Asian Options

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1 Introduction

In finance, options are contracts that allow, but not oblige, to do a transaction on a underlying asset. Typically, a European call option on the underlying S, with maturity T and strike K gives the right to his owner to buy at time T the underlying at price K, whereas its price on the market is S_T. The payoff of this kind of option is $(S_T - K)_+$. Notably, the payoff depends on S only through its final value.

Unlike the European options, Asian ones involve not only the value of the underlying at the term of the contract but its whole trajectory. At time t, a classical Asian call option is defined by an underlying asset S, a term of the contract T and a strike K. The payoff at T is defined by :

$$(\frac{1}{T-t}\int_t^T S_u du - K)_+ \tag{1}$$

Therefore, the classical Asian call option on a given underlying can be seen as an European call option on its average. The same can be observed with the Asian put option. We will assume $t = 0$ for now.

One can also define the floating strike Asian call (resp. put) option, where the strike is not known at the beginning of the contract as it is the value at term of the underlying. In this case, the payoff of the option is :

$$(\frac{1}{T}\int_0^T S_u du - S_T)_+ \tag{2}$$

At an intermediate time $t \in (0, T)$, the price of an Asian option of maturity T depends on S not only through S_t but $(S_{t'})_{0 \leq t' \leq t}$. It is a non-Markovian feature.

The aim of this project is to price numerically these products for an underlying S with a stochastic price. Assumptions are needed on its dynamic. We will work under the Black-Scholes framework, where there is a closed formula for the European calls and puts. In this model, the dynamic of S is :

$$dS_t = \mu S_t dt + \sigma dBt, \tag{3}$$

where B is a standard Brownian motion (under the classical probability $\mathbb{P}$), μ, σ are positive constants. The Black-Scholes model introduces a non-risky asset S^0, whose yield is the interest rate r :

$$dS_t^0 = rS_t^0 dt, \tag{4}$$

Two main approaches are studied : Monte Carlo simulations and Finite Differences. In each one, several methods will be implemented. We will keep the notations of this introduction along the document.

2 Monte Carlo

We study in this part different competitive MC methods to calculate the price of Asian call options, relying on the reference n°13 (B. Lapeyre and E. Temam).

We work under the risk-neutral probability $\mathbb{Q}$, so that the assets are martingales. We introduce the process W :

$$W_t = B_t + \frac{\mu - r}{\sigma} t \tag{5}$$

By Girsanov's theorem, it is a $\mathbb{Q}$-Brownian motion.

One can rewrite the dynamic of S (3) as follows :

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{6}$$

We can easy check that the solution of this SDE on the time interval $[0, T]$ is

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t\right) \tag{7}$$

At an instant t in $[0, T]$, the price of an Asian option with maturity T is

$$p(t, S, Z) = e^{-r(T-t)}\mathbb{E}[f(S_T, A_S(0, T))] \tag{8}$$

$$A_S(t_0, t_1) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} S_u du \tag{9}$$

The function f depends on the Asian option's type :

- for a call with fixed strike : $f(s, a) = (a - K)_+$
- for a put with fixed strike : $f(s, a) = (K - a)_+$
- for a call with floating strike : $f(s, a) = (s - a)_+$
- for a put with floating strike : $f(s, a) = (a - s)_+$

The classical Monte-Carlo estimator of $\mathbb{E}[f(S_T, A_S(0, T))]$ would be :

$$\frac{1}{M} \sum_{j=1}^{M} f(S_T^j, A_{S^j}(0, T)), \tag{10}$$

where the (S^j) are iid, with the same law as S.

We need schemes to approximate

$$Y_T = \int_0^T S_u du \tag{11}$$

2.1 Schemes for the approximation of the integral

2.1.1 Standard scheme

The standard scheme relies on the Riemann sum. Y_T is approximated by

$$Y_T^{r,N} = h \sum_{k=0}^{N-1} S_{t_k}, \tag{12}$$

where $t_k = k\frac{T}{N} = kh$ for $h = \frac{T}{N}$, so that $(t_k)_{k=0,\ldots,N}$ is a subdivision of $[0, T]$. It gives :

$$v \approx \frac{e^{-rT}}{M} \sum_{j=1}^{M} \left(\frac{h}{T}\left(\sum_{k=0}^{N-1} S_{t_k}\right) - K\right)_+ \tag{13}$$

2.1.2 Trapezoidal scheme and equivalent with Taylor expansion

Let $\mathcal{F}_h$ be the σ-field generated by $(S_{t_k})k = 0, \ldots, N$. Knowing $\mathcal{F}_h$, the closest random variable to $\left(\frac{1}{T}\int_0^T S_u du - K\right)_+$ in the subset of L^2 generated by $\mathcal{F}_h$ is

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)_+ \mid \mathcal{F}_h\right] = \left(\mathbb{E}\left[\frac{1}{T}\int_0^T [S_u|\ \mathcal{F}_h\right] du - K\right)_+$$
$$= \left(\frac{1}{T}\int_0^T \mathbb{E}[S_u|\ \mathcal{F}_h]\ du - K\right)_+ \tag{14}$$

$$\begin{aligned}
\int_0^T \mathbb{E}[S_u|\ \mathcal{F}_h]\ du &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}[S_u|W_{t_k}, W_{t_{k+1}}]\ du \\
&= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \exp\left(\left(r-\frac{\sigma^2}{2}\right)u\right)\mathbb{E}[\exp(\sigma W_u) \mid W_{t_k}, W_{t_{k+1}}]\ du \\
&= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{\left(r-\frac{\sigma^2}{2}\right)u}\exp\left(\sigma\frac{t_{k+1}-u}{h}W_{t_k} + \sigma\frac{u-t_k}{h}W_{t_{k+1}} + \frac{\sigma^2}{2}(t_{k+1}-u)(u-t_k\right)\ du \\
&= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} S_{t_k}\exp\left(\left(\sigma\frac{W_{t_{k+1}}-W_{t_k}}{h}+r\right)(u-t_k) - \sigma^2\frac{(u-t_k)^2}{2h}\right)\ du
\end{aligned} \tag{15}$$

Indeed, for u in $[t_k, t_{k+1}]$, $\sigma W_u \sim \mathcal{N}\left(\sigma\frac{t_{k+1}-u}{h}W_{t_k} + \sigma\frac{u-t_k}{h}W_{t_{k+1}}\ ,\ \sigma^2\frac{(t_{k+1}-u)(u-t_k)}{h}\right)$, and for any $X \sim \mathcal{N}(m, s^2)$, $\mathbb{E}\left[e^X\right] = e^{m+\frac{s^2}{2}}$.

With a Taylor expansion of the exponential in (15), we have :

$$\exp\left(\left(\sigma\frac{W_{t_{k+1}}-W_{t_k}}{h}+r\right)(u-t_k) - \sigma^2\frac{(u-t_k)^2}{2h}\right) \simeq 1 + (\sigma(W_{t_{k+1}}-W_{t_k}) + rh)\frac{u-t_k}{h}$$
$$\int_{t_k}^{t_{k+1}} S_{t_k}\exp\left(\left(\sigma\frac{W_{t_{k+1}}-W_{t_k}}{h}+r\right)(u-t_k) - \sigma^2\frac{(u-t_k)^2}{2h}\right)\ du \simeq S_{t_k}\left(h + \sigma(W_{t_{k+1}}-W_{t_k}) + rh)\frac{h}{2}\right) \tag{16}$$

Finally, we can approximate $\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du \mid \mathcal{F}_h\right]$ by

$$Y_T^{l,N} = \frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}\left(1 + \frac{rh}{2} + \sigma\frac{W_{t_{k+1}}-W_{t_k}}{2}\right) \tag{17}$$

This method is equivalent to the trapezoidal approximation, which states

$$\int_a^b f(u)\ du \simeq \frac{f(a)+f(b)}{2}(b-a) \tag{18}$$

2.1.3 Higher accuracy scheme

From the equation (19), a new approximation of $\frac{1}{T}\int_0^T S_u du$ is proposed using a Taylor expansion in (20).

$$\int_0^T S_u du = \sum_{k=0}^{N-1} S_{t_k}\left(\int_{t_k}^{t_{k+1}} \exp\left(\sigma(W_u - W_{t_k}) + \left(r - \frac{\sigma^2}{2}\right)(u-t_k)\right) du\right) \tag{19}$$

$$Y_T^{p,N} = \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\left(h + \frac{rh^2}{2} + \sigma\int_{t_k}^{t_{k+1}} W_u - W_{t_k}\ du\right) \tag{20}$$

The use of this estimator requires to know the law of $\int_{t_k}^{t_{k+1}} W_u\ du$ in order to simulate it. We can compute this law, and the calculations are given in appendix 4. We find that conditionally on $W_{t_k} = x, W_{t_{k+1}} = y$,

$$\int_{t_k}^{t_{k+1}} W_u\ du \sim \mathcal{N}\left(h\frac{x+y}{2}, \frac{h^3}{12}\right) \tag{21}$$

2.2 Variance reduction

This section propose an improvement to the previous methods. The principle of the variance reduction is to write $\mathbb{E}[f(X)] = \mathbb{E}[f(X) - h(X)] + \mathbb{E}[h(X)]$ when $(f-h)(X)$ has a smaller variance than $f(X)$.

For r and σ small, we have

$$A(T) := \frac{1}{T}\int_0^T S_u \, du \simeq \exp\left(\frac{1}{T}\int_0^T \ln(S_u) \, du\right) \tag{22}$$

Letting $C(T) = e^{-rT}(G_T - K)_+$ with $G_T = \exp\left(\frac{1}{T}\int_0^T \ln(S_u) \, du\right)$, we write

$$\mathbb{E}[e^{-rT}(A(T) - K)_+] = \mathbb{E}[e^{-rT}(A(T) - K)_+ - C(T)] + \mathbb{E}[C(T)] \tag{23}$$

Relying on Kemna and Vorst (”A pricing method for options based on average asset values”, 1990, p123-124), we know the expectation of $C(T)$:

$$\mathbb{E}[e^{-rT}(A(T) - K)_+ - C(T)] = e^{-rT}\left(S_0\Phi(d)e^{d^*} - K\Phi(d - \sigma\sqrt{\frac{T}{3}})\right) \tag{24}$$

with $d^* = \frac{1}{2}(r - \frac{\sigma}{6})T$, $d = \frac{\ln(S_0/K) + (r+\sigma^2/6)T/2}{\sigma\sqrt{T/3}}$, Φ the cumulative function of the standard normal distribution.

Now that we know the left term in (24), we focus on C_T.

$$\begin{aligned}
\ln(S_u) &= \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)u + \sigma W_u \\
\int_0^T \ln(S_u) \, du &= \ln(S_0)T + \left(r - \frac{\sigma^2}{2}\right)\frac{T^2}{2} + \sigma\int_0^T W_u \, du \\
C(T) &= e^{-rT}\left(S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_u \, du\right) - K\right)_+ \qquad (25)
\end{aligned}$$

For each scheme, we approximate $\int_0^T W_u \, du$ by :

- Euler : $\int_0^T W_u \, du \simeq \sum_{k=0}^{N-1} hW_{t_k}$
- Trapezoidal : $\int_0^T W_u \, du \simeq \sum_{k=0}^{N-1} h\frac{W_{t_k} + W_{t_{k+1}}}{2}$
- Last scheme (21) : $\int_0^T W_u \, du = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} W_u \, du$

We experimented all these techniques numerically, the results are displayed in 1 for the following setting : $r = 0.1, \sigma = 0.2, T = 1, S_0 = K = 100$, where the true value is 7.04. We used $N = 50$ time steps and $M = 10^5$ Monte Carlo simulations. We see that the true value is always in the 95% confidence interval, except for the Euler scheme which underestimates the price. The variance reduction improves all the methods, dividing by a factor 20 the errors.

	Estimator	Value	Standard deviation	Confidence interval 95%	Error (%)
Without variance reduction	Euler	6,9085	0.026484	[6.8566 , 6.9604]	0.751365
	Trapezoidal	7,0320	0.026933	[6.9792 , 7.0848]	0.7507
	Trapezoidal with Taylor expansion	7,0319	0.026932	[6.9791 , 7.0847]	0.750667
	Higher accuracy scheme	7,0321	0.026933	[6.9793 , 7.0849]	0.750676
With variance reduction	Euler	6,9888	0.001196	[6.9865 , 6.9912]	0.033549
	Trapezoidal	7,0409	0.001266	[7.0384 , 7.0434]	0.035245
	Trapezoidal with Taylor expansion	7,0408	0.001264	[7.0383 , 7.0433]	0.03518
	Higher accuracy scheme	7,0408	0.001264	[7.0383 , 7.0432]	0.035178

Figure 1: Numerical tests for the Monte Carlo methods.

We also recreated the table 1.3 of the reference n°16 (L. C. G. Rogers and Z. Shi) for lower bounds (where the lower bounds are extremely accurate so they can be taken as prices) in 2. To this purpose we set $T = 1, S_0 = 100, r = 0.09$, and we use the higher accuracy scheme with variance reduction.

Interest rate r	Volatility	Strike K	Lower bound	M-C result	Upper bound	Standard deviation
0.09	0.1	90.0	13.385	13.3857	13.3864	0.0004
		100.0	4.9149	4.9156	4.9164	0.0004
		110.0	0.6287	0.6294	0.6302	0.0004
	0.3	90.0	14.9789	14.984	14.9891	0.0026
		100.0	8.824	8.8293	8.8346	0.0027
		110.0	4.6931	4.6985	4.7038	0.0027
	0.5	90.0	18.1756	18.1909	18.2062	0.0078
		100.0	13.0122	13.0274	13.0426	0.0078
		110.0	9.1115	9.127	9.1425	0.0079

Figure 2: Numerical tests for the Monte Carlo methods.

3 Finite Differences

We will now focus on the PDEs to evaluate the price of Asian options, rather than using simulations. Two approaches are discussed in this part, and we will study the Asian call option.

3.1 Change of variables

The first approach is proposed in the reference n°16 (L. C. G. Rogers and Z. Shi). In general, computing the value of an Asian call with maturity T and strike K is equivalent to computing $\mathbb{E}[(Y-K)_+]$ where $Y = \int_0^T S_u \nu(du)$. For a fixed strike, $\nu(du) = \frac{1}{T}\mathbb{1}_{[0,T]}(u)\, du$. For a floating strike, $\nu(du) = \frac{1}{T}\mathbb{1}_{[0,T]}(u)\, du - \delta_T(du)$ and $K = 0$.

The function ϕ is introduced, defined by $\phi(t,x) = \mathbb{E}\left[\left(\int_t^T S_u\nu(du) - x\right)_+ \mid S_t = 1\right]$. Then,

$$M_t \;:=\; \mathbb{E}\left[\left(\int_t^T S_u\nu(du) - K\right)_+ \mid \mathcal{F}_\sqcup\right] = S_t\phi(t,\xi_t)$$

defines a martingale (with $\xi_t = \frac{K - \int_0^t S_u\nu(du)}{S_t}$). Defining ρ_t as the density of the measure ν , *ie.* $\rho_t dt \;=\; \nu(dt)$, it leads to the following PDE for ϕ :

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 \phi}{\partial x^2} - (r\xi + \rho_t)\frac{\partial \phi}{\partial x} + r\phi = 0 \tag{26}$$

With $f(t,x) = e^{-r(T-t)}\phi(t,x)$, the PDE becomes :

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - (rx + \rho_t)\frac{\partial f}{\partial x} = 0 \tag{27}$$

The boundary conditions are :

1. $f(T,x) = (-x)_+ = x_-$ for a fixed strike K, it yields to the following initial price

$$e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u \; du \; - K\right)_+\right] = e^{-rT}M_0 = e^{-rT}S_0\phi(0,\xi_0) = S_0 e^{-rT}\phi\left(0, \frac{K}{S_0}\right) = S_0 f\left(0, \frac{K}{S_0}\right) \tag{28}$$

   With the change of variables $L = \frac{K}{S_0}$, we can then use the classical finite differences method on f.

2. $f(T,x) = (-1-x)_+ = (1+x)_-$ for the floating strike, giving as an initial price (recall that here the measure ν changes so that $K = 0$ in the definition of the martingale M)

$$e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u \; du \; - S_T\right)_+\right] = e^{-rT}M_0 = e^{-rT}S_0\phi(0,0) \tag{29}$$

   The classical finite differences method can be used on ϕ.

To apply the finite differences method we discretize time and space as follows

$$(t_n, x_j) := (nh, \alpha + j\delta), 0 \le n \le m, 0 \le j \le l+1 \tag{30}$$

The time step is $h = \frac{T}{m}$ whereas the space step is $\delta = \frac{\beta - \alpha}{l+1}$

We study now the fixed strike case. Our boundary conditions are

1. right condition $\forall 0 \le j \le l+1, f_j^m = (x_j)_-$ (payoff)
2. bottom condition $\forall 0 \le n \le m, f_0^n = \frac{1-e^{-r(T-nh)}}{r}$; can be obtained by a direct computation considering that S_t follows a log-normal distribution for $L \le 0$
3. top condition $\forall 0 \le n \le m, f_{l+1}^n = 0$ (Dirichlet condition) ; intuitive because if L is large, it means that $S_0 \ll K$, so that the payoff of the Asian call will probably be 0.

To take the linear PDE defined in the course, we need to set

$$\sigma(x) = \sigma x, b(x) = -(\rho_t + rx) \tag{31}$$

We can then define the tridiagonal matrix $A_\delta^n = (a_{i,j}^n)i,j$ of size $l \times l$ for each time $t_n, 0 \le n \le m-1$, which has the following coefficients for $1 < i < l$:

$$a_{i,i+1}^n = \frac{\sigma^2(x_i)}{2\delta^2} + \frac{b(x_i)}{2\delta}, a_{i,i}^n = -\frac{\sigma^2(x_i)}{\delta^2}, a_{i,i-1}^n = \frac{\sigma^2(x_i)}{2\delta^2} - \frac{b(x_i)}{2\delta} \tag{32}$$

and $a_{l,l+1}^n = 0, a_{1,0}^n = \frac{1-e^{-r(T-nh)}}{r}\left(\frac{\sigma^2(x_i)}{2\delta^2} - \frac{b(x_i)}{2\delta}\right)$.

We start from $f^m = (x_i)_{1 \le i \le l}$. We propose to use the θ-scheme, $\theta \in [0,1]$, and solve recursively for $0 \le n \le m-1$,

$$\frac{f^{n+1} - f^n}{h} + \theta A_\delta^n f^n + (1-\theta)A_\delta^n f^{n+1} = 0 \tag{33}$$

We can isolate f^n and get :

$$f^n = (I - \theta h A_\delta^n)^{-1}(I + (1-\theta)hA_\delta^n)f^{n+1} \tag{34}$$

The vector f^0 gives the initial prices for any L in the space grid. To obtain a price for any $L \in [\alpha, \beta]$, we use the linear interpolation. Denoting by $J = \max\{j \mid x_j \leq L\}$, we write the price of an Asian call option :

$$p(L) = f_J^0 + \frac{f_{J+1}^0 - f_J^0}{\delta}(L - x_J) \tag{35}$$

We expose the numerical results obtained with the Crank-Nicholson scheme ($\theta = \frac{1}{2}$) in 3.

σ	Step sizes	Interest rate r	Strike price K	Price	Time (sec.)
0.3	δ = 0.006 h = 0.001	0.02	90.0	13.18	4.48
			95.0	9.95	4.34
			100.0	7.31	4.34
			105.0	5.22	4.34
			110.0	3.64	4.34
		0.09	90.0	14.98	4.4
			95.0	11.65	4.45
			100.0	8.83	4.37
			105.0	6.52	4.34
			110.0	4.7	4.27
		0.28	90.0	19.67	4.38
			95.0	16.36	4.33
			100.0	13.33	4.45
			105.0	10.63	4.37
			110.0	8.31	4.44
0.1	δ = 0.0048 h = 0.0006	0.02	90.0	10.83	10.1
			95.0	6.27	9.88
			100.0	2.8	9.89
			105.0	0.91	9.86
			110.0	0.22	9.96
		0.09	90.0	13.39	10.0
			95.0	8.9	9.79
			100.0	4.9	10.18
			105.0	2.08	9.8
			110.0	0.66	9.85
		0.28	90.0	19.21	9.86
			95.0	15.43	11.15
			100.0	11.66	9.81
			105.0	7.97	9.74
			110.0	4.67	9.77
0.05	δ = 0.0046 h = 0.0005	0.02	90.0	10.81	12.17
			95.0	5.87	12.41
			100.0	1.68	12.76
			105.0	0.2	12.29
			110.0	0.01	11.92
		0.09	90.0	13.4	12.91
			95.0	8.84	12.55
			100.0	4.22	12.24
			105.0	1.01	11.8
			110.0	0.11	11.77
		0.28	90.0	19.22	12.63
			95.0	15.44	12.79
			100.0	11.66	12.65
			105.0	7.89	12.0
			110.0	4.06	11.76

Figure 3: Numerical tests for the Finite Differences method with a change of variables.

3.2 Splitting Method

We focus now on the floating strike Asian call option, relying on the reference n°3 (G. Barles). The following process is introduced, for $0 \leq t \leq T$:

$$Z_t = \frac{1}{T} \int_0^t S_u du \tag{36}$$

The price of the option studied here can be expressed from the function p solving the following PDE in $\mathbb{R} \times \mathbb{R} \times (0, T)$:

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp + \frac{S}{T}\frac{\partial p}{\partial Z} = 0 \tag{37}$$

and with terminal conditions $u(S, Z, T) = (Z - S)_+$. The Asian call option with floating strike and maturity T has a price at time $t \in (0, T)$:

$$P_t = p(S_t, \frac{1}{T} \int_0^t S_u du, t) \tag{38}$$

There is no explicit formula for such prices as there are two space variables in the PDE (37) : S and Z. A splitting method can be used to approximate the solution of this PDE. Each step of a classical FD method is divided in two half-steps, one in S and one in Z. Starting from the terminal conditions $u^m(x, z) = (z - x)_+$, we apply the following process recursively :

1. For all z fixed, we do a one step classical FD method for the following equation in $[0, h] \times \mathbb{R}$:

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2} + rx\frac{\partial w}{\partial x} - rw = 0 \tag{39}$$

   with initial condition $w(x, 0) = u^{n+1}(x, z)$, and we set $u^{n+1/2}(x, z) = w(x, z, h)$.

2. For all x fixed, we do a one step classical FD method for the following equation in $[0, h] \times \mathbb{R}$:

$$\frac{\partial w}{\partial t} + \frac{x}{T}\frac{\partial w}{\partial z} = 0 \tag{40}$$

   with initial condition $w(z, 0) = u^{n+1/2}(x, z)$, and we set $u^n(x, z) = w(x, z, h)$.

The splitting method relies on a first order Taylor expansion. Note that, as a variant, the term $-rp$ in the PDE (38) could be included in the second step of the splitting method rather than in the first one.

The numerical tests do not show up stable results here. Using the implicit scheme ($\theta = 1$) with Dirichlet conditions, and the following values : $r = 0.1, \sigma = 0.2, T = 1, m = 20, l = 200, \alpha = 70, \beta = 130$, we first show the initial price of the Asian call with a floating strike as a function of the spot price in 4. We see big fluctuations even if there is no CFL condition with such a scheme. Afterwards, we show in 5 the evolution of the contract's price within the time interval $[0, T]$ for a particular trajectory (constant) of the spot price ($S_t = 100.15$).

4 Appendix : proof of equation (21)

We first have, denoting γ the covariance function,

$$\mathbb{E}[W_t \mid W_u = x, W_v = y] = \frac{v-t}{v-u}x + \frac{t-u}{v-u}y \ , \ \gamma(W_t, W_s \mid W_u = x, W_v = y) = \frac{(v-t)(s-u)}{v-u}$$

We want to know the law of $\int_{t_i}^{t_{i+1}} W_t \, dt$. It is a Gaussian law because W is a Gaussian process. Let $\mathcal{A} := \{W_{t_i} = x, W_{t_{i+1}} = y\}$.

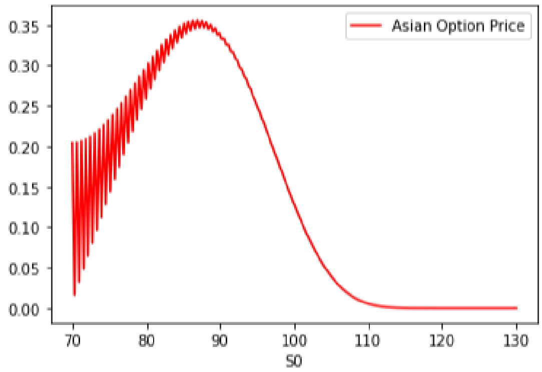

Figure 4: Numerical tests for the Finite Differences method with a change of variables.

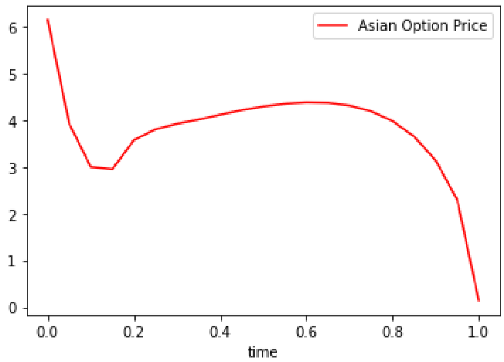

Figure 5: Numerical tests for the Finite Differences method with a change of variables.

$$\begin{aligned}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} W_t\ dt \mid \mathcal{A}\right] &= \int_{t_i}^{t_{i+1}} \mathbb{E}[W_t \mid \mathcal{A}]\ dt \\ &= \int_{t_i}^{t_{i+1}} \frac{t_{i+1}-t}{h}x + \frac{t-t_i}{h}y\ dt \\ &= \frac{h(x+y)}{2}\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} W_t\ dt\right)^2 \mid \mathcal{A}\right] &= 2\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} \mathbb{E}[W_tW_u \mid \mathcal{A}]\ du\ dt \\ &= 2\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} \frac{(t_{i+1}-t)(u-t_i)}{h} + \left(\frac{t_{i+1}-t}{h}x + \frac{t-t_i}{h}y\right)\left(\frac{t_{i+1}-u}{h}x + \frac{u-t_i}{h}y\right)\ du\ dt\end{aligned}$$

since $\gamma(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Moreover,

- $\int_{t_i}^{t_{i+1}} \frac{t_{i+1}-t}{h}\ dt = \frac{t_{i+1}-t_i}{2} = \frac{h}{2}$
- $\int_{t_i}^{t} \frac{u-t_i}{h}\ du = \frac{(t-t_i)^2}{2h}$
- $\int_{t_i}^{t_{i+1}} \frac{t_{i+1}-u}{h}x + \frac{u-t_i}{h}y\ du = \frac{hx}{2} - x\frac{(t_{i+1}-t)^2}{2h} + y\frac{(t-t_i)^2}{2h}$

$$\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} W_t dt\right)^2 |\mathcal{A}\right] = \int_{t_i}^{t_{i+1}} \frac{(t_{i+1}-t)(t-t_i)^2}{h} + \left(\frac{t_{i+1}-t}{h}x + \frac{t-t_i}{h}y\right)\left(y\frac{(t-t_i)^2}{h} - x\frac{(t_{i+1}-t)^2}{h} + hx\right) dt$$

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} \frac{(t_{i+1}-t)(t-t_i)^2}{h}\ dt &= \int_{t_i}^{t_{i+1}} \frac{(t_{i+1}-t_i+t_i-t)(t-t_i)^2}{h}\ dt \\
&= \int_{t_i}^{t_{i+1}} \frac{h(t-t_i)^2+(t_i-t)^3}{h}\ dt \\
&= \left[\frac{(t-t_i)^3}{3} - \frac{(t_i-t)^4}{4h}\right]_{t_i}^{t_{i+1}} \\
&= \frac{h^3}{12}
\end{aligned}$$

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} \left(\frac{t_{i+1}-t}{h}x + \frac{t-t_i}{h}y\right)\left(y\frac{(t-t_i)^2}{h} - x\frac{(t_{i+1}-t)^2}{h} + hx\right) dt &= \int_{t_i}^{t_{i+1}} \left[\left(y\frac{(t-t_i)^2}{2h} - x\frac{(t_{i+1}-t)^2}{2h} + \frac{hx}{2}\right)^2\right]' dt \\
&= \left[\left(y\frac{(t-t_i)^2}{2h} - x\frac{(t_{i+1}-t)^2}{2h} + \frac{hx}{2}\right)^2\right]_{t_i}^{t_{i+1}} \\
&= \frac{h^2(x+y)^2}{4}
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}\left(\int_{t_i}^{t_{i+1}} W_t\ dt \mid \mathcal{A}\right) &= \frac{h^3}{12} + \frac{h^2(x+y)^2}{4} - \left(\frac{x+y}{2}h\right)^2 \\
&= \frac{h^3}{12}
\end{aligned}$$

Hence the result : conditionally on $\mathcal{A}$,

$$\int_{t_i}^{t_{i+1}} W_t\ dt \sim \mathcal{N}\left(\frac{x+y}{2}, \frac{h^3}{12}\right)$$