EQ2410 - Advanced Digital Communications

Project 1 : Channel Equalization

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1 Problem 1

1.a Identification of signals

- `Vect_1`: Random QPSK data symbol sequence
  $\texttt{Vect\_1} \in \{1+i, 1-i, -1+i, -1-i\}^{\texttt{N\_symbols}}$
- `Signal_1`: `Vect_1` upsampled by factor `T_sym`/ `delta_t` and normalized
  $\texttt{Signal\_1}(1+k*\texttt{T\_sym/delta\_t}) = \texttt{Vect\_1}(1+k)/\texttt{delta\_t}$ with $k \in [0, \texttt{N\_symbols}-1]$
- `Signal_2`: Complex baseband data transmitted waveform
  $\texttt{Signal\_2} = \texttt{Signal\_1} \star \texttt{Filter\_1}$
- `Signal_3`: Complex baseband data received waveform
  $\texttt{Signal\_3} = \texttt{Signal\_2} \star \texttt{Filter\_2}$
- `Signal_4`: Complex baseband noise waveform with two-sided variance N_0
- `Signal_5`: Complex baseband total received waveform
  $\texttt{Signal\_5} = \texttt{Signal\_3} + \texttt{Signal\_4}$
- `Signal_6`: Complex baseband total received waveform matched filtered
  $\texttt{Signal\_6} = \texttt{Signal\_5} \star \texttt{Filter\_4}$
- `Vect_2`: Sampler output received symbol sequence after synchronization
  $\texttt{Vect\_2}(k) = \texttt{Signal\_6}(k*\texttt{T\_sample/delta\_t} + \delta_{offset})$ (`Signal_6` downsampled)

1.b Identification of filters

- `Filter_1`: Transmitter filter
- `Filter_2`: Channel filter
- `Filter_3`: Transmitter and Channel chain filter
- `Filter_4`: Matched filter of `Filter_3`, can be used as receiver filter
- `Filter_5`: Transmitter, channel and receiver chain filter, used to simulate the overall system response

1.c Normalization coefficient

In the program, we use a small time resolution (`delta_t`) to represent continuous-time signals. We consider the signals to be constant over these small intervals. With this approximation, the integrals, used in the convolutions for example, are transformed into sums in the following way:

$$(f \star g)(m\delta_t) = \int_{-\infty}^{+\infty} f(t)g(m\delta_t - t)dt = \sum_{k=-\infty}^{\infty} \left(\int_{k\delta_t}^{(k+1)\delta_t} f(t)g(m\delta_t - t)dt \right) \approx \delta_t \sum_{k=-\infty}^{\infty} f(k\delta_t)g((m-k)\delta_t)$$

And we can notice that, compared to the discrete-time convolution formula used by Matlab, the last sum is multiplied by `delta_t`.

The factor 1/`delta_t` when we generate `Signal_1` is another normalization used to keep the signal power indepedant of the time resolution. In the same manner we have:

$$\begin{aligned} P &= \frac{1}{N_{symbols}T_{sym}} \int_0^{N_{symbols}T_{sym}} |S_1(t)|^2 dt \\ &\approx \frac{1}{N_{symbols}T_{sym}} \sum_{k=0}^{(N_{symbols}-1)T_{sym}\delta_t} |S_1(k\delta_t)|^2 \delta_t \\ &\approx \frac{{\delta_t}^2}{N_{symbols}T_{sym}\delta_t} \sum_{k=0}^{(N_{symbols}-1)T_{sym}\delta_t} |S_1(k\delta_t)|^2 \\ &\approx \frac{1}{N_{symbols}T_{sym}\delta_t} \sum_{k=0}^{(N_{symbols}-1)T_{sym}\delta_t} |\delta_t^2 S_1(k\delta_t)|^2 \end{aligned}$$

And by dividing by 1/`delta_t`, the discrete-time signal power of `Signal_1` is independant of `delta_t` and equal to the one of `Vect_1`.

2 Problem 2

We know that $\mathbf{r}[n] = \mathbf{U}\mathbf{b}[n] + \mathbf{w}[n]$, that $\mathbf{R} = \mathbb{E}[\mathbf{r}[n]\mathbf{r}[n]^H]$ and that $\mathbf{p} = \mathbb{E}[b^*[n]\mathbf{r}[n]]$
Therefore:

$$\mathbf{r}[n]^H = \mathbf{b}[n]^H\mathbf{U}^H + \mathbf{w}[n]^H$$

so that:

$$\mathbf{r}[n]\mathbf{r}[n]^H = \mathbf{U}\mathbf{b}[n]\mathbf{b}[n]^H\mathbf{U}^H + \mathbf{U}\mathbf{b}[n]\mathbf{w}[n]^H + \mathbf{w}[n]\mathbf{b}[n]^H\mathbf{U}^H + \mathbf{w}[n]\mathbf{w}[n]^H \tag{1}$$

with

$$\mathbb{E}[\mathbf{w}[n]\mathbf{w}[n]^H] = \mathbf{C_w}$$

As the data values $b[n]$ are independant, we have

$$\mathbb{E}[\mathbf{b}[n]\mathbf{b}[n]^H] = P_s\mathbf{I} \qquad \text{with } P_s = \mathbb{E}[|b[n]|^2]$$

Moreover, as $b[n]$ and $w[k]$ are independant for every n and k, it remains in the expectation of (1):

$$\mathbf{R} = P_s\mathbf{U}\mathbf{U}^H + \mathbf{C_w} \tag{2}$$

We can also write:

$$b^*[n]\mathbf{r}[n] = \mathbf{U}b^*[n]\mathbf{b}[n] + b^*[n]\mathbf{w}[n] \tag{3}$$

and $b[n]$ is independant of $w[k]$ for every k and independant of $b[k]$ for every $k \neq 1$. It gives:

$$\mathbb{E}[b^*[n]\mathbf{b}[n]] = P_s\mathbf{e}$$

Taking the expectation of (3), it remains:

$$\mathbb{E}[b^*[n]\mathbf{r}[n]] = P_s\mathbf{U}\mathbf{e} \tag{4}$$

3 Problem 3

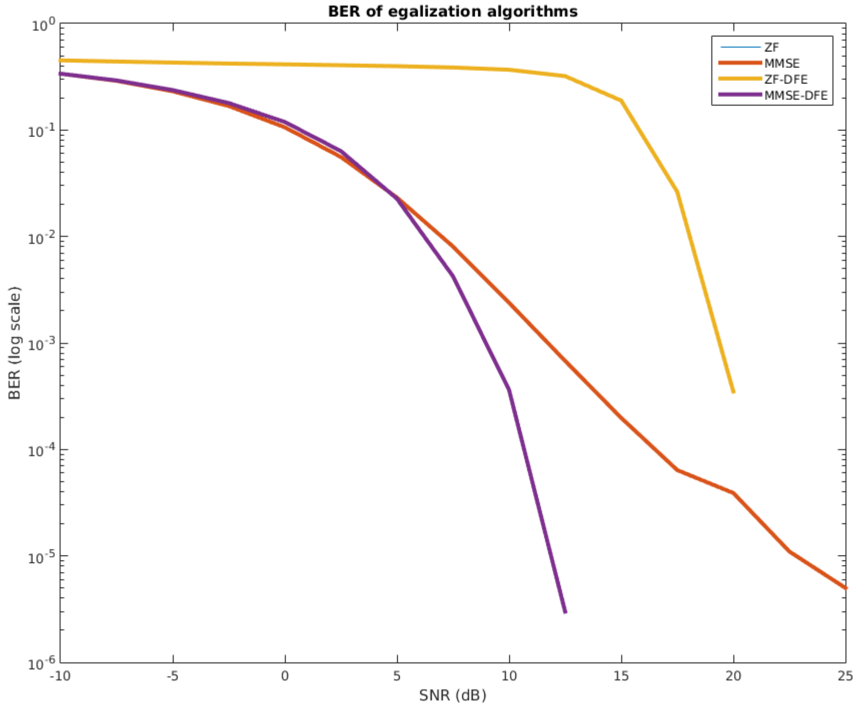


Figure 1: BER comparison for $m = 1$

Figure 1 confirm the results in Figure 5.14 of Upamanyu Madhow, Fundamentals of Digital Communication.

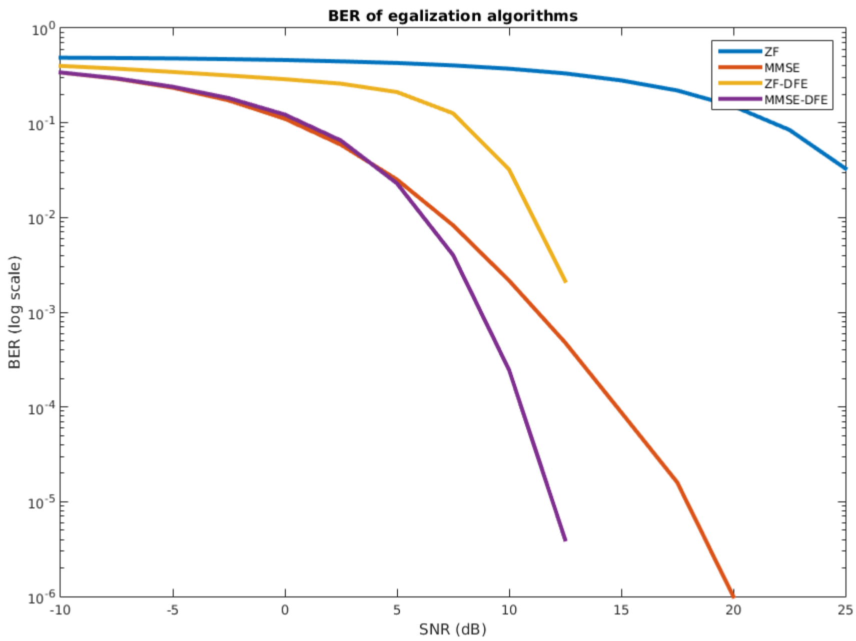


Figure 2: BER comparison for $m = 2$

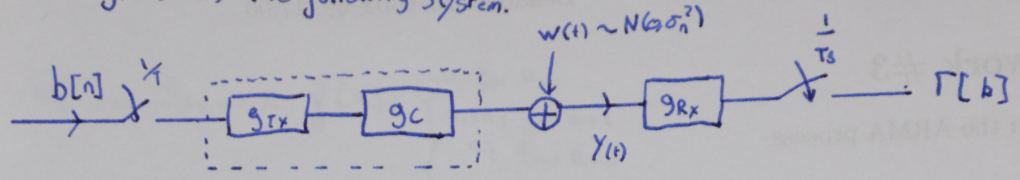
Problem 4. As we don't know our student letter, all the calculations will be held with a_1, a_2.

a) The symbol rate gives $T=2$, $g_{Tx} = I_{[0;2]}$

$$g_c(t) = a_1 \delta_0(t-1) + a_2 \delta_0(t-2)$$

$$g_{Rx} = I_{[0,1]}$$

It gives us the following system.



$P(t) = g_{Tx} * g_c(t)$

We have $p(t) = g_{Tx} * g_c(t) = I_{[0;2]} * (a_1 \delta_0(t-1) + a_2 \delta_0(t-2))$

$$\underline{p(t) = a_1 I_{[0;2]}(t-1) + a_2 I_{[0;2]}(t-2)}$$

We then can write $r[k] = y * g_{rx}(k) = \int_{-\infty}^{+\infty} g_{r_x}(k-t)\, y(t)\, dt = \int_{k-1}^{k} y(t)\, dt$ since $T_S = 1$

$$= \int_{k-1}^{k} \sum_n b[n]\, p(t-2n)\, dt \quad + w[k]$$

$\hookrightarrow w * g_{Rx}(k T_S)$

$$\underline{r[k] = \sum_n \left(\int_{k-1}^{k} p(t-2n)\, dt\, b[n] \right) + w[k]} \qquad (1)$$

Then $w[k] = w * g_{Rx}(t - kT_S)$. $E[w[k]] = E[w] * g_{Rx} = 0$

and $Cov[w[n], w[n+k]] = \sigma_n^2 \int_D g_{R_x}(t)\, g_{R_x}(t - kT_S)\, dt$

D: domain where $g_{R_x}(t)$ and $g_{R_x}(t-kT_S)$ overlapp

Hence, $\underline{Cov[w[n], w[n+k]] = \sigma_n^2 \delta_0(k)}$

<u>So $w[k]$ is white zero mean with variance σ_n^2</u>

b) Starting from (1):

$$r[k] = \sum_n b[n] \int_{k-1}^{k} a_1 I_{[0;2]}(t-2n-1) + a_2 I_{[0;2]}(t-2n-2)\, dt + w[k]$$

Hence as $t \in [k-1;k]$ The integral is non negative for:

$K \in \{2n+2;\ 2n+3;\ 2n+4\}$

So r can take the values:
$$\begin{cases} a_2 b[n-1] + a_1 b[n] + w[k] \\ (a_1 + a_2) b[n] + w[k] \\ a_1 b[n+1] + b[n] \cdot a_2 + w[k] \end{cases}$$

Then taking the $L=3$ block:

$$\underline{\Gamma}[n] = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix} b[n-1] + \begin{pmatrix} a_1 \\ a_1+a_2 \\ a_2 \end{pmatrix} b[n] + \begin{pmatrix} 0 \\ 0 \\ a_1 \end{pmatrix} b[n+1] + \underline{w}[n]$$

$$\underline{\Gamma}[n] = \underbrace{\begin{pmatrix} a_2 & a_1 & 0 \\ 0 & a_1+a_2 & 0 \\ 0 & a_2 & a_1 \end{pmatrix}}_{U} \underline{b}[n] + \underline{w}[n]$$

$\hookrightarrow$ $\underline{w}$ is a zero mean gaussian vector with $C_w = \sigma_n^2 I_3$

We can also write $g[k] = \begin{cases} a_1 \text{ for } k=1 \\ a_1+a_2 \text{ for } k=2 \\ a_2 \text{ for } k=3 \\ 0 \text{ otherwise} \end{cases}$

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c) $Z[n] = \underline{C}^H \underline{\Gamma}[n] = C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} + C^H \underline{w}[n]$

and our decision is $\hat{b}[n] = \text{sign}(Z[n])$

Then the probability of error, $P_e = \frac{1}{2} P_r(Z[n]>0 \mid b[n]=-1) + \frac{1}{2} P_r(Z[n]<0 \mid b[n]=+1)$

$P_e = P_r(Z(n)>0 \mid b[n]=-1)$ due to symmetry.

$= P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| b(n)=-1 \right)$

$= \frac{1}{4} \Bigg(P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| b[n-1]=-1, b(n)=-1, b[n+1]=-1 \right) + P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=1 \\ b[n]=-1 \\ b[n+1]=-1 \end{matrix} \right)$

$+ P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=-1 \\ b[n]=1 \\ b[n+1]=1 \end{matrix} \right) + P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=1 \\ b[n]=-1 \\ b[n+1]=1 \end{matrix} \right) \Bigg)$

$P_e = \frac{1}{4} \sum_{b[n-1]} \sum_{b[n+1]} P_r\left(C^H [w[n]] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1] \\ b[n]=-1 \\ b[n+1] \end{matrix} \right)$

As $\underline{w}$ is gaussian $C^H w$ is also gaussian with $\sigma^2 = C^H C_w C = E[C^H w (C^H w)^H] = C^H E[w w^H] C$

Then $P_r\left(C^H w[n] > -C^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1] \\ b[n] \\ b[n+1] \end{matrix} \right) = Q\left(\frac{-C^H U [b[n-1], b[n], b[n+1]]^T}{\sqrt{C^H C_w C}} \right)$

$$P_e = \frac{1}{4} \sum_{b[n-1] \in \{\pm 1\}} \sum_{b[n+1] \in \{\pm 1\}} Q\left(\frac{-C^H U [b[n-1], b[n], b[n+1]]}{\sqrt{C^H C_w C}} \right)$$

4) d) Let's design a ZF-DFE

Let U^g be $[U_0 U_1] = \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix}$, The The Z_g corelator is given by:

$$C_{zg} = U_g (U_g^H U_g)^{-1} e \quad \text{where } e = (1\,0)^T$$

Then $C_{zg} = \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} a_1^2+a_2^2+a_1a_2 & a_1a_2 \\ a_1a_2 & a_1^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$C_{zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{a_1}{a_2} \\ -\frac{a_1}{a_2} & \frac{2(a_1^2+a_1a_2+a_2^2)}{a_1^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_{zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{a_1}{a_2} \end{pmatrix}$$

$$C_{zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 \\ a_1+a_2 \\ a_2 - \frac{a_1^2}{a_2} \end{pmatrix}$$

The Feedback correlator is then: $-C_{zg}^H U[-1] = \frac{1}{2a_1^2+a_1a_2+a_2^2} (-a_1 a_2)$

Then: $Z_{DFE} = C_{FF}^H \underline{r}^g[n] + C_{FB} \left(\hat{b}(n-1) - b(n-1)\right)$

$$Z_{DFE} = C_{FF}^H U^g \begin{pmatrix} b[n] \\ b[n+1] \end{pmatrix} + C_{FB} \left(\hat{b}(n-1) - b(n-1)\right)$$

Then $P_e = P_r(Z_{DFE} > 0 \mid b[n] = -1)$ (as for 4.c))

$$= \frac{1}{4} \sum_{b_{n-1}} \sum_{b_{n+1}} P_r(Z_{DFE} > 0 \mid b[n-1], b[n+1], b[n])$$

$$= \frac{1}{4} \sum_{b_{n-1}} \sum_{b_{n+1}} P_r(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] \neq b[n-1]) \cdot P_e + (1-P_e) P_r\left(Z_{DFE} > 0 \,\middle|\, \begin{matrix} b_{n-1}, b_n, b_{n+1} \\ \hat{b}[n-1] = b[n-1] \end{matrix}\right)$$

$$\Leftrightarrow P_e = \frac{\sum\sum P_r(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n-1])}{4 + \sum\sum \left(P_r(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n-1]) - P_r(Z_{DFE} > 0 \mid b_n, b_{n-1}, b_{n+1}, \hat{b}[n-1] \neq b[n-1]\right)}$$

if $\hat{b}[n-1] = b[n]$: $P_r(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}) = P_r\left(C_{zg}^H w > -C_{FF}^H U_g \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} \,\middle|\, b_n, b_{n-1}, b_{n+1}\right)$

Using the arguments of 4.c):

$$P_r(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n]) = Q\left(\frac{-C_{FF}^H U_g \begin{pmatrix} b[n] \\ b[n+1] \end{pmatrix}}{\sqrt{C_{FD}^H C_w C_{FD}}}\right) \quad (1)$$

if $\hat{b}[n-1] \neq b[n-1] \Leftrightarrow \hat{b}[n-1] = -b[n-1]$

Then: $\Pr(z_{DFE} > 0 \mid b_n, b_{n-1}, b_{n+1}, \hat{b}[n-1] \neq b[n-1]) = \Pr\left(C_{zf}^H w > -C_{zf}^H U_f \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} + 2C_{FB}\, b[n-1] \,\middle|\, \begin{matrix} b_n, b_{n+1} \\ b_{n-1} \end{matrix}\right)$

$$= Q\left(\frac{-C_{FF}^H U_f \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} + 2C_{FB}\, b[n-1]}{\sqrt{C_{FF}^H C_W C_{FF}}}\right)$$

Then $P_e =$

$$\frac{\sum\sum Q\left(\frac{-C_{FF}^H U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix}}{\sqrt{C_{FF}^H C_W C_{FF}}}\right)}{4 + \sum\sum\left(Q\left(\frac{-C_{FF}^H U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix}}{\sqrt{C^{AF} C_W C_{FF}}}\right) - Q\left(\frac{-C_{FF} U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix} + 2C_{FB}\, b[n-1]}{\sqrt{C_{FF}^H C_W C_{FF}}}\right)\right)}$$