



© International Baccalaureate Organization 2021

All rights reserved. No part of this product may be reproduced in any form or by any electronic or mechanical means, including information storage and retrieval systems, without the prior written permission from the IB. Additionally, the license tied with this product prohibits use of any selected files or extracts from this product. Use by third parties, including but not limited to publishers, private teachers, tutoring or study services, preparatory schools, vendors operating curriculum mapping services or teacher resource digital platforms and app developers, whether fee-covered or not, is prohibited and is a criminal offense.

More information on how to request written permission in the form of a license can be obtained from <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organisation du Baccalauréat International 2021

Tous droits réservés. Aucune partie de ce produit ne peut être reproduite sous quelque forme ni par quelque moyen que ce soit, électronique ou mécanique, y compris des systèmes de stockage et de récupération d'informations, sans l'autorisation écrite préalable de l'IB. De plus, la licence associée à ce produit interdit toute utilisation de tout fichier ou extrait sélectionné dans ce produit. L'utilisation par des tiers, y compris, sans toutefois s'y limiter, des éditeurs, des professeurs particuliers, des services de tutorat ou d'aide aux études, des établissements de préparation à l'enseignement supérieur, des fournisseurs de services de planification des programmes d'études, des gestionnaires de plateformes pédagogiques en ligne, et des développeurs d'applications, moyennant paiement ou non, est interdite et constitue une infraction pénale.

Pour plus d'informations sur la procédure à suivre pour obtenir une autorisation écrite sous la forme d'une licence, rendez-vous à l'adresse <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organización del Bachillerato Internacional, 2021

Todos los derechos reservados. No se podrá reproducir ninguna parte de este producto de ninguna forma ni por ningún medio electrónico o mecánico, incluidos los sistemas de almacenamiento y recuperación de información, sin la previa autorización por escrito del IB. Además, la licencia vinculada a este producto prohíbe el uso de todo archivo o fragmento seleccionado de este producto. El uso por parte de terceros —lo que incluye, a título enunciativo, editoriales, profesores particulares, servicios de apoyo académico o ayuda para el estudio, colegios preparatorios, desarrolladores de aplicaciones y entidades que presten servicios de planificación curricular u ofrezcan recursos para docentes mediante plataformas digitales—, ya sea incluido en tasas o no, está prohibido y constituye un delito.

En este enlace encontrará más información sobre cómo solicitar una autorización por escrito en forma de licencia: <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

**Mathematics: analysis and approaches**  
**Higher level**  
**Paper 1**

Thursday 6 May 2021 (afternoon)

Candidate session number

2 hours

--	--	--	--	--	--	--	--	--	--

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.



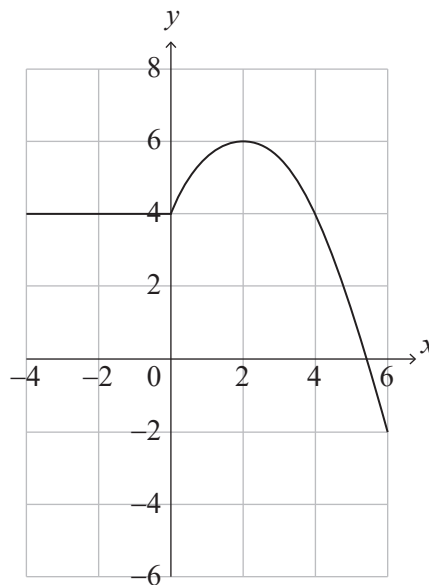
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

### Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of  $y = f(x)$  for  $-4 \leq x \leq 6$  is shown in the following diagram.



(a) Write down the value of

(i)  $f(2)$ ;

(ii)  $(f \circ f)(2)$ .

[2]

(b) Let  $g(x) = \frac{1}{2}f(x) + 1$  for  $-4 \leq x \leq 6$ . On the axes above, sketch the graph of  $g$ .

[3]

.....

.....

.....

.....

.....

.....



## 2. [Maximum mark: 5]

Consider an arithmetic sequence where  $u_8 = S_8 = 8$ . Find the value of the first term,  $u_1$ , and the value of the common difference,  $d$ .

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

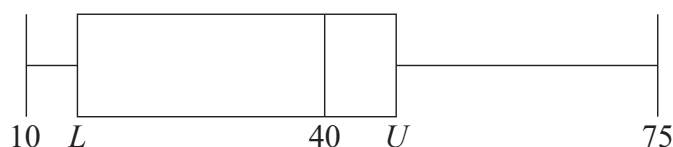
.....

.....



A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where  $L$  and  $U$  are the lower and upper quartiles respectively.

**diagram not to scale**



(a) Find the minimum possible value of  $U$ . [3]

(b) Hence, find the minimum possible value of  $L$ . [2]

[illegible]

4. [Maximum mark: 7]

Consider the functions  $f(x) = -(x - h)^2 + 2k$  and  $g(x) = e^{x-2} + k$  where  $h, k \in \mathbb{R}$ .

(a) Find  $f'(x)$ . [1]

The graphs of  $f$  and  $g$  have a common tangent at  $x = 3$ .

(b) Show that  $h = \frac{e+6}{2}$ . [3]

(c) Hence, show that  $k = e + \frac{e^2}{4}$ . [3]

A series of horizontal dotted lines for writing.



5. [Maximum mark: 8]

(a) Show that  $\sin 2x + \cos 2x - 1 = 2 \sin x (\cos x - \sin x)$ . [2]

(b) Hence or otherwise, solve  $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$  for  $0 < x < 2\pi$ . [6]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



6. [Maximum mark: 4]

It is given that  $\operatorname{cosec} \theta = \frac{3}{2}$ , where  $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ . Find the exact value of  $\cot \theta$ .

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....





7. [Maximum mark: 8]

Consider the quartic equation  $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$ ,  $z \in \mathbb{C}$ .

Two of the roots of this equation are  $a + bi$  and  $b + ai$ , where  $a, b \in \mathbb{Z}$ .

Find the possible values of  $a$ .

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



8. [Maximum mark: 5]

Use l'Hôpital's rule to find  $\lim_{x \rightarrow 0} \left( \frac{\arctan 2x}{\tan 3x} \right)$ .

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



## 9. [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.


Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

- (a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]
- (b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let  $X$  be the score obtained when die A is rolled. The probability distribution for  $X$  is given in the following table.

$x$	1	2	3	4
$P(X=x)$	$p$	$p$	$p$	$\frac{1}{2}p$

(a) Find the value of  $p$ . [2]

(b) Hence, find the value of  $E(X)$ . [2]

A second biased four-sided die, B, is rolled. Let  $Y$  be the score obtained when die B is rolled. The probability distribution for  $Y$  is given in the following table.

$y$	1	2	3	4
$P(Y=y)$	$q$	$q$	$q$	$r$

(c) (i) State the range of possible values of  $r$ .

(ii) Hence, find the range of possible values of  $q$ . [3]

(d) Hence, find the range of possible values for  $E(Y)$ . [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is  $\frac{1}{2}$ .

(e) Find the value of  $E(Y)$ . [6]



Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

Consider the line  $L_1$  defined by the Cartesian equation  $\frac{x+1}{2} = y = 3 - z$ .

(a) (i) Show that the point  $(-1, 0, 3)$  lies on  $L_1$ .

(ii) Find a vector equation of  $L_1$ .

[4]

Consider a second line  $L_2$  defined by the vector equation  $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$ ,

where  $t \in \mathbb{R}$  and  $a \in \mathbb{R}$ .

(b) Find the possible values of  $a$  when the acute angle between  $L_1$  and  $L_2$  is  $45^\circ$ .

[8]

It is given that the lines  $L_1$  and  $L_2$  have a unique point of intersection, A, when  $a \neq k$ .

(c) Find the value of  $k$ , and find the coordinates of the point A in terms of  $a$ .

[7]

**12.** [Maximum mark: 20]

Let  $f(x) = \sqrt{1+x}$  for  $x > -1$ .

(a) Show that  $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$ .

[3]

(b) Use mathematical induction to prove that  $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!} (1+x)^{\frac{1}{2}-n}$   
for  $n \in \mathbb{Z}$ ,  $n \geq 2$ .

[9]

Let  $g(x) = e^{mx}$ ,  $m \in \mathbb{Q}$ .

Consider the function  $h$  defined by  $h(x) = f(x) \times g(x)$  for  $x > -1$ .

It is given that the  $x^2$  term in the Maclaurin series for  $h(x)$  has a coefficient of  $\frac{7}{4}$ .

(c) Find the possible values of  $m$ .

[8]

**References:**

© International Baccalaureate Organization 2021

