

CSE400 – Lecture 5 Scribe

Bayes' Theorem, Random Variables, and Probability Mass Function

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1. Bayes' Theorem

1.1 Weighted Average of Conditional Probabilities

Event Representation Using Set Relationships

Let A and B be events.

The event A can be expressed as:

$$A = AB \cup AB^c$$

This representation follows because for an outcome to belong to event A , it must satisfy one of the following:

- The outcome is in both A and B
- The outcome is in A but not in B

Thus, event A is decomposed into two disjoint components:

- AB
- AB^c

These two events are **mutually exclusive**.

Derivation of Probability as Weighted Conditional Probabilities

Since AB and AB^c are mutually exclusive, by **Axiom 3 of probability**, we have:

$$P(A) = P(AB) + P(AB^c)$$

Using conditional probability definitions:

$$P(AB) = P(A | B)P(B)$$

$$P(AB^c) = P(A \mid B^c)P(B^c)$$

Since:

$$P(B^c) = 1 - P(B)$$

Substituting:

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)[1 - P(B)]$$

Interpretation

The probability of event A is expressed as a **weighted average of conditional probabilities**, where:

- $P(A \mid B)$ is weighted by $P(B)$
- $P(A \mid B^c)$ is weighted by $P(B^c)$

1.2 Learning by Example

Example 3.1 (Part 1)

Given

Population is divided into two classes:

- Accident-prone persons
- Not accident-prone persons

Provided probabilities:

$$P(A_1 \mid A) = 0.4$$

$$P(A_1 \mid A^c) = 0.2$$

$$P(A) = 0.3$$

Let:

- A_1 : Policyholder has an accident within 1 year
- A : Policyholder is accident prone

Required

Find:

$$P(A_1)$$

Solution

Using the weighted conditional probability result:

$$P(A_1) = P(A_1 | A)P(A) + P(A_1 | A^c)P(A^c)$$

Substitute values:

$$P(A_1) = (0.4)(0.3) + (0.2)(0.7)$$

$$P(A_1) = 0.12 + 0.14$$

$$P(A_1) = 0.26$$

Example 3.1 (Part 2)**Given**

A policyholder had an accident.

Required

Find:

$$P(A | A_1)$$

Solution

Using conditional probability:

$$P(A | A_1) = \frac{P(AA_1)}{P(A_1)}$$

Using multiplication rule:

$$P(AA_1) = P(A)P(A_1 | A)$$

Thus:

$$P(A | A_1) = \frac{P(A)P(A_1 | A)}{P(A_1)}$$

Substitute values:

$$P(A | A_1) = \frac{(0.3)(0.4)}{0.26}$$

$$P(A | A_1) = \frac{0.12}{0.26}$$

$$P(A | A_1) = \frac{6}{13}$$

1.3 Formal Introduction: Law of Total Probability

Let:

$$B_1, B_2, \dots, B_n$$

be mutually exclusive events such that:

$$\bigcup_{i=1}^n B_i = B$$

Exactly one of these events occurs.

Derivation

Event A can be expressed as:

$$A = \bigcup_{i=1}^n AB_i$$

Since the events AB_i are mutually exclusive:

$$P(A) = \sum_{i=1}^n P(AB_i)$$

Using conditional probability:

$$P(AB_i) = P(A \mid B_i)P(B_i)$$

Therefore:

$$P(A) = \sum_{i=1}^n P(A \mid B_i)P(B_i)$$

This is the **Law of Total Probability**.

1.4 Derivation of Bayes' Formula

Using:

$$P(AB_i) = P(B_i \mid A)P(A)$$

and substituting into conditional probability:

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^n P(A \mid B_j)P(B_j)}$$

This is known as the **Bayes Formula (Proposition 3.1)**.

Apriori and Posteriori Probabilities

- **Apriori Probability**

$$P(B_i)$$

Probability formed from presupposed models.

- **Posteriori Probability**

$$P(B_i \mid A)$$

Probability obtained after observing event A .

1.5 Example 3.2

Problem Description

Three cards exist:

- Card 1: Both sides red (RR)
- Card 2: Both sides black (BB)
- Card 3: One red and one black (RB)

One card is selected randomly and placed face up.

Let:

- R : Upturned side is red

Find:

$$P(RB \mid R)$$

Solution

Let events:

- RR – all red card
- BB – all black card
- RB – red-black card

Each card chosen with probability $\frac{1}{3}$.

Using Bayes rule:

$$P(RB \mid R) = \frac{P(R \mid RB)P(RB)}{P(R \mid RR)P(RR) + P(R \mid RB)P(RB) + P(R \mid BB)P(BB)}$$

Substitute:

$$P(R \mid RB) = \frac{1}{2}$$

$$P(R \mid RR) = 1$$

$$P(R \mid BB) = 0$$

Thus:

$$P(RB \mid R) = \frac{(\frac{1}{2})(\frac{1}{3})}{(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3}) + (0)(\frac{1}{3})}$$

$$P(RB \mid R) = \frac{1/6}{1/3 + 1/6}$$

$$P(RB \mid R) = \frac{1/6}{1/2}$$

$$P(RB \mid R) = \frac{1}{3}$$

2. Random Variables

2.1 Motivation

Often interest lies in a function of outcomes rather than outcomes themselves.

Examples:

- Dice tossing \rightarrow interest in sum
- Coin tossing \rightarrow interest in number of heads

These functions are called **Random Variables**.

2.2 Formal Definition

A random variable X on sample space Ω is a function:

$$X : \Omega \rightarrow \mathbb{R}$$

It assigns each sample point $\omega \in \Omega$ a real number:

$$X(\omega)$$

2.3 Distribution of a Random Variable

Two components define the distribution:

1. The set of values taken by the random variable
2. The probabilities associated with those values

Let a be a possible value.

Event:

$$\{\omega \in \Omega : X(\omega) = a\}$$

is denoted:

$$X = a$$

Probability:

$$P[X = a]$$

The collection of these probabilities forms the distribution of X .

Visualization

- X-axis: Values taken by random variable
 - Bar height: Probability $P[X = a]$
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2.4 Discrete and Continuous Random Variables

Discrete Random Variable

- Countable support
 - Probabilities assigned to single values
 - Each value has positive probability
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Continuous Random Variable

- Uncountable support

- Probability density function
 - Probabilities assigned to intervals
 - Each exact value has zero probability
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2.5 Random Variable Example

Example: Three Coin Tosses

Let Y denote number of heads.

Possible values:

$$Y \in \{0, 1, 2, 3\}$$

Probabilities:

$$P(Y = 0) = \frac{1}{8}$$

$$P(Y = 1) = \frac{3}{8}$$

$$P(Y = 2) = \frac{3}{8}$$

$$P(Y = 3) = \frac{1}{8}$$

Since Y must take one of these values:

$$1 = \sum_{i=0}^3 P(Y = i)$$

3. Probability Mass Function (PMF)

3.1 Definition

A random variable that takes at most a countable number of values is called **discrete**.

Let X be discrete with range:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The function:

$$p(x_k) = P(X = x_k)$$

is called the **Probability Mass Function (PMF)**.

3.2 PMF Property

Since X must take one of its possible values:

$$\sum_k p(x_k) = 1$$

3.3 PMF Example – Two Independent Coin Tosses

Sample space:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Let:

$$X = \text{Number of heads}$$

PMF:

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = 0 \text{ or } x = 2 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Example probability:

$$P(X > 0) = P(X = 1) + P(X = 2)$$

$$P(X > 0) = \frac{1}{2} + \frac{1}{4}$$

$$P(X > 0) = \frac{3}{4}$$

3.4 PMF Functional Example

Given PMF:

$$p(i) = c \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

Since:

$$\sum_{i=0}^{\infty} p(i) = 1$$

We have:

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using:

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Thus:

$$ce^{\lambda} = 1$$

$$c = e^{-\lambda}$$

Required Probabilities

$$P(X = 0) = p(0) = c$$

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

End of Lecture Topics

- Bayes' Theorem
- Law of Total Probability
- Apriori and Posteriori Probabilities
- Random Variables and Distributions
- Probability Mass Function and Examples