CSE400 — Fundamentals of Probability in Computing

Lecture 9: Uniform, Exponential, Laplace and Gamma Random Variables

(Prepared as an exam-oriented lecture scribe from the provided lecture slides)

1. Lecture Outline

According to the outline slides, the lecture covers:

- Types of Continuous Random Variables
- Uniform Random Variable — Example
- Exponential Random Variable — Example
- Laplace Random Variable — Example
- Gamma Random Variable
  - Graph and Special Cases
  - Example
  - Homework Problem
- Problem Solving
- In-class Activity: Gaussian Density Estimation

Only the material appearing in the provided slides is included in this scribe.
L9_S1_A

2. Types of Continuous Random Variables

2.1 Uniform Random Variable

Definition (PDF)
A continuous random variable X is uniformly distributed over the interval $[a, b]$ if its probability density function (PDF) is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b, \\ 0, & \text{elsewhere.} \end{cases}$$

This definition specifies:

- The density is constant over the interval $[a, b]$.
- The density is zero outside the interval.

Definition (CDF)
The cumulative distribution function (CDF) is given by

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x < b, \\ 1, & x \geq b. \end{cases}$$

The slides show the corresponding PDF as a rectangle over $[a, b]$ and the CDF as a linear increase between a and b (Figure 3.8).
L9_S1_A

3. Example #1 — Uniform Random Variable

Problem Statement
The phase of a sinusoid, θ, is uniformly distributed over $[0, 2\pi)$. The PDF is

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Tasks:
(a) Find $\Pr(\theta > 3\pi/4)$
(b) Find $\Pr(\theta < \pi \mid \theta > 3\pi/4)$
(c) Find $\Pr(\cos\theta < 1/2)$

Given Property for Uniform RV
For a uniform random variable on $[a, b]$,

$$\Pr(a < \theta < b) = \frac{b-a}{2\pi}.$$

This relation is used directly in the solution.

(a) Compute $\Pr(\theta > 3\pi/4)$
Step-by-step as shown in the slides:

- The interval of interest is from $3\pi/4$ to 2π.
- Length of interval: $2\pi - \frac{3\pi}{4}$
- Probability equals interval length divided by total length:

$$\Pr(\theta > 3\pi/4) = \frac{2\pi - 3\pi/4}{2\pi} = \frac{5}{8}.$$

(b) Compute $\Pr(\theta < \pi \mid \theta > 3\pi/4)$
The slides apply the conditional probability formula:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Steps:

- Event B: $\theta > 3\pi/4 \implies \Pr(B) = \frac{5}{8}$.
- Event $A \cap B$: $3\pi/4 < \theta < \pi \implies \Pr(3\pi/4 < \theta < \pi) = \frac{\pi - 3\pi/4}{2\pi} = \frac{1}{8}$.
- Therefore, $\Pr(\theta < \pi \mid \theta > 3\pi/4) = \frac{1/8}{5/8} = \frac{1}{5}$.

(c) Compute $\Pr(\cos\theta < 1/2)$
Steps shown in the slides:

- Solve $\cos\theta = \frac{1}{2}$. This occurs at $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.
- From the diagram in the solution, $\cos\theta < 1/2$ for $\frac{\pi}{3} < \theta < \frac{5\pi}{3}$.
- Probability: $\Pr(\cos\theta < 1/2) = \frac{5\pi/3 - \pi/3}{2\pi} = \frac{4\pi/3}{2\pi} = \frac{2}{3}$.

Applications of Uniform Random Variable
As listed in the slides:

- Phase of a sinusoidal signal when all phase angles between 0 and 2π are equally likely.
- Random number generated by a computer between 0 and 1 for simulations.
- Arrival time of a user within a known time window assuming no time preference.

L9_S1_A

4. Exponential Random Variable

Definition
The exponential random variable has the following PDF and CDF (for any $b > 0$):

PDF
$f_X(x) = \frac{1}{b} \exp(-\frac{x}{b})u(x)$

CDF
$F_X(x) = [1 - \exp(-\frac{x}{b})]u(x)$
where $u(x)$ is the unit step function.

The slides also include plots showing:

- A decreasing exponential PDF.
- A monotonically increasing CDF approaching 1.

L9_S1_A

5. Example #2 — Exponential Random Variable

Problem Statement
Let X be an exponential random variable with PDF

$$f_X(x) = e^{-x}u(x).$$

Tasks:
(a) Find $\Pr(3X < 5)$.
(b) Generalize the answer to find $\Pr(3X < y)$ for an arbitrary constant y.
(Only the problem statements are shown in the slides; solution steps are not provided in the visible material.)