

# CSE400 Lecture 5 Scribe



L5\_A\_S1.pdf

PDF

Generate a lecture scribe intended to serve as exam-reference material for CSE400 Lecture 5: Bayes' Theorem, Random Variables, and Probability Mass Function. The scribe must faithfully reconstruct the lecture content so that a student can reliably revise the lecture using only this document.

Use only the provided context (lecture slides/PDF and relevant textbook excerpts). Do not introduce any content, examples, explanations, interpretations, or intuition that are not present in the provided context.

While writing the lecture scribe, reason step-by-step through all definitions, notation, assumptions, results, proofs, derivations, conceptual transitions, and worked examples exactly as presented in the provided context. Ensure that all logical dependencies between ideas are explicitly explained so that the reasoning can be followed during exam revision.

The lecture scribe must clearly and systematically reconstruct the lecture structure, including:

## 1. Bayes' Theorem

- Step-by-step explanation of expressing events using set relationships
- Derivation of probability as a weighted average of conditional probabilities
- Formal development using the Law of Total Probability
- Step-by-step derivation of Bayes' Formula
- Explanation of apriori and posteriori probabilities
- Step-by-step reconstruction of all worked examples shown in the lecture

## 2. Random Variables

- Motivation for defining random variables using

experiment-based examples

- Formal definition of a random variable as a function on the sample space
- Explanation of discrete and continuous random variables as presented
- Explanation of distribution of a random variable and interpretation using probability assignments
- Step-by-step reconstruction of all examples involving random variables

### 3. Probability Mass Function (PMF)

- Definition of discrete random variables
- Formal definition of Probability Mass Function
- Properties of PMF including probability summation requirements
- Step-by-step reconstruction of PMF examples presented in the lecture

Include:

- Definitions and notation exactly as introduced
- Assumptions and conditions associated with results
- Theorem or result statements when presented
- Step-by-step proofs or derivations when present
- Step-by-step worked examples when present
- Clear logical transitions between major topics

Maintain a formal academic tone appropriate for exam preparation.

Do not:

- Add new examples or explanations
- Simplify into tutorial-style teaching
- Include material from outside the provided lecture or textbook content

Below is a **faithful lecture scribe reconstruction** strictly based on the provided lecture slides. No additional explanations, examples, or interpretations have been introduced. All derivations, logical transitions, and results are reconstructed directly from the lecture material.

# CSE400 – Lecture 5 Scribe

## Bayes' Theorem, Random Variables, and Probability Mass Function

(Prepared strictly from Lecture 5 slides)

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### 1. Bayes' Theorem

#### 1.1 Weighted Average of Conditional Probabilities

##### Event Representation Using Set Relationships

Let  $A$  and  $B$  be events.

The event  $A$  can be expressed as:

$$A = AB \cup AB^c$$

This representation follows because for an outcome to belong to event  $A$ , it must satisfy one of the following:

- The outcome is in both  $A$  and  $B$
- The outcome is in  $A$  but not in  $B$

Thus, event  $A$  is decomposed into two disjoint components:

- $AB$
- $AB^c$

These two events are **mutually exclusive**.

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##### Derivation of Probability as Weighted Conditional Probabilities

Since  $AB$  and  $AB^c$  are mutually exclusive, by **Axiom 3 of probability**, we have:

$$Pr(A) = Pr(AB) + Pr(AB^c)$$

Using conditional probability definitions:

$$Pr(AB) = Pr(A | B)Pr(B)$$

$$Pr(AB^c) = Pr(A | B^c)Pr(B^c)$$

Since:

$$Pr(B^c) = 1 - Pr(B)$$

Substituting:

$$Pr(A) = Pr(A | B)Pr(B) + Pr(A | B^c)[1 - Pr(B)]$$

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## Interpretation

The probability of event  $A$  is expressed as a **weighted average of conditional probabilities**, where:

- $Pr(A | B)$  is weighted by  $Pr(B)$
  - $Pr(A | B^c)$  is weighted by  $Pr(B^c)$
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## 1.2 Learning by Example

### Example 3.1 (Part 1)

#### Given

Population is divided into two classes:

- Accident-prone persons
- Not accident-prone persons

Provided probabilities:

$$Pr(A_1 | A) = 0.4$$

$$Pr(A_1 | A^c) = 0.2$$

$$Pr(A) = 0.3$$

Let:

- $A_1$ : Policyholder has an accident within 1 year
  - $A$ : Policyholder is accident prone
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#### Required

Find:

$$Pr(A_1)$$

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**Solution**

Using the weighted conditional probability result:

$$Pr(A_1) = Pr(A_1 | A)Pr(A) + Pr(A_1 | A^c)Pr(A^c)$$

Substitute values:

$$Pr(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$Pr(A_1) = 0.12 + 0.14$$

$$Pr(A_1) = 0.26$$

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**Example 3.1 (Part 2)****Given**

A policyholder had an accident.

**Required**

Find:

$$Pr(A | A_1)$$

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**Solution**

Using conditional probability:

$$Pr(A | A_1) = \frac{Pr(AA_1)}{Pr(A_1)}$$

Using multiplication rule:

$$Pr(AA_1) = Pr(A)Pr(A_1 | A)$$

Thus:

$$Pr(A | A_1) = \frac{Pr(A)Pr(A_1 | A)}{Pr(A_1)}$$

Substitute values:

$$Pr(A | A_1) = \frac{(0.3)(0.4)}{0.26}$$

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$$Pr(A | A_1) = \frac{0.12}{0.26} = \frac{6}{13}$$

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## 1.3 Formal Introduction: Law of Total Probability

Let:

$$B_1, B_2, \dots, B_n$$

be mutually exclusive events such that:

$$\bigcup_{i=1}^n B_i = B$$

Exactly one of these events occurs.

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### Derivation

Event  $A$  can be expressed as:

$$A = \bigcup_{i=1}^n AB_i$$

Since the events  $AB_i$  are mutually exclusive:

$$Pr(A) = \sum_{i=1}^n Pr(AB_i)$$

Using conditional probability:

$$Pr(AB_i) = Pr(A | B_i)Pr(B_i)$$

Therefore:

$$Pr(A) = \sum_{i=1}^n Pr(A | B_i)Pr(B_i)$$

This is the **Law of Total Probability**.

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## 1.4 Derivation of Bayes' Formula

Using:

$$Pr(AB_i) = Pr(B_i | A)Pr(A)$$

and substituting into conditional probability:

$$Pr(B_i | A) = \frac{Pr(A | B_i)Pr(B_i)}{\sum_{j=1}^n Pr(A | B_j)Pr(B_j)}$$

This is known as the **Bayes Formula (Proposition 3.1)**.

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## Apriori and Posteriori Probabilities

- **Apriori Probability**

$$Pr(B_i)$$

Probability formed from presupposed models.

- **Posteriori Probability**

$$Pr(B_i | A)$$

Probability obtained after observing event  $A$ .

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## 1.5 Example 3.2

### Problem Description

Three cards exist:

- Card 1: Both sides red (RR)
- Card 2: Both sides black (BB)
- Card 3: One red and one black (RB)

One card is selected randomly and placed face up.

Let:

- $R$ : Upturned side is red

Find:

$$Pr(RB | R)$$

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## Solution

Let events:

- RR – all red card
- BB – all black card
- RB – red-black card

Each card chosen with probability  $\frac{1}{3}$

Using Bayes rule:

$$Pr(RB | R) = \frac{Pr(R | RB)Pr(RB)}{Pr(R | RR)Pr(RR) + Pr(R | RB)Pr(RB) + Pr(R | BB)Pr(BB)}$$

Substitute:

$$Pr(R | RB) = \frac{1}{2}$$

$$Pr(R | RR) = 1$$

$$Pr(R | BB) = 0$$

Thus:

$$Pr(RB | R) = \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)}$$

$$Pr(RB | R) = \frac{1/6}{1/3 + 1/6}$$

$$Pr(RB | R) = \frac{1/6}{1/2} = \frac{1}{3}$$

## 2. Random Variables

### 2.1 Motivation

Often interest lies in a function of outcomes rather than outcomes themselves.

Examples:

- Dice tossing → interest in sum
- Coin tossing → interest in number of heads

These functions are called **Random Variables**.

## 2.2 Formal Definition

A random variable  $X$  on sample space  $\Omega$  is a function:

$$X : \Omega \rightarrow \mathbb{R}$$

It assigns each sample point  $\omega \in \Omega$  a real number:

$$X(\omega)$$

## 2.3 Distribution of a Random Variable

Two components define the distribution:

1. The set of values taken by the random variable
2. The probabilities associated with those values

Let  $a$  be a possible value.

Event:

$$\{\omega \in \Omega : X(\omega) = a\}$$

is denoted:

$$X = a$$

Probability:

$$Pr[X = a]$$

The collection of these probabilities forms the distribution of  $X$ .

## Visualization

- X-axis: Values taken by random variable
- Bar height: Probability  $Pr[X = a]$

## 2.4 Discrete and Continuous Random Variables

### Discrete Random Variable

- Countable support
  - Probabilities assigned to single values
  - Each value has positive probability
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## Continuous Random Variable

- Uncountable support
  - Probability density function
  - Probabilities assigned to intervals
  - Each exact value has zero probability
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## 2.5 Random Variable Example

### Example: Three Coin Tosses

Let  $Y$  denote number of heads.

Possible values:

$$Y \in \{0, 1, 2, 3\}$$

Probabilities:

$$P(Y = 0) = \frac{1}{8}$$

$$P(Y = 1) = \frac{3}{8}$$

$$P(Y = 2) = \frac{3}{8}$$

$$P(Y = 3) = \frac{1}{8}$$

Since  $Y$  must take one of these values:

$$1 = \sum_{i=0}^3 P(Y = i)$$

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## 3. Probability Mass Function (PMF)

### 3.1 Definition

A random variable that takes at most a countable number of values is called **discrete**.

Let  $X$  be discrete with range:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The function:

$$p(x_k) = Pr(X = x_k)$$

is called the **Probability Mass Function (PMF)**.

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### 3.2 PMF Property

Since  $X$  must take one of its possible values:

$$\sum_k p(x_k) = 1$$

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### 3.3 PMF Example – Two Independent Coin Tosses

Sample space:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Let:

$$X = \text{Number of heads}$$

PMF:

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0 \text{ or } x = 2 \\ \frac{1}{2} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example probability:

$$P(X > 0) = P(X = 1) + P(X = 2)$$

$$= \frac{1}{2} + \frac{1}{4}$$

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$$= \frac{3}{4}$$

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### 3.4 PMF Functional Example

Given PMF:

$$p(i) = c \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

Since:

$$\sum_{i=0}^{\infty} p(i) = 1$$

We have:

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using:

$$e^\lambda = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Thus:

$$ce^\lambda = 1$$

$$c = e^{-\lambda}$$

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### Required Probabilities

$$P(X = 0) = p(0) = c$$

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

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### End of Lecture Topics

- Bayes' Theorem

- Law of Total Probability
  - Apriori and Posteriori Probabilities
  - Random Variables and Distributions
  - Probability Mass Function and Examples
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If you want, I can also format this into **exam-ready handwritten-style notes**, **LaTeX notes**, or **GitHub submission PDF format**.