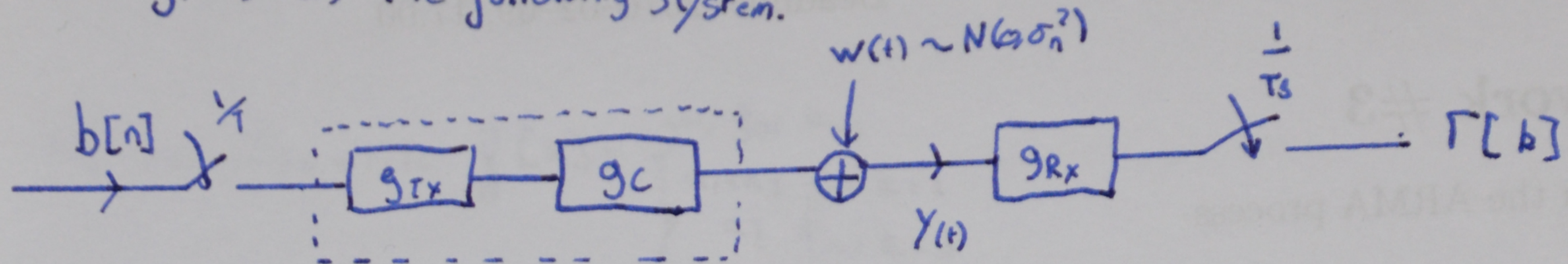
Problem 4. As we don't know our student letter, all the calculations will be held with a_1, a_2.

a) The symbol rate gives $T = 2$, $g_{Tx} = I_{[0;2]}$

$$g_c(t) = a_1 \delta_0(t-1) + a_2 \delta_0(t-2)$$

$$g_{Rx} = I_{[0,1]}$$

It gives us the following system.



$$p(t) = g_{Tx} * g_c(t)$$

We have $p(t) = g_{Tx} * g_c(t) = I_{[0;2]} * \left(a_1 \delta_0(t-1) + a_2 \delta_0(t-2)\right)$

$$\underline{p(t) = a_1 I_{[0;2]}(t-1) + a_2 I_{[0;2]}(t-2)}$$

We then can write $r[k] = y * g_{r_x}(k) = \int_{-\infty}^{+\infty} g_{r_x}(k-t)\, y(t)\, dt = \int_{k-1}^{k} y(t)\, dt$ since $T_S = 1$

$$= \int_{k-1}^{k} \sum_n b[n]\, p(t-2n)\, dt \quad + w[k]$$

$\hookrightarrow w * g_{Rx}(k T_s)$

$$\underline{r[k] = \sum_n \left(\int_{k-1}^{k} p(t-2n)\, dt\, b[n] \right) + w[k]} \quad (1)$$

Then $w[k] = w * g_{Rx}(t - kT_S)$. $E[w[k]] = E[w] * g_{Rx} = 0$

and $Cov[w[n], w[n+k]] = \sigma_n^2 \int_D g_{R_x}(t)\, g_{R_x}(t - kT_S)\, dt$

$\llcorner$ domain where $g_{R_x}(t)$ and $g_{R_x}(t-kT_S)$ overlapp

Hence, $\underline{Cov[w[n], w[n+k]] = \sigma_n^2 \delta_0(k)}$

So $w[k]$ is white zero mean with variance σ_n^2

b) Starting from (1):

$$r[k] = \sum_n b[n] \int_{k-1}^{k} a_1 I_{[0;2]}(t-2n-1) + a_2 I_{[0;2]}(t-2n-2)\, dt + w[k]$$

Hence as $t \in [k-1;k]$ The integral is non negative for:

$k \in \{2n+2; 2n+3; 2n+4\}$

So r can take the values: $\begin{cases} a_2 b[n-1] + a_1 b[n] + w[k] \\ (a_1 + a_2) b[n] + w[k] \\ a_1 b[n+1] + b[n] \cdot a_2 + w[k] \end{cases}$

Then taking the $L=3$ block:

$$\underline{\Gamma}[n] = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix} b[n-1] + \begin{pmatrix} a_1 \\ a_1+a_2 \\ a_2 \end{pmatrix} b[n] + \begin{pmatrix} 0 \\ 0 \\ a_1 \end{pmatrix} b[n+1] + \underline{w}[n]$$

$$\underline{\Gamma}[n] = \underbrace{\begin{pmatrix} a_2 & a_1 & 0 \\ 0 & a_1+a_2 & 0 \\ 0 & a_2 & a_1 \end{pmatrix}}_{U} \underline{b}[n] + \underline{w}[n]$$

↳ $\underline{w}$ is a zero mean gaussian vector with $C_w = \sigma_n^2 I_3$

We can also write $g[k] = \begin{cases} a_1 & \text{for } k=1 \\ a_1+a_2 & \text{for } k=2 \\ a_2 & \text{for } k=3 \\ 0 & \text{otherwise} \end{cases}$

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c) $z[n] = \underline{c}^H \underline{\Gamma}[n] = c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} + c^H \underline{w}[n]$

and our decision is $\hat{b}[n] = \text{sign}(z[n])$

Then the probability of error, $P_e = \frac{1}{2} P_r(z[n]>0 \mid b[n]=-1) + \frac{1}{2} P_r(z[n]<0 \mid b[n]=+1)$

$P_e = P_r(z[n]>0 \mid b[n]=-1)$ due to symmetry.

$$= P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| b(n)=-1\right)$$

$$= \frac{1}{4}\Bigg(P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| b[n-1]=-1, b(n)=-1, b[n+1]=-1\right) + P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=1 \\ b[n]=-1 \\ b[n+1]=-1 \end{matrix}\right)$$

$$+ P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=-1 \\ b[n]=1 \\ b[n+1]=1 \end{matrix}\right) + P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1]=1 \\ b[n]=-1 \\ b[n+1]=1 \end{matrix}\right)\Bigg)$$

$$P_e = \frac{1}{4} \sum_{b[n-1]} \sum_{b[n+1]} P_r\left(c^H[w[n]] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b(n-1) \\ b[n]=-1 \\ b[n+1] \end{matrix}\right)$$

As $\underline{w}$ is gaussian $c^H w$ is also gaussian with $\sigma^2 = c^H C_w c = E[c^H w (c^H w)^H] = c^H E[w w^H] c$

Then $P_r\left(c^H w[n] > -c^H U \begin{bmatrix} b[n-1] \\ b[n] \\ b[n+1] \end{bmatrix} \middle| \begin{matrix} b[n-1] \\ b[n] \\ b[n+1] \end{matrix}\right) = Q\left(\frac{-c^H U [b[n-1], b[n], b[n+1]]^T}{\sqrt{c^H C_w c}}\right)$

$$P_c = \frac{1}{4} \sum_{b[n-1]\in\{\pm 1\}} \sum_{b[n+1]\in\{\pm 1\}} Q\left(\frac{-c^H U[b[n-1], b[n], b[n+1]]}{\sqrt{c^H C_w c}}\right)$$

4) d) Let's design a ZF-DFE

Let u^g be $[u_d u_1] = \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix}$, The The Z_g corelator is given by:

$$C_{Zg} = u_g \left(u_g^H u_g \right)^{-1} e \quad \text{where } e = (1\,0)^T$$

Then $C_{Zg} = \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} a_1^2+a_2^2+a_1a_2 & a_1a_2 \\ a_1a_2 & a_1^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$C_{Zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{a_1}{a_2} \\ -\frac{a_1}{a_2} & \frac{2(a_1^2+a_1a_2+a_2^2)}{a_1^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_{Zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 & 0 \\ a_1+a_2 & 0 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{a_1}{a_2} \end{pmatrix}$$

$$C_{Zg} = \frac{1}{2a_1^2+a_1a_2+a_2^2} \begin{pmatrix} a_1 \\ a_1+a_2 \\ a_2 - \frac{a_1^2}{a_2} \end{pmatrix}$$

The Feedback correlator is then: $-C_{Zg}^H U[\cdot 1] = \frac{1}{2a_1^2+a_1a_2+a_2^2}(-a_1a_2)$

Then: $Z_{DFE} = C_{FF}^H \underline{r}^g[n] + C_{FB}\left(\hat{b}(n-1) - b(n-1)\right)$

$$Z_{DFE} = C_{FF}^H U^g \begin{pmatrix} b[n] \\ b[n+1] \end{pmatrix} + C_{FB}\left(\hat{b}(n-1) - b(n-1)\right)$$

Then $P_e = P_r(Z_{DFE} > 0 \mid b[n] = -1)$ (as for 4.c))

$$= \frac{1}{4} \sum_{b_{n-1}} \sum_{b_{n+1}} P_r\left(Z_{DFE} > 0 \mid b[n-1], b[n+1], b[n]\right)$$

$$= \frac{1}{4} \sum_{b_{n-1}} \sum_{b_{n+1}} P_r\left(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] \neq b[n-1]\right) \cdot P_e + (1-P_e)\, P_r\left(Z_{DFE} > 0 \,\middle|\, \begin{array}{l} b_{n-1}, b_n, b_{n+1} \\ \hat{b}[n-1] = b[n-1] \end{array}\right)$$

$$\Leftrightarrow P_e = \frac{\sum\sum P_r\left(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n-1]\right)}{4 + \sum\sum\left(P_r\left(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n-1]\right) - P_r\left(Z_{DFE} > 0 \mid b_n, b_{n-1}, b_{n+1}, \hat{b}[n-1] \neq b[n-1]\right)\right)}$$

if $\hat{b}[n-1] = b[n]$: $P_r\left(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}\right) = P_r\left(C_{Zg}^H w > -C_{FF}^H u_g \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} \,\middle|\, b_n, b_{n-1}, b_{n+1}\right)$

Using the arguments of 4.c):

$$P_r\left(Z_{DFE} > 0 \mid b_{n-1}, b_n, b_{n+1}, \hat{b}[n-1] = b[n]\right) = Q\left(-\frac{C_{FF}^H u_g \begin{pmatrix} b[n] \\ b[n+1] \end{pmatrix}}{\sqrt{C_{FF}^H C_w C_{FF}}}\right) \quad (2)$$

if $\hat{b}[n-1] \neq b[n-1] \Leftrightarrow \hat{b}[n-1] = -b[n-1]$

Then: $Pr\left(z_{DFE} > 0 \mid b_n, b_{n-1}, b_{n+1}, \hat{b}[n-1] \neq b[n-1]\right) = Pr\left(C_{zf}^{H} w > -C_{zf}^{H} U_f \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} + 2 C_{FB}\, b[n-1] \,\middle|\, \begin{matrix} b_n, b_{n+1} \\ b_{n-1} \end{matrix}\right)$

$$= Q\left(\frac{-C_{FF}^{H} U_f \begin{bmatrix} b[n] \\ b[n+1] \end{bmatrix} + 2 C_{FB}\, b[n-1]}{\sqrt{C_{FF}^{H} C_W C_{FF}}}\right)$$

Then $$P_e = \frac{\sum\sum Q\left(\frac{-C_{FF}^{H} U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix}}{\sqrt{C_{FF}^{H} C_W C_{FF}}}\right)}{4 + \sum\sum\left(Q\left(\frac{-C_{FF}^{H} U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix}}{\sqrt{C^{FF} C_W C_{FF}}}\right) - Q\left(\frac{-C_{FF} U_f \begin{bmatrix} -1 \\ b[n+1] \end{bmatrix} + 2 C_{FB}\, b[n-1]}{\sqrt{C_{FF}^{H} C_W C_{FF}}}\right)\right)}$$