Diploma Programme
Programme du diplôme
Programa del Diploma

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Mathematics: analysis and approaches
Higher level
Paper 2

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

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Please **do not** write on this page.

Answers written on this page will not be marked.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

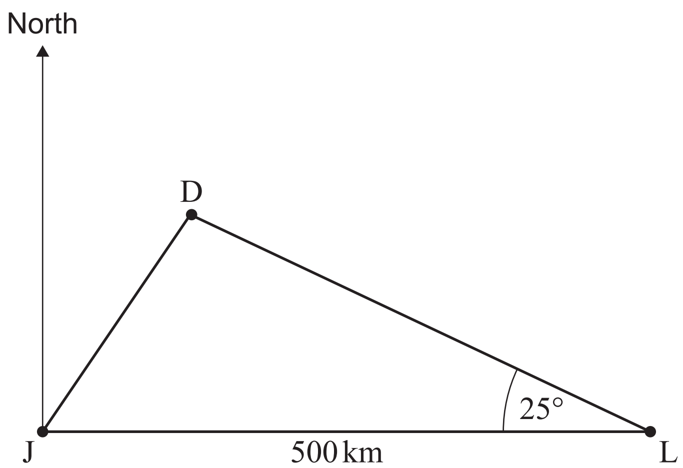
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram. Lucknow lies $500\,\mathrm{km}$ directly east of Jaipur, and $\mathrm{JLD} = 25°$.

diagram not to scale



The bearing of D from J is $034°$.

(a) Find $\mathrm{J\hat{D}L}$. [2]

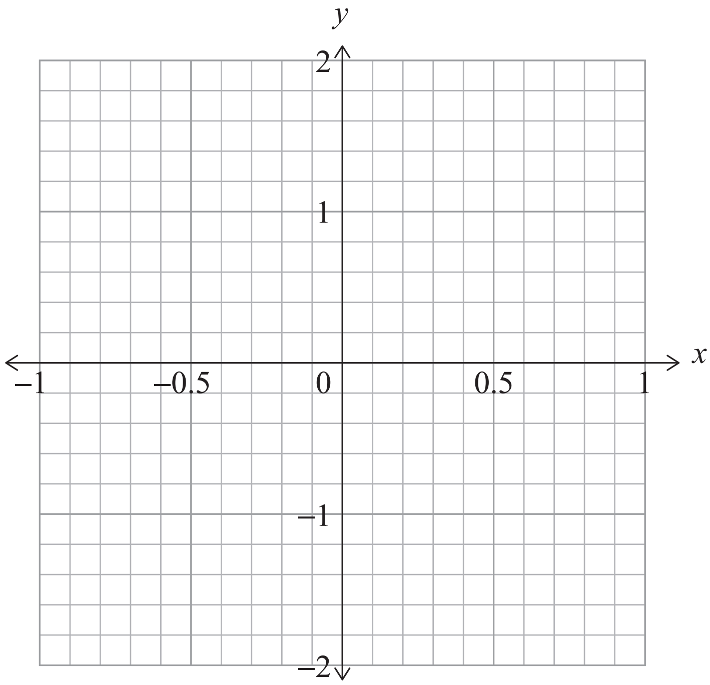
(b) Find the distance between Lucknow and Delhi. [3]

2. [Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \le x \le 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



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3. [Maximum mark: 7]

The total number of children, y, visiting a park depends on the highest temperature, T, in degrees Celsius (°C). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \leq T \leq 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°\text{C}$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, x. The following table shows the data collected on five different days.

Total number of children (y)	81	175	202	346	360
Ice creams sold (x)	15	27	23	35	46

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are y children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°\text{C}$. [2]

4. [Maximum mark: 5]

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, $D\,\text{mm}$, are normally distributed with mean 32 and standard deviation σ. The interquartile range of the diameters is 0.28.

Find the value of σ.

5. [Maximum mark: 7]

The coefficient of x^6 in the expansion of $(ax^3 + b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3 + b)^{10}$ is 2880.

Find the value of a and the value of b, where $a, b > 0$.

6. [Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of n that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

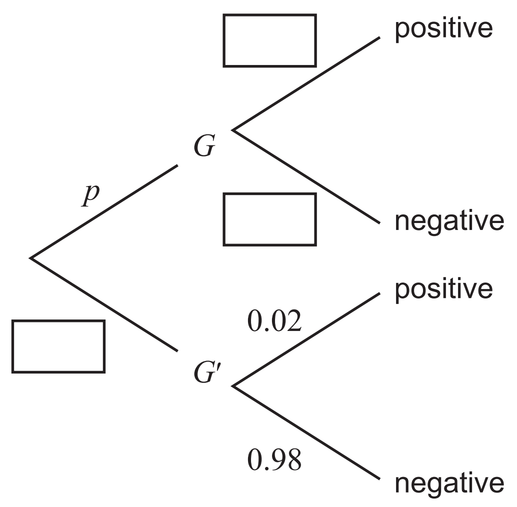
(b) Hence or otherwise, describe a single geometric transformation applied to z on the Argand diagram that results in z^{10}. [3]

7. [Maximum mark: 6]

A new test has been developed to identify whether a particular gene, G, is present in a population of parrots. The test returns a correct positive result 95% of the time for parrots with the gene, and a false positive result 2% of the time for parrots without the gene.

The proportion of the population with the gene is p.

(a) Complete the tree diagram below. [2]



(b) A random sample of the population was tested. It was found that 150 tests returned a positive result. Out of the 150 parrots with a positive test result, 18 did not actually have the gene. Find an estimate for p. [4]

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8. [Maximum mark: 7]

The angle between a line and a plane is α, where $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

Find the value of α.

9. [Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

Do **not** write solutions on this page.

Section B

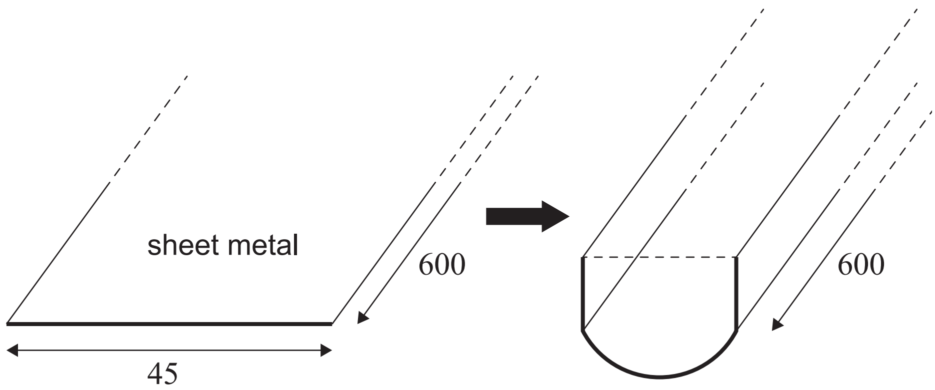
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

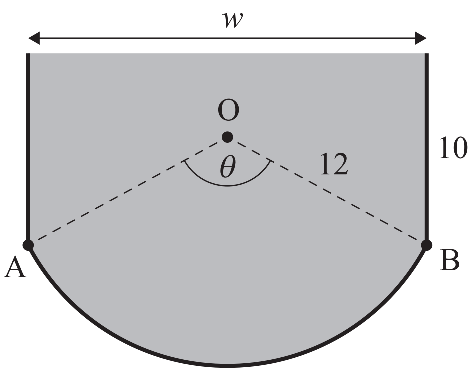
The gutter will be open at the top and is made by folding a piece of sheet metal $45\,\text{cm}$ wide and $600\,\text{cm}$ long.

diagram not to scale



The cross-section of the gutter is shaded in the following diagram.

diagram not to scale



The height of both vertical sides is $10\,\text{cm}$. The width of the gutter is $w\,\text{cm}$.

Arc AB lies on the circumference of a circle with centre O and radius $12\,\text{cm}$.

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

Let $\mathrm{A\hat{O}B} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3, of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000, \; t \geq 0.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

A continuous random variable, X, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{6}{\pi\sqrt{16-x^2}}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X < 0.5)$. [2]

A laboratory trial may require up to 2 millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where X is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X < 0.5$.

(c) Determine the least number of trials required to be 99% sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider n trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of n such that the probability of three consecutive successful trials is more than 0.05. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$.

(ii) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y.

(ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2 - 1}{x^2 + 1} \equiv 1 - \frac{2}{x^2 + 1}$.

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \le x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y. [2]

References:

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