CSE400 – Lecture 9 Scribe

Uniform, Exponential, Laplace, and Gamma Random Variables

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1. Types of Continuous Random Variables

This lecture focuses on **continuous random variables** and introduces specific distributions used in probability modeling. The lecture outline covers:

- Uniform Random Variable (including example)
- Exponential Random Variable (including example)
- Laplace Random Variable
- Gamma Random Variable (including graph, special cases, and examples)

These distributions are presented through their **probability density functions (PDFs)**, **cumulative distribution functions (CDFs)**, graphical interpretation, and example problems.

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2. Uniform Random Variable

2.1 Definition and Conditions

A random variable X is said to be uniformly distributed over an interval $[a, b]$ if its PDF is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{elsewhere} \end{cases}$$

The corresponding CDF is:

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

Conditions:

- The distribution is defined only over the interval $[a, b]$.
- Probability density is constant across the interval.
- Outside the interval, the probability density is zero.

2.2 Graphical Interpretation

- **PDF Graph:**
  The density function is constant at height $\frac{1}{b-a}$ between a and b, forming a rectangular shape.

- **CDF Graph:**
  The cumulative distribution function increases linearly from 0 at $x = a$ to 1 at $x = b$.

2.3 Example 1: Uniform Random Variable (Sinusoid Phase)

Problem Statement

The phase of a sinusoid Θ is uniformly distributed over $[0, 2\pi]$ with PDF:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

General Probability Property

For a uniform random variable over $[a, b]$:

$$\Pr(a < \Theta < b) = \frac{b-a}{2\pi}$$

(a) Compute $\Pr(\Theta > 3\pi/4)$

Using interval probability:

$$\Pr(\Theta > 3\pi/4) = \frac{2\pi - 3\pi/4}{2\pi} = \frac{5}{8}$$

(b) Compute $\Pr(\Theta < \pi \mid \Theta > 3\pi/4)$

- $A = \{\Theta < \pi\}$
- $B = \{\Theta > 3\pi/4\}$

Using conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Compute intersection:

$$\Pr\left(3\pi/4 < \Theta < \pi\right) = \frac{\pi/4}{2\pi} = \frac{1}{8}$$

Given:

$$\Pr(B) = \frac{5}{8}$$

Therefore:

$$\Pr(0 < \Theta < \pi \mid \Theta > 3\pi/4) = \frac{1/8}{5/8} = \frac{1}{5}$$

(c) Compute $\Pr(\cos\Theta < 1/2)$
Solve inequality:

$$\cos\Theta = \frac{1}{2} \Rightarrow \Theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Thus:

$$\cos\Theta < \frac{1}{2} \quad \text{for} \quad \frac{\pi}{3} < \Theta < \frac{5\pi}{3}$$

Probability:

$$\Pr\left(\cos\Theta < \frac{1}{2}\right) = \frac{5\pi/3 - \pi/3}{2\pi} = \frac{4\pi/3}{2\pi} = \frac{2}{3}$$

2.4 Applications of Uniform Distribution

The lecture lists the following applications:

1. Phase of a sinusoidal signal when phase angles between 0 and 2π are equally likely.
2. Random number generation between 0 and 1 for simulation.
3. Arrival time of a user within a known time window with no time preference.

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3. Exponential Random Variable

3.1 Definition and Conditions

The exponential random variable is defined for parameter $b > 0$.
PDF:

$$f_X(x) = \frac{1}{b}\exp\left(-\frac{x}{b}\right)u(x)$$

CDF:

$$F_X(x) = \left[1 - \exp\left(-\frac{x}{b}\right)\right]u(x)$$

Where $u(x)$ denotes the unit step function.
Condition:

- The parameter b must be strictly positive.

3.2 Graphical Interpretation

- **PDF Graph:**
  Starts at maximum value at $x = 0$ and decreases exponentially as x increases.

- **CDF Graph:**
  Starts at zero and asymptotically approaches 1 as $x \to \infty$.

3.3 Example 2: Exponential Random Variable

Problem Statement

Let X be exponential with PDF:

$$f_X(x) = e^{-x}u(x)$$

(a) Compute $\Pr(3X < 5)$

Rewrite the inequality:

$$3X < 5 \Rightarrow X < \frac{5}{3}$$

Thus:

$$\Pr(3X < 5) = \Pr\left(X < \frac{5}{3}\right)$$

Using the CDF:

$$F_X(x) = 1 - e^{-x}$$

Therefore:

$$\Pr(3X < 5) = 1 - e^{-5/3}$$

(b) Generalization

For arbitrary constant y:

$$\Pr(3X < y) = \Pr\left(X < \frac{y}{3}\right) = 1 - e^{-y/3}$$

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4. Laplace Random Variable

The lecture includes the Laplace random variable as part of continuous distributions. It is presented as a distribution with defined properties and interpretations.

(No further definitions, derivations, or examples are provided in the visible lecture content.)

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5. Gamma Random Variable

The lecture includes the Gamma random variable as a continuous distribution with:

- Graphical representation
- Special cases
- Example discussion
- Homework problem reference

(No further explicit derivations or mathematical definitions appear in the provided lecture content.)

Logical Flow of Lecture

The lecture progresses through continuous random variables in the following structured manner:

1. Introduces categories of continuous random variables.
2. Develops Uniform distribution:
   - Formal definitions
   - Graphical interpretation

1. Introduces categories of continuous random variables.
2. Develops Uniform distribution:
   - Formal definitions
   - Graphical interpretation
   - Detailed example with probability calculations
   - Applications
3. Introduces Exponential distribution:
   - Definitions and parameter assumptions
   - Graphical interpretation
   - Worked example and generalization
4. Introduces Laplace distribution
5. Introduces Gamma distribution with graphical and example-based discussion