CSE400 – Fundamentals of Probability in Computing

Lecture 5 Scribe: Bayes' Theorem, Random Variables, and Probability Mass Function

1 Bayes' Theorem

1.1 Weighted Average of Conditional Probabilities

Let A and B be events.

We may express event A as:

$$A = AB \cup AB^c$$

since an outcome in A must either:

be in both A and B, or

be in A but not in B.

The events AB and AB^c are mutually exclusive. By Axiom 3 of Probability:

$$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

Using the definition of conditional probability:

$$\Pr(AB) = \Pr(A \mid B)\Pr(B)$$

$$\Pr(AB^c) = \Pr(A \mid B^c)\Pr(B^c)$$

Hence:

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

Interpretation: The probability of event A is a weighted average of conditional probabilities, where the weights are the probabilities of the conditioning events.

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2 Bayes' Theorem – Learning by Example

Example 3.1 (Part 1)

An insurance company classifies people as:

accident prone

not accident prone

Given:

$$\Pr(\text{accident} \mid \text{accident prone}) = 0.4$$

$$\Pr(\text{accident} \mid \text{not accident prone}) = 0.2$$

$$\Pr(\text{accident prone}) = 0.3$$

Let:
A_1: policyholder has an accident within a year
A: policyholder is accident prone
Then:

$$\Pr(A_1) = \Pr(A_1 \mid A) \Pr(A) + \Pr(A_1 \mid A^c) \Pr(A^c)$$

$$= (0.4)(0.3) + (0.2)(0.7) = 0.26$$

Example 3.1 (Part 2)

Given that a policyholder had an accident, find the probability that they are accident prone.

The required probability is:

$$\Pr(A \mid A_1) = \frac{\Pr(A_1 \cap A)}{\Pr(A_1)}$$

Using:

$$\Pr(A_1 \cap A) = \Pr(A) \Pr(A_1 \mid A)$$

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26} = \frac{6}{13}$$

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3 Law of Total Probability and Bayes Formula

Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events whose union is the sample space.

Law of Total Probability

$$\Pr(A) = \sum_i \Pr(A \mid B_i) \Pr(B_i)$$

Bayes Formula (Proposition 3.1)

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_j \Pr(A \mid B_j) \Pr(B_j)}$$

$\Pr(B_i)$: a priori probability
$\Pr(B_i \mid A)$: a posteriori probability

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4 Bayes Formula – Card Example (Example 3.2)

Three cards:

RR (both sides red)

BB (both sides black)

RB (one red, one black)

One card is randomly selected and placed on the ground.

Let:

R: upper side is red

RR, BB, RB: corresponding card events

We compute:

$$\Pr(RB \mid R) = \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)}$$

Substituting values:

$$= \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)} = \frac{1}{3}$$

5 Random Variables

Concept

A random variable is a real-valued function defined on a sample space.

Values depend on experimental outcomes.

Probabilities are assigned to these values.

Examples:

Sum of dice outcomes

Number of heads in coin tosses

The distribution of a random variable can be visualized using a bar diagram, where:

x-axis: possible values

bar height: $\Pr[X = a]$

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Example: Tossing 3 Fair Coins

Let Y = number of heads.

Possible values: $0, 1, 2, 3$

$$\Pr(Y=0)=\frac{1}{8}, \quad \Pr(Y=1)=\frac{3}{8}, \quad \Pr(Y=2)=\frac{3}{8}, \quad \Pr(Y=3)=\frac{1}{8}$$

Since Y must take one of these values:

$$\sum_y \Pr(Y=y)=1$$

6 Probability Mass Function (PMF)

A random variable is discrete if it can take at most a countable number of values.

Let X be a discrete random variable with range:

$$R_X=\{x_1, x_2, x_3, \ldots\}$$

The function:

$$p(x_k)=\Pr(X=x_k)$$

is called the Probability Mass Function (PMF) of X.

Since X must take one of its possible values:

$$\sum_k p(x_k)=1$$

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7 PMF Example

Given:

$$p(i)=c\lambda^i, \quad i=0,1,2,\ldots$$

where $\lambda>0$.

Using normalization:

$$\sum_{i=0}^{\infty} c\lambda^i=1$$

Required:

$\Pr(X=0)$

$\Pr(X>2)$

(Computed directly from the PMF as shown in lecture slides.)