Diploma Programme
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Mathematics: analysis and approaches
Higher level
Paper 3

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(i) one x-axis intercept; [1]

(ii) three x-axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(i) two x-axis intercepts; [1]

(ii) one x-axis intercept; [1]

(iii) three x-axis intercepts. [1]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0, b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(i) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

(ii) Determine whether each point is located above or below the x-axis. [1]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

(ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

(h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(i) Write down an expression for the gradient of L in terms of x and y. [1]

(ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

(iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a-b,\, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\, -2\sqrt{ab}\right)$. [6]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

(Question 2 continued)

Consider two families of curves, F and G.

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α.

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

(e) (i) Show that $g(x, y) = \frac{y - x}{y + x}$. [2]

(ii) Hence, by solving the homogeneous differential equation $\frac{dy}{dx} = \frac{y - x}{y + x}$, find a general equation that represents this family of curves, G. Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

(f) By considering $\lim_{\alpha \to \frac{\pi}{2}} \tan\alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \to \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}.$$ [2]