

Lecture 5 Scribe CSE 400



L5_A_S1.pdf

PDF

Role: Act as a professional Lecture Scribe for CSE 400: Fundamentals of Probability in Computing. Your task is to generate a comprehensive, exam-ready lecture scribe for Lecture 5: Bayes' Theorem, Random Variables, and PMF, based strictly on the provided PowerPoint (L5_A_S1.pdf).
Mandatory Chain-of-Thought (CoT) Instructions: Step-by-Step Reasoning: While writing the scribe, you must reason step-by-step through every definition, assumption, theorem derivation (specifically the Law of Total Probability), and example solution exactly as presented in the context.
Explicit Logical Flow: You must include these reasoning steps and "thoughts" within the final document. The logical dependencies between concepts must be explicit so a student can follow the full derivation during exam revision.
Context Constraint: All reasoning, examples, and content must come only from the provided lecture materials. Do not introduce outside intuition, external examples, or "common knowledge" not found in the PDF.
Exam Purpose: This scribe is strictly intended as a primary reference for a closed-notes, reading-based exam.
Specific Content & Formatting Requirements:
Header & Objective: Begin the scribe with a section titled 'Objective' that states: "This scribe serves as an academic reference for exam preparation for Lecture 5."
Detailed Derivations: When introducing the Law of Total Probability and Bayes' Theorem, show the intermediate steps (e.g., partitioning the sample space, applying Axiom 3, and using the definition of conditional probability).
Variable Definitions: In the 'Three-Card Problem' example, explicitly define the events/card types (e.g., RR for Red-Red, BB for Blue-Blue, RB for Red-Blue) at the very start of the calculation.
Mathematical Accuracy: Ensure the Probability Mass Function (PMF) properties are rendered using proper LaTeX. Specifically, ensure the property that all probabilities must sum to 1 is written

as: $\sum_{i=1}^{\infty} p(x_i) = 1$. Tone: Maintain a formal, academic, and disciplined tone throughout.

CSE 400: Fundamentals of Probability in Computing

Lecture 5 Scribe: Bayes' Theorem, Random Variables, and Probability Mass Function

Objective

This scribe serves as an academic reference for exam preparation for Lecture 5.

All definitions, derivations, examples, and explanations below are reconstructed strictly from the lecture slides of L5_A_S1.pdf .

The presentation follows an explicit logical flow so that each result can be reproduced during a closed-notes, reading-based examination.

1. Bayes' Theorem as a Weighted Average of Conditional Probabilities

1.1 Decomposition of an Event

Let A and B be two events.

Any outcome that belongs to A must satisfy **one and only one** of the following:

- it lies in both A and B ,
- or it lies in A and not in B .

Hence, we write:

$$A = (A \cap B) \cup (A \cap B^c)$$

The events $A \cap B$ and $A \cap B^c$ are **mutually exclusive**.

1.2 Application of Axiom 3

By **Axiom 3 of Probability** (additivity for mutually exclusive events),

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c)$$

1.3 Use of Conditional Probability

By definition of conditional probability:

$$\Pr(A \cap B) = \Pr(A \mid B) \Pr(B)$$

$$\Pr(A \cap B^c) = \Pr(A \mid B^c) \Pr(B^c)$$

Substituting:

$$\Pr(A) = \Pr(A \mid B) \Pr(B) + \Pr(A \mid B^c) [1 - \Pr(B)]$$

1.4 Interpretation

The probability of event A is a **weighted average** of:

- $\Pr(A \mid B)$,
- $\Pr(A \mid B^c)$,

where the weights are $\Pr(B)$ and $\Pr(B^c)$, respectively .

2. Learning by Example: Insurance Policyholder Problem

2.1 Problem Statement (Example 3.1 – Part 1)

An insurance company classifies people into:

- **Accident prone**
- **Not accident prone**

Given:

- $\Pr(\text{Accident within a year} \mid \text{Accident prone}) = 0.4$
- $\Pr(\text{Accident within a year} \mid \text{Not accident prone}) = 0.2$
- $\Pr(\text{Accident prone}) = 0.3$

Find the probability that a new policyholder has an accident within one year.

2.2 Event Definitions

Let:

- A_1 : policyholder has an accident within one year
 - A : policyholder is accident prone
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2.3 Step-by-Step Solution

Condition on whether the policyholder is accident prone:

$$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$

Substitute given values:

$$\Pr(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$\Pr(A_1) = 0.12 + 0.14 = 0.26$$

2.4 Follow-up Question (Part 2)

Given that a policyholder **did** have an accident within a year, find the probability that the person is accident prone.

2.5 Bayes' Formula Application

We compute:

$$\Pr(A \mid A_1) = \frac{\Pr(A \cap A_1)}{\Pr(A_1)}$$

Using conditional probability:

$$\Pr(A \cap A_1) = \Pr(A)\Pr(A_1 \mid A)$$

Thus,

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26} = \frac{6}{13}$$

3. Formal Introduction: Law of Total Probability

3.1 Partition of the Sample Space

Let B_1, B_2, \dots, B_n be:

- mutually exclusive,
 - exhaustive events (they form a partition of the sample space),
 - with $\Pr(B_i) > 0$.
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3.2 Derivation

Since:

$$A = \bigcup_{i=1}^n (A \cap B_i)$$

and the intersections are mutually exclusive, Axiom 3 gives:

$$\Pr(A) = \sum_{i=1}^n \Pr(A \cap B_i)$$

Using conditional probability:

$$\boxed{\Pr(A) = \sum_{i=1}^n \Pr(A \mid B_i) \Pr(B_i)}$$

This is the **Law of Total Probability**.

4. Bayes' Theorem (Proposition 3.1)

Starting from:

$$\Pr(A \cap B_i) = \Pr(B_i \mid A) \Pr(A)$$

Substitute $\Pr(A)$ from the Law of Total Probability:

$$\boxed{\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_{j=1}^n \Pr(A \mid B_j) \Pr(B_j)}}$$

- $\Pr(B_i)$: a priori probability
 - $\Pr(B_i \mid A)$: a posteriori probability
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5. Bayes' Theorem Example: Three-Card Problem

5.1 Card Definitions (Explicit)

- **RR:** card with both sides red
- **BB:** card with both sides black
- **RB:** card with one red and one black side

Let:

- R : event that the upper side is red

Each card is equally likely:

$$\Pr(RR) = \Pr(BB) = \Pr(RB) = \frac{1}{3}$$

5.2 Required Probability

Find:

$$\Pr(RB \mid R)$$

5.3 Step-by-Step Calculation

Using Bayes' formula:

$$\Pr(RB \mid R) = \frac{\Pr(R \mid RB) \Pr(RB)}{\Pr(R \mid RR) \Pr(RR) + \Pr(R \mid RB) \Pr(RB) + \Pr(R \mid BB) \Pr(BB)}$$

Substitute:

$$= \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)}$$

Simplify:

$$= \frac{1/6}{1/2} = \frac{1}{3}$$

6. Random Variables

6.1 Motivation

In many experiments, interest lies in a **numerical function of outcomes**, not the outcomes themselves.

Examples from lecture:

- sum of two dice,
 - number of heads in coin tosses.
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6.2 Definition

A **random variable** is a **real-valued function** defined on the sample space.

- Values are determined by outcomes.
 - Probabilities are assigned to these values .
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6.3 Example: Three Coin Tosses

Let Y = number of heads.

Possible values:

$$Y \in \{0, 1, 2, 3\}$$

Probabilities:

$$\Pr(Y = 0) = \frac{1}{8}, \quad \Pr(Y = 1) = \frac{3}{8}, \quad \Pr(Y = 2) = \frac{3}{8}, \quad \Pr(Y = 3) = \frac{1}{8}$$

Since Y must take one of these values:

$$\sum \Pr(Y = y) = 1$$

7. Probability Mass Function (PMF)

7.1 Discrete Random Variable

A random variable that takes **countably many values** is called **discrete**.

7.2 Definition of PMF

Let X be a discrete random variable with range:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The **Probability Mass Function (PMF)** is defined as:

$$p(x_i) = \Pr(X = x_i)$$

7.3 PMF Properties

1. $p(x_i) \geq 0$ for all i
- 2.

$$\boxed{\sum_{i=1}^{\infty} p(x_i) = 1}$$

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End of Lecture 5 Scribe

This document is designed to be sufficient, precise, and exam-ready for CSE 400 Lecture 5.