① $AX = B \implies (L \cdot U)X = B \implies L(UX) = B \implies LY = B$

$A = LU$ $\qquad\qquad\qquad\qquad Y = UX$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

$$\implies \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{12}l_{21}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

$$\implies \begin{cases} u_{11} = 1 & u_{12} = 1 & u_{13} = 1 \\ l_{21} = 3 & u_{22} = -2 & u_{23} = -6 \\ l_{31} = 1 & l_{32} = \frac{3}{2} & u_{33} = 3 \end{cases}$$

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \implies \begin{cases} y_1 = 1 \\ y_2 = 2 \\ y_3 = 6 \end{cases}$$

$$UX = Y \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \implies \begin{cases} x_3 = 2 \\ x_2 = -7 \\ x_1 = 6 \end{cases}$$

$$\implies X = \begin{bmatrix} 2 \\ -7 \\ 6 \end{bmatrix}$$

② $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

$\xrightarrow{R_2 -= R_1} \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 -= R_2}$

$\begin{matrix} & & 2 & & 4 & 5 \\ & & \text{free} & & \text{free} & \text{free} \end{matrix}$

$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

pivot's columns

free

متغیرهای آزاد هستند. 5 , 4 , 2 متغیرهای سُتون های

③ $A^T = A^{-1} \implies$ Matrix is Orthogonal

$$A^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & a & b \end{bmatrix} \quad \text{(I)}$$

$A^{-1} = ?$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & b & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & b & 0 & 0 & 1 \end{array}\right]$$

$\xrightarrow{R_3 -= R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a & 0 & 1 & 0 \\ 0 & 0 & b-a & 0 & -1 & 1 \end{array}\right]$

$\xrightarrow{R_3 \,*= 1/(b-a)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/(b-a) & 1/(b-a) \end{array}\right]$

$\xrightarrow[R_2 \,*= \sqrt{2}]{R_2 -= (R_3 a)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dfrac{\sqrt{2}\,b}{b-a} & \dfrac{-\sqrt{2}\,a}{b-a} \\ 0 & 0 & 1 & 0 & -\dfrac{1}{b-a} & \dfrac{1}{b-a} \end{array}\right]$ $\quad$ (II)

$\text{(I)} \overset{?}{=} \text{(II)} \implies \begin{cases} \text{②③} & a = -\dfrac{1}{b-a} \\ \text{③③} & b = \dfrac{1}{b-a} \end{cases} \implies a = -b \overset{\text{②②}}{\implies} \dfrac{\sqrt{2}\,b}{2b} = \dfrac{1}{\sqrt{2}} \;\checkmark$

$\overset{\text{③③}}{\implies} \dfrac{1}{2b} = b \implies b^2 = \dfrac{1}{2}$

$$\implies \boxed{\begin{aligned} b &= -\tfrac{1}{\sqrt{2}} = -a \\ b &= \tfrac{1}{\sqrt{2}} = -a \end{aligned}}$$

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• برای ماتریس $A_{M \times N}$:

* رتبه ماتریس = بعد فضای ستونی = بعد فضای سطری = r = $rank(A)$

* در دستگاه $AX = B$ با m معادله، اگر متغیرهای سازگار، اگر رتبه A برابر r باشد در این صورت در جواب

دستگاه $n-r$ پارامتر وجود دارد . $dim(N(A)) = 5 - 3 = 2$

طبق این نکته اگر متغیرهای معادلات سازگار باشد همیشه بی شمار جواب داریم . (دو متغیر آزاد)

• فضای ستونی A :

$$A_{m \times n} = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad AX = b$$

$$b = x_1 A_1 + x_2 A_2 + x_3 A_3 + \cdots + x_n A_n \in C(A)$$

• فضای پوچی A : مجموعه‌ی همه‌ی جواب‌های $AX = 0$. $(N(A))$

اگر b = AX جواب های محدود داشته باشد که چرا با جواب ها مقایسه می‌کنیم وبایر باکی از نابر

برابر بشد نا درفضای ستونی A باشد،

اگر b = AX جواب های نامحدود داشته باشد باید ببینیم آیا b=فر جواب هست یا نیست باشر مثلا اگروابی

معادلهی صفحه است بایدچک کنیم آیا b در این صفحه است یا نر، جواب، بلی، وجود درفضای ستونی A است.

$$AX = b \Rightarrow \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$$
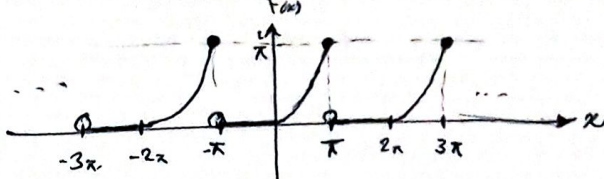
$$\begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 6x_1 + 5x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = C(A)$$

$$\begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix} \xrightarrow[\substack{R_3 -= R_1 \\ R_2 -= 3R_1}]{R_1 * = \frac{1}{2}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \ (rank = 2)$$

$$\begin{cases} 2x_1 + x_2 = 8 \\ 6x_1 + 5x_2 = 28 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases} \Rightarrow 2x_1 + 4x_2 = 14 \ \checkmark$$

بله b در فضای ستونی A هست

⑥ $f_{(x)} = \begin{cases} 0 & -\pi < x \leq 0 \\ x^2 & 0 < x \leq \pi \end{cases}$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{(x)} \, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x^2 \, dx \right] = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{6}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \cos(nx) \, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x^2 \cos(nx) \, dx \right]$

$= \frac{1}{\pi} \left[\frac{(x^2 n^2 - 2) \overset{0}{\sin(\pi n)} + 2xn \overset{(-1)^n}{\cos(\pi n)}}{n^3 \, n^2} \right]_0^{\pi} = \frac{2(-1)^n}{n^2}$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \sin(nx) \, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x^2 \sin(nx) \, dx \right]$

$= \frac{1}{\pi} \left[\frac{2n \overset{0}{\sin(\pi n)} + (2 - \pi^2 n^2) \overset{(-1)^n}{\cos(\pi n)} - 2}{n^3} \right] = \frac{(-1)^n (2 - \pi^2 n^2) - 2}{\pi n^3}$

$f_{(x)} = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{\pi^2}{6} + \sum_{1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \frac{(-1)^n(2 - \pi^2 n^2) - 2}{\pi n^3} \sin(nx)$

$\xrightarrow{x=0} f(0) \overset{0}{=} \frac{\pi^2}{6} + 2 \sum_{1}^{\infty} \frac{2(-1)^n}{n^2} \implies \boxed{\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots}$

$\xrightarrow{x=\pi} f(\pi) \overset{\pi^2}{=} \frac{\pi^2}{6} + 2 \sum_{1}^{\infty} \frac{(-1)^n 1}{n^2} (-1)^n \implies \frac{\pi^2}{12} = -1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \; \overset{\cdot \cdot}{\times}$

$f_{(x)} = \frac{\pi^2}{6} + \sum_{1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \frac{(-1)^n(2 - \pi^2 n^2) - 2}{\pi n^3} (\sin(nx))$

$f_{(-x)} = \frac{\pi^2}{6} + \sum_{1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) - \frac{(-1)^n(2 - \pi^2 n^2) - 2}{\pi n^3} (\sin(nx))$

$f_{(x)} + f_{(-x)} = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{2(-1)^n \cos(nx)}{n^2}$

$\xrightarrow{x=\pi} f_{(\pi)} + f_{(-\pi)} = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{1}{n^2}$

$\implies \pi^2 + 0 = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{1}{n^2} \implies \frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}$

$\implies \boxed{\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots}$

⑦

a) $f_{(x)} = x^2 |x-1|$ $-1 < x \leq 1 \implies f_{(x)} = -x^3 + x^2$ $-1 < x \leq 1$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{(x)} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} -x^3 + x^2 \, dx = \frac{1}{2\pi} \left[-\frac{x^4}{4} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$

$\underbrace{\qquad}_{\frac{2\pi^3}{3}}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \cos(nx) \, dx = \frac{1}{\pi} \int_{\pi}^{\pi} (x^2 - x^3) \cos(nx) \, dx$

$= \frac{1}{\pi} \left[\frac{(2x^2 n^2 - 4) \sin(\pi n) + 4 \pi n \cos(\pi n)}{n^3} \right]$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \sin(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 - x^3) \sin(nx) \, dx$

$= \frac{1}{\pi} \left[-\frac{(6\pi^2 n^2 - 12) \sin(\pi n) + (12\pi n - 2\pi^3 n^3) \cos(\pi n)}{n^4} \right]$

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b) $f_{(x)} = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{(x)} \, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x \, dx \right] = \frac{1}{2}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \cos(nx) \, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x \cos(nx) \, dx \right]$

$= \frac{1}{\pi} \left[\frac{\pi n \sin(\pi n) + \cos(\pi n) - 1}{n^2} \right]$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{(x)} \sin(nx) \, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} x \sin(nx) \, dx \right]$

$= \frac{1}{\pi} \left[\frac{\sin(\pi n) - \pi n \cos(\pi n)}{n^2} \right]$

(8)

a) $f(x) = \begin{cases} -x & -\pi < x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} -x\, dx + \int_{0}^{\pi} x\, dx \right] = \frac{\pi}{2}$

$\qquad\qquad\qquad\qquad\qquad\qquad \underset{\frac{\pi^2}{2}}{} \qquad \underset{\frac{\pi^2}{2}}{}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} -x \cos(nx)\, dx + \int_{0}^{\pi} x \cos(nx)\, dx \right]$

$\cancel{\cos(nx)}$

$\Longrightarrow \frac{2}{\pi} \cdot \left[\frac{\pi n \sin(n\pi) + \cos(\pi n) - 1}{n^2} \right] = \begin{cases} -\frac{4}{\pi n^2} & زوج \\ 0 & زوج \end{cases}$

$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} -x \sin(nx)\, dx + \int_{0}^{\pi} x \sin(nx)\, dx \right]$

$\sin(nx)$

$= \frac{1}{\pi} \left[0 \right] = 0$

$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos nx + b_n \sin(nx) = \frac{\pi}{2} - \frac{4 \cos x}{\pi (1^2)} - \frac{4 \cos 3x}{\pi (3^2)} - \frac{4 \cos 5x}{\pi (5^2)} + \cdots$

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b) $f(x) = \begin{cases} x & -\pi < x \leq 0 \\ \pi - x & 0 < x \leq \pi \end{cases}$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} x\, dx + \int_{0}^{\pi} (\pi - x)\, dx \right] = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad \underset{-\frac{\pi^2}{2}}{} \qquad \underset{\pi - \frac{\pi^2}{2}}{\underset{\frac{\pi^2}{2}}{}}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} x \cos(nx)\, dx + \int_{0}^{\pi} (\pi - x) \cos(nx)\, dx \right]$

$= \frac{2}{\pi} \left[\frac{\pi n \sin(n\pi) + \cos(\pi n) - 1}{n^2} + \frac{\pi \sin(\pi n)}{n} \right] = \begin{cases} -\frac{4}{\pi^2} & زوج \\ 0 & زوج \end{cases}$

$b_n = \frac{1}{\pi} \left[-\frac{\pi (\cos(\pi n) - 1)}{n} \right] = \frac{1 - \cos(\pi n)}{n} = \begin{cases} \frac{2}{n} & زوج \\ 0 & زوج \end{cases}$

$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \left(\frac{-4 \cos x}{\pi^2} + \frac{2}{1} \sin x \right) + \left(-\frac{4 \cos 3x}{\pi^2} + \frac{2}{3} \sin 3x \right) + \cdots$

$N(A):$ $\qquad$ $Ax = 0 \implies A^T A x = 0 \implies x \in N(A^T A) \implies \boxed{N(A) \subseteq N(A^T A)}$ Ⓘ

$A^T A x = 0 \implies x^T A^T A x = 0 \implies \underbrace{(Ax)^T (Ax) = 0}_{} \implies Ax = 0 \implies x \in N(A)$

$\longrightarrow y^T y = 0 \implies$ همهٔ مؤلفه‌های y و $\bar{y}$ $\implies \boxed{N(A^T A) \subseteq N(A)}$ Ⓘ

صفر باشند چون حاصل ضرب

داخلی بر صورت جمع مؤلفه‌ها به توان ۲ روی قطر اصلی است .

Ⓘ , Ⓘ $\implies$ $\boxed{N(A) = N(A^T A)}$

برای $A_{m×n}$

• نادرست ← اگر $n > m$ باشد مقدار مينيمم nullity برابربا $m - n$ است كه در اين مثال برابر 2 است نه 3. ☐

• نادرست ← برای $A_{m×n}$ اگر $m > n$ جمله درست است اما در غير اين حالت جمله درست نيست. ☐