of $\mathrm{O}$, the displacement north of $\mathrm{O}$ and the distance above sea level, all measured in kilometres.

(a) Find the three-figure bearing on which airplane $B$ is travelling. [2]

(b) Show that airplane $A$ travels at a greater speed than airplane $B$. [2]

(c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point $\mathrm{P}$.

(d) (i) Find the coordinates of $\mathrm{P}$.

(ii) Determine the length of time between the first airplane arriving at $\mathrm{P}$ and the second airplane arriving at $\mathrm{P}$. [7]

Let $D(t)$ represent the distance between airplane $A$ and airplane $B$ for $0 \le t \le 2.5$.

(e) Find the minimum value of $D(t)$. [5]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

The population, $P$, of a particular species of marsupial on a small remote island can be modelled by the logistic differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$$

where $t$ is the time measured in years and $k$, $N$ are positive constants.

The constant $N$ represents the maximum population of this species of marsupial that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$. [1]

(b) Show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. [4]

(c) Hence show that the population of marsupials will increase at its maximum rate when $P = \frac{N}{2}$. Justify your answer. [5]

(d) Hence determine the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ in terms of $k$ and $N$. [2]

Let $P_0$ be the initial population of marsupials.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right).$$ [7]

After $10$ years, the population of marsupials is $3P_0$. It is known that $N = 4P_0$.

(f) Find the value of $k$ for this population model. [2]

**References:**

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Diploma Programme
Programme du diplôme
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# Mathematics: analysis and approaches
# Higher level
# Paper 1

Monday 1 November 2021 (afternoon)

Candidate session number

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2 hours

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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
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- The maximum mark for this examination paper is **[110 marks]**.

14 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find $y$ in terms of $x$.

**2.** [Maximum mark: 9]

The function $f$ is defined by $f(x) = \dfrac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of $f$;

(ii) the horizontal asymptote of the graph of $f$. [2]

(b) Find the coordinates where the graph of $f$ crosses

(i) the $x$-axis;

(ii) the $y$-axis. [2]

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**(This question continues on the following page)**

**(Question 2 continued)**

(c) Sketch the graph of $f$ on the axes below. [1]

The function $g$ is defined by $g(x) = \dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of $a$. [4]

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**3.** [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

**4.** [Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let $A$ be the event that "box 1 is chosen" and let $R$ be the event that "a red ball is drawn".

(b) Determine whether events $A$ and $R$ are independent. [2]

**5.** [Maximum mark: 7]

The function $f$ is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of $f$ at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function $g$ is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of $h$ at $x = 4$. [3]

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**6.** [Maximum mark: 7]

(a) Show that $2x-3-\frac{6}{x-1}=\frac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{4}$. [5]

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**7.** [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for $p$. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of $a$. [2]

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**8.** [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

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**9.** [Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of $x$, as far as the term in $x^2$ is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of $a$ and the value of $b$. [6]

(b) State the restriction which must be placed on $x$ for this expansion to be valid. [1]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A particle $P$ moves along the $x$-axis. The velocity of $P$ is $v\,\mathrm{m\,s^{-1}}$ at time $t$ seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, $P$ is at the origin $\mathrm{O}$.

(a) (i) Find the value of $t$ when $P$ reaches its maximum velocity.

(ii) Show that the distance of $P$ from $\mathrm{O}$ at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of $v$ against $t$, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by $P$. [5]

**11.** [Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right) = \left[x^2 + 2nx + n(n-1)\right]\mathrm{e}^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2\mathrm{e}^x$ in ascending powers of $x$, up to and including the term in $x^4$. [3]

(c) Hence or otherwise, determine the value of $\lim\limits_{x \to 0}\left[\frac{\left(x^2\mathrm{e}^x - x^2\right)^3}{x^9}\right]$. [4]

Do **not** write solutions on this page.

**12.** [Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are $\omega_1$, $\omega_2$ and $\omega_3$, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find $\omega_2$ and $\omega_3$, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots $\omega_1$, $\omega_2$ and $\omega_3$ are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 2

Tuesday 2 November 2021 (morning)

Candidate session number

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2 hours

## Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

14 pages

8821 – 7102
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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 7]

In Lucy's music academy, eight students took their piano diploma examination and achieved scores out of 150. For her records, Lucy decided to record the average number of hours per week each student reported practising in the weeks prior to their examination. These results are summarized in the table below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Average weekly practice time $(h)$ | 28 | 13 | 45 | 33 | 17 | 29 | 39 | 36 |
| Diploma score $(D)$ | 115 | 82 | 120 | 116 | 79 | 101 | 110 | 121 |

(a) Find Pearson's product-moment correlation coefficient, $r$, for these data. [2]

(b) The relationship between the variables can be modelled by the regression equation $D = ah + b$. Write down the value of $a$ and the value of $b$. [1]

(c) One of these eight students was disappointed with her result and wished she had practised more. Based on the given data, determine how her score could have been expected to alter had she practised an extra five hours per week. [2]

(d) Lucy asserts that the number of hours a student practises has a direct effect on their final diploma result. Comment on the validity of Lucy's assertion. [1]

Lucy suspected that each student had not been practising as much as they reported. In order to compensate for this, Lucy deducted a fixed number of hours per week from each of the students' recorded hours.

(e) State how, if at all, the value of $r$ would be affected. [1]

**(This question continues on the following page)**

**(Question 1 continued)**

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Answers written on this page
will not be marked.

**2.** [Maximum mark: 5]

Consider a triangle $\mathrm{ABC}$, where $\mathrm{AC} = 12$, $\mathrm{CB} = 7$ and $\mathrm{B\hat{A}C} = 25^{\circ}$.

Find the smallest possible perimeter of triangle $\mathrm{ABC}$.

**3.** [Maximum mark: 7]

A factory manufactures lamps. It is known that the probability that a lamp is found to be defective is $0.05$. A random sample of $30$ lamps is tested.

(a) Find the probability that there is at least one defective lamp in the sample. [3]

(b) Given that there is at least one defective lamp in the sample, find the probability that there are at most two defective lamps. [4]

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**4.** [Maximum mark: 6]

The following diagram shows a semicircle with centre $\mathrm{O}$ and radius $r$. Points $\mathrm{P}$, $\mathrm{Q}$ and $\mathrm{R}$ lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.

(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of $\theta$. [1]

**5.** [Maximum mark: 9]

The sum of the first $n$ terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, $u_1$. [2]

(b) Find $S_\infty$. [3]

(c) Find the least value of $n$ such that $S_\infty - S_n < 0.001$. [4]

**6.** [Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, $\alpha$ and $\beta$.

Consider the equation $x^2 + mx + n = 0$, where $m, n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of $m$ and $n$. [6]

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**7.** [Maximum mark: 6]

A continuous random variable $X$ has a probability density function given by

$$f(x) = \begin{cases} \arccos x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The median of this distribution is $m$.

(a) Determine the value of $m$. [2]

(b) Given that $P(|X - m| \le a) = 0.3$, determine the value of $a$. [4]

**8.** [Maximum mark: 8]

Consider the curve $C$ given by $y = x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln\left(xy\right)\right) = 1$. [3]

(b) Hence find the equation of the tangent to $C$ at the point where $x = 1$. [5]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**9.** [Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin\left(b(t-c)\right) + d$, where $t$ is the number of hours after midnight, and $a$, $b$, $c$ and $d$ are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.

The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\text{m}$ and $6.8\,\text{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

(b) Find the value of $a$. [2]

(c) Find the value of $d$. [2]

(d) Find the smallest possible value of $c$. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than $5$ metres. [3]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 9 continued)**

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

**10.** [Maximum mark: 18]

Consider the function $f(x) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of $f$ crosses the

(i) $x$-axis;

(ii) $y$-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of $f$. [1]

(c) The oblique asymptote of the graph of $f$ can be written as $y = ax + b$ where $a, b \in \mathbb{Q}$.

Find the value of $a$ and the value of $b$. [4]

(d) Sketch the graph of $f$ for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

Do **not** write solutions on this page.

**11.** [Maximum mark: 21]

Three points $\mathrm{A}(3, 0, 0)$, $\mathrm{B}(0, -2, 0)$ and $\mathrm{C}(1, 1, -7)$ lie on the plane $\Pi_1$.

(a) (i) Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$.

(ii) Hence find the equation of $\Pi_1$, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane $\Pi_2$ has equation $3x - y + 2z = 2$.

(b) The line $L$ is the intersection of $\Pi_1$ and $\Pi_2$. Verify that the vector equation of $L$ can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane $\Pi_3$ is given by $2x - 2z = 3$. The line $L$ and the plane $\Pi_3$ intersect at the point $\mathrm{P}$.

(i) Show that at the point $\mathrm{P}$, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of $\mathrm{P}$. [3]

(d) The point $\mathrm{B}(0, -2, 0)$ lies on $L$.

(i) Find the reflection of the point $\mathrm{B}$ in the plane $\Pi_3$.

(ii) Hence find the vector equation of the line formed when $L$ is reflected in the plane $\Pi_3$. [9]

**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

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2 hours

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- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

2221 – 7106
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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.

(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of $g$. [3]

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**2.** [Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, $u_1$, and the value of the common difference, $d$.

**3.** [Maximum mark: 5]

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where $L$ and $U$ are the lower and upper quartiles respectively.

**diagram not to scale**

The interquartile range is $20$ grams and there are no outliers in the results.

(a) Find the minimum possible value of $U$. [3]

(b) Hence, find the minimum possible value of $L$. [2]

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**4.** [Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = \mathrm{e}^{x-2} + k$ where $h,\ k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of $f$ and $g$ have a common tangent at $x = 3$.

(b) Show that $h = \dfrac{\mathrm{e}+6}{2}$. [3]

(c) Hence, show that $k = \mathrm{e} + \dfrac{\mathrm{e}^2}{4}$. [3]

**5.** [Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

**6.** [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

**7.** [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a, b \in \mathbb{Z}$.

Find the possible values of $a$.

**8.** [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0} \left( \frac{\arctan 2x}{\tan 3x} \right)$.

**9.** [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.

Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let $X$ be the score obtained when die A is rolled. The probability distribution for $X$ is given in the following table.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(X=x)$ | $p$ | $p$ | $p$ | $\frac{1}{2}p$ |

(a) Find the value of $p$. [2]

(b) Hence, find the value of $\mathrm{E}(X)$. [2]

A second biased four-sided die, B, is rolled. Let $Y$ be the score obtained when die B is rolled. The probability distribution for $Y$ is given in the following table.

| $y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(Y=y)$ | $q$ | $q$ | $q$ | $r$ |

(c) (i) State the range of possible values of $r$.

(ii) Hence, find the range of possible values of $q$. [3]

(d) Hence, find the range of possible values for $\mathrm{E}(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $\mathrm{E}(Y)$. [6]

Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

Consider the line $L_1$ defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on $L_1$.

(ii) Find a vector equation of $L_1$. [4]

Consider a second line $L_2$ defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of $a$ when the acute angle between $L_1$ and $L_2$ is $45°$. [8]

It is given that the lines $L_1$ and $L_2$ have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of $k$, and find the coordinates of the point A in terms of $a$. [7]

**12.** [Maximum mark: 20]

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$

for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = \mathrm{e}^{mx}$, $m \in \mathbb{Q}$.

Consider the function $h$ defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the $x^2$ term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

(c) Find the possible values of $m$. [8]

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**References:**

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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

Consider two consecutive positive integers, $n$ and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

**2.** [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

**3.** [Maximum mark: 5]

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in $x^5$ is $63$.

Find the possible values of $k$.

**4.** [Maximum mark: 9]

Consider the function $f$ defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of $f$ which crosses the $x$-axis at point A, with coordinates $(a, 0)$. The line $L$ is the tangent to the graph of $f$ at the point B.

(a) Find the exact value of $a$. [3]

(b) Given that the gradient of $L$ is $\frac{1}{3}$, find the $x$-coordinate of B. [6]

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**5.** [Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

**6.** [Maximum mark: 7]

The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$.

**diagram not to scale**

Given that $\cos \hat{\mathrm{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

**7.** [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots $\alpha$, $\beta$ and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of $k$.

**8.** [Maximum mark: 8]

The lines $l_1$ and $l_2$ have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that $l_1$ and $l_2$ do not intersect. [3]

(b) Find the minimum distance between $l_1$ and $l_2$. [5]

**9.** [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

Consider the function $f$ defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.

The graph of $f$ touches the $x$-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the $x$-axis, between the points A and B.

(a) Find the $x$-coordinates of A and B. [3]

(b) Show that the area of the shaded region is $12\pi$. [5]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 10 continued)**

The right cone in the following diagram has a total surface area of $12\pi$, equal to the shaded area in the previous diagram.

The cone has a base radius of $2$, height $h$, and slant height $l$.

**diagram not to scale**

(c) Find the value of $l$. [3]

(d) Hence, find the volume of the cone. [4]

Do **not** write solutions on this page.

**11.** [Maximum mark: 20]

The acceleration, $a\,\mathrm{ms}^{-2}$, of a particle moving in a horizontal line at time $t$ seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\mathrm{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin $\mathrm{O}$ and has initial velocity $v_0\,\mathrm{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time $t$ is given by $v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$. [6]

(b) Initially at $\mathrm{O}$, the particle moves in the positive direction until it reaches its maximum displacement from $\mathrm{O}$. The particle then returns to $\mathrm{O}$.

Let $s$ metres represent the particle's displacement from $\mathrm{O}$ and $s_{\max}$ its maximum displacement from $\mathrm{O}$.

(i) Show that the time $T$ taken for the particle to reach $s_{\max}$ satisfies the equation $\mathrm{e}^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of $v_0$. [7]

Let $v(T-k)$ represent the particle's velocity $k$ seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\mathrm{e}^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \mathrm{e}^k - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity $k$ seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of $k$. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

Do **not** write solutions on this page.

**12.** [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.

(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p$, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

**References:**

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Answers written on this page
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Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 2

Friday 7 May 2021 (morning)

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

14 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 7]

The following table shows the data collected from an experiment.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 3.3 | 6.9 | 11.9 | 13.4 | 17.8 | 19.6 | 21.8 | 25.3 |
| $y$ | 6.3 | 8.1 | 8.4 | 11.6 | 10.3 | 12.9 | 13.1 | 17.3 |

The data is also represented on the following scatter diagram.

**(This question continues on the following page)**

**(Question 1 continued)**

The relationship between $x$ and $y$ can be modelled by the regression line of $y$ on $x$ with equation $y = ax + b$, where $a, b \in \mathbb{R}$.

(a) Write down the value of $a$ and the value of $b$. [2]

(b) Use this model to predict the value of $y$ when $x = 18$. [2]

(c) Write down the value of $\overline{x}$ and the value of $\overline{y}$. [1]

(d) Draw the line of best fit on the scatter diagram. [2]

**2.** [Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean $1000$ and standard deviation $3.5$. A bag of sugar is rejected for sale if its mass is less than $995$ grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of $100$ bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than $1005$ grams. [3]

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**3.** [Maximum mark: 5]

A Ferris wheel with diameter $110$ metres rotates at a constant speed. The lowest point on the wheel is $10$ metres above the ground, as shown on the following diagram. $\mathrm{P}$ is a point on the wheel. The wheel starts moving with $\mathrm{P}$ at the lowest point and completes one revolution in $20$ minutes.

**diagram not to scale**

The height, $h$ metres, of $\mathrm{P}$ above the ground after $t$ minutes is given by $h(t) = a\cos(bt) + c$, where $a, b, c \in \mathbb{R}$.

Find the values of $a$, $b$ and $c$.

**4.** [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\,\mathrm{ms}^{-1}$, of the particle at time $t$ seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of $v$.

(a) Find the smallest value of $t$ for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

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**5.** [Maximum mark: 5]

Consider the expansion of $(3 + x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of $x^4$ is $20412$, find the value of $n$.

**6.** [Maximum mark: 5]

Consider the planes $\Pi_1$ and $\Pi_2$ with the following equations.

$$\Pi_1 : 3x + 2y + z = 6$$

$$\Pi_2 : x - 2y + z = 4$$

(a) Find a Cartesian equation of the plane $\Pi_3$ which is perpendicular to $\Pi_1$ and $\Pi_2$ and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where $\Pi_1$, $\Pi_2$ and $\Pi_3$ intersect. [2]

**7.** [Maximum mark: 7]

A continuous random variable $X$ has the probability density function $f$ given by

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{\left(x^2+k\right)^3}} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}^+$.

(a) Show that $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of $k$. [2]

**8.** [Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of $zw$. [1]

(b) Find the argument of $zw$ in terms of $k$. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of $k$.

(ii) For the value of $k$ found in part (i), find the value of $zw$. [4]

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**9.** [Maximum mark: 7]

Two boats $\mathrm{A}$ and $\mathrm{B}$ travel due north.

Initially, boat $\mathrm{B}$ is positioned $50$ metres due east of boat $\mathrm{A}$.

The distances travelled by boat $\mathrm{A}$ and boat $\mathrm{B}$, after $t$ seconds, are $x$ metres and $y$ metres respectively. The angle $\theta$ is the radian measure of the bearing of boat $\mathrm{B}$ from boat $\mathrm{A}$.
This information is shown on the following diagram.

**diagram not to scale**

(a) Show that $y = x + 50\cot\theta$. [1]

At time $T$, the following conditions are true.

Boat $\mathrm{B}$ has travelled $10$ metres further than boat $\mathrm{A}$.
Boat $\mathrm{B}$ is travelling at double the speed of boat $\mathrm{A}$.
The rate of change of the angle $\theta$ is $-0.1$ radians per second.

(b) Find the speed of boat $\mathrm{A}$ at time $T$. [6]

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

Consider the function $f$ defined by $f(x) = 90\,\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of $f$ and the line $y = x$ intersect at point P.

(a) Find the $x$-coordinate of P. [2]

The line $L$ has a gradient of $-1$ and is a tangent to the graph of $f$ at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of $L$ is $y = -x + 2\ln 45 + 2$. [2]

The shaded region $A$ is enclosed by the graph of $f$ and the lines $y = x$ and $L$.

(d) (i) Find the $x$-coordinate of the point where $L$ intersects the line $y = x$.

(ii) Hence, find the area of $A$. [5]

**(This question continues on the following page)**

Do **not** write solutions on this page.

**(Question 10 continued)**

The line $L$ is tangent to the graphs of both $f$ and the inverse function $f^{-1}$.

(e) Find the shaded area enclosed by the graphs of $f$ and $f^{-1}$ and the line $L$. [2]

**11.** [Maximum mark: 20]

The function $f$ is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of $p$ and the value of $q$. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the $x$-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function $g$ is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Do **not** write solutions on this page.

**12.** [Maximum mark: 20]

The function $f$ has a derivative given by $f'(x) = \frac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where $k$ is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where $a, b \in \mathbb{R}$. Find $a$ and $b$ in terms of $k$. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider $P$, the population of a colony of ants, which has an initial value of $1200$.

The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P(k-P)}{5k}$,

where $t$ is the time measured in days, $t \geq 0$, and $k$ is the upper bound for the population.

(c) By solving the differential equation, show that $P = \frac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of $k$, giving your answer correct to four significant figures. [3]

(e) Find the value of $t$ when the rate of change of the population is at its maximum. [3]

**References:**

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 2

Friday 7 May 2021 (morning)

Candidate session number

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2 hours

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- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
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- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

2221 – 7112

13 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

At a café, the waiting time between ordering and receiving a cup of coffee is dependent upon the number of customers who have already ordered their coffee and are waiting to receive it.

Sarah, a regular customer, visited the café on five consecutive days. The following table shows the number of customers, $x$, ahead of Sarah who have already ordered and are waiting to receive their coffee and Sarah's waiting time, $y$ minutes.

| **Number of customers** ($x$) | 3 | 9 | 11 | 10 | 5 |
|---|---|---|---|---|---|
| **Sarah's waiting time** ($y$) | 6 | 10 | 12 | 11 | 6 |

The relationship between $x$ and $y$ can be modelled by the regression line of $y$ on $x$ with equation $y = ax + b$.

(a) (i) Find the value of $a$ and the value of $b$.

(ii) Write down the value of Pearson's product-moment correlation coefficient, $r$. [3]

(b) Interpret, in context, the value of $a$ found in part (a)(i). [1]

On another day, Sarah visits the café to order a coffee. Seven customers have already ordered their coffee and are waiting to receive it.

(c) Use the result from part (a)(i) to estimate Sarah's waiting time to receive her coffee. [2]

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**2.** [Maximum mark: 5]

An arithmetic sequence has first term $60$ and common difference $-2.5$.

(a) Given that the $k$th term of the sequence is zero, find the value of $k$. [2]

Let $S_n$ denote the sum of the first $n$ terms of the sequence.

(b) Find the maximum value of $S_n$. [3]

**3.** [Maximum mark: 8]

At a school, $70\%$ of the students play a sport and $20\%$ of the students are involved in theatre. $18\%$ of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school $48\%$ of the students are girls, and $25\%$ of the girls are involved in theatre.

A student is selected at random. Let $G$ be the event "the student is a girl" and let $T$ be the event "the student is involved in theatre".

(c) Find $\mathrm{P}(G \cap T)$. [2]

(d) Determine if the events $G$ and $T$ are independent. Justify your answer. [2]

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**4.** [Maximum mark: 6]

The functions $f$ and $g$ are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of $f$. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for $c$. [4]

**5.** [Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, $A$, of carbon-14 present in a plant $t$ years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and $A_0$, $k$ are positive constants.

At the time of death, a plant is defined to have $100$ units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be $5730$ years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest $10$ years, the time taken after the plant's death for $25\%$ of the carbon-14 to decay. [3]

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**6.** [Maximum mark: 6]

A continuous random variable $X$ has the probability density function $f_n$ given by

$$f_n(x) = \begin{cases} (n+1)x^n, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $n \in \mathbb{R}$, $n \ge 0$.

(a) Show that $\mathrm{E}(X) = \dfrac{n+1}{n+2}$. [2]

(b) Show that $\mathrm{Var}(X) = \dfrac{n+1}{(n+2)^2(n+3)}$. [4]

**7.** [Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

**8.** [Maximum mark: 5]

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$.

**9.** [Maximum mark: 8]

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of $t$. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in $x^4$ is $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x\arctan 2x}{\sec x-1}\right)$. [3]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

The flight times, $T$ minutes, between two cities can be modelled by a normal distribution with a mean of $75$ minutes and a standard deviation of $\sigma$ minutes.

(a) Given that $2\%$ of the flight times are longer than $82$ minutes, find the value of $\sigma$. [3]

(b) Find the probability that a randomly selected flight will have a flight time of more than $80$ minutes. [2]

(c) Given that a flight between the two cities takes longer than $80$ minutes, find the probability that it takes less than $82$ minutes. [4]

On a particular day, there are $64$ flights scheduled between these two cities.

(d) Find the expected number of flights that will have a flight time of more than $80$ minutes. [3]

(e) Find the probability that more than $6$ of the flights on this particular day will have a flight time of more than $80$ minutes. [3]

Do **not** write solutions on this page.

**11.** [Maximum mark: 18]

A function $f$ is defined by $f(x) = \dfrac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y = f(x)$, the $x$-axis, the $y$-axis and the line $x = \ln 16$ is rotated $360°$ about the $x$-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\dfrac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\mathrm{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.

The vertical height of the bowl, BO, is measured along the $x$-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of $k$ that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of $f$, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(ii) State the cross-sectional radius of the bowl at this point. [6]

Do **not** write solutions on this page.

**12.** [Maximum mark: 21]

A function $f$ is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that $f$ is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that $f$ is decreasing for $x < 0$. [9]

A function $g$ is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of $g^{-1}$. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

**References:**

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Diploma Programme
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# Mathematics: analysis and approaches
# Higher level
# Paper 1

Specimen paper

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13 pages

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## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Let $A$ and $B$ be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

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**2.** [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

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**3.** [Maximum mark: 5]

Let $f'(x) = \dfrac{8x}{\sqrt{2x^2+1}}$. Given that $f(0) = 5$, find $f(x)$.

**4.** [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the $x$-axis at $x = -1$ and $x = 1$, and the $y$-axis at $y = 2$.

On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.

**5.** [Maximum mark: 5]

The functions $f$ and $g$ are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of $a$. [3]

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**6.** [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3\sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

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**7.** [Maximum mark: 7]

A continuous random variable $X$ has the probability density function $f$ given by

$$f(x) = \begin{cases} \dfrac{\pi x}{36}\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

**8.** [Maximum mark: 7]

The plane $\Pi$ has the Cartesian equation $2x + y + 2z = 3$.

The line $L$ has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line $L$ and the plane $\Pi$ is $30^\circ$.

Find the possible values of $p$.

**9.** [Maximum mark: 8]

The function $f$ is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.

(a) Find the largest value of $a$ such that $f$ has an inverse function. [3]

(b) For this value of $a$, find an expression for $f^{-1}(x)$, stating its domain. [5]

**(This question continues on the following page)**

**(Question 9 continued)**

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Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

The graph of $f$ has exactly one maximum point $\mathrm{P}$.

(b) Find the $x$-coordinate of $\mathrm{P}$. [3]

The second derivative of $f$ is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of $f$ has exactly one point of inflexion $\mathrm{Q}$.

(c) Show that the $x$-coordinate of $\mathrm{Q}$ is $\dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region $R$ is enclosed by the graph of $f$, the $x$-axis, and the vertical lines through the maximum point $\mathrm{P}$ and the point of inflexion $\mathrm{Q}$.

(d) Given that the area of $R$ is $3$, find the value of $k$. [7]

Do **not** write solutions on this page.

**11.** [Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be $u$, $v$ and $w$.

(b) Find $u$, $v$ and $w$ expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

On an Argand diagram, $u$, $v$ and $w$ are represented by the points $\mathrm{U}$, $\mathrm{V}$ and $\mathrm{W}$ respectively.

(c) Find the area of triangle $\mathrm{UVW}$. [4]

(d) By considering the sum of the roots $u$, $v$ and $w$, show that
$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$ [4]

**12.** [Maximum mark: 21]

The function $f$ is defined by $f(x)=\mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the $x^2$ term. [8]

(b) Show that the coefficient of $x^3$ in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x}-1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x}-1\right)$ up to and including the $x^3$ term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x\to 0}\frac{f(x)-1}{\arctan\left(\mathrm{e}^{3x}-1\right)}$. [3]

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# Mathematics: analysis and approaches
# Higher level
# Paper 2

Specimen

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- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 6]

The following diagram shows part of a circle with centre $\mathrm{O}$ and radius $4\,\mathrm{cm}$.

Chord $\mathrm{AB}$ has a length of $5\,\mathrm{cm}$ and $\mathrm{A\hat{O}B} = \theta$.

(a) Find the value of $\theta$, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

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**2.** [Maximum mark: 6]

On $1\text{st}$ January $2020$, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of $5.5\%$, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, $r$.

(a) Find the value of $r$, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

**3.** [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

**4.** [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Test 1 ($x$) | 15 | 23 | 25 | 30 | 34 | 34 | 40 |
| Test 2 ($y$) | 20 | 26 | 27 | 32 | 35 | 37 | 35 |

Let $L_1$ be the regression line of $x$ on $y$. The equation of the line $L_1$ can be written in the form $x = ay + b$.

(a) Find the value of $a$ and the value of $b$. [2]

Let $L_2$ be the regression line of $y$ on $x$. The lines $L_1$ and $L_2$ pass through the same point with coordinates $(p, q)$.

(b) Find the value of $p$ and the value of $q$. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

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**5.** [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time $t$ seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \le t \le 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

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**6.** [Maximum mark: 6]

In a city, the number of passengers, $X$, who ride in a taxi has the following probability distribution.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | 0.60 | 0.30 | 0.03 | 0.05 | 0.02 |

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is \$2.40 per taxi plus \$1.20 per passenger. Let $T$ represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X) = 0.8419$, find $\mathrm{Var}(T)$. [2]

**7.** [Maximum mark: 5]

Two ships, $\mathrm{A}$ and $\mathrm{B}$, are observed from an origin $\mathrm{O}$. Relative to $\mathrm{O}$, their position vectors at time $t$ hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

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**8.** [Maximum mark: 7]

The complex numbers $w$ and $z$ satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find $w$ and $z$ in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

**9.** [Maximum mark: 5]

Consider the graphs of $y = \dfrac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for $m$ such that the two graphs have no intersection points.

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

The length, $X\,\text{mm}$, of a certain species of seashell is normally distributed with mean $25$ and variance, $\sigma^2$.

The probability that $X$ is less than $24.15$ is $0.1446$.

(a) Find $\text{P}(24.15 < X < 25)$. [2]

(b) (i) Find $\sigma$, the standard deviation of $X$.

(ii) Hence, find the probability that a seashell selected at random has a length greater than $26\,\text{mm}$. [5]

A random sample of $10$ seashells is collected on a beach. Let $Y$ represent the number of seashells with lengths greater than $26\,\text{mm}$.

(c) Find $\text{E}(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than $26\,\text{mm}$. [2]

A seashell selected at random has a length less than $26\,\text{mm}$.

(e) Find the probability that its length is between $24.15\,\text{mm}$ and $25\,\text{mm}$. [3]

Do **not** write solutions on this page.

**11.** [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let $x$ grams represent the amount of salt in the tank and let $t$ minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{\mathrm{d}x}{\mathrm{d}t}$, is described by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 10\mathrm{e}^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of $x$ versus $t$ for $0 \le t \le 60$ and hence find the maximum amount of salt in the tank and the value of $t$ at which this occurs. [5]

(d) Find the value of $t$ at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first $60$ minutes. [4]

**12.** [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.
Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]