Conditions on existence of resonances in Hamiltonian system

Consider an analytic Hamiltonian system
$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}},$
with n degrees of freedom, near its equilibrium at the origin $\mathbf{x} = \mathbf{y} = 0$.
The Hamilton function $H(\mathbf{x}, \mathbf{y})$ expands into a convergent power series

$$H(\mathbf{x}, \mathbf{y}) = \sum H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \tag{1}$$

with constant coefficients $H_{\mathbf{pq}}$, where $\mathbf{p}, \mathbf{q} \geq 0$, $|\mathbf{p}| + |\mathbf{q}| \geq 2$, and $|\mathbf{p}| = |p_1| + \dots + |p_n|$.

Linear approximation of the Hamiltonian phase flow is provided by system
$\dot{\mathbf{z}} = B\mathbf{z} \text{ with matrix } B = J \operatorname{Hess} H|_{\mathbf{x}=\mathbf{y}=0}$, where $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ and $\operatorname{Hess} H$ is the Hessian of function (1).

The eigenvalues of matrix B can be reordered is a such way that $\lambda_{j+n} = -\lambda_j$, $j = 1, \dots, n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ the set of *basic eigenvalues* of the linear system with Hamiltonian H_2. In the Hamiltonian case the characteristic polynomial is written in the form

$$f(\mu) = \mu^n + a_1 \mu^{n-1} + a_2 \mu^{n-2} + \dots + a_{n-1}\mu + a_n, \tag{2}$$

where $\mu = \lambda^2$.

According to Theorem 12 in [Bruno1972]: [1] in the case of semi-simple eigenvalues of quadratic form H_2 there exists a canonical formal transformation that reduces the Hamiltonian system to its *normal form* $\dot{\mathbf{u}} = \partial h / \partial \mathbf{v}$, $\dot{\mathbf{v}} = -\partial h / \partial \mathbf{u}$, given by the normalized Hamiltonian $h(\mathbf{u}, \mathbf{v})$

$$h(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum_{|\mathbf{p}|+|\mathbf{q}| \geqslant 2} h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}, \tag{3}$$

containing only the resonant terms $h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}$ satisfying the resonant equation

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{4}$$

Here $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = \sum\limits_{j=1}^{n} p_j \lambda_j$ is the scalar product.

The resonant equation (4) has two kinds of solutions, which correspond to two kinds of resonant terms in the normal form (3):

1. *Secular terms* of the form $h_{\mathbf{pp}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{p}}$, which are always present in the Hamiltonian normal form due to the special structure of the matrix B of the linearized system.
2. *Strictly resonant terms*, which correspond to nontrivial integer solutions of the equation

$$\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0. \tag{5}$$

Following [2, Ch. I, Sec. 3]: [2] we define *resonance multiplicity* $\mathfrak{k}$ as the number of linearly independent solutions $\mathbf{p} \in \mathbb{Z}^n$ to the equation (5), and the *resonance order* $\mathfrak{q} = \min |\mathbf{p}|$ by $\mathbf{p} \in \mathbb{Z}^n$, $\mathbf{p} \neq 0, \langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0.$
If the solution to the equation (5) contains only two eigenvalues, then such resonance is called a *two-frequency resonance*, if more than two, then it is called a *multifrequency resonance*.

Problem. Obtain conditions on the coefficients a_j, $j=1,\ldots,n$, of the polynomial (3) of degrees $n=3$ and $n=4$, under which the multifrequency resonance of multiplicity 1 of order 3 or order 4 takes place.

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1. A.D. Bruno Analytical form of differential equations (II) // *Trans. Moscow Math. Soc.*, 26:199--239, 1972. ↩

2. A.D. Bruno. *The Restricted 3-body Problem: Plane Periodic Orbits*. Walter de Gruyter, Berlin, 1994. ↩