CSE 400: Fundamentals of Probability in Computing

Tutorial-1: Important Formulae and Theories

Q1. Division into Unlabeled Groups (Multinomial Counting)

(a) Total number of ways

Given:

- 20 distinct dishes
- Divided into 4 groups
- Each group contains 5 dishes
- Order within groups and order of groups does not matter

Using the multinomial formula adjusted for unlabeled groups:

$$\text{Total ways} = \frac{20!}{(5!)^4 4!}$$

(b) Probability all platters contain same cuisine

Since:

- Each cuisine has exactly 5 dishes.
- Only one specific grouping (by cuisine) satisfies the condition.
- Platters are unlabeled.

$$P = \frac{1}{\frac{20!}{(5!)^4 4!}} = \frac{(5!)^4 4!}{20!}$$

Q2. Uniform Distribution (Waiting Time)

Arrival is uniformly distributed over 30 minutes.

$$P = \frac{\text{Favorable time}}{30}$$

(a) Waiting ¡ 5 minutes
Total favorable time = 10 minutes.

$$P(\text{wait} < 5) = \frac{10}{30} = \frac{1}{3}$$

(b) Waiting ¿ 10 minutes
Total favorable time = 10 minutes.

$$P(\text{wait} > 10) = \frac{10}{30} = \frac{1}{3}$$

Q3. Conditional Probability & De Morgan's Law

Let L be the event "arrives late" and E be the event "leaves early."
Given: $P(L) = 0.15, P(E) = 0.25, P(L \cap E) = 0.08$.

- $P(E^c) = 1 - P(E) = 0.75$
- $P(L \cup E) = P(L) + P(E) - P(L \cap E) = 0.32$
- **De Morgan's Law:** $P(L^c \cap E^c) = 1 - P(L \cup E) = 0.68$

Conditional probability:

$$P(L^c|E^c) = \frac{P(L^c \cap E^c)}{P(E^c)} = \frac{0.68}{0.75} \approx 0.907$$

Q4. Poisson Distribution ($\lambda = 0.2$)

Poisson PMF: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$
(a) $P(X = 0) = e^{-0.2} \approx 0.8187$
(b) $P(X \geq 2) = 1 - P(0) - P(1)$

$$P(1) = 0.2e^{-0.2} \approx 0.1637$$

$$P(X \geq 2) = 1 - 0.8187 - 0.1637 = 0.0176 \approx 0.0175$$

Q5. Poisson Distribution ($\lambda = 3.5$)

(a) $P(X \geq 2) = 1 - P(0) - P(1)$

- $P(0) = 0.0302$
- $P(1) = 0.1057$

$$P(X \geq 2) = 0.8641$$

(b) $P(X \leq 1) = P(0) + P(1) = 0.1359$

Q6. Poisson Distribution ($\lambda = 3$)

(a) $P(X \geq 3) = 1 - [P(0) + P(1) + P(2)]$

- $P(0) = 0.0498, P(1) = 0.1494, P(2) = 0.2240$

$$P(X \geq 3) = 0.5768$$

(b) Conditional Probability:

$$P(X \geq 3 | X \geq 1) = \frac{P(X \geq 3)}{P(X \geq 1)} = \frac{0.5768}{1 - 0.0498} \approx 0.607$$

Q7. Gaussian Distribution & Q-Function

Given PDF: $f_X(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(x+3)^2}{8}\right)$
Comparing with $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$, we identify:

$$m = -3, \quad \sigma = 2$$

Results:

- $P(X \leq 0) = \Phi\left(\frac{0-(-3)}{2}\right) = 1 - Q(1.5)$
- $P(X > 4) = Q\left(\frac{4-(-3)}{2}\right) = Q(3.5)$
- $P(|X + 3| < 2) = 1 - 2Q(1)$
- $P(|X - 2| > 1) = 1 - Q(2) + Q(3)$

Q8. Binomial Model (Jury Problem)

Let θ be the probability an individual juror makes a correct decision.
If $\alpha = P(\text{guilty})$:

$$\text{Total Correct} = \alpha \sum_{i=8}^{12} \binom{12}{i} \theta^i (1-\theta)^{12-i} + (1-\alpha) \sum_{i=5}^{12} \binom{12}{i} \theta^i (1-\theta)^{12-i}$$

Q9. Poisson PMF from Normalization

Given $p(i) = \frac{c\lambda^i}{i!}$. Since $\sum_{i=0}^{\infty} p(i) = 1$ and $e^\lambda = \sum \frac{\lambda^i}{i!}$:

$$ce^\lambda = 1 \implies c = e^{-\lambda}$$

$$P(X > 2) = 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2}$$

CDF of Discrete Random Variable: $F(a) = \sum_{x \leq a} p(x)$. This is a step function with jumps at x_i equal to $p(x_i)$.

Q10. Binomial Reliability Model

(a) 5-component vs 3-component: A majority must function.

$$P_5 = 10p^3(1-p)^2 + 5p^4(1-p) + p^5$$

$$P_3 = 3p^2(1-p) + p^3$$

The 5-component system is better if $P_5 - P_3 > 0$, which reduces to:

$$3(p-1)^2(2p-1) > 0 \implies p > \frac{1}{2}$$

(b) General Case:

$$P_{2k+1} - P_{2k-1} = \binom{2k-1}{k} p^k (1-p)^k (2p-1)$$

Better performance $\iff p > \frac{1}{2}$.