Diploma Programme
Programme du diplôme
Programa del Diploma

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# Mathematics: analysis and approaches
# Higher level
# Paper 1

Monday 31 October 2022 (afternoon)

Candidate session number

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2 hours

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8822 – 7101

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

The function $g$ is defined by $g(x) = e^{x^2+1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.

**2.** [Maximum mark: 7]

Consider a circle with a diameter $\mathrm{AB}$, where $\mathrm{A}$ has coordinates $(1, 4, 0)$ and $\mathrm{B}$ has coordinates $(-3, 2, -4)$.

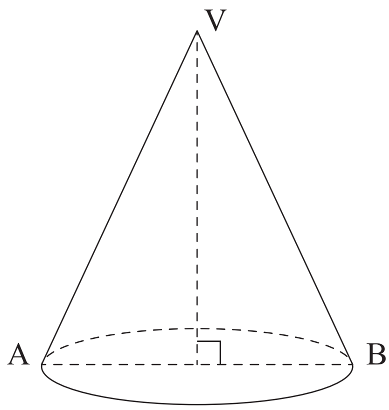
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex $\mathrm{V}$ has coordinates $(-1, -1, 0)$.

**diagram not to scale**




(b) Find the exact volume of the cone. [3]

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**3.** [Maximum mark: 5]

Let $a$ be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length $a$, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of $a$. [2]

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**4.** [Maximum mark: 5]

The derivative of the function $f$ is given by $f'(x) = \dfrac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

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**5.** [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, $\alpha$ and $\alpha^2$, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of $\alpha$. [4]

(b) Find the value of $p$. [3]

**6.** [Maximum mark: 6]

Events $A$ and $B$ are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events $A$ and $B$ are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

**7.** [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

**8.** [Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \leq x \leq a$ and $a, k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of $a$ for which the inverse function $f^{-1}$ exists. [2]

(b) Find the largest value of $a$ for which the inverse function $f^{-1}$ exists in the case where $k = \pi$. [1]

(c) Find the largest value of $a$ for which the inverse function $f^{-1}$ exists in the case where $\pi < k < 2\pi$. Give your answer in terms of $k$. [2]

**9.** [Maximum mark: 10]

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of $m$. [2]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 20]

The function $f$ is defined by $f(x) = \cos^2 x - 3\sin^2 x$, $0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

**11.** [Maximum mark: 16]

Consider a three-digit code $abc$, where each of $a$, $b$ and $c$ is assigned one of the values $1$, $2$, $3$, $4$ or $5$.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, $121$ or $444$);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of $a$, $b$ and $c$ is assigned one of the values $1$, $2$, $3$, $4$ or $5$. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for $b$ in terms of $a$.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

Let $z_n$ be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of $n$. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\dfrac{a+b}{1-ab}\right)$ for $a$, $b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

**References:**