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ECE 141 - Project
Spring 2023

Please carefully justify all your answers.
Submit your MATLAB script separately.
You can refer to the additional material posted under Week 8 for useful MATLAB commands.
If you think that there is something wrong with the questions,
just use your best interpretation and state your justification.
You are not allowed to collaborate with others.

Due: June 9, 2023 11:59 pm, 90 points in total

Figure 1: Hot air balloon system where heat is injected as a control input

In this project we will design a controller for a hot air balloon system based on an approximate differential equation describing its motion. The visual representation of the system is given in Figure 1. The equation of motion for the system is as follows

$$\begin{aligned}
\dot{\theta} &= -\frac{1}{\tau_1}\theta + u, \\
\dot{v} &= -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w, \\
\dot{h} &= v,
\end{aligned} \tag{1}$$

where $\theta \in \mathbb{R}$ is a temperature of air inside the balloon [K]; $u \in \mathbb{R}$ is a control input [K/s]; $h \in \mathbb{R}$ is an altitude change from some reference point [m]; $w \in \mathbb{R}$ is a wind disturbance [m/s]; and $v \in \mathbb{R}$ is balloon's vertical velocity [m/s]; $\sigma \in \mathbb{R}$ [m/s^{2}K], $\tau_1 \in \mathbb{R}$ [s], $\tau_2 \in \mathbb{R}$ [s] are all system parameters. You can assume $\tau_1 \neq \tau_2$ for all the questions below. Output of the system is h.

Part I, Laplace Domain

1. (15 points) Assume that $\sigma, \tau_1, \tau_2 > 0$. Find the expression for the open-loop output response of the system in Laplace domain when both the control input and the disturbance are present. Now, suppose you place the complete system in a closed-loop with a unity negative feedback. Draw a block-diagram of the complete system. Can you make the system stable using a proportional controller for the closed-loop system with a unity negative feedback?

2. (30 points) Assume that w is equal to some constant value for all $t \geq 0$. Find a controller such that (1) the system tracks constant reference signals with zero steady-state error; and (2) the error signal is no less than -0.2 for all $t \geq 0$ with convergence to zero for $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Take both constants for the reference and disturbance signals as 1 for (2). Provide a plot of the error signal.

Part II, State-Space

1. (30 points) Assume the disturbance, w, is zero for all t in (1) for the remainder of the project. Write the system in the state-space matrix form. Compute the controllability matrix of the system using the appropriate matrices found in part 1. Find the determinant of the controllability matrix found in part 2. What condition should be satisfied for the system to be controllable? Suppose now that the condition you found in part 3 is violated, what part of the state can you still control?

2. (15 points) Take the parameters as $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Suppose your controller in state-space is given by $u = Kx$ where $K \in \mathbb{R}^{1\times 3}$ is the controller gain and $x \in \mathbb{R}^3$ is the state of the system. Take $K = \begin{bmatrix} 14 & -10 & -1 \end{bmatrix}$. Verify that your closed-loop system is stable.

Part I:

1. (15 points) Assume that $\sigma, \tau_1, \tau_2 > 0$. Find the expression for the open-loop output response of the system in Laplace domain when both the control input and the disturbance are present. Now, suppose you place the complete system in a closed-loop with a unity negative feedback. Draw a block-diagram of the complete system. Can you make the system stable using a proportional controller for the closed-loop system with a unity negative feedback?

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u,$$
$$\dot{v} = -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w,$$
$$\dot{h} = v,$$

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u$$
$$\dot{V} = -\frac{1}{\tau_2}V + \sigma\theta + \frac{1}{\tau_2}W$$
$$\dot{h} = V$$

$$\mathcal{L}\{\dot{\theta}\} = \mathcal{L}\{-\frac{1}{\tau_1}\theta + u\}$$

$$\Rightarrow s\theta(s) - \theta(0) = -\frac{1}{\tau_1}\theta(s) + U(s) \Rightarrow s\theta(s) + \frac{1}{\tau_1}\theta(s) = U(s)$$

$$\Rightarrow \theta(s)\left(s + \frac{1}{\tau_1}\right) = U(s) \Rightarrow \boxed{\theta(s) = \frac{U(s)}{s + \frac{1}{\tau_1}}}$$

$$\mathcal{L}\{\dot{V}\} = \mathcal{L}\{-\frac{1}{\tau_2}V + \sigma\theta + \frac{1}{\tau_2}W\}$$

$$\Rightarrow sV(s) - V(0) = -\frac{1}{\tau_2}V(s) + \sigma\theta(s) + \frac{1}{\tau_2}W(s)$$

$$\Rightarrow sV(s) + \frac{1}{\tau_2}V(s) = \sigma\theta(s) + \frac{1}{\tau_2}W(s)$$

$$\Rightarrow V(s)\left(s + \frac{1}{\tau_2}\right) = \sigma\theta(s) + \frac{1}{\tau_2}W(s)$$

$$\Rightarrow \boxed{V(s) = \frac{\sigma\theta(s) + \frac{1}{\tau_2}W(s)}{s + \frac{1}{\tau_2}}}$$

$$V(s) = \frac{\sigma\left(\frac{U(s)}{s+\frac{1}{T_1}}\right) + \frac{1}{T_2}W(s)}{s+\frac{1}{T_2}}$$

$$= \frac{\sigma U(s)}{\frac{sT_1+1}{T_1}}\left(\frac{T_2}{sT_2+1}\right) + \frac{W(s)}{T_2}\left(\frac{T_2}{sT_2+1}\right)$$

$$= \frac{\sigma U(s)}{1}\left(\frac{T_1}{sT_1+1}\right)\left(\frac{T_2}{sT_2+1}\right) + \frac{W(s)}{sT_2+1}$$

$$\Rightarrow \boxed{V(s) = \frac{\sigma T_1 T_2 U(s)}{(sT_1+1)(sT_2+1)} + \frac{W(s)}{sT_2+1}}$$

$$\mathcal{L}\{\dot{h}\} = \mathcal{L}\{v\}$$

$$\Rightarrow sH(s) + h(0) = V(s)$$

$$\Rightarrow H(s) = \frac{V(s)}{s}$$

$$\Rightarrow H(s) = \frac{\frac{\sigma T_1 T_2 U(s)}{(sT_1+1)(sT_2+1)} + \frac{W(s)}{sT_2+1}}{s}$$

$$G_1(s) = \frac{\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}$$

$$G_2(s) = \frac{1}{s(sT_2+1)}$$

$$\Rightarrow \boxed{H(s) = \frac{\sigma T_1 T_2 U(s)}{s(sT_1+1)(sT_2+1)} + \frac{W(s)}{s(sT_2+1)}}$$

→ open-loop expression in Laplace domain.

Open-loop diagram:

Closed-loop block diagram:

$$\hat{G}_1(s) = \frac{K(s)G_1(s)}{1+K(s)G_1(s)}, \quad \hat{G}_2(s) = \frac{G_2(s)}{1+K(s)G_1(s)}$$

$$1 + K(s)G_1(s) = 0$$

$$K(s) = K_p$$

$$1 + K_p G_1(s) = 0 \Rightarrow 1 + K_p\left(\frac{\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}\right)$$

$$\Rightarrow 1(s(sT_1+1)(sT_2+1)) + K_p\sigma T_1 T_2 = 0$$

$$\Rightarrow s(T_1T_2s^2 + sT_1 + sT_2 + 1) + K_p\sigma T_1T_2 = 0$$

$$s(T_1T_2s^2 + (T_1+T_2)s + 1) + K_p\sigma T_1T_2 = 0$$

$$\Rightarrow T_1T_2s^3 + (T_1+T_2)s^2 + s + K_p\sigma T_1T_2 = 0$$

$$\Rightarrow \underbrace{0+0}_{R_3} + \underbrace{K_p\sigma T_1T_2}_{R_2} + \underbrace{s}_{R_1} + (T_1+T_2)s^2 + T_1T_2s^3 = 0$$

$$M = \begin{array}{c} \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} T_1+T_2 & T_1T_2 & 0 \\ K_p\sigma T_1T_2 & 1 & T_1+T_2 \\ 0 & 0 & K_p\sigma T_1T_2 \end{bmatrix} \end{array}$$

$$\det \Delta_1 = T_1+T_2 > 0$$

$$\det \Delta_2 = T_1+T_2-K_p\sigma T_1^2T_2^2 > \qquad K_p < \frac{T_1+T_2}{\sigma T_1^2T_2^2}$$

$$\det \Delta_3 = K_p\sigma T_1T_2 \det \Delta_2 = K_pT_1T_2\left(T_1+T_2-K_p\sigma T_1^2T_2^2\right) > 0$$

$$K_pT_1T_2 > 0 \quad \Rightarrow \quad K_p > 0$$

We can make the system stable with $0 < K_p < \dfrac{T_1+T_2}{\sigma T_1^2T_2^2}$

2. (30 points) Assume that w is equal to some constant value for all $t \geq 0$. Find a controller such that (1) the system tracks constant reference signals with zero steady-state error; and (2) the error signal is no less than -0.2 for all $t \geq 0$ with convergence to zero for $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Take both constants for the reference and disturbance signals as 1 for (2). Provide a plot of the error signal.

$$G_2(s) = \frac{1}{s(s\tau_2+1)}$$, a and b are constants. $\tau_1, \tau_2, \sigma > 0$, $a = b = 1$

$$G_1(s) = \frac{\sigma \tau_1 \tau_2}{s(s\tau_1+1)(s\tau_2+1)}$$

$w(t) = a \Rightarrow \mathcal{L}\{w(t)\} = \mathcal{L}\{a\} \Rightarrow W(s) = \frac{a}{s} = \frac{1}{s}$

$r = b \Rightarrow \mathcal{L}\{r\} = \mathcal{L}\{b\} \Rightarrow R(s) = \frac{b}{s} = \frac{1}{s}$

$$\hat{H}(s) = \hat{G}_1(s) R(s) + \hat{G}_2 W(s)$$

$$= \frac{K(s) G_1(s) R(s)}{1 + K(s) G_1(s)} + \frac{G_2(s) W(s)}{1 + K(s) G_1(s)}$$

$$= \frac{K(s)\left(\frac{\sigma \tau_1 \tau_2}{s(s\tau_1+1)(s\tau_2+1)}\right) R(s)}{1 + K(s)\left(\frac{\sigma \tau_1 \tau_2}{s(s\tau_1+1)(s\tau_2+1)}\right)} + \frac{\left(\frac{1}{s(s\tau_1+1)}\right) W(s)}{1 + K(s)\left(\frac{\sigma \tau_1 \tau_2}{s(s\tau_1+1)(s\tau_2+1)}\right)}$$

$$= \frac{K(s)\sigma \tau_1 \tau_2 R(s)}{s(s\tau_1+1)(s\tau_2+1) + K(s)\sigma \tau_1 \tau_2} + \frac{(s\tau_1+1) W(s)}{s(s\tau_1+1)(s\tau_2+1) + K(s)\sigma \tau_1 \tau_2}$$

$$\Rightarrow \boxed{\hat{H}(s) = \frac{K(s)\sigma \tau_1 \tau_2 R(s) + (s\tau_1+1) W(s)}{s(s\tau_1+1)(s\tau_2+1) + K(s)\sigma \tau_1 \tau_2}}$$

$$E(s) = R(s) - \hat{H}(s)$$

$$= R(s) - \left(\hat{G}_1(s)R(s) + \hat{G}_2 W(s)\right)$$

$$= R(s) - \left(\frac{K(s)G_1(s)R(s)}{1+K(s)G_1(s)} - \frac{G_2(s)}{1+K(s)G_1(s)}W(s)\right)$$

$$= \frac{R_s\left(1+\cancel{K(s)G_1(s)}\right) - \cancel{K(s)G_1(s)}R(s)}{1+K(s)G_1(s)} - \frac{G_2(s)W(s)}{1+K(s)G_1(s)}$$

$$= \frac{R(s)}{1+K(s)G_1(s)} - \frac{G_2(s)W(s)}{1+K(s)G_1(s)}$$

$$= \frac{R(s)}{1+K(s)\left(\frac{\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}\right)} - \frac{\left(\frac{1}{s(sT_2+1)}\right)W(s)}{1+K(s)\frac{\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}}$$

$$= \frac{R(s)}{\frac{s(sT_1+1)(sT_2+1)+K(s)\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}} - \frac{\frac{1}{\cancel{s}(s\cancel{T_2}+1)}W(s)}{\frac{s(sT_1+1)(\cancel{sT_2}+1)+K(s)\sigma T_1 T_2}{\cancel{s}(sT_1+1)(\cancel{sT_2}+1)}}$$

$$= \frac{(sT_1+1)(sT_2+1)}{s(sT_1+1)(sT_2+1)+K(s)\sigma T_1 T_2} - \frac{(sT_1+1)W(s)}{s(sT_1+1)(sT_2+1)+K_p\sigma T_1 T_2}$$

$$\Rightarrow E(s) = \frac{s(sT_1+1)(sT_2+1)R(s)}{s(sT_1+1)(sT_2+1)+K(s)\sigma T_1 T_2} - \frac{(sT_1+1)W(s)}{s(s(sT_1+1)(sT_2+1)+K_p\sigma T_1 T_2)}$$

$$\Rightarrow E(s) = \frac{s(sT_1+1)(sT_2+1)R(s) - (sT_1+1)W(s)}{s(sT_1+1)(sT_2+1)+K(s)\sigma T_1 T_2}$$

$K(s) = K_p$, I checked stability in Part 1-1:

$a = b = 1$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} s \left(\frac{s(sT_1+1)\overset{\to 0}{(sT_2+1)}\frac{1}{s} - \overset{\to 0}{(sT_1+1)}\frac{1}{s}}{s\underset{\to 0}{(sT_1+1)}(sT_2+1) + K_p\sigma T_1 T_2} \right)$$

$$= -\frac{1}{K_p \sigma T_1 T_1}$$

Steady-state error is zero if only if $K_p \to \infty$, but $K = \infty$ is not possible, so we cannot us proportional to satisfy the design conditions.

I checked each controller ($\frac{K_i}{s}$, $K_D s$, PI, ID, PID),

So $\frac{K_i}{s}$, $K_D s$ do not satify the condition.

PI controller, the system is stable and the error is so large to satify the second condition.

PID controller is stable and both conditions are satified, so I used PID controller.

$$K(s) = K_p + K_D s + \frac{K_i}{s} \quad (\text{PID controller})$$

$$1 + K(s)\,G_1(s) = 0$$

$$\Rightarrow 1 + \left(K_p + K_D s + \frac{K_i}{s}\right)\left(\frac{\sigma T_1 T_2}{s(sT_1+1)(sT_2+1)}\right) = 0$$

$$\Rightarrow s^2(sT_1+1)(sT_2+1) + \left(K_p + K_D s + \frac{K_i}{s}\right)(\sigma T_1 T_2) = 0$$

$$\Rightarrow s^2(T_1T_2s^2 + T_1 s + T_2 s + 1) + sK_p\sigma T_1T_2 + K_D s^2 \sigma T_1 T_2 + K_i \sigma T_1 T_2 = 0$$

$$\Rightarrow s^2(T_1T_2s^2 + (T_1+T_2)s + 1) + K_p\sigma T_1T_2 s + K_D\sigma T_1T_2 s^2 + K_i\sigma T_1T_2 = 0$$

$$\Rightarrow T_1T_2s^4 + (T_1+T_2)s^3 + s^2 + K_D\sigma T_1T_2 s^2 + K_p\sigma T_1T_2 s + K_i\sigma T_1T_2 = 0$$

$$\Rightarrow T_1T_2s^4 + (T_1+T_2)s^3 + (1+K_D\sigma T_1T_2)s^2 + K_p\sigma T_1T_2 s + K_i\sigma T_1T_2 = 0$$

$$0+0+0+ K_i\sigma T_1T_2 + K_p\sigma T_1T_2 s + (1+K_D\sigma T_1T_2)s^2 + (T_1+T_2)s^3 + T_1T_2 s^4 = 0 + 0$$

R_1

R_2

R_3

R_4

$$M = \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} T_1+T_2 & T_1T_2 & 0 & 0 \\ K_p\sigma T_1T_2 & 1+K_D\sigma T_1T_2 & T_1+T_2 & T_1T_2 \\ 0 & K_i\sigma T_1T_2 & K_p\sigma T_1T_2 & 1+K_D\sigma T_1T_2 \\ 0 & 0 & 0 & K_i\sigma T_1T_2 \end{bmatrix} \end{array}$$

$$\det D_1 = T_1 + T_2 > 0 \quad = 0.1 + 0.1 = 0.2 > 0$$

$$\det D_2 = (T_1+T_2)(1+K_D\sigma T_1T_2) - (K_p\sigma T_1T_2)(T_1T_2) > 0$$

$$= (0.1+0.1)(1+K_D(5)(0.1)(0.1)) - (K_p(5)(0.1)(0.1)(0.1)(0.1) > 0$$

$$= (0.2)(1+0.05K_D) - (0.05K_p)(0.01) > 0$$

$$= 0.2 + 0.01K_D - 5\times10^{-4}K_p > 0$$

$$\det D_3 = (T_1+T_2)\left((1+K_D\sigma T_1T_2)(K_p\sigma T_1T_2) - K_i\sigma T_1T_2(T_1+T_2)\right)$$
$$-T_1T_2(K_p^2\sigma^2T_1^2T_2^2) > 0$$

$$= 0.2((1+0.05K_D)(0.05K_p) - 0.01K_i) - 2.5\times10^{-5}K_p^2 > 0$$

$$= 0.2(0.05K_p + 0.0025K_DK_p - 0.01K_i) - 2.5\times10^{-5}K_p^2 > 0$$

$$= 0.01K_p + 5\times10^{-4}K_DK_p - 0.002K_i - 2.5\times10^{-5}K_p^2 > 0$$

$$\det D_4 = K_i\sigma T_1T_2 \det D_3 > 0$$

$$K_i\sigma T_1T_2 > 0 \quad \Rightarrow \quad K_i > 0$$

$$\lim_{t\to0} e(t) = \lim_{s\to0} sE(s)$$

$$= \lim_{s\to0} s\left(\frac{s(sT_1+1)(sT_2+1)\frac{1}{s} - (sT_1+1)\frac{1}{s}}{s(sT_1+1)(sT_2+1) + (K_p+K_Ds+\frac{K_i}{s})\sigma T_1T_2}\right)$$

$$= \lim_{s\to0}\left(\frac{s^2(sT_1+1)(sT_2+1) - s(sT_1+1)}{s^2(sT_1+1)(sT_2+1) + (K_ps+K_Ds^2+K_i)\sigma T_1T_2}\right)$$

$$= \frac{0}{K_i\sigma T_1T_2} = 0$$

$$\Rightarrow \lim_{s\to0} sE(s) = 0$$

$$E(s) = \frac{s(sT_1+1)(sT_2+1)R(s) - (sT_1+1)W(s)}{s(sT_1+1)(sT_2+1) + K(s)\sigma T_1 T_2}$$

$$= \frac{(sT_1+1)\left(s(sT_2+1)\frac{1}{s} - \frac{1}{s}\right)}{s(sT_1+1)(sT_2+1) + \left(K_p + K_D s + \frac{K_i}{s}\right)\sigma T_1 T_2}$$

$$\Rightarrow \quad E(s) = \frac{1}{s}\left(\frac{(sT_1+1)\left(s(sT_2+1) - 1\right)}{s(sT_1+1)(sT_2+1) + \left(K_p + K_D s + \frac{K_i}{s}\right)\sigma T_1 T_2}\right)$$

$K_p = 86$

$K_D = 63$

$K_i = 26$

$\det \Delta_1 = T_1 + T_2 = 0.1 + 0.1 = 0.2 > 0$

$\det \Delta_1 = 0.2 > 0$

$\det \Delta_2 = 0.2 + 0.01 K_D - 5\times10^{-4} K_p > 0$

$= 0.2 + 0.01(63) - 5\times10^{-4}(86) = 0.79 > 0$

$\det \Delta_2 = 0.79 > 0$

$\det \Delta_3 = 0.01 K_p + 5\times10^{-4} K_D K_p - 0.002 K_i - 2.5\times10^{-5} K_p^2 > 0$

$= 0.01(86) + 5\times10^{-4}(63)(86) - 0.002(26) - 2.5\times10^{-5}(86)^2 = 3.3 > 0$

$\det \Delta_3 = 3.3$

$\det \Delta_4 = K_i \sigma T_1 T_2 \det \Delta_3 = 26(5)(0.1)(0.1)(3.3) = 4.3 > 0$

$\det \Delta_4 = 4.3 > 0$

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```
clc
close all
s = tf('s');
kp = 86;
kd =63;
ki = 26;
T1 = 0.1;
T2 = T1;
segma = 5;
t = 0:0.001:40;
sysE =((s*T1+1)*(s*(s*T2+1)-1))/((s*(s*T1 + 1)*(s*T2+1) + (kp +kd*s +ki/s)*segma*T1*T2));
err = step(sysE,t);
figure(1);
plot(t,err,'LineWidth',2);
grid on
title ('Error Signal');
ylabel ('e(t)');
xlabel ('t(s)');
```

Workspace

Name	Value
err	40001x1...
kd	63
ki	26
kp	86
s	1x1 tf
segma	5
sysE	1x1 tf
t	1x40001...
T1	0.1000
T2	0.1000

Part II, State-Space

1. (30 points) Assume the disturbance, w, is zero for all t in (1) for the remainder of the project. Write the system in the state-space matrix form. Compute the controllability matrix of the system using the appropriate matrices found in part 1. Find the determinant of the controllability matrix found in part 2. What condition should be satisfied for the system to be controllable? Suppose now that the condition you found in part 3 is violated, what part of the state can you still control?

$$\dot{\theta} = -\frac{1}{T_1}\theta + u$$

$$\dot{V} = -\frac{1}{T_2}V + \sigma\theta$$

$$\dot{h} = V$$

$$\theta = x_1 \qquad \dot{\theta} = \dot{x}_1$$
$$V = x_2 \qquad \dot{V} = \dot{x}_2$$
$$h = x_3 \qquad \dot{h} = \dot{x}_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$\dot{X} = AX + Bu$$

$$\boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{T_1} & 0 & 0 \\ \sigma & -\frac{1}{T_2} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u}$$

$$C_{(A,B)} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -\frac{1}{\tau_1} \\ \sigma \\ 0 \end{bmatrix}, \quad A^2B = A(AB) = \begin{bmatrix} \frac{1}{\tau_1^2} \\ -\frac{\sigma}{\tau_1} - \frac{\sigma}{\tau_2} \\ \sigma \end{bmatrix}$$

$$\boxed{C_{(A,B)} = \begin{bmatrix} 1 & -\frac{1}{\tau_1} & \frac{1}{\tau_1^2} \\ 0 & \sigma & -\frac{\sigma}{\tau_1} - \frac{\sigma}{\tau_2} \\ 0 & 0 & \sigma \end{bmatrix}}$$

$$\det C_{(A,B)} = 1(\sigma^2) + \frac{1}{\tau_1}(0) + \frac{1}{\tau_1^2}(0) = \sigma^2$$

$$\Rightarrow \boxed{\det C_{(A,B)} = \sigma^2}$$

System can be controllable if segma should not be equal to zero. ($\sigma \neq 0$).

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u,$$

$$\dot{v} = -\frac{1}{\tau_2}v + \cancel{\sigma\theta}^{\,0} + \cancel{\frac{1}{\tau_2}w}^{\,0},$$

$$\dot{h} = v,$$

$$\Rightarrow \quad \dot{\theta} = -\frac{1}{\tau}\theta + u$$

$$\dot{V} = -\frac{1}{\tau_2}V$$

$$\dot{h} = V$$

$\dot{V}$ and $\dot{h}$ are independent of $\dot{\theta}$,
so I can only control $\dot{\theta}$.

I cannot control $\dot{V}$ and $\dot{h}$ but I can still control $\dot{\theta}$.

$\theta = x_1 \qquad \dot{\theta} = \dot{x}_1$

$V = x_2 \qquad \dot{V} = \dot{x}_2$

$h = x_3 \qquad \dot{h} = \dot{x}_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$\dot{X} = \overset{''}{A} X + \overset{''}{B} u$$

$\sigma = 0, w = 0$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 \\ 0 & -\frac{1}{\tau_2} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\overset{''}{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\overset{''}{B}} u$$

$$\overset{''}{C}_{(\overset{''}{A}, \overset{''}{B})} = \begin{bmatrix} \overset{''}{B} & \overset{''}{A}\overset{''}{B} & \overset{''}{A}^2\overset{''}{B} \end{bmatrix}$$

$$\overset{''}{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overset{''}{A}\overset{''}{B} = \begin{bmatrix} -\frac{1}{\tau_1} \\ 0 \\ 0 \end{bmatrix}, \quad \overset{''}{A}\overset{''}{B} = \overset{''}{A}(\overset{''}{A}\overset{''}{B}) = \begin{bmatrix} \frac{1}{\tau_1^2} \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \overset{''}{C}_{(\overset{''}{A}, \overset{''}{B})} = \begin{bmatrix} 1 & -\frac{1}{\tau_1} & \frac{1}{\tau^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \det(\overset{''}{C}_{(\overset{''}{A}, \overset{''}{B})}) = 0$$

I can only control $\dot{\theta}$ and I cannot control $\dot{V}$ and $\dot{h}$ because $\dot{V}$ and $\dot{h}$ are not depend on $\dot{\theta}$.

2. (15 points) Take the parameters as $\sigma = 5$, $\tau_1 = 0.1$, $\tau_2 = 0.1$. Suppose your controller in state-space is given by $u = Kx$ where $K \in \mathbb{R}^{1\times 3}$ is the controller gain and $x \in \mathbb{R}^3$ is the state of the system. Take $K = \begin{bmatrix} 14 & -10 & -1 \end{bmatrix}$. Verify that your closed-loop system is stable.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 \\ \sigma & -\frac{1}{\tau_2} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} \begin{bmatrix} 14 & -10 & -1 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 \\ \sigma & -\frac{1}{\tau_2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 14 & -10 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A'} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \dot{x} = A'x$$

$$sI - A' = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 4 & -10 & -1 \\ 5 & -10 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s-4 & 10 & 1 \\ -5 & s+10 & 0 \\ 0 & -1 & s \end{bmatrix}$$

$$\det(sI - A') = 0$$

$$\det(SI - A') = (s-4)(s(s+10)) - 10(-5s) - (5) = 0$$

$$= (s-4)(s^2+10s) + 50s + 5 = 0$$

$$= s^3 + 10s^2 - 4s^2 - 40s + 50s + 5 = 0$$

$$= s^3 + 6s^2 + 10s + 5 = 0$$

$$\Rightarrow \underbrace{0+0+}_{R_3} \underbrace{+5+10s}_{R_2} \underbrace{+6s^2+s^2}_{R_1} = 0$$

$$M = \begin{array}{c} \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} \overset{+}{6} & \overset{-}{1} & \overset{+}{0} \\ \overset{-}{5} & \overset{+}{10} & \overset{-}{6} \\ \overset{+}{0} & \overset{-}{0} & \overset{+}{5} \end{bmatrix} \end{array}$$

$$\det \Delta_1 = 6 > 0$$

$$\det \Delta_2 = 60 - 5 = 55 > 0$$

$$\det \Delta_2 = 5 \det \Delta_2 = 5(55) = 275 > 0$$

It is Hurwitz, so it $\boxed{\text{stable}}$