1   Quantum Thermometer [1, 2]
Leon Zaporski - University of Cambridge

In the exercise below you will study the interaction of an electron's spin with a reservoir of many other spins. Towards the end, you will find out how to use the electron as a quantum probe of the reservoir's temperature. If you have not studied quantum information before (but know all about Pauli matrices - like a physicist should do), you can still get full marks here, as there are hints and guides along the lines.

An important tool for describing the state of a system in quantum statistical mechanics is the density operator. Here we list some *potentially* useful facts about density operators:

- To a system in a pure quantum state $|\psi\rangle$ we can assign the operator $\rho = |\psi\rangle\langle\psi|$. However, sometimes we wish to consider the system as being prepared in one of many quantum states $|\psi_n\rangle$ with a *classical* probability p_n. The density operator of such a system is:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

Often, this arises from treating the system together with its reservoir as being in a pure quantum state $\rho_{S+R} = |\psi_{S+R}\rangle\langle\psi_{S+R}|$, and then tracing out the reservoir's degrees of freedom, i.e.:

$$\rho_S = \sum_n \langle R_n| \rho_{S+R} |R_n\rangle$$

where the sum is over all basis states of the reservoir's Hilbert space.

- Density operators for closed systems evolve in time according to :

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho]$$

where H stands for the Hamiltonian.

- For any observable A, its expectation value in a state described by a density operator ρ is

$$\langle A \rangle = \sum_n \langle \psi_n| \rho A |\psi_n\rangle$$

where the sum is over all basis states or Hilbert space in which ρ is defined.

(a) (1 point) Verify, that for a time-independent Hamiltonian H, the density operator ρ for the system and reservoir at time t is given by:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}.$$

(b) (4 points) At $t = 0$ the electron is in the state $|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$. It interacts with a reservoir of spins, that is initially in the state:

$$\rho_R(0) = \int_{-\infty}^{\infty} dI_z p(I_z) |I_z\rangle\langle I_z|,$$

where I_z is a projection of the bath's total spin on the z-axis (using the continuum approximation). At $t = 0$ the density operator of the electron and reservoir is a tensor product of their respective density operators.

Find the *electron's* density operator at time $t = \tau$, if:

$$H = \hbar\omega_e S_z + \hbar\omega_R I_z + \hbar A S_z I_z.$$

Confirm that its off-diagonal elements gain prefactors $e^{\pm i\omega_e \tau} \int_{-\infty}^{\infty} dI_z p(I_z) e^{\pm i A I_z \tau}$ with respect to the state at $t = 0$. Evaluate these for

$$p(I_z) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left\{-\frac{I_z^2}{2\Delta^2}\right\}.$$

(c) (1 point) Now consider an electron that evolves only under a laser drive $H_D = \frac{\hbar}{2}\vec{\Omega}\cdot\vec{\sigma}$, where $\vec{\sigma}$ is a vector of Pauli matrices[1]. After identifying $|\uparrow\rangle$ with $(1,0)^T$, and $|\downarrow\rangle$ with $(0,1)^T$, it is true, that at all times:

$$\rho_e(t) = \frac{1}{2}\left(\mathbb{1} + \vec{v}(t)\cdot\vec{\sigma}\right). \tag{$\star$}$$

Prove that:

$$\dot{\vec{v}} = \vec{\Omega} \times \vec{v}.$$

What does it mean?

(d) (3 points) Finally, consider the following scenario: an electron, initially in state $|\uparrow\rangle$, is driven by a laser pulse sequence:

$$H_D(t) = \begin{cases} \frac{\hbar}{2}\Omega\sigma_y, & 0 < t < \frac{\pi}{2\Omega} \\ 0, & \frac{\pi}{2\Omega} < t < \frac{\pi}{2\Omega} + \tau \\ \frac{\hbar}{2}\Omega\sigma_y, & \frac{\pi}{2\Omega} + \tau < t < \frac{\pi}{\Omega} + \tau. \end{cases}$$

On top of that, it interacts with the reservoir of spins, as in part (b), but $A, \omega_R, \omega_e \ll \Omega$. At $t = \tau + \frac{\pi}{\Omega}$ the electron's spin is measured in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.

Find the electron's *Bloch vector* $\vec{v}$ from Equation ($\star$) after each of the three pulses, but before the measurement (hint: drawing it on/within the sphere may help). Relate the components of $\vec{v}$ to $\langle S_z\rangle$. Plot $\langle S_z\rangle$ as a function of τ for $A = 0$ and $A \neq 0$.

(e) (1 point) Suggest the way to infere the temperature of the reservoir of spins based on measurements of $\langle S_z\rangle$ with the above protocol.

References

[1] G. Éthier-Majcher, D. Gangloff, R. Stockill, E. Clarke, M. Hugues, C. Le Gall, and M. Atatüre. Improving a solid-state qubit through an engineered mesoscopic environment. *Phys. Rev. Lett.*, 119:130503, Sep 2017.

[2] D. A. Gangloff, G. Éthier-Majcher, C. Lang, E. V. Denning, J. H. Bodey, D. M. Jackson, E. Clarke, M. Hugues, C. Le Gall, and M. Atatüre. Quantum interface of an electron and a nuclear ensemble. *Science*, 364(6435):62–66, 2019.

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[1]In the lab frame, this is a valid approximation for strong laser pulses, perfectly tuned to transition frequency, and lasting way shorter than the intrinsic timescales associated with the system dynamics.

2   An oscillation and a gamma function
Marios Kalomenopoulos - University of Edinburgh

An empty container of mass M is connected to a horizontal elastic spring of spring constant k, and performs simple harmonic motion with amplitude A_0, on a smooth surface. The total energy of the system is E_0 and the frequency of oscillation is ω_0 (with $E_0/\omega_0 = 1$). At some point onward, the mass of the container is consecutively increased by Δm every time it passes through the equilibrium position **and** the extreme positions. The mass increment is a result of a plastic collision of the container M and small masses Δm that fall vertically from a suitable position.

You may assume that the time of collision is negligible (at the instances where the mass M passes through the equilibrium position $x = 0$ and the extreme positions $x = \pm A$).

Let E_n and ω_n the energy and frequency of the oscillator after the $n-$th collision. After a lot of increments ($n \gg 1$):

(a) (5 points) With $J_n = E_n/\omega_n$ and $\lambda = \Delta m/M$ show the following:

$$J_{2n+1} = J_0 K \sqrt{1 + (2n + 1)\lambda}$$
$$J_{2n+2} = J_0 K \sqrt{1 + (2n + 2)\lambda},$$

where

$$K = \frac{1 + 2\lambda}{1 + \lambda} \frac{1 + 4\lambda}{1 + 3\lambda} \cdots \frac{1 + 2n\lambda}{1 + (2n - 1)\lambda} \frac{1}{1 + (2n + 1)\lambda}.$$

(b) (5 points) Show that the energy and the frequency are related by:

$$E_n = \frac{\Gamma(u + \frac{1}{2})}{\sqrt{u}\,\Gamma(u)}\,\omega_n,$$

where $u = \frac{M}{2\Delta m}$.

Assume that the first mass increase happens at the equilibrium position.

Hints:

- $\Gamma(z + 1) = z\Gamma(z) = z! = z(z - 1)!$
- $\Gamma(z + n) = (n - 1 + z)(n - 2 + z)...(1 + z)\Gamma(1 + z)$
- $\Gamma(2x) = \frac{1}{\sqrt{2\pi}} 2^{2x-1/2} \Gamma(x)\Gamma(x + 1/2)$
- $\lim\limits_{x \to \infty} \frac{\Gamma(x+a)x^{b-a}}{\Gamma(x+b)} = 1$

3   Minimal breakdown voltage
Dr. Bartlomiej Waclaw - University of Edinburgh

In this problem we will calculate the minimum voltage, $U_{\min}$, that will cause electrical break-down (a spark) in air at pressure $p = 100\text{kPa}$ (1 atm) between two spherical electrodes of radius 1 cm, and the separation distance d (surface-to-surface) between the spheres at which the voltage is minimized.

(a) (3 points) Consider first two parallel plates with distance d and voltage U between them. Electrons present between the plates will accelerate towards the anode. Upon collision with gas molecules, the electrons with sufficient energy will cause impact ionization of the gas which will release more electrons. Additionally, ions colliding with the cathode will release secondary electrons. Show that the relationship between the number α of electrons per unit length that must be produced through impact ionization to sustain the discharge, the number γ of secondary electrons released from the cathode per ion, and the distance d between the plates is

$$\alpha d = ln(1 + 1/\gamma)$$

(b) (3 points) Find α by considering the probability of impact ionization (electron energy larger than the ionization energy E_i). Assume the cross-section for impact ionization is constant and equal σ.
Hint: Consider the probability $P(\Delta)$ that an electron travels *at least* the distance Δ between collisions with gas molecules. Argue that this probability is $P(\Delta) = e^{-\Delta/\lambda}$, where λ is the mean free path between collisions that can be expressed through σ and p (use the ideal gas law). Write the minimum distance Δ for which the electron gains enough energy to ionize a gas molecule as a function of U, d, and calculate α as the probability density (per unit length) of the electron having at least this energy.

(c) (2 points) Combine the two results and show that the breakdown voltage for the parallel plates case has the form

$$U = \frac{Apd}{\ln(Bpd) + C}$$

with constants A, B, C depending on the aforementioned parameters (E_i, σ, γ) and $k_B T$.

(d) (1 point) Find the minimum of $U(d)$ with respect to d (still assuming parallel plates). This should give you the distance at which the required voltage is the smallest.

(e) (1 point) What (if anything) needs to be modified to solve the case of two spherical electrodes?

Approximate solutions are permitted. The final result for $U_{\min}$ and $d_{\min}$ must be accurate to +/-10 %. Assume the following parameters: $k_B T = 4.1 \times 10^{-21}\text{J}$, $\gamma = 0.01$, $\sigma = 4.6 \times 10^{-20}\text{m}^2$, $E_i = 24\text{eV}$.

4   Chern-Simons Electrodynamics in (3+1)D [1]
Christos Kourris - University of Edinburgh

In this problem we will study a Lorentz violating theory for electromagnetism by introducing an extra term to the Maxwell Lagrangian $\mathcal{L}_{EM}$,

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Before that, let us review some of the relativistic notation:

In Lorentz-Heaviside units, the Maxwell equations take the form:

$$\underline{\nabla} \cdot \underline{E} = \rho \qquad\qquad\qquad \underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \qquad\qquad\qquad \underline{\nabla} \times \underline{B} = \frac{1}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

The Maxwell equations can be solved if one is able to find a scalar potential, ϕ and a vector potential $\underline{A}$, such that,

$$\underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t} \qquad\qquad \underline{B} = \underline{\nabla} \times \underline{A}.$$

Using the wave operator, $\partial^2 = -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, and working in Lorenz-gauge $(\partial_\mu A^\mu(x) = 0)$ for charge and current sources ρ and $\underline{J}$, the potentials are solved by

$$\partial^2 \phi = \rho \qquad\qquad \partial^2 \underline{A} = \frac{1}{c}\underline{J}$$

In a more compact notation, $J^\mu = (\rho c, \underline{J})$ and $A^\mu = (\phi(\underline{x}, t), \underline{A}(\underline{x}, t))$, the above become

$$\partial^2 A^\mu = J^\mu/c$$

We define the electromagnetic field tensor as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

so that $E^i = F^{i0}$ and $F^{ij} = -\epsilon^{ijk}B^k$ for $i, j, k = 1, 2, 3$. The dual field strength tensor is defined as

$$F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

The modification involves adding an extra term to the Lagrangian, known as the *Chern-Simons* term, $\mathcal{L}_{CS}$, so that it becomes:

$$\mathcal{L}_p = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}p_\alpha A_\beta F^{*\alpha\beta},$$

where $F^{*\alpha\beta}$ is the dual electromagnetic tensor, A_β is the electromagnetic four-potential and $p_\alpha = (p^0, \mathbf{p})$ is a 4-vector, which couples to the EM field. In a sense, p_α parametrizes the violation of the

Lorentz invariance. For the rest of the question we set $c = 1$.

(a) (1 point) Show that, in Lorenz gauge, Maxwell's equations are written as

$$\partial_\mu F^{\mu\nu} = J^\nu$$
$$\partial_\mu F^{*\mu\nu} = 0$$

(b) (2 points) By considering $A_\mu \to A_\mu + \partial_\mu \chi$, find $\delta \mathcal{L}_{CS}$ and discuss the conditions under which $\mathcal{L}_{CS}$ is gauge invariant.

Hint: You can ignore boundary contributions when calculating the variation of $\mathcal{L}_{CS}$

(c) (2 points) Show that the field equations for $\mathcal{L}_p$, for a general source current four-vector J^ν are given by

$$\partial_\mu F^{\mu\nu} = J^\nu + p_\mu F^{*\mu\nu}$$

and argue that the Chern-Simons addition preserves gauge invariance but violates Lorentz invariance.

We now seek for wave solutions to the source-free (i.e. $\rho = \mathbf{J} = 0$) field equations.

(d) (3 points) Show that the dispersion relations for a propagating wave with a general wave vector $k^\alpha = (\omega, \mathbf{k})$ are

$$(k^\alpha k_\alpha)^2 + (k^\alpha k_\alpha)(p^\beta p_\beta) = (k^\alpha p_\alpha)^2$$

Hint: Write first the equations above - (c) - in terms of the E, B fields.

(e) (2 points) Notice that the above dispersion relation gives two modes of propagation. Take $p_\alpha = (m, 0, 0, 0)$ to be a timelike vector. Find the dispersion relation for that case and discuss what happens at long wavelengths.

The mass of the photon was introduced as a Chern-Simons parameter $m = (p_\alpha p^\alpha)^{1/2}$. Progressing further, one can show that there is a phase difference between the two modes of propagation which can be used to put an upper bound to the mass of the photon:

$$m \leq 10^{-26} \text{ GeV}$$

References

[1] Carroll, Sean M. and Field, George B. and Jackiw, Roman. *Limits on a Lorentz and Parity Violating Modification of Electrodynamics*. Phys. Rev. D41, 1990

5   Gravitational Wave Detection
Prof. Avery Meiksin - University of Edinburgh

The Laser Interferometer Gravitational (Wave) Observatory (LIGO) spectacularly discovered its first gravitational wave source in September 2015, and since then it and VIRGO have been seeing events at regular intervals. In this question, the types of sources are characterized and the requirements for the detectors are estimated.

According to General Relativity, gravity is represented by a modification to the flat spacetime Minkowski metric. For a weak perturbation, with $|h^{\alpha\beta}| \ll 1$,

$$c^2 d\tau^2 \simeq -c^2 dt^2 + dx^2 + dy^2 + dz^2 + h_{\alpha\beta} dx^\alpha dx^\beta,$$

where $x^0 = ct$ is time and the x^i for $i = 1$, 2 and 3 correspond to the spatial coordinates x, y and z. Here, $cd\tau$ is the actual length measured, while the remaining differentials are in terms of the unperturbed coordinates t, x, y and z, which are unaffected by $h^{\alpha\beta}$. The Einstein summation convention is understood, summing indices α and β over 0, 1, 2 and 3.

A system with a changing second mass moment I, with spatial components

$$I^{ij} = c^2 \int d^3x \, \rho(t, \vec{x}) x^i x^j,$$

where $\rho(t, \vec{x})$ is the mass density, will radiate gravitational waves according to

$$h^{ij}(r, t) \simeq -\frac{2}{r} \frac{G}{c^6} \ddot{I}^{ij}$$

at large distances r, and $\ddot{I}^{ij}$ is a function of $\left(t - \frac{r}{c}\right)$.

(a) (4 points) Consider two black holes of equal mass M in a circular orbit of radius R in the $x - y$ plane, with an angular frequency Ω. Treating the black holes as point masses, find the second mass moment of the system and the metric fluctuation spatial components h^{ij}.
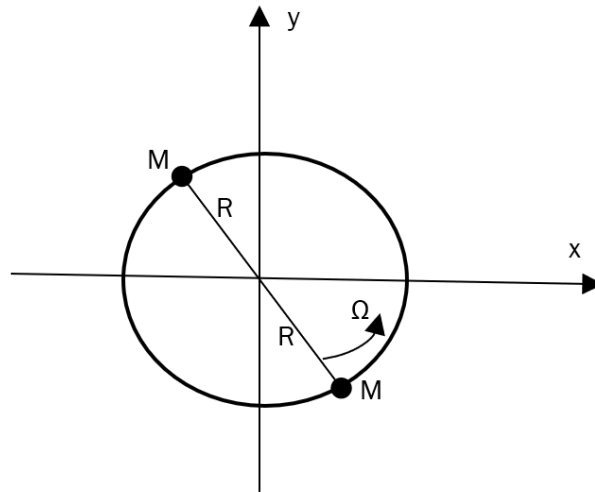
(b) (2 points) The first event is believed to arise from the merger of two black holes of masses about 30 solar masses each. The gravitational wave detected was sinusoidal in shape, with a

frequency of about 150 Hz when the black holes merged. Estimate the size of the binary black hole system.

(c) (1 point) The black holes are estimated to have been at a distance of about a billion light years. Estimate how large the metric fluctuation h^{xx} was in the x-direction when the black holes merged.

(d) (2 points) Consider an interferometer design with two arms at right angles, and a half-silvered mirror at their intersection that functions as a beam splitter. A laser beam shines along one arm and is split by the beam splitter, half continuing along the arm and the other continuing in one direction along the perpendicular arm. At the end of each arm is a mirror which reflects the laser beam back to the beam splitter, where the beams are recombined, producing an interference pattern whenever the total lengths travelled by the beams are not the same.
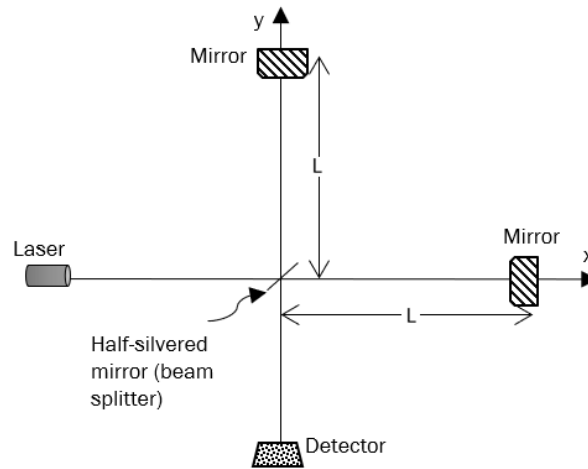
Sketch the expected intensity of the interference pattern in terms of the difference in path lengths $\delta L/\lambda$, measured in units of the wavelength λ of the laser beam light.

(e) (1 point) How small a fractional change in length $\delta L_x/L_x$ must LIGO be sensitive to in the x-direction to have detected the black hole merger event? Using multiple reflections, the effective length of each arm in the detector is $L_x \simeq 1200$ km. How small a change in length is LIGO sensitive to? How small a fraction of a wavelength of optical light does this correspond to?

Some useful constants:
$c = 2.9979 \times 10^8 \, \mathrm{m\,s^{-1}}$
$G = 6.67 \times 10^{-11} \, \mathrm{m^3\,kg^{-1}\,s^{-2}}$
solar mass $M_\odot = 1.989 \times 10^{30}$ kg
light-year $= 9.46 \times 10^{15}$ m

6   Quantum Spin Chain
Contributed by the PLANCKS Preliminary in Portugal

Consider a one-dimensional chain of N spin-1/2 particles coupled through the Hamiltonian

$$H = J \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1},$$

where $\vec{S} = (S_x, S_y, S_z)$ are the usual spin operators for a spin-1/2 particle, $J > 0$ is a *positive* constant and $N \gg 1$.

In a famous 1931 paper, Hans Bethe showed that for this Hamiltonian, the ground state energy per particle $E_{GS}/N \equiv E_0$ was equal to $-\hbar^2 J(\log 2 - \frac{1}{4}) \approx -0.433\hbar^2 J$. We will not reproduce this result in this exercise but we will determine upper and lower bounds for the ground state energy per particle, by restricting our attention to spins in the directions $|\uparrow\rangle$ and $|\downarrow\rangle$.

(a) (2 points) If the spin operators are treated as classical spin vectors with $|\vec{S}| = \hbar/2$, what is the ground state spin configuration and what is the ground state energy per particle E_0?

(b) (5 points) Consider the trial wave function

$$|\Psi\rangle = \bigotimes_{i=odd} |i, i+1\rangle_0 = |1, 2\rangle_0 \otimes |3, 4\rangle_0 \otimes \cdots \otimes |N-1, N\rangle_0 ,$$

where $|i, j\rangle_0$ is the spin singlet state formed form the spins on sites i and j:

$$|ij_0\rangle = \frac{1}{\sqrt{2}} (|i_\uparrow j_\downarrow\rangle - |i_\downarrow j_\uparrow\rangle) .$$

Use this state and the variational method to find an **upper** bound on E_0.

(c) (3 points) Prove the following **lower** bound on E_0:

$$E_0 \geq -\frac{3}{4}\hbar^2 J$$

7 Fermions in a Square Well
Prof. Luigi Del Debbio - The University of Edinburgh

Consider N free identical particles of spin $1/2$ confined in a three-dimensional square infinite potential well of size L. The minimum of the potential is $-U$, with $U > 0$.

(a) (2 points) Show that the energy eigenstates of each particle can be described by three integers n_x, n_y, n_z. Find the corresponding energy eigenvalues E_{n_x,n_y,n_z}.

(b) (3 points) Consider now the ground state of the system. Describe the state of each electron in the cases where $N = 2$ and $N = 9$. Deduce a general expression for N as a sum of single-particle states.

   Hint: No closed form is required for the last part.

(c) (5 points) Assuming very large N and L, and assuming spherical symmetry of the ground state, find the energy of the most energetic particle in the ground state. Show that it is an *intensive* quantity.

8   The colour of the Sun in a 2-dimensional world
Prof. Alexander Morozov - The University of Edinburgh

(a) (10 points) What would the colour of the Sun be if our world would be two-dimensional? Assume that the surface temperature of the Sun would not change.

Hint: Consider a gas of photons in a square cavity with the side L.