CSE400 – Fundamentals of Probability in Computing

Lecture 5: Bayes' Theorem, Random Variables, and Probability Mass Function

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1 Bayes' Theorem

1.1 Weighted Average of Conditional Probabilities

Let A and B be events. Event A can be expressed as:

$$A = AB \cup AB^c$$

because for an outcome to be in A, it must either be in both A and B, or in A and not in B.

The events AB and AB^c are mutually exclusive. Hence, by Axiom 3:

$$\Pr(A) = \Pr(AB) + \Pr(AB^c)$$

Using the definition of conditional probability:

$$\Pr(AB) = \Pr(A \mid B)\Pr(B)$$

$$\Pr(AB^c) = \Pr(A \mid B^c)\Pr(B^c)$$

Thus,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)[1 - \Pr(B)]$$

Conclusion: The probability of event A is a *weighted average* of the conditional probabilities, where the weights are the probabilities of the conditioning events.

1.2 Learning by Example: Example 3.1 (Part 1)

An insurance company classifies people as:

- Accident prone
- Not accident prone

Given:

$$\Pr(\text{Accident} \mid \text{Accident prone}) = 0.4$$

$$\Pr(\text{Accident} \mid \text{Not accident prone}) = 0.2$$

$$\Pr(\text{Accident prone}) = 0.3$$

Let:

$$A_1 = \text{event that policyholder has an accident within a year}$$

$$A = \text{event that policyholder is accident prone}$$

Using conditioning:

$$\Pr(A_1) = \Pr(A_1 \mid A)\Pr(A) + \Pr(A_1 \mid A^c)\Pr(A^c)$$

Substituting values:

$$\Pr(A_1) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

1.3 Learning by Example: Example 3.1 (Part 2)

Given that a policyholder has an accident, find the probability that they are accident prone.

We compute:

$$\Pr(A \mid A_1) = \frac{\Pr(AA_1)}{\Pr(A_1)}$$

Using:

$$\Pr(AA_1) = \Pr(A)\Pr(A_1 \mid A)$$

Thus,

$$\Pr(A \mid A_1) = \frac{(0.3)(0.4)}{0.26} = \frac{6}{13}$$

2 Formal Introduction

2.1 Law of Total Probability

Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive events such that:

$$\bigcup_{i=1}^{n} B_i = B$$

Since exactly one of the events must occur, we write:

$$A = \bigcup_{i=1}^{n} AB_i$$

Using mutual exclusivity:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(AB_i)$$

Applying conditional probability:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \mid B_i)\Pr(B_i)$$

This is known as the **Law of Total Probability**.

2.2 Bayes Formula

Using:

$$\Pr(AB_i) = \Pr(B_i \mid A)\Pr(A)$$

We obtain:

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{n}\Pr(A \mid B_j)\Pr(B_j)}$$

This is known as the **Bayes Formula (Proposition 3.1)**.
Here:

- $\Pr(B_i)$ is the **a priori probability**
- $\Pr(B_i \mid A)$ is the **posteriori probability**

3 Bayes Formula: Example 3.2

Three cards:

- RR (both sides red)
- BB (both sides black)
- RB (one red, one black)

Let:

$$R = \text{event that upturned side is red}$$

We want:

$$\Pr(RB \mid R)$$

Using Bayes formula:

$$\Pr(RB \mid R) = \frac{\Pr(R \mid RB)\Pr(RB)}{\Pr(R)}$$

Where:

$$\Pr(R) = \Pr(R \mid RR)\Pr(RR) + \Pr(R \mid RB)\Pr(RB) + \Pr(R \mid BB)\Pr(BB)$$

Substituting values:

$$\Pr(RB \mid R) = \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)} = \frac{1}{3}$$

4 Random Variables

4.1 Definition

A **random variable** is a real-valued function defined on a sample space Ω:

$$X : \Omega \to \mathbb{R}$$

It assigns a real number $X(\omega)$ to each outcome $\omega \in \Omega$.
In this lecture, we restrict attention to **discrete random variables**.

4.2 Distribution of a Random Variable

Let a be a value in the range of X. The event:

$$\{\omega \in \Omega : X(\omega) = a\}$$

is written as $X = a$.

The probability:

$$\Pr(X = a)$$

defines the distribution of X.

4.3 Example: Tossing 3 Fair Coins

Let Y be the number of heads.

$$\Pr(Y = 0) = \frac{1}{8}, \quad \Pr(Y = 1) = \frac{3}{8}, \quad \Pr(Y = 2) = \frac{3}{8}, \quad \Pr(Y = 3) = \frac{1}{8}$$

Since Y must take one of these values:

$$\sum_{i=0}^{3} \Pr(Y = i) = 1$$

5 Probability Mass Function (PMF)

5.1 Definition

A random variable that takes at most a countable number of values is called **discrete**.

Let X be a discrete random variable with range:

$$R_X = \{x_1, x_2, x_3, \dots\}$$

The function:

$$p_X(x) = \Pr(X = x)$$

is called the **Probability Mass Function (PMF)** of X.

Since X must take one of its values:

$$\sum_x p_X(x) = 1$$

5.2 PMF Example: Two Independent Coin Tosses

Let X be the number of heads.

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = 0 \text{ or } x = 2 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

5.3 PMF Example

Given:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

Since:

$$\sum_{i=0}^{\infty} p(i) = 1$$

We obtain:

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Using:

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Thus:

$$c = e^{-\lambda}$$

Hence:

$$\Pr(X = 0) = e^{-\lambda}$$

$$\Pr(X > 2) = 1 - \sum_{i=0}^{2} e^{-\lambda} \frac{\lambda^i}{i!}$$

End of Lecture 5